Title
Scalar conformal invariants of hypersurfaces

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Publication Date
2016

UNIVERSITY OF CALIFORNIA

SANTA CRUZ

SCALAR CONFORMAL INVARIANTS OF HYPERSURFACES

A dissertation submitted in partial satisfaction

of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

by

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September 2016

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SCALAR CONFORMAL INVARIANTS OF HYPERSURFACES

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Abstract

For a hypersurface in a conformal manifold, by following the idea of Fefferman and Graham's work, we use the conformal Gauss map and the conformal transform to construct the associate hypersurface in the ambient space. By evaluations of scalar Riemannian invariants of associate hypersurface, we find out a way to construct and collect scalar conformal invariants of the given hypersurface. This method provides chances for searching higher order partial differential equations which are similar like the Willmore equation.

1. Introduction

The first issue that we concern in the study of differential geometry is local invariant. Invariant theories has been well studied in Riemannian geometry, for instant, all local Riemannian invariants on a Riemannian manifold are combinations of Riemannian curvature and its covariant derivatives. We can also consider the case for hypersurface, then the local Riemannian invariants would be combinations from ambient curvature tensor, the first and second fundamental forms and their derivatives. There are also corresponding theories for conformal invariants of Rienmannian manifold and hypersurfaces [1-5]. The main result of scalar conformal invariants came from Fefferman and Graham's work [6-7], they realized that the conformal group is also the orthogonal group for Minkowski space-time and this observation provides chance to apply Weyl invariant theorem for orthogonal groups. The constructions of ambient space and ambient metric in their work are powerful tools in the study of scalar invariants in conformal geometry nowadays. In this article we extend the idea of Fefferman and Graham into the study of scalar conformal invariants of hypersurface. Following the homogeneity of conformal Gauss map we set up the associated hypersurface and we find out a nature way to collect and evaluate scalar conformal invariants of hypersurface in high order.

The structure of this article is the following. First we review the construction of ambient space and ambient metric from Fefferman and Graham's work. Then we introduce the new construction of associate hypersurface for any given hypersurface, explain how scalar Riemannian invariants of associate hypersurface can provide scalar conformal invariants of original hypersurface. At the end of the article, we evaluate the general formula for Willmore curvature and some other high order scalar conformal invariants. It is well known that the study of Wiilmore conjecture is a big issue in the filed of geometric analysis [14-16]. We hope we have chance to get similar partial differential equations which like Willmore equation and get further understanding on the conformal geometry of hypersurfaces.

2. Ambient space construction from Fefferman and Graham's work

2.1. **Conformal structure and metric bundle.** Let M^n be a n-dimensional smooth manifold, there is a abstract way to specify its conformal structure: to consider the class $[g]$ of metrics as a subbundle of the bundle of symmetric 2-tensors on M^n. We denote the subbundle of $[g]$ by $\mathcal{G}$ and call $\mathcal{G}$ the metric bundle of the conformal manifold $(M^n, [g])$. The bundle structure is given as:

$$\pi : \mathcal{G} \to M, \ \pi(g|_p) = p.$$

and reflected by the fact that there is a dilation bundle map:

$$\delta_s : \mathcal{G}|_p \to \mathcal{G}|_p, \ \delta_s(g|_p) = s^2 g|_p.$$

Such metric bundle $\mathcal{G}$ is characterized by a tautological symmetric 2-tensor g_0 on $\mathcal{G}$ such that

$$g_0(X, Y) = g|_p(\pi_* X, \pi_* Y).$$

for any $X, Y \in T_z(\mathcal{G})$, where $z = g|_p$ and $p \in M^n$. It is easily seen that $\delta_s^* g_0 = s^2 g_0$, in fact, given a conformal structure $[g]$ on M^n and a representative $g \in [g]$, we may use the following natural coordinates:

$$\mathcal{G} = \{\alpha^2 g|_p : \alpha \in \mathbb{R}^+, \ p \in M^n\} = \{(\alpha, p) : \alpha \in \mathbb{R}^+, \ p \in M^n\} = \mathbb{R}^+ \times M^n.$$

and

$$\pi(\alpha, p) = p, \ \delta_s(\alpha, p) = (s\alpha, p).$$

It is important to realize that g_0 is not Riemannian. It is insightful to see that the metric bundle for the standard conformal structure on the sphere $\mathbb{S}^n$ is the positive light cone $\mathbb{N}_1^{n+1}$ in Minkowski space-time R_1^{n+1}.

2.2. **Ambient space and ambient metric.** Suppose that $(M^n, [g])$ is a conformal manifold and $\mathcal{G}$ is its metric bundle. We then consider the space $\mathcal{G} \times \mathbb{R}$, the identification

$$i(z) = (z, 0) : \mathcal{G} \to \mathcal{G} \times \mathbb{R}, \ \forall z \in \mathcal{G},$$

and the extension of the dilation

$$\delta_s(z,\rho) = (\delta_s(z),\rho),\ \forall (z,\rho) \in \mathcal{G}\times\mathbb{R}.$$

Definition 2.1 (Pre-Ambient Space)**.** Let $\mathcal{G}$ be the metric bundle of a given conformal manifold $(M^n,[g])$. A pre-ambient space $(\tilde{\mathcal{G}},\tilde{g})$ is such that

1) $\tilde{\mathcal{G}}$ is a dilation invariant open neighborhood of the set $\mathcal{G}\times\{0\}\subset\mathcal{G}\times\mathbb{R}$,

2) $\tilde{g}$ is a Lorenzian metric such that $\delta_s^*\tilde{g} = s^2\tilde{g}$,

3) $i^*\tilde{g} = g_0$, where g_0 is the tautological metric on the metric bundle $\mathcal{G}$.

Next, on a pre-ambient space $\tilde{\mathcal{G}}$ of a conformal manifold $(M^n,[g])$, we would like to set a coordinate system in which the pre-ambient space metric is in a normal form and with which we take away the degeneracy caused by the natural symmetry of diffeomorphisms.

Definition 2.2 (Normal Coordinate)**.** Suppose $(\tilde{\mathcal{G}},\tilde{g})$ is a pre-ambient space of a given conformal manifold $(M^n,[g])$. A normal coordinate on $\tilde{\mathcal{G}}$ associated with a choice of representative $g\in[g]$ is a coordinate system such that

1) For each $z\in\mathcal{G}$, the set of all ρ such that $(z,\rho)\in\tilde{\mathcal{G}}$ is an open interval I_z.

2) For each $z\in\mathcal{G}$, the parametrized curve $\{(z,\rho)|\rho\in I_z\}$ is a geodesic for the pre-ambient space metric $\mathcal{G}$.

3) Let (α,x,ρ) be the coordinate under the choice of $g\in[g]$ on $\tilde{\mathcal{G}}$. Then on $\mathcal{G}\times\{0\}$,

$$\tilde{g}|_{\rho=0} = g_0 + 2\alpha d\alpha d\rho. \tag{2.1}$$

Assume that $x=(x_1,\cdots,x_n)$ is a coordinate system on M^n, naturally we define (α,x,ρ) as a coordinate system for $\mathcal{G}\times\mathbb{R}$, and we will also use the notation (x_0,x,x_∞) later, here $x_0=\alpha$ and $x_\infty=\rho$. First we obtain this fact: (x_0,x,x_∞) is normal if and only if

$$\tilde{g}_{0\infty}=\alpha, \tilde{g}_{i\infty}=0 \text{ and } \tilde{g}_{\infty\infty}=0. \tag{2.2}$$

We may ask for more about normal coordinate by the following lemma:

Lemma 2.3. *Assume (α, x, ρ) is the normal coordinate associated with the representative $g \in [g]$, then the following are equivalent:*

1) $\tilde{g}_{00} = 2\rho$ and $\tilde{g}_{0i} = 0$,

2) For each $p \in \tilde{\mathcal{G}}$, the orbit of dilation action $\delta_s(p)$ is a geodesic for $\tilde{g}$.

From these observations, we require the orbit of dilation action to be a geodesic. In conclusion, the ambient metric $\tilde{g}$ we expect has the following expression under normal coordinate:

$$(\tilde{g}_{IJ}) = \begin{pmatrix} 2\rho & 0 & \alpha \\ 0 & \alpha^2 g_{ij} & 0 \\ \alpha & 0 & 0 \end{pmatrix}. \tag{2.3}$$

Here $g_{ij} = g_{ij}(x, \rho)$ and $g_{ij}(x, 0) = g_{ij}$.

We call this normal coordinate as homogeneous normal coordinate. Now we are ready to define what is an ambient space for a given conformal manifold. Basically an ambient space is a Ricci-flat pre-ambient space at $\rho = 0$, when the dimension of manifold is even, we could only ask $\tilde{g}$ to be Ricci-flat to some order at $\rho = 0$.

Definition 2.4 (Ambient Space)**.** A pre-ambient space $(\tilde{\mathcal{G}}, \tilde{g})$ of a given conformal manifold $(M^n, [g])$ is said to be an ambient space if, in addition:

1) when n is odd, $\tilde{Ric} = 0$ to infinite order at $\rho = 0$, or

2) when n is even, $\tilde{Ric} = O(\rho^{\frac{n}{2}-1})$.

Definition 2.5 (Ambient-Equivalent)**.** Let $(\tilde{\mathcal{G}})_1, \tilde{g}_1)$ and $(\tilde{\mathcal{G}})_2, \tilde{g}_2)$ be two pre-ambient spaces for a conformal manifold $(M^n, [g])$. We say that they are ambient-equivalent if there is a diffeomorphism $\phi : \mathcal{U}_1 \subset \tilde{\mathcal{G}}_1 \to \mathcal{U}_2 \subset \tilde{\mathcal{G}}_2$ such that

1) both $\mathcal{U}_1$ and $\mathcal{U}_2$ are dilation invariant open neighborhood of $\mathcal{G} \times \{0\}$;

2) ϕ commutes with dilation;

3) the restriction of ϕ on $\mathcal{G} \times \{0\}$ is the identity;

4) when n is odd, $\tilde{g}_1 - \phi^* \tilde{g}_2$ vanishes to infinite order at $\rho = 0$;

4') when n is even, $\tilde{g}_1 - \phi^* \tilde{g}_2 = O(\rho^{\frac{n}{2}})$.

Fefferman and Graham show that given a conformal manifold $(M^n, [g])$, there is a unique ambient space for the conformal manifold up to a ambient-equivalent. Moreover they provided the power series formula of ρ for g_{ij} in (2.3) at $\rho = 0$, and they proved all coefficients are linear combinations of Riemannian curvature R and its covariant derivatives of any order.

Evaluation of coefficients follows by checking the Einstein equation of (2.3), after tedious calculation, one could obtain the following differential equations:

$$\tilde{R}_{ij} = R_{ij} + \rho \partial_\rho^2 g_{ij} - \rho g^{kl} \partial_\rho g_{ik} \partial_\rho g_{jl} + \frac{1}{2} \rho (g^{kl} \partial_\rho g_{kl}) \partial_\rho g_{ij} \tag{2.4}$$

$$- \frac{1}{2} (g^{kl} \partial_\rho g_{kl}) g_{ij} + (1 - \frac{n}{2}) \partial_\rho g_{ij}. \tag{2.5}$$

$$\tilde{R}_{i\infty} = g^{kl} (\nabla_k \partial_\rho g_{il} - \nabla_i \partial_\rho g_{kl}). \tag{2.6}$$

$$\tilde{R}_{\infty\infty} = -\frac{1}{2} g^{kl} \partial_\rho^2 g_{kl} + \frac{1}{4} g^{kl} g^{pq} \partial_\rho g_{kp} \partial_\rho g_{lq}. \tag{2.7}$$

Notices that all left hand sides of equations turn out to be 0 if you let $\rho = 0$, these would help you find out the term $\partial_\rho g_{ij}|_{\rho=0}$. One should take higher derivatives to get the coefficients of any order. Here we only list the first two terms:

$$\partial_\rho g_{ij}|_{\rho=0} = 2P_{ij} \tag{2.8}$$

$$\partial_\rho^2 g_{ij}|_{\rho=0} = -\frac{2}{n-4} B_{ij} + 2P_i^k P_{kj}, \text{ for } n \neq 4. \tag{2.9}$$

Here

$$P_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{S}{2(n-1)}g_{ij}\right)$$

Define

$$C_{jkl} = \nabla_l P_{jk} - \nabla_k P_{jl}$$

and

$$B_{jk} = \nabla^l C_{jkl} + P^{il} W_{ijkl}.$$

is so-called Bach tensor.

Proof. The key is formula (2.4), let $\rho = 0$ there, we claim:

$$0 = R_{ij} - \frac{1}{2}(g^{kl}\partial_\rho g_{kl}|_{\rho=0})g_{ij} + (1 - \frac{n}{2})\partial_\rho g_{ij}|_{\rho=0}. \tag{2.10}$$

Now take trace for both sides, we will get

$$0 = S - \frac{n}{2}g^{kl}\partial_\rho g_{kl}|_{\rho=0} + (1 - \frac{n}{2})g^{ij}\partial_\rho g_{ij}|_{\rho=0}.$$

which forces:

$$g^{kl}\partial_\rho g_{kl}|_{\rho=0} = \frac{S}{n-1}.$$

Take this back to (2.10)

$$0 = R_{ij} - \frac{S}{n-1}g_{ij} + (1 - \frac{n}{2})\partial_\rho g_{ij}|_{\rho=0}$$

and

$$\partial_\rho g_{ij}|_{\rho=0} = \frac{2}{n-2}\left(R_{ij} - \frac{S}{2(n-1)}g_{ij}\right) = 2P_{ij}.$$

We could take derivative for (2.10) respect to ρ, then restrict $\rho = 0$, the information for second derivative is there and we skip the proof here. $\square$

We need to point out here, $\mathbb{R}^{1,n+1}$ could be view as standard model of ambient space of $\mathbb{S}^n$, since we could understand Minkowski space-time by the following:

$$g_M = -dx_0^2 + \sum_{i=1}^{n+1}|dx_i|^2.$$

In fact, Minkowski space-time is a special case of ambient space, if we choose

$$\begin{cases} \alpha & = \frac{1}{2}(|x| + x_0) \\ \alpha\rho & = (|x| - x_0) \end{cases} \tag{2.11}$$

We therefore arrive at

$$g_M = 2\rho d\alpha^2 + 2\alpha d\alpha d\rho + \alpha^2(1 + \frac{1}{2}\rho)^2 g|_{\mathbb{S}^n}.$$

and

$$g_M|_{\rho=0} = \alpha^2 g|_{\mathbb{S}^n} + 2\alpha d\alpha d\rho. \tag{2.12}$$

Check with formula (2.1), we know the light cone structure provides canonical normal coordinate.

3. Associate hypersurface

We are ready to introduction the idea of associate surface for hypersurfaces. Assume

$$\phi : N^{n-1} \to M^n$$

is a hypersurface in conformal manifold $(M^n, [g])$. First we consider:

$$\phi^{\mathbb{N}} = \alpha\phi : \mathbb{R}^+ \times N^{n-1} \to \mathcal{G}$$

for any $g \in [g]$ as a representative. Notice that the graph of the hypersurface, $\phi^N(\mathbb{R}^+ \times N^{n-1})$ is independent of choices of g. We try to find out a unit normal vector for $\phi^{\mathbb{N}}$ in ambient space $\tilde{\mathcal{G}}$.

3.1. **Conformal Gauss map.** First we need to apply special coordinates, we will use Fermi coordinate on manifold M^n which induced by ϕ, and we will use homogeneous normal coordinate on its ambient space $\tilde{\mathcal{G}}$. We use the notation $\frac{\partial}{\partial x^n}$ to be the unit normal vector for N^{n-1} in M^n, H is the mean curvature for N^{n-1} as a hypersurface.

Definition 3.1.

$$\xi|_{(\alpha,\phi,0)} = H(x)\frac{\partial}{\partial\alpha}|_{(\alpha,\phi,0)} + \alpha^{-1}\frac{\partial}{\partial x^n}|_{(\alpha,\phi,0)}. \tag{3.1}$$

We call ξ as the conformal Gauss map of hypersurface ϕ.

It is necessary to point out that ξ is independent of choice of $g \in [g]$, in other words, consider $g_\lambda = \lambda^2 g$, we obtain:

$$(\xi_\lambda)|_{(\alpha,\phi,0)} = H_\lambda(\frac{\partial}{\partial\alpha})_\lambda|_{(\alpha,\phi,0)} + \alpha^{-1}(\frac{\partial}{\partial x^n})_\lambda|_{(\alpha,\phi,0)} = \xi|_{(\lambda\alpha,\phi,0)}$$

The reason comes from the following formula:

$$H_\lambda = \lambda^{-1}(H - \frac{\partial(\log\lambda)}{\partial x^n}), \quad (\frac{\partial}{\partial\alpha})_\lambda|_{(\alpha,\phi,0)} = \lambda\frac{\partial}{\partial\alpha}|_{(\lambda\alpha,\phi,0)}$$

and

$$(\frac{\partial}{\partial x^n})_\lambda|_{(\alpha,\phi,0)} = \lambda^{-1}\frac{\partial}{\partial x^n}|_{(\lambda\alpha,\phi,0)} + \alpha\frac{\partial(\log\lambda)}{\partial x^n}\frac{\partial}{\partial\alpha}|_{(\lambda\alpha,\phi,0)}.$$

The reason that we call ξ as conformal Gauss map is because ξ is exactly the unit normal vector of $\phi^{\mathbb{N}}$ in the ambient space. On the other hand, it is easy to check $\tilde{\nabla}_{\frac{\partial}{\partial\alpha}}\xi = 0$, which tells us that ξ has homogeneity along the dilation direction.
Consider a special case $M^n = \mathbb{S}^3$ and $x : N^2 \to \mathbb{S}^3$, the ambient space becomes $\mathbb{R}^{1,4}$. In this case, conformal Gauss map ξ has special formula:

$$\xi = H(1, x) + (0, n)$$

Here n is the unit normal vector of x.
This case was first studied by Blaschke in 1929 and he gave the geometric interpretation. Consider ξ as a spacelike vector in $\mathbb{R}^{1,4}$, the corresponding timelike hyperplane has equation as following:

$$\langle (t, z), H(1, x) + (0, n)\rangle = 0$$

Take $t=1$ and $|z|=1$, then z satisfies equation:

$$\Big|z-(x+\frac{1}{H}n)\Big|^2=\frac{1}{H^2}.$$

Which tells us the trace of this hyperplane and $\mathbb{S}^3$ is the mean curvature sphere of x.

3.2. **Conformal transform.** In this section, we discuss another null vector field which is induced by conformal Gauss map, we call it conformal transform. We will continue to use Fermi coordinate in conformal manifold $(M^n,[g])$ and homogeneous normal coordinates in the ambient space $(\tilde{\mathcal{G}},\tilde{g})$.

Definition 3.2. Given homogeneous normal coordinate for the ambient space $(\tilde{\mathcal{G}},\tilde{g})$, a null vector field y^* is called conformal transform of $y=\frac{\partial}{\partial\alpha}$ if it satisfies:
(1) $y^*\perp\xi,\ y^*\perp\tilde{\nabla}_{\frac{\partial}{\partial x^i}}\xi,\ i=1,2,\dots,n-1$ (2) $\tilde{g}\Big(y^*,y\Big)=-1.$
on every point of $\phi^{\mathbb{N}}$.

In fact we can evaluate the formula of y^* from definition:

Lemma 3.3.

$$\begin{aligned} y^*|_{(\alpha,\phi,0)}=&\frac{1}{2}(H^2+|\omega|^2)y|_{(\alpha,\phi,0)}\\ &+\alpha^{-1}(\sum_{i=1}^{n-1}\omega_i\frac{\partial}{\partial x^i}|_{(\alpha,\phi,0)}+H\frac{\partial}{\partial x^n}|_{(\alpha,\phi,0)}-\frac{\partial}{\partial\rho}|_{(\alpha,\phi,0)}) \end{aligned}\tag{3.2}$$

Here

$$\omega_i=\frac{2}{|\overset{\circ}{II}|^2}\overset{\circ}{II}_{ij}(P_{jn}-\frac{\partial H}{\partial x^j})\tag{3.3}$$

$\overset{\circ}{II}$ is the traceless second fundamental form of ϕ in (M^n,g), P_{ij} is the Weyl-Schouten tensor of (M^n,g).

Need to point out that $\tilde{\nabla}_{\frac{\partial}{\partial\alpha}}y^*=0$, which means y^* is homogeneous vector filed. If we construct geodesics along y^*-direction from the surface $\phi^{\mathbb{N}}$, we will generate a space-like hypersurface in the ambient space with homogeneity, this gives us the motivation of the following constructions.

3.3. **Associate hypersurface.** We will use the method of exponential map to extend $\phi^{\mathbb{N}}$ locally and construct the associate surface for ϕ in the ambient space.

Definition 3.4.

$$\tilde{\phi}(\alpha, \phi, \beta) = \exp_{(\alpha,\phi,0)}(\alpha\beta y^*) : \mathbb{R}^+ \times N^{n-1} \times (-\epsilon, +\epsilon) \to \mathcal{G} \tag{3.4}$$

we call ϕ as the associate hypersurface for ϕ.

It is easy to see the homogeneity of the graph of $\tilde{\phi}$, since ξ is the unit normal vector, and the homogeneity of ξ provides the homogeneity of this associate hypersurface.

4. Scalar invariants of hypersurfaces

In this section we discuss the definitions of (local) scalar invariants of hypersurfaces. There are plenty of details for constructions of scalar invariants for Riemannian manifolds in Fefferman and Graham's work. They consider a scalar invariants as a polynomial of metric and its derivatives of any order, such that the value of polynomial is independent of choices of local coordinates. Here we follow the idea and build up the theories for hypersurfaces.

4.1. **Scalar Riemannian invariants.** Assume

$$\phi = \phi(x^1, \cdots, x^{n-1}) : A \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$$

is a local trivialization of ϕ in M^nCA is a domain in $\mathbb{R}^{n-1}$, then ϕ induces a local coordinate of M^n by the following:

$$\tilde{\phi} = \tilde{\phi}(x^1, x^2, \cdots, x^n) : B \subset (-\epsilon, +\epsilon) \times A \to \mathbb{R}^n$$

which satisfies

$$\phi(x^1, \cdots, x^{n-1}) = \tilde{\phi}(x^1, \cdots, x^{n-1}, 0).$$

We will use capital Latin letter for index from 1 to n, and use Latin letter for index from 1 to $n-1$. Then we could write down the ambient metric $\tilde{g}$ as:

$$\tilde{g} = \langle d\tilde{\phi}, d\tilde{\phi} \rangle = \tilde{g}_{IJ} dx^I dx^J. \tag{4.1}$$

The metric of hypersurface ϕ, or we could say, its first fundamental form is:

$$I_\phi = \langle d\phi, d\phi \rangle = \tilde{g}_{ij}\Big|_{x^n=0} dx^i dx^j. \tag{4.2}$$

Let us assume ξ is the unit normal vector, then the second fundamental form could be understood by:

$$II_\phi = -\langle d\phi, d\xi \rangle = h_{ij} dx^i dx^j. \tag{4.3}$$

Definition 4.1. A scalar (pseudo-)Riemannian invariant $I(x, M^n, g)$ for hypersurface $x : N^{n-1} \to M^n$ around a point is a polynomial in the variables that are the coordinate partial derivatives of g of any order and the reciprocal of the determinant of g, such that the value of $I(x, M^n, g)$ at this point is independent of choices of local coordinate $\tilde{\phi}$ of M^n which is induced from ϕ.

A natural way to study scalar Riemannian invariant is by using Fermi coordinates on this hypersurface. We obtain the metric has following expansion:

(4.4)

$$\begin{aligned} \tilde{g}_{IJ}(x^1, \cdots, x^{n-1}, x^n) =& \delta_{IJ} - 2h_{ij}(x^n) - \frac{1}{3}(\tilde{R}_{ikjl} + h_{il}h_{jk} - h_{ij}h_{kl})(x^k)(x^l) \\ & - 2h_{ij,k}(x^k)(x^n) + (h_{ik}h^k_j - \tilde{R}_{ninj})(x^n)(x^n) + O(|x|^3) \end{aligned}$$

By induction one can prove that, any coordinate partial derivative is a linear combination of $R_{IJKL,M_1\cdots M_s}$ and $h_{ij,k_1\cdots k_r}$. From Weyl theorem for invariants of orthogonal groups, up to a orthogonal transform, the only way to get scalar invariant from such linear combination is to take complete contraction of all indices. In conclusion, we obtain the following lemma:

Lemma 4.2. *A scalar Riemannian invariant is a linear combination of terms that are complete contractions of* $R_{IJKL,M_1\cdots M_s}$ *and* $h_{ij,k_1\cdots k_l}$.

4.2. **Scalar conformal invariants.** First let us state the definition.

Definition 4.3. Assume $\phi : N^{n-1} \to M^n$ is hypersuface and assume $[g]$ is the conformal structure. $I_c(\phi, M^n, \tilde{g})$ is called a scalar conformal invariant of ϕ, if it is a scalar Riemaian invariant, and for any positive function λ, it satisfies:

$$I_c(\phi, M^n, \lambda^2\tilde{g}) = \lambda^{-k} I_c(\phi, M^n, \tilde{g}). \tag{4.5}$$

We call k as the order of $I_c(\phi, M^n, \tilde{g})$.

For instant $|\overset{\circ}{II}|^2(\phi, M^n, g)$ is a scalar conformal invariant since one could verifies that

$$|\overset{\circ}{II}|^2(\phi, M^n, \lambda^2\tilde{g}_{IJ}) = \lambda^{-2}\overset{\circ}{II}(\phi, M^n, \tilde{g}_{IJ}).$$

in other words, $|\overset{\circ}{II}|^2$ is a scalar conformal invariant of order 2. Basically this is the only well-known example. The next famous one might be $\mathcal{H} = \delta H + |\overset{\circ}{II}|^2 H$ when you consider $M^n = \mathbb{R}^3$. In fact this is a scalar conformal invariant of order 3, i.e. $\mathcal{H}_\lambda = \lambda^{-3}\mathcal{H}$, but this is tough to verify it directly, we need to know the general formula of $\mathcal{H}$ in the manifold that is not flat. Since this is well-known that Willmore surface satisfies the equation $\mathcal{H} = 0$, we would like to call it Willmore curvature later.

In fact, to verify whether a scalar Riemannian invariant is a scalar conformal invariant is tedious. This is one of the motivations why we would like to introduce the construction of associate hypersurface. The following theorem tells us the relation between scalar conformal invariants of ϕ and scalar Riemannian invariants of associate hypersurface $\tilde{\phi}$.

Theorem 4.4. *Given conformal manifold $(M^n, [g])$ and hypersurface $\phi : N^{n-1} \to M^n$. Assume $(\tilde{\mathcal{G}}, \tilde{g})$ is the corresponding ambient space of $(M^n, [g])$, $\tilde{\phi}$ is the associate hypersurface in $\tilde{\mathcal{G}}$. For any scalar Riemannian invariant $I(\tilde{\phi}, \tilde{\mathcal{G}}, \tilde{g})$, restrict it at $(1, \phi, 0)$, if it is not trivial, it turns out to be a scalar conformal invariant divided by the term $|\overset{\circ}{II}|$ to some power. In other words:*

$$|\overset{\circ}{II}|^p \cdot I(\tilde{\phi}, \tilde{\mathcal{G}}, \tilde{g})\Big|_{\beta=0} = I_c(\phi, M^n, g). \tag{4.6}$$

Here p is some positive integer that large enough. When the dimension n is even, the order of invariants can not larger than n.

In fact, this result is very different from our expect. Recall Fefferman and Graham's work, in their case $\phi = id$ and there is no trouble for the factor $|\overset{\circ}{II}|^p$, more precisely, "polynomial" always goes to "polynomial". Unfortunately in our case, "polynomial" goes to be "rational", and we will point out that we can not conquer this difficulty, the first term $\mathcal{H}$ will be "rational" if the dimension $n > 3$. So far we guess this issue is dimension dependent.

5. Calculation of scalar conformal invariants

Let us evaluate some scalar conformal invariants following the method we mentioned in last section. First we need to check the behavior of the curvature of the ambient metric $\tilde{g}$, in fact when you restrict those curvature at $\rho = 0$ they turn out to be linear combinations of curvature of g. There are details in Fefferman and Graham's work and we have shown the first two coefficient in the second section. Secondly, we need to study the expansion formula of the second fundamental form $II_{\tilde{\phi}}$ in term of β, those are all the gears we need.

5.1. **Geometry of associate hypersurface.** From the discussion above one may see that, to study scalar conformal invariants, the first step is to study the first and second fundamental forms for $\tilde{\phi}$. Since the way we define $\tilde{\phi}$ is by using exponential maps, Jacobi equations would help us find out the expansion formula in term of β of these forms.

We need to introduce some tensors that we would use often:

Definition 5.1.

$$\Omega_{ij} = -\tilde{g}\Big(\tilde{\nabla}_{\frac{\partial}{\partial u^i}} y, \tilde{\nabla}_{\frac{\partial}{\partial u^j}} \xi\Big). \tag{5.1}$$

$$\Omega^*_{ij} = -\tilde{g}\Big(\tilde{\nabla}_{\frac{\partial}{\partial u^i}} y^*, \tilde{\nabla}_{\frac{\partial}{\partial u^j}} \xi\Big). \tag{5.2}$$

$$\omega = (\omega_1, \cdots, \omega_{n-1}), \quad \omega_i = \frac{2}{|\overset{\circ}{II}|^2} \overset{\circ}{II}_{ij}(P_{jn} - \frac{\partial H}{\partial x^j}). \tag{5.3}$$

We will also use the notations of matrices, $\mathbf{\Omega} = (\Omega_{ij})_{(n-1)\times(n-1)}$ and $\mathbf{\Omega}^* = (\Omega^*_{ij})_{(n-1)\times(n-1)}$. In fact we can evaluate their formula from definition.

$$\Omega_{ij} = \overset{\circ}{II}_{ij}, \quad \Omega^*_{ij} = \sum_{k=1}^{n-1} (\frac{1}{2} F_{ik} - \omega_i \omega_k) \overset{\circ}{II}_{kj}. \tag{5.4}$$

Here $F_{ik} = \delta_{ik}(H^2 + |\omega|^2) + \frac{\partial}{\partial u^i}\omega_k + \frac{\partial}{\partial u^k}\omega_i - 2Hh_{ik} - 2P_{ik}$, then we mark $\mathbf{F} = (F_{ij})_{(n-1)\times(n-1)}$. Now restrict the calculation at $\rho = 0$ and we obtain:

$$I_{\tilde{\phi}}|_{\beta=0} = \begin{pmatrix} 0 & 0 & -\alpha \\ 0 & \alpha^2 I_{n-1} & \alpha^2 \omega^t \\ -\alpha & \alpha^2 \omega & 0 \end{pmatrix}, \quad I^{-1}_{\tilde{\phi}}|_{\beta=0} = \begin{pmatrix} |\omega|^2 & \alpha^{-1}\omega & -\alpha^{-1} \\ \alpha^{-1}\omega^t & \alpha^{-2} I_{n-1} & 0 \\ -\alpha^{-1} & 0 & 0 \end{pmatrix}.$$

Here $|\omega|^2 = \sum_{i=1}^{n-1} \omega_i^2$.

Moreover, we could use Jacobi filed equation to get higher order $\beta-$derivative information:

$$\frac{\partial}{\partial \beta}\Big|_{\beta=0} I_{\tilde{\phi}} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & \alpha^2 \mathbf{F} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\frac{\partial}{\partial \beta}\Big|_{\beta=0} I^{-1}_{\tilde{\phi}} = \begin{pmatrix} 2|\omega|^4 - \omega \mathbf{F} \omega^t & \alpha^{-1}(2|\omega|^2\omega - \omega\mathbf{F}) & -2\alpha^{-1}|\omega|^2 \\ \alpha^{-1}(2|\omega|^2\omega^t - \mathbf{F}\omega^t) & \alpha^{-2}(2\omega^t\omega - \mathbf{F}) & -2\alpha^{-2}\omega^t \\ -2\alpha^{-1}|\omega|^2 & -2\alpha^{-2}\omega & 2\alpha^{-2} \end{pmatrix}.$$

To get the second fundamental form for $\tilde{\phi}$, we point out that ξ is the unit normal vetor for whole hypersurface. Easy to see when $\beta = 0$ it is true, since y^* is the direction of exponential map, this property hold along y^*-direction.

Now use ξ as unit normal vector and we obtain:

$$II_{\tilde{\phi}}|_{\beta=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha\mathbf{\Omega} & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha\mathring{II} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \frac{\partial}{\partial\beta}\Big|_{\beta=0} II_{\tilde{\phi}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha\mathbf{\Omega}^* & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the result from above, we obtain:

$$I_{\tilde{\phi}}^{-1} II_{\tilde{\phi}}|_{\beta=0} = \begin{pmatrix} 0 & \omega\mathring{II} & 0 \\ 0 & \alpha^{-1}\mathring{II} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{5.5}$$

By taking the trace and the norm square of this matrix, we obtain important information of $\tilde{\phi}$:

$$H_{\tilde{\phi}}|_{\beta=0} = 0, |\mathring{II}_{\tilde{\phi}}|^2|_{\beta=0} = \alpha^{-2}|\mathring{II}|^2. \tag{5.6}$$

It is obviously that mean curvature and the norm square of the second fundamental forms are scalar Riemannian invariants of $\tilde{\phi}$. Now restrict these two tensors at $\beta = 0$, from the main theorem of last section, we should get scalar conformal invariants of ϕ. In fact, we did get 0 and $|\mathring{II}|^2$ as the answers. Though there is no surprise from these two computations, at least we confirmed our main theorem.

5.2. **Calculation for higher order.** From the calculation above we conclude, if we do not take higher order derivative of $\mathring{II}_{\tilde{\phi}}$, the result should be always trivial. Naturally we need to consider the first scalar Riemannian invariant, which is $\Delta_{\tilde{\phi}} H^{\tilde{\phi}}$. We may foresee that the tensor we will obtain on the right hand side should be a scalar conformal invariant of order 3. In fact, it is Willmore curvature $\mathcal{H}$, which comes from the calculation of $-tr(\Omega^*)$. Let us state the result first, calculation steps will be listed on appendices. We claim that the general formula for Willmore curvature $\mathcal{H}$ is:

$$\mathcal{H} = \Delta_\phi H + |\overset{\circ}{I}I|^2 H + \sum_{ij=1}^{n-1} P_{ij} \overset{\circ}{I}I_{ij} - \sum_{k=1}^{n-1} P_{nk,k} + (n-3) \sum_{ij=1}^{n-1} \omega_i \overset{\circ}{I}I_{ij} \omega_j. \tag{5.7}$$

Recall that

$$\omega_i = \frac{2}{|\overset{\circ}{I}I|^2} \overset{\circ}{I}I_{ij} (P_{jn} - \frac{\partial H}{\partial x^j}).$$

So the term $\sum_{ij=1}^{n-1} \omega_i \overset{\circ}{I}I_{ij} \omega_j$ has order $|\overset{\circ}{I}I|^{-2}$, this shows that the only chance we can achieve a "real" scalar conformal invariant is when $n = 3$, otherwise we need to multiply at least $|\overset{\circ}{I}I|^2$ to cancel the denominator.
Another result is $|\nabla h_{\tilde{\phi}}|^2\Big|_{\beta=0}$ when $n = 3$, we obtain:

$$\begin{aligned} |\nabla h|^2_{\tilde{\phi}}\Big|_{\beta=0} =& \alpha^{-4} [|\nabla \overset{\circ}{I}I|^2 + 4|\nabla H|^2 - 8H_{,i} P_{3i} + 4|P_{3i}|^2 + 3(H^2 + P^k_k)|\overset{\circ}{I}I|^2 \\ &+ 6(H_{,ij} - P_{3i,j}) \overset{\circ}{I}I^{ij}]. \end{aligned} \tag{5.8}$$

More details are shown in the appendices.

A. Calculation of $\mathcal{H}$

Let us start with the definition $\mathcal{H} = -tr(\Omega^*)$, we need to apply Codazzi theorem for benefit.

$$\begin{aligned} \Omega^*_{ij} =& \left[\frac{1}{2}\delta_{ki}(H^2 + |\omega|^2) + \omega_{k,i} - Hh_{ik} - P_{ik} - \omega_i\omega_k \right] \overset{\circ}{I}I_{kj} \\ =& \frac{1}{2}(H^2 + |\omega|^2)\overset{\circ}{I}I_{ij} + \omega_{k,i}\overset{\circ}{I}I_{kj} - H(\overset{\circ}{I}I_{ik} + H\delta_{ik})\overset{\circ}{I}I_{kj} - P_{ik}\overset{\circ}{I}I_{kj} - \omega_i\omega_k\overset{\circ}{I}I_{kj} \\ =& \frac{1}{2}(-H^2 + |\omega|^2)\overset{\circ}{I}I_{ij} + (\omega_k\overset{\circ}{I}I_{kj})_{,i} - \omega_k\overset{\circ}{I}I_{kj,i} \\ &- \frac{1}{2}\delta_{ij}|\overset{\circ}{I}I|^2 H - P_{ik}\overset{\circ}{I}I_{kj} - \omega_i\omega_k\overset{\circ}{I}I_{kj} \\ =& \frac{1}{2}(-H^2 + |\omega|^2)\overset{\circ}{I}I_{ij} + (P_{nj} - H_{,j})_{,i} - \omega_k\overset{\circ}{I}I_{kj,i} \\ &- \frac{1}{2}\delta_{ij}|\overset{\circ}{I}I|^2 H - P_{ik}\overset{\circ}{I}I_{kj} - \omega_i\omega_k\overset{\circ}{I}I_{kj} \end{aligned}$$

Now apply Codazzi theorem we obtain:

$$R_{njki} = h_{ji,k} - h_{jk,i} = \overset{\circ}{II}_{ij,k} + \delta_{ij} H_{,k} - \overset{\circ}{II}_{jk,i} - \delta_{jk} H_{,i}.$$

Furthermore,

$$\overset{\circ}{II}_{jk,i} = -R_{njki} + \overset{\circ}{II}_{ij,k} + \delta_{ij} H_{,k} - \delta_{jk} H_{,i}.$$

Then we have:

$$-\omega_k \overset{\circ}{II}_{kj,i} = \omega_k R_{njki} - \omega_k \overset{\circ}{II}_{ij,k} - \delta_{ij}\omega_k H_{,k} + \omega_j H_{,i}.$$

Now we take trace of index i and j, we obtain

$$\begin{aligned} tr(-\omega_k \overset{\circ}{II}_{kj,i}) &= \omega_k R_{nk} - (n-2)\omega_k H_{,k} \\ &= (n-2)(\omega_k P_{nk} - \omega_k H_{,k}) \\ &= (n-2)\omega_k \omega_i \overset{\circ}{II}_{ik} \qquad \text{(A.1)} \end{aligned}$$

Notice here we have $R_{nk} = (n-2)P_{3k}$ since $g_{nk} = 0$. Use formula (A.1) we obtain:

$$\mathcal{H} = -tr(\Omega^*_{ij}) = \Delta H + |\overset{\circ}{II}|^2 H + P_{ij}\overset{\circ}{II}^{ij} - P_{ni,}{}^{i} + (n-3)\omega_i\omega_j \overset{\circ}{II}_{ij}. \qquad \text{(A.2)}$$

B. Calculation of $\Delta_{\tilde{\phi}} H^{\tilde{\phi}}\Big|_{\beta=0}$

Let us start from $\frac{\partial}{\partial\beta}\Big|_{\beta=0} H^{\tilde{\phi}}$:

$$
\begin{aligned}
\frac{\partial}{\partial\beta}\Big|_{\beta=0} H^{\tilde{\phi}} =& \frac{\partial}{\partial\beta}\Big|_{\beta=0} tr(I_{\tilde{\phi}}^{-1}\cdot II_{\tilde{\phi}}) \\
=& tr(\frac{\partial}{\partial\beta}\Big|_{\beta=0} I_{\tilde{\phi}}^{-1}\cdot II_{\tilde{\phi}}\Big|_{\beta=0}) + tr(I_{\tilde{\phi}}^{-1}\Big|_{\beta=0}\cdot\frac{\partial}{\partial\beta}\Big|_{\beta=0} II_{\tilde{\phi}}) \\
=& tr\left[\begin{pmatrix} * & * \\ * & \alpha^{-2}(2\omega_i\omega_j - F_{ij})\end{pmatrix}\cdot\begin{pmatrix} 0 & 0 \\ 0 & \alpha \mathring{I}I\end{pmatrix}\right] \\
&+ tr\left[\begin{pmatrix} * & * \\ * & \alpha^{-2} I\end{pmatrix}\cdot\begin{pmatrix} 0 & 0 \\ 0 & \alpha\Omega^*\end{pmatrix}\right]. \\
=& \alpha^{-1} tr\left[\begin{pmatrix} 0 & * \\ 0 & -2\Omega^*\end{pmatrix}\right] + \alpha^{-1} tr\left[\begin{pmatrix} 0 & * \\ 0 & \Omega^*\end{pmatrix}\right] \\
=& -\alpha^{-1} tr(\Omega^*) = \alpha^{-1}\mathcal{H}.
\end{aligned}
$$

Here we use results from $(A.2)$

Now recall the formula of Laplacian operator:

$$
\frac{\partial}{\partial\beta}\Big|_{\beta=0} H^{\tilde{\phi}} = \frac{1}{\sqrt{|\det(I_{\tilde{\phi}}|_{\beta=0})|}}\partial_A\left(\sqrt{|\det(I_{\tilde{\phi}})|}\cdot g^{AB}\cdot\partial_B H^{\tilde{\phi}}\right)\Big|_{\beta=0}. \tag{B.1}
$$

If index A or B is ∞, this term turns out to be 0 since $H^{\tilde{\phi}}\Big|_{\beta=0} = 0$. So there are only 3 nonzero terms:

(Case 1) $B = \infty$ and $A \neq \infty$, in this case the index of A can only be 0 Since when $\beta = 0$, the only nonzero term is $g^{0\infty}$, then we can evaluate this $0\infty-$term:

$$
\alpha^{-3}\partial_0\left(\alpha^3\cdot(-\alpha^{-1})\cdot\frac{\partial}{\partial\beta}\Big|_{\beta=0} H^{\tilde{\phi}}\right) = \alpha^{-3}\partial_0(-\alpha^2\cdot\alpha^{-1}\mathcal{H}) = -\alpha^{-3}\mathcal{H}.
$$

(Case 2) $A = \infty$ and $B \neq \infty$, for the same reason the index of B can only be 0, this $\infty 0-$term is:

$$
\alpha^{-3}\alpha^3\cdot(-\alpha^{-1})\cdot\partial_0\left(\frac{\partial}{\partial\beta}\Big|_{\beta=0} H^{\tilde{\phi}}\right) = \alpha^{-3}\mathcal{H}.
$$

(Case 3) A and B are both ∞, this $\infty\infty-$term is:

$$\alpha^{-3}\alpha^{3}\cdot\cdot\frac{\partial}{\partial\beta}\Big|_{\beta=0}g^{\infty\infty}\cdot\frac{\partial}{\partial\beta}\Big|_{\beta=0}H^{\tilde{\phi}}=2\alpha^{-2}\cdot\alpha^{-1}\mathcal{H}=2\alpha^{-3}\mathcal{H}$$

Here we use the result of $\frac{\partial}{\partial\beta}\Big|_{\beta=0}g^{\infty\infty}$ from the matrix $\frac{\partial}{\partial\beta}\Big|_{\beta=0}I_{\tilde{\phi}}^{-1}$. Add all 3 terms and we obtain:

$$\Delta_{\tilde{\phi}}H^{\tilde{\phi}}\Big|_{\beta=0}=2\alpha^{-3}\mathcal{H}. \tag{B.2}$$

We need to point out, from calculation we proved that $\mathcal{H}$ is a scalar conformal invariant of order 3.

C. Calculation of $|\nabla h_{\tilde{\phi}}|^{2}\Big|_{\beta=0}$ when $n=3$

Let us define $\Phi_{B}^{A}=I_{\tilde{\phi}}^{-1}\cdot II_{\tilde{\phi}}$ and compute

$$\Phi_{B}^{A}\Big|_{\beta=0}=\begin{pmatrix}0 & 0 & \omega_{i}\mathring{II}_{i1} & \omega_{i}\mathring{II}_{i2}\\ 0 & 0 & 0 & 0\\ 0 & 0 & \alpha^{-1}\mathring{II}_{11} & \alpha^{-1}\mathring{II}_{12}\\ 0 & 0 & \alpha^{-1}\mathring{II}_{21} & \alpha^{-1}\mathring{II}_{22}\end{pmatrix}. \tag{C.1}$$

$$\frac{\partial}{\partial\beta}\Big|_{\beta=0}\Phi_{B}^{A}=\begin{pmatrix}0 & 0 & -2\omega_{i}\Omega_{i1}^{*} & -2\omega_{i}\Omega_{i2}^{*}\\ 0 & 0 & -2\alpha^{-1}\omega_{i}\mathring{II}_{i1} & -2\alpha^{-1}\omega_{i}\mathring{II}_{i2}\\ 0 & 0 & -\alpha^{-1}\Omega_{11}^{*} & -\alpha^{-1}\Omega_{12}^{*}\\ 0 & 0 & -\alpha^{-1}\Omega_{21}^{*} & -\alpha^{-1}\Omega_{22}^{*}\end{pmatrix}. \tag{C.2}$$

We need to evaluate Christoffel symbols for $I_{\tilde{\phi}}$ and we obtain the following:

$$
\begin{aligned}
\Gamma^k_{\alpha\alpha} &= \Gamma^k_{\beta\beta} = \Gamma^k_{\alpha\beta} = \Gamma^\alpha_{aA} = 0. \\
\Gamma^\beta_{k\alpha} &= \Gamma^\beta_{\alpha\alpha} = 0, \Gamma^\beta_{\alpha\beta} = \alpha^{-1}. \\
\Gamma^i_{j\alpha} &= \frac{1}{2} g^{iA}(\partial_j g_{A\alpha} + \partial_\alpha g_{jA} - \partial_A g_{i\alpha}) = \frac{1}{2} g^{il} \partial_\alpha g_{jl} = \alpha^{-1}\delta_{ij}. \\
\Gamma^\beta_{ij} &= \frac{1}{2} g^{\beta\alpha}(-\partial g_{ij}) = \delta_{ij}. \\
\Gamma^k_{ij} &= \hat{\Gamma}^k_{ij} + \frac{1}{2} g^{k\alpha}(-\partial_\alpha g_{ij}) = \hat{\Gamma}^k_{ij} - \omega_k \delta_{ij}. \\
\Gamma^i_{j\beta} &= \frac{1}{2} g^{il}(\partial_\beta \Big|_{\beta=0} g_{jl} + \partial_j g_{l\beta} - \partial_l g_{j\beta}) - \frac{1}{2} g^{i\alpha}\partial_\alpha g_{j\beta} \\
&= \frac{1}{2} F_{ij} + \frac{1}{2}\omega_{i,j} - \frac{1}{2}\omega_{j,i} - \omega_i\omega_j. \\
\Gamma^\alpha_{ij} &= \frac{1}{2} g^{\alpha\alpha}(-\partial_\alpha g_{ij}) + \frac{1}{2} g^{\alpha\beta}(\partial_i g_{j\beta} + \partial_i g_{j\beta} - \partial_\beta \Big|_{\beta=0} g_{ij}) + \frac{1}{2} g^{\alpha l}\alpha^2 \hat{\Gamma}_{ijl} \\
&= -\frac{1}{2}\alpha\left[2|\omega|^2\delta_{ij} + \omega_{j,i} + \omega_{i,j} - F_{ij} - \omega^l \hat{\Gamma}_{ijl}\right] \\
&= -\frac{1}{2}\alpha\left[(|\omega|^2 - H^2)\delta_{ij} + 2Hh_{ij} + 2P_{ij} - \omega^l \hat{\Gamma}_{ijl}\right].
\end{aligned}
$$

As a result we also get partial derivatives of Φ^A_B

$$
\begin{aligned}
&\Phi^\beta_{\beta,A} = \Phi^\beta_{\alpha,A} = \Phi^\alpha_{\beta,a} = \Phi^\alpha_{\alpha,a} = 0. \\
&\Phi^i_{\alpha,a} = \Phi^i_{\beta,a} = \Phi^\beta_{i,\alpha} = \Phi^i_{\beta,\alpha} = 0. \\
&\Phi^\alpha_{\alpha,i} = -\alpha^{-1}\omega_k \mathring{II}_{ki}. \\
&\Phi^\alpha_{\beta,k} = -\Gamma^l_{k\beta}\Phi^\alpha_l = -(\frac{1}{2}F_{kl} + \frac{1}{2}\omega_{l,k} - \frac{1}{2}\omega_{k,l} - \omega_k\omega_l) \cdot \omega_i \mathring{II}_{il}
\end{aligned}
$$

$$
\begin{aligned}
\Phi^{\alpha}_{\beta,k} &= -\Gamma^{l}_{k\beta}\Phi^{\alpha}_{l} = -(\frac{1}{2}F_{kl} + \frac{1}{2}\omega_{l,k} - \frac{1}{2}\omega_{k,l} - \omega_k\omega_l)\cdot\omega_i \overset{\circ}{II}_{il} \\
&= \frac{1}{2}(\omega_{k,l} - \omega_{l,k})\omega_i \overset{\circ}{II}_{il} - \omega_i \Omega^*_{ik} = -\omega_i\Omega^*_{ik}. \\
\Phi^{i}_{\alpha,k} &= -\alpha^{-1}\Phi^{i}_{k} = -\alpha^{-2}\overset{\circ}{II}_{ik}. \\
\Phi^{\alpha}_{i,k} &= \partial_k\Phi^{\alpha}_{i} + \Gamma^{\alpha}_{kl}\Phi^{l}_{i} - \Gamma^{l}_{ik}\Phi^{\alpha}_{l} \\
&= (\partial_k\Phi^{\alpha}_{i} - \hat{\Gamma}^{l}_{ik}\Phi^{\alpha}_{l}) + \omega_l\delta_{ik}\Phi^{\alpha}_{l} + \Gamma^{\alpha}_{kl}\Phi^{l}_{i} \\
&= (\omega_l \overset{\circ}{II}_{il})_{,k} + \delta_{ik}\omega_l\omega_p \overset{\circ}{II}_{lp} \\
&- \frac{1}{2}\left[(|\omega|^2 - H^2)\delta_{kl} + 2Hh_{kl} + 2P_{kl} - \omega_p\hat{\Gamma}_{klp}\right]\overset{\circ}{II}_{li}. \\
\Phi^{\beta}_{i,k} &= \Gamma^{\beta}_{kl}\Phi^{l}_{i} = \alpha^{-1}\overset{\circ}{II}_{ik}. \\
\Phi^{i}_{\beta,k} &= -\Gamma^{l}_{\beta k}\Phi^{i}_{l} = -(\frac{1}{2}F_{kl} + \frac{1}{2}\omega_{l,k} - \frac{1}{2}\omega_{k,l} - \omega_k\omega_l)\overset{\circ}{II}_{il} \\
&= -\Omega^*_{ik} + \frac{1}{2}(\omega_{k,l} - \omega_{l,k})\overset{\circ}{II}_{il} = -\Omega^*_{ik}. \\
\Phi^{i}_{j,\alpha} &= -\alpha^{-2}\overset{\circ}{II}_{ij}. \\
\Phi^{i}_{j,\beta} &= \partial_\beta\Big|_{\beta=0}\tilde{\Phi}^{i}_{j} + (\Gamma^{i}_{\beta l}\Phi^{l}_{i} - \Gamma^{i}_{\beta j}\Phi^{i}_{l}). \\
&= \alpha^{-1}\left[-\Omega^*_{ij} + \frac{1}{2}(\omega_{i,l} - \omega_{l,i})\overset{\circ}{II}_{lj} - \frac{1}{2}(\omega_{l,j} - \omega_{j,l})\overset{\circ}{II}_{il}\right] = -\alpha^{-1}\Omega^*_{ij}. \\
\Phi^{i}_{j,k} &= \partial_k\Phi^{i}_{j} + \Gamma^{i}_{kl}\Phi^{l}_{j} + \Gamma^{i}_{k\alpha}\Phi^{\alpha}_{j} - \Gamma^{l}_{kj}\Phi^{i}_{l} \\
&= (\partial_k\Phi^{i}_{j} + \hat{\Gamma}^{i}_{kl}\Phi^{l}_{j} - \hat{\Gamma}^{l}_{kj}\Phi^{i}_{l}) - \alpha^{-1}\delta_{ik}\Phi^{\alpha}_{j} - \omega_i\delta_{kl}\Phi^{l}_{j} + \omega_l\delta_{kj}\Phi^{i}_{l} \\
&= \alpha^{-1}\left[\overset{\circ}{II}_{ij,k} - \omega_i\overset{\circ}{II}_{jk} + \delta_{jk}\omega_l\overset{\circ}{II}_{li} + \delta_{ik}\omega_l\overset{\circ}{II}_{lj}\right].
\end{aligned}
$$

Using the following formula to evaluate $|\nabla\tilde{h}|^2$F

$$
\begin{aligned}
|\nabla\tilde{h}|^2 &= \tilde{g}^{EF}\tilde{g}^{AC}\tilde{g}^{BD}(\tilde{h}_{BC})_{,E}(\tilde{h}_{AD})_{,F} \\
&= \tilde{g}^{EF}(\Phi^{A}_{B})_{,E}(\Phi^{B}_{A})_{,F} \\
&= \tilde{g}^{\alpha\alpha}(\Phi^{A}_{B})_{,\alpha}(\Phi^{B}_{A})_{,\alpha} + 2\tilde{g}^{\alpha k}(\Phi^{A}_{B})_{,\alpha}(\Phi^{B}_{A})_{,k} \\
&+ 2\tilde{g}^{\alpha\beta}(\Phi^{A}_{B})_{,\alpha}(\Phi^{B}_{A})_{,\beta} + \tilde{g}^{ij}(\Phi^{A}_{B})_{,i}(\Phi^{B}_{A})_{,j} \\
&= I + II + III + IV.
\end{aligned}
$$

Evaluate term by term:

$$
\begin{aligned}
I &= \tilde{g}^{\alpha\alpha}(\Phi^A_B)_{,\alpha}(\Phi^B_A)_{,\alpha} = \tilde{g}^{\alpha\alpha}(\Phi^i_j)_{,\alpha}(\Phi^j_i)_{,\alpha} \\
&= |\omega|^2\alpha^{-4}\overset{\circ}{II}_{ij}\overset{\circ}{II}_{ji} = \alpha^{-4}|\omega|^2|\overset{\circ}{II}|^2. \qquad \text{(C.3)} \\
II &= 2\tilde{g}^{\alpha k}(\Phi^A_B)_{,\alpha}(\Phi^B_A)_{,k} \\
&= 2\tilde{g}^{\alpha k}(\Phi^\alpha_i)_{,\alpha}(\Phi^i_\alpha)_{,k} + 2\tilde{g}^{\alpha k}(\Phi^i_j)_{,\alpha}(\Phi^j_i)_{,k} \\
&= 2\alpha^{-4}\omega_k\left[\omega_l\overset{\circ}{II}_{li}\overset{\circ}{II}_{ik} - \overset{\circ}{II}_{ij}(\overset{\circ}{II}_{ij,k} - \omega_i\overset{\circ}{II}_{jk} + \delta_{jk}\omega_l\overset{\circ}{II}_{li} - \delta_{ik}\omega_l\overset{\circ}{II}_{lj})\right] \\
&= \alpha^{-4}\left[|\omega|^2|\overset{\circ}{II}|^2 - 2\omega_k\overset{\circ}{II}_{ij}\overset{\circ}{II}_{ij,k} + |\omega|^2|\overset{\circ}{II}|^2 - |\omega|^2|\overset{\circ}{II}|^2 + |\omega|^2|\overset{\circ}{II}|^2\right] \\
&= \alpha^{-4}(-2\omega_k\overset{\circ}{II}_{ij}\overset{\circ}{II}_{ij,k}). \qquad \text{(C.4)} \\
III &= 2\tilde{g}^{\alpha\beta}(\Phi^A_B)_{,\alpha}(\Phi^B_A)_{,\beta} = 2\tilde{g}^{\alpha\beta}(\Phi^i_j)_{,\alpha}(\Phi^j_i)_{,\beta} \\
&= -2\alpha^{-1}(-\alpha^{-2}\overset{\circ}{II}_{ij})(-\alpha^{-1}\Omega^*_{ij}) = \alpha^{-4}(-2\Omega^*_{ij}\overset{\circ}{II}_{ij}). \qquad \text{(C.5)} \\
IV &= \tilde{g}^{ij}(\Phi^A_B)_{,i}(\Phi^B_A)_{,j} = \frac{1}{\alpha^2}(\Phi^A_B)_{,k}(\Phi^B_A)_{,k} \\
&= \frac{1}{\alpha^2}(\Phi^\alpha_\alpha)_{,k}(\Phi^\alpha_\alpha)_{,k} + \frac{2}{\alpha^2}(\Phi^i_\beta)_{,k}(\Phi^\beta_i)_{,k} + \frac{2}{\alpha^2}(\Phi^i_\alpha)_{,k}(\Phi^\alpha_i)_{,k} + \frac{1}{\alpha^2}(\Phi^i_j)_{,k}(\Phi^j_i)_{,k} \\
&= i + ii + iii + iv.
\end{aligned}
$$

Furthermore we obtain:

$$
\begin{aligned}
i =& \alpha^{-2}(\Phi^{\alpha}_{\alpha})_{,k}(\Phi^{\alpha}_{\alpha})_{,k} \\
=& \alpha^{-2}(-\alpha^{-1}\omega_i \mathring{II}_{ik})(-\alpha^{-1}\omega_j \mathring{II}_{jk}) = \alpha^{-4}(\frac{1}{2}|\omega|^2|\mathring{II}|^2). \qquad \text{(C.6)}
\end{aligned}
$$

$$
ii = 2\alpha^{-2}(\Phi^{i}_{\beta})_{,k}(\Phi^{\beta}_{i})_{,k} = \alpha^{-4}(-2\Omega^{*}_{ij}\mathring{II}_{ij}). \qquad \text{(C.7)}
$$

$$
\begin{aligned}
iii =& 2\alpha^{-2}(\Phi^{i}_{\alpha})_{,k}(\Phi^{\alpha}_{i})_{,k} \\
=& -2\alpha^{-4}\mathring{II}_{ik}[(\omega_l \mathring{II}_{il})_{,k} + \delta_{ik}\omega_l\omega_p \mathring{II}_{lp}] \\
& + \alpha^{-4}\cdot\frac{1}{2}|\mathring{II}|^2\delta_{kl}[(|\omega|^2 - H^2)\delta_{kl} + 2Hh_{kl} + 2P_{kl} - \omega_p\hat{\Gamma}_{klp}] \\
=& \alpha^{-4}[-2\mathring{II}_{ik}(\omega_l \mathring{II}_{il})_{,k} + (H^2 + |\omega|^2 + P^k_k)|\mathring{II}|^2] \\
=& \alpha^{-4}[2H_{,ij}\mathring{II}^{ij} - 2P_{3i,j}\mathring{II}^{ij} + (H^2 + |\omega|^2 + P^k_k)|\mathring{II}|^2]. \qquad \text{(C.8)}
\end{aligned}
$$

$$
\begin{aligned}
iv =& \alpha^{-2}(\Phi^{i}_{j})_{,k}(\Phi^{j}_{i})_{,k} \\
=& \alpha^{-4}\left[\mathring{II}_{ij,k} - \omega_i\mathring{II}_{jk} + \delta_{jk}\omega_l\mathring{II}_{li} + \delta_{ik}\omega_l\mathring{II}_{lj}\right] \\
& \cdot\left[\mathring{II}_{ji,k} - \omega_j\mathring{II}_{ik} + \delta_{ik}\omega_l\mathring{II}_{lj} + \delta_{jk}\omega_l\mathring{II}_{li}\right] \\
=& \alpha^{-4}\left[|\nabla\mathring{II}|^2 - 2\omega_j\mathring{II}_{ik}\mathring{II}_{ij,k} + \frac{5}{2}|\omega|^2|\mathring{II}|^2 + 4\omega_l\mathring{II}_{lj}\mathring{II}_{ij,i}\right]. \qquad \text{(C.9)}
\end{aligned}
$$

Apply Codazzi Theorem on (C.9)

$$
-R^N_{3iji} = h_{ij,i} - h_{ii,j} = \mathring{II}_{ij,i} + H_{,i}\delta_{ij} - h_{ii,j}.
$$

Take trace of index i, easy to see $\mathring{II}_{ij,i} = H_{,j} - P_{3j} = -\omega_p\mathring{II}_{pj}$, which means $\omega_l\mathring{II}_{lj}\mathring{II}_{ij,i} = -2|\omega|^2|\mathring{II}|^2$. And also easy to check:

$$
\begin{aligned}
-2\omega_j\mathring{II}_{ik}\mathring{II}_{ij,k} =& -2\mathring{II}_{ik}[(\omega_j\mathring{II}_{ij})_{,k} - \mathring{II}_{ij}\omega_{j,k}] \\
=& 2H_{,ij}\mathring{II}^{ij} - 2P_{3i,j}\mathring{II}^{ij} + div(\omega)|\mathring{II}|^2.
\end{aligned}
$$

In conclusion we have:

$$
iv = \alpha^{-4}[|\nabla\mathring{II}|^2 + \frac{1}{2}|\omega|^2|\mathring{II}|^2 + 2H_{,ij}\mathring{II}^{ij} - 2P_{3i,j}\mathring{II}^{ij} + div(\omega)|\mathring{II}|^2]. \qquad \text{(C.10)}
$$

Now check term (C.4), again we apply Codazzi identity:

$$
\begin{aligned}
-2\omega_k \overset{\circ}{II}_{ij}\overset{\circ}{II}_{ij,k} =& -2\omega_k \overset{\circ}{II}_{ij}(-R^N_{3ijk} + \overset{\circ}{II}_{ik,j} + \delta_{ik}H_{,j} - \delta_{ij}H_{,k}) \\
=& 2\omega_k \overset{\circ}{II}_{ij} R^N_{3ijk} - 2\overset{\circ}{II}_{ij}(\omega_k \overset{\circ}{II}_{ik})_{,j} + 2\overset{\circ}{II}_{ij}\overset{\circ}{II}_{ik}\omega_{k,j} - 2\omega_k \overset{\circ}{II}_{jk} H_{,j} \\
=& 2\omega_k \overset{\circ}{II}_{ij} R^N_{3ijk} + 2H_{,ij}\overset{\circ}{II}^{ij} - 2P_{3i,j}\overset{\circ}{II}^{ij} + div(w)|\overset{\circ}{II}|^2 \\
&+ 2|\nabla H|^2 - 2H_{,j}P_{3j}.
\end{aligned}
$$

where we claim:

$$
\begin{aligned}
\omega_k \overset{\circ}{II}_{ij} R^N_{3ijk} =& \omega_2 \overset{\circ}{II}_{i1} R^N_{3i12} + \omega_1 \overset{\circ}{II}_{i2} R^N_{3i21} \\
=& \omega_2 \overset{\circ}{II}_{11} R^N_{3112} + \omega_2 \overset{\circ}{II}_{21} R^N_{3212} + \omega_1 \overset{\circ}{II}_{12} R^N_{3121} + \omega_1 \overset{\circ}{II}_{22} R^N_{3221} \\
=& (\omega_1 \overset{\circ}{II}_{12} + \omega_2 \overset{\circ}{II}_{22}) R^N_{3121} + (\omega_1 \overset{\circ}{II}_{11} + \omega_2 \overset{\circ}{II}_{21}) R^N_{3212} \\
=& (\omega_1 \overset{\circ}{II}_{12} + \omega_2 \overset{\circ}{II}_{22}) R^N_{32} + (\omega_1 \overset{\circ}{II}_{11} + \omega_2 \overset{\circ}{II}_{21}) R^N_{31} \\
=& (\omega_1 \overset{\circ}{II}_{12} + \omega_2 \overset{\circ}{II}_{22}) P_{32} + (\omega_1 \overset{\circ}{II}_{11} + \omega_2 \overset{\circ}{II}_{21}) P_{31} \\
=& (\omega_k \overset{\circ}{II}_{ki}) P_{3i} = (P_{3i} - H_{,i})P_{3i} = |P_{3i}|^2 - H_{,i}P_{3i}.
\end{aligned}
$$

Here we use the fact that $P_{i3} = R^N_{i3}$ and $\overset{\circ}{II}$ is traceless. Notice that:

$$
|\omega|^2 |\overset{\circ}{II}|^2 = 2|\nabla H|^2 - 4H_{,i}P_{3i} + 2|P_{i3}|^2.
$$

by definition of ω_i, in conclusion we hold:

$$
II = |\omega|^2 |\overset{\circ}{II}|^2 + 2H_{,ij}\overset{\circ}{II}^{ij} - 2P_{3i,j}\overset{\circ}{II}^{ij} + div(\omega)|\overset{\circ}{II}|^2. \tag{C.11}
$$

And finally we claim:

$$
\begin{aligned}
|\nabla h_{ij}|^2\Big|_{\beta=0} =& I + II + III + i + ii + iii + iv \\
=& \alpha^{-4}[|\nabla \overset{\circ}{II}|^2 + 4|\omega|^2|\overset{\circ}{II}|^2 + 6H_{,ij}\overset{\circ}{II}^{ij} - 6P_{3i,j}\overset{\circ}{II}^{ij} \\
&+ (H^2 + P^k_k)|\overset{\circ}{II}|^2 - 4\Omega^*_{ij}\overset{\circ}{II}_{ij} + 2div(\omega)|\overset{\circ}{II}|^2].
\end{aligned}
$$

Recall definition of Ω^*_{ij} we have identity as following:

$$\begin{aligned}2\Omega^*_i j \mathring{II}_{ij} &= [(H^2+|\omega|^2)\delta_{ik}+\omega_{i,k}+\omega_{k,i}-2Hh_{ik}-2P_{ik}]\mathring{II}_{kj}\mathring{II}_{ij}\\ &= [(H^2+|\omega|^2)\delta_{ik}+\omega_{i,k}+\omega_{k,i}-2Hh_{ik}-2P_{ik}]\delta_{ik}\frac{1}{2}|\mathring{II}|^2.\\ &= [|\omega|^2+div(\omega)-H^2-P^k_k]|\mathring{II}|^2\end{aligned}$$

Which implies:

$$-4\Omega^*_{ij}\mathring{II}_{ij}+2div(\omega)|\mathring{II}|^2 = 2(H^2+P^k_k)|\mathring{II}|^2-2|\omega|^2|\mathring{II}|^2.$$

In conclusion we hold:

$$\begin{aligned}|\nabla h|^2_{\tilde{\phi}}\Big|_{\beta=0} =&\alpha^{-4}[|\nabla\mathring{II}|^2+2|\omega|^2|\mathring{II}|^2+3(H^2+P^k_k)|\mathring{II}|^2+6(H_{,ij}-P_{3i,j})\mathring{II}^{ij}]\\ =&\alpha^{-4}[|\nabla\mathring{II}|^2+4|\nabla H|^2-8H_{,i}P_{3i}+4|P_{3i}|^2+3(H^2+P^k_k)|\mathring{II}|^2\\ &+6(H_{,ij}-P_{3i,j})\mathring{II}^{ij}].\end{aligned}\tag{C.12}$$

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