Title
Chaos in the test particle eccentric Kozai-Lidov mechanism

Permalink
https://escholarship.org/uc/item/0204v0f4

Journal
Astrophysical Journal, 791(2)

ISSN
0004-637X

Authors
Li, G
Naoz, S
Holman, M
et al.

Publication Date
2014-08-20

DOI
10.1088/0004-637X/791/2/86

Peer reviewed
CHAOS IN THE TEST PARTICLE ECCENTRIC KOZAI–LIDOV MECHANISM

GONGJIE LI, SMADAR NAOZ, MATT HOLMAN, AND ABRAHAM LOEB
Harvard-Smithsonian Center for Astrophysics, The Institute for Theory and Computation, 60 Garden Street, Cambridge, MA 02138, USA; gli@cfa.harvard.edu
Received 2014 May 2; accepted 2014 June 28; published 2014 July 30

ABSTRACT

The Kozai–Lidov mechanism can be applied to a vast variety of astrophysical systems involving hierarchical three-body systems. Here, we study the Kozai–Lidov mechanism systematically in the test particle limit at the octupole level of approximation. We investigate the chaotic and quasi-periodic orbital evolution by studying the surfaces of section and the Lyapunov exponents. We find that the resonances introduced by the octupole level of approximation cause orbits to flip from prograde to retrograde and back as well as cause significant eccentricity excitation, and chaotic behavior occurs when the mutual inclination between the inner and the outer binary is high. We characterize the parameter space that allows large amplitude oscillations in eccentricity and inclination.

Key words: binaries: close – celestial mechanics

Online-only material: color figures

1. INTRODUCTION

The Kozai–Lidov mechanism (Kozai 1962; Lidov 1962) has proven very useful for interpreting numerous astrophysical systems. For example, it has been shown that it can play a major role in exoplanet configurations and obliquities (e.g., Holman et al. 1997; Wu & Murray 2003; Fabrycky & Tremaine 2007; Veras & Ford 2010; Correia et al. 2011; Naoz et al. 2011, 2012). In addition, close stellar binaries with two compact objects are likely produced through triple evolution, and the Kozai–Lidov mechanism may play a key role in these systems (e.g., Harrington 1969; Maze & Shaham 1979; Soderhjelm 1982; Kiseleva et al. 1998; Ford et al. 2000; Eggleton & Kiseleva-Eggleton 2001; Fabrycky & Tremaine 2007; Perets & Fabrycky 2009; Thompson 2011; Katz & Dong 2012; Shappee & Thompson 2013; Naoz et al. 2013; Naoz & Fabrycky 2014). Furthermore, the Kozai–Lidov mechanism has been proposed as an important element in the growth of black holes at the centers of dense star clusters, the formation of short-period binary black holes (Blaes et al. 2002; Miller & Hamilton 2002; Wen 2003; Ivanova et al. 2010), and tidal disruption events (Chen et al. 2009, 2011; Wegg & Bode 2011; Bode & Wegg 2013; G. Li et al., in preparation).

The Kozai–Lidov mechanism focuses on hierarchical three-body systems, which can be treated as the interaction between two elliptical wires by orbit averaging: the inner wire is composed of the inner two objects, and the outer wire is composed of the outer companion orbiting around the center of mass of the inner two objects. The total angular momentum of this system, the vector sum of the inner orbit’s and the outer orbit’s angular momenta, is conserved.

Kozai (1962) and Lidov (1962) first studied this mechanism by expanding the gravitational potential in a power series of the semi-major axis ratio and considered applications when one of the inner objects is massless (the test particle limit) and the outer orbit is circular. Kozai (1962) considered the secular (long-term) evolution of asteroids under the perturbation of Jupiter, and Lidov (1962) studied the secular evolution of satellites under the perturbation of the Moon. In those cases, the gravitational potential of the inner orbit is axisymmetric, which renders the \( z \) component of the inner orbit’s angular momentum \((\mathbf{J}_z)\) constant, where \( \hat{z} \) is the direction of the total angular momentum of the system. The quadrupole order of approximation \((O(a_1/a_2^2))\) sufficiently describes the orbital evolution of such systems, and the eccentricity and the inclination undergo large amplitude oscillations due to the “Kozai resonance” when \( i > 39/2 \).

Recently, Naoz et al. (2011) considered the case when none of the inner objects is a test particle, and pointed out that \( J_z \) is no longer conserved. In addition, the eccentric Kozai–Lidov mechanism (hereafter EKL) applies to cases when the outer orbit is non-circular, where the \( z \) component of the angular momentum of the inner orbit is also not conserved (Naoz et al. 2011). In this situation, the octupole terms in the potential \((O(a_1/a_2^3))\) need to be taken into account to describe the orbital evolution, where the eccentricity of the inner orbit can be excited to unity, and the inner orbit may flip from prograde to retrograde or vice versa (Naoz et al. 2011; Lithwick & Naoz 2011; Katz et al. 2011; Naoz et al. 2013). As the eccentricity increases, the pericenter distance decreases and causes an enhanced tidal disruption rate (G. Li et al., in preparation). Furthermore, including the octupole effects, oscillations of the eccentricity and the inclination of the inner orbit may still exist when \( i < 39/2 \), and the inner orbit may undergo a coplanar flip from \( \sim 0^\circ \) to \( \sim 180^\circ \) (Li et al. 2014).

Here, we probe the test particle limit, which simplifies the analysis due to its smaller number of degrees of freedom. This approximation was proven to be very useful in a large range of astrophysical settings (Lithwick & Naoz 2011; Katz et al. 2011; Naoz et al. 2012; Li et al. 2014; S. Naoz & J. Silk, in preparation; G. Li et al., in preparation). Importantly, probing this limit can help us gain some basic understanding of the EKL mechanism. The test particle limit has been studied in the literature before to obtain an analytical understanding of the flip of the orbit (Lithwick & Naoz 2011; Katz et al. 2011). Nevertheless, a systematic study of the chaotic behavior and the identification of the underlying resonances are necessary but are uncovered in the literature. We identify the resonances, and characterize the chaotic regions and the initial conditions where high eccentricity and flips may occur in the parameter space. This can help predict the dynamical evolution of systems without doing a large amount of simulations.

This paper is organized as follows. In Section 2, we give a brief overview of the Kozai–Lidov mechanism. In Section 3, we investigate the surface of section systematically for a large
range of orbital parameters. In Section 4, we characterize the initial condition which allows large amplitude oscillations in eccentricity and inclination. Finally, in Section 5, we characterize the chaotic regions.

2. OVERVIEW OF THE ECCENTRIC KOZAI–LIDOV MECHANISM IN THE TEST PARTICLE LIMIT

As mentioned in the introduction, the Kozai–Lidov mechanism describes the dynamical behavior of hierarchical three-body systems (see Figure 1). The inner two objects \((m_1 \text{ and } m_2)\) form an inner orbit, and the outer orbit is formed by the outer object \((m_3)\) orbiting around the center of mass of the inner two objects. The eccentric Kozai–Lidov mechanism describes the dynamics when the outer orbit is eccentric, and the test particle limit requires one of the closely separated objects to be a test particle \(m_3 \to 0\).

In the hierarchical configuration, we average over the mean motion of the two orbits and treat the evolution of the system as the interaction of two elliptical wires, which is known as the secular approximation. This approximation reduces this system from six degrees of freedom to four degrees of freedom. In addition, in the test particle limit, the outer orbit is stationary and reduces the system to two degrees of freedom (Harrington 1968, 1969; Ford et al. 2000). Expanding the Hamiltonian of the interaction energy between the two ellipses in a power series of \(\alpha_1/\alpha_2\), the Hamiltonian can be expressed as the following at the second (quadrupole) and third (octupole) orders (Lithwick & Naoz 2011):

\[
F_{\text{quad}}(J_1, \omega, J_2, \Omega) = \frac{1}{2}(-1 + J_2^2) + \frac{J_1^2}{2J^2} + \frac{3(1 - J_2^2)J_2^2}{2J^2} \quad (1)
\]

\[
+ \frac{1 - J_2^2}{1 - J_2^2/J^2} \cos(2\omega)F_{\text{oct}}(J_1, \omega, J_2, \Omega) = \frac{5}{16} \sqrt{1 - J_2^2}
\]

\[
+ \frac{3}{4}(1 - J_2^3J_3^2/2)\left[\left(1 - \frac{11J_1}{J} - \frac{5J_2}{J^2} + \frac{15J_3^3}{J^2}\right)\cos(\omega - \Omega) + \left(1 + \frac{11J_1}{J} - \frac{5J_2}{J^2} - \frac{15J_3^3}{J^2}\right)\cos(\omega + \Omega)\right]
\]

(2)

where \(H_{\text{quad}} = -F_{\text{quad}}\) and \(H_{\text{oct}} = F_{\text{quad}} - \epsilon F_{\text{oct}}\), and \(\epsilon\) characterizes the importance of the octupole order. The Hamiltonian is scaled with \(m_1\sqrt{Gm_1a_1\omega_1}\), where

\[
t_K = \frac{8}{3}P_{\text{in}}\left(\frac{a_2}{a_1}\right)^3(1 - e_2^2)^{3/2}
\]

(4)

(Lithwick & Naoz 2011). \(J = \sqrt{1 - e_1^2}\) is the specific angular momentum of the inner orbit, \(\omega\) is the argument of the periapsis of the inner orbit, \(J_1 = \sqrt{1 - e_1^2}\cos i_1\) is the \(\hat{z}\) component of the inner orbit’s angular momentum \(J_2\), and \(\Omega\) is the longitude of the ascending node of the inner orbit. Specifically, \(J, \omega, J_1, J_2, \Omega\) are conjugate momentum and coordinate pairs. We denote by \(e_1\) the eccentricity of the inner orbit, and by \(i_1\) the inclination of the inner orbit to the total angular momentum of the system. In the test particle limit, \(i_1 = i\) is the mutual inclination between the two orbits.

The secular approximation breaks down when the change in the angular momentum of the inner binary happens faster than the orbital timescales (see Equation (B14) in Ivanov et al. 2005, Equation 18 in Antonini et al. 2014, and Equation (48) in Bode & Wegg 2014). Since the analysis presented here is for the reduced Hamiltonian (note that \(J\) is the specific angular momentum: \(J = \sqrt{1 - e_2^2}\)), which is independent of the masses, the timescales can be set arbitrarily and the secular approximation is irrelevant here. Nevertheless, applying this study to physical systems requires the correct scaling of the masses to satisfy the criteria for the secular approximation.

In the quadrupole limit, the Hamiltonian is independent of \(\Omega\), so \(J_2\) is constant, and the system is integrable. In addition, the angle \(\omega = \sigma - \Omega\) is the resonant angle of the system, where \(\sigma\) is the longitude of the periapsis. When \(i > 39.2^\circ\), the solution admits a resonant region and \(e_1\) and \(i\) exhibit large amplitude oscillations. Particularly, \(e_1\) may be excited to high values starting from \(e_1 \sim 0\) (e.g., Morbidelli 2002).

As mentioned in the introduction, the octupole order adds variations in \(J_2\) which allows the inner orbit to flip from prograde to retrograde, and the eccentricity to be excited very close to 1 (Lithwick & Naoz 2011; Katz et al. 2011; Naoz et al. 2011, 2012, 2013). In the following sections, we work with the Hamiltonian at the octupole level of approximation to analyze the surface of section and the chaotic behaviors.

3. SURFACES OF THE SECTION

For a two-dimensions of freedom system, the surface of section projects a four-dimensional trajectory on a two-dimensional surface. Specifically, we plot points on a two-dimensional surface composed of one canonically conjugate pair (e.g., \(J_1\) or \(J_2\)) whenever the other angle (\(\Omega\) or \(\omega\)) reaches a fixed value and moves in a fixed direction (see the left panel in Figure 2). The collection of points form the surface of section.

There are three distinct regions in the surface of section: “resonant regions,” “circulation regions,” and “chaotic regions.”
by the red and the green arrows. The chaotic zones are indicated by the gray points are not regular and the trajectories are chaotic.

This set of points forms the "surface of section." Lower panel: illustration of Figure 2.

Figure 2. Upper panel: illustration of the "surface section" for the \( J-\omega \) plane. By recording the point in the trajectory every time \( \Omega = 0, \Omega > 0 \), the trajectory can be represented by a two-dimensional graph, as shown in the left panel. This set of points forms the "surface of section." Lower panel: illustration of the resonant and chaotic regions on the surface of section. We set \( H = -0.1, \epsilon = 0.1 \) in this plot. The resonant and higher-order resonant zones are marked by the red and the green arrows. The chaotic zones are indicated by the gray arrows. In the resonant region, the angle \( \omega \) is constrained in a small region and the trajectories are quasi-periodic. In the chaotic region, the position of the points are not regular and the trajectories are chaotic.

(A color version of this figure is available in the online journal.)

(right panel in Figure 2). The resonant regions are formed by points where the momenta and coordinates (the angles) undergo bound oscillations. The trajectories in this region are quasi-periodic, where the system is in the liberation mode. The circulation region represents trajectories where the coordinates are not constrained to a specific interval. Both resonant and circulatory trajectories map onto a one-dimensional manifold on the surface of section. On the contrary, chaotic trajectories map onto a two-dimensional manifold. In other words, while quasi-periodic trajectories form lines on the section, chaotic trajectories are area-filling. Embedded in the chaotic region, the small islands correspond to the higher order resonances, which are caused by the interaction between the primary resonances. The trajectories in the higher order resonant regions are also quasi-periodic.

We now consider the surface of section in the \( J-\omega \) plane (setting \( \Omega = 0 \) and \( d\Omega/dt > 0 \)). When \( e_1 \) is excited to large values, \( J \to 0 \). When \( \Omega \) is set, for each point in the \( J-\omega \) plane, \( J_{\omega} (\leq J \leq J_{\omega}) \) is unequivocally defined by the conservation of \( H \). There is a finite range of \( H \) that the system can take on, because both actions must have zero imaginary components. Since we plot the sections with constant \( H \) values, we first explore the range of energy \( H \) it can achieve in the \( J-\omega \) plane. This way, we can select the range in \( H \) that we explore below.

We note that the maximum and minimum energy it can reach in the \( J-\omega \) plane when \( \Omega = 0 \) is \( \sim 3 \) and \( \sim -2.4 \) (see Appendix Figures 8 and 9, which show the maximum and the minimum \( H \) in the \( J-\omega \) plane). Thus, we plot six surfaces of section for \( H \) ranging from \( H = -2 \) to \( H = 1.2 \), since when \( H > 1.2 \), the behavior is similar to that of \( H = 1.2 \). Note that the \( H \) admits positive values for this bound system, because it is the interaction energy between the test particle \( (m_0) \) and the outer companion \( (m_2) \), i.e., the disturbing function of this system to the Kepler Hamiltonian of the inner and outer orbits. To investigate the role of the octupole effects, we plot the surface section for two extreme values of \( \epsilon: \epsilon = 0.001 \) and \( \epsilon = 0.1 \). When \( \epsilon < 0.001 \), the octupole effects are negligible. On the other hand, \( \epsilon = 0.1 \) represents the maximal octupole effects, where, when \( \epsilon > 0.1 \), the hierarchical condition may break down and the system may become unstable.

The sections are shown in Figure 3. The empty region (bounded by the black curves) do not have physical solutions. The comparison between the two rows in Figure 3 shows the difference between the octupole and the quadrupole resonances: \( \epsilon = 0.001 \) is dominated by the quadrupole effect and \( \epsilon = 0.1 \) is dominated by both the quadrupole and the octupole effects. For the former, where the quadrupole dominates, there are two resonant regions with fixed points at \( \omega = \pi/2 \) and \( 3\pi/2 \) when \( H \) is high (as shown in Figure 3 at \( \epsilon = 0.001 \), \( H = -0.5, -0.1, 0.5 \) and 1.2). For the latter when the octupole plays an important role (i.e., \( \epsilon = 0.1 \)), we find different resonant regions for different energy levels, and the location of the resonant regions varies according to the energy levels.

The resonant regions are associated with fixed points at \( \omega = \pi, \omega = \pi/2 \) and \( \omega = 3\pi/2 \) depending on the energy level. The resonances at high \( J \) and at \( \omega = \pi/2 \) or \( 3\pi/2 \) correspond to the quadrupole resonances identified in the literature (Kozai 1962; Holman et al. 1997; Morbidelli 2002). The other resonant zones result from the interaction of the resonances associated with the “harmonics” in the octupole level Hamiltonian, i.e., \( 2\omega, \omega \pm \Omega \), and \( 3\omega \pm \Omega \). Moreover, chaotic regions can only be seen for high \( \epsilon \) at \( H = -0.5 \) and \( H = -0.1 \), where reading from the surfaces, the chaotic zones are a result of the overlap of the resonances between the quadrupole and the octupole resonances (e.g., Chirikov et al. 1979; Murray et al. 1997). Embedded in the chaotic region, higher-order resonances can be found at \( H = -0.1 \), where the trajectories are quasi-periodic and the eccentricity cannot be excited.

On the other hand, the comparison between the different energy levels shows that the orbital evolution corresponds to different orbital parameters. The corresponding \( e_1 \) and \( i \) are shown in Figures 10 and 11 in the Appendix. Accordingly, the low \( H \) corresponds to the low inclination (\( i \sim 0°-30° \)) and high eccentricity (\( e_1 \gtrsim 0.6 \)) case, the higher \( H \) corresponds to the high inclination (\( i \sim 30°-60° \)) and low eccentricity (\( e_1 \lesssim 0.6 \)) case, and \( H > 0 \) corresponds to the high inclination (\( i \sim 60°-90° \)) and low eccentricity case (\( e_1 \lesssim 0.3 \)). When \( H \) is low (\( H \sim -2 \)), the evolution is only affected by the octupole resonances, while when \( H \) is higher, octupole and quadrupole resonances both contribute and may overlap to cause the chaotic region as mentioned above. We find that \( e_1 \) can be excited to high values (\( J \to 0 \)) for almost all energy levels but is only excited very close to unity for higher \( \epsilon \). This emphasizes that
Figure 3. Surface of section in the $J-\omega$ plane. In the first row, $\epsilon = 0.001$ and in the second row, $\epsilon = 0.1$. The octupole terms are important when $\epsilon$ is bigger. $H$ varies from $-2 \sim 1$. The corresponding $e_1$ and $i$ in this plane is shown in Figures 10 and 11. There are chaotic regions at $H = -0.5$ and $H = -0.1$. (A color version of this figure is available in the online journal.)

Figure 4. Surface of section in the $J_z-\Omega$ plane. In the first row, $\epsilon = 0.001$ and in the second row, $\epsilon = 0.1$. $H$ varies from $-2 \sim 0$. There are chaotic regions at $H = -0.5$, $H = -0.3$, and $H = -0.1$. All the features are due to the octupole order, as $J_z$ is constant in the quadrupole order. The corresponding $e_1$ and $i$ are shown in Figures 12 and 13. (A color version of this figure is available in the online journal.)

the octupole level of approximation causes large eccentricity excitation, since a larger $\epsilon$ implies that the octupole level is important.

Next, we study the surface section in the $J_z-\Omega$ plane (Figure 4). These sections clearly show the flip of the orbit when $J_z$ changes sign. The maximum and minimum energy that can be reached in the $J_z-\Omega$ plane (with $\omega = 0$) is $\sim 0$ and $\sim -2.4$. Thus, we plot the surface section ranging from $H = -2$ to $H = -0.1$ for two values of $\epsilon = 0.001$ and 0.1. At the quadrupole level, $J_z$ is constant, and there are no resonances in the $J_z-\omega$ plane. Thus, all the resonances originated from the octupole level of approximation, and the fixed points are at $\Omega = \pi$ and $\Omega = 0$. In addition, similar to the surface section on the $J-\Omega$ plane, we see higher-order resonances for $\epsilon = 0.1$ at $H = -0.3$ and $H = -0.1$ embedded in the chaotic region, and the chaotic region is confined to $H = -0.5$ and $H = -0.1$. Since $J_z$ changes sign in all energy levels, the orbit may flip for all energy levels, and the flip parameter space is larger for higher $\epsilon$. The corresponding $e_1$ and $i$ on the surface are shown in Figures 12 and 13.

To summarize, the surfaces of section show that flips and the excitation of $e_1$ can occur for both regular regions and chaotic regions for a wide range of $H$, and they depend sensitively on the initial condition. In addition, the trajectories are chaotic only when $H \lesssim 0$, corresponding to high mutual inclination, low eccentricity cases. Furthermore, it is the octupole resonances...
that cause the flip of the orbit and the excitation of eccentricity very close to unity.

4. THE MAXIMUM ECCENTRICITY AND THE FLIP CONDITION

To apply this mechanism to astrophysical systems with different initial conditions, we investigate the parameter regions that exhibit interesting dynamical behaviors. We create a finer grid of \( H \) and \( \epsilon \) than those presented in Figures 3 and 4, and we monitor the trajectories that start with the selected initial condition in the \( J-\omega \) or the \( J-\Omega \) plane. Of course, some behaviors which do not pass through the selected initial condition will be missed, but this exploration gives a general idea of the behavior of the system as a whole.

We start with the exploration in the \( J-\omega \) plane. To systematically estimate the range in \( J \) that the trajectories may reach, we start at the maximum energy boundary of \( J \) for a given \( H \) and \( \epsilon \), which corresponds to the minimum eccentricity. Accordingly, for \( H < -1 \), \( \omega \) starts at \( \pi \), and for \( H > -1 \), \( \omega \) starts at \( \pi/2 \). The maximum \( e_i \) is recorded after monitoring for \( t = 500t_K \) (we define \( t_K \) in Equation (4)), which is much longer than the Lyapunov timescale (see below).

In Figure 5, we plot \( 1 - e_{i,\text{max}} \) as a function of \( \epsilon \), where each curve represents a fixed \( H \) (\( H \in [-2, 2] \)), and \( \epsilon \) ranges from 0.001 to 0.1. In addition, we use the symbol “\( \times \)” to mark the \( \epsilon \) higher than which the orbit flips. It shows that there are roughly five dynamical regions in \( H \): when \( H \lesssim -1.5 \), \( -0.5 \lesssim H \lesssim 0 \), and \( H \gtrsim 0.5 \), the orbit may flip and \( e_i \) can be excited very close to unity; when \( -1.5 \lesssim H \lesssim -0.5 \) and \( 0 \lesssim H \lesssim 0.5 \), starting with the minimum \( e_1 \), \( e_i \) cannot be excited to unity. Reading from the surface of section in Figure 3, the lack of \( e_i \) excitation at \( 0 \lesssim H \lesssim 0.5 \) and high \( \epsilon \) is due to the quadrupole resonances, which traps the trajectory at low \( e_1 \).

Particularly, \( e_1 \) may be excited and the orbit may flip in three scenarios: when the inner orbit is eccentric and coplanar, when the inner orbit is circular and with high inclination, or when the inner orbit is moderately eccentric and with very high inclination \( \sim 80^\circ-90^\circ \) (see Figure 14). In addition, the maximum change in \( \Delta J \) can be well fit by a power law:

\[
\Delta J = \begin{cases} 
e^{-2.27H-3.62} & (H < -1) \\ e^{2.14H+1.23} & (H > -1). \end{cases}
\]

Next, we explore the \( J-\Omega \) plane. We start the trajectories at the lower energy boundary of \( J \), at \( \Omega = \pi \) for the given \( H \) and \( \epsilon \), and we record the maximum change in \( J \) after \( t = 500t_K \). Figure 6 shows \( \Delta J_c \) as a function of \( \epsilon \), where each curve represents a different \( H \). \( \epsilon \) ranges from 0.001 to 0.1, and \( H \) ranges from \(-2\) to \(0\), since the maximum \( H \) is zero for \( \omega = 0 \). Similarly to the \( J-\omega \) plane, we use the symbol “\( \times \)” to mark the \( \epsilon \) higher than which the orbit flips. As expected, it shows that the orbit may flip when \(-2 < H < -1.5\) and \(-0.5 < H < 0\), where \(-2 < H < -1.5\) corresponds to an eccentric and coplanar inner orbit, and \(-0.5 < H < 0\) corresponds to a circular inner orbit with a high inclination. Moreover, \( \Delta J_c \) can be fit by a power law of \( H \) and \( \epsilon \):

\[
\Delta J_c = \begin{cases} e^{-2.21H+0.06} & (H < -0.5) \\ e^{10.7H+4.23} & (H > -0.5). \end{cases}
\]

Note that Equations (5) and (6) are for the specific initial conditions mentioned above. They show the general dependence of the maximum change in \( J \) and \( J_c \) on \( \epsilon \) and \( H \).

5. CHAOTIC REGIONS

The surfaces of section show that the system is chaotic when \( H \lesssim 0 \) (Figures 3 and 4). To better characterize the chaotic regions, we first calculate the percentage of area that is chaotic in each surface in Figure 3. Specifically, we divide each surface into equally spaced grids in \( J \) and \( \omega \), and count the fraction of grids that has chaotic trajectories. We use the Lyapunov exponent...
are also chaotic when

\[ \lambda(\epsilon) \] to determine whether the trajectories are chaotic, where \( \lambda \) indicates how quickly two closely separated trajectories diverge from each other,

\[ \lambda = \lim_{t \to \infty} \frac{1}{t} \ln \frac{\delta_{\text{trajectories}}(t)}{\delta_{\text{trajectories}}(0)}. \] (7)

We integrate the tangent of the trajectories for 1000\( t_K \) to compute \( \lambda \), and we find that there are chaotic trajectories only when \( \epsilon = 0.1 \), \( H = -0.5 \) or \( -0.1 \). Specifically, 85 out of 276 (~31%) grid cells have chaotic trajectories when \( \epsilon = 0.1 \) and \( H = -0.5 \), and 109 out of 242 (~45%) grid cells have chaotic trajectories when \( \epsilon = 0.1 \) and \( H = -0.1 \). It shows that even when \( H \lesssim 0 \), a large range of orbital parameters would still yield regular trajectories.

Next, we characterize the chaotic region in the parameter space of \( H \) and \( \epsilon \). We arbitrarily select the trajectories starting with \( \Omega = 0 \), \( \omega = \pi/2 \), and the maximum \( J \) for the given \( H \) and \( \epsilon \), where the associated \( e_1 \) and \( i \) of the initial condition are shown in Figure 14. Similarly, we integrate the tangent of the trajectories for 1000\( t_K \) to compute \( \lambda \), and we plot \( \lambda \) as a function of \( H \) and \( \epsilon \) in the left panel of Figure 7. The larger \( \lambda \) corresponds to the more chaotic systems. A large region in the parameter space is regular, and the system is chaotic only when \(-0.6 < H < 0 \) for larger \( \epsilon \). The Lyapunov timescale is \( \sim 6t_K \) when \( \epsilon \gtrsim 0.01 \) and \(-0.6 < H < 0 \) (low \( e_1 \) and \( i \gtrsim 40^\circ \)).

To justify that the regions with smaller \( \lambda \) are regular, we increase the run time to 4000\( t_K \), and we find that the Lyapunov exponents for the regular region decrease, while the Lyapunov exponents in the chaotic region remain at \( \sim 6t_K \). Moreover, to avoid missing chaotic regions due to the specific choice of the initial condition, we vary the initial condition and make several contour plots of \( \lambda \) in the plane of \( H \) and \( \epsilon \). The right panel of Figure 7 shows the case when \( \omega \) starts at 0, where the trajectories are also chaotic when \(-0.6 < H < 0 \).

Figure 7. Lyapunov exponents with different \( H \) and \( \epsilon \). Run time \( t = 1000 \).
Left panel: we consider the following initial condition: \( \omega_0 = \pi/2 \), \( \Omega_0 = 0 \), \( J_0 = 1 \), or the maximum \( J \) at the energy boundary and \(-0.8 < H < 0.3 \). Right panel: we consider the following initial condition: \( \omega_0 = \Omega_0 = 0 \), \( J_0 = 1 \), and \(-0.6 < H < 0 \). Note that for this choice of initial conditions no physical solution exists for \( H > 0 \). The colormap represents the value of the Lyapunov exponents \( \lambda \). The yellow and red colors correspond to large Lyapunov exponents, which are associated with chaotic regions, and the cyan and blue colors represent the regular regions.

(A color version of this figure is available in the online journal.)

6. CONCLUSION

The hierarchical three-body system in the test particle limit is common in a large range of astrophysical settings. The dynamical behavior of such systems may lead to retrograde objects, an enhanced rate for tidal disruption, and merger or collision events (e.g., Holman et al. 1997; Fabrycky & Tremaine 2007; Naoz et al. 2011, 2012; Chen et al. 2011; Bode & Wegg 2013; S. Naoz & Silk, in preparation; G. Li et al., in preparation). Here, we used a large range of the initial condition to systematically study the dynamics, including the underlying resonances, and the chaotic characteristics of the system.

First, we plotted the surface of section on the \( J-\omega \) plane for a large range of energy \( H \) and two different \( \epsilon \) to identify
the underlying resonances (Figure 3). In the quadrupole level, the resonances that occur at high $H$ are centered around fixed points at $\omega = \pi/2$ and $3\pi/2$. On the other hand, the octupole level resonances are centered at $\omega = 0$, $\pi/2$, $\pi$, or $3\pi/2$ depending on the different energy levels, and we can identify resonances in all these energy levels. The octupole resonances cause the excitation of $e_1$ in the high eccentricity coplanar case (corresponding to low $H$), shown in Li et al. (2014). The overlap of the quadrupole and octupole resonances causes the chaos for the low eccentricity and high inclination case (corresponds to higher $H$; e.g., Naoz et al. 2011).

The surfaces of section in the $J_z-\Omega$ plane not only show the octupole resonances but the condition when the orbit flips as $J_z$ changes sign (Figure 4). At the quadrupole level, $J_z$ is constant, and there is no resonant zones in the $J_z-\Omega$ plane. However, at the octupole level, the resonant zones exist and lead to the flip of the orbit. As expected, similarly to the $J-\omega$ plane, it also shows that chaotic behavior exist when $H \lesssim 0$ for high $\epsilon$.

Finally, we calculated the Lyapunov exponent for different $H$ and $\epsilon$ to characterize the region where the evolution is chaotic. Consistently with the surface of section, we have found that the orbital evolution is chaotic when $H \lesssim 0$ (low $e_1$ high $i$ cases). Specifically, the Lyapunov timescale $\sim 6t_K$.

By monitoring the trajectories, we find that the inner eccentricity may be excited and the orbit may flip for a circular high inclination orbit or for an eccentric and nearly coplanar orbit. This agrees with previous discussions in the literature for the flips with high inclination (Naoz et al. 2011; Lithwick & Naoz 2011).
given was supported in part by NSF grant AST-1312034 (for A.L.).

Moreover, our analysis could help predict the enhancement in the rate of tidal disruption events due to eccentricity excitation. In addition, we note that the flips with high inclination are chaotic and the coplanar flips are regular. This analysis can be applied to observed systems. Knowing roughly the orbital elements, one can identify the type of trajectories in the surface of section. Then, one can study the evolution features of the system without doing a large number of simulation for different initial conditions. Moreover, our analysis could help predict the enhancement in the rate of tidal disruption events due to eccentricity excitation.

We thank Konstantin Batygin for helpful remarks. This work was supported in part by NSF grant AST-1312034 (for A.L.).

Figure 10. Eccentricities in the $J$–$\omega$ plane ($\Omega = 0$). Note that these are not the initial conditions, but directly the values of $e_1$ in the $J$–$\omega$ surface at fixed $\Omega = 0$ for the given $H$ and $\epsilon$. Similar to Figure 3, in the first row, $\epsilon = 0.001$ and in the second row, $\epsilon = 0.1$. The octupole terms are more dominant when $\epsilon$ is bigger.

(A color version of this figure is available in the online journal.)

Figure 11. Inclinations in the $J$–$\omega$ plane ($\Omega = 0$). Note that these are not the initial conditions, but directly the values of $i$ in the $J$–$\omega$ surface at fixed $\Omega = 0$ for the given $H$ and $\epsilon$. Similar to Figure 3, in the first row, $\epsilon = 0.001$ and in the second row, $\epsilon = 0.1$. The octupole terms are more dominant when $\epsilon$ is bigger.

(A color version of this figure is available in the online journal.)

2011; Katz et al. 2011) and coplanar flips (Li et al. 2014). In addition, we note that the flips with high inclination are chaotic and the coplanar flips are regular. This analysis can be applied to observed systems. Knowing roughly the orbital elements, one can identify the type of trajectories in the surface of section. Then, one can study the evolution features of the system without doing a large number of simulation for different initial conditions. Moreover, our analysis could help predict the enhancement in the rate of tidal disruption events due to eccentricity excitation.

We thank Konstantin Batygin for helpful remarks. This work was supported in part by NSF grant AST-1312034 (for A.L.).

**APPENDIX**

First, we explore the range $H$ can reach for the surface of section in the $J$–$\omega$ plane with $\Omega = 0$ and in the $J_z$–$\Omega$ plane with $\omega = 0$. We contour plot the maximum and minimum of $H$ as a function of $J$ and $\omega$ while setting $\Omega = 0$ in Figure 8, which depicts that the range of $H$ is $\sim -2.4$ to $\sim 3$. Similarly, we plot the maximum and minimum of $H$ for different $J_z$ and $\Omega$ with $\omega = 0$ in Figure 9. It shows that $H$ ranges from $\sim -2.4$ to 0. Accordingly, we plot the surface of section for $-2 < H < 1.2$ in Figure 3, since when $H > 1.2$ the section are similar to that when $H = \sim 1.2$, and we set $-2 < H < 0$ for the surface of section in Figure 4.
Next, we show the associated eccentricity and inclination for the surface of section (Figures 3 and 4) and the initial condition in Figures 5 and 6. This helps to connect the resulting dynamical behavior to the parameters in \( e_1 \) and \( i \), which can be obtained more directly for observations.

In Figures 10 and 11, we plot the initial condition in the \( J - \omega \) plane corresponding to the surfaces of section in Figure 3. \( e_1 \) can be calculated from the \( J \) value directly as \( e_1 = \sqrt{1 - J^2} \), so a higher \( J \) is associated with lower \( e_1 \). On the other hand, \( i \) is lower for larger \( J \) when \( H = -2 \), -1, -0.5, and \( i \) is higher for larger \( J \) when \( H = -0.1, 0.5, 1.2 \).

Next, in Figures 12 and 13, we plot \( e_1 \) and \( i \) in the \( J \rightarrow \Omega \) plane, corresponding to the surface section in the \( J \rightarrow \Omega \) plane with \( \omega = 0 \) in Figure 4. When \( i > 90^\circ \), \( J_z > 0 \), and when \( i < 90^\circ \), \( J_x < 0 \). We find that \( e_1 \) is higher for lower \( H \) and \( i \) is closer to 90\(^\circ\) for higher \( H \).

Furthermore, we plot the initial condition for the trajectories we selected to investigate the maximum \( e_1 \) in Figure 14. It shows that for the maximum \( e_1 \) plot (Figure 5), when \( H \lesssim -1.2 \), we monitor the trajectories that start with high eccentricity and low inclination. In this case, when \( -1.2 \lesssim H \lesssim 0 \), we monitor trajectories that start with low eccentricity and high inclination. In this case, not much variation is seen unless \( H \lesssim 0 \). When \( H > 0 \), we monitor trajectories starting with high inclination \( i \sim 80^\circ - 90^\circ \).
In the end, we plot the initial condition for the trajectories that are monitored for \( J_i \) or the flip of the orbit in Figure 15. When \( H \lesssim -1 \), we start the trajectories with high \( e_1 \) and low \( i \); when \( H \gtrsim -1 \), we start the trajectories with low \( e_1 \) and high \( i \). The orbit may flip with \(-0.4 \lesssim H \lesssim 0\) at high \( \epsilon \) for trajectories starting with low \( e_1 \) and high \( i \), and the orbit may flip with \( H \lesssim -1.5 \) when the trajectories start with high \( e_1 \) and low \( i \).

REFERENCES

Chirikov, B. V. 1979, PhR, 52, 263
Correia, A. C. M., Laskar, J., Farago, F., & Boué, G. 2011, CeMDA, 111, 105
Harrington, R. S. 1968, AJ, 73, 190
Harrington, R. S. 1969, CeMec, 1, 200
Holman, M., Touma, J., & Tremaine, S. 1997, Nautr, 386, 254
Katz, B., Dong, S., & Malhotra, R. 2011, PhRVL, 107, 181101
Lidov, M. L. 1962, P&SS, 9, 719