Abstract: From an experimental-mathematics perspective we analyze some structurally interrelated $n$-dimensional integrals we call $C_n, D_n, E_n$, where $D_n$ is a magnetic susceptibility integral relevant to the Ising theory of solid-state physics. With a view to closed-form results for such “Ising-class” integrals, we analyze in depth the most tractable of the collection, namely

$$C_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left(\sum_{j=1}^n (u_j + 1/u_j)\right)^2} \frac{du_1}{u_1} \cdots \frac{du_n}{u_n}.$$ 

We first conjectured, on the basis of extreme-precision numerical quadrature, that $C_n$ has a finite large-$n$ limit, namely $C_\infty = 2e^{-2\gamma}$, with $\gamma$ being the Euler constant. On such a numerological clue we are able to prove the conjecture. We also present results relevant to the more recondite integrals $D_n$ and $E_n$; for example, both these integrals are now known to decay exponentially with $n$, in a certain rigorous sense. Also, both integrals now enjoy proven closed forms for $n = 1, 2, 3, 4$. And for $E_5$, we posit a closed form whose discovery involved three-dimensional integration of an intricate integrand, performed via highly parallel numerical quadrature.
Part I. Experimental-mathematics approaches

1 Background and nomenclature

This research began as a quest for a numerical scheme for high-precision values of Ising susceptibility integrals, in our preferred normalization being defined as

\[ D_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{\prod_{i<j} \left( \frac{u_i-u_j}{u_i+u_j} \right)^2}{\left( \sum_{j=1}^n (u_j + 1/u_j) \right)^2} \frac{du_1 \cdots du_n}{u_n}. \]  

The integrals \( D_n \) appear in susceptibility expansions from Ising theory, as detailed in the literature [17, 18, 21, 22, 23, 24]. Very briefly, the importance of \( D_n \) in Ising physics runs as follows [20]. Magnetic susceptibility \( \chi(T) \)—essentially a spin-spin correlation in the 2D Ising model—depends asymptotically on temperature \( T \) as

\[ \chi_{\pm}(T) \sim C_{0\pm} \left( 1 - \frac{T}{T_c} \right)^{-7/4}, \]

where \( T_c \) is the critical temperature and the subscript \( \pm \) indicates whether \( T > T_c \) (plus) or \( T < T_c \) (minus). The connection with our present analysis is that the so-called susceptibility amplitudes

\[ C_{0+} = C_+ \sum_{n=0}^{\infty} I_{2n+1}, \]
\[ C_{0-} = C_- \sum_{n=1}^{\infty} I_{2n}, \]

where \( C_{\pm} \) are explicitly known constants [21], involve integrals \( I_n \) proportional to our \( D_n \); specifically

\[ I_n := 2^{-n} \pi^{1-n} D_n. \]

We have taken the \( D_n \) integral, therefore, as a prime candidate for experimental-mathematics research; i.e. knowing a \( D_n \) in closed form traces immediately back to an important term from a susceptibility expansion.

It was suggested to us by C. Tracy [20] and emphasized by J-M. Maillard [14] that evaluation of the \( D_n \) susceptibility integrals—to sufficient precision—could well lead to experimental-mathematical capture for some \( n > 4 \). In fact, the appearance of Riemann-zeta evaluations is already a known phenomenon in related nonlinear physics [7]. Now, because closed forms for the \( D_n \) are difficult, as are numerical evaluations for large \( n \), we elected to study first some related but simpler integrals. This was the initial motive for defining the entities

\[ C_n := \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{1}{\left( \sum_{j=1}^n (u_j + 1/u_j) \right)^2} \frac{du_1 \cdots du_n}{u_n}. \]
Because these $C_n$ are relatively easy to resolve to extreme\textsuperscript{1} precision, we remain hopeful that finding closed forms experimentally for some $C_n$ will suggest, at least qualitatively, what fundamental constants might appear in the higher $D_n$. Indeed, a mere glance at similarities between closed forms at a given level $n$ vindicates this expectation (see Table 1 in Section 2). In the sense that we are taking not a physics-oriented but an experimental-mathematics approach, the present work is reminiscent of [9, pg. 312–313] and [6, 5, 4]. Moreover, as enunciated in our Abstract, these $C_n$ for large $n$ appeared to approach a positive constant, in fact rather rapidly. The natural conjecture and proof of same are given in a later section.

While only $D_n$, being an Ising-susceptibility component, has immediate physical significance, we assert that the $C_n$, $E_n$ are mathematically natural variants, with the $C_n$ being numerically accessible (and asymptotically well behaved) while the $E_n$ provide important bounds on the elusive $D_n$. In any case, we anticipate experimental-mathematical capture to provide “similar” fundamental constants for all these Ising-class integrals.

We have found the following symbolic machinations particularly useful. For either integral (1) or (2), consider the simplex $u_1 > u_2 > \cdots > u_n$. We may then use the change of variables $u_k := \prod_{i=1}^k t_i$, with $t_1 \in (0, \infty)$ and all other $t_i \in (0, 1)$, to transform the integration domain into a finite one. Define

\[
\begin{align*}
\quad w_k & \quad := \prod_{i=2}^k t_i, \\
\quad v_k & \quad := \prod_{i=k}^n t_i,
\end{align*}
\]

and the functions

\[
\begin{align*}
\quad A_n(t_2, t_3, \ldots, t_n) & \quad := \left( \prod_{\substack{n \geq k > j \geq 1}} \frac{u_k/u_j - 1}{u_k/u_j + 1} \right)^2, \\
\quad B_n(t_2, t_3, \ldots, t_n) & \quad := \frac{1}{(1 + \sum_{k=2}^n w_k)(1 + \sum_{k=2}^n v_k)}. 
\end{align*}
\]

Then the relevant integrals can be cast like so:

\[
\begin{align*}
D_n & \quad = \quad 2 \int_0^1 \cdots \int_0^1 A B dt_2 dt_3 \cdots dt_n, \\
C_n & \quad = \quad 2 \int_0^1 \cdots \int_0^1 B dt_2 dt_3 \cdots dt_n,
\end{align*}
\]

Here, the $1/n!$ normalization has disappeared due to the $n!$ ways of ordering the simplex indices, and we have symbolically integrated over $t_1$. It will turn out

\textsuperscript{1}By “extreme precision” we mean, loosely, “precision sufficient for reasonable confidence in experimental detection,” which in our experience is between 100 and 1000 digits.
to be useful to define also an integral
\[ E_n := 2 \int_0^1 \cdots \int_0^1 A(t_2)dt_2 dt_3 \cdots dt_n. \]  
(5)

It transpires that, for all \( n \geq 1 \), we have
\[ D_n \leq E_n \leq C_n. \]  
(6)

The first inequality is trivial, and also trivial is the implicit relation
\[ D_n \leq C_n, \]
since by their very definitions \( A, B \in [0, 1] \) on the domain of integration. Almost as obvious is the inequality \( E_n \leq n^2 D_n \). But it will require more work to establish the hardest branch \( E_n \leq C_n \) (see text after Theorem 3).

Beyond such inequalities, one can go yet further in the matter of asymptotic analysis. Using representations (3, 5) we shall be able to establish that \( (D_n), (E_n) \) sequences are both strictly monotone decreasing and genuinely exponentially decaying in the sense that for positive constants \( a, b, A, B \) we have
\[ \frac{a}{b^n} \leq D_n \leq E_n \leq \frac{A}{B^n}. \]

In Section 7 we shall not only prove this (Theorem 3) but also give effective \( a, b, A, B \) values.

2 Tabulation of results

Table 1 exhibits known evaluations of \( D_n \) and the structurally related Ising-class integrals \( C_n, E_n \). The reader should beware of varying normalizations in the physics literature; yet every Ising-susceptibility integrand involves, as do our \( D_n \) from (1), some manner of combinatorial entity constructed over \( (i, j) \) index pairs. (For \( n = 1 \) we interpret the \( (i < j) \) product in the definition (1) as unity.)

Our particular normalization for \( D_n \) vs. \( I_n := D_n/(2^n \pi^n - 1) \) means, in reference to our Table 1, that \( I_1 = 1, I_2 = 1/(12 \pi), \) and so on. For example, the ferromagnetic constant of solid-state physics is thus \( I_3 = 3/(8 \pi^2) \approx 0.00081446 \), as in the literature \cite{22} \cite{15}. The entity \( I_4 = D_4/(16 \pi^3) \approx 0.000025448 \) is the McCoy–Tracy–Wu constant resolved in closed form c. 1977 \cite{19}, while \( D_5 \), though still algebraically elusive, was resolved to 30 decimal places by B. Nickel in 1999 \cite{15}—these respective symbolic and numerical achievements being remarkable for their eras. (See Section 12 and Appendix 2 for our recent extreme-precision renditions of \( D_5, E_5 \).)

In the construction of Table 1, we have invoked a Dirichlet L-function that occurs frequently in mathematical physics (see \cite[§2.6]{8}, \cite[Chapter 3]{9}) namely\(^2\)
\[ L_{-3}(2) := \sum_{n \geq 0} \left( \frac{1}{(3n+1)^2} - \frac{1}{(3n+2)^2} \right), \]

\(^2\)Note that some literature treatments (e.g. \cite{19}) use the Clausen function \cite{13} which is algebraically related to the stated L-function.
\[
\begin{array}{ccc}
 n & C_n & D_n & E_n \\
 1 & = 2 & = 2 & = 2 \\
 2 & = 1 & = 1/3 & = 6 - 8 \log 2 \\
 3 & = L_{-3}(2) & = 8 + 4\pi^2/3 - 27 L_{-3}(2) & = 10 - 2\pi^2 - 8 \log 2 + 32 \log^2 2 \\
 4 & = 7\zeta(3)/12 & = 4\pi^2/9 - 1/6 - 7\zeta(3)/2 & = 22 - 82\zeta(3) - 24 \log 2 \\
 & & & + 176 \log^2 2 - 256 (\log^3 2)/3 \\
 & & & + 16\pi^2 \log 2 - 22\pi^2/3 \\
 5 & 0.6657598001\ldots & 0.0024846057\ldots & \gamma - 42 - 1984 \text{Li}_4(1/2) + 189\pi^4/10 \\
 & & & - 74\zeta(3) - 1272\zeta(3) \log 2 \\
 & & & + 40\pi^2 \log^2 2 - 62\pi^2/3 \\
 & & & + 40(\pi^2 \log 2)/3 + 88 \log^4 2 \\
 & & & + 464 \log^2 2 - 40 \log 2 \\
 6 & 0.6486342090\ldots & 0.0004891422\ldots & 0.00068783287\ldots \\
 \ldots & \sim 2e^{-2\gamma} & \Omega \left( \frac{1}{\pi n} \right), O \left( \frac{1}{\pi^2 n} \right) & \Omega \left( \frac{1}{\pi n} \right), O \left( \frac{1}{\pi^2 n} \right) \\
 n & & & \\
\end{array}
\]

Table 1: What is known of Ising-class integrals: ‘=’ connotes proven and ‘? ’ detected experimentally.

and also the standard \textit{polylogarithm}

\[
\text{Li}_s(z) := \sum_{k \geq 1} \frac{z^k}{k^s}.
\]

All the closed forms in Table 1 are proven, except for the one shown for \(E_5\)—an experimental result based on a 240-digit computation. This \(E_5\) relation was found using PSLQ at a confidence level of 190 digits beyond the level that could reasonably be ascribed to numerical round-off error (we will describe the computation of \(E_5\) in Section 12). As for large-\(n\) behavior implied in Table 1, we know \(C_\infty\) rigorously as an exotic constant, while the \(\Omega, O\) notation means both \(D_n, E_n\) decay exponentially but no faster than that (see Theorem 3). Numerical entries here are known to higher precision than is displayed—in fact we know many \(C_n\), as well as \(D_5, E_5\), to extreme precision (see Section 12 and Appendix 1).
3 Bessel-kernel representations for $C_n$

Let us first use the transformation $u_k \to e^{x_k}$ in (1), (2) to achieve the representations

$$D_n = \frac{1}{n!} \int \mathcal{D}\vec{x} \prod_{i<j} \tanh^2 \left( \frac{x_i - x_j}{2} \right) \left( \cosh x_1 + \cdots + \cosh x_n \right)^2,$$

(7)

$$C_n := \frac{1}{n!} \int \mathcal{D}\vec{x} \left( \cosh x_1 + \cdots + \cosh x_n \right)^2,$$

(8)

where here and elsewhere $\int \mathcal{D}\vec{x}$ is interpreted symbolically as the full-space operation $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_n$.

Now $C_n$ can be put in the form

$$C_n = \frac{1}{n!} \int_0^\infty p \int \mathcal{D}\vec{x} e^{-p \sum \cosh x_k} \, dp.$$

which leads to an attractive, 1-dimensional integral

$$C_n = \frac{2^n}{n!} \int_0^\infty p K_0^n(p) \, dp,$$

(9)

where $K_0$ is the standard, modified Bessel function [1]

$$K_0(p) := \int_0^\infty e^{-p \cosh t} \, dt.$$

(10)

In anticipation of experiments and theorems to follow, we state ascending and asymptotic expansions of $K_0$, respectively:

$$K_0^{(asc)}(t) = \sum_{k \geq 0} \frac{t^{2k}}{4^{k!}k!} \left( H_k - \left( \gamma + \log \frac{t}{2} \right) \right),$$

(11)

$$K_0^{(asy)}(t) \sim \sqrt{\frac{\pi}{2t}} e^{-t} \sum_{m=0}^{\infty} \frac{(-1)^m ((2m)!)^2}{m! (32t)^m},$$

(12)

where $\gamma$ denotes the Euler constant and the $H_k := \sum_{m \leq k} 1/m$ are the harmonic numbers, with $H_0 := 0$. It is known [1] that the error accrued in taking terms through index $m = M$ in (12) is no larger than the first dropped term (and with sign of that dropped term). We also make use of the representation

$$K_\nu(x) = \frac{2^\nu \Gamma(\nu + 1/2)}{\Gamma(\nu + 1/2)} \int_0^\infty \frac{\cos(xt) \, dt}{(1 + t^2)^{\nu + 1/2}},$$

(13)

It is a both a convenience and a pleasure to invoke thus the “curly-D” of Feynman path-integral lore, as the present research traces back to solid-state physics, not to mention that we contemplate at one juncture an infinite-dimensional limit.
valid for real \( x > 0 \) and \( \operatorname{Re}(\nu) > -1/2 \) [1]. Observe that in the ascending series (11) the leading term is \( -\gamma - \log(t/2) \), revealing a logarithmic singularity at the origin. It will turn out to be lucrative to define a “pivot point”

\[
p_0 := 2 e^{-\gamma},
\]

such that said leading term vanishes at \( t = p_0 \). To simplify our derivations to follow, we also adopt an “effective big-\(O\)” notation, as

\[
\Theta(f) = g,
\]

meaning \( |f/g| \leq 1 \), equivalent to \( O() \) notation but with implied big-\( O \) multiplier of unity.

Again in anticipation of experiment and theory, we state the next result.

**Lemma 1** For the modified Bessel function \( K_\nu(x) \) with real \( \nu \geq 0 \) and real \( x > 0 \), with pivot point \( p_0 \), we have

\[
0 < K_\nu(p_0) < \Gamma(\nu)\frac{2^{\nu-1}}{p^\nu}; \; \nu > 0,
\]

(14)

\[
K'_0 = -K_1,
\]

(15)

\[
K_0(p) = -\gamma - \log(p/2) + \Theta(p/3); \; p \in (0, p_0),
\]

(16)

\[
K_0(p) < \sqrt{\frac{\pi}{2p}} e^{-p}.
\]

(17)

**Proof.** Relation (14) follows easily from integral (13), since \( |\cos| \leq 1 \). Relation (15) is standard [1]. Relation (16) follows from inspection of the ascending series (11) over the finite interval \((0, p_0)\). (Note that \( \Theta(p/3) \) is simply some function bounded by \( p/3 \) on said interval, and could also be written \( p \Theta(1/3) \).) Relation (17) either follows from general asymptotic theory [1], or from the observation that \( \int_0^\infty e^{-p \cosh x} dx < e^{-p} \int_0^\infty e^{-px^2/2} dx \). \( \text{QED} \)

## 4 Experiment leads to theory

Later in Section 8 we discuss numerical evaluation of \( C_n \) for large \( n \). Even a cursory examination of the high-precision numerical results displayed in Appendix 1 suggests that \( C_n \) appears to approach a definite limit, namely

\[
C_\infty = 0.63047350337438679612204019271087890435458707871273234\ldots
\]

After inserting the numerical value we obtained for \( C_{1024} \) into the smart lookup facility of the CECM *Inverse Symbolic Calculator* at [http://oldweb.cecm.sfu.ca/cgi-bin/isc/](http://oldweb.cecm.sfu.ca/cgi-bin/isc/) we obtained the output:
Mixed constants, 2 with elementary transforms.

\[ 6304735033743867 = \frac{sr(2)^2}{\exp(gamma)^2} \]

In fact, according to our calculations,

\[ 0 < C_{1024} - 2e^{-2\gamma} < 10^{-300}. \]

On the basis of this and other observations, we were convinced of the truth of the following, experimentally motivated conjecture:

**Conjecture 1** The sequence of integrals \((C_n : n = 1, 2, 3, \ldots)\) is strictly decreasing. Moreover, we have the finite limit

\[ \lim_{n \to \infty} C_n = 2e^{-2\gamma}. \]

Indeed, armed with confidence in the above conjecture, we may proceed to prove all aspects of the conjecture, starting with

**Theorem 1** \((C_n : n = 1, 2, 3, \ldots)\) is strictly decreasing.

**Proof.** We may integrate by parts, starting with equation (9), to arrive, via Lemma 1 (15), at

\[ C_n = \frac{2^{n-1}}{(n-1)!} \int_0^\infty p^2K_1(p)K_0^{n-1}(p) \, dp. \quad (18) \]

We may therefore express a difference

\[ C_{n-1} - C_n = \frac{2^{n-1}}{(n-1)!} \int_0^\infty p(1-pK_1(p))K_0^{n-1}(p) \, dp \quad (19) \]

But, by Lemma 1 (14), the integrand in (19) is nonnegative on \(p \in (0, \infty)\), whence \(C_{n-1} - C_n > 0\). \(\Box\)

Our next observation is that certain generating functions can be used to extract limits of monotonic sequences. We have

**Lemma 2** Let \((r_n : n = 1, 2, 3, \ldots)\) be a positive, strictly monotone-decreasing sequence. Denote, then, \(r = \lim_n r_n\), and define a generating function

\[ R(z) := \sum_{n=1}^\infty r_n z^n. \quad (20) \]

Then \(r = \lim_{z \to 1^{-}} (1-z)R(z)\).
Proof. For $z \in (0, 1)$, we have

$$(1 - z) R(z) := rz + T(z)$$

where $T(z) := (1 - z) \sum_{n=1}^{\infty} (r_n - r) z^n$.

Now fix $\epsilon > 0$, and observe that

$$T(z) \leq r_1 N (1 - z) + \frac{\epsilon}{2} z^{N+1},$$

when $N$ is chosen such that $r_M - r < \epsilon/2$ for $M \geq N$.

Set $\delta := \min\{\epsilon/(2(r + r_1 N)), \epsilon/2\}$. It follows that $|(1 - z) R(z) - r| < \epsilon$ for $1 - z \leq \delta$.

QED

Remark 1 Deeper such results obtain in Abelian–Tauberian theory, yet this lemma is quite sufficient for our present purpose.

Now we contemplate the generating function

$$C(z) = \sum_{n=1}^{\infty} C_n z^n,$$

and we use this construct to establish the large-$n$ limit of our $C_n$:

**Theorem 2** The sequence $(C_n : n = 1, 2, 3, \ldots)$ has

$$\lim_{n \to \infty} C_n = 2 e^{-2\gamma}.$$

**Proof.** The generating function (21) at hand may be developed, via the representation (9) and then (16), (17) of Lemma 1, like so:

$$C(z) = \int_0^{p_0} pe^{2zK_0(p)} dp - 1 \int_0^{p_0} pe^{2z(-\gamma - \log(p/2) + \log(1/3))} dp + \Theta(c),$$

where $c$ is a constant independent of $z$. Using the fact that for $x \in [0, 1]$ we have $e^x = 1 + \Theta(x + x^2)$, we obtain

$$C(z) = e^{-2\gamma} \frac{2^2 z}{2 - 2z} p_0^{2 - 2z} + \Theta \left( c + \frac{c_1}{3 - 2z} + \frac{c_2}{4 - 2z} \right),$$

where $c_1, c_2$ are again $z$-independent constants. It follows that

$$\lim_{z \to 1^-} (1 - z) C(z) = 2 e^{-2\gamma},$$

9
and via Lemma 2 the theorem follows. QED

It has become evident—largely on hindsight—that integration of (9) up to only the pivot point \( p_0 \) generally leaves an extremely small residual integral. Indeed, if we interpret the representation (9) as

\[
C_n = \frac{2^n}{n!} \left( \int_{0}^{p_0} + \int_{p_0}^{\infty} \right) p K_0^n(p) \, dp
\]

then the second integral is easily seen—via Lemma 1 (17)—to be factorially minuscule, in the sense that for any \( n > 1 \),

\[
C_n = \frac{2^n}{n!} \int_{0}^{p_0} p K_0^n(p) \, dp + \Theta\left( \frac{1}{n!} \right).
\]

By inserting the ascending series (11) into this pivot integral over \( p \in (0, p_0) \), we obtain—after various manipulations—the asymptotic expansion

\[
C_n \sim \frac{2^n}{n!} \int_{0}^{\infty} \sum_{J=1}^{\infty} e^{-2J\gamma} \sum_{k_1 + \cdots + k_n = J-1} \int_{0}^{\infty} e^{-y} \, dy \prod_{i=1}^{n} \frac{2H_{k_i} + y/J}{k_i!^2},
\]

where the partitions are over nonnegative integers \( k_i \). This attractive expansion is in the spirit of mathematical physics—it is essentially a perturbation expansion with coupling parameter \( e^{-2\gamma} \). Indeed, the first few terms go

\[
C_n \sim 2 e^{-2\gamma} + n + 4 \frac{e^{-4\gamma} + 2n^2 + 23n + 57}{3^n \cdot 6} e^{-6\gamma} + \ldots
\]  \quad (22)

Remarkably, just these displayed terms with \( n = 32 \) yield a \( C_{32} \) value to 17 good decimals—an efficient way to effect quadrature to reasonable precision on a 32-dimensional integral!

5 Further dimensional reduction for \( C_n \)

One way to proceed analytically is to invoke a scaled-coordinate system. Using the representation

\[
C_n = \frac{4}{n!} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \frac{1}{\left( \sum_{j=1}^{n} (u_j + 1/u_j) \right)^2} \, du_1 \cdots du_n, \tag{23}
\]

we let the first coordinate \( u_1 \) be an overall scale. This is much the same as using \( n \)-dimensional “spherical coordinates” involving the scale (radius) \( r \) and \( (n-1) \) angular coordinates. Let us posit, for (5.1),

\[
u_1 = r, \quad u_2 = rx_0, \quad u_3 = rx_1, \ldots, \quad u_n = rx_{n-2}.
\]
It turns out that this scaled-coordinate transformation generally reduces the integral (23) by two dimensions, since one may easily integrate symbolically over \( r \), then almost as easily over \( x_0 \). Inter alia we find, trivially, that

\[
C_1 = 2 \quad \text{and} \quad C_2 = 1,
\]
as start out our Table 1 entries for \( C_n \). Beyond this, the general procedure yields an \((n - 2)\)-dimensional form

\[
C_n = \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \frac{\log P}{Q - 1} \frac{dx_1}{x_1} \cdots \frac{dxn - 2}{x_n - 2}, \tag{24}
\]
for \( n \geq 3 \), where \( P, Q \) are the interesting constructs (here and in what follows, \( P, Q \) are to be written in terms of the available integration variables \( x_1, \ldots \)):

\[
P := 1 + x_1 + \cdots + x_{n-2}, \tag{25}
\]
\[
Q := P \cdot (1 + 1/x_1 + \cdots + 1/x_{n-2}). \tag{26}
\]

Thus, for \( n = 3 \) we only need evaluate a one-dimensional integral:

\[
C_3 = \frac{2}{3} \int_0^\infty \frac{\log(1 + x)}{x^2 + x + 1} \, dx,
\]
which, via the transformation \( x \to 1/t - 1 \) becomes

\[
= \frac{2}{3} \int_0^1 \frac{(1 + t) \log t}{1 + t^3} \, dt
\]
\[
= \frac{2}{3} \sum_{n \geq 0} (-1)^n \left( \frac{1}{(3n + 1)^2} + \frac{1}{(3n + 2)^2} \right)
\]
\[
= L_{-3}(2),
\]
where the factor ‘2/3’ is removed from the final line on the observation that
\[
1/1^2 + 1/2^2 - 1/4^2 - 1/5^2 + \cdots = (1 + 1/2)(1/1^2 - 1/2^2 + 1/4^2 - 1/5^2 + \cdots).
\]

For \( n = 4 \) we had conjectured, on the basis of numerical values, such as those in Appendix 1, and PSLQ integer relation finding facilities [8], that

\[
C_4 \equiv \frac{7}{12} \zeta(3).
\]

This turns out to be true, derivable via the 2-dimensional reduced integral

\[
C_4 = \frac{1}{6} \int_0^\infty \int_0^\infty \frac{\log(1 + x + y)}{(1 + x + y)(1 + 1/x + 1/y) - 1} \frac{dx \, dy}{x \, y},
\]
Indeed performing the internal integration leads to

\[
C_4 = \frac{1}{6} \int_0^\infty \frac{\text{Li}_2(x^{-1}) - \text{Li}_2(x)}{x^2 - 1} \, dx
\]
\[
= \frac{1}{3} \int_0^1 \frac{\text{Li}_2(x^{-1}) - \text{Li}_2(x)}{x^2 - 1} \, dx,
\]
by transforming $x \to 1/x$. Here $\text{Li}_2(x) := \sum x^n/n^2$, is the dilogarithm, [8],
analytically continued. Now, integrating by parts leads to

$$24 C_4 = 8 \int_0^1 \frac{\ln^2 (x+1)}{x} \mathrm{d}x - 8 \int_0^1 \frac{\ln (1+x) \ln (1-x)}{x} \mathrm{d}x$$
$$-4 \int_0^1 \frac{\ln (1+x) \ln (x)}{x} \mathrm{d}x + 4 \int_0^1 \frac{\ln (1+x) \ln (1-x)}{x} \mathrm{d}x$$
$$= 2\zeta(3) + 5\zeta(3) + 3\zeta(3) + 4\zeta(3) = 14\zeta(3),$$

where each integral is an integral multiple of $\zeta(3)$, as can be obtained from
the analysis of the trilogarithm $\text{Li}_3(x) := \sum x^n/n^3$, in [13, §6.4 and Appendix A3.5].

For $n \geq 5$ we may continue the procedure at least once more and write an
$(n-3)$-dimensional integral. One expresses the coordinates $(x_1, \ldots, x_{n-2})$ using
$x_1$ as scale, to arrive at

$$C_n = \frac{4}{n!} \int_0^\infty \cdots \int_0^\infty \mathcal{M}(Q) \frac{dt_1}{t_1} \cdots \frac{dt_{n-3}}{t_{n-3}},$$

(27)

where, here, $Q := Q(t_1, \ldots, t_{n-3})$ is the $Q$-form (25) for $(n-3)$ dimensions, and

$$\mathcal{M}(Q) := \int_0^\infty \frac{\log(1+u)}{u^2 + Qu + Q} \mathrm{d}u.$$

Moreover, $\mathcal{M}(Q)$ is directly expressible in terms of logarithms and dilogarithms.

In fact, with $\alpha := \frac{Q}{2} - 1 - \left(\left(\frac{Q}{2} - 1\right)^2 - 1\right)^{1/2} > 0$ so that the larger quantity

$$1/\alpha = \frac{Q}{2} - 1 + \left(\left(\frac{Q}{2} - 1\right)^2 - 1\right)^{1/2}$$

we have

$$(Q^2 - 4Q)^{1/2} \mathcal{M}(Q) = \text{Li}_2(-\alpha) - \text{Li}_2(-1/\alpha)$$
$$= 2 \text{Li}_2(-\alpha) + \zeta(2) + \frac{1}{2} \log^2(\alpha)$$

where the last equality follows from [13, A.2.1. (5)]. This development, for
example, represents $C_5$ as a double integral, namely

$$C_5 = \frac{1}{30} \int_0^\infty \int_0^\infty \mathcal{M}(Q) \frac{dx \, dy}{x \, y}$$

(28)

$$= \frac{1}{30} \int_0^1 \int_0^1 \mathcal{M}(Q) \frac{dx \, dy}{x \, y},$$

(29)

where $Q := (1+x+y)(1+1/x+1/y)$.

While the details are a bit foreboding, all of this suggests that in general $C_n$
may well be a combination of polylogarithmic constants of order at most $n-1$.
In this language the results we have obtained are $C_3 = (4/3) \Im \text{Li}_2((-1)^{1/3})/\sqrt{3}$
and $C_4 = -(56/3) \Re \text{Li}_3((-1)^{1/2})/3$. 

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On the other hand, there is some theoretical evidence in support of a possible “blockade” against closed forms for \( C_5 \) and beyond. Namely, the Adamchik algorithm [2] for evaluating integrals of argument powers with Bessel-function powers does not extend beyond 4th powers of the Bessel terms [3]. Thus \( C_4 \) can be derived via the Adamchik method, but evidently \( C_5 \) cannot.

To summarize: We have rigorously established closed forms as in Table 1 for \( C_1 \) through \( C_4 \). However, the higher \( C_n \)'s remain elusive. It is pleasing—and justifies our original research motivation—that the above closed forms for the \( C_n \) involve, at least for these small \( n \), similar fundamental constants as appear for the few known \( D_n \) appearing in Table 1.

### 6 Symbolics for the susceptibility integrals \( D_n \)

A first approach to closed forms for \( D_n \) is to exploit various advantages of integral representation (3). We have, with \( A_n B_n \) denoting the integrand with the \((n-1)\) variables \( t_2, t_3, \ldots, t_n \), \( A_1 B_1 := 1 \) and \( A_2 B_2 = (t_2 - 1)^2 / (t_2 + 1)^4 \), while

\[
A_3 B_3 = \frac{(t_2 - 1)^2 (t_2 t_3 - 1)^2 (t_3 - 1)^2}{(t_2 + 1)^2 (t_2 t_3 + 1)^2 (t_3 + 1)^2 (t_2 + t_2 t_3 + 1) (t_2 t_3 + t_3 + 1)}
\]

Hence, \( D_1 = 2 \) while

\[
D_2 = 2 \int_0^1 \frac{(x - 1)^2}{(x + 1)^4} \, dx = \frac{1}{3}
\]

\[
D_3 = \frac{1}{3} \int_0^1 \int_0^1 A_3 B_3(x, y) \, dx \, dy
\]

\[
= \frac{2}{3} \int_0^1 \int_0^x A_3 B_3(x, y) \, dx \, dy,
\]

which integral Maple can reduce\(^4\) to the exact value for \( D_3 \) given in our introduction, at least in the form

\[
18 i \text{Li}_2 \left(\frac{1}{2} - \frac{1}{2} i \sqrt{3}\right) \sqrt{3} - 18 i \text{Li}_2 \left(\frac{1}{2} + \frac{1}{2} i \sqrt{3}\right) \sqrt{3} + 24 + 4 \pi^2.
\]

As noted in our introduction, a closed form for \( D_4 \) is known (see the caption to our Table 1, with reference to the McCoy–Tracy–Wu constant), yet the status of higher values is open. The representation above for \( D_4 \) via \( A_4 B_4 \) was sufficient to compute 14 decimal places in Maple and so to recover this constant with PSLQ. In principle, these methods and especially those of Section Seven allow for a complete symbolic resolution of \( D_4 \) but the details are somewhat daunting.

\(^4\) Adequate Maple code is

\[
p := (x - 1)^2 * (x - y)^2 * (y - 1)^2 / (x + 1)^2 / (x + y)^2 / (y + 1)^2 / (1 + y + x) / (y + x + x * y):
\]

\[
d := \text{int}(\text{int}(p, x = 0..\text{infinity}), y = 0..\text{infinity}) : \text{evalc(value(d))};
\]

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For a second analytical foray, one may envision possible roles of the $C_n$ in $D_n$ analysis. Looking longingly at (7), one may write

$$D_n = \frac{1}{n!} \int \mathcal{D} \vec{x} \prod_{i<j} \left( 1 - \text{sech}^2 \left( \frac{x_i - x_j}{2} \right) \right) \left( \cosh x_1 + \cdots + \cosh x_n \right)^2.$$  \hfill (30)

This form reveals that in a specific sense, $C_n$ amounts to a first term in a finite sum of integrals. Indeed, one might expand the product into partial products of $\text{sech}^2$ terms, and furthermore employ the attractive Fourier identity

$$\text{sech}^2 \left( \frac{z}{2} \right) = 2 \int_{-\infty}^{\infty} \frac{k}{\sinh(\pi k)} e^{ikz} \, dk.$$  \hfill (31)

We also have the convenient integral representation

$$\int_{-\infty}^{\infty} e^{-p \cosh x + ikx} \, dx = 2K_{ik}(p).$$

Now for small $n$ one may extract closed forms for $D_n$ using a $(p,k)$-transform apparatus. For example, we have

$$D_2 = C_2 - 4 \int_{-\infty}^{\infty} \frac{k \, dk}{\sinh \pi k} \int_{0}^{\infty} pK_{ik}^2(p) \, dp$$

$$= C_2 - 2\pi \int_{-\infty}^{\infty} \frac{k^2 \, dk}{\sinh^2 \pi k} = \frac{1}{3}.$$  

Notice the direct involvement of the $C_2$ value as a 1st-order term.

For higher $n$, one can still evaluate the Bessel-$K$ integrals in terms of hypergeometric functions, but it is not clear how to handle the rapidly growing number of $k$ variables. Still, these $(p,k)$-transforms may conceivably give rise to high-precision numerical schemes.

The problem with growing $k$-variable counts is that an appropriate term from the natural expansion of representation (30), say

$$\int_{0}^{\infty} p \, dp \int \mathcal{D} \vec{x} \ e^{-p \sum \cosh x_k} \prod_{(a,b) \in \mathcal{P}} \text{sech}^2((x_a - x_b)/2),$$

where $\mathcal{P}$ is some set of index pairs, has expansion

$$\int_{0}^{\infty} p \, dp \int \mathcal{D} \vec{k} \prod_{q=1}^{c} \frac{k_q}{\sinh(\pi k_q)} K_{i\nu_q}(p),$$

where $c = \text{card}(\mathcal{P})$. Unfortunately, $c$ can be $O(n^2)$.

Still it may somehow be possible to somehow employ a higher-order sech-Fourier transform, namely a generalization of (31) \cite{16}:

$$\text{sech}^{2m}(x/2) = \frac{2^{2m-1}}{(2m-1)!} \int_{-\infty}^{\infty} \frac{k}{\sinh(\pi k)} e^{ikx} \prod_{h=1}^{m-1} (k^2 + h^2) \, dk.$$  

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Likewise, it would be good to know the Fourier transform of
\[ \prod_{(a,b) \in P} \text{sech}^2((x_a - x_b)/2) \]
in terms of at most \( n \) spectral variables \( k_q \), rather than \( c = \text{card}(P) = O(n^2) \) such variables. In any case, it may well be that an appropriate \((k, p)\) transform would lead us back to the highly successful numerical approach that yielded results for the \( C_n \). As interesting as these \((k, p)\) transforms may be, such an approach may be misdirected in the sense that a “perturbation series” for \( D_n \) starting with leading term \( C_n \) is unrealistic, due to the different asymptotic character of \( D_n \), as we next discuss.

7 Asymptotic character of \( D_n \) and \( E_n \)

With a view to proving that \( D_n, E_n \) are genuinely exponentially decaying in a certain sense, we first note the examples

\[
E_1 := 2, \\
E_2 = 2 \int_0^1 A \, dt_2 = 2 \int_0^1 \left( \frac{1 - x}{1 + x} \right)^2 \, dx = 6 - 8 \log 2 \approx 0.454823, \\
E_3 = 2 \int_0^1 \int_0^1 \left( \frac{(1 - x)(1 - xy)(1 - y)}{(1 + x)(1 + xy)(1 + y)} \right)^2 \, dx \, dy \\
= 10 - 2\pi^2 - 8 \log 2 + 32 \log^2 2 \approx 0.0901102,
\]

with \( E_4 \) and \( E_5 \) also enjoying a more extended but similar closed form (see Table 1). Just these few examples suggest exponential decay of the \( E_n \) integrals, with a decay constant about 5 (see Table 2 and Section 11).

For convenience in the theorem to follow, we define

\[ R(x) := \left( \frac{1 - x}{1 + x} \right)^2, \]

and let \( m := n - 1 \), so that \( E_n \) is the integral over the unit \( m \)-cube of the product of (a triangular number) \( m(m + 1)/2 \) instances of \( R \). Specifically, for \( n > 1 \),

\[
E_n = 2 \int_{[0,1]^m} D \prod_{k=1}^m R(x_k) R(x_k x_{k+1}) \cdots R(x_k \cdots x_m).
\]

Observe also that the reduced \( D_n \) integrand is the same \( R \)-product multiplied by the extra factor \( \mathcal{B}_n(x_1, \ldots, x_m) := (1 + x_1 S)^{-1} (T + U x_1)^{-1} \), where

\[
S := 1 + x_2 + x_2 x_3 + \cdots + x_2 \cdots x_m, \\
T := 1 + x_m + x_m x_{m-1} + \cdots + x_m \cdots x_2,
\]

and \( U := x_m \cdots x_2 \).
The sequences \((D_n)\) and \((E_n)\) are both strictly monotone decreasing for \(n \geq 1\). Moreover, \(D_n\) and \(E_n\) enjoy genuine exponential decay; that is, there exist positive constants \(a, b, A, B\) such that for all positive integers \(n\)

\[
\frac{a}{b^n} \leq D_n \leq E_n \leq \frac{A}{B^n},
\]

where effective values are \(\{a, b\} = \{19, 14\}\) and \(\{A, B\} = \{12, 4\}\).

**Proof.** First, monotonicity. By bounding the integral over the first coordinate \(x_1\) we see that

\[
E_n \leq \left( \int_0^1 R(x_1) \, dx_1 \right) E_{n-1} = \frac{E_2}{2} E_{n-1} < 0.26 E_{n-1}.
\]

This establishes strict monotonicity for the sequence \((E_n)\); below we shall tighten this approach to yield better effective constant. As for monotonicity of the \(D_n\), note that for \(m := n - 1\) the \(R\)-product involving the first coordinate \(x_1\) can be bounded as

\[
R(x_1) R(x_1 x_2) \cdots R(x_1 \cdots x_m) \leq e^{-2x_1 S},
\]

where \(S\) is given in the text prior to this theorem. This bound on the \(x_1\)-dependent part can be quickly obtained by taking the logarithm of the \(R\)-product, noting \(\log R(z) = -2(z + z^3/3 + z^5/5 + \cdots) \leq -2z\). Now we obtain an upper bound for the integral over \(x_1\), as

\[
\int_0^1 \frac{e^{-2x_1 S}}{(1 + x_1 S)(1 + U x_1)} \, dx_1 < \frac{0.37}{ST},
\]

where we have used \(\int_0^\infty e^{-2z/(1 + z)} \, dz = e^2 \text{Ei}(1, 2) \approx 0.361\), an exponential integral, [1]. But \(1/(ST)\) is precisely the \(E_n-1\) factor in the integrand for \(D_{n-1} = 2 \int_{[0,1]^{n-2}} A_{n-1} B_{n-1} \, dx\), thus we establish monotonicity in the form \(D_n < 0.37 D_{n-1}\).

Next, for a fundamentally tighter effective upper bound on \(E_n\) (and perforce \(D_n\)—recall the trivial inequality \(D_n \leq E_n\)). For a given \(n\), the integrand for \(E_n/2\) has at least \([n-1]/2\) disjoint triples of the form \(R(x_i) R(x_i x_j) R(x_j)\), as inspection of a few cases suggests. For example, the integrand for \(E_3/2\) with variables \(w, x, y, z\) is

\[
R(w) R(wx) R(wx y) R(wxyz) R(x) R(xy) R(xyz) R(y) R(yz) R(z),
\]

from which one may read off six (underlined) \(R\)'s amounting to \([5-1]/2 = 2\) disjoint triples. Thus the integral for \(E_n/2\) is bounded above by the product of \([n-1]/2\) copies of \(E_3/2\) and so

\[
\frac{1}{2} E_n \leq \left( \frac{2}{E_3} \right)^{-(n-1)/2}.
\]
and the upper bound follows.

Now for the lower bound. The reduced $D_n$ integrand is a product of $m(m+1)/2$ evaluations of $R$ (where $m := n - 1$) times the factor $B_n$. Said integrand is monotone decreasing in all variables $x_1, \ldots, x_m$. That is, the integrand $\iota$ satisfies $\iota(\vec{x}) \leq \iota(\vec{y})$ whenever $x_k \leq y_k$ for all coordinate indices $k$. But this means that for any $\alpha \in [0, 1]$ the integral is bounded below by a natural approximation of the integral over the sub-cube $[0, \alpha]^m$. Namely, we evaluate all the $R$ terms at the corner vector $\vec{\alpha} := (\alpha, \alpha, \ldots, \alpha)$, observing also $B_n(\vec{\alpha}) \geq (1 - \alpha)^2$, and deduce

$$D_n \geq 2(1 - \alpha)^2 \alpha^m \left( \frac{1 - \alpha}{1 + \alpha} \right)^{2m} \left( \frac{1 - \alpha^2}{1 + \alpha^2} \right)^{2m - 2} \left( \frac{1 - \alpha^3}{1 + \alpha^3} \right)^{2m - 4} \cdots \left( \frac{1 - \alpha^m}{1 + \alpha^m} \right)^2$$

(32)

since $\alpha^m$ is the volume of the reduced hyper-cube.

Interestingly, this expression in $\alpha$ may be bounded below by a theta-function term, as we may estimate

$$D_n \geq 2(1 - \alpha)^2 \alpha^m \prod_{k=1}^{\infty} \left( \frac{1 - \alpha^k}{1 + \alpha^k} \right)^{2m} = 2(1 - \alpha)^2 \left( \alpha \theta_4(\alpha) \right)^m,$$

where $\theta_4(q) := \sum_{n \in \mathbb{Z}} (-q)^n^2$ is a Jacobi theta function, see [9]. Now $\alpha \theta_4(\alpha)^2$ has a maximum greater than 0.074 at $\alpha = \alpha_0 > 0.169$ and we conclude that $D_n \geq 2(1 - \alpha_0)^2(0.074)^{n-1}$, leading immediately to the desired lower bound as well as effective constants.

Remark 2 The effective values may be further improved with more aggressive application of the following techniques. For example, $B$ can be $(2/E_p)^{1/(p-1)}$ for any $p > 1$, and so the approximate (nonrigorous) value for $E_8$ in Table 2 yields effective constant $B \approx 4.97$. Likewise, more effort to enhance (32) will presumably improve the lower bound $b$, the remaining inequalities being quite tight.

Corollary 1 For all positive integers $n$, we have $E_n \leq C_n$.

Proof. This follows directly from the observation that even for $n = 2$, Theorem 3 with $A := 12, B := 4.71$ gives us $E_{(n \geq 2)} < 0.54 < 2 e^{-27}$, the right-hand side being inf $n C_n$.

Theorem 3 suggests that $D_n, E_n$ may both follow a truly exponential-decay asymptotic, and numerical work suggests further a universal decay constant, whence we posit:
Conjecture 2 \(D_n, E_n\) both decay exponentially, with the same decay constant. That is, there exist positive constants \(\delta, \Delta, \phi\) such that

\[ D_n \sim \frac{\delta}{\Delta^n} \quad \text{and} \quad E_n \sim \frac{\phi}{\Delta^n}, \]

so that ratios behave as

\[
\lim_{n \to \infty} \frac{D_n}{D_{n+1}} = \lim_{n \to \infty} \frac{E_n}{E_{n+1}} = \Delta,
\]

and

\[
\lim_{n \to \infty} \frac{D_n}{E_n} = \frac{\delta}{\phi}.
\]

Remark 3 If this conjecture is true, we expect, based on the quasi-Monte Carlo (qMC) integrations of Section 8, that \(\Delta \approx 5\) and \(\delta/\phi \approx 0.7\). Moreover, given our rigorous result Theorem 3, is it perhaps reasonable anyway to expect \(\Delta\) to be of order \(b \approx 4.7\).

8 Further dimensional reduction of \(D_n\) and \(E_n\)

We have seen that \(D_n, E_n\) can each be defined by an \((n-1)\)-dimensional integral, via relations (3), (5), and that \(C_n\) can be reduced to an \((n-2)\)-dimensional integral, as in (24) and further to an \((n-3)\)-dimensional form (27). However, it turns out that \(D_n, E_n\) can also be reduced to \((n-2)\)-dimensional forms, albeit with considerable combinatorial complications, as we shall now establish.

We begin by considering the integrand factor \(A\) appearing in (3), (5), and noting the combinatorial recursion that results from an attempt to factor out terms involving only \(t_2\):

\[
A_n(t_2, \ldots, t_n) = \left(\frac{1 - t_2}{1 + t_2}\right)^2 \left(\frac{1 - t_2 t_3}{1 + t_2 t_3}\right)^2 \cdots \left(\frac{1 - t_2 \cdots t_n}{1 + t_2 \cdots t_n}\right)^2 A_{n-1}(t_3, \ldots, t_n).
\]

Observe also that we may write

\[
B_n(t_2, \ldots, t_n) = \frac{b^{-1}}{(1 + t_2(1 + t_3 + t_3 t_4 + \cdots + t_3 \cdots t_n))(1 + (a/b) t_2)}
\]

with

\[
a := t_3 \cdots t_n, \quad b := 1 + t_n + t_n t_{n-1} + \cdots + t_n \cdots t_3.
\]

Next, we observe a key formal identity

\[
\left(\frac{1 - z}{1 + z}\right)^2 = \frac{\partial}{\partial \lambda} \bigg|_{\lambda = 1} \left(\lambda + \frac{4}{1 + \lambda^2}\right)
\]

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which will allow us to create terms \((1-z)^2/(1+z)^2\) via partial differentiation. Now for a parameter vector \(\mathbf{\lambda}\) of dimension \((n-1)\), define

\[
G_n(\mathbf{\lambda}; t_2, \ldots, t_n) := 2 \prod_{k=1}^{n-1} \left( \lambda_k + \frac{4}{1 + \lambda_k \prod_{j=2}^{k+1} t_j} \right).
\]

Putting all this together yields

\[
D_n = \int_0^1 \cdots \int_0^1 \mathcal{A}_{n-1}(t_3, \ldots, t_n) \left( \frac{\partial^{n-1}}{\partial \lambda_1 \cdots \partial \lambda_{n-1}} |_{\lambda_k=1} \int_0^1 G_n B_n \, dt_2 \right) \, dt_3 \cdots dt_n
\]

\[
E_n = \int_0^1 \cdots \int_0^1 \mathcal{A}_{n-1}(t_3, \ldots, t_n) \left( \frac{\partial^{n-1}}{\partial \lambda_1 \cdots \partial \lambda_{n-1}} |_{\lambda_k=1} \int_0^1 G_n \, dt_2 \right) \, dt_3 \cdots dt_n.
\]

Remarkably, as we shall presently show, \(G_n\) and \(G_n B_n\)—for any \(n\)—can each be integrated in closed form with respect to the \(t_2\) coordinate. Furthermore these closed forms may be differentiated with respect to the \(\lambda_k\) and then evaluated at \(\lambda_k = 1\) to provide a legitimate, \((n-2)\)-dimensional integral over \((t_3, \ldots, t_n)\). Indeed, we have a general reduction theorem:

**Theorem 4** For every integer \(n > 2\), each of \(C_n, D_n, E_n\) can be written as an \((n-2)\)-dimensional integral with elementary integrand consisting of algebraic multivariate functions of logarithms.

**Proof.** For a parameter collection \((\sigma_k : k = 1, \ldots, M)\) we know from partial-fraction decomposition that

\[
\int_0^1 \prod_{k=1}^M \frac{1}{1 + \sigma_k t} \, dt = \sum_{i=1}^M \sigma_i^{M-2} \log(1 + \sigma_i) \prod_{j \neq i} (\sigma_i - \sigma_j).
\]

Now the \(t_2\)-dependent part of the product integrand \(G_n B_n\) for \(D_n\) can be written as a product of the type in the integral here, with \(M = n + 1\), \(t := t_2\), and the \(\sigma_k\) involving subsets of variables taken only from \((t_3, \ldots, t_n)\), so immediately we have an algebraic function of logs for an integral over one coordinate \(t_2\). Then we differentiate inside with respect to \(\lambda_1, \ldots, \lambda_{n-1}\) and arrive at an \((n-2)\)-dimensional integral. The same argument goes through for the simpler integrand \(G_n\) of \(E_n\), with \(M = n - 1\). QED

Note that if need be, \(C_n\) can be processed as above, with integrand \(2B_n\)—see (4)—but the previous result (24) gives equivalent reduction. A specific manifestation of the reduction procedure is detailed in Section 12, where we provide some numerical values for \(D_5, E_5, D_6, E_6\).

We were able to reduce \(E_4\) entirely to one-dimensional integrals and ultimately to evaluate it symbolically but for higher dimensions this seems impracticable.
Part II. Various numerical algorithms

9 Algorithm for Bessel-kernel evaluation of $C_n$

As implied in our Abstract and elsewhere, we first approached the $C_n$ integrals experimentally. Our central strategy for a high-precision numerical evaluation scheme for $F(t) = K_0(t)$ in relation (9) is to utilize a combination of an ascending series $F^{(asc)}(t)$ (which is well-suited for small $t$) and an asymptotic series $F^{(asy)}(t)$ (which is well-suited for large $t$), together with a chosen parameter $\lambda$ that is the boundary between the “small” arguments and the “large” $t$.

Given the formulae (11), (12) for the modified Bessel function $K_0$, there are two approaches to computing $C_n$ from (9). The first, suitable for those who have access to symbolic computing software, is simply to write the integral (9) as a sum of two integrals, one from 0 to $\lambda$, and the second from $\lambda$ to $\infty$, and then to symbolically expand suitably truncated versions of (11) and (12) and evaluate the numerous individual integrals that result. We have obtained reliable results by taking $\lambda = D/2$, where $D$ is the desired precision level in digits, and truncating the two series after $3n\lambda$ and $2\lambda$ terms, respectively. This approach suffices to obtain modestly high precision results (at least 30 digits) for $n$ up to eight or so. Beyond this level, the symbolic computing costs become too great to complete in reasonable time.

A second approach is to directly evaluate the integral in (9) using the tanh-sinh numerical quadrature scheme [6], [9, pg. 312–313], where the integrand function is evaluated by either the ascending series (11) or the descending series (12), depending on whether the argument $t$ is less than or greater than $\lambda$. For these calculations, we found it satisfactory to take $\lambda = D$, and to truncate the series summations when the absolute value of the term being added is less than $10^{-D}$ times the absolute value of the current sum.

Tanh-sinh quadrature is remarkably effective in evaluating integrals to very high precision, even in cases where the integrand function has an infinite derivative or blow-up singularity at one or both endpoints. It is well-suited for highly parallel evaluation [4], and is also amenable to computation of provable bounds on the error [5]. It is based on the transformation $x = g(t)$, where $g(t) = \tanh[\pi/2 \cdot \sinh(t)]$. In a straightforward implementation of the tanh-sinh scheme, one first calculates a set of abscissas $x_k$ and weights $w_k$

\[
x_j := \tanh[\pi/2 \cdot \sinh(jh)] \\
w_j := \frac{\pi/2 \cdot \cosh(jh)}{\cosh^2[\pi/2 \cdot \sinh(jh)]},
\]

where $h$ is the interval of integration. Then the integral of the function $f(t)$ on $[-1,1]$ is performed as

\[
\int_{-1}^{1} f(x) \, dx = \int_{-\infty}^{\infty} f(g(t))g'(t) \, dt \approx \sum_{-N}^{N} w_j f(x_j)
\]
where \( N \) is chosen so that the terms \( w_j f(x_j) \) are sufficiently small that they can be ignored for \( j > N \). Full details of a robust implementation are given in [6]. Note that in this particular application, multiple \( C_n \) can be efficiently computed for different \( n \), since the abscissas, weights and \( K_0(t) \) function values at these abscissas are independent of \( n \).

Using this approach, we have been able to evaluate \( C_n \) to very high precision (500-digit accuracy), for \( n \) as large as 1024, which is equivalent to performing a 1024-fold iterated integral in (8). Each of these runs (regardless of \( n \)) requires only about 100 seconds on a 2006-era single-processor computer. Selected high-precision results are exhibited in Appendix 1.

10 Hypergeometric-kernel representation for \( D_n \)

Now to numerical issues for the Ising-susceptibility integrals \( D_n \). It is highly suggestive that we were able to transform the \( C_n \) integral into a 1-dimensional form that admits of arbitrary-precision evaluation. For the \( D_n \), a 1-dimensional form is also possible, at least formally: We do not yet know the precise convergence rate of the approach: consequently, the 1-dimensional representation we achieve below may well not be practical.

A hyperbolic representation for \( D_n \) similar to (8) develops as

\[
D_n := \frac{1}{n!} \int \frac{\mathcal{D}\vec{x}}{(\cosh x_1 + \cdots + \cosh x_n)^2} \prod_{i<j} \tanh^2 \left( \frac{x_i - x_j}{2} \right). \tag{33}
\]

Knowing the identity

\[
\tanh(t - u) = \frac{\tanh t - \tanh u}{1 - \tanh t \tanh u},
\]

we fix \( n \) and ponder the formal power series

\[
\prod_{i<k} \left( \frac{t_i - t_k}{1 - t_i t_k} \right)^2 = \sum_{m_1,\ldots,m_n \geq 0} A(m_1,\ldots,m_n) t_1^{m_1} \cdots t_n^{m_n}.
\]

We intend that this define the set of \( A \) coefficients. So, formally at least, we have

\[
D_n = \int_0^\infty d_n(p) \, dp, \tag{34}
\]

where the kernel \( d_n \) is represented

\[
d_n(p) := \frac{2^n p}{n!} \sum_{m_k \geq 0, \text{ even}} A(m_1,\ldots,m_n) \prod_{k=1}^n T_{m_k}(p),
\]

where

\[
T_m(p) := \int_0^\infty \tanh^m \left( \frac{t}{2} \right) e^{-p \cosh t} \, dt,
\]

21
a confluent hypergeometric function \([1]\) in disguise. In fact,

\[
T_m(p) = e^{-p} \Gamma \left( \frac{m+1}{2} \right) U \left( \frac{m+1}{2}, 1, 2p \right),
\]

where \(U\) is the standard confluent hypergeometric function \([1]\). Still formally, without regard to convergence, we claim a 1-dimensional kernel for the \(D_n\) as

\[
d_n(p) := \frac{2^n p e^{-np}}{n!} \sum_{m_k \geq 0, \text{even}} A(m_1, \ldots, m_n) \prod_{k=1}^{n} \Gamma \left( \frac{m_k + 1}{2} \right) U \left( \frac{m_k + 1}{2}, 1, 2p \right).
\]

(35)

This kernel \(d_n\) is more complicated than the Bessel kernel \(c_n\), which is not unexpected on the basis of the combinatorial product’s stultifying appearance in the original \(D_n\) integrand. As previously intimated, we do not know the convergence rate for \(d_n\), not to mention the efficiency of the integral (34), say in terms of precision vs. a computational bound on the \(m_k\) indices.

It is therefore admitted that this hypergeometric-kernel representation remains of theoretical interest but with as yet untapped numerical power. We do, however, posit the

**Conjecture 3** For fixed \(n\), the 1-dimensional kernel \(d_n(p)\) defined by (35) converges to an integrable function on \(p \in (0, \infty)\), and therefore gives via (34) the correct Ising integral \(D_n\).

In future research it may be useful to analyze the character of the \(A\) tensor. For \(n = 2\), the pattern of the \(A\) coefficients is evident in the small collection:

\[
\{A(2x, 2y)\}_{0 \leq x, y \leq 6} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 3 & 0 & 0 & 0 & 0 \\
0 & 3 & -8 & 5 & 0 & 0 & 0 \\
0 & 0 & 5 & -12 & 7 & 0 & 0 \\
0 & 0 & 0 & 7 & -16 & 9 & 0 \\
0 & 0 & 0 & 0 & 9 & -20 & 11 \\
0 & 0 & 0 & 0 & 0 & 11 & -24
\end{pmatrix}
\]

Useful for calculations on the \(d_n\) kernel may be the ascending and asymptotic series, respectively

\[
\Gamma(a) U(a, 1, z) = \sum_{k \geq 0} \frac{(a)_k z^k}{k!^2} \left( 2\psi(k+1) - \psi(k+a) - \log z \right),
\]

(36)

and

\[
\Gamma(a) U(a, 1, z) \sim \sum_{m \geq 0} \frac{(a)_m (-1)^m \Gamma(m+a)}{m! z^{m+a}}.
\]

(37)
We have shown (Theorem 3) that $D_n, E_n$ are bounded above and below by exponential decay. We also have the decay Conjecture 2 that $D_n, E_n$ share the same decay constant $\Delta$. Contrast this to our proven result $C_n \to \text{constant}$.

The quasi-Monte Carlo (qMC) integrations as shown below in Table 2 suggest that the decay conjecture is true and that $\Delta \approx 5$. Similar theorems and conjectures appear to be reasonable and similar for the related $E_n$, the ratios $E/D$, and so on. Yet, there are interesting open questions, such as: Is $D_{n-1}/D_n$ eventually monotonic decreasing in $n$, as Table 2 suggests? Is the same true for $D_n/E_n$? The qMC algorithm we employed—a “spacefill-Halton hybrid”—is, for some integrands, suitable for high dimensions lying somewhat beyond the reach of the classical Halton sequences, [10, 11]. This qMC approach we employed evidently yields several good decimals even up to dimension $n = 32$. We draw this supposition from the stability of qMC for various $n$-regions, together with tests on the very much more accurately known $C_n$. (See also the recent survey on qMC, [12].)

Referring to Table 2: Rows marked ‘*’ (and two items likewise marked) are exactly known (see closed-form evaluations for $n = 1, 2, 3, 4$ and $E_5$ in Table 1) but all other entities are only numerically understood. Each table entry, for each $n$, involved $2 \cdot 10^9$ qMC points. Errors are not all rigorously known—entries here are to “believed” precision, based on the qMC trends, and we admit to the usual degradation of precision with increasing dimension. Note that all of the tabulated ratios appear to approach respective constants. Though such limits are only conjectured, we have already proven that $D_n, E_n$ themselves decay at least exponentially rapidly to zero as $n \to \infty$.

There is an additional question which further computation may well address. Namely, J-M. Maillard has suggested that ratios $D_n/D_{n+2}$, meaning ratios of consecutive even or odd $D_n$ values, might converge more efficiently (or more
smoothly?) based on general principles of Ising susceptibility expansions [14]. Unfortunately, the qMC values in our Table 2 are evidently too imprecise to decide such an issue. Generally speaking, though, such “parity acceleration” is not uncommon in other fields; for example, the pure-even, pure-odd convergents of continued fractions are examples of split sequences that can each converge efficiently and independently to an actual common limit.

12 Quadrature for higher-dimensional $D_n, E_n$

Compared with the one-dimensional quadrature calculations we described earlier, multi-dimensional extreme-precision quadrature is very expensive indeed. Thus, to perform numerical quadrature for entities such as $D_5, E_5$ and beyond requires a representation in the lowest possible dimension. We have seen in Section 8 that $D_n, E_n$ can each be reduced to an $(n-2)$-dimensional form. The details of this extra reduction can be quite intricate, so we shall summarize the explicit algebra for the elusive $D_5, E_5$, knowing from Theorem 4 that in higher dimensions we can in principle follow the prescription.

For $n = 5$ let us denote variables $w, x, y, z$ and symbolically perform the interior integration over $w$ (which was $t_2$ in Section 8). We use

$$\mathcal{A}_4(x, y, z) := \left(\frac{(1-x)(1-xy)(1-xyz)(1-y)(1-yz)(1-z)}{(1+x)(1+xy)(1+xyz)(1+y)(1+yz)(1+z)}\right)^2$$

$$\mathcal{G}_5 := 2\left(\lambda_1 + \frac{4}{1+\lambda_1w}\right)\left(\lambda_2 + \frac{4}{1+\lambda_2wx}\right)\left(\lambda_3 + \frac{4}{1+\lambda_3wxy}\right)\left(\lambda_4 + \frac{4}{1+\lambda_4wxyz}\right)$$

$$\mathcal{B}_5^{-1} := \frac{zyx}{1+z+zy+zyxw}.$$ 

Then we have, from Section 8,

$$D_5 = \int_0^1 \int_0^1 \int_0^1 \mathcal{A}_4(x, y, z) \left(\frac{\partial^4}{\partial \lambda_1 \partial \lambda_2 \partial \lambda_3 \partial \lambda_4} \bigg|_{\lambda_k = 1}\int_0^1 \mathcal{G}_5 \mathcal{B}_5 \, dw\right) \, dx \, dy \, dz$$

$$E_5 = \int_0^1 \int_0^1 \int_0^1 \mathcal{A}_4(x, y, z) \left(\frac{\partial^4}{\partial \lambda_1 \partial \lambda_2 \partial \lambda_3 \partial \lambda_4} \bigg|_{\lambda_k = 1}\int_0^1 \mathcal{G}_5 \, dw\right) \, dx \, dy \, dz.$$ 

The results for this procedure are two respective integrals for $D_5, E_5$, over the three variables $x, y, z$. As we have intimated, the details are overwhelmingly complicated, producing enormous expressions involving multivariate polynomials, rational functions and logarithms. To give but one example, we present here the stultifying triple integral we used to compute $D_5$:
\[ D_5 = \int_0^1 \int_0^1 \int_0^1 \left[ 2(1-x)^2(1-y)^2(1-xy)^2(1-z)^2(1-yz)^2(1-xyz)^2 \right. \\
\left. \left( -4(y+1)(xy+1) \log(2)(5y^3z^2x^7 - y^4z^2(4y+1)z^3 + 3(y^2+1)z^2 + 4(y+1)z + 5) x^5 + y^2(4y+1)z^3 + 3(y^2+1)z^2 + 4(y+1)z - 1) x^4 + y(z(z^2 + 4z \\
+ 5)y^2 + 4(z^2 + 1)y + 5z + 4) x^3 + \left( (-3z^2 - 4z + 1)y^2 - 4zy + 1) x^2 - (y(5z + 4) \\
+ 4) x - 1) \right) / [(x - 1)^3(xy - 1)^3(xy^2 - 1)] + [(y - 1)^2y^2(z - 1)^2z^2(yz \\
- 1)]^2 + 2y^3z(3z - 1)^2z^3y^5 + z^2(5z^3 + 3z^2 + 3z + 5) y^4 + z(1 - z)^2z \\
(5z^2 + 16z + 5)y^3 + (3z^5 + 3z^4 - 22z^3 - 22z^2 + 3z + 3) y^2 + 3(-2z^4 + z^3 + 2) \\
z^2 + z - 2) y + 3z^3 + 5z^2 + 5z + 3) x^5 + y^2(7z - 1)^2z^2y^6 - 2x^3(3z^3 + 15z^2 \\
+ 6z^2 - 6z + 1) y^3 + (7z^2 - 3z^2 + 15z^2 + 15z^2 + 6z - 5)(5z^2 + 6z - 15z + 7) y + 7z^4 - 2z^2 - 42z^2 - 2z + 7) x^4 - 2y(z^3 - 3z^2 - 9z^2 - 9z + 1) y^6 + z^2(7z^4 - 14z^3 - 18z^2 - 14z + 7) y^5 + z(7z^2 + 14z + 3) + 15z^2 - 6z^2 - 6z^2 - 6z + 11) y^6 + 2z(5z^5 + 13z^4 - 2z^2 + 2z \\
+ 13z + 5) y^5 + (11z^6 + 26z^5 + 44z^4 - 66z^3 + 44z^2 + 26z + 11) y^4 + (6z^5 - 4z^4 - 66z^3 - 42z^2 - 4z + 6) y^3 - 2(33z^4 + 2z^3 - 22z^2 + 2z + 33) y^2 + (6z^3 + 2) \\
6z^2 + 26z + 6) y + 11z^2 + 10z + 11) x^2 - 2(x^2(5z^3 + 3z^2 + 3z + 5) y^5 + z(22z^4 \\
+ 5z^3 - 22z^2 + 5z + 22) y^4 + (5z^5 + 5z^4 - 26z^3 - 26z^2 + 5z + 5) y^3 + (3z^4 - \\
22z^3 - 22z^2 - 22z + 3) y^2 + (3z^3 + 5z^2 + 5z + 3) y + 5z^2 + 22z + 5) x + 15z^2 + 2z \\
+ 2y(z - 1)^2(z + 1) + y^4(z - 1)^2z(z + 1) + y^4z^2(15z^2 + 2z + 15) + y^2(15z^4 \\
- 2z^3 - 9z^2 - 2z + 15) + 15) / [(x - 1)^3(y - 1)^2(xy - 1)^2(z - 1)^2(yz - 1)^2 \\
(xy^2 - 1)^2 - \left( -4(x + 1)(y + 1)(yz + 1) \right) / [(x - 1)^3x(y - 1)^3(yz - 1)^3] - (4(y + 1)(xy \\
+ 1)(z + 1) x^2(z^2 - 4z - 1) y^4 + 4z(x + 1) (z^2 - 1) y^3 - (x^2 + 1) (z^2 - 4z - 1) \\
y^3 - 4(x + 1) (z^2 - 1) y^2 - 4z - 1) \log(xy + 1) / [(x(y - 1)^3yg(x - 1)^3(z - \\
1)^3] - [4(z + 1)(yz + 1) x^2y^5z^7 + x^2y^4(4x(y + 1) + 5) z^6 - xy^3( \left( y^2 + \\
y^2 - 4(y + 1) x^3 - 3) z^5 - y^2(4y(y + 1) x^3 + 5(y^2 + 1) x^2 + 4(y + 1) x + 1) z^4 + \\
y(y^2 x^3 - 4y(y + 1) x^2 - 3) (y^2 + 1) x - 4(y + 1) ) z^3 - (5x^2y^2 + y^2 + 4x(y + 1) \\
y + 1) z^2 + (3x + 4) y + 4z - 1) \log(xy + 1) / [(xy(z - 1)^3z(y - 1)^3(zyz - 1) \\
)^3] / [(x + 1)^2(y + 1)^2(xy + 1)^2(z + 1)^2(yz + 1)^2(xy + 1)^2] dx dy dz \right]. \]
There is a similar integrand for \(E_5\). The corresponding expressions for \(D_6\) and \(E_6\) are several times more complicated still—the parse tree for the \(D_6\) integrand has over 27,000 leaves, even after some simplification! In Appendix 2 we display the numerical results for \(D_5, E_5, D_6, E_6\) obtained in this fashion. Note that we expended more machine work for \(n = 5\) because that \(n\) marks the spot where previous research had reached a kind of blockade.

Based on the numerical value for \(E_5\), we applied a PSLQ integer relation detection program to recognize this constant as exhibited in Table 1, where as one can see the constants \(\pi\), \(\log 2\), \(\zeta(3)\) and \(\text{Li}_4(1/2)\) are involved. Note the use there of the notation \(\hat{a}\), meaning that we have not yet worked out a formal proof. This experimental detection for \(E_5\) is quite strong, though—190 orders of magnitude beyond the level that could reasonably be ascribed to numerical round-off error.

Alas, we still have not been successful in identifying either \(C_5\) or \(D_5\). However, we have established, via a PSLQ computation and based on the 500-digit values given in Appendix 2, that \(\text{neither } C_5 \text{ nor } D_5 \text{ satisfies a integer linear relation}\) with the following set of constants, where the vector of integer coefficients in the linear relation has Euclidean norm less than \(4 \cdot 10^{12}\):

\[
1, \pi, \log 2, \pi^2, \pi \log 2, \log^2 2, \text{Li}_{-3}(2), \pi^3, \pi^2 \log 2, \pi \log^2 2, \log^3 2, \\
\zeta(3), \pi \text{Li}_{-3}(2), \log 2 \cdot \text{Li}_{-3}(2), \pi^4 \pi^3 \log 2, \pi^2 \log^2 2, \pi \log^3 2, G, G\pi^2, \\
\text{Li}_4(1/2), \sqrt{3}\text{Li}_{-3}(2), \log^4 2, \pi \zeta(3), \log 2 \cdot \zeta(3), \pi^2 \text{Li}_{-3}(2), \pi^2 \text{Li}_{-3}(2), \\
\pi \log 2 \cdot \text{Li}_{-3}(2), \log^2 2 \cdot \text{Li}_{-3}(2), \text{Li}_{-3}(2), \text{Im}[\text{Li}_4(e^{2\pi i/5})], \text{Im}[\text{Li}_4(e^{4\pi i/5})], \\
\text{Im}[\text{Li}_4(i)], \text{Im}[\text{Li}_4(e^{2\pi i/3})]
\]

Here \(G = \sum_{n \geq 0} (-1)^n / (2n + 1)^2\) is the Catalan constant. Some constants that may appear to be “missing” from this list are actually linearly redundant with this set, and thus were not included in the PSLQ search. These include \(\text{Re}[\text{Li}_3(i)], \text{Im}[\text{Li}_3(i)], \text{Re}[\text{Li}_3(e^{2\pi i/3})], \text{Im}[\text{Li}_3(e^{2\pi i/3})], \text{Re}[\text{Li}_4(i)], \\
\text{Re}[\text{Li}_4(e^{2\pi i/3})], \text{Re}[\text{Li}_4(e^{2\pi i/5})], \text{Re}[\text{Li}_4(e^{4\pi i/5})], \text{Re}[\text{Li}_4(e^{2\pi i/6})]\) and \(\text{Im}[\text{Li}_4(e^{2\pi i/6})]\).

We should note that computing numerical integrals sufficiently high precision to enable serious PSLQ relation searches, which typically require several hundred to several thousand digits, has only recently been achieved for a wide range of integrand functions, even for one-dimensional integrals [8, 9]. Thus our examples here of 3-dimensional and 4-dimensional quadrature, which require thousands of times as much computation as one-dimensional integrals, truly lie on the edge of currently available numerical techniques and computing technology. Indeed, we are not aware of any other instance of a successful three-dimension quadrature of a nontrivial function to an accuracy of 500 or more digits. In any case, our reductions to \((n - 2)\) dimensions yield dramatic reductions in computational cost, compared to direct quadrature of the original \(n\)-dimensional integral, such as (1).

As we have noted, reasonably extensive—but far from conclusive—PSLQ experiments have failed to identify any evaluations of \(C_n, D_n, E_n\) for \(n > 4\),
except for the experimental evaluation of $E_5$ mentioned above. The profusion of potential polylogarithmic constants of order 4 and higher, such as $\text{Li}_4(1/2)$, is one of the problems. While the numerical values for $D_6$ and $E_6$ in Appendix 2 may not yet be precise enough for experimental-mathematical closed-form capture, the values of $C_n$ and $D_5$ given in Appendix 1 likely are sufficiently accurate, if one could only surmise the right set of test constants.

13 The susceptibility amplitudes

It is interesting that, via Painlevé differential analysis B. Nickel [15], using the differential theory in [21], has resolved numerical values for two infinite sums relating to the susceptibility amplitudes mentioned in the introduction, namely, recalling $I_n := \pi D_n/(2\pi)^n$, 

$$
\sum_{n=1,3,5,...} I_n = 1.0008152604402126471194763630472102369375\ldots
$$

(38)

and

$$
\sum_{n=2,4,6,...} I_n = 0.02655129735925232532107227312986256362526\ldots
$$

(39)

Our qMC values from Table 2, optionally augmented by the above higher precision $D_5, E_5, D_6, E_6$ values, are entirely consistent with these Nickel numbers, in that we get about 20-decimal-place agreement when adding up $D_n$ terms directly. Indeed, it would be wonderful to capture closed forms for these infinite sums.

In the same vein, for comparison we have considered $H_n := \pi C_n/(2\pi)^n$. In this case we may use (9) to write 

$$
\sum_{n=1,3,5,...} H_n = \frac{\pi}{\pi} \int_0^\infty p \sinh(K_0(p)/\pi) \, dp
$$

(40)

$$
= 1.0101422864199451701704796866927057660215362408\ldots
$$

and

$$
\sum_{n=2,4,6,...} H_n = \frac{\pi}{\pi} \int_0^\infty p (\cosh(K_0(p)/\pi) - 1) \, dp
$$

(41)

$$
= 0.81024856380868082565191010347800614283172529480320\ldots
$$

with the values in Table 1 allowing one to confirm these values to about five places. The use of numerical values from (9) and/or estimates from (22) would allow further confirmation.

One might well ask: If the Painlevé analysis leads to high-precision values for the above sums, why does one need a closed form for say $D_5$ or its relatives? One answer, as posited by J-M. Maillard, is that new Ising theoretical avenues involving Fuchsian ODEs might require precise knowledge of these higher $D_n$, starting with $n = 5$ [14].
14 Open problems

• We have in a sense solved what had been an open computational problem, which is to provide a workable quadrature approach for some higher susceptibility integrals $D_{n>4}$. But (referring to Appendix 2) what is a closed form for $D_5$, and how far do we need to take $D_6, E_6$ quadrature to perform successful detection? ($E_6$ may well be easier that $D_5, D_6$ based on our success with $E_5$.)

• Can the the two-dimensional integral (28) for $C_5$ be symbolically resolved? The constants obtained would most likely shed light on those involved in $D_5$.

• Is there a way to calculate the hypergeometric $D_n$-kernel (35) efficiently, say by adroit grouping of the confluent summands?

• Can the methods of the exponential-decay Theorem 3 be extended to find the universal decay constant $\Delta$ in Conjecture 2?

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Appendix 1. Numerical values for $C_n$

Some 500-digit calculated values of $C_n$ are as follows, obtained via the Bessel-kernel method (i.e., quadrature on formula (9), as in Section 9). Note that $C_3, C_4$ are known in closed form, as in Table 1.

$c_{3,}$

0.78130241289645629986771874296240923563651343365452854202221000629668869646516
1821809286957083220966102104235025650903576865870552440307992607844199895749
307569672130980859321609553643633967477285839770325515895664770912428899241
0.02498188853713087884895238876822815932695420227471363581893707479059383766516
146217899177920860361353023942276038250642262683054573101120356257264891045811
1495272539802496679976445479960266333658422275946005535371765622825623963016
98967938757682094583043

$c_{4,}$

0.7.01199860176429999816513927548345827946242003865291014378825073949405620042
0159627543259238778900585282842047235419660786997665892748541369564821704608
427514799373871267055861950857213081216423109122806373935865094725388696550213
2466190696450005659930090049807056428565666666395943538802990788263605649925

28
C_64:
0.6304735033743867964883620886001591690547671669974413289715
4884050887667663801397197313026528924316698018827150469029242813676054813
8258968294288920075747441483449191948683072313004358281951598012303248189040
1547690508198247781473477053899423295297589585411554733649367946428576688768
67306315849054817465842889811317033415809648876677137017861532162324947974232
86709000898784239323763345031914326008811625314333787483540017572553022175851
8690730950772643094149

C_128:
0.6304735033743867961220401927108789043546870778172323415738179837089700038301
81326322260756873250503156116078066412573397680518052712398229192648583019302
31781630022638396370730710220779308440947193909573071733855973855708853267
94060393120609629827920404474663389027960770845068818243593284630885898998
308770561607706525967263895015715572494837496700328739266383238684008584
950094211059621803323245840734579484667306771963654166681617368088576937287069
6032385325056498839156

C_256:
0.6304735033743867961220401927108789043546870778172323415738179837089700038301
81326322260756873250503156116078066412573397680518052712398229192648583019302
31781630022638396370730710220779308440947193909573071733855973855708853267
94060393120609629827920404474663389027960770845068818243593284630885898998
308770561607706525967263895015715572494837496700328739266383238684008584
950094211059621803323245840734579484667306771963654166681617368088576937287069
6032385325056498839156

C_512:
0.6304735033743867961220401927108789043546870778172323415738179837089700038301
81326322260756873250503156116078066412573397680518052712398229192648583019302
31781630022638396370730710220779308440947193909573071733855973855708853267
94060393120609629827920404474663389027960770845068818243593284630885898998
308770561607706525967263895015715572494837496700328739266383238684008584
950094211059621803323245840734579484667306771963654166681617368088576937287069
6032385325056498839156

C_1024:
0.6304735033743867961220401927108789043546870778172323415738179837089700038301
81326322260756873250503156116078066412573397680518052712398229192648583019302
31781630022638396370730710220779308440947193909573071733855973855708853267
94060393120609629827920404474663389027960770845068818243593284630885898998
308770561607706525967263895015715572494837496700328739266383238684008584
950094211059621803323245840734579484667306771963654166681617368088576937287069
6032385325056498839156
Appendix 2. Numerical values for $D_n, E_n$

The values for $D_n, E_n$ below all started with the respective, dimensionally reduced integrands as described in Section 12. Each integral in this Appendix is thus $(n-2)$-dimensional. As intimated in the main text, we expended less effort on $D_6, E_6$; their values below were obtained via direct application of the \texttt{NIntegrate[]} function in \textit{Mathematica}, with a chosen working precision. Even so, $D_6, E_6$ each required more than one CPU day on a 1.5 GHz G5 workstation. This is testimony to the rapidly growing complexity of the integrand with $n$.

As for $D_5, E_5$, these were done by converting the relevant integrands to valid Fortran-90 code, by means of a combination of the \textit{Mathematica} \texttt{FortranForm} function, together with some offline processing to divide the full expression into “chunks” that could be handled by the IBM XLFortran compiler. We then prepared a special three-dimensional, high-precision Gaussian integration program, implemented using the Message Passing Interface (MPI) parallel programming constructs. The resulting programs was then run on the “Bassi” system at the Lawrence Berkeley National Laboratory, which is a large cluster of IBM Power5 nodes.

Computing $E_5$ to 240 digits in this manner required 96 minutes on 64 CPUs. In regard to $D_5$, we were not able to recognize this constant based on a 240-digit value, so we extended the computation to 500 digits (although we were still unsuccessful in recognizing it). The 500-digit run was significantly more expensive, requiring 18.2 hours run time on 256 CPUs.

\begin{verbatim}
D_5 =
0.00248460576234031547995050915390974963506067764248751561587076921618221378569
1543657537926899487245120187068721106392525011862069944997542226566556264670853828
412450011668223000454570326268769738489861519824796130355252585151071543863811369
617492242985578076280428947770278710921198111606340631254136038598401982807864
018693072681098854823037887884878758358351257555236419969469146314091127363094
605240934008771628387064364218612045090299733566341137276112202408345463150171
13540844197840922456685...
\end{verbatim}

\begin{verbatim}
D_6 =
0.000489141700188034...
\end{verbatim}

\begin{verbatim}
E_5 =
0.00349365371172952174068067279184251569632944955141314683698982336992415271
72666766950706520894333290399856686123538476859944386681548777982364143996
61191401373654167274769656668452339750941312947032252211618325511271865089014
602148...
\end{verbatim}
\[ E_b = 0.000687832871826409437004784\ldots \]

References


