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SELECTED TOPICS IN BARYON RESONANCES

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Publication Date
1972-08-01
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A. Barbaro-Galtieri

August 1972

Prepared for the U.S. Atomic Energy Commission under Contract W-7405-ENG-48

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SELECTED TOPICS IN BARYON RESONANCES*
A. Barbaro-Galtieri

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I. Introduction: General Remarks

These lectures will not be a complete review of Baryon Resonances, nor a discussion of Baryon Spectroscopy. Professor Dalitz has already talked about this last topic in his quark model lectures. Instead I will discuss some problems in Baryon Resonances which interest me or which I have worked on recently.

A look at Tables I, II and III, taken from Review of Particle Properties, will clearly show you why I am reluctant to review all the Baryon Resonances in a few lectures. There are 74 reported states, of which only 44 we thought to be reasonably established, therefore worth tabulating in the Tables of Particle Properties. They are divided as follows:

<table>
<thead>
<tr>
<th>Particle</th>
<th>Reported</th>
<th>Established</th>
</tr>
</thead>
<tbody>
<tr>
<td>N</td>
<td>18</td>
<td>12</td>
</tr>
<tr>
<td>Δ</td>
<td>12</td>
<td>9</td>
</tr>
<tr>
<td>Λ</td>
<td>16</td>
<td>8</td>
</tr>
<tr>
<td>Σ</td>
<td>21</td>
<td>12</td>
</tr>
<tr>
<td>Ξ</td>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>Total</td>
<td>74</td>
<td>44</td>
</tr>
</tbody>
</table>

If we try to put all of these states into SU(3) multiplets, given the 12 and 9 established N and Δ states, we would expect 21 Σ, 21 Ξ, and at least 12 Λ states. The above table shows that many Σ, Λ, and Ξ states are missing. The quark model discussed by Professor Dalitz requires even more octets, decuplets, and singlets than just the 21 SU(3) multiplets counted in this fashion. However, even without the quark model, it appears that SU(3) alone suggests that more hyperon resonances should be uncovered in the future.

A question which has been raised during the past few days here at the school is: how much can we trust the partial wave analyses reported so far?

My opinion is that for the π-N analyses below M = 1700 MeV the present solutions are quite good: all the analyses agree on 6 N* and 2 Δ states (P33 and S31). Above M = 1700 MeV, with the exception of the G17 (2190) and F37 (1950), most of the states are not very clear; they appear as a second smaller loop (crosses in Fig. 1) in some analyses and not in others. When they appear in more than one analysis we have accepted them as established and have rated them with
TABLE I. STATUS OF \( N^* \) RESONANCES
THOSE WITH AN OVERALL STATUS OF \( *** \) OR \( **** \) ARE INCLUDED IN THE MAIN JAKYON TABLE, THE OTHERS AWAIT CONFIRMATION.

| PARTICLE | \( L/I/J \) | OVERALL STATUS | TOTAL CR.S. | OTHER | \( \pi N \) | \( \eta \) | \( \eta \) | \( \eta \) | \( \kappa \) | \( \lambda \) | \( \lambda \) | \( \lambda \) | \( \sigma \) | \( \pi \) | \( \delta \) | \( \gamma \) | \( \gamma \) | \( \gamma \) | \( \gamma \) | \( \gamma \) |
|----------|------------|----------------|------------|-------|---------|---------|---------|---------|----------|----------|----------|----------|---------|---------|---------|---------|---------|---------|---------|
| \( N^*(1470) \) | P11 | **** | **** | *** | RHO N |
| \( N^*(1520) \) | D13 | **** | **** | * | RHO N |
| \( N^*(1535) \) | S11 | **** | **** | *** | RH-J N |
| \( N^*(1670) \) | D15 | **** | **** | * | RH-J N |
| \( N^*(1688) \) | F15 | **** | **** | * | RH-J N |
| \( N^*(1700) \) | S11 | **** | **** | * | RH-J N |
| \( N^*(1700) \) | D13 | * | * | * | RH-J N |
| \( N^*(1780) \) | P11 | **** | **** | ** | RH-J N |
| \( N^*(1860) \) | P13 | **** | **** | * | RH-J N |
| \( N^*(1990) \) | F17 | * | * | * | RH-J N |
| \( N^*(2040) \) | D13 | * | * | * | RH-J N |
| \( N^*(2190) \) | G17 | **** | **** | * | RH-J N |
| \( N^*(2220) \) | H19 | **** | **** | * | RH-J N |
| \( N^*(2650) \) | **** | **** | * | RH-J N |
| \( N^*(3030) \) | **** | **** | * | RH-J N |
| \( N^*(3245) \) | * | * | * | RH-J N |
| \( N^*(3690) \) | * | * | * | RH-J N |
| \( N^*(3755) \) | * | * | * | RH-J N |
| \( \text{DE}(1236) \) | P33 | **** | **** | **** | F | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| \( \text{DE}(1650) \) | S31 | **** | **** | * | O | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| \( \text{DE}(1670) \) | D33 | **** | **** | * | R | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| \( \text{DE}(1690) \) | P33 | * | * | * | R | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| \( \text{DE}(1910) \) | P31 | **** | **** | * | I | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| \( \text{DE}(1950) \) | F37 | **** | **** | * | D | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| \( \text{DE}(1960) \) | D35 | * | * | * | F | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| \( \text{DE}(2160) \) | P33 | * | * | * | N | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| \( \text{DE}(2420) \) | H311 | **** | **** | * | F | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| \( \text{DE}(2850) \) | **** | **** | * | F | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |
| \( \text{DE}(3230) \) | **** | **** | * | R | * | * | * | * | * | * | * | * | * | * | * | * | * | * | * |

**** GOOD, CLEAR, AND UNMISTAKABLE.
*** GOOD, BUT IN NEED OF CONFIRMATION OR NOT ABSOLUTELY CERTAIN.
** NEEDS CONFIRMATION.
* WEAK.
# ATTRIBUTED TO THE STATE CLOSEST TO WHERE THE CROSS SECTION PEAKS.
TABLE II. STATUS OF \(Y^*\) RESONANCES

Those with an overall status of **** or ***** are included in the main baryon table. The others await confirmation.

---

**STATUS AS SEEN IN --**

<table>
<thead>
<tr>
<th>PARTICLE</th>
<th>LIJ</th>
<th>STATUS</th>
<th>CR. SEC.</th>
<th>KHA R N</th>
<th>LAM PI</th>
<th>SIG PI</th>
<th>OTHER CHANNELS</th>
</tr>
</thead>
<tbody>
<tr>
<td>LAM(1115)</td>
<td>P01</td>
<td>****</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LAM(1330)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LAM(1405)</td>
<td>S01</td>
<td>****</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LAM(1520)</td>
<td>D03</td>
<td>****</td>
<td>****</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LAM(1670)</td>
<td>S01</td>
<td>****</td>
<td>****</td>
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</tr>
<tr>
<td>LAM(1690)</td>
<td>D03</td>
<td>****</td>
<td>****</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>LAM(1750)</td>
<td>P01</td>
<td>**</td>
<td>**</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>LAM(1815)</td>
<td>F05</td>
<td>****</td>
<td>****</td>
<td></td>
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<td></td>
</tr>
<tr>
<td>LAM(1830)</td>
<td>D05</td>
<td>****</td>
<td>**</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>LAM(1860)</td>
<td>P03</td>
<td>**</td>
<td>**</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>LAM(1870)</td>
<td>S01</td>
<td>**</td>
<td>**</td>
<td></td>
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</tr>
<tr>
<td>LAM(2010)</td>
<td>D03</td>
<td>**</td>
<td>F</td>
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<tr>
<td>LAM(2020)</td>
<td>F07</td>
<td>**</td>
<td>**</td>
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</tr>
<tr>
<td>LAM(2100)</td>
<td>G07</td>
<td>****</td>
<td>****</td>
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<tr>
<td>LAM(2110)</td>
<td></td>
<td>**</td>
<td></td>
<td>B</td>
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<td></td>
</tr>
<tr>
<td>LAM(2350)</td>
<td></td>
<td>****</td>
<td>****</td>
<td></td>
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</tr>
<tr>
<td>LAM(2585)</td>
<td></td>
<td>****</td>
<td>**</td>
<td></td>
<td></td>
<td>D</td>
<td></td>
</tr>
<tr>
<td>SIG(1190)</td>
<td>P11</td>
<td>****</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>SIG(1385)</td>
<td>P13</td>
<td>****</td>
<td>****</td>
<td>****</td>
<td>****</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SIG(1440)</td>
<td>PE</td>
<td>**</td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>SIG(1480)</td>
<td>PE</td>
<td>**</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SIG(1620)</td>
<td>S11</td>
<td>**</td>
<td>**</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SIG(1620)</td>
<td>P11</td>
<td>**</td>
<td>**</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SIG(1620)</td>
<td>PE</td>
<td>**</td>
<td>**</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SIG(1670)</td>
<td>D13</td>
<td>****</td>
<td>**</td>
<td>**</td>
<td>****</td>
<td>****</td>
<td></td>
</tr>
<tr>
<td>SIG(1670)</td>
<td>PE</td>
<td>**</td>
<td>**</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SIG(1750)</td>
<td>S11</td>
<td>**</td>
<td>**</td>
<td>**</td>
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</tr>
<tr>
<td>SIG(1765)</td>
<td>D15</td>
<td>****</td>
<td>****</td>
<td>****</td>
<td>****</td>
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<td></td>
</tr>
<tr>
<td>SIG(1800)</td>
<td>P11</td>
<td>**</td>
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</tr>
<tr>
<td>SIG(1915)</td>
<td>F15</td>
<td>****</td>
<td>****</td>
<td>****</td>
<td>****</td>
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<tr>
<td>SIG(1940)</td>
<td>D13</td>
<td>**</td>
<td>**</td>
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<tr>
<td>SIG(2030)</td>
<td>P17</td>
<td>****</td>
<td>****</td>
<td>****</td>
<td>****</td>
<td>**</td>
<td></td>
</tr>
<tr>
<td>SIG(2070)</td>
<td>F15</td>
<td>**</td>
<td>**</td>
<td></td>
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</tr>
<tr>
<td>SIG(2080)</td>
<td>P13</td>
<td>**</td>
<td>**</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>SIG(2100)</td>
<td>G17</td>
<td>**</td>
<td>**</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>SIG(2250)</td>
<td></td>
<td>****</td>
<td>****</td>
<td>****</td>
<td>****</td>
<td>**</td>
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</tr>
<tr>
<td>SIG(2455)</td>
<td></td>
<td>****</td>
<td>****</td>
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</tr>
<tr>
<td>SIG(2620)</td>
<td></td>
<td>****</td>
<td>****</td>
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</tr>
<tr>
<td>SIG(3000)</td>
<td></td>
<td>****</td>
<td>****</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**** Good, clear, and unmistakable.

*** Good, but in need of clarification or not absolutely certain.

** Needs confirmation.

* Weak.

# Attributed to the state closest to where the cross section peaks.
TABLE III. STATUS OF XI* RESONANCES
THOSE WITH AN OVERALL STATUS OF *** OR **** ARE INCLUDED IN THE MAIN BARYON TABLE. THE OTHERS WAIT CONFIRMATION.

STATUS AS SEEN IN --

<table>
<thead>
<tr>
<th>PARTICLE</th>
<th>LII</th>
<th>OVERALL STATUS</th>
<th>XI PI</th>
<th>LAM K</th>
<th>SIG K</th>
<th>XI* PI</th>
<th>OTHER CHANNELS</th>
</tr>
</thead>
<tbody>
<tr>
<td>XI(1320) P11</td>
<td>****</td>
<td>*****</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>WEAK TO LAM PI</td>
</tr>
<tr>
<td>XI(1530) P13</td>
<td>****</td>
<td>****</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XI(1630)</td>
<td>**</td>
<td>**</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XI(1820)</td>
<td>***</td>
<td>***</td>
<td>***</td>
<td>**</td>
<td>**</td>
<td></td>
<td></td>
</tr>
<tr>
<td>XI(1940)</td>
<td>***</td>
<td>***</td>
<td>***</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XI(2030)</td>
<td>**</td>
<td>**</td>
<td>**</td>
<td></td>
<td></td>
<td></td>
<td>3-BODY DECAYS</td>
</tr>
<tr>
<td>XI(2250)</td>
<td>*</td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>XI(2550)</td>
<td>**</td>
<td>**</td>
<td>**</td>
<td></td>
<td></td>
<td></td>
<td>3-BODY DECAYS</td>
</tr>
</tbody>
</table>

**** GOOD, CLEAR, AND UNMISTAKABLE.
*** GOOD, BUT IN NEED OF CLARIFICATION OR NOT ABSOLUTELY CERTAIN.
** NEEDS CONFIRMATION.
* WEAK.
3 or 4 stars in Table I. In addition, above 1900 MeV the old analyses do not compare well with the new polarization data, as shown in Fig. 2, therefore new analyses are certainly necessary at this point.

The KN analyses so far have been done in an energy-dependent way; that is, the momentum dependence of resonances as well as background waves have been chosen a priori. This is mainly due to lack of data. Many two-body channels are available: \( \bar{K}N \), \( \Lambda \pi \), \( \Sigma \pi \), whereas for the \( \pi N \) system the data on the \( \Lambda K \) and \( \Sigma K \) channels are scarce. I think that for the \( \bar{K}N \) up to a mass \( M = 1900 \) MeV the higher partial waves seem to be fairly well known whereas the lower partial waves (S, P, and perhaps D around 1900 MeV) still need some work (more data as well as analysis). Because of the energy dependence of the partial wave analyses done so far it seems that the results have more predictive power than the \( \pi N \) analyses. In Fig. 3 the predictions of the CHS partial wave analysis are compared with the recent polarization measurements of Anderson et al. for the \( K^-p \) channel. The predictions seem to be somewhat shifted but have the same behavior as the data.

In closing I remind you that at the Amsterdam Conference there has been extensive discussion of the problem of ambiguities in phase shift analysis and that the review talk by C. Schmid includes references to all the papers presented there.
Fig. 1. Argand plots of the latest Saclay phase shift analysis. The partial waves shown are only those which present clear resonant loops. The P33 partial wave, not shown, follows a circular path for the great majority of the unitary circle; the \( \Delta(1236) \) is the corresponding resonance. The crosses indicate the position of some reported states above \( M = 1700 \) MeV.
Fig. 2. Polarization data of Burleson et al.\textsuperscript{3} compared with the phase shift analysis of CERN-experimental\textsuperscript{4} and Berkeley.\textsuperscript{5} Both analyses did not use these data.
Fig. 3. Polarization data of Anderson-Almehed et al.\textsuperscript{7} compared with the partial wave analysis predictions of Armenteros et al.\textsuperscript{6}
II. Resonance formulae: Parameters for $\Delta(1236)$

Many resonance formulae have appeared in the literature and often experimentalists ask which one is the best form to use. In a recent experiment, Carter et al. $^9$ have measured very accurately the $\pi^+p$ cross section in the $\Delta(1236)$ region. We can now use these data to test the various formulae and answer the following questions:

(a) Which resonance formula fits the data best?

(b) Are the resonance parameters dependent upon the formula used?

These problems have been discussed in Ref. 1 in connection with the $\Delta(1236)$ parameters. In fact, for some years the PDG compilation $^4$ has quoted the parameters obtained by Olsson, $^10$ who used a resonance form which we call STANDARD (see later) and obtained

$$M = 1236.0 \pm 0.6 \text{ MeV} \quad \Gamma = 120 \pm 2 \text{ MeV}. \quad (2.1)$$

However, Carter et al., $^9$ who used the LAYSON form, report

$$M = 1231.0 \pm 0.6 \text{ MeV} \quad \Gamma = 111 \pm 3 \text{ MeV}. \quad (2.2)$$

The difference between the above values is certainly quite striking and needs investigation. It turns out that some of the difference is due to the fact that the new data is somewhat in disagreement with the old data, but this is not sufficient to explain the discrepancy between (2.1) and (2.2). As we will see, the values $M$ and $\Gamma$ are not dependent upon the formula used, but upon the way the fit to the data is done: the inclusion of background in the $P_{33}$ partial wave tends to increase the values of $M$ and $\Gamma$. As for the first question we will see that there is no significant difference among the fits made by using different formulae.

The new $\pi^+p$ total cross-section data of Carter et al. $^9$ are shown in Fig. 4. This total cross section can be written in terms of the relevant partial waves as

$$\sigma(\pi^+p) = 4\pi \lambda^2 \left[ \sin^2 \delta_{S31} + \sin^2 \delta_{P31} + 2 \sin^2 \delta_{P33} \right], \quad (2.3)$$

where $\delta_i$ are phase shifts: $\delta_{S31}$ for the $S$-wave, $\delta_{P31}$ for the $P$-wave $J^P = 1/2^+$, and $\delta_{P33}$ for the $P_{33}$ partial wave, that is, $P$-wave $J^P = 3/2^+$. In order to obtain the $P_{33}$ phase shift one needs analyze angular distributions and polarizations. Since Carter et al. have not yet done this analysis they used the phase shift solutions of Donnachie
Fig. 4. The $\pi^+ p$ total cross-section data. The "present results" refers to data of Carter et al., a total of 14 points, of which 9 are below the resonant mass, 5 above. The curve fits these 14 points very well. (The dots in the figure have been enlarged by us).
et al.\textsuperscript{4} for the S31 and P31 partial waves and extracted the $\delta_{p33}$ (called $\delta_{33}$ from now on) from Eq. (2.3).

The 14 values of $\delta_{33}$ obtained as discussed above have been used to test resonance formulae and study the dependence of the parameters on the different types of fits,\textsuperscript{11} which we now discuss. The errors on the $\delta_{33}$ do not include systematic errors or errors due to the uncertainties of the background partial waves. However, they are adequate for the tests discussed here. We have added to the 14 data points of Carter et al.,\textsuperscript{9} five additional points from the latest phase shift analysis of Lovelace et al.\textsuperscript{12} who included the data of Carter et al. in their numerous sets of data. For the purpose of the discussion on resonance formulae or dependence of resonance parameters on type of fit, the inclusion of the phase shifts from these two sources is adequate. Table IV contains a list of the 19 values of $\delta_{33}$ used in the fits discussed in the following sections.

1. Three-Parameter Fits to the $\delta_{33}$ Phase Shifts

The scattering amplitude $T$ can be written as

$$T = \frac{1}{\cot \delta - i} .$$

To fit the resonant behavior of the phase shift $\delta_{33}$ we have tried the four different formulae which we next discuss.

**STANDARD**

$$\tan \delta = \frac{E_R \Gamma(q)}{(E'_R - E^2)}$$

$$\Gamma(q) = \Gamma_R \left( \frac{q}{q_R} \right)^3 \left[ \frac{1 + (qR^2)^2}{1 + (qr)^2} \right],$$

where $E$ is the c.m. energy of the $\pi N$ system, $q$ the c.m. momentum of the decay products, $E_R$ and $\Gamma_R$ the mass and width of the resonance at the resonant energy, and $r$ the radius of interaction in fermi. This is a relativist Breit-Wigner form\textsuperscript{13} with the Blatt-Weisskopf\textsuperscript{14} barrier factor, calculated by using a square-well potential at non-relativistic energies. $E_R$, $\Gamma_R$, and $r$ are the parameters to be determined.
Table IV. Phase shifts used for the fits discussed in the text and shown in Table V. \( N_e \) are the points excluded in some of the fits.

<table>
<thead>
<tr>
<th>N</th>
<th>( E_{c.m.} ) (MeV)</th>
<th>( \delta_{33} \pm d_{33} )</th>
<th>Author</th>
<th>Ref.</th>
<th>( N_e )</th>
</tr>
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<tbody>
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<td>Carter et al.</td>
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<td>9</td>
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<td>44.20±0.11</td>
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<td></td>
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<td>61.31±0.18</td>
<td>Carter et al.</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>8</td>
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<td>9</td>
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</tr>
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</tr>
<tr>
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<td>X</td>
</tr>
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<td>9</td>
<td>X</td>
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<tr>
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<td>1320.4</td>
<td>135.5 ±0.9</td>
<td>Lovelace et al.</td>
<td>12</td>
<td>X</td>
</tr>
</tbody>
</table>
where $E_\pi$ is the pion c.m. energy, $E_0$ is the resonant value of $E_\pi$, $q$ the c.m. momentum of the pion, $\gamma^2$ a reduced width, and $r$ the radius of interaction. The fit to the data will provide $E_0'$, therefore the mass of the resonance $E_R$, and the width at resonance, is given by

$$\Gamma = \Gamma(E_0')E_R/(E_R - E_0').$$

This is also a relativistic form.

Next we fit the Chew-Low equation

$$\tan \delta = \frac{\Gamma(E_\pi)}{2(E_0' - E_\pi)} \quad \Gamma(E_\pi) = \frac{4m_p(qr)\gamma^2}{(E_\pi + E_0')\left[1 + (qr)^2\right]}, \quad (2.6)$$

where $E$ is the c.m. energy of the $\pi N$ system, $q$ the c.m. momentum of the decay products, and $r$ a radius of interaction. The width of the resonance is the value of $\Gamma(q)$ at $E = E_R$.

Finally we use a polynomial form:

$$q^3 \cot \delta = \sum_{n=1}^{N} a_n q^{2n-2}. \quad (2.8)$$

In this case the resonant energy corresponds to the energy ($s = E_R^2$) where $q^3 \cot \delta = 0$. The width is given by

$$\Gamma_0 = \frac{q_0^3}{E_R} \left[\frac{d}{ds} (q^3 \cot \delta)\right]^{-1}_{s = E_R^2}. \quad (2.9)$$

The first part of Table V shows the results of the three-parameter fits of Eqs. (2.5) through (2.8) to the phase shifts of Table IV. The values of $M$ and $\Gamma/2$ can also be found on Table VI. Various fits have been made:

13 points: only $\delta_{33}$ of Carter et al., except point 18 ($E_{c.m.} = 1300.9$ MeV)

14 points: only $\delta_{33}$ of Carter et al.

17 points: all $\delta_{33}$, except point 12 ($E_{c.m.} = 1247.3$ MeV) and 18 ($E_{c.m.} = 1300.9$ MeV)

19 points: all $\delta_{33}$ values of Table IV.
Table V. Fits to the $^{33}_{13}$ phase shift values of Table IV. The results of the fits of Carter et al. and Olsson are included. The $N_p$ column shows the number of parameters used in the fit. The $\chi^2$ column shows the chi-squared obtained for that fit; the degrees of freedom being $N_p$.

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<th>Resonance form</th>
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<th>Points</th>
<th>$\chi^2$</th>
<th>$M$ (MeV)</th>
<th>$\Gamma$ (MeV)</th>
<th>$a^a$ (fermi)</th>
<th>$a^{a+3}_{13}$ (fermi) $^{-3}$</th>
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For the polynomial fits the values reported in these columns are the coefficients of Eq. (2.8), the units are fermi to the appropriate power.
The fit of Carter et al. is equivalent to our fit with the LAYSON form and 13 points only.

Inspection of the three-parameters fits in Tables V and VI shows that:

(a) All the fits with 13 or 17 points are good.
(b) The fits with 14 or 19 points are bad.
(c) The fits to the data for 13 or 17 points are equally good for the four different formulae with perhaps the LAYSON form slightly preferred.
(d) Independently from the goodness of fit, all four formulae used give the same value of $M$ within 0.5 MeV and the same value of $\Gamma$ within 2 MeV. As shown in Table VI, fits to same number of data points with different formulae give $dM < 0.2$ MeV and $d\Gamma \leq 0.6$ MeV.

2. Four-Parameter Fits to the $\delta_{33}$ Phase Shifts

The fits discussed in the previous section show that among the points of Table IV there are at least two points which fit badly: point 12 ($E = 1247.3$ MeV) and point 18 ($E = 1300.9$ MeV). There are various possibilities:

(a) Points 12 and 18 are incorrect.
(b) Points 12 and 18 are correct, but their errors are underestimated.
(c) Points 12 and 18 are correct, but the three-parameter fits are inadequate.

In what follows I assume that the three-parameter fits are inadequate, not because I believe the two phase shifts are correct, but to carry on with the tests of the formulae (2.5) through (2.8) and to study the dependence of the resonance parameters on the type of fit. A fourth parameter can be added in many ways: (a) add a second radius of interaction to the barrier factors, (b) assume that there is some non-resonant "background" in the P33 partial wave, (c) add a fourth term to the polynomial expansion of Eq. (2.8).

A second radius of interaction can be easily added to the STANDARD Breit-Wigner form by replacing the width with the form

$$\Gamma(q) = \Gamma_R \left( \frac{q}{q_R} \right)^3 \frac{1 + (qRr_1)^2 + (qRr_2)^4}{1 + (q r_1)^2 + (q r_2)^4}.$$  

(2.10)
The second radius of interaction can be justified by the fact that, as mentioned earlier, the barrier factor with only one radius is calculated at non-relativistic energies.

The addition of "background" to the P33 partial wave can be physically accepted if it is considered as the lower end of a resonance at higher energy. The \( \Delta(1690) \) is a P33 resonance (see Table I) which could easily produce the small amplitude needed as background in the \( \Delta(1236) \) region. Another way to introduce a "background" is to consider, as it will be discussed in Sec. II-3, that a resonance in the physical region is the manifestation of a pole in the S matrix in the unphysical sheet. In this case the Breit-Wigner form is only the first term of the expansion around the singularity of the S matrix in the unphysical sheet. Keeping more terms in the expansion is equivalent to adding "background" to the resonance in the physical sheet.

Assuming that the \( \pi^+p \) channel is completely elastic, a simple way to combine a background scattering to the resonant amplitude is to multiply the S matrices for the two: the resulting amplitude will preserve unitarity. If \( \delta_R \) is the phase shift for the resonance and \( \delta_B \) is the background phase shift we get

\[
S = e^{2i\delta_B} e^{2i\delta_R} e^{2i\delta} (1 + 2iT_R) = 1 + 2iT,
\]

which gives

\[
T = T_B + e^{2i\delta_B} T_R = \frac{1}{\cot(\delta_B + \delta_R) - i}.
\]

The background phase can simply be parametrized as

\[
\tan \delta_B = a_{33} k^3,
\]

where \( k = q/\hbar c \), \( a_{33} \) is expressed in \( \text{(fermi)}^{-3} \) and the \( k^3 \) dependence has been chosen in order to have the correct threshold behavior for a P-wave amplitude.

The bottom parts of Tables V and VI show the results of the four-parameter fits. We observe the following.

(a) All of the fits with 19 points are not very good; points 1, 12, and 19 tend to fit badly in most of the fits.
### Table VI. Mass and half-width values obtained in the various fits to the $\delta_{33}$ phase shifts of Table IV. $N_p$ indicates the number of parameters used in the fit.

<table>
<thead>
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<th>Points</th>
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<th>STANDARD(2R)</th>
<th>LAYSON</th>
<th>CHEW-LOW</th>
<th>POLYNOMIAL</th>
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### Table VII. Results of the extrapolation to the pole in Sheet II for the various resonance forms discussed in the text, using the parameters of Tables V and VI. $M$ and $\Gamma/2$ (in MeV) represent the pole location in the complex energy plane.

<table>
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<tr>
<th>Points</th>
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<th>STANDARD(2R)</th>
<th>LAYSON</th>
<th>CHEW-LOW</th>
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</table>
(b) The fits with the STANDARD formula [Eq. (2.5)] with two radii in the width [Eq. (2.10)], give the least variation of values of M and \( \Gamma \) obtained for the four types of fits (\( dM = 0.6 \text{ MeV} \) and \( d\Gamma = 1.6 \text{ MeV} \), as shown in Table VI).

(c) For the other fits the values of M and \( \Gamma \) obtained depend more heavily upon the number of points used, since the background is different in the various cases.

(d) For fits to the same number of points, different formulae give different M and \( \Gamma \): \( dM \) varies within 1.3 to 3.6 MeV, \( d\Gamma \) within 3.8 to 10.4 MeV.

The fit of Olsson reported in the last row of Table V has been obtained with a four-parameter fit to the total cross-section data, using Eq. (2.3). For the resonance amplitude the STANDARD resonance form [Eq. (2.5)] was used, for the \( \delta_{P31} \) and \( \delta_{S31} \) the phase shift of Roper\(^{16} \) was used and a fourth parameter, an overall scaling factor for the background cross section, was introduced. This parameter produces shifting of the mass to higher values and tends to widen the width as do the fits discussed in this section. However, by comparing the new cross-section data with the old ones (Fig. 4) it is clear that some of the shift is due to difference in the data.

In conclusion the results of Table V and VI show that:

(a) In absence of background for a given set of data the four formulae used give values of M and \( \Gamma \) in good agreement with each other, M is within 0.2 MeV, \( \Gamma \) is within 0.6 MeV. The fit to the data is equally good (or bad) for the four formulae.

(b) The introduction of a background, however small (see Fig. 5), alters M and \( \Gamma \) considerably. For a given set of data the values obtained depend upon the formula used (\( dM \leq 3.6 \text{ MeV} \), \( d\Gamma \leq 10.4 \text{ MeV} \)).

(c) The results of Table VI show that the parameters of \( \Delta(1236) \) (since we do not know if points 12 and 18 are wrong), are comprised in the intervals

\[
M = 1230.7 \text{ to } 1235.0 \text{ MeV} \quad \text{or} \quad dM = 4.3 \text{ MeV},
\]

\[
\Gamma = 109.2 \text{ to } 123.6 \text{ MeV} \quad \text{or} \quad d\Gamma = 14.4 \text{ MeV}.
\]

3. **Extrapolation to the Pole**

In the previous section I have described some fits to the phase shifts, using different resonance forms. These fits have been carried out in the
Fig. 5. Fit to the total cross-section data of Carter et al. 9 (14 data points shown as dots). The addition of a small background improves the fit considerably (see Table V). The dotted curve ($\sigma_{BR}$) represents the interference term. The background cross section ($\sigma_B$) is consistent with zero.
physical region, and the mass and width of the resonance were defined ac-
cording to Eqs. (2.5) through (2.8), with the addition of some background,
as in Eq. (2.12) for the four-parameter fits. However, since a reso-
nance in the physical region is the manifestation of a pole of the S
matrix in the unphysical sheet, it is of some interest to investigate the
location of the pole which is at the origin of the $\Delta(1236)$.

Professor Coleman, here at the school, has suggested that the
location of the pole should be independent of the formula used to fit
the data in the physical region: all of the formulae used should extrap-
olate to the same pole location. Ball et al. \textsuperscript{17} have subsequently proved
that this is the case. Let us now investigate the analytic continuation of
the resonance forms of the previous sections to the second sheet of the
complex energy plane. Many authors have discussed the relation be-
tween the location of the S-matrix pole in the unphysical sheet and its
manifestation in the physical region; we refer to the articles by
Fraser and Hendry \textsuperscript{18} and by Chew. \textsuperscript{19}

The extrapolation to the pole position is easily done by going to
the complex energy plane. We use the T matrix expression of Eq. (2.11),
with $\delta_B = 0$ for the three-parameters fits, and the forms (2.5) through
(2.8) with the values of the parameters obtained in the fits to the data
(Table V). We then substitute for $E$ in the expression for $\delta$ the complex
variable

$$\sqrt{s} = E - i\Gamma/2.$$ 

The momentum $q$ is now complex too, given by

$$q = \sqrt{\frac{(s - (M_N + M_\pi)^2)(s - (M_N - M_\pi)^2)}{4s}}$$

which can have two values associated with the $\pm$ sign in front of the
square root. Following the convention of Fraser and Henry we have

Sheet I: $\text{Im}q > 0$
Sheet II: $\text{Im}q < 0$.

We then search both sheets for a pole in $T$; that is, for the complex
value of $\sqrt{s}$ where
\[-21-\]

\[\cot \delta - i = 0.\]

The pole turns out to be always in Sheet II, as expected, but actually there is a conjugate pair of poles at \(E \pm i \Gamma/2\).

Table VII shows the location of the pole for the 36 fits of Table V. Comparison of the Table VI and Table VII shows that:

(a) The pole position is at \(\sim 1211\) MeV; that is, about 20 MeV below the mass of the resonance in the physical region. The width is also about 20 MeV smaller than the width in the physical region.

(b) The extrapolations of the three-parameter formulae to the unphysical sheet give, for a given set of data, a pole position within 0.2 MeV and a width within 0.8 MeV, ranges comparable with the physical region values.

(c) For the four-parameter fits, the extrapolation to the pole gives values of \(M\) and \(\Gamma\) much more stable than in the physical region for different formulae used on a given set of data. In fact we get \(dM \leq 0.4\) MeV and \(d\Gamma \leq 2.0\) MeV, compared with \(dM \leq 3.6\) MeV and \(d\Gamma \leq 10.4\) MeV in the physical region.

In conclusion, Table VII shows that the pole position is at

\[
M = 1210.9 - 1212.3 \text{ MeV} \quad \text{or} \quad dM = 1.4 \text{ MeV},
\]
\[
\Gamma = 96.4 - 101.8 \text{ MeV} \quad \text{or} \quad d\Gamma = 5.8 \text{ MeV},
\]

therefore it is much better known than the resonance parameters in the physical region [Eqs. (2.13) and (2.14), i.e., \(dM = 4.3\) and \(d\Gamma = 14.4\) MeV]. This result poses some new problems:

1) For how many resonances can we find the pole location?

2) Which values of \(M\) and \(\Gamma\) should we use for SU(3) fits?

In order to extrapolate to the pole for non-elastic resonances we need to know the complete \(T\) matrix; that is, have good data on all the decay modes of the resonance. This is not the case at the present time, except perhaps for \(\Lambda(1405), \Sigma(1385),\) and \(\Lambda(1520)\). As for the second question, it is irrelevant at the present time in view of the fact that we cannot find the pole location for most of the resonances.
IIII. Decay of $\Lambda(1520)$ and $\Lambda(1690)$ into $\Sigma(1385)+\pi$

There are two known $I = 0$ resonances in the $J^P = 3/2^-$ state: $\Lambda(1520)$ and $\Lambda(1690)$. The study of the two-body decays of these two states into $\bar{K}N$ and $\Sigma\pi$ have led to the conclusion that $\Lambda(1520)$ is very close to being an SU(3) singlet, whereas $\Lambda(1690)$ is a member of the $J^P = 3/2^-$ octet along with $N(1520)$ and $\Sigma(1670)$. If the known $\Xi(1815)$ is added to the octet, the Gell-Mann–Okubo mass formula along with octet-singlet mixing would predict a mixing angle of $\sim 20^\circ$. The two-body decays of the two isospin-singlets also yield a mixing angle of this magnitude therefore it is important to check if this agreement is still good when the decays

$$\Lambda(1520) \to \Sigma(1385) + \pi$$  \hspace{1cm} (3.1)

$$\Lambda(1690) \to \Sigma(1385) + \pi$$  \hspace{1cm} (3.2)

are studied. The $\Sigma(1385)$ is a member of the $J^P = 3/2^+$ decuplet, therefore—since an SU(3) singlet has no coupling with a decuplet (see Table IX in Sec. IV-1), one can directly measure the mixing angle of the octet-singlet states.

It turns out that the mixing angle calculated from the decay modes (3.1) and (3.2) does not agree with the value $\theta \sim 20^\circ$ discussed above, as will be discussed in Sec. IV-2. In this section we will discuss two recent, yet unpublished, experiments which measured the widths for the decays of Eqs. (3.1) and (3.2).

1. $\Lambda(1520)$

The experiment$^{20}$ has been done at LBL in the 25-inch hydrogen bubble chamber. The reaction studied is

$$K^- p \to \Lambda\pi^+\pi^- \hspace{1cm} (9000 \text{ events})$$  \hspace{1cm} (3.3)

for $K^-$ laboratory momenta in the 300-470 MeV/c range.

The total cross section for this channel as function of laboratory momentum is shown in Fig. 6. The curve is the result of a multichannel analysis which included, along with the $\Lambda\pi\pi$, the $\bar{K}N$, $\Sigma\pi$, and $\Lambda\pi$ channels in order to satisfy the unitarity of each partial wave.$^{21}$ For the $\Lambda\pi\pi$ channel only the total cross section was included in the multichannel analysis; the resulting fit required contribution from the P01 P03, D13 in addition to the large D03 due to the $\Lambda(1520)$ resonance.

In order to fit the energy dependence of the total cross section, each
Fig. 6(a). Cross section for $K^- p \rightarrow \Lambda \pi^+ \pi^-$ as a function of $K^-$ momentum. The curve is the result of the multichannel analysis described in the text. The $\bar{\Lambda}(1520)$ resonance is also indicated, as obtained from the isobar analysis of the $\Lambda \pi \pi$ events. The two dotted points are from Armenteros et al., Nucl. Phys. B21, 15 (1970).

(b). Schematic diagram of the isobar-model description of the reaction $K^- p \rightarrow \Lambda \pi^+ \pi^-$. The isobar can be a $Y^*$, an S-wave $\Lambda \pi$ system, a $\sigma$, or a $\rho$ meson.
partial wave had the appropriate kinematical factors. For the D03, which is the dominant partial wave, the momentum used in the kinematical factors was calculated by assuming that all the partial wave is explained by the decay $\Lambda(1520) \rightarrow \Sigma(1385) + \pi$. As we will see later the partial wave analysis of the $\Lambda\pi\pi$ channel alone confirms the results of the multichannel analysis.

**Qualitative features of the data.** Just by looking at the data the following observations can be made.

(a) The cross-section data (Fig. 6) clearly show an enhancement at 395 MeV/c, the background at the peak being of the order of 15%. The crosses reported in Fig. 6 are actually the result of the partial wave analysis discussed later.

(b) The distributions of the production angle ($\theta$) of the $\Lambda$ in the reaction c.m. system are shown in Fig. 7 for nine momentum intervals. It is clearly seen that a rapid change takes place from one interval to the next: below the resonant energy (390-400 MeV/c interval) the $\Lambda$ is produced mainly forwards; above the resonant energy the $\Lambda$ is produced mainly backwards. The coefficients of the Legendre polynomials expansion of the differential cross sections ($A_i$) and the $\Lambda$ polarization ($B_i$) are shown in Fig. 8 a-c. Only $i = 1, 2$ are shown, since higher coefficients are consistent with being zero. As expected from the asymmetry seen in Fig. 7, the $A_1$ coefficient shows change of sign while going through the resonant energy. This change is clearly due to interference of the D03 resonant wave with some background. The partial wave analysis shows that the P01 amplitude is responsible for this behavior (see Fig. 10) as expected since $A_1 \sim \text{Re}(P1^*D3)$.

(c) The study of the Dalitz-plot behavior through the resonance region provides information about how much of the resonance decays through the $\Sigma(1385)\pi\pi$ mode. Actually, the $\Sigma(1385)$ is centered outside the kinematically allowed region for the invariant mass of the $\Lambda\pi$ system at a $\Lambda\pi\pi$ invariant mass of 1520 MeV. This is shown in Fig. 9 where the Dalitz plot and its projections are plotted for events in the 390-400 MeV/c region. Enhancements in the high $\Lambda\pi$ invariant masses indicate the presence of $\Sigma(1385)$, which the partial wave analysis finds to be 82% of the resonant cross section. The increased density at low $\pi\pi$ mass indicates constructive interference between $\Sigma(1385)^-$ and $\Sigma(1385)^+$, therefore dominance of symmetric $I = 0$ production.
Fig. 7. Distributions of the \( \Lambda \) production angle in c.m. system for nine momentum intervals. The curves are the results of the Legendre polynomial expansion fit. \(^{20}\)
Fig. 8. Legendre polynomial coefficients for the $\Lambda$ angular distributions (a) and (b) and polarization (c) as a function of incident momentum. (d) Charge asymmetry of the Dalitz plot as a function of incident momentum. The dots are the results of the partial wave analysis. 20
The variation of charge asymmetry of the Dalitz plot as function of incident momentum provides information about the small $I = 1$ contribution. Figure 8d shows the variation with momentum of the quantity

$$
\alpha = \frac{N^- - N^+}{N^- + N^+},
$$

where $N^-$ is the number of events with a $\Lambda\pi^-$ mass larger than the $\Lambda\pi^+$ mass and $N^+$ are the remaining events. This asymmetry is explained by two effects: the mass difference between $\Sigma(1385)^-$ and $\Sigma(1385)^+$ and the $I = 0$ and $I = 1$ interference. The $I = 1$ amplitude needed is about 10% as large as the resonant $I = 0$ amplitude.

**Energy-independent partial wave analysis.** The details of this analysis are reported in Ref. 20; here we only outline the procedure. Since there is enough data it was possible to perform the analysis in each of the nine momentum intervals by using the isobar model of Deler and Valladas modified to include the $\Lambda$ polarization. In the resonance region each momentum interval is of 10 MeV/c, corresponding to c.m. system mass interval of about 3.5 MeV.

The data in each bin was fitted by an event-by-event maximum-likelihood fit using up to 17 partial waves chosen to describe the data. These include $S$, $P_1$, $P_3$, and $D_3$ partial waves for the incoming state (in two isospin states) and four types of isobars: (a) "Y", a $\Lambda\pi$ system resonating as $\Sigma(1385)$; (b) "$\Lambda\pi$", a $\Lambda\pi$ system in a relative $S$ wave; (c) "$\sigma$", an $S$-wave $\pi\pi$ system; and (d) "$\rho$", a $P$-wave $\pi\pi$ system. Many of these amplitudes were found to be zero within errors for most of the momentum intervals and so were discarded. In this manner, by successive elimination and remaximization, a satisfactory set of solutions at nine momenta was found, using only six of the amplitudes. These were

\begin{align}
K^- p &\to \Sigma(1385) + \pi & \text{DS03, PP01, DS13, PP03,} \\
K^- p &\to \Lambda \sigma & \text{DP03, PS01,}
\end{align}

where the second symbol refers to the angular momentum between the isobar and the third particle ($L'$) and the other symbols refer to the quantum numbers of the incoming $K^- p$ system, according to the usual convention ($LL' IZJ$). A schematic diagram of the isobar model is
Fig. 9. Dalitz plot and projections for events in the momentum interval 390-400 MeV/c corresponding to c.m. energy in the 1520-MeV region. The symbol $\Sigma(1385)$ in the Dalitz plot indicates $\Sigma(1385)$ bands (plot from Ref. 20).
shown in Fig. 6b. Argand plots of these amplitudes are shown in Fig. 10.

The overall magnitude of the waves was fixed to agree with the measured cross sections (Fig. 6). The relative amounts of each wave have been freely chosen by the fit. In addition, the overall free phase, inherent in any reaction process, has been fixed to be the one of the dominant DS03 resonant Breit-Wigner form. This form, drawn as a solid curve in Fig. 10, was calculated as explained earlier for the multichannel analysis; that is, including a momentum dependence for the partial width into $\Lambda\pi\pi$ corresponding to a decay $\Lambda(1520) \rightarrow \Sigma(1385) + \pi$. Since the magnitude of the DS03 was left free to vary, it is interesting to note that the results of the isobar model are in very good agreement with the calculated shape. The DS03 amplitude contributes 77\% of the cross section at resonance (1.96 mb), the $\Lambda ''\sigma''$ decay of $\Lambda(1520)$ was found to be small (0.11 mb), the remaining 0.45 mb of the resonant cross section coming from the interference term. Using the five points with higher cross section in Fig. 6, it is found:

$$R = \frac{\Lambda(1520) \rightarrow \Sigma(1385) + \pi}{\Lambda(1520) \rightarrow \Lambda\pi\pi} = 0.82 \pm 0.10.$$  \hspace{1cm} (3.6)

The remaining four partial waves are small and are sufficient to explain all the qualitative features of the data. Using the values from the Particle Data Group, \(^1\)

$$\Gamma(1520) = 16 \pm 2 \text{ MeV}, \quad \Lambda(1520) \rightarrow \Lambda\pi\pi = (9.6 \pm 0.6)\%,$$

$$\Sigma(1385) \rightarrow \Lambda\pi = (90 \pm 3)\%.$$

The result of the analysis gives

$$\Gamma [\Lambda(1520) \rightarrow \Sigma(1385) + \pi] = 1.44 \pm 0.26 \text{ MeV}.$$

The ratio $R$ of Eq. (3.6) is in disagreement with the previously reported result of Burkhardt et al. \(^{23}\) of $R = 0.39 \pm 0.10$. This value was found in a production experiment with many fewer events (206 events) than for the formation experiment described in this section. The data of Ref. 23 were divided into three mass intervals centered around $\Lambda(1520)$. The data in the central interval (with about 15\% background) agree with the result (3.6), whereas the data in the side intervals (background $\sim 30\%$) yield lower branching ratios, thus lowering the value for the entire sample.
Fig. 10. Partial wave amplitudes for the reaction $K^- p \rightarrow \Lambda \pi^+ \pi^-$ (result of the analysis of Ref. 20). The errors are only statistical, and the numbers 1 through 9 refer to the momentum intervals indicated in Fig. 7. The curve is explained in the text.
2. \( \Lambda(1690) \)

Information about this decay mode (3.2) comes from the energy-dependent partial wave analysis of Prevost et al., who analyzed the reaction

\[
K^- p \rightarrow \Lambda \pi^+ \pi^- \quad (11000) \quad (3.7)
\]

\[
K^- n \rightarrow \Lambda \pi^- \pi^0 \quad (2600) \quad (3.8)
\]

in the experiment done by the CHS collaboration.\(^{24}\) The incident momentum interval is 600 to 1200 MeV/c, corresponding to c.m. system energies from 1615 to 1865 MeV, — that is, a 250-MeV region—whereas the \( \Lambda(1520) \) experiment had 9000 events in a 60-MeV mass interval. In addition, the analysis here is complicated by the fact that in this mass region there are many more resonances: Table II lists for this mass region at least 10 reported states. The cross sections do not show any narrow enhancements, as can be seen in Fig. 11a-b, which show \((\sigma_0 + \sigma_1)/4\pi \chi^2\) and \(\sigma_1/4\pi \chi^2\) respectively, where the subscript refers to the isospin state.

The Dalitz plot for about one-half of the data of reaction (3.7) is shown in Fig. 12. Here, in contrast with the \( \Lambda(1520) \) situation (Fig. 9) the \( \Sigma(1385) \) is well within the kinematical limits. Actually the two \( \Sigma(1385) \) bands cross inside the Dalitz plot and a problem of interference arises. From the analysis of this plot the authors concluded that all the events could be accounted for as being \( \Sigma(1385) + \pi \) events, therefore no additional amplitudes were added to the partial wave analysis.

**Input Data.** The analysis has not been carried out on the Dalitz plot (event by event) as for the LBL analysis described earlier, but on the calculated moments of the Legendre polynomials. The events of reaction (3.7) have been divided in 30 momentum intervals and at each momentum the events were divided into two categories: \( N^+ \), events with \( M(\Lambda \pi^+) < M(\Lambda \pi^-) \), and \( N^- \), events with \( M(\Lambda \pi^-) < M(\Lambda \pi^+) \). Then the coefficients of the following expansion were calculated:

\[
\frac{d^2 \sigma}{d \cos \theta d \phi} = \chi^2 \sum_{l, m} (a_l^m) \pm Y_l^m (\theta, \phi),
\]
Fig. 11. Data of Prevost et al. \(^{24}\) for the reaction \(K^- p \rightarrow \Sigma(1385)\pi\).

(a) The sum of the \(I=0,1\) cross sections \((\sigma_0 + \sigma_1)/4\pi\lambda^2\);
(b) the \(I=1\) cross section \(\sigma_1/4\pi\lambda^2\); (c-f) coefficients of the spherical harmonic expansion discussed in the text.
Fig. 12. Dalitz plot for part ($P_{K^-} > 770$ MeV/c) of the data of Prevost et al.\textsuperscript{24} for the reaction $K^- p \rightarrow \Lambda \pi^+ \pi^-$. $\Sigma(1385)$ bands $(M \pm \Gamma/2)$.\textsuperscript{24}
where $\theta$ is the polar angle ($\cos \theta = \hat{\xi} \cdot \hat{K}$, $\xi$ being the normal to the production plane and $\hat{K}$ the direction of the $K^-$) and $\phi$ the azimuth angle ($\cos \phi = \hat{\pi}^+ \cdot \hat{K}_p$, where $\hat{K}_p$ is the projection of $K$ on the production plane and $\hat{\pi}^+$ is in the overall c.m. system). Since $l_{\text{max}}$ was taken to be 4, eliminating the illegal moments they get 900 points for the 30 momenta. Reaction (3.8) contributes 8 additional points, since only the cross sections were used (Fig. 11b).

In order to isolate the contributions from isospin 0 and 1 they actually fit the quantities

$$A^m_l = (a^m_l)^+ + (a^m_l)^-$$

$$D^m_l = (a^m_l)^- - (a^m_l)^+.$$  \hspace{1cm} (3.9)

The $A^m_l$ contain only $l=0$ and $l=1$ terms with no interference terms, and $D^m_l$ include only interference terms between the two isospin states. Some of these coefficients, chosen among the 30 used, are shown in Fig. 11. The cross sections (11a and 11b) do not show any structure except for a possible broad peak in the 1800-MeV region. The other coefficients show structure but, in view of the many resonant states present, it is very difficult to relate the effects to a particular state.

Energy-dependent partial wave analysis. The coefficients (3.9) are then expressed in terms of the partial waves to be determined. The partial waves included are:

$$\text{SD01, PP01, PP03, DS03, DD05, FP05, SD11, PP11, PP13, DS13, DD15, FP15,}$$

with the usual symbolism (LL' I2J, where L' refers to the angular momentum between $\Sigma(1385)$ and the $\pi$, the rest refers to the incoming partial wave). Some of these amplitudes are parametrized as resonances, the others have for form

$$T = f(kr) \left(a + b \, P_{\text{inc}}\right)$$

where $P_{\text{inc}}$ is the incident $K^-$ momentum; $f(kr)$ is a factor dependent on the c.m. system momentum of $\Sigma(1385)$ and the radius of interaction,
necessary to take into account correctly the threshold dependence of
the various partial waves. The resonances used and the corresponding
partial waves are:

\[ \Sigma(1660), \Lambda(1690), \Sigma(1765), \Lambda(1815), \Lambda(1830), \]

\[ \text{DS13, DS03, DD15, FP05, DD05}. \]

Among the low angular momentum partial waves, only the S wave has
been parametrized as resonant in some fits in both \( I = 0 \) and \( I = 1 \).

Table VIII shows the results for the five resonances in the higher
partial waves. The results indicate smaller values for the branching
ratios into \( \Sigma(1385) + \pi \) than previously reported. However, this is the
first complete analysis in this energy region with considerable sta­
tistics.

A previous experiment has reported an upper limit for \( \Lambda(1690) \)
based only on cross-section data, where an enhancement was seen in
the \( I = 0 \) cross section at 1690 MeV. However, the CHS experiment with
much improved data does not show any such enhancement (Fig. 11a
and b). In addition, the partial wave analysis requires only a small
amplitude in the DS03 partial wave; therefore, at present the value
for the decay rate of reaction (3.2) is

\[ \Gamma \left[ \Lambda(1690) \rightarrow \Sigma(1385) + \pi \right] = 0.9 \pm 0.8 \text{ MeV}, \]

where the errors are calculated by using

\[ \Gamma(1690) = 50 \pm 20 \text{ MeV}, \]

\[ t = 0.06 \pm 0.02, \]

\[ x = 0.20 \pm 0.06. \]
### Table VIII. Results of the CHS analysis\(^\text{24}\) of \(Y^* \rightarrow \Sigma(1385)\pi\).

<table>
<thead>
<tr>
<th>Resonance</th>
<th>Partial wave</th>
<th>Amplitude (t=\pm \sqrt{xx}^T)</th>
<th>Elasticity (x)</th>
<th>Branching ratios (%)</th>
<th>This expert.</th>
<th>PDG(^1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Lambda(1690))</td>
<td>DS03</td>
<td>-0.06</td>
<td>0.20</td>
<td>1.8</td>
<td>&lt; 20</td>
<td></td>
</tr>
<tr>
<td>(\Sigma(1660))</td>
<td>DS13</td>
<td>-0.01</td>
<td>0.08</td>
<td>0.1</td>
<td>&lt; 11</td>
<td></td>
</tr>
<tr>
<td>(\Sigma(1765))</td>
<td>DD15</td>
<td>-0.12</td>
<td>0.44</td>
<td>3.3</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>(\Lambda(1815))</td>
<td>FP05</td>
<td>0.11</td>
<td>0.62</td>
<td>2.0</td>
<td>17</td>
<td></td>
</tr>
<tr>
<td>(\Lambda(1830))</td>
<td>DD05</td>
<td>0.11</td>
<td>0.10</td>
<td>1.2</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>

\(^a\)From Review of Particle Properties,\(^1\) except for \(\Lambda(1690)\) which (in that compilation) should be updated to 0.20.

### Table IX. SU(3) ISOSCALAR FACTORS

Adapted from J. J. de Swart, Rev. Mod. Phys. 35, 916 (1963)

\[[8] \otimes [8] = [27] \otimes [10] \otimes [10^*] \otimes [\bar{8}], \otimes [\bar{8}], \otimes [11].\]

Five single-coefficient tables are omitted. The one involving \(\{10^*\}\) has a negative coefficient, i.e. \((NK|10^*) = -1\). The others, involving \(\{27\}\) and \(\{10\}\), are all +1.

The phase factor \(\xi_1 = \pm 1\), from de Swart's Table I, enters in his symmetry formula (14.3):

\[
\langle \mu_1\mu_2|\xi |\mu \rangle = \xi_1(-1)^{n_1^2+n_2^2-n_3^2} \langle \mu_1\mu_2|\xi |\mu \rangle.
\]

This factor is irrelevant if you are doing your own self-consistent calculations; it enters when you try to check someone else who chose \(\mu_2 \otimes \mu_1\) instead of \(\mu_1 \otimes \mu_2\).
IV. SU(3) Fits

The classification of the known baryon states into quark model supermultiplets has been extensively studied, and so has the classification into SU(3) multiplets. Since Professor Dalitz has talked about the general classification of states in the quark model, I will discuss only a few problems which arise when detailed calculations are done in the framework of SU(3) symmetry.

1. General Considerations

Exact SU(3) symmetry predicts that all members of a multiplet have the same mass and the same couplings for decays into members of other multiplets. The members of the octet of Stable Baryons, however, lie within 20% of their mean mass, therefore some symmetry-breaking interaction must be considered. Gell-Mann and Okubo derived a mass formula (here called GMO formula) with the assumption that the symmetry is broken by a medium-strong interaction which conserves I-spin and hypercharge (Y). Since the and meson octets do not obey the GMO formula, it was soon realized that another symmetry-breaking interaction has to be introduced to account for mixing of a singlet and the \( I=0 \) member of an octet.

A. Mass Formulae

In first-order perturbation theory the mass formula is

\[
M = a + bY + c \left[ I(I+1) - \frac{Y^2}{4} \right]
\]

with \( a, b, c \) constants. For decuplets and octets it becomes

Decuplet: \( \Delta - \Sigma = \Sigma - \Xi^* = \Xi^* - \Omega \) GMO, \hspace{1cm} (4.1)

Octet: \( 2(N + \Xi) = 3\Lambda + \Sigma \) GMO, \hspace{1cm} (4.2)

where the particle symbol indicates its mass. For octet-singlet mixing one has

\[
\sin^2 \theta = \frac{\Lambda - M_8}{\Lambda - \Lambda^'} \hspace{1cm} \text{Mixing angle} \hspace{1cm} (4.3)
\]

\[
M_8 = \frac{2(N + \Xi) - \Sigma}{3} \hspace{1cm} \text{GMO,} \hspace{1cm} (4.4)
\]
where $\Lambda$ stands for the mostly octet particle and $\Lambda'$ for the mostly singlet. Formula (4.3) was originally written in this form for mesons with mass squared and then extended to baryons with mass instead of mass squared.

B. Decay Rates

For a two-body decay of a resonance of mass $M_R$, the width can be written as

$$\Gamma \propto \frac{|T|^2 R_2}{M_R},$$

where $R_2$ is the two-body phase space, $R_2 = k/M_R$. Since $|T|^2$ has an extra factor $M_R$ coming from spin sums, in terms of a dimensionless $T$, we introduce $M_N$ (nucleon mass) to fix the dimensions, and write

$$\Gamma = \frac{|T|^2 k}{M_R} M_N.$$

In addition $|T|^2$ has centrifugal barrier factors. We then have:

\[
\begin{align*}
\text{Decuplet singlet:} & \quad \Gamma = (c_g)^2 B_f(k) \frac{M_N}{M_R} k, \\ 
\text{Octet:} & \quad \Gamma = (c_D g_D + c_F g_F)^2 B_f(k) \frac{M_N}{M_R} k. 
\end{align*}
\tag{4.5}
\tag{4.6}
\]

For singlet-octet mixing, defining

$$\begin{cases}
G_8 = c_D g_D + c_F g_F \\
G_1 = c_1 g_1
\end{cases}
\tag{4.7}$$

we get

$$\begin{cases}
\text{Octet-singlet mixing} & \quad \begin{align*}
G_8 &= \Lambda \cos \theta - \Lambda' \sin \theta \\
G_1 &= \Lambda \sin \theta + \Lambda' \cos \theta,
\end{align*} \\
\end{cases}
\tag{4.8}
$$

where $\Gamma$ is the measured decay width, $c_i$ are SU(3) coefficients with the appropriate signs (see Sec. IV-Ac), $g_1$ are the relevant
couplings, \( k \) is the c.m. system momentum for the decay being considered, \( B_{\ell}(k) \) is the barrier factor for the decay orbital angular momentum \( \ell \), and the particle symbols \( (\Lambda, \Lambda') \) stand for the couplings of the physical states.

Exact SU(3) predicts that for all the decay modes of members of the same multiplets the \( g \) are the same. Equation (4.8) takes care of the symmetry-breaking introduced because of singlet-octet mixing. However, the strong-medium interaction which produces 20% splitting in the masses of the stable-baryon octet, produces also breaking in the rates.

For broken SU(3), formulae analogous to the GMO formula for the width have been calculated and should be used in checking decay rates.

For the \( 3/2^+ \) decuplet \( \{10\} \rightarrow \{8\} + \{8\} \) the relation found by Becchi, Eberle, and Morpurgo and by Gupta and Sing independently is

\[
2(\Delta + \Xi) = 3\Sigma^*(\Lambda\pi) + \Sigma^*(\Sigma\pi),
\]

where the particle symbol stands for the coupling \( g_{10} \) as calculated from Eq. (4.5). \( \Sigma^*(\Lambda\pi) \) and \( \Sigma^*(\Sigma\pi) \) stand for \( g_{10} \) calculated for the \( \Lambda\pi \) and \( \Sigma\pi \) decay mode of the \( \Sigma \) component of the decimet. See Sec. IV-3 for the experimental check of the formula.

The relation among \( g_D \), \( g_F \) of Eq. (4.6) and the \( D \) (symmetric) and \( F \) (antisymmetric) couplings is as follows:

\[
\frac{F}{D} = \frac{\sqrt{5}}{3} \frac{g_F}{g_D}.
\]

Also the relation between \( g_D \), \( g_F \) and Gell-Mann's \( \alpha = D/(F+D) \) is

\[
\alpha^{-1} = \frac{F + D}{D} = 1 + \frac{\sqrt{5}}{3} \frac{g_F}{g_D}.
\]

**Barrier Factor.** The form of barrier factor \( B_{\ell}(k) \) has been a source of discussion among the various authors of SU(3) fits. Two forms of \( B_{\ell} \) have been used:

(a) The form

\[
B_{\ell}(k) = (kr)^{2\ell} D_{\ell}(kr),
\]

where \( D_{\ell}(kr) \) are the Blatt-Weisskopf \(^{14}\) polynomials in \( kr \), and \( r \) is
a radius of interaction usually taken to be 1 fermi. This form for the barrier factor is often used, along with the Breit-Wigner form we called STANDARD (Eq. 2.5 in Sec. II-1), to fit data from which resonance parameters are obtained. For Λ(1236) the best fit in Sec. II-1 gave \( r = 1.05 \) to 1.10.

(b) The form

\[ B_k(k) = k^{2f}, \tag{4.12} \]

in order to avoid the introduction of a radius of interaction. We will discuss some of the consequences of the use of (4.11) or (4.12) in Secs. IV-2 and IV-3.

C. Sign Conventions

The Clebsh-Gordan coefficients \( c_D', c_F', c_1', \text{ and } c_{10} \) in Eqs. (4.5) through (4.7) have some definite signs. If we use the de Swart tables of SU(3) coefficients with the convention that the first particle must be a baryon and the second particle a meson, we get the coefficients of Table IX for \( \Xi \times \Xi \) decays of baryon states. We reproduce here only those coefficients which are relevant to some of the discussion in these lectures, and refer the reader to the article by Levi-Setti for a more extensive discussion of the sign conventions adopted here, and to Ref. 1 for more complete tables as well as a sign-convention discussion.

The determination of the relative signs of resonant amplitudes can be used to determine the SU(3) assignment of the resonances. The resonant amplitude for an inelastic channel is

\[ T' = \frac{\beta \sqrt{x x_1}}{\epsilon^i} \]

\[ \epsilon = \frac{E_F - E}{\Gamma / 2}, \tag{4.13} \]

where \( x \) is the elasticity and \( x_1 \) the branching ratios into the inelastic channel in consideration (\( x_1 = \Gamma_i / \Gamma \)) and \( \beta = \pm 1 \). This amplitude can be written as

\[ T' = \frac{M_N}{M_R} \frac{\sqrt{B_e^k_k}}{\sqrt{B_{i1}^k_{i1}}} \frac{\beta G_{1i}}{\epsilon - i}, \tag{4.14} \]

where \( G_{1i} \) are the couplings of Eqs. (4.5) through (4.8) and the subscript \( e \) refers to the elastic channel. Let us consider two cases:
(a) \( \Sigma^* \to \Sigma \pi \).

From Table IX and Eq. (4.7) we have

\[
\text{Octet} \quad \left( - \frac{\sqrt{30}}{10} g_D + \frac{\sqrt{6}}{6} g_F \right) \left( \frac{\sqrt{6}}{3} g_F \right) = \beta g_e G_i \begin{cases} +1 & \text{if } g_D < \frac{\sqrt{5}}{3} g_F \\ -1 & \text{if } g_D > \frac{\sqrt{5}}{3} g_F \end{cases}
\]

\[
\text{Decuplet} \quad \left( - \frac{\sqrt{6}}{6} g_{10} \right) \left( \frac{\sqrt{6}}{6} g_{10} \right) = \beta g_e G_i \quad \beta = -1.
\]

In the decuplet case the sign is immediately determined; in the octet case one has to fit the other members of the octet to determine the \( g_D/g_F \) ratio. All the \( g_i \) except either \( g_D \) or \( g_F \) are assumed to be positive, since Eqs. (4.5) through (4.8) have an intrinsic overall sign ambiguity. (See more on this point in Sec. IV-2A.)

(b) \( \Lambda^* \to \Sigma \pi \)

From Table IX and Eq. (4.7) we have

\[
\text{Octet} \quad \left( \frac{\sqrt{20}}{10} g_D + \frac{\sqrt{2}}{2} g_F \right) \left( - \frac{\sqrt{45}}{5} g_D \right) = \beta g_e G_i \quad \beta = -1,
\]

\[
\text{Singlet} \quad \left( \frac{1}{2} g_1 \right) \left( \frac{\sqrt{6}}{4} g_1 \right) = \beta g_e G_i \quad \beta = +1.
\]

In both cases we expect a definite sign, if we assume that the mixing angle of Eq. (4.8) is very small, otherwise we have to include it in the calculations.

For both cases (a) and (b), however, we have to take into account an additional overall phase in determination intrinsic in every experiment, therefore we can use these signs only when they are taken relative to another \( \Lambda \) and \( \Sigma \) state of known SU(3) assignment.

D. Multiplet Assignments

Using the established states of Tables I, II, and III, a few multiplets seem to be complete or partially complete with satisfactory agreement according to Eqs. (4.5) through (4.8). Table X shows the present assignment according to the analysis of Refs. 27, 38, and 39 which assume exact SU(3). As discussed in (C) above, the relative signs of the various \( \Lambda \) and \( \Sigma \) states also have some definite predictions which should be satisfied. Figure 13, adapted from Levi Setti,\(^{39}\) shows the signs expected for various \( Y^* \) resonances once the signs of \( \Sigma(1385) \), decuplet, and \( \Lambda(1405) \), singlet, have been fixed according to Table IX. All the
Fig. 13. Signs of the amplitudes for $\Sigma\pi$ and $\Lambda\pi$ decays of hyperon resonances. The arrows indicate the phases predicted by $SU(3)$, the crosses show the measured phases, and the dots show the phases chosen as reference. Plot adapted from Levi Setti, the expected amplitudes for $\Sigma(1915)$ have been changed from his original display.
measured signs agree with the expected ones, including those of the S01 and D03 states, since the mixing angles are not large enough to change the expected sign for the amplitudes. The most recent partial wave analysis by Kim included data from K−p threshold to 1900 MeV and it is the only analysis to extend over such a wide region. All the signs of the resonant amplitudes agree with the signs of Fig. 13.

In addition to the states of Table X there are the following states for which the quantum numbers are known: N(1470)P11, N(1780)P11, N(1700)S11, N(1860)P13, N(2190)G17, N(2220)H19; Δ(1910)P31, Δ(1650)S31, Δ(1670)D33, Δ(1890)F35, Δ(2420)J^P = 11/2^+; Λ(2100)G17.

Clearly, a lot of Λ and Σ states are missing, as already pointed out in Sec. I.

2. Problems with the \( J^P = 3/2^- \) States

As mentioned in Sec. III there are two ways to calculate the mixing angle between \( \Lambda(1520) \) and \( \Lambda(1690) \), the two isospin-singlets with \( J^P = 3/2^- \). One is through the decay rates for two-body decays, the other through the decay rates for quasi-two-body decay into \( \Sigma(1385) \pi^+ \). We will discuss here two problems.

(a) Does the mixing angle depend upon the form of the barrier factor \( B_f \) (Eq. 4.11 or 4.12)?

(b) Does the mixing angle for the two-body decays agree with the one for quasi-two-body decays?

We will discuss separately the two-body and quasi-two-body decays.

A. Two-Body Decays

The mixing angle can easily be calculated from Eqs. (4.7) and (4.8) by using two decay modes of \( \Lambda(1520) \) and \( \Lambda(1690) \), that is, the \( \overline{K}N \) and \( \Sigma\pi \) decay modes. Assuming that \( \Lambda(1520) \) is the mostly singlet member of the multiplet and using the SU(3) coefficients of Table IX we obtain:

\[
\frac{1}{2} g_4 = \frac{\overline{K}N}{1520} \cos \theta + \frac{\overline{K}N}{1690} \sin \theta ,
\]

\[
\frac{\sqrt{6}}{4} g_4 = \frac{\Sigma\pi}{1520} \cos \theta + \frac{\Sigma\pi}{1690} \sin \theta ,
\]
Table X. SU(3) baryon multiplets with two or more known members. The coupling constants are those for decay into baryon \((1/2^+)\) octet \(\otimes (0^-)\) meson octet. Values of \(\theta\) and \(F/D\) taken from Plane et al. \(^{27}\) The analysis reported by Samios et al. finds similar results. \(^{27}\)

<table>
<thead>
<tr>
<th>(J^P)</th>
<th>Octet members(^a)</th>
<th>Singlet</th>
<th>(\theta) (deg)(^b)</th>
<th>(F/D)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1/2^-)</td>
<td>N'(1535) (\Lambda(1670)) (\Sigma(1750)) ([\Xi(1825)])</td>
<td>(\Lambda(1405))</td>
<td>(18 \pm 17)</td>
<td>([-2.9, -0.2])</td>
</tr>
<tr>
<td>(3/2^-)</td>
<td>N(1520) (\Lambda(1690)) (\Sigma(1670)) ([\Xi(1815)])</td>
<td>(\Lambda(1520))</td>
<td>(-25 \pm 6)</td>
<td>(2.3)</td>
</tr>
<tr>
<td>(5/2^-)</td>
<td>N(1670) (\Lambda(1830)) (\Sigma(1765))</td>
<td>(\Lambda(1915))</td>
<td>(-0.19)</td>
<td></td>
</tr>
<tr>
<td>(5/2^+)</td>
<td>N(1688) (\Lambda(1815)) (\Sigma(1915))</td>
<td></td>
<td>(0.82)</td>
<td></td>
</tr>
</tbody>
</table>

Decuplet members

| \(3/2^+\) | \(\Delta(1236)\) \(\Sigma(1385)\) \(\Xi(1530)\) | \(\Omega^-\) | 0.93 to 1.52 |
| \(7/2^+\) | \(\Delta(1950)\) \(\Sigma(2030)\) | | 0.25 to 0.97 |

\(^a\)Masses in parentheses are the nominal masses used in Tables I, II, III. The \(\Xi\) members have masses as calculated by using formulae (4.3) and (4.4) with the mixing angle \(\theta\) derived from the decay widths.

\(^b\)Values calculated from the decay widths, Eqs.(4.5) through (4.8), and the barrier factor (4.12). See Sec.IV-2 for a discussion of the \(3/2^-\) mixing angle.

\(^c\)Values calculated by us, using Eq. (4.11) for the barrier factor \(B_4\). See Sec.IV-3 for a discussion of SU(3) breaking effects for decuplet decay rates.

\(^d\)Second value of \(F/D\) obtained by using 15\% for the N(1525) \(\rightarrow\) N\(_2\) branching ratio instead of 60\%, which is the most likely one. Samios et al. \(^{27}\) report only the second value. The 15\% branching ratio (Diem et al., Kiev Conference Report, 1970) has now been retracted by the authors.

\(^e\)Samios\(^{27}\) finds \(F/D = 1.5\).

---

Table XI. Effect of barrier factor on mixing angle for \(J^P = 3/2^-\) states.

<table>
<thead>
<tr>
<th>(\Gamma) (MeV)</th>
<th>(K)N</th>
<th>(\Sigma\pi) (\sqrt{x^2+y^2})</th>
<th>(\Sigma\pi) (%)</th>
<th>(\theta_1) (B_4=BW)</th>
<th>(\theta_2) (B_2=k^4)</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (\Lambda(1518))</td>
<td>16 (\pm 2)</td>
<td>46 (\pm 1)</td>
<td>0.43 (\pm 0.01)</td>
<td>41 (\pm 1)</td>
<td>(-16.1 \pm 1.2)</td>
<td>(-27.5 \pm 3.1)</td>
</tr>
<tr>
<td>(\Lambda(1690))</td>
<td>50 (\pm 10)</td>
<td>20 (\pm 6)</td>
<td>-0.35 (\pm 0.05)</td>
<td>61 (\pm 25)</td>
<td>(-16.1 \pm 1.4)</td>
<td>(-27.5 \pm 3.6)</td>
</tr>
<tr>
<td>2. (\Lambda(1518))</td>
<td>16 (\pm 2)</td>
<td>46 (\pm 2)</td>
<td>0.43 (\pm 0.02)</td>
<td>41 (\pm 2)</td>
<td>(-16.1 \pm 1.4)</td>
<td>(-27.5 \pm 3.6)</td>
</tr>
<tr>
<td>(\Lambda(1690))</td>
<td>50 (\pm 20)</td>
<td>20 (\pm 6)</td>
<td>-0.35 (\pm 0.05)</td>
<td>61 (\pm 25)</td>
<td>(-24.7 \pm 5.1)</td>
<td>(-4.7)</td>
</tr>
<tr>
<td>3. (\Lambda(1518))</td>
<td>16 (\pm 4)</td>
<td>45 (\pm 5)</td>
<td>0.43 (\pm 0.05)</td>
<td>41 (\pm 5)</td>
<td>(-14.3 \pm 1.9)</td>
<td>(-24.7 \pm 5.1)</td>
</tr>
<tr>
<td>(\Lambda(1690))</td>
<td>50 (\pm 20)</td>
<td>18 (\pm 5)</td>
<td>-0.36 (\pm 0.05)</td>
<td>72 (\pm 28)</td>
<td>(-24.4 \pm 4.9)</td>
<td>(-24.7 \pm 5.1)</td>
</tr>
</tbody>
</table>

\(^a\)The barrier factor here is from Blatt-Weisskopf \(B_4 = (kr)^4/[9+3(kr)^2+(kr)^4]\) with \(r = 1\) fermi, except for case 4, where \(r = 0.15\) fermi.
which gives

\[ \tan \theta = \frac{\frac{4}{\sqrt{6}} G_{1520}^{\Sigma \pi} - 2 G_{1520}^{KN}}{2 G_{1690}^{KN} - \frac{4}{\sqrt{6}} G_{1690}^{\Sigma \pi}} \]  

(4.16)

The \( G_i \) are measured constants related to the widths \( \Gamma_i \) according to Eq. (4.5):

\[ G_i = \pm \sqrt{\frac{\Gamma_i M_R}{B(k) k M_N}} ; \]

they have an intrinsic sign as discussed earlier. In practice, except for the elastic channel, only products are measured [see Eq. (4.13) and (4.15)]. Figure 13 shows that \( G_{1690}^{KN} G_{1690}^{\Sigma \pi} \) is negative, whereas \( G_{1520}^{KN} G_{1520}^{\Sigma \pi} \) is positive. Since only products are measured, there are four possible sign combinations for the four couplings. However, for the \( G_8 \) couplings we can write

\[ \frac{\sqrt{10}}{10} g_D + \frac{\sqrt{5}}{2} g_F = G_{1690}^{KN} \cos \theta - G_{1520}^{KN} \sin \theta, \]

\[ -\frac{\sqrt{15}}{5} g_D = G_{1690}^{\Sigma \pi} \cos \theta - G_{1520}^{\Sigma \pi} \sin \theta, \]

and we can eliminate the sign ambiguities by assuming that \( g_1 \) and \( g_F \) are positive.

Table XI shows the values of \( \theta \) calculated from (4.16), with the only sign combination left after these assumptions are made, using different input data and the two barrier factors discussed in Sec. IV-1, Eqs. (4.11) and (4.12):

\[ B_1 = \frac{(kr)^4}{9+3(kr)^2+(kr)^4}, \]  

(4.18)

\[ B_2 = k^4. \]  

(4.19)

The differences between \( \theta_1 \) and \( \theta_2 \) appear to be of a few standard deviations in general. Examples 3 and 4 show that the form (4.19) used by Plane et al. \(^{27}\) corresponds to (4.18) when a radius of interaction \( r \approx 0.15 \) fermi is chosen. This value of \( r \) would certainly give a bad fit
to resonance shapes, as discussed in Sec. II. For the data on $\Delta(1236)$ the value $r = 0.2$ fermi gave a $\chi^2$ of 6970 for the 17-point, three-parameter fit of Table V; it also gave $M = 1227$ and $\Gamma = 119$ MeV. We will come back to this point in Sec. IV-3B, for now we refer to this angle as $\theta = (15 \pm 3)^o$, as the $\theta_1$ values of cases 1 to 3 of Table XI would suggest. This mixing angle implies that $\Lambda(1520)$ is the mostly singlet state and $\Lambda(1690)$ is the mostly octet state.

As shown in Table X, Plane et al. $^{27}$ have fitted the entire octet by assuming that $\Lambda(1690)$ is the mostly octet member, and by assuming exact SU(3) symmetry. From the two isospin singlets they find values of $g_F$ and $g_D$ which agree with the values obtained from the other members of the octet. The fit is done by using Eqs. (4.5) through (4.8) in a way which is more related to experiments; in fact, the experimental results are usually expressed in terms of the amplitude at resonance, $t = \pm \sqrt{x e_i}$. At resonance ($\epsilon = 0$) Eqs. (4.13) and (4.14) give

$$t = \beta \sqrt{x e_i} = \beta \frac{G e_i}{G m_i} \sqrt{B k B_m k_m} \frac{M_N}{M_R \Gamma},$$

where $\beta$ is $\pm 1$, as discussed earlier. If instead of $\Gamma$'s, $t$'s are used as inputs, Eq. (4.6) gives hyperbolae instead of straight lines in a $g_D$ versus $g_F$ plot.

The fit of Plane et al. for the $J^P = 3/2^-$ states is shown in Fig. 14. It is evident that a common intersection for the hyperbolae is satisfactorily found when a mixing angle is introduced. The change in barrier factor discussed above, (4.18) versus (4.19), affects only slightly the goodness of the fit, once all the states are treated equally. Figure 14 uses $B = k^4$. The value of $\alpha$ defined in Eq. (4.10) is found to be

$$\alpha = \left[1 + \frac{\sqrt{5} g_F}{3 g_D}\right]^{-1} = 0.30 \pm 0.05.$$

Direct evidence for the SU(3) singlet nature of $\Lambda(1520)$ has been observed by Tripp et al. $^{41}$ The reactions studied are

$$K^- p \rightarrow \Sigma^\pm \pi^\mp$$  \hspace{1cm} (4.20.1)

$$K^- p \rightarrow \Sigma^0 \pi^0$$  \hspace{1cm} (4.20.2)
Fig. 14. The fit to the decay rates of the $J^P = 3/2^-$ octet states by Plane et al.\textsuperscript{27} $\Sigma$ stands for $\Sigma(1660)$, $N$ for $N(1520)$, $\Lambda_1$ for $\Lambda(1520)$, $\Lambda_8$ for $\Lambda(1690)$. The importance of the octet-singlet mixing is seen by the comparison of plot (a) ($\theta = 0^\circ$) and plot (b) ($\theta = 25^\circ$). The $\downarrow$ in (b) represents the fitted values of $g_D$ and $g_F$. 
Fig. 15. Evidence for singlet assignment of Λ(1520) from Tripp et al. 41 (a) The difference between the $A_3/A_0$ coefficient of the Legendre polynomial expansion of the $\Sigma^+\pi^-$ and $\Sigma^-\pi^+$ angular distributions. (b) The $A_2/A_0$ coefficient for $\Sigma^0\pi^0$ data. Curves show the best fit obtained from a multichannel analysis to all $K^-p$ data in the 300- to 465-MeV/c region; 21 the dashed curves are the fit with the P13 reversed in (a) and S01 reversed in (b). The arrows in the circle at right indicate the SU(3) predictions for the signs of the amplitudes.
in a formation experiment in the 300- to 450-MeV/c interval (same experiment discussed in Sec. III-1). Reaction (4.20.2) involves only isospin \( I = 0 \) amplitudes, whereas reaction (4.20.1) involves both \( I = 0 \) and \( I = 1 \) amplitudes. The Legendre polynomial coefficients for the above reactions contain information about the following three resonances:

\[
\begin{align*}
\text{S01} & \quad \Lambda(1405) \\
\text{P13} & \quad \Sigma(1385) \\
\text{D03} & \quad \Lambda(1518) 
\end{align*}
\]

In particular,

\[
\begin{align*}
\Sigma^+ \pi^- & \quad A^+_3/A_0 \propto \Re [ (P03 + P13)^* D03 ], \\
\Sigma^- \pi^+ & \quad A^-_3/A_0 \propto \Re [ (P03 - P13)^* D03 ].
\end{align*}
\]

Therefore

\[
(A^+_3 - A^-_3)/A_0 \propto 2 \Re [ (P13)^* (D03) ] \tag{4.21}
\]

for the \( \Sigma^0 \pi^0 \) events of reaction (4.20):

\[
A^-_2/A_0 \propto \Re [ (S01)^* (D03) ] \tag{4.22}
\]

The coefficients (4.21) and (4.22) are shown in Fig. 15. From the \( A_3/A_0 \) coefficients one can deduce that \( \Lambda(1520) \) and \( \Sigma(1385) \) have opposite signs; from the \( A_2/A_0 \) coefficient one can conclude that \( \Lambda(1405) \) and \( \Lambda(1520) \) have the same sign. From Table IX we see that for \( \Lambda(1520) \) pure-singlet we expect exactly these relative signs, whereas for \( \Lambda(1520) \) pure-octet we would expect the opposite to be true.

In conclusion: from the two-body decays of \( \Lambda(1520) \) and \( \Lambda(1690) \) every analysis points to a singlet assignment of \( \Lambda(1520) \).

B. \( \Sigma(1385) + \pi \) Decay Modes

The de Swart tables\(^{35,4}\) show that only members of a \( \{27\} \) or an \( \{8\} \) multiplet can decay into \( \Sigma(1385) \), member of a \( \{10\} \), and a \( \pi \), member of an \( \{8\} \) multiplet. That is,
so Eq. (4.8) now becomes

\[ 0 = G_{1520}^{* \pi} \cos \theta + G_{1690}^{* \pi} \sin \theta, \]

\[ \tan \theta = - \frac{G_{1520}^{* \pi}}{G_{1690}^{* \pi}}. \tag{4.23} \]

Since the relative signs of these two couplings have not been measured (neither of the two experiments \textsuperscript{20}, \textsuperscript{24} of Sec. III could determine the relative sign), and \( t = 0 \), therefore \( B_0 = 1 \), from (4.5) and (4.23) one has

\[ \tan^2 \theta = \frac{\Gamma_{1520}}{\Gamma_{1690}} \times \frac{1.518}{1.690} \times \frac{\langle p \rangle_{1690}}{\langle p \rangle_{1520}}. \tag{4.24} \]

The two widths have been discussed in Sec. III:

\[ \Gamma_{1520} = 1.44 \pm 0.26 \text{ MeV}, \]

\[ \Gamma_{1690} = 0.9 \pm 0.8 \text{ MeV}. \]

The phase space ratio can be calculated:

\[ \frac{\langle p \rangle_{1690}}{\langle p \rangle_{1520}} = \frac{\int_{\Lambda(1690)}^{M} F(M) \int_{\Sigma + \pi}^{M} p(m) f(m) \, dmdM}{\Lambda(1520) \int_{\Sigma + \pi}^{M} p(m) f(m) \, dmdM} = 10.6 , \tag{4.25} \]

where the \( F(M) \) and \( f(m) \) are the Breit-Wigner form for \( \Lambda(1520) \) or \( \Lambda(1690) \) and \( \Sigma(1385) \) respectively, and the lower limit \( \Sigma + \pi \) stands for \( M_{\Sigma + M_\pi} \). With these values we get

\[ \tan^2 \theta = 15.2 \pm 13.8 , \quad \theta = (75^{+5}_{-10})^\circ. \tag{4.26} \]

This value of \( \theta \) is clearly in disagreement with the value \( (15 \pm 3)^\circ \) discussed in Sec. IV-2A.
A previous calculation\textsuperscript{23} of the mixing angle \((4.26)\) has yielded a value of \(\theta\) in agreement with the two-body mixing angle. The discrepancy between the evaluation of Burkhard et al.\textsuperscript{23} and the above value \((75^{+5}_{-10})\) is due to the following factors:

(a) A factor larger than 2 in \(\Gamma_{1520}\) as discussed in Section III-1.
(b) A factor larger than 10 in \(\Gamma_{1690}\) as seen in Sec. III-2.
(c) A factor 3 in the ratio \((4.25)\).

This last factor 3 is due to the fact that Burkhard et al. have normalized the \(\Sigma(1385)\) Breit-Wigner differently for the numerator and denominator, having used as normalization only the part of \(\Sigma(1385)\) kinematically allowed by the \(\Lambda(1520)\) decay in the denominator.

C. Disagreement Between the Two Mixing Angles

The experimental data at present show a large disagreement between the two-body and the quasi-two-body decays of \(\Lambda(1520)\) and \(\Lambda(1690)\). Possible causes of this disagreement, assuming the experiments are correct, are:

(a) The assignment obtained from the two-body decays is erroneous. This is possible if either of the two \(\Lambda\)'s belongs to another as-yet-un-detected octet or to a 27-plet. This last assignment would imply existence of exotic states, already suggested by the possible existence of \(Z^*\) states.\textsuperscript{42} In addition, the octet assignment of \(\Lambda(1690)\) is based on the validity of exact SU(3), which may not be the case.

(b) Mixing among two octets and a singlet should be considered. The quark model in fact predicts two octets with \(J^P = 3/2^-\) and a singlet all being part of the 70(\(L=1^-\)) configuration. Faiman\textsuperscript{43} has recently done calculations of this type by using the harmonic oscillator quark model (see Prof. Dalitz' lectures for some details on this model), suggesting mixing of two octets and a singlet.

3. The \(J^P = 3/2^+\) Decuplet

How well does the \(3/2^+\) decuplet fit SU(3) predictions? This question has been asked often during this school and has been a source of much discussion. Let us try to analyze the situation. We will find that:

(a) Exact SU(3) fits very badly (confidence level \(CL < 10^{-4}\)) if the Blatt-Weisskopf barrier factor, Eq. (4.18), is used. This form is
experimentally needed to fit resonance shapes, as seen in Sec. II.

(b) Exact SU(3) fits better (CL = 0.10) if the form $k^2$ is used. However, for resonance shapes this form is completely wrong, as already mentioned in Sec. IV-2A.

(c) Broken SU(3) fits perfectly even with the use of the experimentally sound Blatt-Weisskopf barrier factor.

A. Data

The members of this decuplet are: $\Delta(1236)$, $\Sigma(1385)$, and $\Xi(1530)$. There is some uncertainty for some of the parameters so we discuss here the values we used for the fits. Of course one could increase the errors of some of the widths and improve the SU(3) fit accordingly, but this approach is artificial and I prefer to use uncertainties as suggested by the experimental data available. The values used here are shown in Table XII.

$\Delta(1236)$

In Sec. II we have discussed the new data and have seen that the mass and width for this state depend upon the parametrization used. This is true for most of the resonances, in general, and the errors are supposed to include this type of uncertainty. Reasonable values for $\Delta(1236)$ are:

$$M = 1233 \pm 3 \text{ MeV},$$
$$\Gamma = 115 \pm 5 \text{ MeV},$$
$$x_e = (99.4 \pm 0.6)\%$$

where the $x_e$ takes into account the $\gamma N$ decay.\(^1\)

$\Sigma(1385)$

For the mass and width we use the PDG values;\(^1\) for the branching ratios we calculate a new world average by using only the most recent experiments listed in the PDG compilation. We get:

$$M = 1385 \pm 3 \text{ MeV},$$
$$\Gamma = 36 \pm 6 \text{ MeV},$$
$$\Sigma(1385) \to \Lambda\pi = (88 \pm 5)\%,$$
$$\Sigma(1385) \to \Sigma\pi = (12 \pm 5)\%.$$
In SU(3) fits we have to ignore mass differences among different charge states, so $M = 1530$ MeV is a reasonable value. As for the width, the published experiments give an average $\Gamma = 7.3 \pm 1.7$ MeV; a new recent value $11 \pm 2$ MeV by Badier et al.\textsuperscript{44} increases the average. We do the calculations by using both values; the conclusions do not differ very much.

$$M = 1530 \text{ MeV}, \quad \Gamma = 7.3 \pm 1.7 \text{ MeV} \quad \text{or} \quad 8.8 \pm 1.8 \text{ MeV},$$

$$\Xi(1530) \rightarrow \Xi\pi = 100\%.$$

B. Exact SU(3)

From Eq. (4.5) the coupling for each decay is given by

$$g_i = \frac{1}{c_i} \sqrt{\frac{\Gamma_i M_R}{M_N B(k_i) k_i}} = \frac{1}{c_i} \sqrt{\frac{\Gamma_i}{F_i}}, \quad (4.27)$$

where $c_i$ is a SU(3) coefficient from Table IX, $\Gamma_i$ is the measured width for the decay $i$, $M_R$ and $M_N$ the masses of the resonance and of the nucleon, $k_i$ the c.m. system momentum of the decay products, and $B(k_i)$ a barrier factor. According to Eqs. (4.11) and (4.12) the two forms for the barrier factor are

$$B_1 = \frac{(kr)^2}{1+(kr)^2}, \quad (4.28)$$

$$B_2 = k^2. \quad (4.29)$$

The factor $F_1 = B_1 k_i M_N/M_R$ is shown in Fig. 16 as a function of the radius of interaction for the four decays of the decuplet. The values of $F_2$ for the four decays are shown as straight lines in Fig. 16. The graph clearly shows that for $r \approx 0.2$ fermi the two values of the factors are equal. Therefore, using $B_2$ [Eq. (4.29)] is equivalent to using the Blatt-Weisskopf barrier factor [Eq. (4.28)] with $r = 0.2$ fermi.

Table XIII shows the calculated values of $g$ and $g^2$ for the two barrier factors, ($r = 1$ fermi in $B_1$) and for all the decay widths of Table XII. Exact SU(3) predicts that the coupling for each decay of
Table XII. Data and factors used for SU(3) fit to \( J^P = 3/2^+ \) decuplet. Here \( c \) are the SU(3) coefficients from Table IX; the barrier factors are \( B_1 = (kr)^2/[1+(kr)^2] \) with \( r = 1 \) fermi and \( B_2 = k^2 \).

<table>
<thead>
<tr>
<th>Decay</th>
<th>( M^* ) (MeV)</th>
<th>( \Gamma^* ) (MeV)</th>
<th>Fraction (%)</th>
<th>Momentum ( k ) (GeV/c)</th>
<th>( F_1 = B_1 k M^* / M^* ) (GeV/c)</th>
<th>( F_2 = B_2 k M^* / M^* ) (GeV/c)</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta(1236) \to \Lambda^* )</td>
<td>1233</td>
<td>11.5±5</td>
<td>99.4±0.6</td>
<td>0.229</td>
<td>0.0997</td>
<td>0.00909</td>
<td>-1/(\sqrt{2} )</td>
</tr>
<tr>
<td>( \Sigma(1385) \to \Lambda^* )</td>
<td>1385</td>
<td>36±6</td>
<td>88±8 ( \dagger )</td>
<td>0.209</td>
<td>0.0748</td>
<td>0.00617</td>
<td>-1/2</td>
</tr>
<tr>
<td>( \Sigma(1385) \to \Sigma^* )</td>
<td>1385</td>
<td>36±6</td>
<td>12±5 ( \dagger )</td>
<td>0.121</td>
<td>0.0226</td>
<td>0.00121</td>
<td>1/(\sqrt{2} )</td>
</tr>
<tr>
<td>( \Xi(1530) \to \Xi^\ast (\text{a}) )</td>
<td>1530</td>
<td>7.3±1.7 ( \dagger )</td>
<td>100</td>
<td>0.150</td>
<td>0.0336</td>
<td>0.00206</td>
<td>1/2</td>
</tr>
<tr>
<td>or (b)</td>
<td></td>
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</tbody>
</table>

\( \dagger \) World average, taken from Review of Particle Properties. \( \dagger \)

Value obtained by including in world average \( \dagger \) the new result of Badier et al. \( \dagger \)

\( \dagger \) New evaluation of the world average with more conservative errors.

Table XIII. Calculated couplings for the \( J^P = 3/2^+ \) decuplet using data and factors of Table XII. Averages (a) and (b) refer to average couplings obtained when values (a) or (b) for the width of \( \Xi(1530) \to \Xi^* \) are used. Chi-squared and confidence levels for each fit are also given.

<table>
<thead>
<tr>
<th>Decay</th>
<th>( F_1 )</th>
<th>( x_a )</th>
<th>( x_b )</th>
<th>( x_a )</th>
<th>( x_b )</th>
<th>( \chi^2 )</th>
<th>( \chi^2 )</th>
<th>( \chi^2 )</th>
<th>( \chi^2 )</th>
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<th>( \chi^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta(1236) \to \Lambda^* )</td>
<td>1.52±0.03</td>
<td>1.3</td>
<td>3.0</td>
<td>2.29±0.10</td>
<td>10.9</td>
<td>7.8</td>
<td>5.02±0.11</td>
<td>0.9</td>
<td>0.6</td>
<td>25.2±1.1</td>
<td>1.6</td>
<td>0.7</td>
<td>9.9</td>
<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>( \Sigma(1385) \to \Lambda^* )</td>
<td>1.30±0.12</td>
<td>1.6</td>
<td>1.7</td>
<td>1.70±0.30</td>
<td>0.8</td>
<td>1.1</td>
<td>4.53±0.40</td>
<td>0.9</td>
<td>1.0</td>
<td>20.5±3.6</td>
<td>0.8</td>
<td>1.0</td>
<td></td>
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<td></td>
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</tr>
<tr>
<td>( \Sigma(1385) \to \Sigma^* )</td>
<td>1.07±0.24</td>
<td>2.4</td>
<td>2.5</td>
<td>1.15±0.52</td>
<td>2.5</td>
<td>2.6</td>
<td>4.63±1.04</td>
<td>0.1</td>
<td>0.1</td>
<td>21.4±9.6</td>
<td>0.1</td>
<td>0.1</td>
<td></td>
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<tr>
<td>( \Xi(1530) \to \Xi^* (\text{a}) )</td>
<td>0.93±0.11</td>
<td>22.4</td>
<td>0.87±0.20</td>
<td>29.9</td>
<td>3.77±0.44</td>
<td>6.8</td>
<td>14.2±3.3</td>
<td>8.4</td>
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<tr>
<td>or (b)</td>
<td>1.02±0.11</td>
<td>16.6</td>
<td>1.05±0.22</td>
<td>20.1</td>
<td>4.14±0.42</td>
<td>3.5</td>
<td>17.1±3.5</td>
<td>4.0</td>
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<tr>
<td>Average (a)</td>
<td>1.447±0.030</td>
<td>30.5</td>
<td>1.96±0.09</td>
<td>43.1</td>
<td>4.91±0.10</td>
<td>8.7</td>
<td>23.8±1.0</td>
<td>10.9</td>
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<tr>
<td>or (b)</td>
<td>1.452±0.030</td>
<td>24.4</td>
<td>2.01±0.09</td>
<td>51.9</td>
<td>4.93±0.10</td>
<td>5.2</td>
<td>24.1±1.0</td>
<td>6.0</td>
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<tr>
<td>Confidence Level</td>
<td>10^{-6}</td>
<td>2×10^{-5}</td>
<td>&lt;10^{-7}</td>
<td>5×10^{-7}</td>
<td>0.034</td>
<td>0.16</td>
<td>0.012</td>
<td>0.11</td>
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</table>
Fig. 16. Factor $F = B k M_N/M_R$ plotted versus the radius of interaction for the two barrier factor forms $B_1 = (kr)^2/[1+(kr)]^2$ and $B_2 = k^2$ for all four decays of members of the $3/2^+$ decuplet. For $r = 0.2$ fermi the values of $F_1$ and $F_2$ coincide. The units are indicated in Table XII.
the members of the decuplet should be the same. It is not clear if this applies to $g$ or $g^2$; therefore, in Table XIII we have calculated both. In the past, $g^2$ was used\textsuperscript{27, 38, 39} to test the validity of exact SU(3), probably because $g^2$ is more related to experiments since for the members of this decuplet $\Gamma$ and not $\sqrt{\Gamma}$ is measured. A look at Table XIII shows that exact SU(3) fits better for $g$ than for $g^2$. In Table XIII the average $g$ and $g^2$ are also calculated as well as a chi-squared ($\chi^2$) for each set of four decay widths $(a, b)$ and the confidence level for the four decay rates to obey exact SU(3). The results reported in Table XIII lead to the following considerations:

(a) Using the Blatt-Weisskopf barrier factor, both $g_1$ and $g_1^2$ fit exact SU(3) badly, even for the new increased width for $\Xi(1530)$, fit (b). The major contribution to $\chi^2$ comes in both cases from the $\Xi(1530)$ width.

(b) Using the $k^2$ barrier factor, both $g$ and $g^2$ fit better; still, the major contribution to $\chi^2$ comes from the width of $\Xi(1530)$. As seen earlier the form $k^2$ is equivalent to the Blatt-Weisskopf barrier factor with $r = 0.2$ fermi.

Seeing the results (a) and (b), some authors take the point of view that the barrier factor $k^{2f}$ is best suited to describe SU(3) couplings because "exact SU(3) fits better" in spite of the fact that resonance shapes would never fit with such a barrier factor. My point of view is that exact SU(3) does not have to fit the measured widths of Table XII because, since SU(3) is broken for the masses (the real part of the pole position $M + i\Gamma/2$), it is probably broken for the widths. Therefore I do not believe in using an unreasonable barrier factor (corresponding to a radius of interaction of 0.2 fermi) to fit exact SU(3) when we can use a perfectly reasonable approach to fitting the couplings, that is, broken SU(3).

C. Broken SU(3)

The formula to be used for a decuplet is Eq. (4.9):

$$\Delta + \Xi = \frac{1}{2} \left[ 3\Sigma^*(\Lambda\pi) + \Sigma^*(\Sigma\pi) \right].$$

From the calculated couplings reported in Table XIII we find for the two different barrier factors:
The $\chi^2$ for the fit of the couplings to Eq. (4.9) is very good in all cases. The goodness of the fit seems, with present data, to be independent of the barrier factor used and of the width used for $\Xi(1520)$. In Fig. 17 the left- and right-hand side of the sum rule (4.9) are shown for cases (4.30.1) and (4.30.3).
Fig. 17. SU(3) fits for the $J^P = 3/2^+$ decuplet for: (a) Blatt-Weisskopf barrier factor, (b) simplified barrier factor $k^2$. The lower part of the graph shows exact SU(3) fits; that is, all four couplings which are supposed to be the same. The upper part of the graph shows broken SU(3) fits; that is, data points for the left- and right-hand side of the sum rule for the couplings.
V. Study of Λ(1405): CDD Pole Versus Bound State

The Λ(1405) state has been discovered in a production experiment \(^{45}\) in the reaction \(K^- p \rightarrow \Sigma^+ \pi^+ \pi^-\). It is produced only in the \(\Sigma \pi\) channel, because its mass (1405 MeV) is below the \(K^- p\) threshold (1432 MeV) and there is no other isospin \(I = 0\) channel open.

Studies of \(\bar{K}N\) interactions from threshold to 300 MeV/c \(^{46, 47}\) have indicated that \(Λ(1405)\) can be interpreted as an \(S\)-wave bound state of the \(\bar{K}N\) system, coupled to both \(\bar{K}N\) and \(\Sigma \pi\) channels. The evidence for \(Λ(1405)\) as a bound state has come from partial wave analyses, which have shown a pole at 1405 MeV for the reduced \(K\) matrix for the \(I = 0\) \(\bar{K}N\) \(S\)-wave amplitude. The reduced \(K\)-matrix formalism was first discussed by Dalitz and Tuan, \(^{48}\) and these types of poles have been called virtual bound states.

In recent years, however, two alternative interpretations have been proposed for the nature of \(Λ(1405)\):

(a) An \(S\)-wave bound-state of the \(\bar{K}N\) system resulting from the attraction generated by the exchange of a vector meson. A potential model calculation of this sort has been carried out by Dalitz, Wong, and Rajasekaran \(^{49}\) who conclude that \(Λ(1405)\) can arise from such a model.

(b) An isoscalar unitary singlet, member of an \(SU(6)-L\) supermultiplet with orbital angular momentum \(L = -1\). The symmetric quark model \(^{50, 51}\) naturally includes one such state. This interpretation implies that \(Λ(1405)\) is due to the strong forces which produce the entire supermultiplet and not to particular forces, like vector exchange in the \(\bar{K}N\) system, which are only relevant to the \(Λ(1405)\) state.

Dalitz has recently pointed out that these two interpretations are opposite, essentially because the dynamics involved in the two cases are different. \(^{47}\) He also suggests a method to experimentally distinguish between these two possibilities, by studying the energy dependence of the \(K\) matrix used to fit the data. The second interpretation, in fact, would require \(Λ(1405)\) to appear as a pole in the entire \(K\) matrix and not as a pole in the reduced \(K\) matrix (See Sec. V-2).

In recent months two papers have been published on the subject. Martin, Martin, and Ross \(^{52}\) have analyzed existing low-energy \(\bar{K}N\) scattering data and concluded that interpretation (a) is strongly favored. Cline, Laumann, and Mapp \(^{53}\) have analyzed their own data for the reaction \(K^- d \rightarrow Y \pi N\) and concluded that interpretation (b) is strongly favored by the data.
I have done my own analysis on old as well as new unpublished KNN data, and conclude that the low-energy KNN data cannot distinguish between the two models by studying the energy dependence of the K matrix, but, however, the bound state interpretation seems more plausible. In addition I find that the data of Cline et al. unless completely mislabelled, do not seem to agree with any experiment done so far in the region where they can be checked, therefore I think that their method of extracting data below the KNN threshold from K\-'d interactions has to be reexamined.

1. Formalism for Low-Energy KNN Scattering

A partial wave analysis was performed by using the differential cross section and polarization data for the K\-'p reactions

\[ K^- p \rightarrow K^- p, ~ K^0 n, \]
\[ K^- p \rightarrow \Sigma^+ \pi^-, \Sigma^0 \pi^0, \Sigma^- \pi^+, \]
\[ K^- p \rightarrow \Lambda \pi^0, \]
\[ K^- p \rightarrow \Lambda \pi^+ \pi^-, \]

and the K\(^0\)\-'p reactions

\[ K^0 p \rightarrow K^+_p, ~ K^0 p, \Sigma^0 \pi^+, \Lambda \pi^+. \]

The differential cross sections and polarizations are expanded in terms of \( f(\theta) \), the non-spin-flip amplitude, and \( g(\theta) \), the spin-flip amplitude as

\[
\frac{d\sigma}{d\Omega} = \lambda^2 \left( |f(\theta)|^2 + |g(\theta)|^2 \right),
\]

\[
\vec{P} \frac{d\sigma}{d\Omega} = 2 \lambda^2 \text{Re} [f^*(\theta) g(\theta)] \hat{n},
\]

where \( \vec{P} \) is the polarization and \( \hat{n} \) the unit vector perpendicular to the plane of scattering. The \( f(\theta) \) and \( g(\theta) \) are then expanded in terms of the Legendre polynomials \( P^1_{\ell}(\theta) \) and the first associated Legendre functions \( P^1_{\ell}(\theta) \):

\[
f(\theta) = \sum_{\ell} \left[ (\ell + 1) T_{\ell+} + \ell T_{\ell-} \right] P^1_{\ell}(\cos \theta),
\]

\[
g(\theta) = i \sum_{\ell} (T_{\ell+} - T_{\ell-}) P^1_{\ell}(\cos \theta),
\]
where $T_{\ell^+}$ is the amplitude for total angular momentum $J = \ell + \frac{1}{2}$, and $T_{\ell^-}$ is the amplitude for $J = \ell - \frac{1}{2}$. Appropriate isospin factors have to be introduced for each channel, and unitarity has to be imposed on each partial wave.

The analysis discussed here is the same multichannel analysis mentioned earlier, in Sec. III-1, extended to lower momenta, down to $K^-p$ at rest. Since some of the fits included data from 0-465 MeV/c, I describe here all the analysis.

The partial waves included were:

- $S_0^1$, $P_0^1$, $P_0^3$, $D_0^3$, $D_0^5$
- $S_1^1$, $P_1^1$, $P_1^3$, $D_1^3$, $D_1^5$

where the convention is as usual, $\ell I2J$; $\ell$ is the orbital angular momentum, $I$ is the isospin, and $J$ is the total angular momentum of the $K^-p$ system. For these partial waves the following parametrizations were used:

a. $P_0^1$, $P_1^1$, $P_0^3$, $D_1^3$, $D_0^5$, $D_1^5$. A constant scattering length, $A$, parametrization of the type

$$T_e = \frac{k^{2\ell+1}}{A} e^{i\Phi_e}, \quad A = a + ib,$$

where the subscript $e$ refers to the elastic channel and $c$ to the other channels. $\Phi_c$ is an arbitrary phase introduced for each nonelastic channel. Unitarity is imposed by requiring $\Sigma r_i = 1$.

b. $D_0^3$. This amplitude resonates at 390 MeV/c, where $\Lambda(1520)$ is formed. It was parametrized as the Breit-Wigner form we have called STANDARD (see Sec. II-1) with a radius of interaction of 1 fermi. A constant-scattering-length background term, Eqs. (5.3) and (5.4), was added to the resonance, but it turned out to be extremely small.

c. $S_0^1$, $S_1^1$, $P_1^3$. The K-matrix parametrization has been used for these partial waves in order to introduce the correct threshold behavior. Both the $P_1^3$ and $S_0^1$ resonate below the $K^-p$ threshold as $\Sigma(1385)$ and $\Lambda(1405)$ respectively; in addition, the $S_1^1$ is also large at threshold and should be properly parametrized. The K-matrix formalism has been discussed by Dalitz and Tuan and by Dalitz.
and Shaw\textsuperscript{55, 56} have developed an effective range formalism for systems with many coupled channels which gives an explicit energy dependence for the K-matrix elements.\textsuperscript{57} They define a real and symmetric matrix
\begin{equation}
\mathbf{M} = k^{\ell + \frac{1}{2}} \mathbf{K}^{-1} k^{\ell + \frac{1}{2}},
\end{equation}
where $k^{\ell + \frac{1}{2}}$ is a diagonal matrix having elements $k_{i}^{\ell + \frac{1}{2}}$, the c.m. momentum of the $i$th channel to the $(\ell + \frac{1}{2})$ power, $\ell_{i}$ being the orbital angular momentum of channel $i$. $\mathbf{M}$ can then be expanded in terms of $k^{2}$ as follows:
\begin{equation}
\mathbf{M}(E) = \mathbf{M}(E_{0}) + \frac{1}{2} \mathbf{R} (k^{2} - k_{0}^{2})
\end{equation}
where $k_{0}$ is the momentum corresponding to $E_{0}$. In this case $E_{0}$ is the $K^{-}p$ threshold energy 1432 MeV, and $\mathbf{R}$ is a real energy-independent matrix, usually assumed to be diagonal. For the S01 partial wave (see Sec. V-2 below) the non-diagonal term has been included. Finally, the relation between $\mathbf{T}$ and $\mathbf{M}$ is
\begin{equation}
\mathbf{T} = k^{\ell + \frac{1}{2}} (\mathbf{M} - i k^{2\ell + 1})^{-1} k^{\ell + \frac{1}{2}}.
\end{equation}
For the S11 and P13 partial waves $\mathbf{T}$ is a 3X3 matrix, therefore nine parameters are needed to fit all three channels ($\overline{K}N, \Sigma \pi, \Lambda \pi$). The S01 will be discussed in Sec. V-2.

In order to fit the data at threshold there are two corrections to be introduced in the formalism: one for the $K^{-}p - \overline{K}^0n$ mass difference, the other for the Coulomb interaction in the $K^{-}p$ channel. The formalism of Ross and Shaw\textsuperscript{56} has been followed here too. For each isospin state $I = 0, 1$ a complex scattering length has been calculated according to the definition
\begin{equation}
(T_{11})_{I} = \frac{A_{1}k_{1}}{1 - i A_{0}k_{1}},
\end{equation}
where $T_{11}$, the elastic amplitude, is the corresponding element of the matrix (5.7) and $k_{1}$ is the c.m. momentum for the $K^{-}p$ system. The amplitude for each channel is then obtained by $T_{1j}$ from Eq. (5.7) multiplied by a factor, a function of $A_{1}$ and $A_{0}$, which can be found in Ross and Shaw.\textsuperscript{56} Notice, however, that the definition (5.8) of the scattering length has a sign opposite to the one of Ref. 56. The $K^{-}p$ amplitude included, in addition to the mentioned factor, another amplitude for the Coulomb scattering.\textsuperscript{56}
For the $K^0 \bar p$ data the same amplitudes as for $K^- p$ were used, with the appropriate Clebsch-Gordan coefficients, except that for the $K^0 p \rightarrow K_1^0 p$ also a strangeness $+1$ amplitude contributes. In fact,

$$|K_2^0 p\rangle = \frac{1}{\sqrt{2}} |K^0 p\rangle + \frac{1}{\sqrt{2}} |\bar K^0 p\rangle.$$ 

The $\bar K^0 p$ state is pure $I=1$, whereas the $K^0 p$ state includes both $I=0, 1$. The strangeness $+1$ amplitudes included only $S$ and $P$ waves for the $I=0$ state and $S$-wave for the $I=1$. They were parametrized according to Eq. (5.3) with $b=0$, since only the elastic channel is open, and with constant scattering length for S01 and P01 and effective range for S11. The parameters used were taken from Stenger et al. 58 for the S01 and P01, and from the analysis of Martin and Perrin 59 for the S11. Expressed in fermi, there were as follows:

$$a_{0}^S = 0.04 \text{ fermi},$$

$$a_{1}^S = -0.32 \text{ fermi} + \frac{1}{2} rk^2, \quad r = 0.31 \text{ fermi},$$

$$a_{0}^P = 0.11 \text{ fermi}.$$

2. Formalism for the S01 Partial Wave

The $I=0$ S-wave is best described by the K-matrix formalism, because the K matrix is immune from the singularity of the T matrix at $\bar K N$ threshold, but allows one at the same time to find the poles of the T matrix. Since we have to go across the threshold to extrapolate down to the $\Lambda(1405)$ location, it is best to use non-dimensionless T and K matrices and define instead of (5.5) and (5.7),

$$M = K^{-1},$$

$$T = K (1-ikK)^{-1}.\tag{5.10}$$

For the S01 we have only a two-channel system, the $\bar K N$, called channel 1, and the $\Sigma\pi$, called channel 2. The scattering amplitude for the $\Sigma\pi \rightarrow \Sigma\pi$, where $\Lambda(1405)$ occurs, is

$$T_{22} = \frac{k_2 K_{22}^2 (1-k_1 K_{11}^1) + ik_1 k_2 K_{12}^2}{(1-ik_1 K_{11}^1) (1-ik_2 K_{22}^2) + k_1 k_2 K_{12}^2}.\tag{5.11}$$

Below $\bar K N$ threshold, it is appropriate to use the single channel reduced K matrix 47, 48, 54.
where \( k_1 \) is the momentum for the \( K^-p \) system, which is purely imaginary below threshold so \( ik_1 = -|k_1| \). The reduced \( K \) matrix, \( K_R \), has a pole any time that

\[
1 + |k_1|K_{11} = 0.
\]

However, this same condition can also be obtained from (5.11) for the real part of the denominator \( D(s) \). If we write it as \( (E - E - \Gamma) \), at \( E = E^* \), \( \text{Re}D(s) = 0 \).

Dalitz\(^{47}\) suggested that a way to distinguish between the two interpretations for \( \Lambda(1405) \) is to study the energy dependence of the \( K \) matrix. In fact the two models would predict different behavior for the \( K \) matrix.

a. \( \Lambda(1405) \) due to \( K^+N \) interactions. In this case the \( K \)-matrix energy dependence is expected to be like Eq. (5.6):

\[
K^{-1}_R = K_0^{-1} + \frac{1}{2} R (k^2 - k_0^2)
\]

and the two following things would happen:

1a. The \( K \) matrix elements would not have poles.

2a. The reduced \( K \) matrix (5.12) would have a pole around 1405 MeV.

b. \( \Lambda(1405) \) member of a supermultiplet. In this case the \( K \)-matrix energy dependence would have an explicit pole term:

\[
K = K_0 + \frac{cc}{E^* - E}
\]

and the following would be true:

1b. Each of the \( K \)-matrix elements would have a pole, this is called CDD pole because the \( K \) matrix has a background, \( K_0 \), added to an explicit pole.

2b. The reduced \( K \) matrix (5.12) would still have a pole where condition (5.13) is verified, that is,

\[
1 + |k_1| \left( K_0^{11} + \frac{c_1^2}{E^* - E} \right) = 0,
\]

which has a pole at

\[
E = E^* + \frac{c_1^2 |k_1|}{1 + |k_1|K_{11}^0}.
\]
Both expressions for the $K$ matrix, (5.14) and (5.15), can be expressed in terms of six variables: three variables for $K_0$ in both cases, and three effective ranges in (5.14); $c_1$, $c_2$ and $E^*$ in (5.15). The next question is: can the available data distinguish between the energy dependence (5.14) and (5.15)?

3. Data Used in the Fits

All available $KN$ data up to 280 MeV/c from Refs. 60-64 have been included in the fit. Whenever two experiments measured the same quantity, the two results have been averaged and the error obtained was multiplied by a factor $s = \sqrt{\chi^2}$ whenever the chi-squared turned out to be $\chi^2 > 1$. A total of 163 points (out of 170) above 280 MeV/c are from Ref. 21 and have not been used previously in any other analysis. All the $A_i/A_0$ and polarization data, except for four values of $A_i/A_0$ from Ref. 64, also come from Ref. 21. The results at rest have been averaged and provide three data points. Table XIV shows what kind of data points have been used and the corresponding references. The last column shows the individual channel's contribution to the total chi-squared for fit $a$ as shown in Table XV and discussed in the next section.

Some fits were done, as discussed later, up to 465 MeV/c. All the data points used for $p_K > 320$ MeV/c are from Ref. 21.

4. Results of the Analysis

The analysis was done in two parts: first all the data from zero to 465 MeV/c were used for the partial wave analysis, then all the D waves were fixed to the values obtained in the fit and a new fit was performed, using only data in the 0-320 MeV/c region. This was done for two reasons:

a. The high partial waves contribute very little below 320 MeV/c, so they cannot be determined from these data; however, their contribution is not zero and for reasons of continuity they should be included in the analysis.

b. The fit up to 465 MeV/c extends too far from the region of interest for the study of $\Lambda(1405)$ and a new resonance, $\Lambda(1520)$, dominates around the 385 MeV/c region. Although the interference of the S01-D03 waves is essential to an understanding of the relative signs of the $\Lambda(1405)$ and $\Lambda(1520)$ couplings to $\Sigma\pi$, it may be considered too far away from 1405 MeV to contribute to the knowledge of detailed energy dependence of the amplitude below threshold, which is what we want to study.
Table XIV. Data used and chi-squared obtained in the analysis for $P_K < 320$ MeV/c.

<table>
<thead>
<tr>
<th>Reaction</th>
<th>No. of data points</th>
<th>References</th>
<th>Total data points</th>
<th>Chi-squared $\chi^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K^- p \rightarrow K^- p$</td>
<td>13 16</td>
<td>60, 61, 21</td>
<td>29</td>
<td>32.2</td>
</tr>
<tr>
<td>$K^0 \rightarrow K^0 n$</td>
<td>15 12</td>
<td>60, 62, 63, 21</td>
<td>27</td>
<td>32.3</td>
</tr>
<tr>
<td>$\Sigma^+ \pi^-$</td>
<td>15 16</td>
<td>49 60, 61, 21</td>
<td>80</td>
<td>84.4</td>
</tr>
<tr>
<td>$\Sigma^- \pi^+$</td>
<td>15 16</td>
<td>60, 61, 21</td>
<td>31</td>
<td>63.7</td>
</tr>
<tr>
<td>$\Sigma^0 \pi^0$</td>
<td>7 8 8</td>
<td>60, 64, 21</td>
<td>23</td>
<td>10.8</td>
</tr>
<tr>
<td>$\Lambda \pi^0$</td>
<td>16 16</td>
<td>8 60, 64, 21</td>
<td>40</td>
<td>26.2</td>
</tr>
<tr>
<td>$\Lambda \pi^+ \pi^-$</td>
<td>3</td>
<td>21</td>
<td>3</td>
<td>1.3</td>
</tr>
<tr>
<td>$K^0_S p$, total $\sigma$</td>
<td>10</td>
<td>65</td>
<td>10</td>
<td>5.5</td>
</tr>
<tr>
<td>$K^0_S p \rightarrow \text{channels}^c$</td>
<td>8</td>
<td>64</td>
<td>8</td>
<td>4.9</td>
</tr>
<tr>
<td>$K^0_S p$ at rest$^d$</td>
<td>3</td>
<td>60, 66</td>
<td>3</td>
<td>3.3</td>
</tr>
</tbody>
</table>

| Totals                    | 105 84 65          | 254        | 264.5            |

$^a A_i/A_0$ are the coefficients of the Legendre polynomials expansion

$$\frac{d\sigma}{d\Omega} = \kappa^2 \sum_{i=1}^{4} A_i P_i(\theta).$$

$^b$ Polarization data are expressed as $B_i$, coefficients of the expansion

$$\frac{d\sigma}{d\Omega} = \kappa^2 \sum_{i=1}^{4} B_i P_i(\theta)$$

for the $\Sigma^0 \pi^0$ and $\Lambda \pi^0$ channels. For the $\Sigma^+ \pi^-$ channel $P(\theta)$ has been directly fitted.

$^c$ These data points are

$$\epsilon = \frac{\sigma_\Lambda}{(\sigma_\Lambda + \sigma_{\Sigma^0})} \quad \text{and} \quad R = \frac{\sigma_{Kp}}{(\sigma_\Lambda + \sigma_{\Sigma^0})}.$$

$^d$ These are ratios of production intensities at rest,

$$r_1 = \frac{N(\Lambda)}{N(\Sigma^+ + \Lambda)} \quad , \quad r_2 = \frac{N(\Sigma^+ + \Sigma^-)}{N(\Sigma^0 + \Lambda)} \quad \text{and} \quad r_3 = \frac{N(\Sigma^-)}{N(\Sigma^+)}.$$
The results of the fits to data with $P_K < 320$ MeV/c are shown in Table XV. In all fits the $P_{13}$ amplitude below threshold was constrained to resonate at 1385 MeV; whenever the constraint was not imposed the $\Sigma(1385)$ would appear at about 10 MeV lower mass. I now discuss the details of the fit.

**Constant K-matrix.** Fit a and e are of this type. They are essentially the same, and the only reason for listing both is to provide values for the $K^0$ and $M^0$ matrix elements. The $\chi^2$ of 264 for 254 data points is somewhat larger than obtained in previous fits; however, the new data have smaller errors and the old data, being averaged as explained in the previous section, also have smaller errors than used previously. Figure 18 shows the cross-section data with the fitted curves for channels 1 through 6 (see Table XIV). Figure 19 shows some $K^0\pi$ data with the fitted curves. The last column of Table XV shows $E_R$, the position of the pole in the reduced K matrix, which for this fit turns out to be at 1411 MeV.

**CDD pole parametrization.** Three fits have been done and are reported in Table XV as b, c, and d. The chi-squared for all three fits is quite good and the difference 264 to 271 is not large enough to exclude any of the fits. For fit b the reduced K matrix has two poles, one close to the energy $E^*$, the other at 1414 MeV, very close to the pole obtained for the constant K-matrix fit. Inspection of Eq. (5.16) and of Table XV shows that this is not surprising: the term in $c_2^*$ is very small compared with the constant K-matrix term. However, the $c_2^*$ term varies very rapidly around $E^*$, and there is a value of $E$ in this region where relation (5.16) is satisfied. In addition, $c_4^2$ being small, the constant K-matrix term, $K^0$, also has a pole very close to where it was in fit a. Fits c and d, done without a constant K-matrix term, also have a pole in the reduced K matrix, but it is somewhat far from $E^*$, closer to the location of fit a than to $E^*$.

**Effective range parametrization.** Fit f is as good as the others. Three new parameters have been added and the chi-squared has decreased only by 2. As expected, this parametrization shows a pole in the reduced K matrix at $M = 1417$ MeV, a slightly higher mass than for the constant K-matrix.

**Fits to data up to 465 MeV/c.** These fits give results which slightly prefer the effective range parametrization (5.14) over the CDD pole parametrization (5.15). In fact for 878 data points the overall fits gave
Fig. 18. Cross-section data from Refs. 60-64 and 21 for various channels. The curves are the result of a constant K-matrix fit.
Fig. 19. $K_2^0$ data from Ref. 64-65. The curves are the result of the fit to a constant $K$ matrix.
Table XV. Results of different fits to data with $P_K < 320$ MeV/c. The chi-squared and the values of the parameters are shown. $E_R$ is the position of the pole in the reduced K matrix. The value in square brackets has been kept constant.

<table>
<thead>
<tr>
<th>Fit</th>
<th>$K^0_{11}$ (fm)</th>
<th>$K^0_{12}$ (fm)</th>
<th>$K^0_{22}$ (fm)</th>
<th>$c_{11}$</th>
<th>$c_{22}$</th>
<th>$\chi^2$</th>
<th>$E_R$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a $K^0$</td>
<td>-1.74</td>
<td>-1.32</td>
<td>-0.24</td>
<td></td>
<td></td>
<td>264</td>
<td>1441</td>
</tr>
<tr>
<td>b $K^0 + \frac{c c}{E^*-E}$</td>
<td>-1.62</td>
<td>-1.23</td>
<td>-0.19</td>
<td>-0.013</td>
<td>-0.008</td>
<td>1390</td>
<td>264</td>
</tr>
<tr>
<td>c $\frac{c c}{E^*-E}$</td>
<td></td>
<td></td>
<td></td>
<td>-0.050</td>
<td>-0.030</td>
<td>[1405]</td>
<td>271</td>
</tr>
<tr>
<td>d $\frac{c c}{E^*-E}$</td>
<td></td>
<td></td>
<td></td>
<td>-0.056</td>
<td>-0.033</td>
<td>1397</td>
<td>267</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$M^0_{11}$ (fm)$^2$</th>
<th>$M^0_{12}$ (fm)$^2$</th>
<th>$M^0_{22}$ (fm)$^2$</th>
<th>$r_1$ (fm)</th>
<th>$r_{12}$ (fm)</th>
<th>$r_2$ (fm)</th>
<th>$\chi^2$</th>
<th>$E_R$ (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>$M^0$</td>
<td>0.18</td>
<td>-1.00</td>
<td>1.31</td>
<td></td>
<td></td>
<td>264</td>
<td>1441</td>
</tr>
<tr>
<td>f</td>
<td>$M^0 + \frac{1}{2} R (k^0 - k_0^2)$</td>
<td>0.17</td>
<td>-1.00</td>
<td>1.40</td>
<td>-0.30</td>
<td>-0.52</td>
<td>0.77</td>
<td>262</td>
</tr>
</tbody>
</table>
$\chi^2_E = 1309$ and $\chi^2_{CDD} = 1430$. However, as stated earlier, these fits extend too far away from the region of interest and should not be used to decide on the behavior of the $S01$ partial wave below threshold.

5. **Conclusions**

Dalitz$^{47}$ and more recently Rajasekaran$^{67}$ have suggested that by studying the energy dependence of the $S01$ amplitude in $\bar{K}N$ interactions, one could distinguish among the two possible interpretations for $\Lambda(1405)$: (a) a $\bar{K}N$ bound state due to strong $\bar{K}N$ interactions, or (b) a member of a supermultiplet and therefore a bound state of three heavy quarks.

The analysis, discussed in the previous sections, shows that any type of parametrization suggested fits the data equally well, so there seems to be no particular energy dependence preferred. However, no matter what parametrization is used, the data require a pole in the reduced $K$ matrix approximately at the same location where the $\bar{K}N$ bound-state fit finds it. This seems to me an indication that the bound-state hypothesis is more plausible.

This conclusion agrees with a recent analysis by Dobson and McElhaney$^{68}$; it disagrees with Martin, Martin, and Ross$^{52}$ only in their statement that the CDD pole parametrization gives a bad fit to the data, but agrees with their general conclusion that the data favors the $\bar{K}N$ bound-state interpretation. As for the analysis of Cline et al.$^{53}$ Fig. 20 shows the ratio of $I = 0$ to $I = 1$ cross section for the $\Sigma\pi$ invariant mass. In this approach secondary interactions are not taken into consideration at all, so the sample at a given energy can be badly contaminated. The points on the low curve are ratios of cross sections from threshold up to $P_{K^-} = 445$ MeV/c obtained in $K^-p$ interactions (Ref. 60, 61, 65, and 21). The disagreement is very clear, unless what is plotted here is not the ratio of $I = 0$ to $I = 1$ $\pi\Sigma$ cross sections, which is what the authors say it is. One more comment on Ref. 53. They claim that the two models predict two completely different curves, drawn in Fig. 20, but in view of the results of the analysis of Martin, Martin, and Ross$^{52}$, Dobson and McElhaney$^{68}$, and myself I do not see how this difference can be obtained.
Fig. 20. Plot taken from Cline et al. The lower points have been added: ▼ = data from Refs. 60 and 61, ■ = data from Refs. 60 and 66, ✶ = data from Ref. 21.
Footnotes and References

*Work done under the auspices of the U. S. Atomic Energy Commission.

1. Review of Particle Properties, Particle Data Group, Rev. Mod. Phys. 43, S1(1971).
11. Some of these results have been reported in the PDG compilation of Ref. 1. For details see: T. Lasinski, Parameters for \( \Delta(1236) \) Resonance, Lawrence Berkeley Laboratory, Group A Memo 742 (1971).
12. S. Almehed and C. Lovelace. New \( \pi N \) Phase Shift Analysis, Lund University Preprint (1971), to be published in Nuclear Physics B.


36. C. Becchi, E. Eberle, and G. Morpurgo, Phys. Rev. 136, B808 (1964); I thank Professor Morpurgo for drawing my attention to this paper and Ref. 37, which had been forgotten by all the experimentalists fitting data into SU(3) multiplets (see Refs. 27, 38, 39).
42. A discussion on the possible existence of $Z^*$ can be found in Ref. 1.
57. For a concise description of the K and M matrix formalism see: A. Barbaro-Galtieri, Ref. 13.
68. P. N. Dobson, Jr., and R. McElhaney, "Interpretation of the \( \gamma_0^{*}(1405) \) Resonance," University of Hawaii Report UH-511-129-72 (1972); to be published in Phys. Rev.
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