Title

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Recent policy applications of control theory methods in dynamic economic models raise the issue of imposing the transversality condition for determining a unique optimal control policy. In a stochastic framework this issue involves alternative methods of estimation, which are discussed here both theoretically and empirically. The economic implications of the alternative methods are analyzed here in some detail through several recent dynamic models in economic growth and exchange rate instability.

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1. Introduction

There is a large body of recent literature in dynamic economics, which deals with the uniqueness of solutions to 'forward looking' dynamic models, more commonly known as *rational expectations* (RE) models. Here the dynamic control-theoretic models are 'forward looking', because the future behavior depends explicitly on the agent's future expectations. The rationality part emphasizes the point that the expectations of the future outcomes should be formed on some rational or consistent basis. A simple example of such a model is given by the scalar difference equation

$$y_t = a\, y_{t-1} + b\, y_t^e + \varepsilon_t$$

Here a and b are constant parameters, $\{\varepsilon_t\}$ is a sequence of independent zero-mean random variables with finite variance and $y_t^e = E_{t-1}(y_{t+1}) = E\{y_{t+1} \mid \Omega_t\}$ is the conditional expectation of the future state y_{t+1} based on some information Ω_t available to the agent at time t. The subscript t-1 used for the expectation term E captures the assumption that this information Ω_t is based on the past values of the relevant state variables.

A second type of forward looking models in recent economic literature arises through the terminal conditions imposed on the optimal control model, which are also known as the transversality conditions. Problems of estimating these terminal conditions are important in economic policy models for several reasons. First of all, the optimal control or decision rules are not unique without these boundary conditions. For the infinite horizon case this implies that there may not be any unique convergence to the steady state and the steady state model which is frequently used in modern growth theory in economics may not then be very meaningful. Secondly, the terminal boundary condition may not be known in the stochastic case. This implies that the so-called 'perfect foresight condition' may not hold, thus rendering the chosen control rule as suboptimal. Finally, the problems of estimation of the boundary conditions play a critical role when there are different phases in the optimal control map.

Our object here is to discuss the various implications of the terminal conditions in linear quadratic control models that are frequently used in recent economic models. To be more specific we would discuss three types of recent control-theoretic economic models and suggest some new methods of estimating the transversality conditions. These economic models deal with the following economic systems: (1) a dynamic adjustment model, (2) a model of optimal growth and (3) a model of exchange rate volatility.

2. Estimating transversality conditions

An optimal control system generally leads to two-point boundary conditions. For a discrete time model the initial and the terminal conditions provide these two sets of boundary points. In the framework of linear quadratic Gaussian (LQG) control the optimal control system, after the optimal control is applied to the output process is given by a set of difference equations. In the scalar case this difference equation, also known as the Euler equation is of second order and we need two boundary conditions. One is provided by the initial value of output at time zero and the other by the terminal value, provided these two values are fixed and known. When the terminal value is free, e.g., infinite horizon case, then fixing its level beforehand may not be feasible. Transversality conditions refer to those conditions by which the free terminal value would lie on the optimal trajectory. These conditions, often stated in terms of the adjoint variables or the Lagrange multipliers associated with the dynamic equations of motion seek to guarantee three things: uniqueness of the optimal trajectory, stability in the sense of convergence of the optimal path and a condition of perfect foresight of the future. In deterministic linear quadratic models of optimal control, these transversality conditions can be imposed fairly easily but difficulties arise as soon as we enter the field of stochastic control, where the stochasticity may enter as disturbances in the difference equations of motion, or as the standard errors in the parameter estimates of the model.

Two most important classes of examples involving the terminal conditions occur in linear dynamic models of economic systems. One is in estimating the optimal Euler equations where the terminal conditions have to be estimated. The other is in dynamic rational expectations models, where there are expectational variables which are unobserved.

We consider the case of estimating the Euler equations first and discuss the various approaches of incorporating the transversality conditions as terminal conditions. We start with the approach of Minford et al. (1979), who in their Liverpool macrodynamic model suggested the imposition of terminal conditions only if these characterize the equilibrium properties of the model. For example in the Liverpool model they obtained such conditions by the analysis of a small linear system on which the complete nonlinear model is based. An alternative method is to set the terminal value by some type of automatic rule. For instance Fisher (1992) has advocated this procedure and considered three types of impositions as follows:

Let $\hat{x}(t)$ denote the solution of the optimal Euler equation in the output space, when a terminal condition is imposed in a scalar model. Then these choices are as follows:

(i) *Fixed value* $\hat{x}(T+1) = d_{T+1}$,

when d_{T+1} is a preassigned value. If $\hat{x}(t)$ is already measured as the deviation from some equilibrium value, then d_{T+1} may be set to zero. It is clear that in case of an estimated model, the choice of this germinal value may lead to an error $e(T) = x(T) - \hat{x}(T)$, where $x(T)$ is the equilibrium population value resulting from the dynamic model;

(ii) *Constant level* $\hat{x}(T+1) = \hat{x}(T)$

This may sometimes run counter to the analytical or equilibrium solution of the optimal Euler equation. Only if the constant level happens to equal the equilibrium solution, that the error $e(T)$ would be zero. Also for nonlinear models this may be more difficult to prescribe;

(iii) *Constant growth* $\hat{x}(T+1) = \gamma\, \hat{x}(T)$

Here γ may be preassigned in terms of the ratio $\hat{x}(T)/\ \hat{x}(T-1)$. In many turnpike models of growth theory such preassigned growth rates were at one time advocated by the national planning models. Here also there is the scope of divergence of $\hat{x}(T)$ from its optimal population value $x(T)$ and the sensitivity of the optimal trajectory due to such errors may need to be analyzed by simulation experiments.

Recently Fisher (1992) has considered the empirical application of three large-scale econometric models, e.g., the LPL (Liverpool) model, the LBS (London Business School) model and the NIESR (National Institute of Economic and Social Research) model and the implications of specifying the alternative terminal conditions. His experiences with the simulation experiments for the above models reached several broad conclusions. First of all, the imposition of a fixed terminal value works well for stable linear dynamic systems where the characteristic roots are all less than unity in absolute value. However if one is not sure whether the controlled model is stable or not, then sensitivity tests must be used not only to validate the choice of terminal condition in a specific form but also to establish the stability of the model. This recommendation is most important for nonlinear systems for which we cannot obtain the characteristic roots for testing their stability. Secondly, for models with unstable roots near unity, the effectiveness of the terminal condition is considerably improved if the long-run solution is known a priori, since these are found to provide better terminal value than those constructed from the solution itself. Finally, the three models exhibit different patterns of sensitivity of the optimal trajectory in respect of the fixed terminal condition. For example for the LPL model several solutions over a short period are compared with a longer run solution, in which the equilibrium conditions were applied. The terminal condition applied to the shorter period forecast did not perform well, displaying a substantial divergence for up to 6 years. And the terminal conditions based on constant growth or constant levels do much better. The LBS model exhibits a seasonal pattern for the sensitivity of the optimal trajectory, i.e., the trajectory depends on which quarter of the year the simulation considers at the terminal period. The NIESR model exhibits a seasonal pattern of sensitivity as in the LBS model but what is most important is that it reflects the pervasive influence of the exchange rate equation coupled with a unit root. Because the unit root allows any choice of terminal value, no specific form of the terminal condition is attempted here. We need here sufficient exogenous information to tie down the terminal value.

The rational expectations model postulates a set of expectational variables which contain the agent's expectations of future outcomes. For example consider the linear dynamic RE model

$$y_t = A\, y_{t-1} + B\, u_t + D\, y_t^e + \varepsilon_t \tag{1}$$

where y_t^e is a vector of rational expectations of endogenous variables, e.g., expectations about the future period t + 1 formed in the current period and ε_t is the random shock. Since y_t^e is not observable, one has to hypothesize as to how it can be related to the observed y_t. Two approaches are possible. One is the backward looking view, e.g., $y_t^e = E(y_t | \Omega_{t-1})$ where Ω_{t-1} is the information set available up to period t - 1. In this case y_t^e can be expressed as a linear combination of lagged values of y_t, e.g., $y_t^e = \alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i}$. The second case is the forward looking view, e.g., $y_t^e = E(y_{t+1} | \Omega_t)$. In this case y_t^e would be a linear combination of future state variables $y_{t+1}, y_{t+2}, \ldots, y_{t+p}$.

The most important implication of dynamic RE models arises in the forward looking case. A simple scalar example of such a forward looking model is

$$y_t = a\, y_{t-1} + b\, u_t + d\, y_{t+1} + \varepsilon_t \tag{2}$$

The effect of the coefficient d in (2) is very critical in the long run as the terminal time $T \to \infty$. If it makes the system unstable in the sense that at least one characteristic root exceeds one in absolute value, then imposing a fixed terminal value may not be consistent with the optimal trajectory. On the other hand if the coefficient d preserves the stability of the system, then a fixed terminal condition can be added without much difficulty, see e.g., Holly and Hallett (1989).

To illustrate some of the basic issues posed by the terminal conditions we consider now two economic examples, one from the dynamic adjustment models discussed in terms of LQG models and the second from the neoclassical theory of optimal growth.

3. A dynamic model of adjustment

Consider a scalar control model where the objective function already incorporates the control variable

$$\text{Minimize } J = E_t\left[\sum_{t=0}^{\infty} r^t\left\{\alpha\left(y_t - y_t^0\right)^2 + \left(y_t - y_{t-1}\right)^2\right\}\right] \tag{3}$$

where y_t is the output (state) variable, E_t is expectation as of time t, r > 1 is the exogenous rate of discount and y_t^0 is the desired target level. The first component of the loss function above is the disequilibrium cost due to deviations from desired target and the second component characterizes the agent's aversion to output fluctuations.

The first order condition for this optimization problem yields the Euler equation

$$\Delta y_t = r\, E_t(\Delta y_{t+1}) - \alpha(y_t - y_t^0) \tag{4}$$

where $\Delta y_t = y_t - y_{t-1}$ and $E_t(\ )$ is the conditional expectation with respect to information available up to t, i.e., Ω_t. The RE hypothesis involves the expectational variable $E_t(\Delta y_{t+1})$. Several types of formulations are possible. For example the backward looking view would replace this by a term involving a linear combination of lagged values of $y_t, y_{t-1}, \ldots, y_{t-p}$. But a forward looking view would consider future values of y_t. But since the future values are not known as of time t, a compromise is needed for all practical situations. A common assumption frequently followed, e.g., see Kennan (1979) is to consider

$$\Delta y_{t+1} = y_{t+1} - y_t + \zeta_{t+1} \tag{5}$$

where ζ_{t+1} is the error term assumed to be white noise with zero means, in which case $E_t(\Delta y_{t+1}) = y_{t+1} - y_t$. On using this assumption in (4) one can solve the second order difference equation

$$r\, y_{t+1} - (1 + r + \alpha)\, y_t + y_{t-1} = -\alpha\, y_t^0 \tag{6}$$

in terms of the characteristic equation of the homogenous part, i..e.,

$$r\,\lambda^2 - (1 + r + \alpha)\,\lambda + 1 = 0 \tag{7}$$

If the desired target level y_t^0 is a constant y^0 and λ_1, λ_2 are the two characteristic roots of (7) assumed to be distinct, the solution can be written very simply as

$$y_t = A_1 \lambda_1^t + A_2 \lambda_2^t + y^0 \tag{8}$$

where for a finite terminal time T we have

$$A_1 = \left(\lambda_2^T - \lambda_1^T\right)^{-1}\left[\lambda_2^T(y_0 - y^0) - (y_T - y^0)\right]$$

$$A_2 = \left(\lambda_2^T - \lambda_1^T\right)^{-1}\left[(y_T - y^0) - \lambda_1^T(y_0 - y^0)\right]$$

y_0, y_T = initial and terminal values of y_t

In case the desired levels y_t^0 are time varying, and the time horizon T is infinite we have to make sure that the two roots of (7) are stable. If both roots are less than one in absolute value, then the optimal solution defines a stable system and the estimates of the roots from the difference equation (6) parameters can be utilized. However in this case the two roots are

$$\lambda_1,\lambda_2 = (2r)^{-1}[(1 + r + \alpha) \pm \{(1 + r + \alpha)^2 - 4r\}]$$

where the product of the roots is 1/r. Hence the roots are positive with one root greater than one and the second less than one. Let λ_1 be the stable root (i.e., less than one). In this case the unstable root is greater than one and it would make the optimal trajectory explosive, i.e., unbounded as $t \to \infty$. Hence one has to choose the stable root in order to assure stability of the optimal path. Thus on using the stable root the second order Euler equation is transformed to a first order as follows:

$$y_t = \lambda_1 y_{t-1} + d$$

where $d = (1 - \lambda_1)(1 - r\lambda_1)\, E_t \left[\sum_{t=0}^{\infty} (r\lambda_1)^t y_t^0 \right]$. Note that the main effect of the transversality conditions is to reduce the second order Euler equation to the first order and thus maintain stability, i.e., for example if d is a constant $\bar{d}$ in the limit, then the steady state equation

$$y_t = \lambda_1 t_{t-1} + \bar{d} \tag{9}$$

converges to the unique steady state solution

$$y_t = \left(y_0 - \frac{\bar{d}}{1-\lambda_1} \right) \lambda_1^T + \frac{\bar{d}}{1-\lambda_1} \to \frac{\bar{d}}{1-\lambda_1} \text{ as } t \to \infty$$

Two general methods of estimation are possible here due to the expectational variable $E_t(\ )$. One is to replace Δy_{t+1} by the stochastic term $(y_{t+1} - y_t + \zeta_{t+1})$ as in (5) where $E_t\, \zeta_{t+1} = 0$ for all t and then rewrite (4) as

$$\Delta y_t = r\Delta y_{t+1} - \alpha(y_t - bz_t) + v_t \quad (10)$$

where the new error term ν_t is $(r\zeta_{t+1} + \alpha e_t)$ and it is assumed that $y_t^0 = bz_t + e_t$. Here the stochastic target variable y_t^0 is assumed for simplicity to be linearly related to an observed exogenous variable z_t and e_t is assumed to be a white noise error component, independent of ζ_t thus implying $E_{t-1}(\nu_t) = 0$. Hence one could apply the method of instrumental variables to estimate the coefficients of (9) consistently, i.e., by ordinary least squares when the errors are Gaussian. Kennan (1979) has used several exogenous variables including time in place of only one, i.e., z_t in (10).

A second method of estimation employs an assumption about how z_t is generated and then derives an estimating equation. For example assume that z_t follows a first order process:

$$(1\text{-}L)\, z_t = \varphi_t,\ E_{t-1}(\varphi_t) = 0 \quad (11)$$

where L is the lag operator, i.e., $Lz_t = z_{t-1}$ and φ_t is a white noise error term. This yields the estimating equation

$$y_t = \lambda_1\, y_{t+1} + (1 - \lambda_1)\, bz_t + (1 - r\lambda_1)(1 - \lambda_1)\, e_t$$

In case z_t follows a second order process, i.e., $(1 - L)^2\, z_t = \varphi_t$, $E_{t-1}(\varphi_t) = 0$

then the estimation equation becomes

$$\Delta y_t = (\lambda_1 - 1)(y_{t+1} - bz_t) + (1 - r\lambda_1)(1 - \lambda_1)\, b\Delta z_t + (1 - r\lambda_1)(1 - \lambda_1)\, e_t$$

and so on for the third order process.

While these two methods of estimation can be employed to any given data set, the simulation studies by Gregory, Pagan and Smith (1993) found the first method to be more robust.

Note that this model (8.4.3) can be easily generalized to the case when y_t and y_t^0 are vectors. Imposing transversality conditions would imply rejecting all the unstable roots and thereby reduce the order of the system of Euler equations containing only the stable roots.

4. A model of optimal economic growth

We now consider a neoclassical model of optimal growth in a finite horizon case

$$\max_{c_t} J = \sum_{t=0}^{T} r^{-t} U(c_t)$$

$$\text{s.t.} \qquad c_t = f(k_t) + k_t - (1 + n)\, k_{t+1}$$

where c_t is consumption (i.e., the control variable) in capita terms, k_t is capital stock per capita and growth of labor $(L_{t+1} - L_t)/L_t = n$ is constant. The production function $f(k_t)$ is nonlinear and concave satisfying the neoclassical assumption of constant returns to scale. The two boundary conditions are $k(0) = k_0$, $K(T) = k_T$.

On assuming a logarithm form for the concave utility function

$$U(c_t) = \ln(c_t - c_0)$$

where c_0 is a given level of initial consumption, the Euler equation for the optimal trajectory can be easily derived as

$$(c_t - c_0)^{-1} [1 + f_k(k_t)] - (1 + n)\, r(c_{t-1} - c_0)^{-1} = 0 \tag{13}$$

where $f_k(k_t) = \partial f(k_t)/\partial k_t$ is the marginal productivity of capital. The steady state solution or the turnpike exists when we have solutions for the nonlinear equation

$$1 + f_k(\bar{k}) = (1 + n)\, r \tag{14}$$

where $\bar{k}$ is the steady state value of k. Note that if this steady state equation (14) has more than one solution, then even if the dynamic system (13) is convergent, it may not converge to a unique steady state, since there are multiple steady states.

Two types of approaches are possible in such a situation. One is to adopt the proposal of Minford and Peel (1983) to accept the criterion of the 'most stable path' (i.e., the roots with smallest modulus for a linearized version of the nonlinear path) and which is free of extraneous variables. However for nonlinear optimal trajectories there may be difficulties when there may be more than one way to linearize. A second approach is to prescribe the terminal conditions exogenously by some outside agencies. A simulation approach which evaluates the sensitivity of the optimal trajectory is also adopted by many applied practitioners. Minford and Peel have discussed some closely related methods other than those discussed here in case of general dynamic models of rational expectations, e.g., the methods of undetermined coefficients by Muth and Lucas, Sargent method of forward substitution and the method of equilibrium solution. Recently Hall and Henry (1988) examined the problems of specifying the terminal conditions empirically in terms of the nonlinear econometric model prepared by the UK National Institute of Economic and Social Research (NIESR) over quarterly data for the period 1983-87. They studied the imposition of the following four terminal conditions:

(i) setting the terminal conditions to exogenously determined levels, i.e., the exogeneity rule,

(ii) setting the terminal conditions so that they lie on the previous year's growth path, i.e., the output level rule,

(iii) setting the terminal conditions so that the levels of the state or output variables are projected flat, i.e., the sensitivity rule and

(iv) setting the terminal conditions so as to project a constant rate of growth from the final quarter of the solution period, i.e., the constant growth rule.

Their empirical experience is that the early parts of the computation run are not substantially affected by the form of the end point terminal conditions as above. However the picture becomes altered radically when a forward looking exchange rate equation is added to the NIESR model. In this case it proved almost impossible to solve the model except with fixed terminal conditions, i.e., the exogeneity rule above.

We may propose a somewhat new approach to estimate the terminal condition in the simple case where the Euler equation is linear with constant coefficients. For example consider the growth model (12) with a linear production function $f(k_t) = \beta k_t$ and assume for simplicity that c_t is measured as a deviation from c_0. Then the homogenous part of the Euler equation would appear as

$$(1+n)\,k_{t+1} + \frac{(1+\beta)(1+r)}{r}\,k_t - \frac{(1+\beta)^2}{r(1+n)}\,k_{t-1} = 0 \tag{15}$$

The two roots are

$$\lambda_1,\lambda_2 = \left[-\frac{(1+\beta)(1+r)}{r} \pm \frac{1+\beta}{r}\sqrt{(1+r)^2+4r}\right]\left[2(1+n)\right]^{-1} \tag{16}$$

For a finite terminal time T+1 the optimal solution is

$$k_t = A_1\,\lambda_1^t + A_2\lambda_2^t \tag{17}$$

where $A_1 = \left(\lambda_2^{T+1} - \lambda_1^{T+1}\right)^{-1}\left[k_0\lambda_2^{T+1} - k_{T+1}\right]$

$$A_2 = \left(\lambda_2^{T+1} - \lambda_1^{T+1}\right)^{-1}\left[k_{T+1} - k_0\lambda_1^{T+1}\right]$$

Assuming a known value for the discount rate r and n we estimate the parameter β from (15) and using this estimate compute the two characteristic roots λ_1, λ_2. Finally we estimate the two parameters k_0 and k_{T+1} from (17). If k_0 is assumed known, then one could apply ordinary least squares to estimate the parameter k_{T+1} from the following

$$k_t = \gamma_{0t} + k_{T+1}\,\gamma_{1t}$$

where γ_0 and γ_1 are known or estimated time series as follows

$$\gamma_{0t} = \left(\lambda_2^{T+1} - \lambda_1^{T+1}\right)^{-1} k_0 \left(\lambda_1^t - \lambda_2^t\right)$$

$$\gamma_{1t} = \left(\lambda_2^{T+1} - \lambda_1^{T+1}\right)^{-1} \left(\lambda_2^t - \lambda_1^t\right)$$

For the infinite horizon case however we have to reject the unstable root and thereby reduce the second order Euler equation to first order.

Clearly the problems of efficient estimation of the transversality condition or its impact on the Euler equation provides an open field of research for the applied econometrician. There is a considerable scope of Monte Carlo studies and simulation here.

5. A model of exchange rate volatility

Modelling the foreign exchange market has posed some challenges for financial researchers, because of its volatility in recent years. This market is by far the largest financial market in the world with a daily turnover exceeding $1000 billion in US dollars in 1995. The role of central banks and hence the government is not a passive one is these international exchange markets. One type of modelling this market is to assume an adjustment process initiated by government's policy aspiration, where an orderly market is considered desirable. This type of approach has been followed by Hall and Henry (1988) and more recently by Fisher (1992) in connection with the NIESR (National Institute of Economic and Social Research) econometric model for the UK over the period 1973 to 1984 and later.

Consider a simple adjustment cost function $C(E_t)$ representing government aspirations

$$C(E_t) = \sum_{t=0}^{T} (1/2)[a_1(E_t - E_t^0)^2 + a_2(E_t - E_{t-1})^2] \tag{17}$$

where E_t is the real exchange rate and E_t^0 is the rate which the market would set in the absence of government intervention. We further suppose an inverse intervention function

$$E_t = g(I_t)$$

representing government intervention either through direct intervention or through official financing flows on the foreign exchange market. If we minimise this function (17) with respect to the control variable I_t we obtain the first-order condition

$$a_1 g'(g(I_t) - E_t^0) + a_2 g'(g(I_t) - g(I_{t-1})) - a_2 g'(g(I_{t+1}) - g(I_t)) = 0$$

$$t=0,1,\ldots,T-1$$

where g' denotes the partial derivative of g with respect to I_t. On dividing through by g' this yields the optimising condition

$$E_t = (a_1 + 2a_2)^{-1} \{a_1 E_t^0 + a_2(E_{t-1} + E_{t+1})\} . \tag{18}$$

To estimate this model one needs to specify the determinants of E_t^0. A common hypothesis frequently used in the finance literature is to assume

$$E_t^0 = b_0 + b_1 E_{t-1} \tag{19}$$

This finally yields the estimable equation

$$E_t = \beta_0 + \beta_1 E_{t+1} + \beta_2 E_{t-1} \tag{20}$$

where $\beta_0 = (a_1 + 2a_2)^{-1} (b_0 a_1)$

$\beta_1 = (a_1 + 2a_2)^{-1} b_1(a_1 + a_2)$

$\beta_2 = (a_1 + 2a_2)^{-1} a_2$

So long as the coefficients β_0,β_1,β_2 are constants, this equation may be transformed to a causal form, suitable for ordinary least squares (OLS) estimation as follows:

$$E_t = \gamma_0 + \gamma_1 E_{t-1} - \gamma_2 E_{t-2} \tag{21}$$

where $\gamma_0 = -\beta_0/\beta_1$, $\gamma_1 = 1/\beta_1$, $\gamma_2 = \beta_2/\beta_1$

This is a backward looking view, since the current level of exchange rate is determined by the lagged exchange rates. By comparison the equation (20) provides a forward looking view of exchange rate determination. Note that by adding an error term on the right hand side of (21), this equation may be estimated by the OLS method and if the error term is a white noise process, the estimated coefficients $\gamma = (\gamma_0, \gamma_1, \gamma_2)$ would have optimum properties of unbiasedness and minimum variance.

The characteristic equation for the second order process (21) is

$$\lambda^2 - \gamma_1 \lambda + \gamma_2 = 0 \tag{22}$$

If the natural exchange rate E_t^0 is assumed to be a constant in the short run (i.e., $b_1 = 0$), then the exchange rate equation (21) becomes simpler

$$E_t = -\alpha_0/\alpha_1 + (1/\alpha_1)\, E_{t-1} - E_{t-2} \tag{23}$$

The characteristic equation now takes the form

$$\lambda^2 - (1/\alpha_1)\, \lambda + 1 = 0 \tag{24}$$

Since the constant term here is unity, the two characteristic roots are reciprocal. Hence if one is positive but less than one, the other must be positive and greater than one. The latter root may contribute an explosive component to the time series of exchange rates in the sense that as $T \rightarrow \infty$, E_T may tend to be unbounded. It is clear that a similar phenomenon may occur with the second order process (22) also, if the coefficient γ_2 is positive and one of the two roots exceeds unity.

A second way of modelling exchange rate volatility is to postulate a model of time-varying variances of exchange rates. The random shocks to the mean exchange rates, which may cause volatility are represented here by what is known as the Arch (autoregressive conditional heteroscedastic) model and its various generalizations in the modern financial literature. The simplest version of the Arch model uses the following specification for the conditional variance

$$y_t = E_{t-1}(y_t) + e_t;\ \sigma_t^2 = E_{t-1}(e_t^2) \tag{25}$$

$$\sigma_t^2 = w + \alpha\, e_{t-1}^2 + \beta\sigma_{t-1}^2$$

where y_t is the real exchange rate and $E_{t-1}(y_t)$ is its conditional mean as of time t-1. The variance term σ_t^2 is conditional on information up to time point t-1, i.e., $E_{t-1}(e_t^2)$ is the conditional expectation of the error term e_t^2. Note that the random shock to mean exchange rate is e_t and the shock to variance is $[E_t(e_t^2) - E_{t-1}(e_t^2)] = \xi_t$, where ξ_t is a random term. A more general model for variance is

$$\sigma_t^2 = \theta(B)\ \sigma_t^2 + w$$

where $\theta(B)$ is a lag polynomial in B, the backshift operator with $\theta(0) = 0$. Clearly this conditional variance model (25) can be estimated by the maximum likelihood method (or by least squares if ξ_t is approximately normally distributed), since an estimated series $\left\{\hat{\sigma}_t^2\right\}$ can be constructed from the estimated residuals $\hat{e}_t$ obtained from (17). If the sum of the estimated coefficients α and β in (25) equals one or more in a statistically significant sense, then the variance shocks are persistent or permanent.

6. Empirical applications

Two empirical applications are considered briefly in this section for illustrative purposes. Other types of applications, e.g., dynamic portfolio models in finance and limit pricing models of market competition are discussed elsewhere by Sengupta and Fanchon (1996).

The first application is in explaining the pattern of input and output growth in Japan over the period 1965-1990. The second is in explaining the role of past trends and future expectations in foreign exchange market instability. Consider first the economic growth process in Japan. Modern growth theory seeks to identify the sources of sustained growth in terms of the major inputs such as capital, labor and the externality factor due to exports. Technological innovation is subsumed in the two inputs, labor and capital. We assume the industrial producer to be the dynamic agent, who plans the dynamic expansion path over time. This planning involves a two-step decision process. In the first step the producer decides the optimal inputs by minimizing a steady state cost function, which gives rise to the long-run target value. The second step then postulates an optimal adjustment rule towards the target value. Thus the first step optimization problem is

$$\text{Min } C_t = w_t' X_t \qquad \text{subject to } Y_t = F(X_t) \tag{27}$$

where X_t and w_t are column vectors of m inputs and their prices, prime denotes a transpose and Y_t is the given level of output subject to the production function F( ) which may exhibit increasing returns to scale. Let X_t^* and Y_t^* be the optimal inputs and outputs. The second step of the optimization process then assumes a short-run adjustment behavior for the producer, who finds that his current factor uses are inconsistent with the long run equilibrium path (X_t^*, Y_t^*) as above. The short-run adjustment behavior is modelled as a quadratic loss function, which minimizes the expected present value of a quadratic loss function L as:

$$\underset{X_t}{\text{Min}}\, \varphi = \sum_{t=0}^{\infty} r^t \left[\left(X_t - X_t^*\right) \Lambda \left(X_t - X_t^*\right) + \left(X_t - X_{t-1}\right) \psi \left(X_t - X_{t-1}\right) \right]$$

where the diagonal matrices Λ, ψ have positive weights in the diagonal. The first component of the loss function is the disequilibrium cost due to deviations from the steady state equilibrium, whereas the second component characterizes the producer's aversion to fluctuations in input levels, i.e., the smoothness objective. It is clear that this formulation is very similar to the one considered in (3) through (6). Sengupta and Okamura (1996) have applied this control-theoretic model to analyze the path of convergence of the different inputs to their steady states for Japan over the period 1965-90. Five inputs are distinguished here, e.g., capital, male labor skilled and unskilled, female labor and the exports as the externality factor. Three broad conclusions emerge from this study. First, capital augmented by technological know-how plays a much more dominant role in Japan's growth process. Also the export externality has a favorable impact on the employment of female labor. Second, the future expectations of demand have a much more prominent role than the past trends. Finally, the capital adjustment equation shows a faster speed of convergence to the steady state than labor.

The second example considers the exchange rate equation (20) in an enlarged logarithmic form, where the interest-rate differential is also allowed as an explanatory variable, e.g.,

$$\ln E_t = \beta_0 + \beta_1 \ln E_{t-1} + \beta_2 \ln E_{t+1} + \beta_3(i_d - i_u)_t + \beta_4(i_d - i_u)_{t-1} + \varepsilon_t \qquad (28)$$

where i_d is the domestic real interest rate and i_u is the US and ε_t is the error term. This type of specification has been applied by Hall and Henry (1988) for monthly UK time series data over 1973(3) though 1984(6), where the international interest rate is proxied by the US real rate of interest, which is the monetary interest rate adjusted for the rate of inflation. The real exchange rate is defined as $E_t = S_t P_t^* / P_t$, where S_t is the exchange rate expressed in the currency of the domestic country per unit of currency of the foreign country, which is taken to be the US here and P_t^* and P_t are the consumer price indices of the foreign and domestic country. Based on the monthly IMF data for two periods: February 1988 to January 1991 and February 1991 to August 1995 the estimated results for the four countries: France (FR), Japan (JP), United Kingdom (UK) and Germany (G) are as follows:

$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$	$\hat{\beta}_4$	$\bar{R}^2$	n

FR	88-91	0.003	0.517**	0.483**	-0.003	0.00	0.91	34
	91-95	-0.007	0.439**	0.567**	0.001	-0.002	0.84	53
JP	88-91	-0.104	0.535**	0.485**	-0.012	0.010	0.91	34
	91-95	-0.166	0.557**	0.476**	0.003	-0.005	0.98	53
UK	88-91	-0.029	0.457**	0.498**	0.006	-0.005	0.88	34
	91-95	-0.016	0.476**	-0.492	-0.009	0.005	0.93	53
G	88-91	-0.010	0.504**	0.511**	-0.001	0.001	0.89	34
	91-95	-0.023	0.463**	0.577**	0.008	-0.009	0.91	53

Here one and two asterisks denote significant t-statistics at 5 and 1% levels respectively and $\bar{R}^2$ denotes squared multiple correlation coefficient adjusted for degrees of freedom. In order to check whether the sum of the parameters $\hat{\beta}_1$ and $\hat{\beta}_2$ equals one, we ran the same equation (28) with the restriction $\hat{\beta}_1 + \hat{\beta}_2 = 1.0$ included and computed an F-statistic, which is a function of the difference between the sum of squared residuals of the equation without and with the restriction. At the 5% level in order to reject the hypothesis that the sum of $\hat{\beta}_1$ and $\hat{\beta}_2$ is different from one, the p value must be less than 0.05. In all four equations for the four countries and for all the sample, the null hypothesis was not rejected. Thus the unit root cannot be rejected, implying a random walk process. This has been discussed in some detail by Sengupta and Sfeir (1996) elsewhere. Furthermore it is clear that $\hat{\beta}_2$ is not always higher than $\hat{\beta}_1$, i.e., future expectations do not always have a stronger effect than the past trend. For example, the case of Germany exhibits the stronger effect of future expectations along with France for the 91-95 period but the cases of UK and Japan are in the reverse direction. Finally, the interest rate differential has a negligible impact in all the cases examined here.

Note that in this case the characteristic equation (22) can be solved for two roots. For example in case of UK (88-91) the two roots are $\lambda_1 = 0.762$, $\lambda_2 = 1.425$. Hence the solution of the homogeneous system for the two constants of integration A_1, A_2 may be written as

$$\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{bmatrix} 1 & 1 \\ 0.762^T & 1.425^T \end{bmatrix}^{-1} \begin{pmatrix} \ln E_0 \\ \ln E_T \end{pmatrix}$$

However if $T \rightarrow \infty$ then one has to impose the transversality condition $A_2 = 0$, which implies a first order difference equation $\ln E_t = \tilde{\beta}_0 + \tilde{\beta}_1 \ln E_{t-1}$. Clearly the OLS estimate of $\tilde{\beta}_1$ need not equal $\hat{\beta}_1 = 0.762$ for example. This shows that the imposed terminal condition need not be satisfied by the estimated data.

7. Conclusion

Recent control theory models of dynamic economic systems which involve intertemporal optimiztion have to impose one or more terminal conditions for deriving a unique optimal policy rule. However this raises the new issue of estimating the impact of these terminal conditions. Three types of economic models are discussed here in order to illustrate various aspects of this transversality problem.

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