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TOPOLOGICAL CLASSIFICATION OF DUALITY DIAGRAMS

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ABSTRACT

It is pointed out that Veneziano's topological expansion contains certain irregular diagrams which require a slight refinement of the original topological classification.

Recently, Veneziano (1,2) proposed a new "topological" expansion for the unitarization of planar dual models which is based on a small parameter $1/N$ and is related to the topology of duality diagrams in such a way that each order of the expansion contains diagrams with the same topological structure. This structure is defined (3) in terms of the minimum number of handles a 2-dimensional surface must have if the diagram is to be embedded in it.

In this note, we wish to point out that there are certain "irregular" diagrams to which Veneziano's original topological classification cannot be applied; diagrams which contribute, for example, to the "cylinder" level of the expansion but can only be embedded in a torus; or diagrams which contribute to the "torus" level but can only be embedded in a double torus, etc. Nevertheless, it turns out that a somewhat refined notion of the topological structure of a diagram makes it possible to classify also these irregular diagrams in such a way that all diagrams contributing to a certain order of the expansion are topologically equivalent.

Duality diagrams can be classified (2) in terms of the number $l$ of loops, the number $b$ of boundaries (lines to which the external legs are attached), and the number $w$ of windows (lines with no external legs attached to them). In a dual model with internal symmetry $SU(N)$, a contribution to the $n$-point function exhibiting $l$ loops and $w$ windows depends on the coupling constant $g$ and on $N$ in the following way

$$A_n = \frac{g^{2n-2l-2}N^{2l-w}}{(g^2 N)^w}.$$  \hspace{1cm} (1)

The factor relevant to the topological expansion is $(g^2)^{l-w}$. Since the planar unitarity equations seem to require $g^2 N$ to be of the order of one (4), the topological expansion becomes an expansion in $1/N$, each graph being suppressed with respect to the corresponding planar graph by a factor $(1/N)^{l-w}$.

Any duality diagram can be embedded in a closed 2-dimensional orientable surface which can be characterized by the number of its handles, known to mathematicians as the genus of the surface (5). Figure 1, for example, shows two duality diagrams; one embedded in a cylinder, the other in a torus. The genus $h$ of a diagram is defined as the minimum genus among all surfaces in which the diagram can be embedded. It is related to the number of loops, boundaries and windows of the diagram by (2).
Because of Eq. (2), which will be derived below, the power of $g^2$ in (1) can be written

$$ (g^2)^{l-w} = (g^2)^{b+2h-1} . \quad (3) $$

Thus each order of the expansion contains diagrams with a fixed number of boundaries and handles but an arbitrary number of internal closed lines. Once $b$ (i.e., the way in which the external legs are attached to the diagrams) has been fixed, the expansion becomes an expansion in $h$; hence the name "topological expansion".

The topological classification of duality diagrams in terms of their genus $h$ is based on Eq. (2). However, there are certain irregular diagrams for which this relation does not hold. These diagrams appear in the unitarity sum when crossed intermediate states occur. The simplest example is shown in Fig. 2. This diagram, which we shall label with the letter $X$, has one loop, no windows and two boundaries, and should therefore contribute to the cylinder ($h = 0$). However, it can only be embedded in a torus (see Fig. 3). The crucial point is that the two twists in the diagram have the same orientation. This makes it impossible to draw it on a cylinder.

We shall now derive Eq. (2) in order to find out why it does not hold for diagrams like the one shown in Fig. 2. In doing so, we shall see that diagram $X$ can still be classified topologically as a cylinder, although it cannot be embedded in a cylinder.

The embedding surface $S$ of a diagram can be characterized by its Euler characteristic

$$ \chi(S) = r + v - e \quad (4) $$

where $r$, $v$, and $e$ refer to the regions, vertices, and edges of any division of the surface. The Euler characteristic of a surface is independent of the particular division that has been chosen and is related to the genus $h$ of the surface by

$$ \chi = 2 - 2h \quad (5) $$

Any orientable surface is uniquely determined by its Euler characteristic or by its genus. Now, the diagram which is embedded in $S$ can be used to divide $S$ into regions. However, care must be taken that every region is encircled by a closed line of edges. Only then does Eq. (4) hold. For example, imagine the diagram shown in Fig. 4 drawn on a sphere. The Euler characteristic of a sphere is 2, but in our example $r + v - e = 4 + 4 - 4 = 4$. The reason for this discrepancy is that the region $r_4$ is not well defined. In order to encircle it by a closed line of edges, we have to draw two additional edges, e.g., as in Fig. 5. Now we have $r + v - e = 2$.

If the regions which are not part of the diagram ($r_1$, $r_2$, $r_3$ in our example) are cut out, one obtains a bordered surface $S_b$ with a number of boundary components equal to the number of regions that have been cut out. The Euler characteristic of such a bordered surface with $k$ boundary components is therefore

$$ \chi(S_b) = \chi(S) - k \quad (6) $$

which, because of Eq. (5), can be written

$$ \chi(S_b) = 2 - 2h - k \quad (7) $$
Any bordered surface is uniquely defined by its Euler characteristic and the number of its boundary components.

The Euler characteristic of a duality diagram with \( l \) loops is easily seen to be

\[ \chi = 1 - l \]  

Combining (7) and (8), we have

\[ l = 2h + k - 1 \]  

and if we divide the boundary components into "boundaries" and "windows", \( k = b + w \), we obtain Eq. (2).

The reason why Eq. (2) does not hold for diagrams like diagram X in Fig. 2 is now apparent. These diagrams cannot be constructed by cutting out well-defined regions from a closed surface. When diagram X is drawn on a torus (see Fig. 6), it divides the surface into two regions, \( r_1 \) and \( r_2 \), but these are not surrounded by closed lines of edges. To turn \( r_1 \) into a properly defined region, one has to add further edges, e.g., in the way shown in Fig. 7. In this way, the region \( r_1 \) becomes completely encircled by one boundary component and can therefore be cut out, whereby a bordered surface with \( l - w = 2 \), \( b = 1 \), and \( h = 1 \) is created.

For irregular diagrams, then, the Euler characteristic of the diagram is not related to its genus in the usual way, i.e., by Eq. (7). Thus diagram X has \( \chi = 0 \), according to Eq. (8), and \( k = 2 \), but has \( h = 1 \). Nevertheless, it can be classified topologically as a cylinder.

The relevant quantities for the topological classification of a bordered surface are its Euler characteristic and the number of its boundary components, but not its genus. Since diagram X has two boundaries and \( \chi = 0 \), it is topologically equivalent to a cylinder, i.e., it is homeomorphic to a bordered surface \( X' \) which can be embedded in a cylinder. The difference between X and \( X' \) is, of course, merely the orientation of one of the twists (see Fig. 8). It is evident that the two diagrams are topologically equivalent, since all points on \( X \) can be mapped onto \( X' \) by a continuous transformation.

Irregular diagrams seem to occur in the unitarity sum only when there are crossed intermediate states. In the case of three intermediate states, for example, crossed states can occur in four different ways (see Figs. 9-12), three of which give rise to irregular diagrams. Note that two of them (Figs. 10 and 11) contain "double twists", i.e., propagators with two consecutive twists of the same orientation. When the diagrams are embedded in closed surfaces, these double twists require the extra handle which invalidates Eq. (2). As far as the topological expansion is concerned, however, the orientation of the twists--and thus all double twists--may be ignored, since they are irrelevant for the topological classification of the diagrams. When this is done in the diagrams shown in Figs. 10, 11, and 12, they are seen to be equivalent to the diagrams shown in Figs. 1a, 1b, and 7, respectively.

In conclusion, we see that Eq. (2) may still be used for the topological classification of duality diagrams. In the case of irregular diagrams, however, the number \( h \), strictly speaking, will not be the genus of the diagram itself, but the genus of another, topologically equivalent, diagram.
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FIGURE CAPTIONS
Fig. 1. Two examples of duality diagrams embedded in closed surfaces; a cylinder (1a) and a torus (1b).
Fig. 2. The irregular diagram X.
Fig. 3. Diagram X embedded in a torus.
Fig. 4. A diagram with an improperly defined region, \( r_4 \).
Fig. 5. The diagram of Fig. 4 with two additional edges.
Fig. 6. Diagram X drawn on a torus, exhibiting improperly defined regions.
Fig. 7. A diagram embedded in a torus with a properly defined region \( r_1 \).
Fig. 8. Diagram X (8a) and diagram X' (8b), embedded in closed surfaces.
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