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HOW ROY'S EQUATIONS RESOLVE THE UP-DOWN AMBIGUITY AND 
REPRODUCE THE $S^*$ RESONANCE

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ABSTRACT
By making simple Regge pole and resonance estimates of the high energy and intermediate energy $\pi\pi$ scattering amplitude and using recent $\pi\pi$ experimental data, we are able to show that both the rapid rise of the $I = 0$ s-wave phase shift near $KK$ threshold and the consequent resolution of the up-down ambiguity are consistent with the rigorous equations of Roy.

I. INTRODUCTION
In a previous paper$^1$ (hereafter referred to as I) we showed that the recent unambiguous Berkeley $\pi\pi$ phase shifts (solution 1 of Ref. 2) were consistent with the crossing and analyticity properties built into Roy's equations$^3$ at least in the energy region from 500-800 MeV. In particular, Roy's equations provided us with a method of continuing these experimentally determined phase shifts down to threshold consistent with crossing and the analyticity of twice-subtracted dispersion relations. In this way we were able to calculate the s- and p-wave scattering lengths for these particular phase shifts. These calculations required us to make some estimate of the intermediate and high energy $\pi\pi$ amplitude, which we did in the simplest possible way. Whilst these crudely estimated terms contribute little in the very low energy region, they become increasingly important if we wish to use Roy's equations much beyond 800 MeV. In this paper we shall make somewhat more reliable estimates of these intermediate and high energy contributions. The purpose of this is twofold.

Experimentally the $\pi^+\pi^-$ total cross section has a very interesting behavior close to $KK$ threshold.$^1$ This cross section falls rapidly within 60 MeV by an amount corresponding to the imaginary part of the s-wave amplitude falling from its unitarity limit to zero. This behavior, which has been confirmed by the more recent CERN-Munich experiment,$^5$ allows the up-down ambiguity for the $I = 0$ s-wave phase shift, $\delta_0^0$, to be resolved in favor of the down solution. It also led to the discovery of a $\pi\pi$ resonance, the $S^*$, near the $KK$ threshold.$^2,4$ Since it is this $S^*$ resonance which is responsible for the rapid rise of $\delta_0^0$ from $\pi/2$ to $\pi$ and which consequently
resolves the up-down ambiguity, it is natural to ask whether this behavior of the I = 0 s-wave amplitude is automatically consistent with crossing and analyticity properties. Making simple Regge pole estimates of the high energy behavior and resonance pole estimates of the intermediate energy amplitude we find that in fact it is.

Experimentally this S* resonance also solves the up-down ambiguity and so the second aim of this paper is to show that knowledge of the same intermediate and high energy contributions which generate this S* from the low energy down solution fails to produce a self-consistent up-type solution. This will be done in Sec. II. In Sec. III we shall discuss how our aims and results differ from those of Basdevant, Froggatt, and Petersen, who have also studied solutions to Roy's equations, and give our conclusions in Sec. IV.

II. ROY'S EQUATIONS APPLIED

II.A. Contributions to Roy's Equations

We begin by recalling the general form of Roy's equations for the partial wave amplitudes. They can be written schematically as, using pion mass units,

\[
\frac{r}{s} = \text{S.T.} + \sum_{I'=0}^{2} \int_{s-\frac{1}{2}}^{s} dt \int_{0}^{\infty} dy \left( \begin{array}{c} \mathcal{I}^{I}_{0} (s,t,y) \text{Im} \mathcal{F}^{I}_{1}(y,t) \\ \mathcal{K}^{I}_{0} (s,t,y) \text{Im} \mathcal{F}^{I}_{1}(y,t) \end{array} \right)
\]

or

\[
\frac{r}{s} = \text{S.T.} + \sum_{I'=0}^{2} \sum_{\ell'=0}^{\infty} \int_{s-\frac{1}{2}}^{s} dt \int_{0}^{\infty} dy \left( \begin{array}{c} \mathcal{G}^{I}_{\ell\ell'} (s,t,y) \text{Im} \mathcal{F}^{I}_{1}(y,t) \\ \mathcal{G}^{I}_{\ell\ell'} (s,t,y) \text{Im} \mathcal{F}^{I}_{1}(y,t) \end{array} \right)
\]  

(2.1)

(2.2)

where S.T. are the subtraction terms which are present only in the s and p waves, and depend on two constants which we take to be the I = 0 s-wave scattering length, \( a_0 \), and the combination of s-wave scattering lengths, \( 2a_0 - 5a_0^2 \). For different purposes either Eq. (2.1) or Eq. (2.2) may be more useful. The functions \( j^{I}_{\ell} \), \( k^{I}_{\ell} \), and \( g^{I}_{\ell\ell'} \) can be found explicitly in Ref. 3.

As in I we divide the y-integration of Eqs. (2.1, 2.2) into four distinct parts, which we refer to as high energy (HE), intermediate energy-resonance (IER), intermediate energy-background (IEB), and low energy (LE) terms. We now discuss how we calculate the contributions to Eqs. (2.1, 2.2) in each of these regions:
We assume that Regge asymptotics is valid for values of 
\( y \geq \frac{1}{3}(a_r^2 + m_g^2) = 1.15 \). We take the Regge exchange amplitudes with 
t-channel isospin equal to 0 or 1, in Eq. (2.1), to be parametrized by

\[
\text{Im} F^0(y,t) = \beta_p e^{bt/2} y + \beta_f \sin \gamma_p(t) \Gamma(1 - \alpha_p(t)) \left( \frac{y}{y_0} \right)^{\alpha_p(t)}
\]

(2.3)

\[
\text{Im} F^1(y,t) = \beta_p \sin \gamma_p(t) \Gamma(1 - \alpha_p(t)) \left( \frac{y}{y_1} \right)^{\alpha_p(t)}
\]

(2.4)

where

\[
\beta_p = \frac{3}{32\pi} \sigma_{\text{tot}}^O(t)
\]

(2.5)

and the \( f_0 \) and \( p \) Regge residues are determined by extrapolation 
to the respective resonance poles giving

\[
\beta_f = \frac{15 m_r^2 \Gamma_f \alpha_f(t) Y_0^2}{(m_r^2 - 4)^{5/2}}
\]

(2.6)

\[
\beta_p = \frac{3 m_p^2 \Gamma_p \alpha_p(t) Y_1}{(m_p^2 - 4)^{5/2}}
\]

(2.7)

The parameters \( y_0, y_1 \) take into account the variation of the residues 
from \( t = m_r^2 \) or \( m_p^2 \) to \( t < h \). In the Lovelace-Veneziano model
\( y_0 = y_1 = 1/\alpha' = 1/\alpha_p(t) \approx 60 m_\pi^2 \). However, as emphasized, by Schmid,\(^7\) 
these Regge residues at \( t = 0 \) may in fact be up to a factor of 2 smaller than what such a simple extrapolation from the resonance pole 
to \( t = 0 \) gives with \( y_0 = y_1 \approx 60 m_\pi^2 \). Because of this we choose

\[
y_0 = (35 \pm 15) m_\pi^2, \quad y_1 = (35 \pm 15) m_\pi^2.
\]

(2.8)

Numerically we take the other parameters in Eqs. (2.6, 2.7) from 
Ref. 8 (except the value of \( \Gamma_p \), which we take from the data of Ref. 2 
for consistency):

\[
\begin{align*}
m_r &= 1269 \pm 10 \text{ MeV}, \quad m_p = 765 \pm 10 \text{ MeV} \\
\Gamma_f &= 150 \pm 25 \text{ MeV}, \quad \Gamma_p = 150 \pm 20 \text{ MeV} \\
\alpha_f(t) &= 0.6 + t/60, \quad \alpha_p(t) = 0.5 + (t - 4)/53
\end{align*}
\]

(2.9)

(pion mass units being used except where stated).

As a guide to the values of both \( \sigma_{\text{tot}}^O(t) \) and \( b \), the slope of the 
forward \( \pi \pi \) diffraction peak, we use factorization. However, we 
treat these values, obtained from \( \pi N \) and \( NN \) scattering, with 
some scepticism and assign large uncertainties to these \( \pi \pi \) parameters, 
i.e., we take

\[
\sigma_{\text{tot}}^O(t) = (20 \pm 10) \text{ mb}, \quad b = (8 \pm 3) \text{(GeV/c)}^{-2}.
\]

(2.10)

In this region with \( y \in \{4, 115\} \) in Eq. (2.2) we have the 
\( f_0 \)-resonance in the \( I = 0 \) d-wave and the tail of the g-resonance in 
the \( I = 1 \) f-wave. We parametrize these partial wave amplitudes by 
the following modified Breit-Wigner formula
where

\[ r^I_E(y) = \left( \frac{m^2_g}{y - \eta} \right)^{1/2} \frac{m_x r^I_g(y)}{m^2_g - y - 2m^2_x r^I_g(y)} \]  

(2.11)

and \( x = x^I_E \) with \( \Gamma^I \) the total width. The parameters \( m^I_g, \Gamma^I \) have been given above, Eq. (2.9), and we take \( x^I_g = 1 \). For the g-resonance we have

\[ m^I_g = 1680 \pm 30 \text{ MeV}, \quad \Gamma^I_g = 160 \pm 30 \text{ MeV}, \quad x^I_g = 0.4 \pm 0.05. \]  

(2.13)

We shall also consider the possible existence of an s-wave resonance in the \( s' \)-region, the \( e^I \). This will contribute in the region \( 115 \leq y \leq Y \), where \( Y \) is the cut-off for the so-called \( \text{LE} \) contributions to be discussed below. To avoid possible inaccuracies in parametrizing this resonance by a Breit-Wigner form, we shall use the explicit data on \( s^0_1(y) \) and \( \eta^0_1(y) \) for

\[ 1020 \text{ MeV} < |y|^3 < 1500 \text{ MeV} \]  

obtained by Grayer et al. We remark here that for \( s < 60 \) the contribution to the s and p waves of the tail of the g-resonance [calculated from Eq. (2.2) with \( m^2_g = 146 \)] is very small and has little effect. Because of this we shall neglect the contribution of the tail of the highly inelastic \( \rho' \) resonance completely.

\[ I_E: \text{This is the contribution of the intermediate energy background amplitude which we estimate by appealing to the Freund-Harari conjecture. We shall assume the background in the region } y \in (Y, 115) \text{ to be given by the Pomeron exchange amplitude [i.e., by the first term of Eq. (2.3)]}. \]  

When we use a specific set of data for the \( I = 0 \) s-wave in this region we subtract out the \( I = 0 \) s-wave component of the background to avoid "double counting."

\[ \text{LE: In this region, } \frac{1}{4} < y < Y, \text{ we use in Eq. (2.2) the } I = 0 \text{ s-wave and } I = 1 \text{ p-wave data of Protopopescu et al.} \]  

and the \( I = 2 \) s-wave data of Baton et al. together with their continuations down to threshold as discussed in I. As in I we firstly set \( Y = 55 \), however, we also considered a number of modifications, such as using different forms for the p-wave amplitude in the region \( y \in (55, 65) \). These modifications make insignificant differences for the computed \( I = 0,1 \) s- and p-wave amplitudes with \( |s|^3 < 1 \text{ GeV} \) and will not be discussed further.

II.B. Results for the Down Solution

We denote the sum of \( \text{HE}, \text{IER}, \text{IEB} \) contributions to Roy's equations for \( f^I_E(s) \) by \( g^I_E(s) \) and illustrate \( g^0_0(s) \) and \( g^1_1(s) \) in Fig. 1. The values shown are to be compared with the scale of the total \( \text{Re } f^I_E(s) \) to which they contribute. The scale is \( \frac{1}{2} \left( \frac{s}{s - \eta} \right)^{1/2} \) as implied by unitarity. To give an indication of the relative importance of the various contributions we note that the ratio of \( \text{IER:IEB:HE} \) terms for the s-wave is 11:3:2 at \( s = 30 \) and 20:6:2 at \( s = 50 \), whilst for the p-wave the ratio is -1:4:3 at \( s = 30 \) and -1:15:5 at \( s = 50 \).
The values of $g^0_0(s)$ and $g^1_0(s)$ calculated in detail here agree extremely well with those cruder estimates given by Eqs. (3.7, 3.9) of I, for $s < 30$. Because of this we take the subtraction constants of Eqs. (2.1), (2.2) to be those calculated in I, viz

$$a^0_0 = 0.15 \pm 0.07, \quad 2a^0_0 - 5a^2_0 = 0.57 \pm 0.04 \quad (2.14)$$

For $s > 30$ the HE, IER, IEB contributions are of course different than our earlier estimates.

The results of calculating $\text{Re} f^0_0(s)$ and $\text{Re} f^1_1(s)$ from the sum of the four LE, IER, IEB, HE contributions and the subtraction terms defined above, by using Eqs. (2.1, 2.2), are shown in Figs. 2, 3 together with the data of Ref. 2. In these figures we actually plot the real parts of $\hat{f}^0_0(s)$ and $\hat{f}^1_1(s)$, where

$$\hat{f}^I_0(s) = \left(\frac{s - i}{s}\right)^{1/2} \hat{f}^I_1(s).$$

This is because unitarity is simply expressed in terms of $\text{Re} \hat{f}^I_0(s)$ without kinematic factors as $|\text{Re} \hat{f}^I_0(s)| \leq 1/2$ and so is easy to see in such figures. No attempt has been made to fit any of the parameters given in Eqs. (2.8, 2.9, 2.10, 2.13). Our aim is to see how simple Regge pole plus resonance estimates agree with the data, and as can be seen from Figs. 2, 3, both the computed $f^0_0(s)$ and $f^1_1(s)$ agree very well. The $s$-wave above 1 GeV is very important for this agreement. For example, if we replace the data of Grayer et al. by the $e^+ e^-$-in the region from 1050 MeV to 1500 MeV by just the background given by the pomeron amplitude, discussed in Sec. II.A. (i.e., $\xi^0 \sim 30^0$) we find that the computed real part of $\hat{f}^0_0(s)$ is that shown in Fig. 4.

II.C. The Up Solution

We next turn to a discussion of the "up" solution. We have replaced the imaginary parts of $f^0_0(y)$ in the energy region from 720 to 950 MeV by the "up" solution obtained from the phase shifts of Ref. 2 by the usual transformation

$$\xi^0_0(\text{up}) = \xi^1_1 - \xi^0_0(\text{down}) + \pi/2 \quad (2.15)$$

We first redetermined the subtraction terms by the method used in I. However, in determining the central value of $a^0_0$ we used the best $\chi^2$ [as given in Ref. 1 and Eq. (2.17) here] defined in the region $s \in (14, 32)$ and not just $s \in (14, 26)$. This guarantees the "up" nature of the solution at least in this energy region. We find

$$a^0_0 = 0.17 \pm 0.07, \quad 2a^0_0 - 5a^2_0 = 0.58 \pm 0.04 \quad (2.16)$$

We have then obtained the continuation of the imaginary parts of these $s$- and $p$-wave amplitudes down to threshold. Including the high energy and intermediate energy contributions described above, namely $g^0_0(s)$, $g^1_1(s)$ of Fig. 1, we compute the real parts of $f^0_0(s)$ and $f^1_1(s)$. The real part of $\hat{f}^1_1(s)$ is negligibly changed from that shown in Fig. 3. However, the computed $\text{Re} f^0_0(s)$ (see Fig. 5) fails to be consistent even in shape with the "up solution data" except below $s = 52$ (which was already guaranteed by our choice of subtraction constants). Since the high and intermediate energy contributions to Eqs. (2.1, 2.2) are smooth functions of $s$, for $s < 50$, the fact that the shape of the $\text{Re} f^0_0(s)$ computed via Roy's equations from the imaginary parts is inconsistent with the $\text{Re} f^0_0(s)$ obtained via unitarity from the imaginary parts is conclusive evidence for the unacceptability of the up solution. As a numerical indication of the
inconsistency of this up solution we have calculated \( (\chi_0^0)^2 \) defined by

\[
(\chi_0^0)^2 = \int_{s_1}^{s_2} ds \left( \frac{\text{Re} f_e(s) - \text{Re} f_c(s)}{\delta(s)} \right)^2
\]

(2.17)

for both the up and down solutions, where \( \text{Re} f_e , \text{Re} f_c \) are the experimental and computed real parts, respectively, for the \( I = 0 \) s-wave and \( \delta^2 \) is the sum of the squares of the experimental and computed errors in these real parts. We choose \( (s_1, s_2) = (35, 45), i.e., \)
\( (s)^{3/2}(825, 935) \text{MeV} \), for defining \( \chi^2 \) since it is this region that \( \varepsilon_0^0 \) for the up and down solutions are most different. We find that

\[
(\chi_0^0)^2(\text{up}) = 85, \quad (\chi_0^0)^2(\text{down}) = 12.
\]

(2.18)

Similarly with \( (s_1, s_2) = (12, 52) \) we have

\[
(\chi_0^0)^2(\text{up}) = 117, \quad (\chi_0^0)^2(\text{down}) = 23
\]

(2.19)

whilst in both cases \( (\chi_1^1)^2 \) is essentially the same. What \( \varepsilon_0^0(s) \) would have to be to obtain a self-consistent up solution is shown as a dashed line in Fig. 1, where it can be compared with the actual \( \varepsilon_0^0(s) \) we obtained from our estimates of the high and intermediate energy \( \pi\pi \) amplitude.

The reason we claim we can distinguish between up and down solutions for \( \varepsilon_0^0 \) beyond the \( \rho \)-region using Roy's equations, whilst Bonnier and Gauron\(^{13} \) concluded that they could not, is partly because we have assigned errors to our computations. This provides us with a measure, whether by eye or by \( \chi^2 \), of how consistent our output real parts are with the data. We are thus able to show that the down solution for \( \varepsilon_0^0 \) beyond the \( \rho \)-region is in very much better agreement than the corresponding up solution. We have also used Roy's equations up to a higher energy than Bonnier and Gauron, who stop at \( s = 40 \).

This enables us to highlight the differing degrees of consistency between the up and down solutions, which is particularly apparent for \( 40 < s < 45 \).
III. DISCUSSION

Our aim has been to take the imaginary parts of the partial wave amplitudes from the data and to use them in Roy's equations together with simple estimates of the intermediate and high energy \( \pi \pi \) amplitude and compute the corresponding real parts for the s- and p-wave amplitudes. To be able to compare these real parts with those obtained via unitarity from the "experimental" imaginary parts we have to obtain values for the subtraction constants \( a_0 \) and \( L = (2a_0 - 5a_0^2)/6 \). In Ref. 1 we determined \( L \) by equating the representation of Roy and the Froissart-Gribov representation for \( f_1(s)/(s - 4) \) evaluated at \( s = 4 \). We then fixed \( a_0 \) by demanding consistency of the input and output \( I = 0 \) s-wave phase shifts in the energy region from 520 MeV to 700 MeV only.

Having determined these subtraction terms and having estimated the intermediate and high energy \( \pi \pi \) amplitudes we were able, in Sec. II, to compute the real parts of the s- and p-wave amplitudes from threshold to 1 GeV.\(^{14}\) We have seen that our results are in very good agreement with experiment if we choose the down solution for the \( I = 0 \) s-wave phase shift beyond the \( \rho \)-resonance region, and reject the up solution, which can also be discarded for the reasons given in the Introduction and in Refs. 4,2. Indeed we find that with our estimates of the intermediate and high energy \( \pi \pi \) amplitude the rapid rise of the \( a_0 \) phase shift near \( \bar{K}K \) threshold is naturally consistent with the crossing and analyticity properties embodied in Roy's equations.

Basdevant, Froggatt, and Petersen\(^ {6} \) have applied Roy's equations to the data in a different way. They have sought general parametric solutions to Roy's equations and nonlinear unitarity, which agree with the data from 500-1000 MeV, and they have found a much larger range of s-wave scattering lengths than we obtain, e.g., they have \( a_0 \in (-0.05, 0.6) \) [to be compared with Eq. (2.14)]. In their search for general solutions, they demand, from the outset, consistency between the input and output phase shifts from \( \pi \pi \) threshold to 1.5 GeV, which includes \( \bar{K}K \) threshold. In contrast our aim has been to demand consistency only in a minimal energy region so as to do nothing more than reasonably determine the subtraction constants and have an analytic continuation of the phase shifts down to threshold. Then we ask whether we automatically have consistency in the whole energy region up to 1 GeV. We find that we do. Thus our aims are somewhat different than those of Basdevant et al.\(^ {6} \)

Further the apparent greater precision in determining the s-wave scattering lengths in our case is due to the fact that we have restricted ourselves to one specific set of data between 500 MeV and 1 GeV. The data we have used for \( a_0 \) has a statistical uncertainty of approximately \( \pm^\circ \) in this energy region. We have assumed that this is the total uncertainty, i.e., we have ignored any possible systematic error. Such systematic uncertainties are allowed in the treatment of Basdevant, Froggatt, and Petersen\(^ {6} \) and so they obtain a larger range for \( a_0 \) than we have found. Better knowledge of the high energy \( \pi \pi \) amplitude would, in principle, restrict the behavior of the very low energy amplitude and in particular the s-wave scattering lengths. For example, one might use an unsubtracted fixed-t dispersion relation for \( F^0(s,t) - F^0_{\text{Regge}}(s,t) \), evaluated at \( t = 0, s = 4 \), to obtain an expression for \( a_0 + 5a_0^2 \).\(^ {15} \) However, the present
errors on the high energy amplitude are such that one obtains

$$a_0^0 + 5a_0^2 = -0.5 \pm 1.6.$$  (3.1)

This result can be combined with a similar unsubtracted dispersion relation for $F^{t1}(s,t)$ which gives

$$2a_0^0 - 5a_0^2 = 0.57 \pm 0.04$$  (3.2)

to fix

$$a_0^0 = 0.03 \pm 0.55.$$  (3.4)

This is to be compared with the result we obtained$^1$ of

$$a_0^0 = 0.15 \pm 0.07.$$  [Eq. (2.14)]. It is clear that a much better knowledge of the high energy terms is necessary before such sum rules can be used to differentiate between various solutions to Roy's equations. However, there is no guarantee that such sum rules are, in fact, satisfied by solutions to Roy's equations, which know only of twice-subtracted dispersion relations.

### IV. CONCLUSIONS

We have shown that, with simple Regge pole and resonance estimates of the high and intermediate energy $\pi\pi$ amplitude, the data of Protopopescu et al.$^2$ are perfectly consistent with crossing and analyticity requirements, provided $a_0^0 = 0.15 \pm 0.07$, $a_0^2 = -0.053 \pm 0.028$. In particular we find that the rigorous equations of Roy, applied to the experimental data, reproduce the rapid rise of the $I = 0$ s-wave phase shift near $K\bar{K}$ threshold. We also show that we cannot reproduce an up-type solution without drastically changing our estimates of the $\pi\pi$ amplitude above 1 GeV.
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FOOTNOTES AND REFERENCES

* Work supported by the U. S. Atomic Energy Commission.


2. S. D. Protopopescu et al., A \( \pi \pi \) Partial Wave Analysis From Reactions \( \pi^+ p \rightarrow \pi^+ \Delta^{++} \) and \( \pi^+ p \rightarrow K^+ K^+ \Delta^{++} \) at 7.1 GeV/c, Lawrence Berkeley Laboratory Report LBL-970, Sept. 1972.


11. A sentence stating that the pomeron amplitude was used as an estimate of the background in the intermediate energy region was inadvertently omitted in the preprint of Ref. 1 (corrected in the version to be published). However, such a term was included in all our calculations in that paper, e.g., in Eqs. (3.7-9) of I.


14. The \( I = 2 \) s wave is essentially just as calculated previously in Ref. 1.

15. We are grateful to Dr. D. Morgan for suggesting this.

16. This calculation can be done in the following way. To make explicit the s-u symmetry of \( F^\text{to} \) we use the variable \( z = (-su)^{1/2} \) and assume

\[
F^\text{to}_{\text{Regge}}(s,t = 0) = 1 \beta_p z + \beta_p \pi (\alpha_c(0))^{-1} \chi \left( 1 - \cot \frac{\pi}{2} \alpha_t \right) \left( \frac{2}{y_0} \right) \alpha_t(0).
\]

We then use this Regge form in the following dispersion relation

\[
\frac{1}{3}(a_0^0 + 5a_0^2) = \frac{1}{\pi} \int_{4}^{115} ds \frac{(2s - 4)}{s(s - 4)} \\
\chi \left( \frac{1}{3} \text{Im} F^s(0) + \text{Im} F^t(0) + \frac{5}{3} \text{Im} F^u(0) - \text{Im} F^\text{to}_{\text{Regge}}(s,0) \right).
\]

Evaluating the right-hand side of this equation with the contributions we have discussed in Sec. II we find

\[
\frac{1}{3}(a_0^0 + 5a_0^2) = -0.16 \pm 0.55.
\]

The major part of the uncertainty comes from \( \sigma_{\text{tot}} \) and \( y_0 \) of Eqs. (2.5, 2.6, 3.1).

Fig. 1. A plot of $g_0^0$ and $g_1^1$ as functions of $s$, for the central values of our parameters. The error bars represent the range of values obtained by varying our parameters as given by Eqs. (2.8)-(2.10), (2.13). The dashed line is the sum of high and intermediate energy contributions to $f_0^0(s)$, which would be required, instead of $g_0^0(s)$, to obtain a self-consistent "up" solutions for $g_0^0$.

Fig. 2. The real part of the $I = 0$ s-wave amplitude $f_0^0(s)$: the unshaded band is computed from Eqs. (2.1), (2.2) using the experimental "down" solution of Ref. 2, which is shown as the hatched band.

Fig. 3. The real part of the $I = 1$ p-wave amplitude $f_1^1(s)$: the unshaded band is computed from Eqs. (2.1), (2.2) using the experimental data of Ref. 2, which is shown as the hatched band.

Fig. 4. The real part of the $I = 0$ s-wave amplitude $f_0^0(s)$: the unshaded band is computed as for Fig. 2 but with the $\epsilon'$ resonance replaced by just the background amplitude. The shaded band is the data of Ref. 2.

Fig. 5. The real part of the $I = 0$ s-wave amplitude $f_0^0(s)$: the unshaded band is that computed from Eqs. (2.1), (2.2) using the "up" solution, shown as the shaded band, corresponding to the phase shifts of Ref. 2.
Fig. 4
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