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Given topological amplitudes with specific genus and boundary structure we define a product of these amplitudes, also specified by a genus and a boundary structure. We show that this product is independent of the particular graphical representation of the members of the product; we describe an explicit construction to obtain the boundary structure of this product without using graphs and finally we derive the general formula to compute the genus of the product.

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Then the physical amplitude can be written as:

\[ M = \sum_{h} \sum_{i=1}^{N!} M_h(A^i_b) \]

where \( \sum_{i=1}^{N!} \) stands for the sum over all possible partitions of the \( N \) particles.*

The purpose of such a "topological expansion" of the physical amplitude is to obtain discontinuity formulae that are easier to handle. The normal threshold discontinuity formula for the physical amplitude can symbolically be written as:

\[ \text{disc } M = M^* \times M \]

where the symbol "disc" refers to a specific discontinuity (i.e., set of intermediate particles) in a specific channel. Therefore:

\[ \sum_{h,1} \text{disc } M_h(A^1_b) = \left( \sum_{h_1} M_{h_1}(A^1_{b_1}) \right) \left( \sum_{h_2} M_{h_2}(A^2_{b_2}) \right) \]

which is then decomposed into the infinite set of equations:

\[ \text{disc } M_h(A^i_b) = \sum (M_{h_1}(A^1_{b_1}) M_{h_2}(A^2_{b_2})) \]  

* The fact that there exist \( N! \) different partitions of \( N \) was pointed out by Y. Eylon.

The sum on the right hand side includes all terms resulting in a complex function whose singularities are characterized by \( h \) and the boundary structure (poles) \( A^i_b \).

Several conjectures were made along the line:

1. **Existence:** The product on the right hand side of (2) can be characterized by a genus and a boundary structure (Veneziano).
2. **Uniqueness:** The product on the right hand side (2) is independent of the particular (graphical) representation used to express the amplitudes (Chew).
3. **Usefulness:** The resulting discontinuity formulae will be easier to use (i.e., the number of terms on the right hand side is finite).

The objective of this paper is to prove these conjectures and to describe a general method to obtain the boundary structure and the genus (\( h \)) of a product of two "topological amplitudes" without using a graphical representation.

In the next section we discuss various equivalent representations. In Section 3 we describe the method to obtain the boundary structure of a product, first on a simple example, then for the general case. In Section 4 we derive the formula for the genus of the product of two general amplitudes. Finally, in the Appendix we discuss some aspects of graphical amplitude representation.
II. Representation of Amplitudes

II.1 Graphical Representation

Chronologically, the first representation of topological amplitude was given in terms of quark diagrams as shown on Fig. 1a (for h = 1, b = 1). Later, when the concept of order became clearer\(^2\) the same amplitude was represented in terms of particle diagrams as shown in Fig. 1c. Here the lines represent mesons and not quarks. A hybrid representation was also used from time to time as shown in Fig. 1b. There, certain edges represent mesons (edges connecting the "bubbles") whereas others represent quarks (edges forming the rings). Although this hybrid notation may appear confusing it helps an intuitive understanding of the equivalence of quark diagrams with particle diagrams. This equivalence is formally shown in the Appendix. Each vertex of Fig. 1c is ordered in the sense that edges sticking out of it have a definite cyclic order on the imbedding two-dimensional surface.

II.2 Edmond's Permutation Technique

In order to find the boundary structure and the genus of an amplitude or a product of amplitudes in a discontinuity formula represented by a particle diagram (or a hybrid diagram) one can apply Edmond's permutation technique.\(^2,3\) The method consists of finding the orbits of the graph (corresponding to the windows and boundaries of quark diagrams) as follows: We start from one vertex (or ring in the hybrid notation) and turn clockwise around it until we reach a connecting edge. We then follow the edge to the next vertex (or ring) and continue clockwise around the latter until we reach a connecting edge (i.e., particle edge) and so on until we return to the starting point. This path is called an "orbit". An example is given in Fig. 2 for a cylinder (b = 2, h = 0).

The genus of the graph is then readily obtained from Euler's formula:

\[
h = \frac{1}{2}(2 - v + e - f)
\]

where \(v\) is the number of vertices (or rings), \(e\) the number of connecting edges (i.e., particle edges only in the hybrid diagrams) and \(f\) the number of orbits. Note that this method is equivalent to following the quark lines in a quark diagram. An orbit is either a boundary or a window. That is, an orbit need not contain twigs (or vertices) representing external particles (window). If it does contain external particles (boundary) then the cyclic order of the external particles must be retained.

II.3 Algebraic notation

Since the assumption is that the number of handles and the boundary structure are sufficient to define the amplitude, Chew\(^2\) has proposed an algebraic notation whereby the particles of a given boundary are listed sequentially modulo a cyclic permutation. Sets of particles belonging to different boundaries are separated by
a + sign. For instance, a 4-point cylinder amplitude is represented by $M_0(AB + CD)$ while a 4-point torus amplitude (one handle) is denoted by $M_1(ABCD)$. The subscript always refers to the genus.

The advantage of such a notation is its independence of any particular graphical representation.

### III. The Boundary Structure

#### III.1 An Example

We shall now describe a method to obtain the boundary structure of the product of two amplitudes using Chew's notation. Since the general case is quite difficult to manipulate we start with a specific simple example defined in Fig. 3.

The external particles are labelled $a$, $b$, $c$, $d$, $e$ and the intermediate particles are labelled $i_1$, $i_2$, $i_3$. By applying Edmond's permutations technique on $P = A \times B$ one can see that the resulting discontinuity is $M_2(a^+ + b + ced)$.

#### III.2 Constructing the Boundary Structure of the Product

We now apply the algebraic method to this particular case:

1. We start from an external particle, say $a$, and record it in the final product.
2. We continue to record all external particles following $a$ on the same boundary until we reach an intermediate particle (if there is no intermediate particle on the boundary carrying $a$, we write a + sign thus terminating the list of this boundary and start on another boundary of $A$ or $B$). In our example we reach $i_1$. We slash it and jump to $i_1$ in $B$; we also slash $i_1$ here and continue on the same boundary.

If there were any external particles between $i_1$ and $i_2$ on $B$ we would have listed them after $a$. Here we reach immediately another intermediate particle $i_2$, which we slash and jump back to $i_2$ on $A$. We slash $i_2$ on $A$ and continue on the same boundary on $A$. We find a which means we are back to the starting point. This completes the first boundary of the product (+ sign).

In our example, the situation after this step is as follows:

$$M_1\left(\frac{1}{a} b x + i_3 c\right) \otimes M_0\left(\frac{2}{b} d + i_3 e\right) = M_0(a +$$

We then write the next external particle, $b$, and repeat the process. The various steps in our example are detailed below. The analysis is completed when each intermediate particle has been slashed twice:

$$M_1\left(\frac{1}{a} b x + c i_3\right) \otimes M_0\left(\frac{2}{b} d + i_3 e\right) = M_0(a + b +$$

and finally:
The value of $h$ is obtained using Eq. (9) derived in Section 4:

$$2h = 2(1 + 0) + (2 + 2 - 3) + (3 - 2) = 4,$$

hence

$$h = 2.$$

III.3 The General Product

We can now write the general product as follows:

$$\prod = M_{h_1} \left( \sum_{a=1}^{b_1} a_1^{a_1} i_1^{a_1} \ldots a_m^{a_1} i_m^{a_1} \right) \otimes M_{b_2} \left( \sum_{\beta=1}^{b_2} b_1^{\beta} i_1^{\beta} \ldots b_m^{\beta} i_m^{\beta} \right)$$

$$= M_h \left( \sum_{\gamma=1}^{g} c_1^{\gamma} c_2^{\gamma} \ldots c_m^{\gamma} \right),$$

where $i_{k\gamma}^a$ stands for an intermediate particle in boundary $a$.

The "length" of the boundary (i.e., the number of intermediate particles in $a$) is $m_a$.

Similarly $a_{k\gamma}$ stands for the group of adjacent external particles lying between $i_{k-1}^a$ and $i_{k}^a$ (the subscripts are understood modulo $m_a$); $a_{k\gamma}$ may represent one, several or no external particle whereas $i_{k\gamma}^a$ always represents one particle. Finally, the $c_{k\gamma}^b$ represent some $a_{k\gamma}^a$ or $b_{k\gamma}^b$ or a combination thereof.

The procedure to obtain the boundary structure of the product is as explained in the particular case discussed above. We start from some $a_{k\gamma}$ which we list in the product, we then slash the following $i_{k\gamma}^a$ and go the corresponding particle in the other amplitude. We slash it and continue. The principle is to always remain within a boundary in one member of the product until we encounter an intermediate particle. We slash it before and after the "jump" to the other amplitude and continue on whatever boundary we have "landed" until we return to the starting point. This constitutes an orbit.

In the event a given boundary $a$ or $b$ has only external particles say $a_{1}^a$ (or $b_{1}^b$) then this boundary is recorded by itself in the final product.

In the event no external particles are found in a particular orbit of the product, a zero must be entered (This corresponds to a window.). This point needs to be clarified: Although amplitudes never contain "windows" as part of the boundary structure, products of amplitudes—which enter in discontinuity formulae for a specific set of intermediate particles may contain "windows" characteristic
of the particular discontinuity involved. In other words, one must be careful not to associate immediately a product of amplitudes with an amplitude but rather with an amplitude discontinuity.

IV. The Genus of the Product

We now derive the formula to compute $h$. This number will be seen to depend on the number of boundaries $b_1$ and $b_2$ and on the number of orbits in the product but not on the boundary structure. Let us consider the particle diagram (Fig. 1c) for a moment and let us assume that the first amplitude $A$ is represented by $v_1$ vertices. We thus have, using Euler formula (Eq. 3):

\begin{equation}
(5) \quad v_1 - e_1 + b_1 = 2 - 2h_1
\end{equation}

where $e_1$ is the number of edges connecting the $v_1$ vertices and forming a graph with $b_1$ orbits and $h_1$ handles. Similarly let B be represented by a graph with $v_2$ vertices then:

\begin{equation}
(6) \quad v_2 - e_2 + b_2 = 2 - 2h_2
\end{equation}

where $e_2$ is defined similarly.

Now, the product $P = A \times B$ will have $v$ vertices, $e$ connecting edges, $f$ orbits and $h$ handles. These quantities are again related by the Euler formula:

\begin{equation}
(7) \quad v - e + f = 2 - 2h,
\end{equation}

and furthermore we have

\begin{equation}
(8) \quad v = v_1 + v_2 \\
\quad e = e_1 + e_2 + I,
\end{equation}

where $I$ is the number of edges between $A$ taken as a whole and $B$ taken as a whole. In other words, $I$ is the number of intermediate particles in the product under consideration.

Inserting (5), (6) and (8) into (7) we get:

\begin{equation}
(9) \quad 2h = 2(h_1 + h_2) + (b_1 + b_2 - f) + (1 - 2)
\end{equation}

which is the general formula giving $h$ in terms of $h_1$, $h_2$, $b_1$, $b_2$ and $f$.

It is crucial to realize that $f$ is the number of orbits involved in the product (not the number of boundaries). This product may contain some "windows" as a result of the particular discontinuity involved in the product (not the number of boundaries). This product may contain some "windows" as a result of the particular discontinuity involved in the product (not the number of boundaries). This product may contain some "windows" as a result of the particular discontinuity involved in the product (not the number of boundaries). This product may contain some "windows" as a result of the particular discontinuity involved in the product (not the number of boundaries). This product may contain some "windows" as a result of the particular discontinuity involved in the product (not the number of boundaries). This product may contain some "windows" as a result of the particular discontinuity involved in the product (not the number of boundaries).
(i. e., intermediate state). Although these "windows" should be dropped in characterizing the amplitude whose discontinuity contains the product, they must be taken into account when computing f.

Conclusions

We have completed the task set at the beginning of this paper. We have shown that given topological amplitudes with specified genus and boundary structure we can define a product also specified by a genus and a boundary structure. We have shown that this product is independent of the graphical representation of the members of the product and we have described an explicit construction to obtain both the boundary structure and the genus h.

The "usefulness" of the topological expansion also stems from Eq. (9). Suppose we are interested in a particular discontinuity of a topological amplitude: disc $M_{h}^{1}(A_{b})$, then owing to the inequality

$$h \geq h_{1} + h_{2}$$

we see that only a finite number of terms can participate in the discontinuity formula. Furthermore one can readily describe a systematic way to generate all terms that should participate.

We first fix one member of the product (which yields $h_{1}$ and $b_{1}$) and then, for each boundary structure of the other member,

Equation (9) will yield the appropriate genus ($h_{2}$). By summing first over all possible boundary structures of the second member and then over all possible graphs (first member) with $h_{1} \leq h$ we get all possible terms.

After completion of this work, it was pointed out to the author that Ciafaloni, Marchesini and Veneziano (1975) have discussed some of the problems addressed in this paper.*

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These authors proposed an inequality, $2h \geq 2(h_{1} + h_{2}) + (b_{1} + b_{2} - 2)$, but counterexamples can easily be found.
Appendix

We first establish the equivalence of the three graphical notations used in the text (cf. Fig. 1).

1) The equivalence of the quark diagrams and hybrid diagrams is trivial. The only difference between the two systems of graphs is that the "gap" between quark lines in the quark diagram is transformed into a vertex in the hybrid diagram and internal (intermediate) particles are represented by a single line instead of two quark-lines.

Edmond's rule insures that boundaries, windows and handles are in complete correspondence.

2) The equivalence of hybrid diagrams and particle diagrams is also trivial and is obtained by shrinking the rings of the hybrid representation into simple vertices.

Formally, the equivalence can be deduced from Euler's formula. For each ring of the hybrid representation, the number of vertices $v$ equals the number of quark edges: $e_q$. Thus, for a hybrid diagram we have

$$v - (e_q + e_p) + f = 2 - 2h,$$

where $e_p =$ number of intermediate particle edges. Hence,

$$-e_p + f = 2 - 2h.$$

Now, let $f = f_{\text{int}} + f_{\text{ext}}$ where $f_{\text{int}}$ refers to the faces inside the rings; so we have:

$$f_{\text{int}} - e_p + f_{\text{ext}} = 2 - 2h.$$

On the other hand, for particle diagrams, Euler's formula yields:

$$v - e_p + g = 2 - 2h,$$

when $v =$ number of vertices; $g =$ number of orbits. The equivalence is obtained by realizing that:

$$v = f_{\text{int}}$$

and

$$f_{\text{ext}} = g.$$

Graphical Representation of Amplitudes

Ordered amplitudes are usually represented by a ring whereas higher order amplitudes are generally represented by two connected rings. In fact, it is easy to see that all amplitudes can
be represented by one ring (or vertex). Using Euler's formula again we get:

\[ 2h = e - b + (2 - v). \]

If \( v = 2 \), the minimum number of edges for a given amplitude is

\[ e = 2h + b. \]

If \( v = 1 \), this number is

\[ e = 2h + b - 1, \]

which is always feasible since \( 2h + b \geq 1 \). We have represented a cylinder and a torus (one handle) 4-point amplitude in Fig. 4a and 4b respectively.

REFERENCES


FIGURE CAPTIONS

Fig. 1: Amplitude representation: (a) quark diagrams, (b) hybrid diagrams, (c) particle diagrams.

Fig. 2: Orbits according to Edmond's method. Cylinder in particle diagram notation (a) and in hybrid notation (b).

Fig. 3: Product of two amplitudes.

Fig. 4: One-vertex representation of amplitudes: (a) a cylinder; (b) a torus.
Fig. 1
\[ A \equiv M_1 (ai_1 b i_2 + i_3 c) \otimes B \equiv M_0 (i_1 i_2 + di_3 e) \]

\[ P = A \otimes B \]

Fig. 3
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