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Integrable Representations for Equivariant Map Algebras Associated with Borel-de Siebenthal Pairs

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Integrable Representations for Equivariant Map Algebras Associated with Borel-de Siebenthal Pairs

A Dissertation submitted in partial satisfaction of the requirements for the degree of

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by

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I would like to thank Professor Vyjayanthi Chari for her guidance and instruction throughout my graduate career in both my mathematical research and professional development. I would like to thank Dr. Deniz Kus for his guidance, encouragement, and patience while working with me. I am also appreciative of my fellow graduate students at UCR for creating an enjoyable environment in which to work. Lastly, I would like to thank my family for their love, encouragement, and support.
To my family.
Borel and de-Siebenthal classified the maximal connected subgroups of maximal rank of a connected compact Lie group. This result can be rephrased in terms of automorphisms of the semisimple Lie algebra and the subalgebra of fixed points. They also give rise in a natural way to a maximal parabolic subalgebra of an affine Lie algebra. In the case when the automorphism is non-trivial, we shall see that the parabolic subalgebra is isomorphic to an equivariant map algebra.

We develop the theory of integrable representations of such Lie algebras. In particular, we define and study global Weyl modules. These are closely related to the module category of a commutative associative algebra. In this dissertation, we give a presentation of this algebra, and give partial results toward a dimension formula for local Weyl modules.
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Introduction

The category of integrable representations of the current algebra, \( \mathfrak{g}[t] \), has been intensively studied in recent years. Two important families of representations in this category are the global and local Weyl modules. The global Weyl modules, indexed by dominant integral weights \( \lambda \in P^+ \), are universal objects in the category and are naturally a right module for a commutative associative algebra, \( A_\lambda \) (see [1] for a nice description of this relationship).

It is known that for the current algebra, \( A_\lambda \) is a polynomial algebra in a finite number of variables depending on the weight \( \lambda \). Moreover, the global Weyl module is always infinite dimensional for nontrivial weights.

The local Weyl modules are indexed by dominant integral weights and maximal ideals in the corresponding algebra \( A_\lambda \). In the case of the current algebra, these modules are finite dimensional, and the dimension of these modules depends only on the weight, and not on the choice of maximal ideal in \( A_\lambda \), and so the global Weyl module is a free \( A_\lambda \)-module of finite rank.

Another family of Lie algebras that are of interest in representation theory are equivariant map algebras. Equivariant map algebras are Lie algebras of the form \( (\mathfrak{a} \otimes A)^\Gamma \), where \( \mathfrak{a} \) is
a Lie algebra, $A$ is a finitely generated commutative associative algebra, and $\Gamma$ is a finite abelian group acting on $\mathfrak{a}$ by Lie algebra automorphisms and $A$ by algebra automorphisms. [5] studies equivariant map algebras with an extra assumption on the action of the group, that being the group acts freely on the set of maximal ideals of $A$. Under this assumption, the finite dimensional representation theory of the equivariant map algebra coincides with the finite dimensional representation theory of the map algebra, $\mathfrak{a} \otimes A$. In [14], the authors ask about the representation theory of equivariant map algebras without this assumption. This dissertation is a first attempt at answering this question.

This dissertation is organized as follows: In Chapter 1, we recall a result of Borel and de Siebenthal which realizes all maximal semisimple subalgebras, $\mathfrak{g}_0$, of maximal rank, of a fixed simple Lie algebra $\mathfrak{g}$ as the set of fixed points of an automorphism of $\mathfrak{g}$. We prove some results of root systems that we will need later in the dissertation, and discuss the running example of the paper, which is the case where $\mathfrak{g}$ is of type $B_n$, and $\mathfrak{g}_0$ is of type $D_n$.

In Chapter 2 we extend the automorphism of $\mathfrak{g}$ to an automorphism of $\mathfrak{g}[t]$. We then study the corresponding equivariant map algebra, which is the set of fixed points of this automorphism. We discuss ideals of this equivariant map algebra, and show that in this case, the equivariant map algebra is not isomorphic to an equivariant map algebra where the action of the group is free, which makes the representation theory of these equivariant map algebras much different from that of the map algebra $\mathfrak{g} \otimes \mathbb{C}[t]$. We conclude the section by making the connection between these equivariant map algebras and maximal parabolic subalgebras of the untwisted affine Kac-Moody algebra $\hat{\mathfrak{g}}$. 
In Chapter 3 we develop the representation theory of the equivariant map algebra $\mathfrak{g}[t]^\tau$. Following [1], [2], we define the notion of global Weyl modules, the associated commutative algebra and the local Weyl modules associated to maximal ideals in this algebra. In the case of $\mathfrak{g}[t]$ it was shown in [4] that the commutative algebra associated with a global Weyl module is a polynomial ring in finitely many variables. This is no longer true for $\mathfrak{g}[t]^\tau$; however in Chapter 4 we see that modulo the Jacobson radical, the algebra is a Stanley–Reisner ring, i.e. a quotient of a finitely generated polynomial ring by a squarefree monomial ideal. By making the connection to Stanley-Reisner theory, we are able to determine the Hilbert series of the commutative algebra. In the case when $a_j^\vee(\alpha_0) = 1$ we also determine the Krull dimension of the commutative algebra, and we give a sufficient condition for the commutative algebra to be Koszul and Cohen-Macaulay.

In Chapter 5 we examine an interesting consequence of determining this presentation of the commutative associative algebra which differs from the case of the current algebra greatly. More specifically, we see that for nontrivial weights, a global Weyl module can be finite dimensional and irreducible, and we give necessary and sufficient conditions for this to be the case.

We conclude this dissertation in Chapter 6 by determining the dimension of the local Weyl module in certain cases. We also discuss other features not seen in the case of the current algebra. Namely, we give an example of a weight where the dimension of the local Weyl module depends on the choice of maximal ideal in $A_\lambda$. We also show an example where the global Weyl module is not a free $A_\lambda$–module, and show that the tensor product dimension formula does not hold in general.
Chapter 1

Borel-de Siebenthal Pairs

1.1 Notation

We denote the set of complex numbers, the set of integers, non-negative integers, and positive integers by $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ respectively. Unless otherwise stated, all the vector spaces considered in this paper are $\mathbb{C}$-vector spaces and $\otimes$ stands for $\otimes_{\mathbb{C}}$. Given any Lie algebra $\mathfrak{a}$ we let $\mathfrak{U}(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$. We also fix an indeterminate $t$ and let $\mathbb{C}[t]$ and $\mathbb{C}[t, t^{-1}]$ be the corresponding polynomial ring, respectively Laurent polynomial ring with complex coefficients.

Let $\mathfrak{g}$ be a complex simple finite-dimensional Lie algebra of rank $n$ with a fixed Cartan subalgebra $\mathfrak{h}$. Let $I = \{1, \ldots, n\}$ and fix a set $\{\alpha_i : i \in I\}$ of simple roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let $R$, $R^+$ be the corresponding set of roots and positive roots respectively. Let $a_i$, $i \in I$ be the labels of the Dynkin diagram of $\mathfrak{g}$; equivalently the highest root of $R^+$ is $\theta = \sum_{i=1}^{n} a_i \alpha_i$. Fix a Chevalley basis $\{x^\pm_\alpha, h_i : \alpha \in R^+, i \in I\}$ of $\mathfrak{g}$, and set $x^\pm_i = x^\pm_{\pm\alpha_i}$. Let
Let \( Q \) be the root lattice with basis \( \alpha_i, i \in I \). Define \( a_i : Q \to \mathbb{Z}, i \in I \) by requiring
\[
\eta = \sum_{i=1}^{n} a_i(\eta)\alpha_i, \text{ and set } \text{ht}(\eta) = \sum_{i=1}^{n} a_i(\eta).
\]
For \( \alpha \in R \) set \( d_{\alpha} = 2/(\alpha, \alpha) \), \( a_i^\vee(\alpha) = a_i(\alpha)d_{\alpha}d_{\alpha}^{-1} \), and \( h_\alpha = \sum_{i=1}^{n} a_i^\vee(\alpha)h_i \). Let \( W \) be the Weyl group of \( g \) and fix a set of simple reflections \( s_i, i \in I \).

## 1.2 Borel-de Siebenthal pairs

From now on we fix an element \( j \in I \) with \( a_j \geq 2 \) and also fix \( \zeta \) to be a primitive \( a_j \)-th root of unity. The following is well-known (see for instance [10]). Set \( I(j) = I \setminus \{j\} \).

**Proposition 1.2.1.** The assignment
\[
x_i^\pm \to x_i^\pm, \ i \in I(j), \ x_j^\pm = \zeta_j^\pm x_j^\pm,
\]
defines an automorphism \( \tau : g \to g \) of order \( a_j \). Moreover, the set of fixed points \( g_0 \) is a semisimple subalgebra with Cartan subalgebra \( h \) and
\[
R_0 = \{ \alpha \in R : a_j(\alpha) \in \{0, \pm a_j\}\},
\]
is the set of roots of the pair \((g_0, h)\). The set \( \{\alpha_i : i \neq j\} \cup \{-\theta\} \) is a simple system for \( R_0 \).

**Remark 1.2.2.** The automorphism above is motivated by the following: We call \((g, a)\) a Borel–de Siebenthal pair if \( a \) is a maximal (proper) semi–simple subalgebras of \( g \) with rank equal to the rank of \( g \). The algebraic version of a result of Borel and de Siebenthal states
that \((\mathfrak{g}, \mathfrak{g}_0)\) is a Borel–de Siebenthal pair if \(a_j\) is prime and any such pair is of that form. If \(a_j\) is not prime we can find a chain of semi–simple subalgebras
\[
\mathfrak{g}_0 \subset \mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_\ell \subset \mathfrak{g},
\]
such that the successive inclusions are Borel–de Siebenthal pairs.

### 1.3 The simple system

For our purposes we will need a different simple system for \(R_0\) which we choose as follows. The subgroup of \(W\) generated by the simple reflections \(s_i, i \in I(j)\) is the Weyl group of the semisimple Lie algebra generated by \(\{x_i^\pm : i \in I(j)\}\). Let \(w_0\) be the longest element of this group.

**Lemma 1.3.1.** The set
\[
\Delta_0 = \{\alpha_i : i \in I(j)\} \cup \{w_0^{-1}\theta\},
\]
is a set of simple roots for \((\mathfrak{g}_0, \mathfrak{h})\) and the corresponding set \(R_0^+\) of positive roots is contained in \(R^+\).

**Proof.** Since \(w_0\) is the longest element of the Weyl group generated by \(s_i, i \in I(j)\), it follows that for \(i \in I(j)\),
\[
w_0\alpha_i \subset \{-\alpha_p : p \in I(j)\}.
\]
Hence
\[
\Delta_0 = -w_0^{-1} (\{\alpha_i : i \in I(j)\} \cup \{-\theta\}).
\]
Since \( w_0 \) is an element of the Weyl group of \( g_0 \) it follows from Proposition 1.2.1 that \( \Delta_0 \) is a simple system for \( R_0 \). Moreover \( w_0^{-1} \theta \in R^+ \) since \( w_0 \alpha_j \in R^+ \) and \( a_j(\theta) = a_j \). Hence \( \Delta_0 \subset R^+ \) thus proving the Lemma. \( \square \)

Let \( Q_0 \) be the weight lattice of \( g_0 \) determined by \( \Delta_0 \); clearly \( Q_0 \subset Q \) and set \( Q_0^+ = Q_0 \cap Q^+ \). Then \( Q_0^+ \) is properly contained in \( Q^+ \) and we see an example of this at the end of this section.

**Remark 1.3.2.** We isolate some immediate consequences of the Lemma which we will use repeatedly. From now on we set \( \alpha_0 = w_0^{-1} \theta, x_0^\pm = x_0^\pm, \) and \( h_0 = h_{\alpha_0} \). Then,

(i) \( \alpha_0 \) is a long root.

(ii) \( (\alpha_0, \alpha_i) \leq 0 \) if \( i \in I(j) \) and since \( \alpha_0 \in R^+ \) it follows that \( (\alpha_0, \alpha_j) > 0 \).

(iii) \( a_j(\alpha_0) = a_j \).

(iv) If \( \alpha \in R_0^+ \) is such that \( a_j(\alpha) = a_j \) and \( \alpha \neq \alpha_0 \), then \( \text{ht} \alpha > \text{ht} \alpha_0 \).

**Example 1.3.3.** Our running example through this dissertation will be the Lie algebra of type \( B_n \) where we drop the node \( j = n \). Recall that the positive roots of \( B_n \) are of the form

\[
\alpha_{r,s} := \alpha_r + \cdots + \alpha_s, \quad \alpha_{r,1,2} := \alpha_r + \cdots + \alpha_{s-1} + 2\alpha_s + \cdots + 2\alpha_n.
\]

Moreover, \( \theta = \alpha_{1,2} \) and so \( a_j = 2 \). In this case, \( g_0 \) is of type \( D_n \) and \( \alpha_0 = \alpha_{n-1} + 2\alpha_n \). The simple system for \( D_n \) is \( \Delta_0 = \{ \alpha_i \mid 0 \leq i \leq n - 1 \} \), and the root system for \( D_n \) is the set of all long roots of \( B_n \). We note that \( \alpha_n \in Q^+ \setminus Q_0^+ \) as mentioned earlier in this chapter.
A decomposition of $\mathfrak{g}$

For $1 \leq k < a_j$ set

$$R_k = \{ \alpha \in R : a_j(\alpha) \in \{k, -a_j + k\}\}, \quad \mathfrak{g}_k = \bigoplus_{\alpha \in R_k} \mathfrak{g}_\alpha.$$ 

Equivalently

$$\mathfrak{g}_k = \{ x \in \mathfrak{g} : \tau(x) = \zeta^k x \}.$$ 

Setting $R_k^+ = R_k \cap R^+$, we observe that

$$[x_0^+, R_k^+] = 0, \quad 1 \leq k < a_j.$$

Proposition 1.4.1. We have,

(i) $\mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_{a_j-1}]$.

(ii) For all $1 < k < a_j$ the subspace $\mathfrak{g}_k$ is an irreducible $\mathfrak{g}_0$–module.

(iii) For all $1 \leq m < k < a_j$, we have $\mathfrak{g}_k = [\mathfrak{g}_{k-m}, \mathfrak{g}_m]$.

Proof. Each connected component of the Dynkin diagram of the semisimple algebra $\mathfrak{g}_0$ contains some simple root $\alpha_i$ with $\alpha_i(h_j) < 0$. Since $0 \neq [x_{a_j}, x_{-a_j}] \in [\mathfrak{g}_1, \mathfrak{g}_{a_j-1}]$ it follows that $[\mathfrak{g}_1, \mathfrak{g}_{a_j-1}]$ intersects each simple ideal of $\mathfrak{g}_0$ non–trivially, which proves (i).

If $a_j = 2$, the proof of the irreducibility in part (ii) of the proposition can be found in [12, Proposition 8.6]. If $a_j \geq 3$ then $\mathfrak{g}$ is of exceptional type and the proof is done in a case by case fashion. One inspects the set of roots to notice that for $1 \leq k < a_j$ there exists a unique root $\theta_k \in R_k^+$ such that $\text{ht} \theta_k$ is maximal. This means that $x_{\theta_k}^+$ generates an irreducible $\mathfrak{g}_0$–module and a calculation proves that the dimension of this module is
precisely dim \( g_k \) and establishes part (ii). Part (iii) is now immediate if we prove that the \( g_0 \)-module \([g_m, g_{k-m}]\) is non–zero and this is again proved by inspection. We omit the details.

Part (ii) of the proposition implies that \( R^+_k \) has a unique element \( \theta_k \) such that the following holds:

\[
(\theta_k, \alpha_i) \geq 0 \quad \text{and} \quad [x^+_i, \theta_k] = 0, \quad i \in I(j) \cup \{0\}.
\]

Since \( \theta_k \neq \theta \) it is immediate that

\[
[x^+_j, \theta_k] \neq 0, \quad \text{i.e.,} \quad \theta_k + \alpha_j \in R^+.
\]

Notice that \( x^-_k \in g_{a_j-k} \) and \([x^-_i, x^-_k] = 0 \) for all \( i \in I(j) \cup \{0\} \). Moreover

\[
a_i(\theta_k) > 0, \quad i \in I, \quad 1 \leq k < a_j. \quad (1.4.2)
\]

To see this note that the set \( \{i : a_i(\theta_k) = 0\} \) is contained in \( I(j) \). Since \( \Delta \) is irreducible there must exist \( i, p \in I \) with \( a_i(\theta_k) = 0 \) and \( a_p(\theta_k) > 0 \) and \( (\alpha_i, \alpha_p) < 0 \). It follows that \( (\alpha_i, \theta_k) < 0 \) and hence \( \alpha_i + \theta_k \in R^+_k \) which contradicts the maximality of \( \text{ht} \theta_k \). As a consequence of (1.4.2) we get,

\[
(\theta, \theta_k) > 0, \quad 1 \leq k < a_j, \quad \text{and hence} \quad \theta - \theta_k \in R^+_j - k. \quad (1.4.3)
\]

Finally, we note that since \( (\theta_k + \alpha_j, \alpha_0) = (\theta_k, \alpha_0) + (\alpha_j, \alpha_0) > 0 \) (see the Remark in Section 1.3) we now have

\[
\theta_k + \alpha_j - \alpha_0 \in R, \quad k \neq a_j - 1, \quad \theta_{a_j-1} + \alpha_j - \alpha_0 \in R^+_0 \cup \{0\}.
\]

**Example 1.4.2.** In the case of \( B_n \), where we drop the node \( j = n \), we recall \( a_j = 2 \). In this case, \( R_1 \) is the set of all short roots of \( B_n \), and \( \theta_1 = \alpha_1 + \cdots + \alpha_n \). When \( n \geq 4 \), \( g_1 \) is the
natural representation of $D_n$. When $n = 3$, $g_1$ is the second fundamental representation of $A_3$. 
Chapter 2

Equivariant Map Algebras

In this chapter we define the current algebra version of the pair \((g, g_0)\); namely we extend the automorphism \(\tau\) to the current algebra and study its fixed points. The fixed point algebra is an example of an equivariant map algebra studied in [15]. We show that our examples are particularly interesting since they can also be realized as a maximal parabolic subalgebra of the affine Lie algebra. We also show that our examples never arise from a fixed point free action of a finite abelian group on the current algebra. This fact makes the study of its representation theory quite different from that of the usual current algebra.

2.1 Equivariant map algebras associated with Borel-de Sieben-thal pairs

Let \(g[t] = g \otimes \mathbb{C}[t]\) be the Lie algebra with the Lie bracket given by extending scalars. Recall the automorphism \(\tau : g \rightarrow g\) defined in Section 1. It extends to an automorphism of
\( g[t] \) (also denoted as \( \tau \)) by

\[
\tau(x \otimes t^r) = \tau(x) \otimes \zeta^{-r}t^r, \quad x \in g, \quad r \in \mathbb{Z}_+.
\]

Let \( g[t]^\tau \) be the subalgebra of fixed points of \( \tau \); clearly

\[
g[t]^\tau = \bigoplus_{k=0}^{a_j-1} g_k \otimes t^k \mathbb{C}[t^{a_j}].
\]

Further, if we regard \( g[t] \) as a \( \mathbb{Z}_+ \)-graded Lie algebra by requiring the grade of \( x \otimes t^r \) to be \( r \) then \( g[t]^\tau \) is also a \( \mathbb{Z}_+ \)-graded Lie algebra, i.e.,

\[
g[t]^\tau = \bigoplus_{s \in \mathbb{Z}_+} g[t]^\tau[s].
\]

A graded representation of \( g[t]^\tau \) is a \( \mathbb{Z}_+ \)-graded vector space \( V \) which admits a compatible Lie algebra action of \( g[t]^\tau \), i.e.,

\[
V = \bigoplus_{s \in \mathbb{Z}_+} V[s], \quad g[t]^\tau[s]V[r] \subset V[r + s], \quad r, s \in \mathbb{Z}_+.
\]

### 2.2 Ideals

Given \( z \in \mathbb{C} \), let \( ev_z : g[t] \to g \) be defined by \( ev_z(x \otimes t^r) = z^r x, \quad x \in g, \quad r \in \mathbb{Z}_+ \). It is easy to see that

\[
ev_0(g[t]^\tau) = g_0, \quad ev_z(g[t]^\tau) = g, \quad z \neq 0. \quad (2.2.1)
\]

More generally, one can construct ideals of finite codimension in \( g[t]^\tau \) as follows. Let \( f \in \mathbb{C}[t^{a_j}] \) and \( 0 \leq k < a_j \). The ideal \( g \otimes t^k f \mathbb{C}[t] \) of \( g[t] \) is of finite codimension and preserved by \( \tau \). Hence, \( i_{k,f} = (g \otimes t^k f \mathbb{C}[t^{a_j}])^\tau \) is an ideal of finite codimension in \( g[t]^\tau \). Notice that

\[
\ker ev_0 = i_{1,1}, \quad \ker ev_z = i_{0,(t^{a_j}-z^{a_j})}.
\]

We now prove,
Proposition 2.2.1. Let $i$ be a non-zero ideal in $\mathfrak{g}[t]^\tau$. Then there exists $0 \leq k < a_j$ and $f \in \mathbb{C}[t^{a_j}]$ such that $i_{k,f} \subset i$. In particular, any non-zero ideal in $\mathfrak{g}[t]^\tau$ is of finite codimension.

Proof. For $0 \leq m < a_j$, set

$$S_m = \{ g \in \mathbb{C}[t^{a_j}] : x \otimes t^m g \in i \text{ for all } x \in \mathfrak{g}_m \}.$$ 

We claim that $S_m$ is an ideal in $\mathbb{C}[t^{a_j}]$ for all $0 \leq m < a_j$ and

$$t^{a_j}S_{a_j-1} \subset S_0 \subset S_1 \subset \cdots \subset S_{a_j-1}. \quad (2.2.2)$$

Let $0 \leq m < a_j$, $g \in S_m$ and $f \in \mathbb{C}[t^{a_j}]$. By Proposition 1.4.1 we have $[\mathfrak{g}_0, \mathfrak{g}_m] = \mathfrak{g}_m$ and hence any $x \in \mathfrak{g}_m$ can be written as $x = \sum_{s=1}^{r} [z_s, y_s]$ with $z_s \in \mathfrak{g}_0$ and $y_s \in \mathfrak{g}_m$. Therefore

$$x \otimes t^m fg = \sum_{s=1}^{r} [z_s \otimes f, y_s \otimes t^m g] \in i,$$

which proves that $fg \in S_m$ and hence $S_m$ is an ideal in $\mathbb{C}[t^{a_j}]$. A similar argument using $[\mathfrak{g}_m, \mathfrak{g}_{k-m}] = \mathfrak{g}_k$ proves the desired inclusions (2.2.2).

We now prove that $S_k \neq 0$ for some $0 \leq k < a_j$. If $i \subset \mathfrak{g}_0 \otimes \mathbb{C}[t^{a_j}]$ then $[x^+_j, i] = 0$. This would imply that $(x^+_i \otimes g) \notin i$ for any $i \in I(j) \cup \{0\}$ and $g \in \mathbb{C}[t^{a_j}]$. This is because if $(x^+_i \otimes g) \in i$ then $(a \otimes g) \in i$ for the simple ideal $a$ of $\mathfrak{g}_0$ containing $x^+_i$. But $a$ contains a simple root vector $x^+_k$ with $[x^+_k, x^+_j] \neq 0$ and hence we have a contradiction. In other words we have proved that $i$ must contain an element of the form $(x \otimes t^kg)$ for some root vector $x \in \mathfrak{g}_k$, $k > 0$ and $0 \neq g \in \mathbb{C}[t^{a_j}]$. Since $\mathfrak{g}_k$ is an irreducible $\mathfrak{g}_0$-module we have $(\mathfrak{g}_k \otimes t^kg) \in i$, i.e. $S_k \neq 0$ and we are done.
Using (2.2.2) we also see that $S_r \neq 0$ for all $0 \leq r < a_j$; let $f_r \in \mathbb{C}[t^{a_j}]$ be a non-zero generator for the ideal $S_r$. By (2.2.2) there exist $g_0, \ldots, g_{a_j-1} \in \mathbb{C}[t^{a_j}]$ such that

$$f_r = g_r f_{r+1}, \quad 0 \leq r \leq a_j - 2, \quad t^{a_j} f_{a_j-1} = g_{a_j-1} f_0.$$ 

This implies

$$g_{a_j-1} f_0 = g_0 \cdots g_{a_j-1} f_{a_j-1} = t^{a_j} f_{a_j-1}.$$ 

Hence there exists a unique $m \in \{0, \ldots, a_j - 1\}$ such that $g_m = t^{a_j}$ and $g_p = 1$ if $p \neq m$.

Taking $f = f_{m+1}$, where we understand $f_{a_j} = f_0$, we see that

$$i_{k,f} \subset i, \quad k = m + 1 - a_j \delta_{m,a_j-1}.$$ 

\[\square\]

### 2.3 $\mathfrak{g}[t]^\Gamma$ is not an equivariant map algebra with free action

We now show that $\mathfrak{g}[t]^\Gamma$ is never a current algebra or more generally an equivariant map algebra with free action. For this, we recall from [15] the definition of an equivariant map algebra. Thus, let $\mathfrak{a}$ be any complex Lie algebra and $A$ a finitely generated commutative associative algebra. Assume also that $\Gamma$ is a finite abelian group acting on $\mathfrak{a}$ by Lie algebra automorphisms and on $A$ by algebra automorphisms. Then we have an induced action on the Lie algebra $(\mathfrak{a} \otimes A)$ (the commutator is given in the obvious way) such that $\gamma(x \otimes f) = \gamma x \otimes \gamma f$. An equivariant map algebra is defined to be the fixed point subalgebra:

$$(\mathfrak{a} \otimes A)^\Gamma := \{ z \in (\mathfrak{a} \otimes A) \mid \gamma(z) = z \, \forall \, \gamma \in \Gamma \}.$$ 

The finite-dimensional irreducible representations of such algebras (and hence for $\mathfrak{g}[t]^\Gamma$) were given in [15] and generalized earlier work on affine Lie algebras. In the case when $\Gamma$
acts without fixed points on \( A \), many aspects of the representation theory of the equivariant map algebra are the same as the representation theory of \( a \otimes A \) (see for instance [5]). The importance of the following proposition is now clear.

**Proposition 2.3.1.** The Lie algebra \( g[t]^{\tau} \) is not isomorphic to an equivariant map algebra \((a \otimes A)^\Gamma\) with \( a \) semisimple and \( \Gamma \) acting without fixed points on \( A \).

*Proof.* Recall our assumption that \( a_j > 1 \) and assume for a contradiction that

\[
g[t]^{\tau} \cong (a \otimes A)^\Gamma
\]

where \( a \) is semi–simple. Write \( a = a_1 \oplus \cdots \oplus a_k \) where each \( a_s \) is a direct sum of copies of a simple Lie algebra \( g_s \) and \( g_s \not\cong g_m \) if \( m \neq s \). Clearly \( \Gamma \) preserves \( a_s \) for all \( 1 \leq s \leq k \) and hence

\[
g[t]^{\tau} \cong (a \otimes A)^\Gamma \cong \oplus_{s=1}^k (a_s \otimes A)^\Gamma.
\]

Since \( g[t]^{\tau} \) is infinite–dimensional at least one of the summands \((a_s \otimes A)^\Gamma\) is infinite–dimensional, say \( s = 1 \) without loss of generality. But this means that \( \oplus_{s=2}^k (a_s \otimes A)^\Gamma \) is an ideal which is not of finite codimension which contradicts Proposition 2.2.1. Hence we must have \( k = 1 \), i.e. \( a = a_1 \). It was proven in [15, Proposition 5.2] that if \( \Gamma \) acts freely on \( A \) then any finite–dimensional simple quotient of \((a \otimes A)^\Gamma\) is a quotient of \( a \); in particular in our situation it follows that all the finite–dimensional simple quotients of \((a \otimes A)^\Gamma\) are isomorphic. On the other hand, (2.2.1) shows that \( g[t]^\tau \) has both \( g_0 \) and \( g \) as quotients. Since \( g_0 \) is not isomorphic to \( g \) we have the desired contradiction. \( \square \)
2.4 The connection with maximal parabolic subalgebras of $\hat{g}$

We now make the connection of $\mathfrak{g}[t]^{\tau}$ with a maximal parabolic subalgebra of the untwisted affine Lie algebra $\hat{g}$ associated to $\mathfrak{g}$.

Fix a Cartan subalgebra $\hat{h}$ of $\hat{g}$ containing $\mathfrak{h}$ and recall that $\hat{h} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$, where $c$ spans the one-dimensional center of $\hat{g}$ and $d$ is the scaling element. Let $\delta \in \hat{h}^*$ be the unique non-divisible positive imaginary root, i.e., $\delta(d) = 1$ and $\delta(\mathfrak{h} \oplus \mathbb{C}c) = 0$. Extend $\alpha \in \mathfrak{h}^*$ to an element of $\hat{h}^*$ by $\alpha(c) = \alpha(d) = 0$. The elements $\{\alpha_i : 1 \leq i \leq n\} \cup \{-\theta + \delta\}$ is a set of simple roots for $\hat{g}$. We define a grading on $\hat{g}$ as follows: for $r \in \mathbb{Z}$ and for each $x_\alpha \in \hat{g}_\alpha$, $x_\alpha \in \hat{g}[r]$ iff $\alpha = \sum_{i=0}^{n} r_i \alpha_i$ and $r = r_j$. The following is not hard to prove.

**Proposition 2.4.1.** Let $\hat{p}$ be the maximal parabolic subalgebra generated by the elements $x_i^\pm$, $i \in I(j)$, $x_{\pm(\delta-\theta)}$ and $x_j^+$. Then there exists an isomorphism of graded Lie algebras

$$\hat{p} \cong \mathfrak{g}[t]^{\tau}.$$ 

**Example 1.** We consider our running example of $B_n$ with $j = n$. Recall the affine Lie algebra $B_n^{(1)}$ can be realized as an extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$. For $r \in \mathbb{Z}_+$, the elements $x_{\alpha_{i,j}}^+ \otimes t^r$, $x_{\alpha_{i,j}}^- \otimes t^{(r-1)}$ and $h_i \otimes t^r$ for $1 \leq i \leq j \leq n$ form a graded basis of $\hat{p}[2r]$, and the elements $x_{\alpha_{i,n}}^- \otimes t^{r+1}$ and $x_{\alpha_{i,n}}^+ \otimes t^r$ for $1 \leq i \leq n$ form a basis of $\hat{p}[2r+1]$.

The map

$$x \otimes t^k \mapsto x \otimes t^{2r+s}, \text{ if } x \otimes t^k \in \hat{p}[2r+s]$$

gives the isomorphism in Proposition 2.4.1.
Chapter 3

The Category of Integrable Representations

In this chapter we develop the representation theory of $\mathfrak{g}[t]^{\tau}$. Following [1], [2], we define the notion of global Weyl modules, the associated commutative algebra and the local Weyl modules associated to maximal ideals in this algebra. In the case of $\mathfrak{g}[t]$ it was shown in [4] that the commutative algebra associated with a global Weyl module is a polynomial ring in finitely many variables. This is no longer true for $\mathfrak{g}[t]^{\tau}$; however we shall see that modulo the Jacobson radical, the algebra is a quotient of a finitely generated polynomial ring by a squarefree monomial ideal. As a consequence we see that under suitable conditions a global Weyl module can be finite dimensional and irreducible. More precise statements can be found in Chapter 5.
3.1 Dominant integral weights

Fix a set of fundamental weights \( \{ \lambda_i : i \in I(j) \cup \{ 0 \} \} \) for \( \mathfrak{g}_0 \) with respect to \( \Delta_0 \) and let \( P_0, P_0^+ \) be their \( \mathbb{Z} \) and \( \mathbb{Z}_+ \)-span respectively. Note that the subset

\[
P^+ = \{ \lambda \in P_0^+ : \lambda(h_j) \in \mathbb{Z}_+ \}
\]

is precisely the set of dominant integral weights for \( \mathfrak{g} \) with respect to \( \Delta \). Also note that \( P^+ \) is properly contained in \( P_0^+ \). For example, in the \( B_n \) case, \( \lambda_{n-1} \in P_0^+ \), and \( \lambda_{n-1}(h_n) = -1 \).

It is the existence of these types of weights that cause the representation theory of \( \mathfrak{g}[t]^\tau \) to be different from that of \( \mathfrak{g}[t] \).

For \( \lambda \in P_0^+ \) let \( V_{\mathfrak{g}_0}(\lambda) \) be the irreducible finite–dimensional \( \mathfrak{g}_0 \)-module with highest weight \( \lambda \) and highest weight vector \( v_{\lambda} \); if \( \lambda \in P^+ \) the module \( V_{\mathfrak{g}}(\lambda) \) and the vector \( v_{\lambda} \) are defined in the same way.

3.2 The category \( \tilde{I} \)

Let \( \tilde{I} \) be the category whose objects are \( \mathfrak{g}[t]^\tau \)-modules with the property that they are \( \mathfrak{g}_0 \) integrable and where the morphisms are \( \mathfrak{g}[t]^\tau \)-module maps. In other words an object \( V \) of \( \tilde{I} \) is a \( \mathfrak{g}[t]^\tau \)-module which is isomorphic to a direct sum of finite–dimensional \( \mathfrak{g}_0 \)-modules.

It follows that \( V \) admits a weight space decomposition

\[
V = \bigoplus_{\mu \in P_0} V_{\mu}, \quad V_{\mu} = \{ v \in V : hv = \mu(h)v, \ h \in \mathfrak{h} \},
\]

and we set \( \text{wt } V = \{ \mu \in P_0 : V_{\mu} \neq 0 \} \). Note that

\[
w \text{wt } V \subset \text{wt } V, \quad w \in W_0,
\]
where $W_0$ is the Weyl group of $g_0$. If $\text{wt } V$ is a finite set, we define the $h$–character of $V$ by

$$\text{ch}_V = \sum_{\mu \in P_0} \dim V_{\mu} e(\mu) \in \mathbb{Z}[R_0].$$

For $\lambda \in P_0^+$ we let $\tilde{I}^\lambda$ be the full subcategory of $\tilde{I}$ whose objects $V$ satisfy the condition that $\text{wt } V \subset \lambda - Q^+$; note that this is a weaker condition than requiring the set of weights be contained in $\lambda - Q_0^+$ (see Section 1.3).

**Lemma 3.2.1.** Suppose that $V$ is an object of $\tilde{I}^\lambda$ and let $\mu \in \text{wt } V$ and $\alpha \in R^+$. Then $\mu - s\alpha \in \text{wt } V$ for only finitely many $s$.

**Proof.** If $\alpha \in R_0^+$ the result is immediate since $V$ is a sum of finite–dimensional $g_0$–modules. Since $\alpha \in P_0$, it follows that there exists $w \in W_0$ such that $w\alpha$ is in the anti–dominant chamber for the action of $W_0$ on $h$. This implies that $w\alpha = -r_0\alpha_0 - \sum_{i \in I(j)} r_i \alpha_i$ where the $r_i$ are non–negative rational numbers. Since $W_0$ is a subgroup of $W$ it follows that $-w\alpha \in R^+$. This shows that if $\mu \in \text{wt } V$ is such that $\mu - s\alpha \in \text{wt } V$, then

$$w\mu - sw\alpha \in \text{wt } V \subset \lambda - Q^+.$$  

This is possible only for finitely many $s$ and hence the Lemma is established.

### 3.3 Global Weyl modules

Let

$$g = n^- \oplus h \oplus n^+, \quad n^\pm = \bigoplus_{\alpha \in R^+} g_{\pm \alpha},$$

be the triangular decomposition of $g$. Since $\tau$ preserves the subalgebras $n^\pm$ and $h$ we have

$$g[\ell]^\tau = n^-[\ell]^\tau \oplus h[\ell]^\tau \oplus n^+[\ell]^\tau.$$
Further $\mathfrak{h}[t]^\tau \cong \mathfrak{h} \otimes \mathbb{C}[t^{\tau_0}]$ is a commutative subalgebra of $\mathfrak{g}[t]^\tau$.

For $\lambda \in P^+_0$ the global Weyl module $W(\lambda)$ is the cyclic $\mathfrak{g}[t]^\tau$–module generated by an element $w_\lambda$ with defining relations: for $h \in \mathfrak{h}$ and $i \in I(j) \cup \{0\}$,

$$hw_\lambda = \lambda(h)w_\lambda, \quad n^+[t]^\tau w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}w_\lambda = 0.$$  \hfill (3.3.1)

It is elementary to check that $W(\lambda)$ is an object of $\tilde{I}_j^\lambda$, one just needs to observe that the elements $x_i^\pm, i \in I(j) \cup \{0\}$ act locally nilpotently on $W(\lambda)$. Moreover, if we declare the grade of $w_\lambda$ to be zero then $W(\lambda)$ acquires the structure of a $\mathbb{Z}_+$ graded $\mathfrak{g}[t]^\tau$–module.

### 3.4 The algebra $A_\lambda$

As in [1] one checks easily that the following formula defines a right action of $\mathfrak{h}[t]^\tau$ on $W(\lambda)$:

$$(uw_\lambda)a = uaw_\lambda, \quad u \in \mathbf{U}(\mathfrak{g}[t]^\tau), \quad a \in \mathfrak{h}[t]^\tau.$$  

Moreover this action commutes with the left action of $\mathfrak{g}[t]^\tau$. In particular, if we set

$$\Ann_{\mathfrak{h}[t]^\tau}(w_\lambda) = \{a \in \mathbf{U}(\mathfrak{h}[t]^\tau) : aw_\lambda = 0\}, \quad A_\lambda = \mathbf{U}(\mathfrak{h}[t]^\tau)/\Ann_{\mathfrak{h}[t]^\tau}(w_\lambda),$$

we get that $\Ann_{\mathfrak{h}[t]^\tau}(w_\lambda)$ is an ideal in $\mathbf{U}(\mathfrak{h}[t]^\tau)$ and that $W(\lambda)$ is a bi–module for $(\mathfrak{g}[t]^\tau, A_\lambda)$. It is clear that $\Ann_{\mathfrak{h}[t]^\tau}(w_\lambda)$ is a graded ideal of $\mathbf{U}(\mathfrak{h}[t]^\tau)$ and hence the algebra $A_\lambda$ is a $\mathbb{Z}_+$ graded algebra with a unique graded maximal ideal $I_0$.

It is obvious from the definition that we have an isomorphism of right $A_\lambda$–modules $W(\lambda)_\lambda \cong A_\lambda$. We now prove,

**Proposition 3.4.1.** For all $\lambda \in P^+_0$ the algebra $A_\lambda$ is finitely generated and $W(\lambda)$ is a finitely generated $A_\lambda$–module.
The proof of the proposition is very similar to the one given in [1] but we sketch the proof below for the reader’s convenience and also to set up some further necessary notation. Unlike in the case of $g[t]$ we will later see that the global Weyl module is not a free $A_\lambda$ module in general (see Chapter 6).

3.5 The algebra $U(h[t]^\tau)$

We need an additional result to prove Proposition 3.4.1. For $\alpha \in R^+$ and $r \in Z_+$, define elements $P_{\alpha,r} \in U(h[t]^\tau)$ recursively by

$$P_{\alpha,0} = 1, \quad P_{\alpha,r} = -\frac{1}{r} \sum_{p=1}^{r} (h_\alpha \otimes t^{p} \otimes) P_{\alpha,r-p}.$$ 

Equivalently $P_{\alpha,r}$ is the coefficient of $u^r$ in the formal power series

$$P_{\alpha}(u) = \exp \left( - \sum_{r \geq 1} \frac{h_\alpha \otimes t^{r} \otimes}{r} u^r \right).$$

Writing $h_\alpha = \sum_{i=1}^{n} a_\alpha^i h_i$, we see that

$$P_{\alpha}(u) = \prod_{i=1}^{n} P_{\alpha_i}(u)^{a_\alpha^i}, \quad \alpha \in R^+.$$ 

Set $P_{\alpha,r} = P_{i,r}$, $i \in I \cup \{0\}$. The following is now trivial from the Poincare–Birkhoff–Witt theorem.

**Lemma 3.5.1.** The algebra $U(h[t]^\tau)$ is the polynomial algebra in the variables

$$\{P_{i,r} : i \in I(\tau) \cup \{0\}, \quad r \in N\},$$

and also in the variables

$$\{P_{i,r} : i \in I, \quad r \in N\}.$$
The comultiplication \( \Delta : U(\mathfrak{g}[t]^+) \to U(\mathfrak{g}[t]^+) \otimes U(\mathfrak{g}[t]^+) \) satisfies
\[
\Delta(P_\alpha(u)) = P_\alpha(u) \otimes P_\alpha(u), \quad \alpha \in R^+.
\] (3.5.1)

For \( x \in U(\mathfrak{g}[t]^+) \), \( r \in \mathbb{Z}_+ \), set
\[
x^{(r)} = \frac{1}{r!} x^r.
\]

3.6 A Garland identity

The following can be found in [4, Lemma 1.3] and is a reformulation of a result of Garland, [9].

Lemma 3.6.1. Let \( x^+, h \) be the standard basis of \( \mathfrak{sl}_2 \) and let \( V \) be a representation of the subalgebra of \( \mathfrak{sl}_2[t] \) generated by \( (x^+ \otimes 1) \) and \( (x^- \otimes t) \). Assume that \( 0 \neq v \in V \) is such that \( (x^+_\alpha \otimes t^r)v = 0 \) for all \( r \in \mathbb{Z}_+ \). For all \( r \in \mathbb{Z}_+ \) we have
\[
(x^+ \otimes 1)^{(r)}(x^- \otimes t)^{(r)}v = (x^+ \otimes t)^{(r)}(x^- \otimes 1)^{(r)}v = (-1)^r P_r v,
\] (3.6.1)

where
\[
\sum_{r \geq 0} P_r u^r = \exp \left( - \sum_{r \geq 1} \frac{h \otimes t^r}{r} u^r \right).
\]

Further,
\[
(x^+ \otimes 1)^{(r)}(x^- \otimes t)^{(r+1)}v = (-1)^r \sum_{s=0}^{r} (x^- \otimes t^{s+1}) P_{r-s} v.
\] (3.6.2)

3.7 Proof of Proposition 3.4.1

Given \( \alpha \in R^+ \), it is easily seen that the elements \( (x^+_\alpha \otimes t^{a_j(\alpha)}) \) and \( (x^-_\alpha \otimes t^{a_j(\alpha)}) \) generate a subalgebra of \( \mathfrak{g}[t]^+ \) which is isomorphic to the subalgebra of \( \mathfrak{sl}_2[t] \) generated by
Using the defining relations of $W(\lambda)$ and equation (3.6.1) we get that

$$P_{\alpha,r}w_\lambda = 0, \quad r \geq \lambda(h_\alpha) + 1, \quad \alpha \in R_0^+.$$  \hfill (3.7.1)

It also follows from Lemma 3.2.1 that $P_{j,r}w_\lambda = 0$ for all $r >> 0$. Using Lemma 3.5.1 we see that $A_\lambda$ is finitely generated by the images of the elements

$$\{P_{i,r} : i \in I(j) \cup \{0\}, \quad r \leq \lambda(h_i)\}.$$

We now prove that $W(\lambda)$ is a finitely generated $A_\lambda$-module. Fix an enumeration $\beta_1, \ldots, \beta_M$ of $R^+$. Using the Poincare–Birkhoff–Witt theorem it is clear that $W(\lambda)$ is spanned by elements of the form $X_1X_2\cdots X_M U(h[t]^r)w_\lambda$ where each $X_p$ is either a constant or a monomial in the elements $\{(x^-_{\beta_p} \otimes t^s) : s \in a_j\mathbb{Z}_+ - a_j(\beta_p)\}$. The length of each $X_r$ is bounded by Lemma 3.2.1 and equation (3.6.2) prove that for any $\gamma \in R^+$, the element $(x^-_{\beta_p} \otimes t^{ra_j-aj(\gamma)}) U(h[t]^r)w_\lambda$, $r \in \mathbb{Z}_+$ is in the span of elements $\{(x^-_{\gamma} \otimes t^{sa_j-aj(\gamma)}) U(h[t]^r)w_\lambda : 0 \leq s \leq N\}$ for some $N$ sufficiently large. An obvious induction on the length of the product of monomials shows that the values of $s$ are bounded for each $\beta$ and the proof is complete.

**Remark 3.7.1.** Notice that the preceding argument proves that the set wt $W(\lambda)$ is finite. This is not obvious since wt $W(\lambda)$ is not a subset of $\lambda - Q_0^+$.

### 3.8 Local Weyl modules

Let $\lambda \in P_0^+$. Given any maximal ideal $I$ of $A_\lambda$ we define the local Weyl module,

$$W(\lambda, I) = W(\lambda) \otimes_{A_\lambda} A_\lambda/I.$$
It follows from Proposition 3.4.1 that \(W(\lambda, I)\) is a finite–dimensional \(g[t]^\tau\)–module in \(\tilde{L}\) and \(\dim W(\lambda, I)_\lambda = 1\). A standard argument now proves that \(W(\lambda, I)\) has a unique irreducible quotient which we denote as \(V(\lambda, I)\). Moreover, \(W(\lambda, I_0)\) is a \(\mathbb{Z}_+\)–graded \(g[t]^\tau\)–module

\[
V(\lambda, I_0) \cong \text{ev}_0^* V_{g_0}(\lambda),
\]  

(3.8.1)

where \(\text{ev}_0^* V\) is the representation of \(g[t]^\tau\) obtained by pulling back a representation \(V\) of \(g_0\).

### 3.9 Irreducible \(g[t]^\tau\)–modules

We now construct an explicit family of representations of \(g[t]^\tau\) which will be needed for our further study of \(A_\lambda\). Given non–zero scalars \(z_1, \ldots, z_k\) such that \(z_r^{a_j} \neq z_s^{a_j}\) for all \(1 \leq r \neq s \leq k\) we have a canonical surjective morphism

\[
g[t]^\tau \to g_0 \oplus g^\otimes k \to 0, \quad (x \otimes t^r) \to (\delta_{r,0} x, z_1^r x, \ldots, z_k^r x).
\]

Given a representation \(V\) of \(g\) and \(z \neq 0\), we let \(\text{ev}_z^* V\) be the corresponding pull–back representation of \(g[t]^\tau\); note that these representations are cyclic \(g[t]^\tau\)–modules. Using the recursive formulae for \(P_{\alpha, r}\) it is not hard to see that the following hold in the module \(\text{ev}_z^* V_{g}(\lambda), \lambda \in P^+\) and \(\text{ev}_0^* V_{g_0}(\mu), \mu \in P_0^+:\)

\[
n^+[t] v_\lambda = 0, \quad P_{t, r} v_\lambda = \left(\frac{\lambda(h_i)}{r}\right) (-1)^r z^{a_j} v_\lambda, \quad i \in I, \quad r \in \mathbb{N}
\]

\[
n^+[t] v_\mu = 0, \quad P_{t, r} v_\mu = 0, \quad i \in I, \quad r \in \mathbb{N}.
\]

The preceding discussion together with equation (3.5.1) now proves the following result.
Proposition 3.9.1. Suppose that $\lambda_1, \ldots, \lambda_k \in P^+$ and $\mu \in P_0^+$. Let $z_1, \ldots, z_k$ be non-zero complex numbers such that $z_r^{a_j} \neq z_s^{a_j}$ for all $1 \leq r \neq s \leq k$. Then

$$\text{ev}_0^* V_{\theta}(\mu) \otimes \text{ev}_{z_1}^* V_{\theta}(\lambda_1) \otimes \cdots \otimes \text{ev}_{z_k}^* V_{\theta}(\lambda_k)$$

is an irreducible $\mathfrak{g}[t]^\tau$-module. Moreover,

$$n^+[t]^\tau(v_\mu \otimes v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_k}) = 0, \quad (P_{i,r} - \pi_{i,r}) (v_\mu \otimes v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_k}) = 0, \quad i \in I, \quad r \in \mathbb{Z}_+,$$

where

$$\sum_{r \in \mathbb{Z}_+} \pi_{i,r} u^r = \prod_{s=1}^k (1 - z_s^{a_j} u)^{\lambda_s(h_i)}, \quad i \in I.$$

Remark 3.9.2. In particular, the modules constructed in the preceding proposition are modules of the form $V(\lambda, I)$ where $\lambda = \mu + \lambda_1 + \cdots + \lambda_k$.

The converse statement is also true; this follows from the work of [15].

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Chapter 4

The Algebra $A_\lambda$ as a Stanley–Reisner Ring

For the rest of this chapter we denote by $\text{Jac}(A_\lambda)$ the Jacobson radical of $A_\lambda$, and use freely the fact that the Jacobson radical of a finitely–generated commutative algebra coincides with its nilradical.

4.1 A presentation of $A_\lambda$

The main result of this chapter is the following.

**Theorem 4.1.1.** The algebra $A_\lambda/\text{Jac}(A_\lambda)$ is isomorphic to the algebra $\tilde{A}_\lambda$ which is the quotient of $U(\mathfrak{h}[t]^+)$ by the ideal generated by the elements

$$P_{i,s}, \ i \in I(j), \ s \geq \lambda(h_i) + 1,$$

(4.1.1)
and
\[ P_{1,r_1} \cdots P_{n,r_n}, \quad \sum_{i=1}^{n} a_i^\gamma(\alpha_0) r_i > \lambda(h_0). \] (4.1.2)

Moreover, \( \text{Jac}(A_\lambda) \) is generated by the images of the elements in (4.1.2) and \( \text{Jac}(A_\lambda) = 0 \) if \( a_j^\gamma(\alpha_0) = 1 \).

**Example 4.1.2.** We discuss the statement of the theorem in the case of \( B_n \) with \( j = n \).

Recall that in this case, \( h_0 = h_{n-1} + h_n \) and so \( a_j^\gamma(\alpha_0) = 1 \). Thus, \( \text{Jac}(A_\lambda) = 0 \) and \( A_\lambda \) is isomorphic the quotient of the polynomial algebra in the variables \( P_{i,r}, \ 1 \leq i \leq n-2, \ 1 \leq r \leq \lambda(h_i), P_{n-1,r}, \ 1 \leq r \leq \min\{\lambda(h_0), \lambda(h_{n-1})\}, P_{n,r}, \ 1 \leq r \leq \lambda(h_0) \) by the ideal generated by the elements
\[ P_{n-1,r_{n-1}} P_{n,r_n}, \quad r_{n-1} + r_n > \lambda(h_0). \]

Before proving Theorem 4.1.1 we note several interesting consequences.

### 4.2 Stanley–Reisner theory

We recall the definition of a Stanley–Reisner ring, and the correspondence between Stanley–Reisner rings and abstract simplicial complexes (for more details, see [7]).

Given a monomial \( m = x_{i_1} \cdots x_{i_\ell} \) we say that \( m \) is squarefree if \( i_1 < \cdots < i_\ell \). We say an ideal of \( \mathbb{C}[x_1, \ldots, x_n] \) is a squarefree monomial ideal if it is generated by squarefree monomials. Finally, we call a ring a Stanley-Reisner ring if it is a quotient of a polynomial ring by a squarefree monomial ideal.

The following is a consequence of Theorem 4.1.1.
Corollary 4.2.1. $A_\lambda / \text{Jac}(A_\lambda)$ is a Stanley–Reisner ring. If in addition we have $a_j(\alpha_0) = 1$, and $|\{i : a_i(\alpha_0) > 0\}| = 2$, then the algebra $A_\lambda$ is Koszul and Cohen-Macaulay.

To see how Theorem 4.1.1 implies Corollary 4.2.1, we need to understand Stanley–Reisner rings in terms of abstract simplicial complexes.

4.3 Abstract simplicial complexes

Let $k \in \mathbb{N}$ and let $X = \{x_1, \ldots, x_k\}$. An abstract simplicial complex $\Delta$ on the set $X$ is a collection of subsets of $X$ such that if $A \in \Delta$ and if $B \subset A$, then $B \in \Delta$. If $A \in \Delta$, we call $A$ a simplex, and a simplex of $\Delta$ not properly contained in another simplex of $\Delta$ is called a facet. Let $\mathcal{F}(\Delta)$ denote the set of facets of $\Delta$. The simplicial complex $\Delta$ is said to be pure if all elements of $\mathcal{F}(\Delta)$ have the same cardinality, and an enumeration $F_0, F_1, \ldots, F_p$ of $\cdot$ is called a shelling if the collection of subsets

$$\left\{ \left( \bigcup_{i=0}^{r-1} F_i \right) \cap F_r : 1 \leq r \leq p \right\}$$

is a pure abstract simplicial complex of cardinality 1 less than the cardinality of any element of $\mathcal{F}(\Delta)$.

There is a natural correspondence between squarefree monomials in $\mathbb{C}[x_1, \ldots, x_k]$ and subsets of $X$. Let $m \in \mathbb{C}[x_1, \ldots, x_k]$ be a squarefree monomial. Then we set

$$A_m := \{x_i \mid x_i \text{ divides } m\} \subset X.$$  

Conversely, for $A \subset X$, we denote by $m_A \in \mathbb{C}[x_1, \ldots, x_n]$ the squarefree monomial

$$m_A = \prod_{x_i \in A} x_i.$$
The following proposition can be found in [7], and gives a correspondence between Stanley–Reisner rings and abstract simplicial complexes.

**Proposition 4.3.1.** If \( J \subset \mathbb{C}[x_1, \ldots, x_k] \) is a squarefree monomial ideal, then the set of squarefree monomials not contained in the generating set of \( J \) by squarefree monomials forms an abstract simplicial complex, \( \Delta_J \). Conversely, for an abstract simplicial complex, which is a subset of the power set of \( X \), the Stanley–Reisner ideal \( J_{\Delta} \subset \mathbb{C}[x_1, \ldots, x_k] \) corresponding to \( \Delta \) is the squarefree monomial ideal generated by monomials \( m_A \) such that \( A \notin \Delta \). The Stanley–Reisner ring of \( \Delta \) is the quotient of \( \mathbb{C}[x_1, \ldots, x_k] \) by the Stanley–Reisner ideal \( J_{\Delta} \).

### 4.4 A shelling

By the above discussion, there is a natural correspondence between \( A_{\lambda}/\text{Jac}(A_{\lambda}) \) and an abstract simplicial complex, which we denote by \( \Delta_{\lambda} \). In order to discuss a consequence of this identification, we recall the following result from [7].

**Proposition 4.4.1.** If \( \Delta \) is pure and shellable, then the Stanley–Reisner ring of \( \Delta \) is Cohen-Macaulay.

This motivates the following Lemma.

**Lemma 4.4.2.** Assume that \( \{i : a_i(\alpha_0) > 0\} = \{s, j\} \). Then the simplicial complex \( \Delta_{\lambda} \) is pure and \( \{F_0, \ldots, F_{\min\{\lambda(h_0), \lambda(h_s)\}}\} \) defines a shelling, where

\[
F_r = \left( \prod_{i:a_i(\alpha_0) = 0 \atop 1 \leq r_i \leq \lambda(h_i)} P_{r_i} \right) P_{j, 1} \cdots P_{j, \lambda(h_0) - r} P_{s, 1} \cdots P_{s, r}, \quad 0 \leq r \leq \min\{\lambda(h_0), \lambda(h_s)\}.
\]
Proof. Let $F$ a facet of $\Delta_\lambda$, i.e., $F$ is not contained properly in another simplex of $\Delta_\lambda$. It is clear that the cardinality of $F$ is less or equal to $\sum_{i:a_i(\alpha_0)=0} \lambda(h_i) + \lambda(h_0)$. If it is strictly less, then $P_{i,r}F$ is a face of $\Delta_\lambda$ for some $i$ and $r$, which is a contradiction. Hence all facets have the same cardinality (equal to the Krull dimension of $A_\lambda$). The shelling property is straightforward to check. 

4.5 Results on Stanley–Reisner rings

Next, we collect a few immediate consequences of the correspondence between $A_\lambda/Jac(A_\lambda)$ and $\Delta_\lambda$.

Proposition 4.5.1. $A_\lambda/Jac(A_\lambda)$ is a Stanley-Reisner ring with Hilbert series

$$
\mathbb{H}(A_\lambda/Jac(A_\lambda)) = \sum_{\sigma \in \Delta_\lambda} \prod_{P_{i,r} \in \sigma} \frac{t^{a_j r}}{1-t^{a_j r}}.
$$

Moreover, if $a_j^\vee(\alpha_0) = 1$, the Krull dimension of $A_\lambda$ is given by

$$
d_\lambda = \lambda(h_0) + \sum_{i:a_i(\alpha_0)=0} \lambda(h_i).
$$

If in addition we have $|\{i : a_i(\alpha_0) > 0\}| = 2$, then the algebra $A_\lambda$ is Koszul and Cohen-Macaulay.

Proof. The statement of the Hilbert series and Krull dimension are immediate consequences of the correspondense between $A_\lambda/Jac(A_\lambda)$ and $\Delta_\lambda$. If $a_j^\vee(\alpha_0) = 1$ and $|\{i : a_i(\alpha_0) > 0\}| = 2$, $A_\lambda$ is a quotient of a polynomial algebra by a quadratic monomial ideal, and hence Koszul (see [8]). By combining Lemma 4.4.2 and [7, Theorem 5.5] we see that $A_\lambda$ is Cohen-Macaulay. 

\[\square\]
Example 4.5.2. We discuss the example $(B_n, D_n)$. Since $\alpha_0 = \alpha_{n-1} + 2\alpha_n$ and $h_0 = h_{n-1} + h_n$ we have that $A_\lambda$ is Koszul and Cohen–Macaulay.

4.6 A finite-dimensionality criterion for global Weyl modules

In this section, we note another interesting consequence of Theorem 4.1.1.

Proposition 4.6.1. Let $\lambda \in P_0^+$. Then $A_\lambda/Jac(A_\lambda)$ is either infinite–dimensional or isomorphic to $\mathbb{C}$. Moreover, the latter is true iff the following two conditions hold:

(i) for $1 \leq i \neq j \leq n$, we have $\lambda(h_i) > 0$ only if $a_i^\vee(\alpha_0) > 0$,

(ii) $\lambda(h_0) < a_i^\vee(\alpha_0)$ if $i = j$ or if $1 \leq i \neq j \leq n$ and $\lambda(h_i) > 0$.

Proof. Suppose that $\lambda$ satisfies the conditions in (i) and (ii). To prove that $\dim A_\lambda/Jac(A_\lambda) = 1$ it suffices to prove that the elements $P_{i,s} \in Jac(A_\lambda)$ for all $i \in I$ and $s \geq 1$. Assume first that $i \neq j$. If $\lambda(h_i) = 0$ then equation (3.7.1) gives $P_{i,s}w_\lambda = 0$ for all $s \geq 1$. If $\lambda(h_i) > 0$ then the conditions imply that $\lambda(h_0) < a_i^\vee(\alpha_0)$ and hence equation (4.1.2) shows that $P_{i,s} \in Jac(A_\lambda)$ for all $s \geq 1$. If $i = j$ then again the result follows from (4.1.2) and condition (ii).

We now prove the converse direction. Suppose that (i) does not hold. Then, there exists $i \neq j$ with $a_i(\alpha_0) = 0$ and $\lambda(h_i) > 0$. Equation (4.1.2) implies that the preimage of $Jac(A_\lambda)$ is contained in the ideal of $U(h[t])$ generated by the elements $\{P_{i,s} : i \in I, a_i^\vee(\alpha_0) > 0\}$. Hence, using Lemma 3.5.1 we see that the image of the elements $\{P_{i,1}^r : r \in \mathbb{N}\}$ in $A_\lambda/Jac(A_\lambda)$ must remain linearly independent showing that the algebra is infinite–dimensional.
Suppose that (ii) does not hold. Then either \( \lambda(h_0) \geq a_j^\vee(\alpha_0) \) or \( \lambda(h_0) \geq a_i^\vee(\alpha_0) \) for some \( 1 \leq i \neq j \leq n \) with \( \lambda(h_i) > 0 \). In either case (4.1.2) and Lemma 3.5.1 show that the image of the elements \( \{P_{i,1}^r : r \in \mathbb{N}\} \) in \( A^\lambda/Jac(A^\lambda) \) must remain linearly independent showing that the algebra is infinite–dimensional.

**Corollary 4.6.2.** The algebra \( A_\lambda \) is finite dimensional iff it is a local ring. It follows that \( W(\lambda) \) is finite–dimensional iff \( A_\lambda \) is a local ring.

**Proof.** If \( A_\lambda \) is finite–dimensional then so is \( A_\lambda/Jac(A_\lambda) \) and the corollary is immediate from the proposition. Conversely suppose that \( A_\lambda \) is a local ring. By the proposition and equation (3.7.1), we have

\[
P_{i,s}w_\lambda = 0, \quad \text{if} \quad a_i^\vee(\alpha_0) = 0, \quad s \in \mathbb{N}.
\]

If \( a_i^\vee(\alpha_0) \neq 0 \) we still have from (3.7.1) that \( P_{i,s}w_\lambda = 0 \) if \( s \) is sufficiently large. Otherwise, equation (4.1.2) shows that there exists \( N \in \mathbb{Z}_+ \) such that

\[
P_{i,s}w_\lambda = 0, \quad \text{for all} \quad i \in I, \quad s \in \mathbb{N}.
\]

This proves that \( A_\lambda \) is generated by finitely many nilpotent elements and since it is a commutative algebra it is finite–dimensional. The second statement of the corollary is now immediate from Proposition 3.4.1.

### 4.7 Proof of Theorem 4.1.1

We turn to the proof of Theorem 4.1.1. It follows from equation (3.7.1) that the elements in (4.1.1) map to zero in \( A_\lambda \). Until further notice, we shall prove results which are needed to show that the elements in (4.1.2) are in \( Jac(A_\lambda) \).
Given \( \alpha, \beta \in R \), with \( \ell \alpha + \beta \in R \), let \( c(\ell, \alpha, \beta) \in \mathbb{Z}\{0\} \) be such that

\[
ad_{\alpha}^\ell(x_\beta) = c(\ell, \alpha, \beta)x_{\ell\alpha+\beta}.
\]

Next, we need the following Lemma:

**Lemma 4.7.1.** Let \( \gamma \in \Delta \) and \( \beta \in R^+ \setminus \Delta \) be such that \( \beta + \gamma \notin R \) and \( (\beta, \gamma) > 0 \). Given

\( m, n, s, p, q \in \mathbb{Z}_+ \) we have

\[
(x_\gamma^+ \otimes t^p)^{(s+d)}(x_{\beta-\gamma}^+ \otimes t^m)^{(q+s)} = C(x_{s\gamma}(\beta) \otimes t^{n+d})^q(x_\gamma^+ \otimes t^p)^{(s)}(x_\beta^- \otimes t^{m+n})^s + X
\]

where \( X \in U(g[t]_\gamma) U(n_\gamma [t]_\gamma) \) and \( (d_\gamma)^q C = c(d_\gamma, \gamma, -\beta)^q c(1, \beta - \gamma, -\beta)^s \).

**Proof.** We first induct on \( s \) and show

\[
(x_{\beta-\gamma}^+ \otimes t^m)^{(k+s)}(x_\beta^- \otimes t^n)^{(s)} = c(1, \beta-\gamma, -\beta)^s(x_\beta^- \otimes t^n)^{(k)}(x_\gamma^- \otimes t^{m+n})^s \mod U(g[t]_\gamma) U(n_\gamma [t]_\gamma).
\]

(4.7.1)

For the \( s = 1 \) case, we have

\[
(x_{\beta-\gamma}^+ \otimes t^m)(x_\beta^- \otimes t^n)^{(k+1)} = (x_\beta^- \otimes t^n)^{(k+1)}(x_{\beta-\gamma}^+ \otimes t^m) + c(1, \beta-\gamma, -\beta)(x_\beta^- \otimes t^n)^{(k)}(x_\gamma^- \otimes t^{n+m})
\]

(4.7.2)

since \( \beta + \gamma \notin R^+ \). For the inductive step, let \( s > 1 \) and consider

\[
(x_{\beta-\gamma}^+ \otimes t^m)^{(s)}(x_\beta^- \otimes t^n)^{(k)} = \frac{1}{s}(x_{\beta-\gamma}^+ \otimes t^m)(x_{\beta-\gamma}^+ \otimes t^m)^{(s-1)}(x_\beta^- \otimes t^n)^{(s-1+k+1)}.
\]

(4.7.3)

By the inductive hypothesis, (4.7.3) equals

\[
\frac{c(1, \beta-\gamma, -\beta)^{s-1}}{s}(x_{\beta-\gamma}^+ \otimes t^m)(x_\beta^- \otimes t^n)^{(k+1)}(x_\gamma^- \otimes t^{m+n})^{(s-1)} \mod U(g[t]_\gamma) U(n_\gamma [t]_\gamma).
\]

(4.7.4)
Finally, since $\beta \notin \Delta$, (4.7.4) equals

$$c(1, \beta - \gamma, -\beta) s(x_{\beta} \otimes t^n)^{(k)}(x_{\gamma} \otimes t^{m+n})^{(s)} \mod U[g[t]^\tau]U(n^+ [t]^\tau)_+,$$

(4.7.5)

proving (4.7.1).

The proof of the lemma now follows from the equality (modulo $U[g[t]^\tau]U(n^+ [t]^\tau)_+$)

$$(x_{\gamma}^+ \otimes p)^{(s+d,k)}(x_{\beta} \otimes t^n)^{(k)}(x_{\gamma}^- \otimes t^{m+n})^{(s)} = c(d_{\gamma}, \gamma, -\beta)^k \frac{(s+d_{\gamma})!}{d_{\gamma}!} (x_{\gamma}^- \otimes t^{n+d_{\gamma}p})^{(k)}(x_{\gamma}^+ \otimes t^p)^{(s)}(x_{\gamma}^- \otimes t^{m+n})^{(s)},$$

(4.7.6)

which we prove by induction on $k$. For the case $k = 1$ consider

$$(x_{\gamma}^+ \otimes t^p)^{(s+d_{\gamma})}(x_{\beta} \otimes t^n)(x_{\gamma}^- \otimes t^{m+n})^{(s)}.$$

(4.7.7)

Recall from [11] that for any two elements $Z, Y \in g[t]^\tau$ and $\ell \in \mathbb{N}$ we have the following identity in $U[g[t]^\tau]$:

$$[Y^\ell, Z] = \sum_{q=1}^{\ell} \binom{\ell}{q} \text{ad}_{Y^q}(Z)Y^{\ell-q}.$$

(4.7.8)

Applying (4.7.8) to (4.7.7) gives

$$\frac{1}{(s+d_{\gamma})!} \binom{s+d_{\gamma}}{d_{\gamma}} c(d_{\gamma}, \gamma, -\beta)(x_{\gamma}^- \otimes t^{n+d_{\gamma}p})^{(k)}(x_{\gamma}^+ \otimes t^p)^{(s)}(x_{\gamma}^- \otimes t^{m+n})^{(s)} \mod U[g[t]^\tau]U(n^+ [t]^\tau)_+$$

(4.7.9)

which equals

$$\frac{c(d_{\gamma}, \gamma, -\beta)}{(d_{\gamma}!)} (x_{\gamma}^- \otimes t^{n+d_{\gamma}p})(x_{\gamma}^+ \otimes t^p)^{(s)}(x_{\gamma}^- \otimes t^{m+n})^{(s)} \mod U[g[t]^\tau]U(n^+ [t]^\tau)_+,$$

(4.7.10)

proving the base case. For the inductive step, let $k > 1$ and consider

$$(x_{\gamma}^+ \otimes t^p)^{(s+d_{\gamma}k)}(x_{\beta} \otimes t^n)^{(k)}(x_{\gamma}^- \otimes t^{m+n})^{(s)}$$

(4.7.11)
equals
\[ \frac{1}{(s + d_r k)!} (x_\gamma^+ \otimes t^p)^{s + d_r k} (x_\beta^- \otimes t^n)^{k-1} (x_\gamma^- \otimes t^{m+n})^s. \]  (4.7.12)

By (4.7.8), (4.7.12) equals
\[ \frac{c(d, \gamma, -\beta)}{(s + d_r k)!} (x_\gamma^- \otimes t^{n+d_r p}) (x_\gamma^+ \otimes t^p)^{s + d_r (k-1)} (x_\gamma^- \otimes t^n)^{k-1} (x_\gamma^- \otimes t^{m+n})^s \]  (4.7.13)

modulo \( U(g[t]^-) U(n^+[t]^-)_+ \) which equals
\[ \frac{c(d, \gamma, -\beta)}{d_r!} (x_\gamma^- \otimes t^{n+d_r p}) (x_\gamma^+ \otimes t^p)^{s + d_r (k-1)} (x_\gamma^- \otimes t^n)^{k-1} (x_\gamma^- \otimes t^{m+n})^s, \]  (4.7.14)

modulo \( U(g[t]^-) U(n^+[t]^-)_+ \). Applying the induction hypothesis to (4.7.14) yields
\[ \frac{c(d, \gamma, -\beta)}{(d_r!^k)} (x_\gamma^- \otimes t^{n+d_r p}) (x_\gamma^+ \otimes t^p)^{s + d_r (k-1)} (x_\gamma^- \otimes t^n)^{k-1} (x_\gamma^- \otimes t^{m+n})^s, \]  (4.7.15)

modulo \( U(g[t]^-) U(n^+[t]^-)_+ \), and we have (4.7.6), finishing the lemma.

\[ \square \]

It is immediate that under the hypothesis of the Lemma we have for all \( P \in U(h[t]^-) \) that
\[ (x_\gamma^+ \otimes t^p)^{s + d_r q} (x_\beta^- \otimes t^m)^{q+s} (x_\gamma^- \otimes t^{m+n})^s P w_\lambda \]  (4.7.16)

= \[ C (x_{s_\gamma(\beta)}^- \otimes t^{n+d_r p})^{q} (x_\gamma^+ \otimes t^p)^{s} (x_\gamma^- \otimes t^{m+n})^s P w_\lambda, \]

for some \( C \neq 0. \)

### 4.8 The recursion

Recall that given any root \( \beta \in R^+ \) we can choose \( \alpha \in \Delta \) with \( (\beta, \alpha) > 0 \). Moreover if \( \beta \notin \Delta \) and \( \beta \) is long then \( \beta + \alpha \notin R \). Setting \( \alpha_{i_0} = \alpha_j \), \( \beta_0 = \alpha_0 \), we set \( \beta_1 = s_{i_0} \beta_0 \) and
note that $\beta_1 \in R^+$. If $\beta_1 \notin \Delta$ then we choose $\alpha_{i_1} \in \Delta$ with $(\beta_1, \alpha_{i_1}) > 0$ and set $\beta_2 = s_{i_1} \beta_1$. Repeating this if necessary we reach a stage when $k \geq 1$ and $\beta_k \in \Delta$. In this case we set $\alpha_{i_k} = \beta_k$. We claim that

$$|\{0 \leq r \leq k : i_r = i\}| = a_i^\vee(\alpha_0), \quad 1 \leq i \leq n. \quad (4.8.1)$$

To see this, notice that since the $\beta_p$ are long roots, we have $h_{\beta_p} = h_{\beta_{p-1}} - h_{i_{p-1}}$. Hence,

$$h_0 = \sum_{s=0}^{k} h_{i_s} = \sum_{i=1}^{n} a_i^\vee(\alpha_0) h_i.$$

Equating coefficients gives (4.8.1).

### 4.9 \textbf{Ann}_{\mathfrak{h}[t]^*} w_\lambda

Retain the notation of Section 4.8. We now prove that

$$P_{i_k, s_k} \cdots P_{i_0, s_0} w_\lambda = 0, \quad \text{if} \quad (s_0 + \cdots + s_k) \geq \lambda(h_0) + 1. \quad (4.9.1)$$

We begin with the equality

$$w = (x_0^- \otimes 1)^{(s_0 + \cdots + s_k)} w_\lambda = 0, \quad (s_0 + \cdots + s_k) \geq \lambda(h_0) + 1,$$

which is a defining relation for $W(\lambda)$. Recalling that $j = i_0$ and setting

$$X_1 = (x_j^+ \otimes t)^{(s_0 + d_{\alpha_j}(s_1 + \cdots + s_k))}(x_{\alpha_0 - \alpha_j} \otimes t_{\alpha_j}^{-1})^{(s_0)}$$

by applying (4.7.16) we get

$$0 = X_1 w = (x_{\beta_1}^- \otimes t_{\alpha_j})^{(s_1 + \cdots + s_k)} P_{i_0, s_0} w_\lambda.$$

More generally, if we set

$$X_{r+1} = (x_{\alpha_i}^+ \otimes t_{\delta_{i,r}})^{(s_r + d_{\alpha_i}(s_{r+1} + \cdots + s_k))}(x_{\beta_r - \alpha_i} \otimes t_{\alpha_i}^m)^{(s_r)},$$
where \( m_r = a_j - \delta_{i_r,j} - d_{a_j} \{ 0 \leq q < r \mid i_q = j \} \) we find after repeatedly applying (4.7.16) that

\[
0 = (x^+_{\beta_k} \otimes t^{s_k}) X_k \cdots X_1 w = P_{i_k,s_k} \cdots P_{i_0,s_0} w_\lambda = 0.
\]

This proves the assertion.

4.10 Surjectivity

We can now prove that

\[
P_{1,r_1} \cdots P_{n,r_n} \in \text{Jac}(A_\lambda) \quad \text{if} \quad \sum_{i=1}^n a^\vee_i(\alpha_0) r_i > \lambda(h_0).
\]

Taking \( s_p = r_m \) whenever \( i_p = m \) in (4.9.1) and using (4.8.1) we see that

\[
P_{1,r_1} P_{n,r_n}^a \cdots P_{n,r_n} \in \text{Jac}(A_\lambda) \quad \text{if} \quad \sum_{i=1}^n a^\vee_i(\alpha_0) r_i > \lambda(h_0).
\]

Multiplying through by appropriate powers of \( P_{i,r_i} \), \( 1 \leq i \leq n \) we get that for some \( s \geq 0 \) we have

\[
P_{1,r_1}^s \cdots P_{n,r_n}^s w_\lambda = 0, \quad \text{if} \quad \sum_{i=1}^n a^\vee_i(\alpha_0) r_i > \lambda(h_0).
\]

Hence \( P_{1,r_1}^s \cdots P_{n,r_n}^s = 0 \) in \( A_\lambda \) proving that \( P_{1,r_1} \cdots P_{n,r_n} \in \text{Jac}(A_\lambda) \). This argument proves that there exists a well-defined morphism of algebras

\[
\varphi : \tilde{A}_\lambda \to A_\lambda / \text{Jac}(A_\lambda).
\]

We now prove,

Lemma 4.10.1. If \( a^\vee_j(\alpha_0) = 1 \) the map \( \varphi \) factors through \( A_\lambda \), i.e., we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{A}_\lambda & \to & A_\lambda / \text{Jac}(A_\lambda) \\
\downarrow & & \\
A_\lambda & \to & A_\lambda \end{array}
\]

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Proof. Using (4.10.1) it suffices to prove that if \( a_j^\vee(\alpha_0) = 1 \) then
\[
a_i^\vee(\alpha_0) \leq 1 \quad \forall i \in I.
\]
Since \( a_j(\alpha_0) = a_j \geq 2 > a_j^\vee(\alpha_0) = 1 \) we see that \( g \) cannot be of simply laced type and hence \( \alpha_j \) is short. It follows that \( s_{\alpha_0} \alpha_j = \alpha_j - \alpha_0 \) is also short and so \( h_{\alpha_0 - \alpha_j} = d_j h_0 - h_j \). If \( a_j^\vee(\alpha_0) > 1 \) for some \( i \neq j \), then we would have
\[
a_i^\vee(\alpha_0 - \alpha_j) = d_j a_i^\vee(\alpha_0) \geq 2d_j.
\]
Since \( \alpha_j \) is short this is impossible unless \( g \) is of type \( F_4 \) and \( j = 4 \). This case can be handled by an inspection.

\[\square\]

4.11 Injectivity

Using Lemma 4.10.1 and (4.10.2) we see that the proof of Theorem 4.1.1 is complete if we show that the map (4.10.2) is injective. Since \( \tilde{A}_\lambda \) is a quotient of \( U(h[t])^\pi \) by a square–free ideal, it has no nilpotent elements and thus \( \text{Jac}(\tilde{A}_\lambda) = 0 \). So if \( f \) is a nonzero element in \( \tilde{A}_\lambda \), there exists a maximal ideal \( \tilde{I}_f \) of \( \tilde{A}_\lambda \) so that \( f \notin \tilde{I}_f \). Therefore, by Lemma 3.5.1 we can choose a tuple \( (\pi_{i,r}) \), \( i \in I, \ r \in \mathbb{N} \) satisfying the relations (4.1.1) and (4.1.2) such that under the evaluation map sending \( P_{i,r} \) to \( \pi_{i,r} \) the element \( f \) is mapped to a non–zero scalar.

Define \( z_1, \ldots, z_k \) and \( \lambda_1, \ldots, \lambda_k \in P^+ \) by
\[
\pi_i(u) = 1 + \sum_{r \in \mathbb{N}} \pi_{i,r} u^r = \prod_{s=1}^{k} (1 - z_a^s u)^{\lambda_s(h_i)}, \quad i \in I
\]
and set \( \mu = \lambda - (\lambda_1 + \cdots + \lambda_k) \in P_0 \). In what follows we show that \( \mu \in P_0^+ \). Since \( (\pi_{i,r}) \) satisfies the relations in (4.1.1) we have that \( \mu(h_i) \in \mathbb{Z}_+ \) for \( i \in I(j) \). Moreover, since

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\((\pi_{i,r})\) satisfies (4.1.2) we get \(\mu(h_0) \in \mathbb{Z}_+\). To see this, note that the coefficient of \(u^r\) in
\[\prod_{i \in I} \pi_i(u)^{a_i^\vee(\alpha_0)}\]
is given by
\[
\sum_{(r_{ik})} \prod_{i \in I} \prod_{k=1}^{a_i^\vee(\alpha_0)} \pi_{i, r_{ik}},
\]
where the sum runs over all tuples \((r_{ik})\) such that \(\sum_{i \in I} \sum_{k=1}^{a_i^\vee(\alpha_0)} r_{ik} = r\). Set \(r_i = \max\{r_{ik}, 1 \leq k \leq a_i^\vee(\alpha_0)\}\), \(i \in I\) and observe that if \(r > \lambda(h_0)\), then
\[
\sum_{i \in I} a_i^\vee(\alpha_0) r_i \geq r > \lambda(h_0)
\]
and hence (4.11.1) vanishes. It follows that
\[
\mu(h_0) = \lambda(h_0) - \deg(\prod_{i \in I} \pi_i(u)^{a_i^\vee(\alpha_0)}) \in \mathbb{Z}_+.
\]

Now using Proposition 3.9.1 we have a quotient of \(W(\lambda)\) where \(f\) acts by a non–zero scalar on the highest weight vector. Hence \(f^N \notin \text{Ann}_{\mathfrak{n}\mathfrak{g}[t]}(w_\lambda)\) for all \(N \geq 1\), i.e. the image of \(f\) under the map (4.10.2) is non–zero. This proves the map (4.10.2) is injective, and so Theorem 4.1.1 is established.
Chapter 5

Finite–Dimensional Global Weyl Modules

In this chapter we give necessary and sufficient conditions for the global Weyl module to be irreducible.

5.1 Finite–dimensional global Weyl modules

Recall the elements $\theta_k \in R^+, 0 \leq k \leq a_j - 1$ defined in Section 1.3.

Theorem 5.1.1. Given $\lambda \in P_0^+$, the module $W(\lambda)$ is an irreducible $g[t]$–module and hence isomorphic to $ev^*_0 V_{\theta_0}(\lambda)$ iff the following hold:

$$\lambda(h_0) = 0 \text{ and } \lambda(h_i) > 0 \text{ only if } a_i(\theta_{a_j-1}) = a_i(\alpha_0).$$

(5.1.1)

The proof of the theorem can be found in the rest of the chapter.
Example 5.1.2. We discuss the finite dimensional irreducible global Weyl modules for the example \((B_n, D_n)\). In this case recall \(\theta_1 = \alpha_1 + \cdots + \alpha_n\) and \(\alpha_0 = \alpha_{n-1} + 2\alpha_n\). Thus, \(W(\lambda)\) is an irreducible \(\mathfrak{g}[t]^\tau\)-module and hence isomorphic to \(\text{ev}^*_0 V_{\mathfrak{g}_0}(\lambda)\) if and only if \(\lambda = r\lambda_{n-1}\) for \(r \in \mathbb{Z}_+\).

5.2 Irreducibility and the grading

Suppose that \(\lambda\) satisfies the conditions of the theorem. Notice that \(W(\lambda) \cong \text{ev}^*_0 V_{\mathfrak{g}_0}(\lambda) \iff \mathfrak{g}[t]^\tau[s]w_\lambda = 0, \ s \in \mathbb{N}\).

Recall from Section 1.4 that \(\theta_{a_j-1} + \alpha_j - \alpha_0 = \sum_{i \in I(j) \cup \{0\}} s_i \alpha_i \in \mathbb{R}_0^+ \cup \{0\}\). It follows that if \(s_i > 0\) then \(a_i(\theta_{a_j-1} - \alpha_0) \neq 0\) and hence by our assumptions on \(\lambda\) we have \(\lambda(h_i) = 0\), i.e. \((\lambda, \theta_{a_j-1} + \alpha_j - \alpha_0) = 0\). If \(\theta_{a_j-1} + \alpha_j - \alpha_0 \in \mathbb{R}_0^+\), using the defining relations of \(W(\lambda)\) we get

\[
(x_{\theta_{a_j-1} - \alpha_0 + \alpha_j} - 1)w_\lambda = 0.
\]

Since \(\lambda(h_0) = 0\) we now have

\[
0 = (x_{\theta_{a_j-1} - \alpha_0 + \alpha_j} + 1)(x_{\alpha_j} \otimes t^{ra_j+1})(x_{\alpha_0} \otimes 1)w_\lambda = (x_{\theta_{a_j-1} - \alpha_0 + \alpha_j} - 1)(x_{\alpha_0 - \alpha_j} \otimes t^{ra_j+1})w_\lambda = (x_{\theta_{a_j-1}} \otimes t^{ra_j+1})w_\lambda.
\]

By Proposition 1.4.1 and the discussion in Section 1.4 we know that \(\mathfrak{g}_1\) is an irreducible \(\mathfrak{g}_0\)-module generated by \(x_{\theta_{a_j-1}}\) by applying elements \(x_i^+\), \(i \in I(j) \cup \{0\}\) and so

\[
(\mathfrak{g}_1 \otimes t\mathbb{C}[t^{a_j}])w_\lambda = 0.
\]

Assume that \((\mathfrak{g}_m \otimes t^m\mathbb{C}[t^{a_j}])w_\lambda = 0\) for all \(m\) with \(1 \leq m < k \leq a_j\). Since \(1 \leq k - m < k\), we also have \((\mathfrak{g}_{k-m} \otimes t^{k-m}\mathbb{C}[t^{a_j}])w_\lambda = 0\) by our induction hypothesis. Now by Proposition 1.4.1
we have $\mathfrak{g}_k = [\mathfrak{g}_{k-m}, \mathfrak{g}_m]$ if $k \leq a_j - 1$ and $\mathfrak{g}_k = [\mathfrak{g}_1, \mathfrak{g}_{a_j-1}]$ if $k = a_j$ and hence using the induction hypothesis we obtain

$$ (\mathfrak{g}_k \otimes t^k \mathbb{C}[t^{a_j}]) w_\lambda = 0, $$

proving that $W(\lambda)$ is irreducible.

5.3 Proof of Theorem 5.1.1, the forward direction

We prove the forward direction of Theorem 5.1.1 in the rest of the chapter for which we need some additional results.

Lemma 5.3.1. Let $\mu, \nu \in P_0^+$ and assume that $W(\nu)$ is reducible. Then $W(\mu + \nu)$ is also reducible.

Proof. It is easily seen using the defining relations of $W(\mu + \nu)$ that we have a map of $\mathfrak{g}[t]^{\tau}$-modules $W(\mu + \nu) \rightarrow \text{ev}_0^* V_{\mathfrak{g}_0}(\mu) \otimes W(\nu)$ extending the assignment $w_{\mu+\nu} \rightarrow v_\mu \otimes w_\nu$.

Since $W(\nu)$ is reducible there exists $x \in \mathfrak{g}[t]^\tau[s]$, $s \geq 1$ with $xw_\nu \neq 0$. Since $xv_\mu = 0$, we now get $x(v_\mu \otimes w_\nu) = v_\mu \otimes xw_\nu \neq 0$. Hence $0 \neq xw_{\mu+\nu} \in W(\mu + \nu)[s]$ and the result follows. 

5.4 Reducible global Weyl modules

Suppose that $\lambda, \mu \in P_0^+$ are such that there exists $0 \neq \Phi \in \text{Hom}_{\mathfrak{g}_0}(\mathfrak{g}_1 \otimes V_{\mathfrak{g}_0}(\lambda), V_{\mathfrak{g}_0}(\mu))$. Then $V := V_{\mathfrak{g}_0}(\lambda) \oplus V_{\mathfrak{g}_0}(\mu)$ admits a $\mathfrak{g}[t]^{\tau}$-structure extending the canonical $\mathfrak{g}_0$-structure as follows:

$$(x \otimes 1)(v_1, v_2) = (xv_1, xv_2), \quad (y \otimes t)(v_1, v_2) = (0, \Phi(y \otimes v_1)), \quad \mathfrak{g}[t]^\tau[s](v_1, v_2) = 0, \quad s \geq 2,$$
where \((v_1, v_2) \in V, x \in \mathfrak{g}_0\) and \(y \in \mathfrak{g}_1\). It is easily checked that if \(\lambda - \mu \in Q^+\) then \(V\) is a quotient of the global Weyl module \(W(\lambda)\).

**Proposition 5.4.1.** The global Weyl module \(W(\lambda_i)\) is not irreducible if \(i = 0\) or \(i \in I(j)\) with \(a_i(\theta_{a_j-1}) \neq a_i(\alpha_0)\).

**Proof.** Recall that \(w_0\) is the longest element in the Weyl group defined by the simple roots \(\{\alpha_i : i \in I(j)\}\) and note that \(w_0\theta_{a_j-1} \in R_{a_j-1}^+\). It follows that \(\alpha_0 + w_0\theta_{a_j-1} \notin R\) and hence \(0 \leq w_0\theta_{a_j-1}(h_0) \leq 1\). Setting \(\mu_0 = \lambda_0 - w_0\theta_{a_j-1}\) we see that \(\mu_0(h_i) = -w_0(\theta_{a_j-1})(h_i) \geq 0\) for \(i \in I(j)\) and \(\mu_0(h_0) = 1 - w_0\theta_{a_j-1}(h_0) \geq 0\), i.e. \(\mu_0 \in P_0^+\). Since \(\mathfrak{g}_1\) is an irreducible \(\mathfrak{g}_0\)-module of lowest weight \(-\theta_{a_j-1}\), the PRV theorem \([?], [?]\) implies that \(V_{\mathfrak{g}_0}(\mu_0)\) is a direct summand of \(\mathfrak{g}_1 \otimes V_{\mathfrak{g}_0}(\lambda_0)\). It follows from the discussion preceding the proposition that \(W(\lambda_0)\) is not irreducible.

It remains to consider a node \(i \in I(j)\) with \(a_i(\theta_{a_j-1}) \neq a_i(\alpha_0)\). By way of contradiction suppose that \(W(\lambda_i)\) is irreducible. Using Proposition 4.6.1(ii) we can assume that \(a_i(\alpha_0) > 0\). Moreover, by Section 1.3 we know that \(\theta_{a_j-1} - \alpha_0 + \alpha_j \in R_0^+\) and hence \(a_i(\theta_{a_j-1}) > a_i(\alpha_0) > 0\). If \(\mathfrak{g}\) is of classical type, this is only possible if \(a_i(\theta_{a_j-1}) = 2, a_i(\alpha_0) = 1\) and hence we have a pair of roots of the form

\[
\alpha_0 = \cdots + \alpha_i + \cdots + 2\alpha_j + \cdots, \quad \theta_{a_j-1} = \cdots + 2\alpha_i + \cdots + \alpha_j + \cdots
\]

which is a contradiction. In other words, there is no such node if \(\mathfrak{g}\) is of classical type. If \(\mathfrak{g}\) is of exceptional type, a case by case analysis shows that \(W(\lambda_i)\) is not irreducible using the discussion preceding the proposition. \(\square\)
5.5 Proof of Theorem 5.1.1, the backward direction

Assume that $\lambda$ violates one of the conditions in (5.1.1), i.e. $\lambda(h_i) > 0$ where $i = 0$ or $i \in I(j) \text{ and } a_i(\theta_{a_{i-1}}) \neq a_i(\alpha_0)$. Now setting $\mu = \lambda - \lambda_i$ and $\nu = \lambda_i$ in Lemma 5.3.1 and using Proposition 5.4.1 we see that $W(\lambda)$ is not irreducible which completes the proof of Theorem 5.1.1.
Chapter 6

The Structure of Local Weyl Modules

Recall from Chapter 2 that the equivariant map algebra $\mathfrak{g}[t]^\tau$ is not isomorphic to an equivariant map algebra where the group, $\Gamma$, acts with fixed points on the commutative algebra, $A$. When $\Gamma$ acts without fixed points on $A$, the finite dimensional representation theory of the equivariant map algebra is closely related to that of the map algebra $\mathfrak{g} \otimes A$ (see for instance [5]). We have already seen a major difference between the finite dimensional representation theory of $\mathfrak{g}[t]^\tau$ and that of $\mathfrak{g}[t]$. Specifically, in Section 5 we showed that unlike in the case of the current algebra, the global Weyl module for $\mathfrak{g}[t]^\tau$ can be finite dimensional and irreducible for nontrivial dominant integral weights.

In this chapter we discuss the structure of local Weyl modules for the case of $(B_n, D_n)$ where $\lambda$ is a multiple of a fundamental weight, in which case $A_\lambda$ is a polynomial algebra. We finish the chapter by discussing the complications in determining the structure of local
Weyl modules for an arbitrary weight $\lambda \in P_0^+$. The simplest example is the case of $\omega_{n-1} = \lambda_0 + \lambda_{n-1}$. Note that in this case $A_{\omega_{n-1}}$ is not a polynomial algebra.

6.1 Local Weyl modules for the current algebra

Recall that we have a well-established theory of local Weyl modules for the current algebra $g[t]$. Given $\lambda \in P^+$ we denote by $W^g_{\text{loc}}(\lambda)$, $\lambda \in P^+$ the $g[t]$–module generated by an element $w_\lambda$ and defining relations

$$n^+[t]w_\lambda = 0, \quad (h \otimes t^r)w_\lambda = \delta_{r,0}\lambda(h)w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}w_\lambda = 0.$$ 

We remind the reader that $\{\omega_i : 1 \leq i \leq n\}$ is a set of fundamental weights for $g$ with respect to $h$. The following was proved in [3],[6] and [13].

$$\dim W^g_{\text{loc}}(\lambda) = \prod_{i=1}^n \dim (W^g_{\text{loc}}(\omega_i))^{m_i}, \quad \lambda = \sum_{i=1}^n m_i\omega_i \in P^+ \quad (6.1.1)$$

We can clearly regard $W^g_{\text{loc}}(\lambda), \lambda \in P^+$ as a graded $g[t]^\tau$ module by restriction, however it is not the case that this restriction gives a local Weyl module for $g[t]^\tau$. The relationship between local Weyl modules for $g[t]^\tau$ and the restriction of local Weyl modules for $g[t]$ is more complicated, as we now explain.

6.2 An isomorphism of quotients

Given $z \in \mathbb{C}^\times$ we have a isomorphism of Lie algebras $\eta_z : g[t] \to g[t]$ given by $(x \otimes t^r) \to (x \otimes (t-z)^r)$ and let $\eta_z^*V$ be the pull–back through this homomorphism of a representation $V$ of $g[t]$. Suppose that $V$ is such that there exists $N \in \mathbb{Z}_+$ with $(g \otimes t^m)V = 0$ for all $m > N$. Then $(g \otimes (t-z)^m)\eta_z^*V = 0$ for all $m > N$. In particular we can regard the module
\( \eta^*_z V \) as a module for the finite-dimensional Lie algebra \( g \otimes \mathbb{C}[t]/(t - z)^N \). Following [5], since \( z \in \mathbb{C}^\times \) we have

\[
\frac{g[t]}{g \otimes (t - z)^N \mathbb{C}[t]} \cong \frac{g[t]^\tau}{(g \otimes (t - z)^N \mathbb{C}[t])^\tau},
\]

so if \( V \) is a cyclic module for \( g[t] \) then \( \eta^*_z V \) is a cyclic module for \( g[t]^\tau \).

We now need a general construction. Given any finite-dimensional cyclic \( g[t]^\tau \)–module \( V \) with cyclic vector \( v \) define an increasing filtration of \( g_0 \)–modules

\[
0 \subset V_0 = U(g[t]^\tau)[0]v \subset \cdots \subset V_r = \sum_{s=0}^r U(g[t]^\tau)[s]v \subset \cdots \subset V.
\]

The associated graded space \( \text{gr} V \) is naturally a graded module for \( g[t]^\tau \) via the action

\[
(x \otimes t^s)\overline{w} = (x \otimes t^s)w, \quad \overline{w} \in V_r/V_{r-1}.
\]

Suppose that \( v \) satisfies the relations

\[
n^+[t]^\tau v = 0, \quad (h \otimes t^{2k})v = d_k(h)v, \quad d_k(h) \in \mathbb{C}, \quad k \in \mathbb{Z}_+, \quad h \in \mathfrak{h}.
\]

Then since \( \dim V < \infty \) it follows that \( d_0(h) \in \mathbb{Z}_+ \); in particular there exists \( \lambda \in P_0^+ \) such that \( d_0(h) = \lambda \) and a simple checking shows that \( \text{gr} V \) is a quotient of \( W_{\text{loc}}(\lambda) \).

The following is now immediate.

**Lemma 6.2.1.** Let \( \lambda \in P^+ \) and \( z \in \mathbb{C}^\times \). The \( g[t]^\tau \)–module \( \text{gr} \left( \eta^*_z W_{\text{loc}}^q(\lambda) \right) \) is a quotient of \( W_{\text{loc}}(\lambda) \) and hence

\[
\dim W_{\text{loc}}(\lambda) \geq \dim W_{\text{loc}}^q(\lambda).
\]

\[\square\]
6.3 Multiples of fundamental weights for the case \((B_n, D_n)\)

Until stated otherwise, we consider the case of \((B_n, D_n)\), and study local Weyl modules corresponding to weights \(r\lambda_i \in P_0^+\), where \(r \in \mathbb{Z}_+\), and \(0 \leq i \leq n - 2\) (the \(i = n - 1\) case is discussed in Chapter 5, where these local Weyl modules are shown to be finite dimensional and irreducible). In particular, we show the reverse of the inequality in Lemma 6.2.1, which proves the following proposition.

**Proposition 6.3.1.** Assume that \((g, g_0)\) is of type \((B_n, D_n)\). For \(0 \leq i \leq n - 2\) and \(r \in \mathbb{Z}_+\) we have an isomorphism of \(g[t]^r\)-modules

\[
W_{loc}(r\lambda_i) \cong \text{gr} \left( \eta_\tau^r W_{loc}(\lambda) \right).
\]

6.4 Results on local Weyl modules for the current algebra

We recall standard results for local Weyl modules for the current algebra \(g[t]\).

**Proposition 6.4.1.** (i) Let \(x, y, h\) be the standard basis for \(sl_2\) and set \(y \otimes t^r = y^r\). For \(\lambda \in P^+\) the local Weyl module \(W_{sl_2}(\lambda)\) has basis

\[
\{ w_{\lambda}, \ y_{r_1} \cdots y_{r_k} w_{\lambda} : 1 \leq k \leq n, \ 0 \leq r_1 \leq \cdots \leq r_k \leq \lambda(h) - k \}.
\]

(ii) Assume that \(g\) is of type \(B_n\) (resp. \(D_n\)) and assume that \(i \neq n\) (resp. \(i \neq n - 1, n\)). Then

\[
W_{sl_2}(\omega_i) \cong V_0(\omega_i) \oplus V(\omega_{i-2}) \oplus \cdots \oplus V(\omega_1),
\]

where

\[
V(\omega_i) = V(\omega_i), \ \text{i odd}, \ V(\omega_i) = \mathbb{C}, \ \text{i even}.
\]
(iii) Assume that \( g \) is of type \( B_n \) (resp. \( D_n \)), and let \( i = n \) (resp. \( i \in \{ n - 1, n \} \)). Then

\[
W^g_{\text{loc}}(\omega_i) \cong g V^g_{\text{loc}}(\omega_i).
\]

We remind the reader that

\[
\text{dim } V^g_{\text{loc}}(\omega_i) = \begin{cases} 
(2n+1), & g = B_n, \ i \neq n \\
C(n,i), & g = D_n, \ i \neq n - 1, n.
\end{cases}
\]

Moreover, if \( g \) is of type \( B_n \),

\[
\text{dim } V^g_{\text{loc}}(\omega_n) = 2^n,
\]

and if \( g \) is of type \( D_n \) and \( i \in \{ n - 1, n \} \), then

\[
\text{dim } V^g_{\text{loc}}(\omega_i) = 2^{n-1}.
\]

We remind the reader of our assumption that \((g, g_0)\) is of type \((B_n, D_n)\), and \(0 \leq i \leq n-2\). Our goal is to prove that

\[
\text{dim } W^g_{\text{loc}}(r\omega_i) \geq \text{dim } W^g_{\text{loc}}(r\omega_i), \quad r \in \mathbb{N}.
\]

The proof needs several additional results, and we consider the cases \(1 \leq i \leq n - 1\) and \(i = 0\) separately.

Recall that \( g_0[t^2] \subset g[t]^r \), and so \( W_{\text{loc}}(r\omega_i) \) can be regarded at a \( g_0[t^2] \)-module by pulling back along the inclusion map \( g_0[t^2] \hookrightarrow g[t]^r \).
Lemma 6.4.2.  (i) For $1 \leq i \leq n - 2$, $W_{\text{loc}}(r\omega_i)$ is generated as a $\mathfrak{g}_0[t^2]$-module by $w_r$ and $YW_r$ where $Y$ is a monomial in the the elements

$$(x_p^- \otimes t^{2s+1})w_r, \ p \leq i, \ 0 \leq s < r.$$

(ii) $W_{\text{loc}}(r\omega_0)$ is generated as a $\mathfrak{g}_0[t^2]$-module by $w_r$ and $YW_r$ where $Y$ is a monomial in the the elements

$$(x_{p,n}^- \otimes t^{2s+1})w_r, \ p \leq n, \ 0 \leq s < r.$$

Proof. For ease of notation we denote the element $w_{r\omega_i}$ by $w_r$. First, for $1 \leq i \leq n - 1$ the defining relation $x_0^-w_r = 0$ implies that

$$(x_0^- \otimes t^{2s})w_r = (x_{n-1}^- \otimes t^{2s})w_r = (x_n^- \otimes t^{2s+1})w_r = 0, \ s \geq 0.$$ 

Since $x_p^-w_r = 0$ if $p \neq i$ it follows that

$$(x_j^- \otimes t^{2p+1})w_r = 0, \ s \geq 0, \ p > i.$$ 

Observe also that

$$(x_1^-)^{r+1}w_r = 0 \implies (x_i^- \otimes t^{2s})w_r = 0, \ s \geq r,$$

and hence we also have that

$$(x_{p,n}^- \otimes t^{2s+1})w_r = 0, \ s \geq r, \ p \leq i.$$ 

A simple application of the PBW theorem now gives (i).

For the case $i = 0$, we have

$$(x_k^- \otimes t^{2s})w_r = 0, \ 1 \leq k \leq p \leq n - 1, \ s \geq 0.$$ 

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The relation \((x_0^-)^{t+1}w_r = 0\) for \(s \geq r\) implies that

\[(x_0^- \otimes t^{2s})w_r = 0, \quad s \geq r\]

and so

\[(x_n^- \otimes t^{2s+1})w_r = 0, \quad s \geq r;\]

and

\[(x_{p,n}^- \otimes t^{2s+1})w_r = 0, \quad 1 \leq p \leq n, \quad s \geq r.\]

(ii) is now clear.

\[\square\]

### 6.5 A generating set

**Lemma 6.5.1.**  
(i) For \(1 \leq i \leq n - 2\), suppose that \(Y = (x_{p_1,n}^- \otimes t^{2s_1+1}) \cdots (x_{p_k,n}^- \otimes t^{2s_k+1})\)

where \(p_1 \leq \cdots \leq p_k \leq i\). Then \(Yw_\lambda\) is in the \(g_0[t^2]\)–module generated by elements \(Zw_r\)

where \(Z\) is a monomial in the elements \((x_{i,n}^- \otimes t^{2s+1})\) with \(0 \leq s < r\).

(ii) For \(i = 0\), suppose that \(Y = (x_{p_1,n}^- \otimes t^{2s_1+1}) \cdots (x_{p_k,n}^- \otimes t^{2s_k+1})\) where \(p_1 \leq \cdots \leq p_k \leq n\).

Then \(Yw_\lambda\) is in the \(g_0[t^2]\)–module generated by elements \(Zw_r\) where \(Z\) is a monomial in the elements \((x_n^- \otimes t^{2s+1})\) with \(0 \leq s < r\).

**Proof.** First, let \(1 \leq i \leq n - 1\). The proof proceeds by an induction on \(k\). If \(k = 1\) and \(p_1 < i\) then by setting

\[x_{p_1,n}^- = [x^-_{p_1,i-1}, x^-_{i,n}]\]

we have

\[x^-_{p_1,i-1}(x_{i,n}^- \otimes t^{2s_1+1})w_r = (x^-_{p_1,n} \otimes t^{2s_1+1})w_r,\]

for \(0 \leq s < r\).
hence induction begins. For the inductive step, we observe that

$$(x_{p,n}^- \otimes t^{2^{s+1}})U(g_0[t^2]) \subset U(g_0[t^2]) \oplus \sum_{m \geq 0} \sum_{p=1}^{n} U(g_0[t^2])(x_{p,n}^- \otimes t^{2m+1}),$$

and hence it suffices to prove that for all $1 \leq p \leq n$ and $Z$ a monomial in $(x_{i,n}^- \otimes t^{2s+1})$ we have that $(x_{p,n}^- \otimes t^{2m+1})Zw_r$ is in the $g_0[t^2]$-submodule generated by elements $Z'w_r$ where $Z'$ is a monomial in $(x_{i,n}^- \otimes t^{2s+1})$. Denote this submodule by $M$ and consider

$$(x_{p,i-1}^- \otimes t^{2^m})(x_{i,n}^- \otimes t)^{\ell+1}w_r = (\ell+1)(x_{p,n}^- \otimes t^{2m+1})(x_{i,n}^- \otimes t)^{\ell}w_r + (x_{p,i}^- \otimes t^{2m+2})(x_{i,n}^- \otimes t)^{\ell-1}w_r.$$ 

Since

$$(x_{p,i-1}^- \otimes t^{2^m})(x_{i,n}^- \otimes t)^{\ell+1}w_r \in M,$$

and

$$(x_{p,i}^- \otimes t^{2m+2})(x_{i,n}^- \otimes t)^{\ell-1}w_r \in M,$$

we have,

$$(x_{p,n}^- \otimes t)(x_{i,n}^- \otimes t)^{\ell}w_r \in M.$$ 

In order to show

$$(x_{p,n}^- \otimes t^{2m+1})(x_{i,n}^- \otimes t^{2r_1+1}) \cdots (x_{i,n}^- \otimes t^{2r_\ell+1})w_r \in M$$

we let $h \in \mathfrak{h}$ with $[h, x_{p,n}^-] = 0$ and $[h, x_{i,n}^-] \neq 0$. Then

$$(h \otimes t^{2s})(x_{p,n}^- \otimes t^{2m+1})(x_{i,n}^- \otimes t) \cdots (x_{i,n}^- \otimes t)w_r \in M$$

for all $s \geq 0$. An induction on $\{|1 \leq t \leq \ell : r_t \neq 0|\}$ finishes the proof for $1 \leq i \leq n-1$. The $i = 0$ case is identical.
6.6 More on the $g_0[t^2]$–structure

Observe that the subalgebra $a[t^2]$ generated by the elements $x_i^+ \otimes t^{2s}$, $s \in \mathbb{Z}_+$ is isomorphic to the current algebra $sl_2[t^2]$. Hence $U(a[t^2])w_r \subset W_{\text{loc}}(r\omega_i)$ is a quotient of the local Weyl module for $a[t^2]$ with highest weight $r$ and we can use the results of Proposition 6.4.1(i).

We now prove,

Lemma 6.6.1.  (i) For $1 \leq i \leq n - 1$, as a $g_0[t^2]$–module $W_{\text{loc}}(r\omega_i)$ is spanned by $w_r$ and elements

$$Y(i, s)w_r := (x_i, n \otimes t^{2s_1+1}) \cdots (x_i, n \otimes t^{2s_k+1})w_r,$$

for $k \geq 1, s \in \mathbb{Z}_+^k, 0 \leq s_1 \leq \cdots \leq s_k \leq r - k$.

(ii) For $i = 0$, as a $g_0[t^2]$–module $W_{\text{loc}}(r\lambda_0)$ is spanned by $w_r$ and elements

$$Y(n, s)w_r := (x_n \otimes t^{2s_1+1}) \cdots (x_n \otimes t^{2s_k+1})w_r,$$

for $k \geq 1, s \in \mathbb{Z}_+^k, 0 \leq s_1 \leq \cdots \leq s_k \leq r - k$.

Proof. First, we consider the case $1 \leq i \leq n - 2$ and suppose that $Y$ is an arbitrary monomial in the elements $(x_{i,n}^- \otimes t^{2s+1})$, $s \in \mathbb{Z}_+$. We proceed by induction on the length $k$ of $Y$. If $k = 1$, then we have

$$(x_{i,n}^- \otimes t^{2s+1})w_r = (x_{i+1,n}^- \otimes t)(x_i^- \otimes t^{2s})w_r = 0, \quad s \geq r,$$

by Proposition 6.4.1(i). This shows that induction begins. Suppose now that $k$ is arbitrary and $s \in \mathbb{Z}_+^k$. Then, by induction on $k$

$$(x_{i+1,n}^- \otimes t^k(x_i^- \otimes t^{2s_1}) \cdots (x_i^- \otimes t^{2s_k}) = A(x_{i,n}^- \otimes t^{2s_1+1}) \cdots (x_{i,n}^- \otimes t^{2s_k+1}) + X + Z, \quad (6.6.1)$$

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where $A$ is a non-zero complex number and $X \in \sum_{m<k} \sum_{p \in \mathbb{Z}} U(g_0[t^2]Y(i; p))$, and $Z \in U(g_0[t^2]Y(i + 1; s'))$ and so $Zw_r = 0$.

To see (6.6.1) we proceed by induction on $k$. For the base case, we have

\[
(x_{i+1,n}^- \otimes t)(x_i^- \otimes t^{2s_1}) = x_{i,n}^- \otimes t^{2s_1 + 1} + (x_i^- \otimes t^{2s_1}) (x_{i+1,n}^- \otimes t),
\]

so induction begins. For the inductive step, we have

\[
(x_{i+1,n}^- \otimes t)^k (x_i^- \otimes t^{2s_1}) \cdots (x_i^- \otimes t^{2s_k}) = (x_{i+1,n}^- \otimes t)^{k-1} (x_{i+1,n}^- \otimes t)(x_i^- \otimes t^{2s_1}) \cdots (x_i^- \otimes t^{2s_k})
\]

\[
= (x_{i+1,n}^- \otimes t)^{k-1} \sum_{m=1}^{k} (x_i^- \otimes t^{2s_1}) \cdots (x_i^- \otimes t^{2s_m})(x_i^- \otimes t^{2s_m})(x_{i,n}^- \otimes t^{2s_{m+1}}).
\]

Applying the inductive hypothesis finishes the proof of (6.6.1).

To finish the proof of the Lemma for $1 \leq i \leq n - 2$, we use (6.6.1) to write

\[
(x_{i,n}^- \otimes t^{2s_1}) \cdots (x_{i,n}^- \otimes t^{2s_k})w_r = (x_{i+1,n}^- \otimes t)^k (x_i^- \otimes t^{2s_1}) \cdots (x_i^- \otimes t^{2s_k})w_r - Xw_r.
\]

The inductive hypothesis applies to $Xw_r$. By Proposition 6.4.1 we can write

\[
(x_{i+1,n}^- \otimes t)^k (x_i^- \otimes t^{2s_1}) \cdots (x_i^- \otimes t^{2s_k})w_r
\]

as a linear combination of elements where $s_p \leq r - k$. Applying (6.6.1) once again to each summand finishes the proof for $1 \leq i \leq n - 2$.

The case $i = 0$, is similar, using the identity

\[
(x_n^- \otimes t^{2s+1})w_r = (x_{n-1,n}^+ \otimes t)(x_0^- \otimes t^{2s})w_r = 0, \quad s \geq r,
\]

for the induction to begin, and

\[
(x_{n-1,n}^+ \otimes t)^k (x_0^- \otimes t^{2s_1}) \cdots (x_0^- \otimes t^{2s_k})w_r = A(x_n^- \otimes t^{2s_1+1}) \cdots (x_n^- \otimes t^{2s_k+1})w_r
\]

for the inductive step.
6.7 Proof of Proposition 6.3.1

We now prove Proposition 6.3.1, first for $1 \leq i \leq n - 2$. Fix an ordering on the elements $Y(i; s)w_r$, $s \in Z^k_+$ and $s_p \leq r - k$ as follows: the first element is $w_r$ and an element $Y(i; s)$ precedes $Y(i; s')$ if $s \in Z^k_+$ and $s' \in Z^m_+$ if either $k < m$ or $k = m$ and $s_1 + \cdots + s_k > s'_1 + \cdots + s'_k$ and let $u_1, \ldots, u_\ell$ be an ordered enumeration of this set. Denote by $U_p$ the $g_0[t^2]$–submodule of $W_{\text{loc}}(r\omega_i)$ generated by the elements $u_m$, $m \leq p$. It is straightforward to see that we have an increasing filtration of $g_0[t^2]$–modules:

$$U_1 \subset \cdots \subset U_\ell = W_{\text{loc}}(r\omega_i).$$

Moreover $U_p/U_{p+1}$ is a submodule of the local Weyl module for $g_0[t^2]$ with highest weight $(r - i_p)\omega_i + i_p\omega_{i-1}$, if $u_p = Y(i, s)$, $s \in Z_{i_p}^+$. Using equation (6.1.1) and Proposition 6.4.1(ii) we get

$$\dim U_p/U_{p+1} \leq \binom{2n-i_p}{i} \binom{2n}{i-1}^{i_p}.$$

Summing we get

$$\dim W_{\text{loc}}(r\omega_i) \leq \sum_{s=0}^r \binom{r}{s} \binom{2n-i_p}{i-s} \binom{2n}{i-1}^s = \left(\frac{2n-i_p}{i} + \frac{2n}{i-1}\right)^r = \binom{2n}{i}^r.$$

For the $i = 0$ case, $U_p/U_{p+1}$ is a submodule of the local Weyl module for $g_0[t^2]$ with highest weight $(r - i_p)\omega_n + i_p\omega_{n-1}$, if $u_p = Y(n, s)$, $s \in Z_{i_p}^+$. Using equation (6.1.1) and Proposition 6.4.1(iii) we get

$$\dim U_p/U_{p+1} \leq (2^{n-1})^{r-i_p}(2^{n-1})^{i_p}.$$

Summing we get

$$\dim W_{\text{loc}}(r\omega_i) \leq \sum_{s=0}^r \binom{r}{s} (2^{n-1})^{r-s}(2^{n-1})^s = (2^{n-1})^r 2^r = (2^n)^r.$$
Since we have already proved that the reverse equality holds the proof of Proposition 6.3.1 is complete.

6.8 Fundamental weights when $a_j^\vee(\alpha_0) = 1$

Our goal is to prove the following proposition:

**Proposition 6.8.1.** For $a_j^\vee(\alpha_0) = 1$, and for $i \in I(j)$ such that $\lambda_i \in P_0^+$ we have

$$\dim W_{\text{loc}}(\lambda_i) \leq \dim W_{\text{loc}}^g(\lambda_i),$$

and so

$$W_{\text{loc}}(\lambda_i) \cong \text{gr} \left( \eta_2 W_{\text{loc}}^g(\lambda) \right),$$

as in Lemma 6.2.1.

We prove the proposition on a case-by-case basis. For notational convenience, in sections 6.8-6.10 we set $\lambda_j = \lambda_0$.

First, consider the $(C_n, C_j \times C_{n-j})$ case and fix $j \in \{1, \ldots, n-1\}$. Recall that the positive roots of $g$ are of the form

$$\alpha_{r,s} := \alpha_r + \cdots + \alpha_s, \quad \alpha_{i,p} := \alpha_i + \cdots + \alpha_{p-1} + 2\alpha_p + \cdots + 2\alpha_{n-1} + \alpha_n,$$

where $1 \leq r \leq s \leq n-1$ and $1 \leq i \leq p \leq n$. We record the following lemma:

**Lemma 6.8.2.** Let $k \in \{0, \ldots, n\}$ and set $\lambda = \lambda_k + \sum_{i=j+1}^n m_i \lambda_i$ for $m_i \in \mathbb{Z}_+$. We further assume that $k < j$ if $m_i \neq 0$ for some $i \in \{j+1, \ldots, n\}$. Then we have

1. $(x_{\alpha_{i,j}} \otimes t^2)w_\lambda = 0$, and $(x_{\alpha_{i,p}} \otimes t^2)w_\lambda = 0$, if $j \notin \{i, \ldots, p\}$.
(2) \((x_{\alpha, i}^\sim \otimes t)w_\lambda = 0\), if \(i \leq p \leq j\) and \(k \notin \{i, \ldots, p\}\).

(3) \((x_{\alpha, i}^\sim \otimes t^3)w_\lambda = (x_{\alpha, i}^\sim \otimes t)^2w_\lambda = 0\), if \(i \leq k \leq j \leq p\).

(4) \((x_{\alpha, i}^\sim \otimes t^2)w_\lambda = 0\), if \(j < i\) or \(p \leq j \leq n - 1\)

(5) \((x_{\alpha, i}^\sim \otimes t^3)w_\lambda = (x_{\alpha, i}^\sim \otimes t)^2w_\lambda = 0\), if \(i \leq k \leq j \leq p - 1\)

(6) \((x_{\alpha, i}^\sim \otimes t)w_\lambda = 0\), if \(i \leq j \leq p - 1\), \(k \notin \{i, \ldots, j\}\).

Proof. Part (1) of the lemma follows immediately from the Garland identities if \(j \geq i\) or \(j < i\) and \(m_j + 1 = \cdots = m_n = 0\). Otherwise it follows from (recall that \(k < j\) in that case)

\[
0 = (x_{\alpha, i, j}^+ \otimes t)(x_{\alpha, j}^+ \otimes 1)w_\lambda = (x_{\alpha, i}^\sim \otimes t^2)w_\lambda
\]

and

\[
0 = (x_{\alpha, j}^+ \otimes t^2)(x_{\alpha, i, j}^+ \otimes t)(x_{\alpha, j}^+ \otimes 1)w_\lambda = (x_{\alpha, i}^\sim \otimes t^2)w_\lambda.
\]

The second part of the lemma is implied by

\[
0 = (x_{\alpha, i, j}^\sim \otimes t)(x_{\alpha, i, j}^+ \otimes t^2)(x_{\alpha, j}^+ \otimes 1)w_\lambda = (x_{\alpha, i}^\sim \otimes t)w_\lambda,
\]

Let \(\epsilon = 2\) if \(k \neq j\) and \(\epsilon = 0\) otherwise. The first statement in part (3) follows from

\[
0 = (x_{\alpha, i, j}^\sim \otimes t^3)(x_{\alpha, j}^+ \otimes t^2)(x_{\alpha, j}^+ \otimes t^{2\delta_{k,j}})w_\lambda = (x_{\alpha, i}^\sim \otimes t^3)w_\lambda,
\]

and the second statement from the following calculations: if \(k = j\)

\[
0 = (x_{\alpha, i, j}^\sim \otimes 1)^2(x_{\alpha, j}^+ \otimes t^2)(x_{\alpha, j}^+ \otimes 1)^2w_\lambda = (x_{\alpha, i, j}^\sim \otimes 1)^2(x_{\alpha, j}^+ \otimes t)(x_{\alpha, j}^+ \otimes t)(x_{\alpha, j}^+ \otimes 1)w_\lambda
\]

\[
= (x_{\alpha, i, j}^\sim \otimes 1)^2(x_{\alpha, j}^+ \otimes t)^2w_\lambda = (x_{\alpha, i}^\sim \otimes t)^2w_\lambda
\]
and if $k \neq j$

$$0 = (x_{\alpha_{j,p}}^+ \otimes t)(x_{\alpha_{j,\tau}}^- \otimes 1)w_{\lambda} = (x_{\alpha_{j,\mu}}^+ \otimes t)(x_{\alpha_{j,\tau}}^- \otimes 1)w_{\lambda} = (x_{\alpha_{j,\mu}}^- \otimes t^2)w_{\lambda}.$$ 

Now we prove part (4). If $j < i$ and $k \neq j$ we have

$$0 = (x_{\alpha_{i-1,p}}^+ \otimes t)(x_{\alpha_{j,\tau}}^- \otimes 1)w_{\lambda} = (x_{\alpha_{i,\mu}}^- \otimes t^2)w_{\lambda}$$

and if $k = j < i$ (recall that $m_i = 0$ in this case) we obviously have $(x_{\alpha_{i,\mu}}^- \otimes t^2)w_{\lambda_k} = 0$.

Now suppose that $p \leq j \leq n - 1$. The statements is clear from (1), since

$$0 = [(x_{\alpha_{i,j}}^- \otimes t^2), (x_{\alpha_{i,p-1}}^+ \otimes 1)]w_{\lambda} = (x_{\alpha_{i,\mu}}^- \otimes t^2)w_{\lambda}.$$ 

Part (5) is a consequence of

$$0 = (x_{\alpha_{i,j-1}}^- \otimes t^2(1-\delta_{k,j}))(x_{\alpha_{i,p-1}}^+ \otimes t)(x_{\alpha_{j,\tau}}^- \otimes t^{2\delta_{k,j}})w_{\lambda} = (x_{\alpha_{i,\mu}}^- \otimes t^3)w_{\lambda}$$

and

$$0 = (x_{\alpha_{i,p}}^+ \otimes t)^2(x_{\alpha_{i,j}}^- \otimes 1)^2w_{\lambda_k} = (x_{\alpha_{i,p}}^+ \otimes t)(x_{\alpha_{i,\mu}}^- \otimes t)(x_{\alpha_{j,\tau}}^- \otimes 1)w_{\lambda} = (x_{\alpha_{i,\mu}}^- \otimes t^2)w_{\lambda},$$

since $(x_{\alpha_{i,p}}^- \otimes t^2)w_{\lambda} = 0$ by part (1). The last part can be deduced from

$$0 = (x_{\alpha_{i,j-1}}^- \otimes 1)(x_{\alpha_{i,p-1}}^+ \otimes t)(x_{\alpha_{j,\tau}}^- \otimes 1)w_{\lambda} = (x_{\alpha_{i,\mu}}^- \otimes t)w_{\lambda_k}.$$ 

The lemma gives the following trivial proposition,

**Proposition 6.8.3.** Let $\lambda$ is as in Lemma 6.8.2, then

(1) The space $U(n^-[t]^\tau)_+w_{\lambda}$ is contained in

$$\sum_{p=j+1}^{n-1} U(n^-[t]^\tau)(x_{k,p}^- \otimes t)w_{\lambda} + \sum_{p=j+1}^{n} U(n^-[t]^\tau)(x_{k,p}^- \otimes t)w_{\lambda} + U(n^-[t]^\tau)(x_{k,j}^- \otimes t)w_{\lambda}$$

(6.8.1)
(2) Fix \( s \in \mathbb{N} \) and \( \gamma_1, \ldots, \gamma_s \in \mathbb{R}^+ \) such that \( \alpha_k, \alpha_j \in \text{supp}(\gamma_i) \) for each \( 1 \leq i \leq s \). Then for any permutation \( \sigma \) we have

\[
\prod_{i=1}^{s} (x_{\gamma_i}^- \otimes t) w_\lambda = \prod_{i=1}^{s} (x_{\gamma_{\sigma(i)}^-} \otimes t) w_\lambda.
\]

(3) \( W(\mu, I_0) \) is an irreducible \( g_0 \)-module if \( \mu(h_i) = 0 \) for \( 0 \leq i \leq j \).

**Remark 6.8.4.** If \( \lambda \) is as in Lemma 6.8.2, then by Proposition 6.8.3(1) we obtain

\[
U(\mathfrak{g}[t]^\ast)w_\lambda \subset U(\mathfrak{g}[t]^\ast)Xw_\lambda,
\]

where \( X = (x_{k,j}^- \otimes t) \) if \( m_p = 0 \) for all \( p > j \) and else \( X = (x_{k,p}^- \otimes t) \) with \( p = \min\{p : m_p \neq 0\} \).

To be more precise, it is easy to show that (6.8.1) is contained in \( U(\mathfrak{g}[t]^\ast)Xw_\lambda \).

We now determine the \( g_0 \)-structure of \( W(\lambda_k, I_0) \) for \( 1 \leq k \leq p \) (recall part (3) of Proposition 6.8.3). For any \( 0 \leq \ell \leq k \) we define a set \( Q_\ell := \{(q, \ell) \in \mathbb{Z}_+^2 : q \leq \min\{\lfloor \frac{\ell}{2} \rfloor, n - j - \lceil \frac{\ell}{2} \rceil\}\} \). Note that \( Q_\ell \) can be the emptyset. We further define for all \((q, \ell) \in Q_\ell\) vectors as follows:

\[
X(q, \ell)w_{\lambda_k} := (x_{k-\ell+1,j+2q+\text{res}_2(\ell)+1}^- \otimes t)(x_{k-\ell+2,j+2q+\text{res}_2(\ell)+2}^- \otimes t) \ldots \]

\[
\ldots (x_{k-\lfloor \frac{\ell}{2} \rfloor-q,j+q+[\frac{\ell}{2}]}^- \otimes t)(x_{k-\lfloor \frac{\ell}{2} \rfloor-q+1,j+q+1+[\frac{\ell}{2}]-1}^- \otimes t) \ldots (x_{k-1,j+1}^- \otimes t) (x_{k,j}^- \otimes t) w_{\lambda_k}
\]

Note that \( X(q, \ell)w_{\lambda_k} \) has weight \( \lambda_k - \ell + (1 - \delta_{q+\text{res}_2(\ell)}, 0)\lambda_j + 2q + \text{res}_2(\ell) \). Our aim is to show that these vectors parametrize the \( g_0 \)-highest weight vectors in \( W(\lambda_k, I_0) \). First we prove,

**Proposition 6.8.5.** Let \( 1 \leq k \leq j \). We have

\[
W(\lambda_k, I_0) = \sum_{0 \leq \ell \leq k, \ 0 \leq q \leq \min\{\lfloor \frac{\ell}{2} \rfloor, n - j - \lceil \frac{\ell}{2} \rceil\}} U(\mathfrak{g}_0)X(q, \ell)w_{\lambda_k}.
\]
Proof. We define a partial order $\prec$ on $Q_\ell$. Say $(q, \ell) \prec (q', \ell)$ if and only if $\lambda_{j+2q+\mathrm{res}_2(\ell)} - \lambda_{j+2q' + \mathrm{res}_2(\ell)} \in Q^+$. The following three properties are easily proven using Lemma 6.8.2

1. $U(b[t]t^\tau)X(q, \ell)w_\lambda = 0$, $\forall (q, \ell) \in Q_\ell$

2. $YX(q, \ell)w_\lambda \neq 0$, with $Y \in n^+[t]^\tau$ implies $\deg(Y) = 0$ and $\alpha_j \notin \text{supp}(Y)$

3. $U(n^+[t]^\tau)X(q, \ell)w_{\lambda_k} = 0$, if $q + 1 + \text{res}_2(\ell) > \lceil \frac{\ell}{2} \rceil$.

We claim that

$$U(n^+[t]^\tau)X(q, \ell)w_{\lambda_k} \in \sum_{(q', \ell) \prec (q, \ell)} U(n^-[t]t^\tau)X(q', \ell)w_{\lambda_k}. \quad (6.8.3)$$

The proof of (6.8.3) is by induction on $\ell$, where the $\ell = 0$ case is clear; so assume $\ell > 0$ and fix $Y \in n^+[t]^\tau$ arbitrary. Let $p_i(X(q, \ell)w_{\lambda_k})$ be the vector obtained from $X(q, \ell)w_{\lambda_k}$ be removing the first $i$ root vectors. It is easy to check that $p_i(X(q, \ell)w_{\lambda_k}) = X(q, \ell - i)w_{\lambda_k}$ for some $(q, \ell - i) \in Q_{\ell-i}$. We abbreviate $p_i(X(q, \ell)w_{\lambda_k}) = X(q, \ell - 1)w_{\lambda_k}$ and set $Y' := [Y, (x_{k-\ell+1, j+2q+\mathrm{res}_2(\ell) + 1}^\tau \otimes t)]$. Our induction hypothesis gives

$$YX(q, \ell)w_{\lambda_k} = Y'X(q, \ell - 1)w_{\lambda_k} + \sum_{(q', \ell-1) \prec (q, \ell-1)} U(n^-[t]t^\tau)X(q', \ell - 1)w_{\lambda_k}. \quad (6.8.4)$$

By (2) we can assume that $Y' \in n^-[t]^\tau$ and by (3) we suppose $q + 1 + \text{res}_2(\ell) \leq \lceil \frac{\ell}{2} \rceil$.

By (1) and our induction hypothesis we know that $X(q, \ell - 1)w_{\lambda_k}$ satisfies the defining relations of $W(\lambda_{k-\ell+1} + \lambda_{j+2q+\mathrm{res}_2(\ell) - 1}, I_0)$ modulo lower terms. Hence we get with Proposition (6.8.3)(1)

$$Y'X(q, \ell - 1)w_{\lambda_k} \in U(n^-[t]^\tau)(x_{k-\ell+1, j+2q+\mathrm{res}_2(\ell) - 1}^\tau \otimes t)X(q, \ell - 1)w_{\lambda_k} \quad (6.8.4)$$

$$+ U(n^-[t]^\tau)(x_{k-\ell+1, j+2q+\mathrm{res}_2(\ell) - 1}^\tau \otimes t)X(q, \ell - 1)w_{\lambda_k} + \text{"lower terms"}. \quad (6.8.5)$$
Note that we can ignore the action with \((x_{k-\ell+1,j}^- \otimes t)X(q', \ell - 1)w_{\lambda_k}\), since it vanishes by Lemma 6.8.2. We know that the weight of \(X(\bar{q}, \ell - 1)w_{\lambda_k}\) is given by \(\lambda_{k-\ell+1} + \lambda_j + 2\bar{q} + \text{res}_{2(\ell - 1)}\) which is only possible if \(X(\bar{q}, \ell - 1)w_{\lambda_k}\) starts with a root vector of the form \((x_{\bullet,j}^- + 2\bar{q} + \text{res}_{2(\ell - 1)} \otimes t)\) or contains a root vector of the form \((x_{\bullet,j}^- + 2\bar{q} + \text{res}_{2(\ell - 1)} \otimes t)\) and starts with \((x_{\bullet,j}^- + 2\bar{q} + \text{res}_{2(\ell - 1)} + 1 \otimes t)\). In the latter case, the vector in (6.8.5) vanishes by Lemma 6.8.2 and the vector in (6.8.4) is of the form \(X(T)\) for some element \(T \in \mathcal{Q}_\ell\). In the first case both vectors are of the form \(X(T)\) for some element \(T \in \mathcal{Q}_\ell\). By weight reasons we must have \(T \prec (q, \ell)\) in either case.

Therefore,

\[
YX(q, \ell)w_\lambda \subseteq \sum_{(q', \ell) \prec (q, \ell)} \mathbf{U}(n^-[t]^+)X(q', \ell)w_{\lambda_k} + \sum_{(q', \ell - 1) \prec (\bar{q}, \ell - 1)} \mathbf{U}(n^-[t]^+)X(q', \ell - 1)w_{\lambda_k},
\]

where the terms in the first sum have the property \(p_1(X(q', \ell)w_{\lambda_k}) = p_1(X(q, \ell)w_{\lambda_k})\). By repeating the above procedure (finitely many times) with each summand of the second sum we finish the proof of (6.8.3). Especially we proved the following: \(X(q, \ell)\) satisfies the defining relations of the local Weyl module modulo lower terms and

\[
\mathbf{U}(n^-[t]^+)X(q, \ell)w_{\lambda_k} \subseteq \sum_{T \in \mathcal{Q}_{\ell + 1}} \mathbf{U}(n^-[t]^+)X(T)w_{\lambda_k} + "\text{lower terms}".
\]

So (6.8.3) and Lemma 6.8.2 implies

\[
\mathbf{U}(g[t]^+)X(q, \ell)w_{\lambda_k} \subseteq \mathbf{U}(g_0)X(q, \ell)w_{\lambda_k} + \mathbf{U}(g[t]^+)X(q, \ell)w_{\lambda_k} = \mathbf{U}(g_0)X(q, \ell)w_{\lambda_k} + \sum_{T \in \mathcal{Q}_{\ell + 1}} \mathbf{U}(g[t]^+)X(T)w_{\lambda_k} + \sum_{(q', \ell) \prec (q, \ell)} \mathbf{U}(g[t]^+)X(q', \ell)w_{\lambda_k}
\]

We repeat the above procedure with the third sum we get

\[
\mathbf{U}(g[t]^+)X(q, \ell)w_{\lambda_k} \subseteq \sum_{T \in \mathcal{Q}_\ell} \mathbf{U}(g_0)X(T)w_{\lambda_k} + \sum_{T' \in \mathcal{Q}_{\ell + 1}} \mathbf{U}(g[t]^+)X(T')w_{\lambda_k}.
\]

Repeating the above procedure, together with Proposition 6.8.3 finishes the proof. \qed
Next, we need a bit more notation. Recall that $g_0$ is a finite-dimensional semisimple Lie algebra and hence we have

$$W(\lambda_k, I_0) \cong_{g_0} \bigoplus_{\lambda \in P(k)} V_{g_0}(\lambda).$$

**Proposition 6.8.6.** Let $k \in I$. We have

1. $P(k) = P(k)[0] = \{\lambda_k\}$, $j < k \leq n$

2. If $1 \leq k \leq j$, then

$$P(k)[\ell] = \begin{cases} \{\lambda_k - \ell + (1 - \delta_{q + res_2(\ell), 0})\lambda_j + 2q + res_2(\ell) : 0 \leq q \leq \min\{\lfloor \frac{\ell}{2} \rfloor, n - j - \lceil \frac{\ell}{2} \rceil\}\}, & \text{if } 0 \leq \ell \leq k, \\ \emptyset, & \text{else.} \end{cases}$$

Note that Proposition 6.8.1 is a consequence of the proposition.

**Proof.** From Proposition 6.8.3 we immediately get $P(k) = \{\lambda_k\}$ for $k > j$. So assume from now on $1 \leq k \leq j$. By degree reasons we obtain from (6.8.3)

$$U(n^+_0)X(q, \ell)w_{\lambda_k} \in \sum_{(q', \ell') < (q, \ell)} U(n^-_0)X(q', \ell)w_{\lambda_k}. \quad (6.8.6)$$

Hence $X(q, \ell)w_{\lambda_k}$ satisfies the defining relations of the irreducible $g_0$-module $V_{g_0}(\mu)$ up to a filtration, where $\mu = \lambda_k - \ell + (1 - \delta_{q + res_2(\ell), 0})\lambda_j + 2q + res_2(\ell)$. So using Proposition 6.8.5 we can construct a filtration of $W(\lambda_k, I_0)[\ell]$ such that the successive quotients are quotients of $V_{g_0}(\mu)$, where $\mu$ runs over $P(k)[\ell]$. In order to show that $W(\lambda_k, I_0)[\ell]$ can be filtered by $\bigoplus_{\mu \in P(k)[\ell]} V_{g_0}(\mu)$ it will be enough to prove

$$\dim W(\lambda_k, I_0) \geq \sum_{\mu \in P(k)} \dim V_{g_0}(\mu). \quad (6.8.7)$$
In order to show (6.8.7) it is enough to show

\[ \sum_{\mu \in P(k)} \dim V_{g_0}(\mu) = \binom{2n}{k} - \binom{2n}{k - 2}. \]

To see this, first note that

\[ P(k) = \bigcup_{p=0}^{\lfloor k/2 \rfloor} \{ \lambda_{k-i-2p} + (1 - \delta_{i,0})\lambda_{j+i} : 0 \leq i \leq \min\{k-2p, n-j\} \}. \]

Then, in \( C_j \times C_{n-j} \),

\[ \sum_{\mu \in P(k)} \dim V_{g_0}(\mu) = \sum_{p=0}^{\lfloor k/2 \rfloor} \sum_{i=0}^{\min\{k-2p, n-j\}} \dim V_{g_0}(\lambda_{k-i-2p} + (1 - \delta_{i,0})\lambda_{j+i}) \]

\[ = \sum_{p=0}^{\lfloor k/2 \rfloor} \sum_{i=0}^{\min\{k-2p, n-j\}} \left( \binom{2j}{k-i-2p} - \binom{2j}{k-i-2p-2} \right) \left( \binom{2(n-j)}{i} - \binom{2(n-j)}{i-2} \right). \]

By distributing and applying the Chu-Vandermonde identity, we get

\[ \sum_{p=0}^{\lfloor k/2 \rfloor} \left( \binom{2n}{k-2p} - 2\binom{2n}{k-2p-2} + \binom{2n}{k-2p-4} \right) \]

which equals

\[ \binom{2n}{k} - \binom{2n}{k-2}, \]

completing the proof.

\[ \square \]

6.9 The case of \( (F_4, B_4) \)

We consider the \( (F_4, B_4) \) case, where \( j = 4 \). Recall that in this case the simple system for \( g_0 \) is

\[ \{ \alpha_1, \alpha_2, \alpha_3, \alpha_2 + 2\alpha_3 + 2\alpha_4 \} \]
Lemma 6.9.1. Let \( \alpha \) be a positive root of \( g \) such that \( x^-_\alpha \otimes t^r \in g[t]^r \).

(1) If \( k \neq 4 \) and if \( a_1(\alpha) = 0 \), then for \( r \geq 1 \), \( (x^-_\alpha \otimes t^r)w_{\lambda_k} = 0 \).

(2) If \( k \in \{2, 3\} \), and if \( a_2(\alpha) \leq 1 \), then for \( r \geq 1 \), \( (x^-_\alpha \otimes t^r)w_{\lambda_2} = 0 \).

(3) If \( k = 3 \), and if \( a_3(\alpha) \leq 2 \), then for \( r \geq 1 \), \( (x^-_\alpha \otimes t^r)w_{\lambda_3} = 0 \).

(4) If \( k = 1 \), and if \( a_1(\alpha) = 0 \) or \( a_4(\alpha) = 0 \) then for \( r \geq 1 \), \( (x^-_\alpha \otimes t^r)w_{\lambda_1} = 0 \).

(5) If \( k = 4 \) and if \( a_4(\alpha) \in \{0, 2\} \), then for \( r \geq 2 \), \( (x^-_\alpha \otimes t^r)w_{\lambda_4} = 0 \).

Proof. For part (1), from the defining relations of \( W(\lambda_k, I_0) \), we have \( (x^-_\alpha \otimes t^r)^2(x^-_\alpha \otimes t^r)w_{\lambda_k} = 0 \),

Then

\[
0 = (x_1^+ + a_4 \otimes t)^2(x^-_\alpha \otimes t^r)w_{\lambda_k} = (x^-_\alpha \otimes t^r)w_{\lambda_k},
\]

\[
0 = (x_4^+ \otimes t)(x_1^+ + a_4 \otimes t)(x^-_\alpha \otimes t^r)w_{\lambda_k} = (x^-_\alpha \otimes t^r)w_{\lambda_k},
\]

and

\[
0 = (x_4^+ \otimes t)(x_1^+ + a_4 \otimes t)(x^-_\alpha \otimes t^r)w_{\lambda_k} = (x_4^+ \otimes t)w_{\lambda_k}.
\]

Also,

\[
0 = (x_1^+ + a_4 \otimes t)(x^-_\alpha \otimes t^r)(x^-_\alpha \otimes t^r)w_{\lambda_3} = (x^-_\alpha \otimes t^r)w_{\lambda_3},
\]

\[
0 = (x_2^+ + a_3 \otimes t)(x^-_\alpha \otimes t^r)(x^-_\alpha \otimes t^r)w_{\lambda_3} = (x^-_\alpha \otimes t^r)w_{\lambda_3},
\]

\[
0 = (x_4^+ \otimes t)^2(x^-_\alpha \otimes t^r)w_{\lambda_3} = (x^-_\alpha \otimes t^r)w_{\lambda_3},
\]

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\[ 0 = (x_{\alpha_1+\alpha_4}^+ \otimes t)(x_{\alpha_2+2\alpha_3+2\alpha_4}^- \otimes 1)w_{\lambda_3} = (x_{\alpha_2+2\alpha_3+\alpha_4}^- \otimes t)w_{\lambda_3}, \]

and

\[ 0 = (x_4^+ \otimes t)(x_{\alpha_2+2\alpha_3+2\alpha_4}^- \otimes 1)w_{\lambda_3} = (x_{\alpha_2+2\alpha_3+\alpha_4}^- \otimes t)w_{\lambda_3}. \]

For part (2), since \( (x_1^- \otimes 1)w_{\lambda_k} = 0 \), use the fact that

\[ (x_{\gamma+\alpha_1}^- \otimes t^r)w_{\lambda_k} = (x_1^- \otimes 1)(x_\gamma^- \otimes t^r)w_{\lambda_k}, \]

and apply part (1).

For part (3), we know \( (x_{\alpha_1+\alpha_2}^- \otimes 1)w_{\lambda_3} = 0 \), and \( (x_{\alpha_2+2\alpha_3+2\alpha_4}^- \otimes 1)w_{\lambda_3} = 0 \), so

\[ 0 = (x_{\alpha_1+\alpha_2}^- \otimes 1, x_{\alpha_2+2\alpha_3+2\alpha_4}^- \otimes 1)]w_{\lambda_3} = (x_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4}^- \otimes 1)w_{\lambda_3}. \]

Then

\[ 0 = (x_4^+ \otimes t)(x_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4}^- \otimes 1)w_{\lambda_3} = (x_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4}^- \otimes t)w_{\lambda_3}, \]

and

\[ 0 = (x_4^+ \otimes t)^2(x_{\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4}^- \otimes 1)w_{\lambda_3} = (x_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4}^- \otimes t^2)w_{\lambda_3}. \]

For part (4), from the defining relations of \( W(\lambda_1, I_0) \) and Garland, we have

\[ (x_1^- \otimes t^2)w_{\lambda_1} = 0. \]

Then

\[ 0 = (x_2^- \otimes 1)(x_1^- \otimes t^2)w_{\lambda_1} = (x_{\alpha_1+\alpha_2}^- \otimes t^2)w_{\lambda_1}, \]

\[ 0 = (x_3^- \otimes 1)(x_{\alpha_1+\alpha_2}^- \otimes t^2)w_{\lambda_1} = (x_{\alpha_1+\alpha_2+\alpha_3}^- \otimes t^2)w_{\lambda_1}, \]
\[ 0 = (x_3^- \otimes 1)(x_{\alpha_1 + \alpha_2 + \alpha_3}^- \otimes t^2)w_{\lambda_1} = (x_{\alpha_1 + \alpha_2 + 2\alpha_3}^- \otimes t^2)w_{\lambda_1}, \]

and

\[ 0 = (x_2^- \otimes 1)(x_{\alpha_1 + \alpha_2 + \alpha_3}^- \otimes t^2)w_{\lambda_1} = (x_{\alpha_1 + 2\alpha_2 + \alpha_3}^- \otimes t^2)w_{\lambda_1}. \]

The rest of part (4) is just a restatement of part (1).

For part (5), if \( a_4(\alpha) = 0 \), the statement clearly follows from the defining relations of \( W(\lambda_0, I_0) \). If \( a_4(\alpha) = 2 \), by the defining relations and Garland, we have

\[ (x_{\alpha_2 + 2\alpha_3 + \alpha_4}^- \otimes t^2)w_{\lambda_0} = 0. \]

Use the fact that \((x_i^- \otimes 1)w_{\lambda_0} = 0\) for \( i \in \{1, 2, 3\} \) to finish part (5).

\[ \square \]

**Proposition 6.9.2.** \( P(1) = \{ \lambda_1, \lambda_3, 0 \} \), \( P(2) = \{ \lambda_2, \lambda_3, 0, \lambda_0 \} \), \( P(3) = \{ \lambda_3, 0 \} \), and \( P(4) = \{ \lambda_0, \lambda_3, 0 \} \), which proves Proposition 6.8.1.

**Proof.** For \( k = 3 \), we have

\[ W(\lambda_3, I_0) = U(\mathfrak{g}[t]^\tau)w_{\lambda_3} = U(\mathfrak{g}_0)w_{\lambda_3} \oplus U(n^-[t]^\tau)_+w_{\lambda_3}. \] \( (6.9.1) \)

Then, by part (3) of Lemma 6.9.1, we have

\[ U(n^-[t]^\tau)_+w_{\lambda_3} = \sum_{a_3(\alpha) \geq 3} U(\mathfrak{g}[t]^\tau)(x_\alpha^- \otimes t^{a_3(\alpha)})w_{\lambda_3} = U(\mathfrak{g}[t]^\tau)(x_{\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4}^- \otimes t)w_{\lambda_3}. \]

Again by part (3) of Lemma 6.9.1, \((x_{\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4}^- \otimes t)w_{\lambda_3}\) is a highest weight vector of weight 0. Thus,

\[ U(\mathfrak{g}[t]^\tau)(x_{\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4}^- \otimes t)w_{\lambda_3} = U(\mathfrak{g}_0)(x_{\alpha_1 + 2\alpha_2 + 3\alpha_3 + \alpha_4}^- \otimes t)w_{\lambda_3}, \]
which implies (6.9.1) equals

\[ U(g_0)w_{\lambda_3} \oplus U(g_0)(x_{-\alpha_1+2\alpha_2+3\alpha_3+\alpha_4} \otimes t)t_{-\lambda_3}, \]

so

\[ \dim W(\lambda_3, I_0) \leq 17. \]

Next we consider \( k = 4. \) We have

\[ W(\lambda_0, I_0) = U(g[t]^\tau)w_{\lambda_0} = U(g_0)w_{\lambda_0} \oplus U(n^-[t]^\tau)_+w_{\lambda_0}. \tag{6.9.2} \]

By Lemma 6.9.1, (6.9.2) equals

\[ U(g_0)w_{\lambda_0} \oplus \sum_{\alpha(\alpha) = 1} U(g[t]^\tau)(x_{-\alpha} \otimes t)t_{-\lambda_0}. \tag{6.9.3} \]

Now, if \( a_4(\alpha) = 1, \)

\[ U(g[t]^\tau)(x_{-\alpha} \otimes t)t_{-\lambda_0} \subset U(g[t]^\tau)(x_{-\alpha} \otimes t)t_{-\lambda_0}, \]

so (6.9.3) equals

\[ U(g_0)w_{\lambda_0} \oplus U(g[t]^\tau)(x_{-\alpha} \otimes t)t_{-\lambda_0}. \]

Since \( (x_{-\alpha} \otimes t)t_{-\lambda_0} \) satisfies the defining relations of \( W(\lambda_3, I_0) \) we have a surjective map

\[ W(\lambda_3, I_0) \twoheadrightarrow U(g[t]^\tau)(x_{-\alpha} \otimes t)t_{-\lambda_0}, \]

and

\[ \dim W(\lambda_0, I_0) \leq 26. \]

Moreover, by Lemma 6.2.1, \( \dim W(\lambda_0, I_0) \geq 26, \) so we have

\[ W(\lambda_0, I_0) \cong g_0 V_{g_0}(\lambda_0) \oplus V_{g_0}(\lambda_3) \oplus V_{g_0}(0); \]

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and
\[ W(\lambda_3, I_0) \simeq g_0 V_{g_0}(\lambda_3) \oplus V_{g_0}(0). \]

For \( k = 1 \), we have
\[ W(\lambda_1, I_0) = U(\mathfrak{g}[t]^{\tau})w_{\lambda_1} = U(\mathfrak{g}_0)w_{\lambda_1} \oplus U(n^-[t]^{\tau})w_{\lambda_1}. \] (6.9.4)

By part (4) of Lemma 6.9.1,
\[ U(n^-[t]^{\tau})w_{\lambda_1} = U(\mathfrak{g}_0)(x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \otimes t)w_{\lambda_1}. \]

Again by part (4) of Lemma 6.9.1, \((x^-_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \otimes t)w_{\lambda_1}\) satisfies the defining relations of \( W(\lambda_3, I_0) \) so we have a surjective map
\[ W(\lambda_3, I_0) \twoheadrightarrow U(\mathfrak{g}[t]^{\tau})(x_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} \otimes t)w_{\lambda_1}, \]
and
\[ \dim W(\lambda_1, I_0) \leq 53. \]

Also, by Lemma 6.2.1, \( \dim W(\lambda_1, I_0) \geq 53 \), so we have
\[ W(\lambda_1, I_0) \simeq g_0 V_{g_0}(\lambda_1) \oplus V_{g_0}(\lambda_3) \oplus V_{g_0}(\lambda_0). \]

Finally we consider \( k = 2 \). We have
\[ W(\lambda_2, I_0) = U(\mathfrak{g}[t]^{\tau})w_{\lambda_2} = U(\mathfrak{g}_0)w_{\lambda_2} \oplus U(n^-[t]^{\tau})w_{\lambda_2}. \] (6.9.5)

By part (2) of Lemma 6.9.1, (6.9.5) equals
\[ U(\mathfrak{g}_0)w_{\lambda_2} \oplus (U(\mathfrak{g}[t]^{\tau})(x_{\alpha_1+2\alpha_2+2\alpha_3} \otimes t^2)w_{\lambda_2} + \sum_{\substack{a_2(\alpha) \geq 2 \\alpha \geq 1}} U(\mathfrak{g}[t]^{\tau})(x_{\alpha} \otimes t^{a_4(\alpha)})w_{\lambda_2}). \] (6.9.6)
Again by part (2) of Lemma 6.9.1, we have

\[ (U(g[t])^\bigotimes(x_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4} \otimes t^2)h_{\lambda_2} + \sum_{\alpha_2(\alpha) \geq 2 \atop \alpha_4(\alpha) \geq 1} U(g[t])^\bigotimes(x_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4} \otimes t)h_{\lambda_2}) \subset U(g[t])^\bigotimes(x_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4} \otimes t)h_{\lambda_2}. \]  

(6.9.7)

This gives us the filtration

\[ W(\lambda_2, I_0) \supset U(g[t])^\bigotimes(x_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4} \otimes t)h_{\lambda_2} \supset U(g[t])^\bigotimes(x_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4} \otimes t^2)h_{\lambda_2}. \]  

(6.9.8)

Define \( M_0 := W(\lambda_2, I_0) \), \( M_1 := U(g[t])^\bigotimes(x_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4} \otimes t)h_{\lambda_2} \) and \( M_2 := U(g[t])^\bigotimes(x_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4} \otimes t^2)h_{\lambda_2} \). Then by (6.9.7) we have shown

\[ M_0/M_1 \cong g_0 V_{g_0}(\lambda_2). \]

Next, by part (2) of Lemma 6.9.1, \( (x_{\alpha_1+2\alpha_2+2\alpha_3+\alpha_4} \otimes t)h_{\lambda_2} \) satisfies the defining relations of \( W(\lambda_3, I_0) \) (modulo \( M_2 \)), so we have a surjective map

\[ W(\lambda_3, I_0) \twoheadrightarrow M_1/M_2. \]

Finally, by Lemma 6.9.1, \( (x_{\alpha_1+2\alpha_2+2\alpha_3} \otimes t^2)h_{\lambda_2} \) satisfies the defining relations of \( W(\lambda_0, I_0) \), so we have a surjective map

\[ W(\lambda_0, I_0) \twoheadrightarrow M_3. \]

Since (again by Lemma 6.9.1)

\[ (x_4^{-1} \otimes t)(x_{\alpha_1+2\alpha_2+2\alpha_3} \otimes t^2)h_{\lambda_2} = 0, \]

this map factors through to a map of \( g_0 \)-modules

\[ V_{g_0}(\lambda_4) \twoheadrightarrow M_3. \]

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Thus,
\[ \dim W(\lambda_2, I_0) \leq 110. \]

In order to show
\[ \dim W(\lambda_2, I_0) \geq 110, \]
we define a \( g[t]^\tau \)-module structure on
\[ V_{g_0}(\lambda_2) \oplus V_{g_0}(\lambda_3) \oplus V_{g_0}(0) \oplus V_{g_0}(\lambda_0) \]
so that we have a surjective map of \( g[t]^\tau \)-modules
\[ W(\lambda_2, I_0) \twoheadrightarrow V_{g_0}(\lambda_2) \oplus V_{g_0}(\lambda_3) \oplus V_{g_0}(0) \oplus V_{g_0}(\lambda_0), \]
with
\[ w_{\lambda_2} \mapsto (v_{\lambda_2}, 0, 0, 0), \]
where \( v_{\lambda_2} \) is a highest weight vector of \( V_{g_0}(\lambda_2) \).

\[ \square \]

6.10 The case of \( (G_2, A_2) \)

We consider the \((G_2, A_2)\) case, where \( j = 2 \). Recall that in this case the simple system for \( g_0 \) is
\[ \{\alpha_1, \alpha_1 + 3\alpha_2\} \]
and the set of positive roots for \( g \) is
\[ \{\alpha_1, \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}. \]
We record the following lemma:
Lemma 6.10.1. Let $k \in \{1, 2\}$, and let $\alpha$ be a positive root of $g$ such that $x_\alpha^- \otimes t^r \in g[t]^r$.

(1) If $r \geq 3$, then $(x_\alpha^- \otimes t^r)w_{\lambda_k} = 0$.

(2) If $k = 1$, then for $r \geq 1$, $(x_\alpha^- \otimes t^r)w_{\lambda_k} = 0$.

(3) If $k = 2$, then for $r \geq 1$, $(x_\alpha^- \otimes t^r)(x_2^- \otimes t^2)w_{\lambda_k} = 0$.

Proof. Part (1) of the lemma follows immediately from the Garland identities if $ht_2(\alpha) \in \{0, 3\}$. Thus,

\[(x_{\alpha_1+3\alpha_2}^- \otimes t^r)w_{\lambda_k} = 0 \quad (6.10.1)\]

for $r \geq 3$. Then,

\[(x_2^+ \otimes t)(x_{\alpha_1+3\alpha_2}^- \otimes t^r)w_{\lambda_k} = 0\]

gives

\[(x_{\alpha_1+2\alpha_2}^- \otimes t^{r+1})w_{\lambda_k} = 0. \quad (6.10.2)\]

Applying $x_2^+ \otimes t$ to (6.10.2) gives

\[(x_{\alpha_1+\alpha_2^-}^- \otimes t^{r+2})w_{\lambda_k} = 0.\]

Meanwhile, applying $x_{\alpha_1+2\alpha_2}^+ \otimes t^2$ to (6.10.1) gives

\[(x_2^- \otimes t^{r+2})w_{\lambda_k} = 0,\]

finishing part (1).

For part (2), since $k = 1$,

\[(x_{\alpha_1+3\alpha_2}^- \otimes 1)w_{\lambda_k} = 0.\]

Applying the same vectors as in part (1) finishes part (2).
For part (3), if \( r \geq 3 \), the equality is clear. Consider

\[
(x_{\alpha_1 + 2\alpha_2}^+ \otimes t^2)(x_{\alpha_1 + 3\alpha_2}^+ \otimes 1)^2 w_{\lambda_k} = 0
\]

which gives

\[
(x_{\alpha_1 + 3\alpha_2}^- \otimes 1)(x_2^- \otimes t^2)w_{\lambda_k} = 0. \tag{6.10.3}
\]

Applying \((x_{\alpha_1 + 2\alpha_2}^+ \otimes t^r)\) to (6.10.3) gives

\[
(x_2^- \otimes t^r)(x_2^- \otimes t^2)w_{\lambda_k} = 0. \tag{6.10.4}
\]

Also, applying \((x_2^+ \otimes t^r)\) to (6.10.3) gives

\[
(x_{\alpha_1 + 2\alpha_2}^- \otimes t^r)(x_2^- \otimes t^2)w_{\lambda_k} = 0. \tag{6.10.5}
\]

Finally, applying \((x_2^+ \otimes t)\) to (6.10.5) gives

\[
(x_{\alpha_1 + \alpha_2}^- \otimes t^r)(x_2^- \otimes t^2)w_{\lambda_k} = 0.
\]

We now prove Proposition 6.8.1 in the \((G_2, A_2)\) case:

**Proof.** For \( k = 1 \), we have

\[
W(\lambda_1, L_0) = U(g[t]^r)w_{\lambda_1} = U(g_0)w_{\lambda_1} \oplus U(n^-[t]^r)w_{\lambda_1}.
\]

By part (2) of Lemma 6.10.1,

\[
U(n^-[t]^r)w_{\lambda_1} = 0.
\]
For \( k = 2 \),

\[
W(\lambda_2, I_0) = U(g[t]^{-})w_{\lambda_2} = U(g_0)w_{\lambda_1} \oplus U(n^-[t]^{-})w_{\lambda_2}. \tag{6.10.6}
\]

By part (1) of Lemma 6.10.1, (6.10.6) equals

\[
U(g_0)w_{\lambda_1} \oplus (U(g[t]^{-})(x_{\alpha_1+2\alpha_2} \otimes t)w_{\lambda_2} + U(g[t]^{-})(x_2^- \otimes t^2)w_{\lambda_2} + U(g[t]^{-})(x_{\alpha_1+2\alpha_2} \otimes t^2)w_{\lambda_2}). \tag{6.10.7}
\]

Now, since

\[
(x_{\alpha_1+2\alpha_2} \otimes t^2)w_{\lambda_2} = (x_1^- \otimes 1)(x_2^- \otimes t^2)w_{\lambda_2},
\]

we have

\[
U(g[t]^{-})(x_{\alpha_1+2\alpha_2} \otimes t^2)w_{\lambda_2} \subset U(g[t]^{-})(x_2^- \otimes t^2)w_{\lambda_2},
\]

so 6.10.7 equals

\[
U(g_0)w_{\lambda_1} \oplus (U(g[t]^{-})(x_{\alpha_1+2\alpha_2} \otimes t)w_{\lambda_2} + U(g[t]^{-})(x_2^- \otimes t^2)w_{\lambda_2}. \tag{6.10.8}
\]

Next, since

\[
(x_2^- \otimes t^2)w_{\lambda_2} = (x_{\alpha_1+2\alpha_2}^+ \otimes t)(x_{\alpha_1+2\alpha_2}^- \otimes t)w_{\lambda_2},
\]

we get

\[
U(g[t]^{-})(x_2^- \otimes t^2)w_{\lambda_2} \subset U(g[t]^{-})(x_{\alpha_1+2\alpha_2}^- \otimes t)w_{\lambda_2}.
\]

Define \( M_0 = W(\lambda_2, I_0), M_1 = U(g[t]^{-})(x_{\alpha_1+2\alpha_2} \otimes t)w_{\lambda_2}, M_2 = U(g[t]^{-})(x_2^- \otimes t^2)w_{\lambda_2}, M_3 = 0. \) Then we have a filtration

\[ M_0 \supset M_1 \supset M_2 \supset M_3. \]

We have shown that \( M_0/M_1 \sim_{g_0} V_{g_0}(\lambda_2) \). Since \( (x_{\alpha_1+2\alpha_2}^- \otimes t)w_{\lambda_2} \) is a highest weight vector modulo \( M_2 \) of weight \( \lambda_0 \), we have \( M_1/M_2 \sim_{g_0} V_{g_0}(\lambda_0) \). By part (3) of Lemma 6.10.1,
Thus $7 \geq \dim W(\lambda_2, I_0)$, and we have

$$W(\lambda_2, I_0) \cong g_0 V_{g_0}(\lambda_2) \oplus V_{g_0}(\lambda_1) \oplus V_{g_0}(\lambda_0).$$

6.11 Other differences from $\mathfrak{g}[t]$

We recall that in the case of the current algebra, $\mathfrak{g}[t]$, for $\lambda \in P^+$ with $\lambda = \sum_{i=1}^{n} r_i \omega_i$ we have

$$\dim W^g_{\text{loc}}(\lambda) = \prod_{i=1}^{n} (\dim W^g_{\text{loc}}(\omega_i))^{r_i}. \quad (6.11.1)$$

By results in Chapter 5, for the weight $r\lambda_{n-1} \in P^+_0$ the global and local Weyl module coincide and we have

$$\dim W_{\text{loc}}(r\lambda_{n-1}) = \dim V_{g_0}(r\lambda_{n-1})$$

and so (6.11.1) does not hold in general.

Moreover, in the case of the current algebra, the dimension of the local Weyl module only depends on the weight $\lambda \in P^+$ and not on the maximal ideal in $A_\lambda$. In the case of $(D_3, B_3)$ and $\lambda = \lambda_0 + \lambda_2$, we show that the dimension of the local Weyl module depends on the choice of ideal. Recall that in this case,

$$A_\lambda = \mathbb{C}[P_{2,1}, P_{3,1}]/(P_{2,1} P_{3,1}),$$

so for $a \in \mathbb{C}^\times$ we let $I_{(a,0)}$ denote the maximal ideal corresponding to $(P_{2,1} - a, P_{3,1})$ and we let $I_{(0,a)}$ denote the maximal ideal corresponding to $(P_{2,1}, P_{3,1} - a)$. In the case of $I_{(a,0)}$, by Section 6.2 the local Weyl module $W(\lambda, I_{(a,0)})$ is a pullback of a local Weyl module for
the current algebra, $\mathfrak{g}[t]$, and so

$$\dim W(\lambda, I_{(0,0)}) = 22.$$  

Meanwhile, in the case of $I_{(0,a)}$, the local Weyl module $W(\lambda, I_{(0,a)})$ is an extension of the pullback of a local Weyl module for the current algebra by an irreducible $\mathfrak{g}_0$–module, and so

$$\dim W(\lambda, I_{(0,a)}) = 32.$$  

To see this, we clearly have a surjective map

$$W(\lambda, I_{(0,a)}) \twoheadrightarrow \mathrm{ev}_0(V_{\mathfrak{g}_0}(\lambda_2)) \otimes \mathrm{ev}_{-\sqrt{a}}(V_{\mathfrak{g}}(\omega_3)).$$

This gives

$$\dim W(\lambda, I_{(0,a)}) \geq 32,$$

and using techniques from this chapter we have

$$\dim W(\lambda, I_0) \leq 32.$$  

The surjective map

$$W(\lambda, I_{(0)}) \twoheadrightarrow \mathrm{gr} W(\lambda, I_{(0,a)})$$

then proves the equality. Finally, the fact that the dimension of the local Weyl module depends on the choice of maximal ideal shows that the global Weyl module is not a projective $\mathbf{A}_\lambda$–module, and hence not free.
Conclusion

In this dissertation, we used a family of automorphisms coming from Borel-de Sienthal theory to give a realization of maximal parabolic subalgebras of untwisted affine Lie algebras as equivariant map algebras. We then showed that the representation theory of these equivariant map algebras differs from that of equivariant map algebras appearing in the literature in many important ways.

First, for this family of equivariant map algebras, the global Weyl module can be finite dimensional and irreducible for nontrivial weights. Second, the commutative associative algebra associated to each weight is non necessarily a polynomial algebra. We show that this commutative associative algebra (modulo the Jacobson radical) is a Stanley–Reisner ring, and use this fact to determine the Krull dimension of the algebra in some cases. We also give a sufficient condition for the algebra to be Koszul and Cohen-Macaulay using combinatorial methods. Finally, we see the local Weyl module exhibits many features not seen for equivariant map algebras in the literature, and show that one can utilize the Borel-de Siebenthal subalgebra to determine the dimension.
Bibliography


