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Author
Schechter, M

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EXISTENCE OF PERIODIC SOLUTIONS OF COMPLETE SECOND ORDER HAMILTONIAN SYSTEMS

Martin Schechter *
Department of Mathematics, University of California,
Irvine, CA 92697-3875, U.S.A.
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Abstract
We study the existence of periodic solutions of a second order non-autonomous dynamical system including both the kinetic and potential terms. We assume little concerning the gradient of the potential other than continuity. This allows both sublinear and superlinear problems. We also study the existence of ground state solutions.

1 Introduction
We consider the system

\[ -\ddot{x}(t) = B(t)x(t) + \nabla_x V(t, x(t)), \]

where

\[ x(t) = (x_1(t), \cdots, x_n(t)) \]

is a map from \( I = [0, T] \) to \( \mathbb{R}^n \) such that each component \( x_j(t) \) is a periodic function in \( H^1 \) with period \( T \), and the function \( V(t, x) = V(t, x_1, \cdots, x_n) \) is continuous from \( \mathbb{R}^{n+1} \) to \( \mathbb{R} \) with

\[ \nabla_x V(t, x) = (\partial V/\partial x_1, \cdots, \partial V/\partial x_n) \in C(\mathbb{R}^{n+1}, \mathbb{R}^n). \]

For each \( x \in \mathbb{R}^n \), the function \( V(t, x) \) is periodic in \( t \) with period \( T \).

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We shall study this problem under several sets of assumptions. First, we make no assumption on $\nabla_x V(t, x)$ other than (3). This allows both sublinear and superlinear problems. The elements of the symmetric matrix $B(t)$ are to be real-valued functions $b_{jk}(t) = b_{kj}(t)$. Our assumption on $B(t)$ is

(B1) Each component of $B(t)$ is an integrable function on $I$, i.e., for each $j$ and $k$, $b_{jk}(t) \in L^1(I)$.

This assumption implies that there is an extension $D$ of the operator 

$$D_0 x = -\ddot{x}(t) - B(t)x(t)$$

having a discrete, countable spectrum consisting of isolated eigenvalues of finite multiplicity with a finite lower bound $-L$

(4) $-\infty < -L \leq \lambda_0 < \lambda_1 < \lambda_2 < ... < \lambda_l < ... .$

Let $\lambda_l$ be the first positive eigenvalue of $D$. We allow $\lambda_{l-1} = 0$. Let $H$ be the set of vector functions $x(t)$ described above. It is a Hilbert space with norm satisfying

$$\|x\|^2_H = \sum_{j=1}^{n} \|x_j\|^2_H.$$

We also write

$$\|x\|^2 = \sum_{j=1}^{n} \|x_j\|^2,$$

where $\|\cdot\|$ is the $L^2(I)$ norm. Define the subspaces $M$ and $N$ of $H$ as,

$$N = \bigoplus_{k<l} E(\lambda_k), \quad M = N^\perp, \quad H = M \oplus N,$$

where $E(\lambda_k)$ is the eigenspace of $\lambda_k$. Let

(5) $$G(x) = d(x) - 2 \int_I V(t, x) \, dt,$$

where $d(x) = \langle Dx, x \rangle$ (cf. Section 4). Let

$$x(t) = w(t) + v(t), \ w(t) \in M, \ v(t) \in N.$$

We write

(6) $$G_\lambda(x) = \lambda d(w) + d(v) - 2 \int_I V(t, x) \, dt, \quad 0 < \lambda < \infty.$$

We let $D_\lambda$ be the operator corresponding to $d_\lambda(x) = \lambda d(w) + d(v)$. 

2
Solutions of

\[ D_\lambda x(t) = \nabla_\lambda V(t, x(t)) \tag{7} \]

are generalized solutions of (1) when \( \lambda = 1 \). We introduced the parameter \( \lambda \) to make use of the monotonicity trick of Section 12. This requires us to work in an interval of a parameter. This allows us to obtain solutions under very weak hypotheses. However, we obtain solutions for almost every value of the parameter. We can then obtain solutions for all values of the parameter by introducing appropriate mild assumptions.

2 The case \( \lambda_0 \leq 0 \).

In this case \( l > 0 \) and \( \dim N > 0 \). We have

**Theorem 2.1.** Assume

1. \[ 2V(t, x) \geq \lambda_{l-1} |x|^2 - W(t), \quad t \in I, \ x \in \mathbb{R}^n \]
   and \[ 2V(t, x) + \nu_l |x|^2 \to \infty, \quad |x| \to \infty, \]
   where \( \nu_l = \min[\lambda_l, |\lambda_{l-1}|] \) and \( W(t) \in L^1(I) \).

2. \[ 2V(t, x) \leq \mu |x|^2 + W(t), \quad t \in I, \ x \in \mathbb{R}^n, \]
   where \( \mu > 0 \) and \( W(t) \in L^1(I) \).

Then the system

\[ D_\lambda x(t) = \nabla_\lambda V(t, x(t)) \tag{8} \]

has a solution for almost all \( \lambda \in [\mu/\lambda_l, \infty) \).

**Theorem 2.2.** Assume

1. \[ 2V(t, x) \geq \lambda_{l-1} |x|^2, \quad t \in I, \ x \in \mathbb{R}^n, \]
   and \[ 2V(t, x) + \nu_l |x|^2 \to \infty, \quad |x| \to \infty, \]
   where \( \nu_l = \min[\lambda_l, |\lambda_{l-1}|] \).
2. There are positive constants $\mu$ and $m$ such that

$$2V(t, x) \leq \mu|x|^2, \quad |x| \leq m, \quad x \in \mathbb{R}^n.$$  

Then the system (8) has a solution for almost all $\lambda \in [\mu/\lambda, \infty)$.  

**Theorem 2.3.** If, in addition, we assume

There are a constant $\gamma > \mu$ and a function $W(t) \in L^1(I)$ such that

$$2V(t, x) \geq \gamma|x|^2 - W(t), \quad t \in I, \quad x \in \mathbb{R}^n,$$

then the system (8) has a nontrivial solution for almost all $\lambda \in (\mu/\lambda, \gamma/\lambda_i)$.  

The advantage of these theorems is that we obtain solutions under very weak hypotheses. In fact, we make no assumption on $\nabla_x V(t, x)$ other than (3). The disadvantage is that we do not obtain a solution for any particular value of $\lambda$. If we wish to prove existence for every such $\lambda$, we will have to make assumptions concerning $\nabla_x V(t, x)$ as well. We now present additional hypotheses which guarantee existence of solutions for all values of $\lambda$ in a given interval. We have  

**Theorem 2.4.** Assume

1. 

$$2V(t, x) \geq \lambda_{i-1}|x|^2, \quad t \in I, \quad x \in \mathbb{R}^n.$$  

2. 

$$V(t, x)/|x|^2 \to \infty, \quad \text{as } |x| \to \infty.$$  

3. There are positive constants $\mu$ and $m$ such that

$$2V(t, x) \leq \mu|x|^2, \quad |x| \leq m, \quad x \in \mathbb{R}^n.$$  

4. There is a function $W(t) \in L^1(I)$ such that

$$2V(t, x + y) - 2V(t, x) - (2ry - (r - 1)^2x) \cdot \nabla_x V(x, t) \geq -W(t), \quad t \in I, \quad x, y \in \mathbb{R}^n, \quad r \in [0, 1].$$

Then the system (8) has a nontrivial solution for all values of $\lambda$ satisfying $\lambda \in [\mu/\lambda_i, \infty)$.  

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**Theorem 2.5.** The conclusions of Theorem 2.4 hold if we replace Hypothesis 4 with
4'. There are a constant $C$ and a function $W(t) \in L^1(I)$ such that

$$H(t, \theta x) \leq C(H(t, x) + W(t)), \quad 0 \leq \theta \leq 1, \quad t \in I, \quad x \in \mathbb{R}^n,$$

where

$$H(t, x) = \nabla_x V(t, x) \cdot x - 2V(t, x).$$

Let $\mathcal{M}_\lambda$ be the set of all solutions of

$$D_\lambda x(t) = \nabla_x V(t, x(t)).$$

A solution $x$ is called a “ground state solution” if it minimizes the functional

$$G_\lambda(x) = d_\lambda(x) - 2 \int_I V(t, x) \, dt$$

over the set $\mathcal{M}_\lambda$.

We have

**Theorem 2.6.** Under the hypotheses of Theorem 2.4, system (11) has a ground state solution for each $\lambda \in [\mu/\lambda_l, \infty)$.

**Remark 2.7.** Since $\lambda_{l-1} \leq 0$, these theorems allow linear, sublinear and superlinear growth at infinity for the problem (11). In particular, potentials of the form $b(t)|x|^p$ are included. The original problem (1) corresponds to the problem (11) when $\lambda = 1$. It will not be included in the interval $\lambda \in [\mu/\lambda_l, \infty)$ unless $\mu \leq \lambda_l$. The monotonicity trick enables one to solve problems such as (11) for almost all $\lambda$ in an interval when the hypotheses are too weak to solve it for definite values of $\lambda$. By adding hypotheses, one is able to solve them for specific values of $\lambda$.

### 3 The case $\lambda_0 > 0$.

In this case $l = 0$ and $N = \{0\}$, $M = H$. We have

**Theorem 3.1.** Assume

1. $2V(t, x) + \lambda_0 |x|^2 \to \infty, \quad |x| \to \infty$. 
2. \[ 2V(t,x) \leq \mu |x|^2 + W(t), \quad t \in I, \ x \in \mathbb{R}^n, \]

where \( \mu > 0 \) and \( W(t) \in L^1(I) \).

Then the system \((8)\) has a solution for almost all \( \lambda \in [\mu/\lambda_0, \infty) \).

**Theorem 3.2.** Assume

1. \[ 2V(t,x) + \lambda_0 |x|^2 \to \infty, \quad |x| \to \infty. \]
2. There are positive constants \( \mu \) and \( m \) such that
   \[ 2V(t,x) \leq \mu |x|^2, \quad |x| \leq m, \quad x \in \mathbb{R}^n. \]

Then the system \((8)\) has a solution for almost all \( \lambda \in [\mu/\lambda_0, \infty) \).

**Theorem 3.3.** If, in addition, we assume

There are a constant \( \gamma > \mu \) and a function \( W(t) \in L^1(I) \) such that
\[ 2V(t,x) \geq \gamma |x|^2 - W(t), \quad t \in I, \ x \in \mathbb{R}^n, \]

then the system \((8)\) has a nontrivial solution for almost all \( \lambda \in (\mu/\lambda_0, \gamma/\lambda_0) \).

**Theorem 3.4.** Assume

1. \[ V(t,x)/|x|^2 \to \infty, \quad \text{as} \ |x| \to \infty. \]
2. There are positive constants \( \mu \) and \( m \) such that
   \[ 2V(t,x) \leq \mu |x|^2, \quad |x| \leq m, \quad x \in \mathbb{R}^n. \]
3. There is a function \( W(t) \in L^1(I) \) such that
   \[ 2V(t,x + y) - 2V(t,x) - (2ry - (r - 1)^2x) \cdot \nabla_x V(x,t) \]
   \[ \geq -W(t), \quad t \in I, \ x, y \in \mathbb{R}^n, \ r \in [0,1]. \]

Then the system \((8)\) has a nontrivial solution for all values of \( \lambda \) satisfying \( \lambda \in [\mu/\lambda_0, \infty) \).
Theorem 3.5. The conclusions of Theorem 3.4 hold if we replace Hypothesis 3 with $3'$. There are a constant $C$ and a function $W(t) \in L^1(I)$ such that

$$H(t, \theta x) \leq C(H(t, x) + W(t)), \quad 0 \leq \theta \leq 1, \quad t \in I, \quad x \in \mathbb{R}^n,$$

where

$$H(t, x) = \nabla_x V(t, x) \cdot x - 2V(t, x). \quad (14)$$

Let $\mathcal{M}_\lambda$ be the set of all solutions of

$$D_\lambda x(t) = \nabla_x V(t, x(t)). \quad (15)$$

Theorem 3.6. Under the hypotheses of Theorem 3.4, system (11) has a ground state solution for each $\lambda \in [\mu/\lambda_0, \infty)$.

We shall prove Theorems 2.1 – 3.3 in Section 9 and Theorems 2.4 - 2.6, 3.4 - 3.6 in Section 13. We use linking and sandwich methods of critical point theory and then apply the monotonicity trick introduced by Struwe in [32, 33] for minimization problems. (This trick was also used by others to solve Landesman-Lazer type problems, for bifurcation problems, for Hamiltonian systems and Schrödinger equations.)

The theory of sandwich pairs began in [31] and [25, 26] and was developed in subsequent publications such as [27, 28].

The periodic non-autonomous problem

$$\ddot{x}(t) = \nabla_x V(t, x(t)), \quad (16)$$

has an extensive history in the case of singular systems (cf., e.g., Ambrosetti-Coti Zelati [1]). The first to consider it for potentials satisfying (3) were Berger and the author [5] in 1977. We proved the existence of solutions to (8) under the condition that

$$V(t, x) \rightarrow \infty \quad \text{as} \quad |x| \rightarrow \infty$$

uniformly for a.e. $t \in I$. Subsequently, Willem [40], Mawhin [19], Mawhin-Willem [20], Tang [34, 35], Tang-Wu [37, 38], Wu-Tang [41] and others proved existence under various conditions (cf. the references given in these publications).
Most previous work considered the case when \( B(t) = 0 \). Ding and Girardi [9] considered the case of (1) when the potential oscillates in magnitude and sign,

\[
-\ddot{x}(t) = B(t)x(t) + b(t)\nabla W(x(t)),
\]

and found conditions for solutions when the matrix \( B(t) \) is symmetric and negative definite and the function \( W(x) \) grows superquadratically and satisfies a homogeneity condition. Antonacci [3, 4] gave conditions for existence of solutions with stronger constraints on the potential but without the homogeneity condition, and without the negative definite condition on the matrix. Generalizations of the above results are given by Antonacci and Magrone [2], Barletta and Livrea [6], Guo and Xu [13], Li and Zou [18], Faraci and Livrea [12], Bonanno and Livrea [7, 8], Jiang [16, 17], Shilgba [29, 30], Faraci and Iannizzotto [11] and Tang and Xiao [39].

Some authors considered the second order system (1) where the potential function \( V(t, x) \) is quadratically bounded as \( |x| \to \infty \). Berger and Schechter [5] considered the case of (1) where \( B(t) \) is a constant symmetric matrix that is positive definite, and showed existence of solutions when the magnitude of \( \nabla_x V(t, x) \) is uniformly bounded, the potential is strictly convex, and if \( y(t) \) is a \( T \)-periodic solutions of the linear system \(-\ddot{y} = Ay\), then there exists a function \( x(t) \) which is weakly differentiable with \( \dot{x} \in L^2(\mathbb{R}, \mathbb{R}^n) \) and satisfies

\[
\int_0^T \langle \nabla_x V(t, x(t)), y(t) \rangle dt = 0.
\]

Han [14] gave conditions for existence of solutions when \( B(t) \) was a multiple of the identity matrix, the system satisfies the resonance condition, and the potential has upper and lower subquadratic bounds. Li and Zou [18] considered the case where \( B(t) \) is continuous and nonconstant and the system satisfies the resonance condition, and showed existence of solutions when the potential is even and grows no faster than linearly. Tang and Wu [36] required the function that satisfies the resonance condition to pass through the zero vector, and gave upper and lower conditions for subquadratic growth of the magnitude of \( V(t, x) \) without the requirement that the potential be even. Faraci [10] considered the case where for each \( t \in I \), \( B(t) \) is negative definite with elements that are bounded but not necessarily continuous and the potential has an upper quadratic bound as \( |x| \to \infty \), showing existence of a solution when the gradient of the potential is bounded near the origin and exceeds the matrix product in at least one direction. In [21] the hypotheses are

1. There exist functions \( W_1, W_2 \in L^1(I) \) and consecutive eigenvalues \( \lambda_t, \lambda_{t+1} \) of \( D \) such that for any \( t \in I \) and any \( x \in \mathbb{R}^n \),

\[
\lambda_t|x|^2 - W_1(t) \leq 2V(t, x) \leq \lambda_{t+1}|x|^2 + W_2(t).
\]

2. There exists a function \( W_0(t) \in L^1(I) \) such that for any \( t \in I \) and any \( x \in \mathbb{R}^n \),

\[
H(t, x) = 2V(t, x) - x \cdot \nabla_x V(t, x) \geq -W_0(t).
\]

3. \( H(t, x) \to \infty \) uniformly in \( t \) as \( |x| \to \infty \).


4 The operator $\mathcal{D}$

In proving our theorems we shall make use of the following considerations.

We define a bilinear form $a(\cdot, \cdot)$ on the set $L^2(I, \mathbb{R}^n) \times L^2(I, \mathbb{R}^n)$,

(19) \[ a( u, v ) = ( \dot{u}, \dot{v} ) + ( u, v ) . \]

The domain of the bilinear form is the set $D(a) = H$, consisting of those periodic $x(t) = (x_1(t), \ldots, x_n(t)) \in L^2(I, \mathbb{R}^n)$ having weak derivatives in $L^2(I, \mathbb{R}^n)$. $H$ is a dense subset of $L^2(I, \mathbb{R}^n)$. Note that $H$ is a Hilbert space with scalar product $(u,v)_H = a(u,v)$. Thus we can define an operator $A$ such that $u \in D(A)$ if and only if $u \in D(a)$ and there exists $g \in L^2(I, \mathbb{R}^n)$ such that

(20) \[ a( u, v ) = ( g, v ) , \quad v \in D(a) . \]

If $u$ and $g$ satisfy this condition we say $A u = g$.

Lemma 4.1. The operator $A$ is a self-adjoint Fredholm operator from $L^2(I, \mathbb{R}^n)$ to $L^2(I, \mathbb{R}^n)$. It is one-to-one and onto.

Proof. Let $f \in L^2(I, \mathbb{R}^n)$. Then

\[ (v, f) \leq \|v\| \cdot \|f\| \leq \|v\|_H \|f\| , \quad v \in H. \]

Thus $(v, f)$ is a bounded linear functional on $H$. Since $H$ is complete, there is a $u \in H$ such that

\[ (u, v)_H = (f, v) , \quad v \in H. \]

Consequently, $u \in D(A)$ and $A u = f$. Moreover, if $A u = 0$, then

\[ (u, v)_H = 0 , \quad v \in H. \]

Thus, $u = 0$. Hence, $A$ is one-to-one and onto.

For any two functions $x, y \in D(A)$,

(21) \[ ( A x, y ) = ( \dot{x}, \dot{y} ) + ( x, y ) = ( x, A y ) . \]

Thus, $A$ is symmetric. It is now easy to show that $D(A) \subset D(a)$ is also a dense subset of $L^2(I, \mathbb{R}^n)$. In fact, if $f \in L^2(I, \mathbb{R}^n)$ satisfies $(f, v) = 0 \forall v \in D(A)$, then $w = D^{-1} f$ satisfies $(w, Dv) = (Dw, v) = 0 \forall v \in D(D)$. Since $D$ is onto, $w = 0$. Hence, $f = Dw = 0$.

Next, we show that $A$ is self-adjoint. Consider any $u, f \in L^2(I, \mathbb{R}^n)$, and suppose for any $v \in D(A)$,

(22) \[ ( u, A v ) = ( f, v ) . \]
Since $\mathcal{A}$ is onto and $f \in L^2(I, \mathbb{R}^n)$, there exists $w \in D(\mathcal{A})$ such that $\mathcal{A}w = f$. Then using (21),

$$ (u - w, \mathcal{A}v) = (f, v) - (\mathcal{A}w, v) = 0 . $$

Since $u - w \in L^2(I, \mathbb{R}^n)$, we can find a $v \in D(\mathcal{A})$ such that $\mathcal{A}v = u - w$, and

$$ \|u - w\|^2 = 0. $$

This implies $u = w$ in the space $L^2(I, \mathbb{R}^n)$, and therefore $u \in D(\mathcal{A})$. Hence, $\mathcal{D}u = \mathcal{A}w = f$. \hfill $\square$

**Lemma 4.2.** The essential spectrum of $\mathcal{A}$ is the null set.

**Proof.** By (4.1), $\mathcal{A}$ is linear, self-adjoint, and onto $L^2(I, \mathbb{R}^n)$.

Next, we note that

$$ \|\mathcal{A}^{-1}u\| \leq \|u\| . $$

To see this, let $f = \mathcal{D}u$. Then $u = \mathcal{D}^{-1}f$, and

$$ (u, v)_H = (f, v), \quad v \in H. $$

Thus,

$$ \|u\|_H^2 \leq \|f\| \cdot \|u\| \leq \|f\| \cdot \|u\|_H. $$

Hence, $\|u\| \leq \|f\|$

Now we show that the inverse operator $\mathcal{A}^{-1}$ is compact on $L^2(I, \mathbb{R}^n)$. Let $(u_k)$ be a bounded sequence in $L^2(I, \mathbb{R}^n)$, and let $C > 0$ satisfy for each $k$, $\|u_k\| \leq C$. By applying the inverse operator, let $(x_k)$ be the sequence such that for each $k$, $\mathcal{A}x_k = u_k$. From the above statements, for each $k$, $\|x_k\| \leq C$. From the definition of the operator $\mathcal{A}$, for any $x \in D(\mathcal{A})$

$$ (\mathcal{A}x, x) = (\dot{x}, \dot{x}) + (x, x) = \|x\|_H^2 \geq 0. $$

Hence, $\mathcal{K} = \mathcal{A}^{-1}$ is a positive compact operator, and the eigenvalues $\mu_k$ of $\mathcal{K}$ are denumerable and have 0 as their only possible limit point. The eigenfunctions $\phi_k$ of $\mathcal{K}$ are also eigenfunctions of $\mathcal{K}^{-1} = \mathcal{A}$ and satisfy

$$ \mathcal{A} \phi_k = \frac{1}{\mu_k} \phi_k. $$

Since the values $\mu_k$ are bounded and have no limit point except 0, there are no limit points of the set $(1/\mu_k)$ and the essential spectrum of $\mathcal{A}$ is the null set. \hfill $\square$

We will use two theorems of Schechter [22] on bilinear forms to prove Lemma 4.5.
Theorem 4.3. Let \( a(\cdot, \cdot) \) be a closed Hermitian bilinear form with dense domain in \( L^2(I, \mathbb{R}^n) \). If for some real number \( N \),

\[
a(u, u) + N\|u\|^2 \geq 0,
\]

then the operator \( A \) associated with \( a(\cdot, \cdot) \) is self-adjoint and \( \sigma(A) \subset [-N, \infty) \).

Theorem 4.4. Suppose \( a(\cdot, \cdot) \) is a bilinear form satisfying the hypotheses of Theorem 4.3. Let \( b(\cdot, \cdot) \) be a Hermitian bilinear form such that \( D(a) \subset D(b) \) and for some positive real number \( K \), for any \( u \in D(a) \),

\[
|b(u, u)| \leq Ka(u, u).
\]

Assume that every sequence \( (u_k) \subset D(a) \) which satisfies

\[
\|u_k\|^2 + a(u_k, u_k) \leq C
\]

has a subsequence \( (v_j) \) such that

\[
b(v_j - v_k, v_j - v_k) \to 0.
\]

Assume also that if (25),(26) hold and \( v_j \to 0 \) in the \( L^2(I, \mathbb{R}^n) \) norm, then \( b(v_j, v_j) \to 0 \). Set

\[
c(u, v) = a(u, v) + b(u, v).
\]

and let \( A, C \) be the operators associated with \( a, c \), respectively. Then

\[
\sigma_e(A) = \sigma_e(C).
\]

Let

\[
b(u, v) = -\sum_{j=1}^n \sum_{k=1}^n \int_0^T (b_{jk}(t) + \delta_{jk}) u_k(t)v_j(t) dt
\]

and

\[
d(u, v) = a(u, v) + b(u, v).
\]

We shall prove

Lemma 4.5. The operator \( D \) associated with the bilinear form \( d(\cdot, \cdot) \) under assumption (B1) is self-adjoint. Its essential spectrum is the null set and there exists a finite real value \( L \) such that \( \sigma(D) \subset [-L, \infty) \). \( D \) has a discrete, countable spectrum consisting of isolated eigenvalues of finite multiplicity with a finite lower bound \(-L\)

\[
-\infty < -L \leq \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_l < \ldots.
\]
To show the bilinear form $b(\cdot, \cdot)$ is Hermitian, we can use the symmetry of the matrix $B(t) + I$ to rearrange the order of the finite summation,

\[
b(u, v) = -\sum_{j=1}^{n} \sum_{k=1}^{n} \int_{0}^{T} \left( b_{jk}(t) + \delta_{jk} \right) u_k(t)v_j(t) dt
\]

\[
= -\sum_{k=1}^{n} \sum_{j=1}^{n} \int_{0}^{T} \left( b_{jk}(t) + \delta_{jk} \right) v_j(t)u_k(t) dt
\]

\[
= -\sum_{k=1}^{n} \sum_{j=1}^{n} \int_{0}^{T} \left( b_{kj}(t) + \delta_{kj} \right) v_j(t)u_k(t) dt
\]

\[
= b(v, u).
\]

Also the magnitude of $b(u) = b(u, u)$ is bounded by a multiple of the bilinear form $a(\cdot, \cdot)$ and satisfies (24),

\[
|b(u)| \leq K_B \|u\|^2_{L^\infty(I, \mathbb{R}^n)}
\]

\[
\leq K_B (M \|u\|_H)^2
\]

(31)

Consider a sequence $(x_k) \subset D(A)$ which is bounded by a constant $C$ in the $H$ norm. Then each term of the sequence satisfies

\[
\|x_k\|^2 + a(x_k) = 2(x_k, x_k) + (x_k, x_k) \leq 2\|x_k\|_H^2 \leq 4C^2.
\]

Since,

(32)

\[
\|u\|_{L^\infty(I, \mathbb{R}^n)} \leq C \|u\|_H, \quad u \in H,
\]

we can find a subsequence $(x_{\bar{k}})$ which converges weakly in $H$ and strongly in $L^\infty(I, \mathbb{R}^n)$ and $L^2(I, \mathbb{R}^n)$ to some function $x \in H$. Because the subsequence is convergent in $L^\infty(I, \mathbb{R}^n)$ it is also Cauchy under this norm. As $j, \bar{k} \to \infty$ we can apply (31) to show this subsequence satisfies (26),

(33)

\[
|b(x_{\bar{j}} - x_{\bar{k}})| \leq K_B \|x_{\bar{j}} - x_{\bar{k}}\|^2_{L^\infty(I, \mathbb{R}^n)} \to 0.
\]

If in addition the subsequence $(x_{\bar{k}})$ converges to zero in $L^2(I, \mathbb{R}^n)$, the subsequence must also converge in $L^\infty(I, \mathbb{R}^n)$ to the zero function, and

\[
b(x_{\bar{k}}) \to 0.
\]

Then the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ satisfy the conditions of Theorem 4.4. The bilinear form $d(\cdot, \cdot)$ is the sum of these two bilinear forms as in (27). By this theorem, the operator $D$ associated with this bilinear form has the same essential spectrum as the operator $A$ associated with the bilinear form $a(\cdot, \cdot)$. Now we show that for any constant $\epsilon > 0$ there exists a positive constant $K_\epsilon$ such that

(34)

\[
|b(x)| \leq \epsilon \|\dot{x}\|^2 + K_\epsilon \|x\|^2 \quad x \in D(A).
\]

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We can use (31) to find a constant \( K_B \), and for any \( \epsilon > 0 \), let \( \xi = \epsilon / K_B \). Then there is a constant \( C_\xi \) which satisfies

\[
|b(x)| \leq K_B \|x\|_2^{\infty(I,\mathbb{R}^n)} \\
\leq K_B \left( \frac{\epsilon}{K_B} \|x\|^2 + C_\xi \|x\|^2 \right) \\
\leq \epsilon \|x\|^2 + \left( K_B \cdot C_\xi \right) \|x\|^2 .
\]

Setting \( K_\epsilon = K_B \cdot C_\xi \) gives the stated inequality. To show \( d(\cdot,\cdot) \) is closed, first apply (34) with \( \epsilon = 1/2 \). Thus there is a constant \( C_0 \) such that

\[
|b(u)| \leq \frac{1}{2} a(u) + C_0 \|u\|^2 .
\]

Now suppose a sequence \( (u_k) \subset D(d) \) satisfies

\[
d(u_j - u_k) \to 0 ,
\]

and \( (u_k) \to u \) in \( L^2(I,\mathbb{R}^n) \). The sequence is Cauchy in \( L^2(I,\mathbb{R}^n) \) and as \( j,k \) increase

\[
\|u_j - u_k\|^2 \to 0 .
\]

Suppose that \( u \notin D(d) \). Because the domains of \( d(\cdot,\cdot) \) and \( a(\cdot,\cdot) \) are the same, \( u \notin D(a) \). We have shown above that \( a(\cdot,\cdot) \) is closed, so the sequence cannot be Cauchy and as \( j,k \) increase \( a(u_j - u_k) \) does not approach zero. But by (36),

\[
a(u_j - u_k) - b(u_j - u_k) \to 0 .
\]

Applying the inequality in (35) bounds the magnitude of each \( b(\cdot,\cdot) \) term, and since \( a(u,u) \geq 0 \), the following inequality is satisfied,

\[
a(u_j - u_k) - b(u_j - u_k) \geq \frac{1}{2} a(u_j - u_k) - C_0 \|u_j - u_k\|^2 .
\]

Adding the last term to both sides leaves only the positive bilinear form on the right side,

\[
a(u_j - u_k) - b(u_j - u_k) + C_0 \|u_j - u_k\|^2 \\
\geq \frac{1}{2} a(u_j - u_k) \\
\geq 0 .
\]

As \( j,k \) increase the left side of this equation approaches zero so the center term must also approach zero, a contradiction to the statement above. Therefore, \( u \in D(a) = D(d) \), and \( d(\cdot,\cdot) \) is also a closed bilinear form.

Next we show that there exists a positive constant \( N \) such that for any \( x \in D(a) \),

\[
d(x) + N \|x\|^2 \geq 0 .
\]
For any positive constant $\epsilon > 0$ we can find $K_\epsilon$ which satisfies (34) and thereby find a lower bound for $b(x, x)$,

$$a(x) + b(x) + N\|x\|^2 \geq a(x) - \epsilon\|\dot{x}\|^2 - K_\epsilon\|x\|.$$ 

We have shown that $d(\cdot, \cdot)$ is closed, and as the sum of two Hermitian bilinear forms, $d(\cdot, \cdot)$ is clearly Hermitian. Its domain is dense in $L^2(I, \mathbb{R}^n)$ and the $N$ in (37) satisfies the conditions of Theorem 4.3, so the operator $D$ associated with this bilinear form is self-adjoint and has its spectrum bounded below by $-N$. We have shown that the essential spectrum of this operator is the null set, so the spectrum is discrete and we can number the eigenvalues in increasing order, and each eigenvalue is of finite multiplicity.

5 Flows

Let $E$ be a Banach space, and let $\Sigma$ be the set of all continuous maps $\sigma = \sigma(t)$ from $E \times [0, 1]$ to $E$ such that

1. $\sigma(0)$ is the identity map,
2. for each $t \in [0, 1]$, $\sigma(t)$ is a homeomorphism of $E$ onto $E$,
3. $\sigma'(t)$ is piecewise continuous on $[0, 1]$ and satisfies

$$\|\sigma'(t)u\| \leq \text{const.}, \quad u \in E.$$ 

The mappings in $\Sigma$ are called flows. We note the following.

**Remark 5.1.** If $\sigma_1$, $\sigma_2$ are in $\Sigma$, define $\sigma_3 = \sigma_1 \circ \sigma_2$ by

$$\sigma_3(s) = \begin{cases} \sigma_1(2s), & 0 \leq s \leq \frac{1}{2}, \\ \sigma_2(2s - 1)\sigma_1(1), & \frac{1}{2} < s \leq 1. \end{cases}$$

Then $\sigma_1 \circ \sigma_2 \in \Sigma$.

6 Sandwich systems

Let $E$ be a Banach space. We define a nonempty collection $K$ of nonempty subsets $K \subset E$ to be a sandwich system if $K$ has the following property:

$$\sigma \in \Sigma, \ K \in K \implies \exists \overline{K} \in K : \overline{K} \subset \sigma(1)K.$$ 

We have

**Theorem 6.1.** Let $K$ be a sandwich system, and let $G(u)$ be a $C^1$ functional on $E$. Define

$$a := \inf_{K \in \mathcal{K}} \sup_{\overline{K} \subset \sigma(1)K} G,$$

(39)
and assume that the quantity \( a \) is finite. Assume, in addition, that there is a constant \( C_0 \) such that for each \( \delta > 0 \) there is a \( K \in \mathcal{K} \) satisfying

\[
\sup_K G \leq a + \delta,
\]

such that

\[
u \in K, \ G(u) \geq a - \delta \implies \|u\| \leq C_0.
\]

Then there is a bounded sequence \( \{u_k\} \subset E \) such that

\[
G(u_k) \to a, \quad \|G'(u_k)\| \to 0.
\]

**Theorem 6.2.** Let \( K \) be a sandwich system, and let \( G(u) \) be a \( C^1 \) functional on \( E \). Assume that there are subsets \( A,B \) of \( E \) such that

\[
a_0 := \sup_A G < \infty, \quad b_0 := \inf_B G > -\infty,
\]

\( A \in \mathcal{K} \) and

\[
B \cap K \neq \emptyset, \quad K \in \mathcal{K}.
\]

Assume, in addition, that there is a constant \( C_0 \) such that for each \( \delta > 0 \) there is a \( K \in \mathcal{K} \) satisfying (40) such that (41) holds. Then the quantity \( a \) given by (39) satisfies \( b_0 \leq a \leq a_0 \) and there is a bounded sequence \( \{u_k\} \subset E \) such that

\[
G(u_k) \to a, \quad \|G'(u_k)\| \to 0.
\]

**Definition 6.3.** We shall say that sets \( A,B \) in \( E \) form a sandwich pair if there is a sandwich system \( K \) such that \( A \in \mathcal{K} \) and (44) holds.

We have

**Theorem 6.4.** Let \( N \) be a finite dimensional subspace of a Banach space \( E \), and let \( p \) be any point of \( N \). Let \( F \) be a continuous map of \( E \) onto \( N \) such that \( F = I \) on \( N \). Then \( A = N \) and \( B = F^{-1}(p) \) form a sandwich pair.

**Corollary 6.5.** Let \( N \) be a closed subspace of a Hilbert space \( E \) and let \( M = N^\perp \). Assume that at least one of the subspaces \( M,N \) is finite dimensional. Then \( M,N \) form a sandwich pair.

**Remark 6.6.** Corollary 6.5 includes the case \( N = \{0\}, \ M = H \). The sandwich system can be taken as the collection of sets, each consisting of a single element of \( H \). In this case

\[
a = \inf_H G
\]

and \( \sup_A G = G(0) \).
7 Linking systems

Let $E$ be a Banach space. We define a nonempty collection $\mathcal{K}$ of nonempty subsets $K \subset E$ each containing the set $A$ to be a **linking system** for $A \notin \mathcal{K}$ if $\mathcal{K}$ has the following property:

$$\forall \sigma \in \Sigma, \ K \in \mathcal{K} \implies \exists \tilde{K} \in \mathcal{K} : \tilde{K} \subset \bigcup_{t \in [0,1]} \sigma(t)A \cup \sigma(1)K.$$  

We have

**Theorem 7.1.** Let $\mathcal{K}$ be a linking system for a set $A$, and let $G(u)$ be a $C^1$ functional on $E$. Define

$$a := \inf_{K \in \mathcal{K}} \sup_{K} G,$$

and assume that the quantity $a$ is finite. If $a = a_0 = \sup_A G$, we add the hypothesis

$$g_K = \{ u \in K \setminus A : G(u) \geq a_0 \} \neq \phi, \ K \in \mathcal{K}.$$  

Assume, in addition, that there is a constant $C_0$ such that for each $\delta > 0$ there is a $K \in \mathcal{K}$ satisfying

$$\sup_K G \leq a + \delta,$$

such that

$$u \in K, \ G(u) \geq a - \delta \implies \|u\| \leq C_0.$$  

Then there is a bounded sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to a, \ \|G'(u_k)\| \to 0.$$  

If $a = a_0$, we can require that $d(u_k, B) \to 0$.

**Theorem 7.2.** Let $\mathcal{K}$ be a linking system for a set $A$, and let $G(u)$ be a $C^1$ functional on $E$. Assume that the quantity $a$ given by (46) is $< \infty$ and there is a subset $B$ of $E$ such that

$$a_0 := \sup_A G \leq b_0 := \inf_B G$$

and

$$B \cap K \neq \phi, \ K \in \mathcal{K}.$$  

Assume, in addition, that there is a constant $C_0$ such that for each $\delta > 0$ there is a $K \in \mathcal{K}$ satisfying (40) such that (41) holds. Then the quantity $a$ given by (46) satisfies $a_0 \leq b_0 \leq a$ and there is a bounded sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to a, \ \|G'(u_k)\| \to 0.$$  

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Definition 7.3. We shall say that set $A$ links a set $B$ in $E$ if there is a linking system $K$ for $A$ such that (51) holds.

A very useful linking system is the following. Let $E$ be a Banach space. The set $\Phi$ of mappings $\Gamma(t) \in C(E \times [0,1], E)$ is to have following properties:

a) for each $t \in [0,1)$, $\Gamma(t)$ is a homeomorphism of $E$ onto itself and $\Gamma(t)^{-1}$ is continuous on $E \times [0,1)$

b) $\Gamma(0) = I$

c) for each $\Gamma(t) \in \Phi$ there is a $u_0 \in E$ such that $\Gamma(1)u = u_0$ for all $u \in E$ and $\Gamma(t)u \to u_0$ as $t \to 1$ uniformly on bounded subsets of $E$.

d) For each $t_0 \in [0,1)$ and each bounded set $A \subset E$ we have

$$\sup_{0 \leq t \leq t_0} \{ \|\Gamma(t)u\| + \|\Gamma^{-1}(t)u\| \} < \infty.$$

It is easy to see that

$$K = \{ \Gamma(t)A : \Gamma \in \Phi, t \in [0,1] \}$$

is a Sandwich System. A subset $A$ of $E$ links a subset $B$ of $E$ if $A \cap B = \phi$ and, for each $\Gamma(t) \in \Phi$, there is a $t \in (0,1]$ such that $\Gamma(t)A \cap B \neq \phi$.

Theorem 2.1.1 of [23] states: Let $G$ be a $C^1$-functional on $E$, and let $A,B$ be subsets of $E$ such that $A$ links $B$ and

$$a_0 := \sup_A G \leq b_0 := \inf_B G.$$

Assume that

$$a := \inf_{\Gamma \in \Phi} \sup_{0 \leq s \leq 1} \sup_{u \in A} G(\Gamma(s)u)$$

is finite. Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to a, \ G'(u_k) \to 0.$$

If $a = b_0$, then we can also require that

$$d(u_k, B) \to 0.$$
8 The parameter problem

Let $E$ be a reflexive Banach space with norm $\|\cdot\|$. Suppose that $G \in C^1(E, \mathbb{R})$ is of the form: $G(u) := I(u) - J(u), u \in E$, where $I, J \in C^1(E, \mathbb{R})$ map bounded sets to bounded sets. Define

$$G_\lambda(u) = \lambda I(u) - J(u), \quad \lambda \in \Lambda,$$

where $\Lambda$ is an open interval contained in $(0, +\infty)$. Assume one of the following alternatives holds.

$$(H_1) \quad I(u) \geq 0 \text{ for all } u \in E \text{ and } I(u) + |J(u)| \to \infty \text{ as } \|u\| \to \infty.$$

$$(H_2) \quad I(u) \leq 0 \text{ for all } u \in E \text{ and } |I(u)| + |J(u)| \to \infty \text{ as } \|u\| \to \infty.$$

Furthermore, we suppose that

$$(H_3) \quad \text{There is a sandwich system } K \text{ such that } a(\lambda) := \inf_{K \in K} \sup_{K} G_\lambda \text{ is finite for each } \lambda \in \Lambda.$$

**Theorem 8.1.** Assume that $(H_1)$ or $(H_2)$ and $(H_3)$ hold. Then we have

1. For almost all $\lambda \in \Lambda$ there exists a constant $k_0(\lambda) := k_0$ (depending only on $\lambda$) such that for each $\delta > 0$ there exists a $K \in K$ such that

   $$\sup_{K} G_\lambda \leq a(\lambda) + \delta$$

   and

   $$(53) \quad \|u\| \leq k_0 \text{ whenever } u \in K \text{ and } G_\lambda(u) \geq a(\lambda) - \delta.$$

2. For almost all $\lambda \in \Lambda$ there exists a bounded sequence $u_k(\lambda) \in E$ such that

   $$\|G'_\lambda(u_k)\| \to 0, \quad G_\lambda(u_k) \to a(\lambda) := \inf_{K \in K} \sup_{K} G_\lambda, \quad \text{as } k \to \infty.$$

**Corollary 8.2.** The conclusions of Theorem 8.1 hold if we replace hypothesis $(H_3)$ with

$$(H'_3) \quad \text{There is a sandwich pair } A, B \text{ such that }$$

$$a_0 := \sup_A G_\lambda < \infty, \quad b_0 := \inf_B G_\lambda > -\infty$$

for each $\lambda \in \Lambda$. 

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Corollary 8.3. The conclusions of Theorem 8.1 hold if we replace hypothesis \((H_3)\) with
\[(H_3') \text{ There is a linking system } K \text{ such that } a(\lambda) := \inf_{K \in K} \sup_K G_\lambda \text{ is finite for each } \lambda \in \Lambda.\]

Corollary 8.4. The conclusions of Theorem 8.1 hold if we replace hypothesis \((H_3)\) with
\[(H_3'') \text{ There are sets } A, B \text{ such that } A \text{ links } B \text{ and }\]
\[a_0 := \sup_A G_\lambda \leq b_0 := \inf_B G_\lambda\]
for each \(\lambda \in \Lambda\).

9 Proofs of the theorems

We now give the proof of Theorem 2.1.

Proof. Let
\[I(x) = d(w), \quad J(x) = -d(v) + 2 \int_I V(t, x(t)) dt.\]
Thus,
\[G_\lambda(x) = \lambda I(x) - J(x), \quad x \in H.\]
By Hypothesis 1,
\[I(x) + J(x) \to \infty \text{ as } \|x\|_\infty \to \infty.\]
To see this, note that
\[d(w) - d(v) + 2 \int_I V(t, x) \to \infty, \quad \|x\|_H \to \infty.\]
This follows from the fact that
\[2V(t, x) + \nu_t |x|^2 \to \infty, \quad |x| \to \infty,\]
where \(\nu_t = \min[\lambda_t, |\lambda_{t-1}|].\) Hence,
\[d(w) - d(v) + 2 \int_I V(t, x) \geq \nu_t \|x\|^2 + 2 \int_I V(t, x) dt \to \infty, \quad \|x\|_H \to \infty.\]

We note that Hypothesis 1 implies
\[G_\lambda(v) \leq Q, \quad v \in N,\]
where
\[ Q = \int_I W(t) \, dt. \]

In fact, we have
\[ G_\lambda(x) = d(x) - 2 \int_I V(t, x) \, dt \leq \int_I [\lambda t - 1|x|^2 - 2V(t, x)] \, dt \leq Q, \quad x \in N. \]

If \( x \in M \), we have by Hypothesis 2 that
\[
G_\lambda(x) \geq \lambda d(x) - \int_I \mu |x(t)|^2 \, dt - Q \geq (\lambda \lambda t - \mu) \|x\|^2 - Q \geq -Q,
\]
provided \( \lambda \geq \mu / \lambda t \).

By Corollary 6.5, \( M \) and \( N \) form a sandwich pair. Then by Corollary 8.2, for almost every \( \lambda \in [\mu / \lambda t, \infty) \) there is a bounded sequence \( \{x^{(k)}\} \subset H \) such that
\[
G_\lambda(x^{(k)}) = d_\lambda(x^{(k)}) - 2 \int_I V(t, x^{(k)}(t)) \, dt \to c,
\]
\[
(G'_\lambda(x^{(k)}), z) / 2 = d_\lambda(x^{(k)}), z) - \int_I \nabla_x V(t, x^{(k)}(t)) \cdot z(t) \, dt \to 0, \quad z \in H
\]
and
\[
(G'_\lambda(x^{(k)}), x^{(k)}) / 2 = d_\lambda(x^{(k)}), x^{(k)}) - \int_I \nabla_x V(t, x^{(k)}(t)) \cdot x^{(k)}(t) \, dt \to 0,
\]
where \( -Q \leq c \leq Q \). Since
\[ \rho_k = \|x^{(k)}\|_H \leq C, \]
there is a renamed subsequence such that \( x^{(k)} \) converges to a limit \( x \in H \) weakly in \( H \) and uniformly on \( I \). From (61) we see that
\[ (G'_\lambda(x), z) / 2 = d_\lambda(x, z) - \int_I \nabla_x V(t, x(t)) \cdot z(t) \, dt = 0, \quad z \in H, \]
from which we conclude easily that \( x \) is a solution of (8).

Proof of Theorem 2.2. We note that Hypothesis 1 implies
\[
G_\lambda(v) \leq 0, \quad v \in N.
\]
In fact, we have
\[ G_\lambda(x) = d(x) - 2 \int_I V(t, x) \, dt \leq \int_I [\lambda_{l-1}|x|^2 - 2V(t, x)] \, dt \leq 0, \quad x \in N. \]

Note that there is a positive \( \rho > 0 \) such that
\[ |x(t)| < m \]
when \( \|x\|_H = \rho \). In fact, we have \( |x(t)| \leq c_0 \|x\|_H \). If \( \lambda > \mu/\lambda_{l-1} \) and \( x \in M \), then
\[ G_\lambda(x) = d_\lambda(x) - 2 \int_I V(t, x) \, dt \geq d(x) \left[ \lambda - \frac{\mu \|x\|^2}{d(x)} \right] > \varepsilon > 0. \]

Take
\[ A = \partial B_\rho \cap M, \]
\[ B = N, \]
where
\[ B_\rho = \{ x \in H : \|x\|_H < \rho \}. \]

By By Example 8, p. 22 of [28], \( A \) links \( B \). Moreover,
\[ \sup_A [-G_\lambda] \leq 0 \leq \inf_B [-G_\lambda]. \]

Hence, we may apply Corollary 8.3 to conclude that for a.e. \( \lambda \in [\mu/\lambda_{l-1}, \infty) \) there is a bounded sequence \( \{x^{(k)}\} \subset H \) such that
\[ G_\lambda(x^{(k)}) = d_\lambda(x^{(k)}) - 2 \int_I V(t, x^{(k)}(t)) \, dt \to c \leq 0, \]
(64)

\[ (G'_\lambda(x^{(k)}), z)/2 = d_\lambda(x^{(k)}), z) - \int_I \nabla_x V(t, x^{(k)}) \cdot z(t) \, dt \to 0, \quad z \in H \]
(65)

and
\[ (G'_\lambda(x^{(k)}), x^{(k)})/2 = d_\lambda(x^{(k)}) - \int_I \nabla_x V(t, x^{(k)}) \cdot x^{(k)} \, dt \to 0. \]
(66)

Since \( \rho_k = \|x^{(k)}\|_H \leq C \),
there is a renamed subsequence such that \( x^{(k)} \) converges to a limit \( x \in H \) weakly in \( H \) and uniformly on \( I \). From (60) we see that
\[ (G'_\lambda(x), z)/2 = d_\lambda(x), z) - \int_I \nabla_x V(t, x(t)) \cdot z(t) \, dt = 0, \quad z \in H, \]
(67)
from which we conclude easily that \( x \) is a solution of (8).

Proof of Theorem 2.3 Let

\[
y(t) = v + sw_0,
\]

where \( v \in N, s \geq 0, \) and \( w_0 \in M \) is an eigenfunction of \( D \) corresponding to \( \lambda_l \). Consequently,

\[
G_\lambda(y) = s^2 \lambda d(w_0) + d(v) - 2 \int_\Omega V(t, y(t)) \, dt \\
\leq \lambda \lambda_l s^2 \|w_0\|^2 + \lambda_{l-1} \|v\|^2 - \gamma \int_\Omega |y(t)|^2 \, dt + Q \\
\leq (\lambda_{l-1} - \gamma)\|v\|^2 + (\lambda \lambda_l - \gamma) s^2 \|w_0\|^2 + Q
\]

\[
\to -\infty \text{ as } s^2 + \|v\|^2 \to \infty,
\]

provided \( \lambda < \gamma / \lambda_l \), where

\[
Q = \int_\Omega W(t) \, dt.
\]

Take

\[
A = \{ v \in N : \|v\|_H \leq R \} \cup \{ sw_0 + v : v \in N, s \geq 0, \|sw_0 + v\|_H = R \},
\]

\[
B = \partial B_\rho \cap M, \ 0 < \rho < R.
\]

By Example 3, p.38, of [23], \( A \) links \( B \). Moreover, if \( R \) is sufficiently large,

\[
\sup_A G_\lambda \leq 0 < \varepsilon \leq \inf_B G_\lambda.
\]

We may now apply Corollary 8.3 to conclude that that for almost all \( \lambda \in [\mu / \lambda_l, \gamma / \lambda_l] \) there is a bounded sequence \( \{x^{(k)}\} \subset H \) such that

\[
G_\lambda(x^{(k)}) = d_\lambda(x^{(k)}) - 2 \int_\Omega V(t, x^{(k)}(t)) \, dt \to c \geq \varepsilon > 0,
\]

\[
(G'_\lambda(x^{(k)}), z) / 2 = d_\lambda(x^{(k)}, z) - \int_\Omega \nabla x V(t, x^{(k)}(t)) \cdot z(t) \, dt \to 0, \quad z \in H
\]

and

\[
(G'_\lambda(x^{(k)}), x^{(k)}) / 2 = d_\lambda(x^{(k)}) - \int_\Omega \nabla x V(t, x^{(k)}(t)) \cdot x^{(k)}(t) \, dt \to 0.
\]

Since

\[
\rho_k = \|x^{(k)}\|_H \leq C,
\]
there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in H$ weakly in $H$ and uniformly on $I$. From (69) we see that

$$G_\lambda(x) = \frac{1}{2} d_\lambda(x, z) - \int_I \nabla x V(t, x(t)) \cdot z(t) \, dt = 0, \quad z \in H,$$

from which we conclude easily that $x$ is a solution of (8). Moreover, (70) implies

$$d_\lambda(x^{(k)}) \to \int_I \nabla x V(t, x) \cdot x \, dt = d_\lambda(x).$$

Consequently,

$$x^{(k)} \to x$$

strongly in $H$. This means that

$$G_\lambda(x) = d_\lambda(x) - 2 \int_I V(t, x) \, dt = c \geq \varepsilon > 0.$$

But

$$G_\lambda(0) = -2 \int_I V(t, 0) \, dt \leq 0.$$

Hence, $x(t) \neq 0$.

The proofs of Theorems 3.1, 3.2 and 3.3 are similar to those of Theorems 2.1, 2.2 and 2.3 taking into account that $\lambda_0 > 0$ and $N = \{0\}$.

10 Some lemmas

Before giving the remaining proofs, we shall prove a few lemmas.

Lemma 10.1.

$$\int_I |V(t, u) - V(t, rw) + (r^2 w - \frac{1 + r^2}{2} u)V(t, u)| \leq C,$$

$$u \in D, w \in M, r \in [0, 1], \|w\|_D \leq \|u\|_D,$$

where the constant $C$ does not depend on $u, w, r$.

Proof. This follows from (9) if we take $t = u$, and $s = rw - u$. \hfill \Box

Lemma 10.2. If $u$ satisfies $G_\lambda'(u) = 0$ for some $\lambda > 0$, then there is a constant $C$ independent of $u, \lambda, r$ such that

$$G_\lambda(rw) - r^2 (Dv, v) - G_\lambda(u) \leq C$$

for all $r \in [0, 1]$, where $w, v$ are the projections of $u$ onto $M, N$, respectively.
Proof. For such $u$, let $u = v + w$, where $v \in N$, $w \in M$. Then
\[
\frac{(G'_\lambda(u), g)}{2} = \lambda(Dw, g_1) + (Dv, g_2) - \int gV(t, u) = 0
\]
for every $g \in D$, where $g_1, g_2$ are the projections of $g$ onto $M, N$, respectively. Take
\[
g = (r^2 + 1)v - (r^2 - 1)w = (r^2 + 1)u - 2r^2w.
\]
Then we have
\[
G_\lambda(rw) - r^2(Dv, v) - G_\lambda(u) = \lambda(r^2 - 1)(Dw, w) - (Dv, v)
\]
\[
+ \lambda(Dw, g_1) + (Dv, g_2) - r^2(Dv, v)
\]
\[
+ \int [2V(t, u) - 2V(t, rw) - gV(t, u)] dx
\]
\[
= \int [2V(t, u) - 2V(t, rw) - ((r^2 + 1)u - 2r^2w)V(t, u)] dx
\]
\[
\leq C
\]
by Lemma 10.1. \hfill \square

11 Finding the sequences

We proceed to the proof of Theorem 6.1. Let $M = C_0 + 1$. Then
\[
\|\sigma(1)v\| \leq M
\]
whenever $\sigma \in \Sigma$ satisfies $\|\sigma'(t)\| \leq 1$ and $v \in E$ satisfies $\|v\| \leq C_0$. If the theorem were false, then there would be a $\delta > 0$ such that
\[
\|G'(u)\| \geq 3\delta
\]
when
\[
u \in \{u \in E : \|u\| \leq M + 2, |G(u) - a| \leq 3\delta\}.
\]
Take $\delta < 1/3$. Since $G \in C^1(E, \mathbb{R})$, for each $\theta < 1$ there is a locally Lipschitz continuous mapping $Y(u)$ of $\hat{E} = \{u \in E : G'(u) \neq 0\}$ into $E$ such that
\[
\|Y(u)\| \leq 1, \theta\|G'(u)\| \leq (G'(u), Y(u)), \quad u \in \hat{E}
\]
(cf., e.g., [23]). Take $\theta > 2/3$. Let
\[
Q_0 = \{u \in E : \|u\| \leq M + 1, |G(u) - a| \leq 2\delta\},
\]
\[
Q_1 = \{u \in E : \|u\| \leq M, |G(u) - a| \leq \delta\},
\]
\[
Q_2 = E \setminus Q_0,
\]
\[
\eta(u) = d(u, Q_2)/[d(u, Q_1) + d(u, Q_2)].
\]
It is easily checked that $\eta(u)$ is locally Lipschitz continuous on $E$ and satisfies

$$
\begin{align*}
\eta(u) &= 1, & u &\in Q_1, \\
\eta(u) &= 0, & u &\in \bar{Q}_2, \\
\eta(u) &\in (0,1), & \text{otherwise}.
\end{align*}
$$

Let $W(u) = -\eta(u)Y(u)$. Then

$$
\|W(u)\| \leq 1, \quad u \in E.
$$

By Theorem 4.5 of [28], for each $v \in E$ there is a unique solution $\sigma(t)v$ of

$$
\sigma'(t) = W(\sigma(t)), \quad t \in \mathbb{R}^+, \quad \sigma(0) = v.
$$

We have

$$
\begin{align*}
&dG(\sigma(t)v)/dt = -\eta(\sigma(t)v)(G'(\sigma(t)v), Y(\sigma(t)v)) \\
&\leq -\theta\eta(\sigma)\|G'(\sigma)\| \\
&\leq -3\theta\delta\eta(\sigma).
\end{align*}
$$

Let $K \in \mathcal{K}$ satisfy the hypotheses of the theorem and let $v$ be any element of $K \cap Q_1$. Then $\|v\| \leq C_0$ since $G(v) \geq a - \delta$. If there is a $t_1 \leq 1$ such that $\sigma(t_1)v \notin Q_1$, then

$$
G(\sigma(1)v) < a - \delta,
$$

since $\|\sigma(1)v\| \leq M$,

$$
G(\sigma(1)v) \leq G(\sigma(t_1)v)
$$

and the right hand side cannot be greater than $a + \delta$ by (80). On the other hand, if $\sigma(t)v \in Q_1$ for all $t \in [0,1]$, then we have by (80)

$$
G(\sigma(1)v) \leq a + \delta - 3\delta\theta < a - \delta.
$$

If $v \in K \setminus Q_1$, then we must have

$$
G(\sigma(1)v) \leq G(v) < a - \delta,
$$

since $G(v) \geq a - \delta$ would put $v$ into $Q_1$. Hence

$$
G(\sigma(1)v) < a - \delta, \quad v \in K.
$$

By hypothesis, $\exists \bar{K} \in \mathcal{K} : \bar{K} \subset \sigma(1)K$. This means that

$$
G(w) < a - \delta, \quad w \in \bar{K}.
$$

But this contradicts the definition (39) of $a$. Hence (75) cannot hold for $u$ satisfying (76). This proves the theorem.
Proof of Theorem 6.2. Since $A \in K$, clearly $a \leq a_0$. Moreover, for any $K \in K$, we have

$$b_0 = \inf_B G \leq \inf_{B \cap K} G \leq \sup_{B \cap K} G \leq \sup_K G.$$  

Hence, $b_0 \leq a$. Apply Theorem 6.1.

Proof of Theorem 7.1. Assume first that $a_0 < a$. Let $\delta < a - a_0$. Then

$$\|\sigma(1)v\| \leq M$$

whenever $\sigma \in \Sigma$ satisfies $\|\sigma'(t)\| \leq 1$ and $v \in E$ satisfies $\|v\| \leq C_0$. If the theorem were false, then there would be a $\delta > 0$ such that

(84) \[\|G'(u)\| \geq 3\delta\]

when

(85) \[u \in \{u \in E : \|u\| \leq M + 2, |G(u) - a| \leq 3\delta\} . \]

Take $\delta < 1/3$. Since $G \in C^1(E, \mathbb{R})$, for each $\theta < 1$ there is a locally Lipschitz continuous mapping $Y(u)$ of $\mathring{E} = \{u \in E : G'(u) \neq 0\}$ into $E$ such that

(86) \[\|Y(u)\| \leq 1, \theta\|G'(u)\| \leq (G'(u), Y(u)), \quad u \in \mathring{E} \]

(cf., e.g., [23]). Take $\theta > 2/3$. Let

$$Q_0 = \{u \in E : \|u\| \leq M + 2, |G(u) - a| \leq 2\delta\},$$

$$Q_1 = \{u \in E : \|u\| \leq M, |G(u) - a| \leq \delta\},$$

$$Q_2 = E \setminus Q_0,$$

$$\eta(u) = d(u, Q_2)/(d(u, Q_1) + d(u, Q_2)).$$

It is easily checked that $\eta(u)$ is locally Lipschitz continuous on $E$ and satisfies

(87) \[
\begin{align*}
\eta(u) &= 1, \quad u \in Q_1, \\
\eta(u) &= 0, \quad u \in Q_2, \\
\eta(u) &\in (0, 1), \quad \text{otherwise}.
\end{align*}
\]

Let

$$W(u) = -\eta(u)Y(u).$$

Then

$$\|W(u)\| \leq 1, \quad u \in E.$$  

By Theorem 4.5 of [28], for each $v \in E$ there is a unique solution $\sigma(t)v$ of

(88) \[\sigma'(t) = W(\sigma(t)), \quad t \in \mathbb{R}^+, \quad \sigma(0) = v. \]
We have

\[
\begin{align*}
\frac{dG(\sigma(t)v)}{dt} &= -\eta(\sigma(t)v)(G'(\sigma(t)v), Y(\sigma(t)v)) \\
&\leq -\theta \eta(\sigma) \|G'(\sigma)\| \\
&\leq -3\theta \delta \eta(\sigma).
\end{align*}
\]

Let \( K \in \mathcal{K} \) satisfy the hypotheses of the theorem and let \( v \) be any element of \( K \cap Q_1 \). Then \( \|v\| \leq C_0 \) since \( G(v) \geq a - \delta \). If there is a \( t_1 \leq 1 \) such that \( \sigma(t_1)v \notin Q_1 \), then

\[
G(\sigma(1)v) < a - \delta,
\]

since \( \|\sigma(1)v\| \leq M \),

\[
G(\sigma(1)v) \leq G(\sigma(t_1)v)
\]

and the right hand side cannot be greater than \( a + \delta \) by (89). On the other hand, if \( \sigma(t)v \in Q_1 \) for all \( t \in [0,1] \), then we have by (80)

\[
G(\sigma(1)v) \leq a + \delta - 3\delta \theta < a - \delta.
\]

If \( v \in K \setminus Q_1 \), then we must have

\[
G(\sigma(1)v) \leq G(v) < a - \delta,
\]

since \( G(v) \geq a - \delta \) would put \( v \) into \( Q_1 \). Hence

\[
G(\sigma(1)v) < a - \delta, \quad v \in K.
\]

Moreover,

\[
G(\sigma(t)v) \leq G(u) \leq a_0 < a - \delta, \quad u \in A, t \in [0,1].
\]

By hypothesis, \( \exists \tilde{K} \in \mathcal{K} : \tilde{K} \subset \sigma(1)K \cup_{t \in [0,1]} \sigma(t)A \). This means that

\[
G(w) < a - \delta, \quad w \in \tilde{K}.
\]

But this contradicts the definition (46) of \( a \). Hence (84) cannot hold for \( u \) satisfying (85). This proves the theorem for the case \( a > a_0 \).

Now assume \( a_0 = a \). In this case we assume

\[
g_K = \{ u \in K \setminus A : G(u) \geq a_0 \} \neq \phi, \quad K \in \mathcal{K}.
\]

Let

\[
B = \bigcup_{K \in \mathcal{K}} g_K.
\]

Then

\[
a_0 := \sup_{A} G \leq b_0 := \inf_{B} G \leq a.
\]
Note that

\[(93)\quad B \cap A = \phi, \quad B \cap K \neq \phi, \quad K \in \mathcal{K}.
\]

If there did not exist a sequence satisfying the conclusion of the theorem, then there would be positive numbers \(\delta, T\) such that \(2/3\theta < T \leq 1\) and \(\|G'(u)\| \geq 3\delta\) holds whenever

\[u \in Q = \{ u \in E : \|u\| \leq M + 2, \ d(u, B) \leq 4T, \ |G(u) - a| \leq 3\delta\}.
\]

Let

\[
Q_0 = \{ u \in E : \|u\| \leq M + 1, \ d(u, B) \leq 3T, \ |G(u) - a| \leq 2\delta\}
\]

\[
Q_1 = \{ u \in E : \|u\| \leq M, \ d(u, B) \leq 2T, \ |G(u) - a| \leq \delta\}.
\]

Since \(a = a_0\), we see that \(Q_1 \neq \phi\). Define \(Q_2\) and \(\eta(u)\) as before and let \(\sigma(t)\) be the flow generated by the mapping given by \((88)\) with everything now with respect to the new sets \(Q_j\). Let \(u\) be any element in \(K \cap Q_1\), where \(K\) satisfies the hypotheses of the theorem. Then \(\|u\| \leq C_0\). If there is a \(t_1 \leq T\) such that \(\sigma(t_1)u \notin Q_1\), then either

\[(94)\quad G(\sigma(t_1)u) < a - \delta \]

or

\[(95)\quad d(\sigma(t_1)u, B) > 2T.
\]

Since

\[
\|\sigma(t)u - \sigma(t')u\| \leq |t - t'|,
\]

\[(95)\) implies

\[(96)\quad d(\sigma(t)u, B) > T, \quad 0 \leq t \leq T.
\]

On the other hand, if \(\sigma(t)u \in Q_1\) for all \(t \in [0, T]\), then

\[(97)\quad G(\sigma(T)u) \leq G(u) - 3\theta \delta T \leq a + \delta - 2\delta = a - \delta.
\]

Thus we have either

\[(98)\quad G(\sigma(T)u) < a - \delta \]

or \((96)\) holds. If \(u \in K \setminus Q_1\), then either \(G(u) < a - \delta\) or \(d(u, B) > 2T\). Again it follows that either \((96)\) or \((98)\) holds. Since \(b_0 = a\), this shows that

\[(99)\quad \sigma(T)K \cap B = \phi.
\]

We also note that

\[(100)\quad \sigma(t)A \cap B = \phi, \quad 0 \leq t \leq T.
\]
For we have by (89)
\[ G(\sigma(t)u) \leq a_0 - 3\delta \int_0^t \eta(\sigma(\tau)u) d\tau, \quad u \in A. \]

If \( \sigma(t)u \in B \), we must have \( G(\sigma(t)u) \geq b_0 \geq a_0 \). The only way this can happen is if
\[ \eta(\sigma(\tau)u) \equiv 0, \quad 0 \leq \tau \leq t. \]

But this implies \( \sigma(\tau)u \in Q_2 \) for such \( \tau \), and this in turn implies either
\[ G(\sigma(\tau)u) < a - \delta, \quad 0 \leq \tau \leq t \]
or
\[ d(\sigma(\tau)u, B) > 2T, \quad 0 \leq \tau \leq t. \]

In either case we cannot have \( \sigma(t)u \in B \). Thus (100) holds. By hypothesis, there is a \( \tilde{K} \in \mathcal{K} \) such that \( \tilde{K} \subset \bigcup_{t \in [0,1]} \sigma(t)A \cup \sigma(T)K \). By (99) and (100), \( \tilde{K} \cap B = \emptyset \). This violates (93). This completes the proof of the theorem.

**Proof of Theorem 7.2.** Let \( B \) satisfy (50) and (51). Then \( a_0 \leq b_0 \leq a < \infty \). If \( a_0 < a \), we follow the proof of Theorem 7.2. If \( a_0 = a \), for any \( K \in \mathcal{K} \), we have
\[ a_0 \leq b_0 = \inf_B G \leq \inf_{B \cap K} G \leq \sup_{B \cap K} G \leq \sup_{K \cap A} G \]
since \( A \cap B = \emptyset \). Hence,
\[ g_K \neq \emptyset, \quad K \in \mathcal{K}. \]

We can now follow the remainder of the proof of Theorem 7.2.

**Proof of Theorem 6.4.** We let \( \mathcal{K} = \{ \sigma(1)N : \sigma \in \Sigma \} \). If we can show that \( B \) satisfies (44), then the result will follow from Theorem 6.2. Now (44) is equivalent to
\[ F^{-1}(p) \cap \sigma(1)N \neq \emptyset, \quad \sigma \in \Sigma. \]

Let \( \Omega_R(p) \) be a ball in \( N \) with radius \( R \) and center \( p \), and let \( \sigma(t) \) be any flow in \( \Sigma_B \). Since
\[ (101) \quad \sigma(t)u - u = \int_0^t \sigma'(\tau) ud\tau, \]
we have
\[ \| \sigma(t)u - \sigma(s)u \| \leq \int_s^t \rho(d(\sigma(r)u, B)) dr, \]
where \( \rho = C(1 + |t|) \). If \( u \in A_R = \partial \Omega_R(p) \), and \( v \in B \), we have
\[ h(s) := d(\sigma(s)u, B) \leq \| \sigma(s)u - v \| \leq \| \sigma(t)u - v \| + \int_s^t \rho(d(\sigma(r)u, B)) dr. \]
This implies,

\[(102) \quad h(s) \leq h(t) + \int_{s}^{t} \rho(h(r)) \, dr.\]

Moreover, by Lemma 4.8 of [26], \(h(s)\) satisfies

\[h(s) \geq m(R), \quad 0 \leq s \leq 1, \ u \in \partial \Omega_{R}(p),\]

where \(m(R)\) is given by

\[\int_{m(R)}^{R} \frac{d\tau}{\rho(\tau)} = 1.\]

Note that \(m(R) \to \infty\) as \(R \to \infty\). Thus,

\[\|\sigma(s)u - F^{-1}(p)\| \geq h(s) \geq m(R) \to \infty, \ u \in A_{R}.\]

Consequently,

\[(103) \quad F^{-1}(p) \cap \sigma(1)A_{R} = \phi, \ \sigma \in \Sigma_{B},\]

for \(R\) sufficiently large. Now \(A_{R}\) links \(B\) (cf., e.g., [23]). For \(\Gamma \in \Phi\), define

\[\Gamma_{1}(s) = \begin{cases} 
\sigma(2s), & 0 \leq s \leq \frac{1}{2}, \\
\sigma(1)\Gamma(2s - 1), & \frac{1}{2} < s \leq 1.
\end{cases}\]

Clearly, \(\Gamma_{1} \in \Phi\). Consequently, there is a \(t_{0} \in [0, 1]\) such that

\[\Gamma_{1}(t_{0})A_{R} \cap B \neq \phi.\]

If \(t_{0} \leq \frac{1}{2}\), then

\[\sigma(2t_{0})A_{R} \cap B \neq \phi,\]

contradicting (103). If \(t_{0} > \frac{1}{2}\), then

\[\sigma(1)\Gamma(2t_{0} - 1)A_{R} \cap B \neq \phi.\]

Take \(\Gamma(s)u = (1 - s)u\). Then \(\Gamma \in \Phi\) and \(\Gamma(2t_{0} - 1)A_{R} \subset N\). Hence,

\[\sigma(1)N \cap B \neq \phi.\]

Thus (44) holds, and the theorem is proved.  \(\square\)
12 The monotonicity trick

We now give the proof of Theorem 8.1.

Proof. We prove conclusion (1) assuming the first alternative hypothesis \( (H_1) \). By \( (H_1) \), the map \( \lambda \mapsto a(\lambda) \) is nondecreasing. Hence, \( a'(\lambda) := da(\lambda)/d\lambda \) exists for almost every \( \lambda \in \Lambda \). From this point on, we consider those \( \lambda \) where \( a'(\lambda) \) exists. For fixed \( \lambda \in \Lambda \), let \( \lambda_n \in (\lambda, 2\lambda) \cap \Lambda, \lambda_n \to \lambda \) as \( n \to \infty \). Then there exists \( \tilde{n}(\lambda) \) such that

\[
(104) \quad a'(\lambda) - 1 \leq \frac{a(\lambda_n) - a(\lambda)}{\lambda_n - \lambda} \leq a'(\lambda) + 1 \quad \text{for } n \geq \tilde{n}(\lambda).
\]

Next, we note that there exist \( K_n \in K, k_0 := k_0(\lambda) > 0 \) such that

\[
(105) \quad \|u\| \leq k_0 \quad \text{whenever } G_\lambda(u) \geq a(\lambda) - (\lambda_n - \lambda).
\]

In fact, by the definition of \( a(\lambda_n) \), there exists \( K_n \) such that

\[
(106) \quad \sup_{K_n} G_\lambda(u) \leq \sup_{K_n} G_{\lambda_n}(u) \leq a(\lambda_n) + (\lambda_n - \lambda).
\]

If \( G_\lambda(u) \geq a(\lambda) - (\lambda_n - \lambda) \) for some \( u \in K_n \), then, by (104) and (106), we have that

\[
(107) \quad I(u) = \frac{G_{\lambda_n}(u) - G_\lambda(u)}{\lambda_n - \lambda} \leq \frac{a(\lambda_n) + (\lambda_n - \lambda) - a(\lambda) + (\lambda_n - \lambda)}{\lambda_n - \lambda} \leq a'(\lambda) + 3,
\]

and it follows that

\[
(108) \quad J(u) = \lambda_n I(u) - G_{\lambda_n}(u) \leq \lambda_n (a'(\lambda) + 3) - G_\lambda(u) \leq \lambda_n (a'(\lambda) + 3) - a(\lambda) + (\lambda_n - \lambda) \leq 2\lambda(a'(\lambda) + 3) - a(\lambda) + \lambda.
\]

On the other hand, by \( (H_1), (104), \) and \( (106), \)

\[
(109) \quad J(u) = \lambda_n I(u) - G_{\lambda_n}(u) \geq -G_{\lambda_n}(u) \geq -(a(\lambda_n) + (\lambda_n - \lambda)) \geq -a(\lambda) + (\lambda_n - \lambda)(a'(\lambda) + 2) \geq -a(\lambda) - \lambda|a'(\lambda) + 2|.
\]

Combining (107)–(109) and \( (H_1) \), we see that there exists \( k_0(\lambda) := k_0 \) (depending only on \( \lambda \)) such that (105) holds.
By the choice of $K_n$ and (104), we see that
\[ G\lambda(u) \leq G\lambda_n(u) \leq \sup_{K_n} G\lambda_n(u) \leq a(\lambda_n) + (\lambda_n - \lambda) \leq (a'(\lambda) + 1)(\lambda_n - \lambda) + a(\lambda) + (\lambda_n - \lambda) \leq a(\lambda) + (a'(\lambda) + 2)(\lambda_n - \lambda) \]
for all $u \in K_n$. We take $n$ sufficiently large to ensure that $|a'(\lambda) + 2|(\lambda_n - \lambda) < \delta$. This proves conclusion (1). Conclusion (2) now follows from Theorem 6.1. The proof under hypothesis ($H_2$) is similar, and is omitted.

\section{The remaining proofs}

Proof of Theorem 2.4. Let $\bar{\lambda} = \mu/\lambda_1$ and $\nu < \infty$. By Theorem 2.3, for a.e. $\lambda \in (\bar{\lambda}, \nu)$, there exists $u_\lambda$ such that $G\lambda'(u_\lambda) = 0$, $G\lambda(u_\lambda) = a(\lambda) \geq a(\lambda_0)$. Let $\lambda$ satisfy $\lambda_0 \leq \lambda < \nu$. Choose $\lambda_n \to \lambda$, $\lambda_n > \lambda$ such that there exists $x_n$ satisfying
\[ G\lambda_n' (x_n) = 0, \quad G\lambda_n (x_n) = a(\lambda_n) \geq a(\lambda_0). \]
Therefore,
\[ \int_I 2V(t, x_n) \frac{dt}{\|x_n\|_H^2} \leq C. \]

Now we prove that $\{x_n\}$ is bounded. If $\|x_n\|_H \to \infty$, let $\tilde{x}_n = x_n/\|x_n\|_H$. Then there is a renamed subsequence such that $\tilde{x}_n \to \tilde{x}$ weakly in $H$, strongly in $L^\infty(I)$ and a.e. in $I$. Let $\Omega_0$ be the set where $\tilde{x} \neq 0$. Then $|x_n(t)| \to \infty$ for $t \in \Omega_0$. If $\Omega_0$ had positive measure, then we would have
\[ C \geq \int_I \frac{2V(t, x_n)}{\|x_n\|_H^2} dt = \int_I \frac{2V(t, x_n)}{|x_n|^2} |\tilde{x}_n|^2 dt \geq \int_{\Omega_0} \frac{2V(t, x_n)}{|x_n|^2} |\tilde{x}_n|^2 dt + \lambda_{-1} \int_{I \setminus \Omega_0} |\tilde{x}_n|^2 dt \to \infty, \]
showing that $\tilde{x} = 0$ a.e. in $L^\infty(I)$. Hence, $\tilde{x}_n \to 0$ in $L^\infty(I)$. Since
\[ \|\tilde{x}_n\|_H^2 = d(\tilde{w}_n) - d(\tilde{v}_n) + \|\tilde{g}_n\|^2 = 1, \]
and $d(\tilde{v}_n) \to 0$, $\|\tilde{g}_n\|^2 \to 0$, we have $d(\tilde{w}_n) \to 1$. For any $s > 0$ and $h_n = s\tilde{x}_n$, we have
\[ (110) \int_I V(t, h_n) dt \to \int_I V(t, 0) dt = 0. \]
Take $r_n = s/R_n \to 0$. By Lemma 10.2
\[ (111) G\lambda_n (r_n w_n) - r_n^2 (Dv_n, v_n) - G\lambda_n (x_n) \leq C. \]
Hence,
\begin{equation}
G_{\lambda_n}(s\tilde{w}_n) - s^2(\tilde{D}\tilde{v}, \tilde{v}_n) \leq C'.
\end{equation}

But
\begin{align*}
G_{\lambda_n}(s\tilde{w}_n) - s^2(\tilde{D}\tilde{v}, \tilde{v}_n) &= \lambda_n s^2(\tilde{D}\tilde{w}, \tilde{w}_n) - s^2(\tilde{D}\tilde{v}, \tilde{v}_n) \\
&\geq s^2(\lambda d(\tilde{w}_n) - d(\tilde{v}_n)) - 2\int_I V(t, s\tilde{w}_n) \\
&\to \lambda s^2
\end{align*}

by (110). This implies
\begin{equation*}
G_{\lambda_n}(s\tilde{w}_n) - s^2(\tilde{D}\tilde{v}, \tilde{v}_n) \to \infty \text{ as } s \to \infty,
\end{equation*}
contrary to (112).

This contradiction shows that \( \|x_n\|_H \leq C \). Then there is a renamed subsequence such that \( x_n \to x \) weakly in \( H \), strongly in \( L^\infty(I) \) and a.e. in \( I \). It now follows that for the bounded renamed subsequence,
\begin{equation*}
G'_\lambda(x_n) \to 0, \quad G_\lambda(x_n) \to a(\lambda) \geq a(\lambda_0).
\end{equation*}

We can now apply Theorem 3.4.1 in [23, p. 64] to obtain the desired solution.

**Proof of Theorem 2.5.** We follow the proof of Theorem 2.4 until the statement:

Since
\begin{equation*}
\|\tilde{x}_n\|_H^2 = d(\tilde{w}_n) - d(\tilde{v}_n) + \|\tilde{g}_n\|^2 = 1,
\end{equation*}
and \( d(\tilde{v}_n) \to 0, \|\tilde{g}_n\|^2 \to 0 \), we have \( d(\tilde{w}_n) \to 1 \). We define \( \theta_n \in [0, 1] \) by
\begin{equation*}
G_{\lambda_n}(\theta_n x_n) = \max_{\theta \in [0, 1]} G_{\lambda_n}(\theta x_n).
\end{equation*}

For any \( c > 0 \) and \( h_n = c\tilde{x}_n \), we have
\begin{equation*}
\int_I V(t, h_n) dt \to \int_I V(t, 0) dt = 0.
\end{equation*}

Thus,
\begin{equation*}
G_{\lambda_n}(\theta_n x_n) \geq G_{\lambda_n}(c\tilde{x}_n) = c^2 \lambda_n d(\tilde{w}_n) - 2\int_I V(t, h_n) dt \to \lambda c^2, \quad n \to \infty.
\end{equation*}

Hence, \( G_{\lambda_n}(\theta_n x_n) \geq \lambda c^2 / 2 \) for \( n \) sufficiently large. That is, \( \lim_{n \to \infty} G_{\lambda_n}(\theta_n x_n) = \infty \). If there is a renamed subsequence such that \( \theta_n = 1 \), then
\begin{equation}
G_{\lambda_n}(x_n) \to \infty.
\end{equation}
If $0 \leq \theta_n < 1$ for all $n$, then we have $(G'_{\lambda_n}(\theta_n x_n), x_n) \leq 0$. Therefore,

$$
\int_I H(t, \theta_n x_n) dt = \int_I \left( \nabla_x V(t, \theta_n x_n) \theta_n x_n - 2V(t, \theta_n x_n) \right) dt
= G_{\lambda_n}(\theta_n x_n) - (G'_{\lambda_n}(\theta_n x_n), \theta_n x_n)
\geq G_{\lambda_n}(\theta_n x_n) \to \infty.
$$

By hypothesis,

$$
G_{\lambda_n}(x_n) = \int_I H(t, x_n) dx
\geq \int_I H(t, \theta_n x_n) dt/C - \int_I W(t) dt \to \infty.
$$

Thus, (113) holds in any case. But

$$
G_{\lambda_n}(x_n) = a(\lambda_n) \leq a(\nu) < \infty.
$$

This contradiction shows that $\|x_n\|_H \leq C$. It now follows that for a renamed subsequence,

$$
G'_{\lambda_n}(x_n) \to 0, \quad G_{\lambda_n}(x_n) \to a(\lambda) \geq a(\lambda_0).
$$

We can now apply Theorem 3.4.1 in [23, p. 64] to obtain the desired solution. 

Proof of Theorem 2.6. We may assume $\lambda = 1$. By the previous proof, $\mathcal{M}_a \neq \emptyset$. Let

$$
\alpha = \inf_{\mathcal{M}_a} G(x).
$$

There is a sequence $\{x^{(k)}\} \in \mathcal{M}_a$ such that

$$
G(x^{(k)}) = d(x^{(k)}) - 2 \int_I V(t, x^{(k)}(t)) dt \to \alpha,
$$

(114) and

$$
(G'(x^{(k)}), z)/2 = d(x^{(k)}, z) - \int_I \nabla_x V(t, x^{(k)}(t)) \cdot z(t) dt = 0, \quad z \in H
$$

(115) and

$$
(G'(x^{(k)}), x^{(k)})/2 = d(x^{(k)}) - \int_I \nabla_x V(t, x^{(k)}(t)) \cdot x^{(k)}(t) dt = 0.
$$

(116) By the previous proof, there is a renamed subsequence such that

$$
\rho_k = \|x^{(k)}\|_H \leq C.
$$

Hence, there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in H$ weakly in $H$ and uniformly on $I$. Moreover,

$$
\int_I H(t, x^{(k)}(t)) dt = G(x^{(k)}) \to \alpha.
$$
From (114) and (115) we see that

\[(117) \quad G(x) = d(x) - 2 \int_I V(t, x(t)) \, dt \leq \alpha,\]

and

\[\frac{(G'(x), z)}{2} = d(x, z) - \int_I \nabla_x V(t, x(t)) \cdot z(t) \, dt = 0, \quad z \in H,\]

from which we conclude easily that $x$ is a solution of (1). Hence, $x \in M_\lambda$ and $G(x) = \alpha$. This completes the proof.

The proofs of Theorems 3.4, 3.5 and 3.6 are similar to those of Theorems 2.4, 2.5 and 2.6 taking into account that $\lambda_0 > 0$ and $N = \{0\}$.

References


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