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Essays in Microeconomics

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Economics

by

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2017
The dissertation of Zachary Breig is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2017
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VITA

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Each of this dissertation's chapters studies a different problem in microeconomics. The approach used in each is uniquely tailored to deal with a particular problem.

The first chapter shows how to optimally delegate actions to an informed agent when the principal cannot require payments. Instead, they are able to commit to actions in the future that affect both parties' payoffs. I use these results to study a model of delegating decisions sequentially, and a model of delegation in which the principal can use measures that are costly to both parties. I characterize optimal mechanisms in both of these settings and show how the
mechanisms vary with the model’s parameters.

The second chapter compares the predictive performance of several models of risk aversion and time preferences in experimental settings. Models are evaluated on the basis of out of sample prediction rather than in sample fit. Some models predict behavior better than others, and these are often not the models which have the best in sample fit. More complicated models improve predictive power in choices over risk, and “behavioral” parameters improve prediction further. This contrasts with time preferences, where adding the present bias parameter $\beta$ worsens prediction for all sample sizes, despite improving fit significantly. The methodology is easy to implement and interpret, and results can be used by both experimentalists and applied modelers.

In the third chapter, I study a dynamic model of monopoly sales in which consumers are finitely lived. I characterize the equilibrium when the monopolist can only commit to a contract in the current period, and compare it to the equilibrium when the monopolist can fully commit and when she can sign long term commitments with renegotiation. I then extend the model to a repeated setting, in which one long-term monopolist interacts with a sequence of short-term consumers. I then characterize how the equilibrium payoff set of the repeated game is affected by exogenously provided commitment.
Chapter 1

Delegation with Continuation Values

1.1 Introduction

Consider the problem of a decision maker working with a better-informed agent: Congress is a principal collaborating with an agent, the Environmental Protection Agency, about how many SO$_2$ permits to allow in a given year. EPA officials are much more informed about factors affecting the policy’s implementation, but they might have different preferences than Congress; they could favor stricter environmental regulations, or prefer to allocate these permits in a different location than Congress wants.

In this case, the principal wants to use information possessed by the agent to implement a better policy, but contingent payments are not possible. In cases such as this, the mechanism design literature has focused on the delegation problem (Holmström, 1977). Instead of deciding how to make transfers to the agent, the principal decides what actions to make available to the agent. However, focusing on delegation alone misses an important part of the story: Congress and EPA officials will likely need to make a different decision next year. While Congress won’t be able to make contingent transfers in either period, it may give the EPA more flexibility or a higher budget in the second period, depending on what EPA officials chose in the first period. The principal will likely be able to use promises about other matters to incentivize a better decision.

This paper studies delegation with continuation values. In the model, an uninformed
principal and an informed agent must implement some action, and have divergent preferences. They are unable to make payments to each other, but are able to commit to actions in the future that affect both parties’ payoffs. The paper gives necessary and sufficient conditions for a mechanism to be optimal. It then applies these results to two canonical settings: dynamic delegation and delegation with money burning.

The model allows the way in which the principal and agent interact in the future to be complex: it could involve the principal promising to play a particular strategy in another game, or to make more options available when the agent makes his next decision. Abstracting from this, I’ll assume that there is some relationship between the principal’s and agent’s future payoffs. Thus, any commitment by the principal to provide the agent with a continuation value affects the principal’s own continuation value. The principal’s problem is to design an incentive scheme which improves the action that the agent takes, but isn’t too costly to implement.

The necessary and sufficient conditions in this paper focus on how the action and continuation values vary with each other and the state. The principal’s primary tradeoff is between incentivizing a better action and promising a continuation contract which is more favorable for herself. These conditions are derived from the calculus of variations, which is used to characterize an indirect utility function for the agent which is optimal for the principal.

To demonstrate the usefulness of these conditions, they are applied to two canonical settings from the delegation literature: dynamic delegation and delegation with money burning. In dynamic delegation, the principal delegates two choices to the agent sequentially. In delegation with money burning, the principal can commit to making choices in the future that hurt both the principal and the agent. Both models can be formulated as special cases of delegation with continuation values.

The paper characterizes the solution to a dynamic delegation problem. In the first period of optimal dynamic delegation, the mechanism implements an action which is strictly increasing and continuous in the state. The principal prevents the agent from taking actions that are too biased by changing the amount of options that the agent gets in the future.

Thus, when a principal delegates dynamically, rules that restrict which options are
available should be more likely to bind at the end of a relationship, while earlier actions are more responsive to the agent’s private information. The form of delegation in the first period contrasts with the results from a static delegation model. When one studies this problem from a static point of view, optimal delegation usually involves the principal placing restrictions on the choices that the agent can make, with the agent often being bound by these restrictions (for a nontrivial proportion of the realizations of the agent’s private information, the agent chooses an action at the bound).

Increasing the bias of an agent in the first period of a dynamic delegation game makes the principal worse off. This is intuitive: an action which is good from the principal’s point of view looks worse to an agent with a stronger bias than to an agent with a weaker bias. To incentivize the agent to take these actions, the principal would have to commit to worse allocations in the future, and would prefer not to do so.

On the other hand, increasing the bias from the second period has a dual effect. Increasing this bias makes actions taken in the second period become worse, but can also make it easier to induce large changes in the agent’s continuation value at small cost to the principal. For a strong enough bias in the second period, the latter effect dominates, and increasing the bias improves the principal’s payoffs.

The optimal mechanism in this dynamic delegation problem is deterministic. This is in accordance with recent work which shows that stochastic delegation is suboptimal when loss functions are quadratic (Goltsman, Hörner, Pavlov, & Squintani, 2009; Kováč & Mylovanov, 2009). However, the intuition shows that randomization is even less likely in dynamic delegation models than in static. In both the static and dynamic models, randomization is costly to both agent and the principal, and can be used to incentivize a choice which is better from the point of view of the principal. Randomization is not optimal in the static case because the changes in the choice it can induce are not worth the costs to the principal. In dynamic delegation, the costs from randomization must be low enough that the changes induced in the action must be worth these costs and the costs from randomization must be lower than those from giving less discretion in later periods. The principal imposes costs on the agent in the way which is least damaging to
herself.

Even if there is not a second period in which the principal can delegate, the principal may be able to commit to measures that are costly to both the principal and the agent. For instance, the principal could demand that the agent fill out tedious paperwork that the principal must then review, or could require that the principal and agent spend time meeting before the decision is made, wasting both parties’ time. These measures are known as “money burning,” because they lower both players’ payoffs (Amador & Bagwell, 2013; Ambrus & Egorov, in press; Amador & Bagwell, 2016).

The theory can be used to determine under what conditions money burning is used in the optimal mechanism. The paper gives conditions for which the principal would rather restrict the options and not use money burning, and relates these conditions to those found in previous work by Amador and Bagwell (2013). The paper also shows that in the standard uniform-quadratic setting, the amount of money burning moves in an intuitive way. The principal uses more money burning when it becomes less costly to do so, and uses less money burning for a more biased agent (who is harder to incentivize).

There are many cases in which one might expect delegation to be an important policy tool, and for which there are implicitly continuation values. Previously, models of delegation have considered situations including a regulator delegating a pricing decision to a monopolist (Alonso & Matouschek, 2008), a legislature delegating the drafting of legislation to a committee (Gilligan & Krehbiel, 1987), or a school delegating grading decisions to a teacher (Frankel, 2014). Importantly, all of these situations are inherently dynamic. Regulators and monopolists need to make pricing decisions repeatedly, a given legislature delegating choices to a committee will convene for multiple years, and a school delegates grading decisions to teachers every semester. We might expect that the dynamic nature of these interactions changes the way in which a principal delegates.

The paper will proceed as follows. Section 1.2 presents the model and restates the problem in a way that allows for results from the calculus of variations to be used. Section 1.3 characterizes general results about the existence and properties of solutions to this problem.
Sections 1.4 and 1.5 apply these results to the problems of dynamic delegation and delegation with money burning, respectively. Section 1.6 reviews the previous literature considering the delegation problem, including multidimensional and dynamic delegation, as well as delegation with money burning, and Section 1.7 is a conclusion. Proofs of all results are in the appendices.

1.2 Model

A principal (she) and an agent (he) are contracting to make a decision about what action to take. While the principal has full control over what action is taken, the agent has private information about the state of the world, $\theta$. This random variable is distributed on a compact interval $\Theta$, which without loss of generality can be set to be the unit interval. The agent wants to match the action to the state, and has utility function

$$-\frac{1}{2}(a - \theta)^2 + \omega,$$

where $\omega$ is the continuation value that the principal promises to the agent. For any continuation value that the principal promises to the agent, she receives $\gamma(\omega)$, and thus has utility function

$$u^P(a, \theta) + \gamma(\omega).$$

The function $\gamma$, which is the mapping between the agent’s continuation value and the principal’s, is key to much of the following analysis. The slope of $\gamma$ captures how costly (or beneficial) it is to either punish or reward the agent. As a baseline, it is useful to compare this to the “standard” case of mechanism design, in which there is transferable utility. In this case, the function is $\gamma(\omega) = -\omega$: one extra unit of utility for the agent comes at a cost of one unit for the principal.

For the remainder of this paper, a number of assumptions will be maintained:

**Basic Assumptions** The following properties will be assumed for the entirety of the paper:

(i) $\theta$ is distributed according to a distribution function $F(\theta)$ and has a density function $f(\theta)$
which is strictly positive and continuously differentiable on $(0,1)$; (ii) $u^p(\cdot, \cdot)$ is twice continuously
differentiable, bounded above, and for all $\theta$, concave in $a$, with first derivative $u^p_a(a, \theta)$ which
converges uniformly to $\infty (\rightarrow -\infty)$ as $a \to -\infty (a \to \infty)$; (iii) $\gamma(\cdot)$ is concave, attains its maximum $\bar{\gamma}$,
is upper semi-continuous, and if $\gamma(\cdot) > -\infty$ on some set $[a, b]$, then $\gamma$ is continuously differentiable
on $(a, b)$.

A mechanism is a message space and a pair functions $a(\cdot)$ and $\omega(\cdot)$, which map the
message space to actions and continuation values, respectively. The revelation principle implies
that we can focus on direct and truthful mechanisms, which map states to actions and continuation
values, subject to incentive compatibility constraints. Thus, the principal is solving the problem

$$
\max_{\{a(\cdot), \omega(\cdot)\}} \int_0^1 \left[ u(a(\theta), \theta) + \gamma(\omega(\theta)) \right] f(\theta) d\theta
$$

(P')

subject to

$$
\theta \in \arg\max_{\hat{\theta} \in [0,1]} -\frac{1}{2} (a(\hat{\theta}) - \theta)^2 + \omega(\hat{\theta})
$$

(IC)

A key difference between this setting and standard mechanism design problems is that
there is no participation (“individual rationality”) constraint. This raises both technical and
interpretative issues. On the technical side, this is what requires us to assume that $\gamma(\cdot)$ attains its
maximum (ruling out, for instance, transferable utility). Otherwise, the maximization problem has
no solution. Regarding interpretation, one should focus on cases in which the agent would always
rather participate than not participate, for instance when a legislature delegates to a committee. In
some cases, this model will also be applicable when there are individual rationality constraints
which can be satisfied with lump sum transfers at the beginning of the relationship.\footnote{In particular, it’s straightforward to see how the results will apply if the individual rationality constraint always binds. If this is true, then in searching for the optimal mechanism, one can restate the principal’s preferences as

$$
\hat{u}^p(a, \theta) + \hat{\gamma}(\omega)
$$

where $\hat{u}^p(a, \theta) = u^p(a, \theta) - \frac{1}{2} (a - \theta)^2$, $\hat{\gamma}(\omega) = \gamma(\omega) + \omega$, and the individual rationality constraint is exactly fulfilled.
This paper follows much of the literature on mechanism design in restating the problem as one of optimal control. In particular, it will define the agent’s value function $W(\theta)$, and show that the principal’s payoffs from an incentive compatible mechanism can be written as a function of $W(\theta)$, its derivative, and the model’s primitives. When stated in this way, the problem can be solved using results from the calculus of variations (Clarke, 1990).

For a given mechanism we can define the agent’s value function $W(\theta)$ as the utility that he receives from that mechanism, i.e.

$$W(\theta) = \sup_{\hat{\theta} \in \Theta} -\frac{1}{2} (a(\hat{\theta}) - \theta)^2 + \omega(\hat{\theta}).$$

which allows for a useful result:

**Lemma 1.1** $W$ is absolutely continuous, and where the derivative exists,

$$W'(\theta) = a(\theta) - \theta$$

(1.1)

and

$$\omega(\theta) = W(\theta) + \frac{1}{2} W'(\theta)^2.$$

As in similar settings, incentive compatibility is equivalent to the agent’s value function satisfying equation (1.1), the envelope condition, plus the monotonicity constraint that the action is increasing in the state. Thus, as a function of the state and the agent’s value function, the principal’s losses are

$$L(\theta, W(\theta), W'(\theta)) = \left[ -u(W'(\theta) + \theta, \theta) - \gamma \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) \right] f(\theta).$$

and we can write the problem as

by the lump-sum transfer.
where $W(\theta)$ is absolutely continuous, and the action implied by $W(\theta)$ is monotonic. Similar to the rest of the literature, we’ll solve problem (P) ignoring the monotonicity constraint, and then discuss the condition which ensures it.

### 1.3 Main Results

#### 1.3.1 Existence

A key condition for the existence of solutions to problem (P) is what I will refer to as the convexity condition:

**Convexity Condition:** For each $(\theta, s)$, $L(\theta, s, \cdot)$ is convex.

Since this condition is critical for all of the following results, it is instructive to consider it further. In a sense, it requires that the second derivative of $u^p$ with respect to $a$ be “large enough” as compared to the slope of $\gamma$. This is similar to a second-order condition, and it ensures that the conditions which we later show imply optimality are valid.

The convexity condition implicitly requires that the function $\gamma$ be greater than $-\infty$ for all $\omega$ less than some cutoff $\bar{\omega}$. Thus, if the principal is able to promise some continuation value, she must be able to promise any value below it. This rules out any continuation games in which payoffs are bounded below, for instance. However, two facts make this constraint less important. First, in many cases one may extend $\gamma$ in a way that allows it to satisfy the convexity condition, and if an optimal mechanism is found in which $\omega$ lies completely in the region in which $\gamma$ is well defined, then it is an an optimal mechanism of the original problem.\(^2\) Second, Clarke (1977) and the references therein show the conditions under which it is sufficient to solve an adjusted

\(^2\)For a more concrete example, assume that in the future, both players are playing a simple coordination game. If they cooperate, they both receive payoffs of $M$, but if either deviates, they both receive a payoff of 0. In the first period,
problem in which one replaces \( L(\theta, s, \cdot) \) with a loss function whose epigraph is a convex hull of the epigraph of \( L(\theta, s, \cdot) \).

With the convexity condition and the basic assumptions, results from the calculus of variations (Clarke, 1990) imply that a solution to problem (P) exists.

**Proposition 1.1** Assume that the convexity condition holds. Then a solution to problem (P) exists.

### 1.3.2 Characterization

There are many cases, including those described in Sections 1.4 and 1.5 below, in which a principal may be unable to promise a continuation value above some cutoff \( \bar{\omega} \). This is accounted for in the model by setting \( \gamma(\omega) > -\infty \) for \( \omega \leq \bar{\omega} \) and \( \gamma(\omega) = -\infty \) for \( \omega > \bar{\omega} \) (in what follows, let \( \bar{\omega} = \infty \) if there is no such restriction). An optimal mechanism for which \( \omega(\theta) = \bar{\omega} \) for some \( \theta \) is in a sense a “corner solution,” and has different conditions for optimality than “interior solutions.” In Propositions 1.2 and 1.3, as well as Theorem 1.1 below, I focus on the case in which this constraint is not binding. The more general results are presented in Propositions 1.A.1 and 1.A.2 in Appendix 1.A, and an example in which this constraint is binding is considered explicitly in Section 1.5.

The problem (P) is somewhat nonstandard, since the loss function may not be differentiable at all points. Clarke (1990) provides conditions which are necessary for optimality in such cases. If the function \( \gamma(\cdot) \) is twice differentiable in the promised continuation value at some point, then at that point a solution to the problem will be a solution to a differential equation, which the principal can commit to the probability of cooperating in the coordination game. Thus, in this case

\[
\gamma(\omega) = \begin{cases} 
\omega & \text{if } 0 \leq \omega \leq M \\
-\infty & \text{otherwise}
\end{cases},
\]

where \( \omega \) will be equal to the probability of cooperating multiplied by \( M \). This is ruled out from the above analysis, because \( \gamma = -\infty \) for \( \omega < 0 \), but \( \gamma > -\infty \) for \( \omega = 0 \). However, if the principal solves an adjusted problem with

\[
\gamma(\omega) = \begin{cases} 
\omega & \text{if } \omega \leq M \\
-\infty & \text{otherwise}
\end{cases}
\]

and none of the continuation values in the optimal mechanism are less than 0, then the mechanism also solves the original problem.
optimally trades off between payoffs in the first period and the continuation value.

**Proposition 1.2** Assume that the convexity condition holds, and that $W(\theta)$ solves problem (P). Define $a(\theta) = W'(\theta) + \theta$ and $\omega(\theta) = W(\theta) + \frac{1}{2}W'(\theta)^2$. Then for $\theta \in [0, 1]$ such that $\omega(\theta) < \bar{\omega}$,

1. if $\theta \in \{0, 1\}$,

$$W'(\theta)\gamma'(\omega(\theta)) = -u^p_a(a(\theta), \theta),$$

and

2. if $\gamma''(\omega(\theta))$ exists, then

$$W''(\theta) = -1 + \frac{2\gamma'(\omega(\theta)) - u^p_{a\theta}(a(\theta), \theta) - [u^p_a(a(\theta), \theta) + W'(\theta)\gamma'(\omega(\theta))] l'(\theta)}{u^p_{aa}(a(\theta), \theta) + W''(\theta)2\gamma''(\omega(\theta)) + \gamma'(\omega(\theta))}$$

almost everywhere.

Proposition 1.2 provides the necessary conditions for optimality, in particular for the instances when $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$. The necessary conditions for optimality when $W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega}$, but $W(\theta) < \bar{\omega}$ are omitted here for clarity, but are given in Proposition 1.A.1 in Appendix 1.A.

Condition (1) from Proposition 1.2 is known as the “free endpoint condition” from the calculus of variations, and essentially states that the marginal value of changing the action taken when $\theta$ is 0 or 1 must be equal to the marginal cost of changing this action. Condition (2), known as the Euler-Lagrange equation, is also standard, and is a necessary condition for a function to minimize a functional as the solution to problem (P) must.

If $W(\theta)$ is absolutely continuous and the implied continuation values are everywhere strictly below $\bar{\omega}$, then the necessary conditions from Proposition 1.2 are also sufficient.

**Proposition 1.3** Assume that the convexity condition holds. If $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$, $\gamma''(W(\theta) + \frac{1}{2}W'(\theta)^2)$ exists, $W(\theta)$ is absolutely continuous, and $W(\theta)$ satisfies conditions (1) and (2) from Proposition 1.2 almost everywhere, then $W(\theta)$ solves problem (P).
Again, the more general conditions for when \( W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega} \) are left for the Appendix in Proposition 1.A.2 and Corollary 1.A.1.

Combining the conditions given in Proposition 1.3, along with the envelope theorem and monotonicity constraint allows us to provide sufficient conditions for a mechanism to be optimal:

**Theorem 1.1** Suppose that the convexity condition holds, and the following is true:

1. for \( \theta \in \{0,1\} \),
   \[-u_a(a(\theta),\theta) = (a(\theta) - \theta)\gamma'(\omega(\theta));\]

2. \( \gamma''(\omega(\theta)) \) exists almost everywhere, with
   \[
   a'(\theta) = \frac{2\gamma'(\omega(\theta)) - u^p_{a\theta}(a(\theta),\theta) - \left[u^p_a(a(\theta),\theta) + (a(\theta) - \theta)\gamma'(\omega(\theta))\right] f'(\theta)}{u^p_{ia}(a(\theta),\theta) + (a(\theta) - \theta)^2 \gamma''(\omega(\theta)) + \gamma'(\omega(\theta))},
   \]
   and \( \omega(\theta) \) is derived from the envelope condition;

3. \( a(\theta) \) is continuous and monotonically increasing.

Then the mechanism defined by \( a(\theta) \) and \( \omega(\theta) \) is optimal.

Condition (1) is an analog of the “no distortion at the top” condition from the solution to a standard mechanism design problem. If it were the case that \( \gamma \) were linear with a slope of -1 (i.e. transferable utility), then this condition would require that the marginal gains to the principal for raising the action at the endpoint are equal to the marginal losses for the agent. Because the transfer of utility has to first pass through the function \( \gamma \), the agent’s marginal utility has to be scaled by this factor. Another difference between this and the standard mechanism design setting is the fact that the condition must hold for both endpoints, not just the “top.” This is due to the fact that there is no individual rationality constraint, so the condition can hold at both endpoints.

The equality from condition (2) can be derived from the equality

\[
\frac{d}{d\theta} \left[(u^p_a(a(\theta),\theta) + (a(\theta) - \theta)\gamma'(\omega(\theta))) f(\theta)\right] = \gamma'(\omega(\theta)) f(\theta).
\]
This equation can be understood as a generalization of more standard principles of mechanism design. For instance, the first order conditions for an optimal mechanism presented by Fudenberg and Tirole (1991) are

$$\frac{\partial U^P}{\partial a} + \frac{\partial U^A}{\partial a} = \frac{1 - F(\theta)}{f(\theta)} \frac{\partial^2 U^A}{\partial a \partial \theta},$$

(1.3)

where $U^P$ is the principal’s utility function and $U^A$ is the agent’s utility function. The left hand side of equation (1.3) is the marginal total surplus, since it is the sum of the agent’s and principal’s marginal utilities. The right hand side can be interpreted as the marginal information rents that the principal must leave to higher types. Similarly, the first term in the brackets in equation (1.2) is the principal’s marginal utility, and the second term is the agent’s marginal utility, multiplied by the rate at which utility can be transferred between the principal and the agent, $\gamma'(\omega(\theta))$.

Treating the right hand side as the information rents which must be left to the agent, this gives a clearer interpretation for equation (1.2): the principal maximizes the total surplus which can be appropriated, minus the information rents that must be left to the agent.

To make the relationship between classic optimality conditions from mechanism design and those found here, one can substitute the utility functions used in this paper in equation (1.3), multiply both sides by $f(\theta)$, and totally differentiate. This results in the equality

$$\frac{d}{d\theta} \left[ (u^P_\alpha(a(\theta), \theta) - (a(\theta) - \theta)) f'(\theta) \right] = -f(\theta),$$

which is equivalent to substituting $\gamma'(\omega) = -1$ into equation (1.2). Not surprisingly, the equality from condition (2) can also be related to the conditions presented by Fudenberg and Tirole. In particular, with the assumption of quadratic losses for the agent, one can manipulate the first order conditions there (equation (7.12)) to obtain

$$\left[ u^P_{\alpha\alpha}(a(\theta), \theta) - 1 \right] a'(\theta) = \left[ -u_\alpha(a(\theta), \theta) + (a(\theta) - \theta) \right] \frac{f'(\theta)}{f(\theta)} - 2 - u_{\alpha\theta}(a(\theta), \theta),$$

3 Notation has been changed slightly for comparability.
which is the same as the equation from condition (2), but with \( \gamma(\omega) = -\omega \). Thus, the condition generalizes the optimality condition from standard mechanism design with transfers, but does so under a functional form assumption on the agent’s utility.

Condition (3) from Theorem 1.1 ensures that the agent’s value function is continuous at points where \( \gamma''(\omega(\theta)) \) does not exist, and that the action satisfies the monotonicity constraint. At points where \( a(\theta) \) is continuously differentiable, the derivative is positive if and only if

\[
2\gamma'(\omega(\theta)) - u^P_{a\theta}(a, \theta) - \left[ u^P_{a\theta}(a(\theta), \theta) + (a(\theta) - \theta) \gamma'(\omega(\theta)) \right] \frac{f'(\theta)}{f(\theta)} \leq 0. \tag{1.4}
\]

Classical results from mechanism design show that studying the “relaxed problem” is sufficient in mechanism design under standard assumptions along with the monotone hazard rate condition. If we were to assume \( \gamma'(\omega) = -1 \) and \( u^P_{a\theta}(a, \theta) \geq 0 \), and again substituted in equation (7.12) from Fudenberg and Tirole (1991), we would find that the monotonicity condition holds when

\[
\left[ \frac{1 - F(\theta)}{f(\theta)} \right] \frac{f'(\theta)}{f(\theta)} \geq 0,
\]

but the monotone hazard rate implies that

\[
-1 + \left[ \frac{1 - F(\theta)}{f(\theta)} \right] \frac{f'(\theta)}{f(\theta)} \geq 0.
\]

Thus, Theorem 1.1 can be seen as an extension of the classical mechanism design results, which relaxes both the usual individual rationality constraint and the assumption of transferrable utility, while imposing additional structure on the agent’s preferences.

### 1.4 Dynamic Delegation

Suppose that the principal and agent are Congress and the EPA from Section 1.1, and that they need to make the decision about how much they should reduce SO₂ emissions in each of two
years. The “states” in each of the years, \( \theta_1 \) and \( \theta_2 \), can be thought of as factors that determine how much environmental damage \( \text{SO}_2 \) will cause. As experts in the field, EPA officials observe this, but Members of Congress do not. For the purposes of this example, let them be uniformly and independently distributed.

Assume that the EPA favors stricter environmental regulations than Members of Congress: no matter what the conditions are, EPA officials would like to reduce emissions more than Congress would (it is easy to restate the problem and get symmetric conclusions if the EPA instead wants looser environmental regulations than Members of Congress). Defining \( a_t \) as the emissions reductions in period \( t \), the principal has per-period utility function \(-\frac{1}{2}(a_t - \theta_t + b_t)^2\), and the agent has per-period utility function \(-\frac{1}{2}(a_t - \theta_t)^2\), and both discount at rate \( \delta \). Assume that the principal and agent have different preferences, so \( b_1 \neq 0 \) and \( b_2 \neq 0 \), and without loss of generality let both be greater than 0, since the EPA wants higher emissions reductions.

This is a dynamic delegation problem, in which the principal sequentially delegates choices to a biased agent. Here, a direct mechanism is a pair of functions \( a_1 : \Theta \rightarrow \mathbb{R} \) and \( a_2 : \Theta \times \Theta \rightarrow \mathbb{R} \), which choose the amount of emissions reductions for any report of the state. Thus, the principal’s problem is

\[
\max_{a_1(\theta_1), a_2(\theta_1, \theta_2)} \mathbb{E}\left[ -\frac{1}{2}(a_1(\theta_1) - \theta_1 + b_1)^2 - \frac{1}{2} \delta (a_2(\theta_1, \theta_2) - \theta_2 + b_2)^2 \right] \tag{PDD’}
\]

subject to \( \forall \theta_1, \theta'_1 \in \Theta \),

\[
\mathbb{E}\left[ -\frac{1}{2}(a_1(\theta_1) - \theta_1)^2 - \frac{1}{2} \delta (a_2(\theta_1, \theta_2) - \theta_2)^2 | \theta_1 \right] \geq 0 \tag{IC1DD}
\]

\[
\mathbb{E}\left[ -\frac{1}{2}(a_1(\theta'_1) - \theta_1)^2 - \frac{1}{2} \delta (a_2(\theta'_1, \theta_2) - \theta_2)^2 | \theta_1 \right] \geq 0
\]

\[4\text{A natural question is how the model extends to more periods. If the interaction is infinitely repeated, the model is similar to that of Guo and Hörner (2015), albeit with an allocation which affects payoffs nonlinearly. Although it’s not shown here, it’s likely that the } T \text{ period problem could be solved similarly to the model here: first find the optimal way to promise } \omega \text{ utility in period } T, \text{ then for } T - 1, \text{ etc.} \]
and \( \forall \theta_1, \theta_2, \theta_2' \in \Theta, \)

\[-\frac{1}{2} (a_2(\theta_1, \theta_2) - \theta_2)^2 \geq -\frac{1}{2} (a_2(\theta_1, \theta_2') - \theta_2')^2 \tag{IC2DD} \]

This is clearly a problem in which the principal is delegating in the first period, and there is a continuation value associated with the continuation contracts in the second period. In fact, because the two states \( \theta_1 \) and \( \theta_2 \) are independent, the fact that the second period must carry out the continuation value from the first period is the only connection between the periods: Congress offers choices other than optimal delegation set in the second period only if it wants to reward or punish EPA officials for their first period choice. Thus, in solving for the optimal mechanism, it will be necessary to solve for the optimal way to promise any level \( \omega \) of utility.

**Proposition 1.4** In the uniform-quadratic setting, the principal’s problem can be written as

\[
\max_{a_1(\theta_1), \omega(\theta_1)} \mathbb{E} \left[ -\frac{1}{2} (a_1(\theta_1) - \theta_1 + b_1)^2 + \gamma(\omega(\theta_1)) \right] \tag{PDD'}
\]

subject to \( \forall \theta_1, \theta_1' \in \Theta, \)

\[-\frac{1}{2} (a_1(\theta_1) - \theta_1)^2 + \omega(\theta_1) \geq -\frac{1}{2} (a_1(\theta_1') - \theta_1)^2 + \omega(\theta_1') \tag{IC1'} \]

where

\[
\gamma(\omega) = \begin{cases} 
\omega - \frac{\delta}{2} b_2^2 + \frac{\delta b_2}{6} (-\frac{72}{\delta} \omega - 3)^{\frac{1}{2}} & \text{if } \omega < -\frac{\delta}{6} \\
\omega - \frac{\delta}{2} b_2^2 + \frac{\delta b_2}{2} (\frac{\delta}{6} \omega)^{\frac{1}{2}} & \text{if } -\frac{\delta}{6} \leq \omega \leq 0 \\
-\infty & \text{otherwise}
\end{cases}
\]

Thus, this uniform-quadratic dynamic delegation setting is equivalent to the more general setting of delegation with continuation values considered in Section 1.3, where \( \gamma(\cdot) \) is as given in Proposition 1.4. Proposition 1.4 greatly simplifies Congress’ problem. Instead of a dynamic
problem requiring it to choose an emissions reduction in the first year and a function for the second year when the EPA makes its first report, she can simply choose the emissions reductions and a “pseudo-transfer” for each type, where this pseudo-transfer is denoted by \( \omega(\theta) \). In choosing to transfer some \( \omega(\theta) \), Congress receives \( \gamma(\omega(\theta)) \) in expected utility. Once found, this function \( \gamma(\cdot) \) is what makes the results from delegation with continuation values applicable to this dynamic delegation problem.

For this specification of payoffs and distribution of states, the options that Congress gives to the EPA in the second period take a specific form. In particular, the actions that the principal makes available to the agent can be interpreted as a simple cap on the emissions reductions that the EPA can require in the second period. This is the same general form as the optimal mechanism of the static version of the game (Melumad & Shibano, 1991). The level of this cap is determined from how much expected utility the principal is promising the agent, so I’ll refer to the cap conditional on a promise of \( \omega \) as \( \gamma(\omega) \).

The function \( \gamma(\cdot) \) in Proposition 1.4 is defined piecewise, with the domain split at \(-\frac{\delta}{b}\). This is the expected continuation value for the agent if he is being forced to take the action \( a_2 = 0 \) regardless of the state in the second period. When the cap on the second period action is above 0, lowering the cap has two effects: it enlarges the set of states which are pooled at the cap, and lowers the action taken for all of those states. When the cap is below 0, lowering it further only changes the action taken (since the agent takes the maximum action available for all values of the state already). The difference between these two regions leads to a discontinuity of the second derivative of \( \gamma(\cdot) \), which will be discussed below.

A version of the function \( \gamma \) when \( b_2 = 0.5 \) can be seen in Figure 1.1. This function provides the mapping between the EPA’s continuation value and Congress’ continuation value which is optimal from Congress’ point of view. This function is continuously differentiable, strictly concave, and has a maximum at \( \omega = -\frac{4}{3}\delta b_2^3 \) for \( b_2 \leq \frac{1}{2} \), and at \( \omega = -\frac{1}{2}\delta b_2^2 - \frac{1}{24}\delta \) for \( b_2 > \frac{1}{2} \). This maximum is the payoff that Congress would receive if it used the solution to the one period delegation problem in the second period. If the principal were only considering a single period, setting \( \omega \) less than the maximizing point would be giving the agent too little discretion,
leading to the principal not taking full advantage of the agent’s information. Alternatively, setting \( \omega \) greater than the maximizing point gives the agent too much discretion, implying that the principal doesn’t properly account for the agent’s bias. Thus, the principal’s problem in the first period is clear: she wants to incentivize the agent to take lower actions using \( \omega \), while keeping \( \omega \) near the value that maximizes \( \gamma(\cdot) \).

As noted, with this simplification the state variable in the second period, \( \theta_2 \), no longer enters into the problem, as it has been dealt with while finding the function \( \gamma(\cdot) \). Thus, from this point on the state will be referred to as \( \theta \), which should be interpreted as the state in period 1. The loss function for any value of the state is

\[
L(\theta, s, v) = \left[ \frac{1}{2} (v + b_1)^2 - \gamma \left( s + \frac{1}{2} v^2 \right) \right]
\]

with \( \gamma \) defined as before, and the principal is looking for a monotonic action rule which is implied
by the $W(\theta)$ that solves

$$
\min_{W(\theta)} \int_0^1 \left[ \frac{1}{2} (W'(\theta) + b_1)^2 - \gamma \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) \right] d\theta. \tag{PDD}
$$

among absolutely continuous functions. $L(\theta, s, \cdot)$ is continuously differentiable and convex for all $(\theta, s)$ such that $s + \frac{1}{2} v^2 \leq 0$. Thus, the problem satisfies all of the requirements necessary for the use of Proposition 1.3, implying that if there is a solution to the differential equation and endpoints conditions found in Proposition 1.3, then this solution solves Problem (PDD). Proposition 1.5 shows that such a solution does exist, and is proven in Appendix 1.B.

**Proposition 1.5** An interior solution to problem (PDD) exists, i.e. there exists a $W(\theta)$ which solves (PDD) such that $\forall \theta$, $W(\theta) + \frac{1}{2} W'(\theta)^2 < \bar{\varepsilon} < 0$.

The discontinuity of the second derivative of $\gamma$ leads to a differential equation which is discontinuous, so many of the standard results regarding the existence and uniqueness of solutions to differential equations do not apply. Instead, the proof uses results from the theory of differential equations with discontinuous right-hand sides (Filippov & Arscott, 1988). In particular, the proof first shows that solutions to this differential equation exist, are unique, and are continuous in initial conditions. It then continuously varies the initial conditions along the curve on which the endpoints must lie, and uses continuity and uniqueness to show that there exist values of $W(0)$ and $W'(0)$ on this curve such that $W(1)$ and $W'(1)$ will also be on the curve.

This discontinuity arises from the piecewise nature of the function $\gamma(\cdot)$ that results from delegation in the second period. Since the action in the first period is a function of $W'(\theta)$, this discontinuity does not result in an action which is discontinuous in the state, but rather a jump in the rate at which the action changes with the state. This captures the fact that the rate of tradeoff between payoffs in the first and second period changes when the principal changes from setting a cap above the agent’s lowest preferred action to one below it.

With the existence of a solution to the differential equation with the appropriate endpoints, it remains to show that the action implied by this function is monotonic. This result and some of
the mechanism’s properties are given in Theorem 1.2.

**Theorem 1.2** In the dynamic delegation game, an optimal mechanism exists and has the following properties:

- the action is strictly increasing and continuous in the first period’s state (no pooling in the first period);
- the maximum action available in the second period is strictly decreasing in the first period’s state;
- the actions taken in the first period are a strict subset of those the principal would take if she were fully informed.

Relating this back to the example of Congress and the EPA, this result says that as the danger of environmental damage increases, when Congress is delegating optimally, they allow the EPA to impose higher emissions reductions. Usually, in the one period problem, Congress would simply set a cap, and for all values of the state above some cutoff, the EPA would set emissions reductions at that cap. In the first period of dynamic delegation, such pooling does not happen. Congress can vary the delegation set in the second period slightly, taking second order losses, but will be making first order gains from the less biased actions in the first period.

Given that the action taken in the first period is strictly increasing in the state, it is perhaps not surprising that the cap on actions in the second period is strictly decreasing. Recall that this cap is the means by which the principal incentivizes the agent to take a given action in the first period. Since the agent would in general prefer to be taking a higher action, agents who take the higher action must be given a lower continuation value, $\omega$. This lower continuation value is in turn associated with less discretion in the second period—a lower cap.

Figure 1.2 shows the optimal spread of $\omega$ when $\delta = 1$ and $b_1 = b_2 = 0.5$. The principal is giving the agent the smallest continuation value when the agent reports that the state is 1, and the highest continuation value when the agent reports that the state is 0.

It may be surprising that the highest action in the first period is strictly less than the action preferred by both the principal and the agent (i.e. the actions taken in the first period are a strict
subset of those the principal would take if she were fully informed). To understand this point, it will first be useful to consider the principal’s incentives more carefully. She is using discretion in the second period, captured by \( \omega(\theta) \), to incentivize the agent to choose the action closer to her own preferred action in the first period. Because there is an optimal level of discretion (the point where \( \gamma(\cdot) \) is maximized) in the second period any deviation away from this comes at a cost. Thus, she’ll be committing to give “too much” discretion if the agent reports a low state, and “too little” if the agent reports a high state.

![Figure 1.2](image)

**Figure 1.2:** A plot of \( \gamma(\cdot) \) with the optimal spread (in solid red) of \( \omega \) for \( \delta = 1 \) and \( b_1 = b_2 = 0.5 \)

If the principal were allowing the agent to take actions higher than \( 1 - b_1 \) (the highest action the principal would ever want to take) for high values of the state, then she could lower all of these actions, and raise the continuation values promised to the agent for these states. *Both* of these changes improve the principal’s payoffs, since the action taken in these states becomes closer to the principal’s preferred action, and the principal is promising more discretion in the second period, which she preferred to do anyway. Similarly, the principal would never have the agent take actions below \(-b_1\).
Figure 1.3 shows the optimal action and cap for various values of $\delta$, $b_1$, and $b_2$, in which the characteristics described in Theorem 1.2 can be noted. These graphs also show the consequences of the discontinuity in the second derivative of $\gamma$ (i.e. the fact that the principal shifts from delegating a set to delegating a point). At the $\theta$ where $y(\omega(\theta)) = 0$, both $a(\theta)$ and $y(\theta)$ are kinked.

The generality of these results is an interesting point to consider. They are derived using quadratic losses and a uniform distribution, both of which are restrictive assumptions. Here, the principal does not need to pool different states into a single action, because she can use continuation values to vary the action smoothly. Furthermore, simply raising or lowering the cap on the second period choices is the optimal way to change the utility promise. In other settings, delegation sets at the end of the relationship are likely to be more complex, but it is to be expected
that the principal will tend to make these delegation sets smaller or larger to punish or reward the agent. With the mapping from agent’s continuation value to principal’s that this provides, it’s plausible that the principal will use these continuation values to vary the action continuously with the state as is the case in this model.

1.4.1 Comparative Statics and the Value of Delegating Dynamically

Intuitively, we might expect that larger differences between the principal’s and agent’s preferred decisions would lead to lower payoffs for the principal; it seems natural that Congress would prefer EPA officials who have preferences similar to their own. This is the case for the one period, uniform-quadratic delegation problem, until the bias becomes so high that delegation has no value at all (Melumad & Shibano, 1991).

Theorem 1.2 demonstrates that a two period delegation problem is fundamentally different than a static one: when delegation is dynamic and biases are strictly positive, delegation is always valuable (at least in the first period). This is because the ability of the principal to incentivize first period actions using continuation values allows for the principal to utilize the agent’s information to some extent, however small.

Even though delegation continues to be valuable in dynamic delegation, an agent who is more biased in the first period is worse for the principal, as noted in the next result.

**Theorem 1.3** In the dynamic delegation game, increasing the agent’s bias in the first period makes the principal strictly worse off.

Increasing the first period’s bias is unambiguously bad for the principal. The higher level of bias makes it necessary to increase the spread of the agent’s continuation values to incentivize actions that are as good for the principal, and increasing this spread makes the principal worse off due to the concavity of $\gamma(\omega)$. Figure 1.4 shows how the principal’s payoffs vary with $b_1$ for two different values of $b_2$, and provides the comparison to the payoffs she would receive if the

---

5The model above has restricted attention to cases in which both bias parameters are strictly positive. Using the result from Section 1.5, it can be checked that the only case in which increasing $b_1$ does not make the principal strictly worse off is if $b_2 = 0$ and $b_1 \geq 0.5$, for which increasing $b_1$ does not change the allocation or continuation value.
optimal static mechanism for each period were used.

![Figure 1.4: Payoffs for the Principal, with the optimal dynamic mechanism (solid), or repeating the optimal static mechanism (dashed)]

Comparative statics for $b_2$ are not as obvious, because increasing $b_2$ has two effects. A higher $b_2$ means that for any delegation set in the second period, the agent will be choosing something which is weakly worse for the principal. From the point of view of period 1, though, a higher $b_2$ makes the agent easier to incentivize to take an action which is good for the principal, since lowering the maximum available action in period 2 is much worse for an agent with a higher bias. In this way, increasing $b_2$ is similar to increasing $\Delta$ in the setting of Koessler and Martimort (2012), which also has this dual effect. Theorem 1.4 shows that for high enough $b_2$, increasing $b_2$ further eventually improves the principal’s payoffs.

**Theorem 1.4** *In the dynamic delegation game, for high enough second period bias, further increasing this bias improves the principal’s payoffs. As bias in the second period becomes arbitrarily large, the principal’s payoffs approach those she would receive if she had full information in the first period, and had to make the optimal uniformed decision in the second period.*

The proof of Theorem 1.4 shows that increasing $b_2$ increases the principal’s payoffs if $b_2 \geq \frac{1}{2}$. For all values of $b_2$ greater than this, if the principal were considering the second period problem in isolation, she would prefer to not give the agent any discretion. Since the optimal static mechanism is the same in all of these cases, the maximum continuation value that the
principal can receive in the first period no longer falls as the second period bias increases.

For an agent that is this biased in the second period, the principal varies the action that will be taken in the second period around her preferred uninformed action (perhaps eventually giving the agent some discretion). This change in the action taken has a higher effect on agents whose bias is higher, because the agent’s loss function is convex. Thus, an equivalent change in the principal’s payoff is associated with a larger change in the agent’s payoff when that agent has a higher bias.

The implications of this can be seen in Figure 1.5. In this figure, two versions of the function $\gamma$ are centered such that their maxima are at 0. The re-centered version of $\gamma$ in which $b_2 = 0.7$ is everywhere above the re-centered version of $\gamma$ in which $b_2 = 0.5$. Thus, as $b_2$ increases for values above 0.5, the principal can incentivize the exact same action profile, with the agent’s continuation values simply shifted downward, and this gives a higher payoff to the principal. Figure 1.6 illustrates these comparative static results, and provides the comparison of what the principal would receive if she used the optimal static mechanism in each period.

![Figure 1.5: The function $\gamma$, centered around it’s maximum, for $b_2 = 0.5$ and $b_2 = 0.7$](image)

Theorem 1.4 also states that as the second period bias becomes arbitrarily large, the payoffs that arise from any two fixed first period biases converge to each other. This relates to the way in which it becomes easier to incentivize the agent as $b_2$ increases: small changes in the promised action create large swings in the agent’s continuation value, and for high enough $b_2$ the
principal can implement her optimal action profile while essentially receiving the payoff from the optimal static mechanism in the second period.

1.4.2 Stochastic Mechanisms and Dynamic Delegation

The mechanism described in Theorem 1.2 is a deterministic one, which is to say that when the agent announces the state, a given action is executed with probability one. A natural question is whether the principal can improve her expected payoffs by instead committing to a mechanism which is stochastic.

We should first consider what it means for a mechanism to be stochastic in a delegation setting. In the original delegation problem, the setting was interpreted as the principal choosing which actions to make available to the agent. Using the idea of a delegation function through the revelation principle was simply an analytical tool to characterize the optimal mechanism. To motivate the possibility of a stochastic mechanism in their setting, Goltsman et al. (2009) consider contracting with an arbitrator, who decides which action to implement after being sent a message by the agent. Since the arbitrator does not have preferences over the implemented action, she can commit to randomizing over which action is chosen after a given report. Kováč and Mylovanov (2009), who consider a wider variety of distributions and biases, focus on stochastic mechanisms in general settings without monetary transfers rather than delegation per se.
There are three points at which a principal may incorporate randomness into the setting at hand. First, she could randomize the action taken after the report in the first period. Second, she could randomize over the delegation set that the agent will choose from in the second period. Finally, she could randomize over the action taken after the report in the second period. The next result shows that none of these will improve the principal’s payoffs.

**Theorem 1.5** The optimal dynamic delegation mechanism is deterministic.

The reason why the principal cannot improve her payoffs by randomizing over second period delegation sets is simple. To incentivize actions in the first period, the principal commits to leave some expected utility for the agent in the second period. Thus, the principal must decide whether it is more efficient to promise this utility through a single delegation set, or by randomizing over delegation sets (and thus over promises of expected utility). Because the mapping between the agent’s utility and the principal’s utility is concave, Jensen’s Inequality implies that it is better to commit to a single delegation set.

The non-optimality of mechanisms which randomize after the agent’s report arises due to the principal’s aversion to variance. Quadratic losses allow payoffs to be written as a function of the average and variance of an action given some report. Thus, variance is essentially “money burning” from the point of view of both players: lowering the agent’s utility by one unit by increasing the variance also lowers the principal’s utility by one unit. In the first period, it’s more efficient to punish the agent by reducing discretion in the second period rather than increasing variance in the first period (the slope of $\gamma$ derived in Proposition 1.4 is always greater than 1). Similarly, the cost of incentivizing a better action in the second period by increasing variance is not worth the gains that come from that better action.

The fact that the proofs of these results rely on properties of the quadratic loss function suggests that they may not hold in more general settings. Kováč and Mylovanov (2009) demonstrate the non-optimality of deterministic mechanisms in a situation where the principal has an absolute value loss function while the agent has a quadratic loss function. In this case, the principal can use a stochastic mechanism to achieve strictly higher payoffs than any deterministic
mechanism, because the principal can increase the variance, strictly lowering the agent’s payoffs, at arbitrarily small cost to herself. Thus, there are likely combinations of preferences for which stochastic mechanisms dominate deterministic ones in dynamic settings.

1.5 Delegation with Money Burning

Another tool Congress might use to incentivize EPA officials is committing to some other inefficient action if the EPA chooses something that seems more biased to Congress. If the cost of this action for Congress is linearly related to the cost to the EPA, then we are in the setting of delegation with money burning (Amador & Bagwell, 2013; Ambrus & Egorov, in press; Amador & Bagwell, 2016). The action could entail Congress reducing the budget on a program that it and EPA officials agree upon, or enacting environmentally damaging policy that it otherwise wouldn’t. In this case, we define the function $\gamma$ to be

$$
\gamma(\omega) = \begin{cases} 
k\omega & \text{for } \omega \leq 0 \\
-\infty & \text{otherwise} \end{cases}
$$

The fact that $\gamma(\omega) = -\infty$ for $\omega > 0$ reflects the fact that there is no way for the principal to improve the agent’s payoffs above 0: she can only punish the agent. Here, $k$ is a parameter which determines how costly money burning is for the principal, relative to the costs the agent faces. A high value of $k$ indicates that punishing the agent is particularly difficult for the principal, while a low value of $k$ implies the opposite. Amador and Bagwell (2013) interpret money burning as “wasteful administrative costs,” and Ambrus and Egorov (in press) refer to it as “paperwork,” but for any given setting there are a variety of other interpretations, such as the payoffs arising from a simple coordination game in the future.

---

6It is equivalent to divide the principal’s utility function by $k$, and have each unit of money burning lower the principal’s payoff by one. The choice is one of interpretation: here, a change in $k$ can be interpreted as a change in the ease of punishing the agent.
When \( s + \frac{1}{2}v^2 \leq 0 \), the loss function is

\[
L(\theta, s, v) = \left[ -u^P(v + \theta, \theta) - ks - \frac{k}{2}v^2 \right] f(\theta),
\]

and it is otherwise infinite. Note that this function’s convexity will depend on the concavity of \( u^P \), since the second derivative is

\[
L_{vv}^{MB}(\theta, s, v) = \left[ -u^P_{vv}(v + \theta, \theta) - k \right] f(\theta).
\]

### 1.5.1 Interval Delegation

Amador and Bagwell (2013) follows the previous literature on delegation in focusing on interval delegation, in which the principal allows the agent to make choices from a single interval (Holmström, 1977; Alonso & Matouschek, 2008). In particular, they provide conditions both in delegation settings with and without money burning for when interval delegation is optimal. For the setting with money burning, interval delegation is defined as allowing the agent to choose from an interval and there being no money burning. In a sense, this is a “corner solution,” in which the principal is not utilizing one of the instruments that is available to her: money burning is at the upper bound of 0 for all values of the state.

When looking for conditions that ensure such interval delegation is optimal when money burning is a possibility, it is obvious that the main results from Section 1.3 won’t apply, since those refer to solutions such that the continuation value is never at its upper bound. The continuation values in interval delegation are always at their upper bound: by definition, interval delegation includes no money burning. Instead, I’ll use the more general results found in Proposition 1.A.2 in Appendix 1.A, which account for these corner solutions.

**Theorem 1.6** Assume that the convexity condition holds. When money burning is feasible and equally costly to the principal and the agent, the following conditions imply that the optimal mechanism is one in which money burning is everywhere 0, and the actions taken are on an interval \([\theta_L, \theta_H]\).
\[
\forall \theta \in [0, \theta_L], \int_0^\theta p_a(\theta_L, z)f(z)dz \geq \left[ p_a(\theta_L, \theta) + (\theta - \theta_L) \right] (\theta - \theta_L)f(\theta), \text{ with equality at } \theta_L.
\]

(2) If \( \theta_L = 0 \), \( p_a(0, 0) \leq 0 \),

(3) \( F(\theta) - p_a(\theta, \theta)f(\theta) \) is nondecreasing for \( \theta \in [\theta_L, \theta_H] \),

(4) If \( \theta_H = 1 \), \( p_a(1, 1) \geq 0 \), and

(5) \( \forall \theta \in [\theta_H, 1], \frac{1}{\theta} \int_\theta^1 p_a(\theta, z)f(z) \leq \left[ -p_a(\theta_H, \theta) - (\theta_H - \theta) \right] (\theta - \theta_H)f(\theta), \text{ with equality at } \theta_H. \)

The conditions of Theorem 1.6 ensure that the principal cannot improve on the interval she is delegating. Conditions (2) and (4) are the easiest to interpret. They state that if the principal is allowing the agent to have flexibility at the “extremes,” then the principal must want even more extreme actions when the state is at its highest or lowest. For instance, if the principal wanted an action lower than one when the state is equal to one, then she could simply place a cap on the available actions which was slightly less than one, improving her payoffs without affecting incentive compatibility.

Condition (3) establishes that there is no profitable way to change incentives for the agent on the interval where the principal is giving the agent full discretion. The condition is equivalent to the quantity

\[
\frac{d}{d\theta} \left[ p_a(\theta, \theta)f(\theta) \right] - f(\theta)
\]

being weakly negative. Intuitively, the principal is considering starting to use money burning to incentivize the agent to take actions lower than the agent’s preferred action. Thus, at a given value of the state, the gains from using money burning are related to how costly the agent taking his preferred action is over the states immediately above the current one: if costs are sharply increasing, money burning is valuable. Money burning at a given state comes at a cost of \( f(\theta) \) to the principal, so the principal doesn’t have an incentive to deviate from giving the agent discretion over the interval as long as the benefits are lower than the cost.

Conditions (1) and (5) ensure optimality on the intervals in which the principal is giving the agent no discretion (the floor and the cap, respectively). To interpret this, consider condition
(5). Where $\theta \neq \theta_H$, the condition can be restated as

$$\left[u_p^P(\theta_H, \theta) + (\theta_H - \theta)\right] f(\theta) \leq -\frac{1}{\theta} \int_{\theta}^{\theta_H} u_p^P(\theta, z) f(z) \, dz.$$ 

The principal could deviate from the proposed cap in an incentive compatible way by allowing the agent to take a slightly higher action, but require money burning for the agent to do so. $u_p^P(\theta_H, \theta)$ is the benefit of raising the action one unit, and $\theta - \theta_H$ is the money burning required to do so. For this change to be incentive compatible without using money burning at any other values of the state, the actions taken at all states above $\theta$ must increase by an amount proportional to the inverse of $(\theta - \theta_H)$. Thus, condition (5) states that the benefit of increasing the action taken has to be weighed against the costs of burning money and the changes in payoffs that arise from the incentive compatibility constraint.

These results are closely related to the sufficiency conditions found by Amador and Bagwell. When the settings are made comparable, conditions (2)-(4) here are the same as conditions (c1), (c2'), and (c3') there. In fact, the conditions in Theorem 1.6 imply the conditions found in Amador and Bagwell (2013).

**Remark 1.1** The conditions from Theorem 1.6 imply those in used in Proposition 1(b) by Amador and Bagwell (2013) when applied to this setting.

It’s unclear how much more general Amador and Bagwell’s result is. The results are derived using similar tools (the costate equation used in Proposition 1.A.2 can be interpreted as Lagrange multipliers). One possible difference is that the conditions in Theorem 1.6 guarantee that the interval described is optimal without imposing monotonicity. In particular, conditions (1) and (5) ensured that even variations that lead to a non-monotonic action profile aren’t beneficial to the principal. Since these are precisely the conditions for which Theorem 1.6 and Amador and Bagwell’s Proposition 1(b) differ, in the future I plan on establishing whether the sufficient conditions would be the same if this constraint were invoked.
1.5.2 Using Money Burning

While Theorem 1.6 gives conditions under which the principal prefers not to use money burning, there are also conditions under which the principal does use money burning. Ambrus and Egorov (in press) and Amador and Bagwell (2016) both study settings in which this is the case. Intuitively, we might expect that money burning becomes more prevalent as the cost of incentivizing the agent goes down for the principal, or the importance of the action goes up.

The simplest case in which to see this is again when the principal has quadratic losses, the distribution is uniform, and the principal’s and agent’s preferred decisions are separated by a constant bias $b$. In this case the principal is solving

$$
\max_{a(\theta), \omega(\theta)} \int_{0}^{1} \left[ -\frac{1}{2} (a(\theta) - \theta + b)^2 + k\omega(\theta) \right] d\theta
$$

subject to

$$
-\frac{1}{2} (a(\theta) - \theta)^2 + \omega(\theta) \geq -\frac{1}{2} (a(\theta') - \theta)^2 + \omega(\theta')
$$

for all $\theta, \theta'$ and

$$
\omega(\theta) \leq 0.
$$

The fact that $u''_{a\theta}(a, \theta), u''_{\omega\theta}(a, \theta)$, and $\gamma' (\cdot)$ are constant in this setting implies that when the principal is using money burning, the action varies with the state linearly, with slope $\frac{2k-1}{k-1}$.

These leads to an optimal mechanism which takes one of four forms, as given in Theorem 1.7.

**Theorem 1.7** In the uniform-quadratic delegation setting with money burning, the optimal mechanism can take one of four forms:

1. Discretion for low actions, with a cap preventing high actions when $b \leq \frac{1}{2}$ and $k \geq \frac{1}{2}$
2. Discretion for low actions, and money burning for high actions when $b \leq k < \frac{1}{2}$
3. Constant action for low states and money burning for high states when $b \geq k$ and $k \leq \frac{1}{2b+1}$
4. The optimal uninformed action otherwise

As $b$ and $k$ vary, the form of the optimal mechanism changes qualitatively. All four forms can be seen in Figure 1.7. When $k$ is high enough, the principal doesn’t use any money burning, because the cost of incentivizing the agent are not worth the benefits of a lower action. In this case, the principal sets an upper bound on the action which can be taken. For high enough bias, this bound is always binding and is set at $\frac{1}{2} - b$, the principal’s optimal uninformed action. When the agent is less biased, the principal takes advantage of the agent’s superior information by giving him discretion to choose among low actions, and prevents him from taking actions above $1 - 2b$.

When $k$, the price of burning money, is low enough, the principal does use money burning. She uses money burning at higher values of the state to incentivize better actions, and doesn’t use any money burning at lower values of the state. Money burning isn’t used at lower states because using money burning to lower actions at these values of $\theta$ forces the principal to burn more money at higher $\theta$. Thus, at low values of the state, the principal gives the agent discretion when $b \leq k$, and holds the agent at a single action otherwise.

The way in which the optimal mechanism varies with the parameters can be seen in Figure 1.8. The comparative statics that are implicit within the figure are intuitive. In all cases, when you hold $b$ constant and lower $k$, the cost of money burning, the principal uses money burning more. Holding $k$ constant and lowering the agent’s bias leads to the principal giving the agent more discretion, and using less money burning.

1.6 Related Literature

The delegation problem was originally posed by Holmström (1977), and the optimal mechanism in the uniform-quadratic case was first found by Melumad and Shibano (1991). In this case, delegation takes the form of a “cap,” taking advantage of the agent’s information when the state is low, and pooling agents when the state is high. Since then, the literature on delegation has grown dramatically. Alonso and Matouschek (2008) solve for optimal delegation with generalized quadratic loss functions and minimal conditions on the bias and state distribution. They are able
Figure 1.7: The four possible forms of the optimal delegation with money burning mechanism with quadratic losses, a uniform distribution, cost of money burning $k$, and a constant bias $b$. 

(a) Discretion with a cap: $b = \frac{1}{3}, k = 1$

(b) Optimal uninformed action: $b = \frac{2}{3}, k = 1$

(c) Discretion with money burning: $b = \frac{1}{3}, k = \frac{2}{3}$

(d) Floor with money burning: $b = \frac{3}{5}, k = \frac{3}{5}$
Figure 1.8: The form of the optimal mechanism by $b$ and $k$
to turn the problem into a finite one, making it possible to use standard methods of optimization. Delegation has also been used to understand optimal commitments when an agent has time inconsistent preferences (Amador, Werning, & Angeletos, 2006; Halac & Yared, 2014; Galperti, 2016).

As mentioned in Section 1.5, Amador and Bagwell (2013), Ambrus and Egorov (in press), and Amador and Bagwell (2016) all study delegation with money burning, although in different settings. The models of these papers are closely connected to delegation with continuation values — they both essentially restrict the transfer function to be linear with slope 1, but generalize on the model here by allowing for the agent to have a wider variety of preferences in the first period. Amador and Bagwell (2013) derive conditions for which money burning is sub-optimal, even when it is possible, and they apply these results to a model of trade agreements and tariff caps. Ambrus and Egorov (in press) and Amador and Bagwell (2016) instead find a variety of settings when money burning *is* optimal, with the former focusing on how money burning interacts with participation constraints and ex-ante monetary transfers.

The delegation literature has turned to characterizing optimal delegation when there are multiple decisions to be made, whether they’re simultaneous or dynamic. Frankel (2014) solves for the max-min optimal mechanism, which maximizes the principal’s payoff against the “worst case” of the agent’s possible preferences. In a related paper, Frankel (2016) shows a way to cap the actions in each dimension against the agent’s bias in that dimension. This mechanism is optimal in the quadratic loss, i.i.d. normal case, and they show that it does well in a more general setting. The same mechanism is then applied to the dynamic setting and it is shown that the principal’s per-period payoffs converge to her full information payoffs as the number of periods tends to infinity.

In a technical sense, this paper is most similar to work by Koessler and Martimort (2012), which characterizes the solution to the delegation problem when there are two actions to be taken, but only one state variable. They use the agent’s incentive compatibility constraints to demonstrate that the principal’s problem can be reduced to solving for the optimal average action and the squared difference between the actions, which they call the spread. This spread variable enters
the agent’s utility function linearly and the principal’s nonlinearly, so they also use optimality conditions from Clarke (1990) to define the optimal mechanism, and find that optimal delegation uses the spread of the two actions to elicit information about the state and that there’s no pooling in the equilibrium.

This paper’s motivation is closely tied to that of Guo (2016), Guo and Hörner (2015), and Lipnowski and Ramos (2016), all of which study infinitely-repeated delegation problems. The first considers how to dynamically delegate experimentation to an agent who prefers to experiment for longer than the principal would like. The model is studied in continuous time, and optimal delegation involves involving pooling those types whose beliefs are initially high by stopping experimentation if posteriors fall below a cutoff. Guo and Hörner (2015) and Lipnowski and Ramos (2016) both study the repeated provision of a good to an agent whose valuation is unobserved. Guo and Hörner consider a model with commitment, and Lipnowski and Ramos consider it without commitment. In both models, the optimal mechanism can be defined as a “budget” mechanism, which restricts the number of occasions in which you can claim the good in the future. The dynamic delegation game studied here differs from those models in that it is finitely repeated and the state space is continuous, rather than binary.

### 1.7 Conclusion

Dynamic contracts can overcome the barriers faced in a static setting. This paper considers the classic problem of an uninformed principal delegating an action to an informed agent when she has some means of rewarding or punishing the agent in the future, but may face a cost of doing so. In characterizing the optimal mechanism, the paper showed that the principal trades off between incentivizing a better action today and giving the agent a continuation value that is better from the point of view of the principal.

Because utility is not transferable in this setting, standard mechanism design tools cannot be used. Instead, the paper identifies techniques from the calculus of variations which can overcome these difficulties. The paper applies these results to two canonical problems from
the literature, which can both be formulated as delegation problems with continuation values: dynamic delegation and delegation with money burning.

The first application is to a model in which a principal delegates two decisions dynamically. The paper first shows the optimal way to promise any continuation value, and then applies the more general results to solve for the optimal mechanism. This mechanism has no pooling in the first period, and caps the agent’s action in the second period. Furthermore, one can show that the principal prefers a less biased agent in the first period, but that increasing the bias in the second period eventually improves the principal’s payoffs. Finally, the optimal mechanism is deterministic.

The second application is to a delegation problem in which the principal has the option to “burn money.” The paper provides conditions under which the principal delegates an interval without using money burning, and relates these conditions to those previously found in the literature. The paper then shows a case in which money burning is used in the optimal mechanism, and provides intuitive comparative statics which describe how this mechanism varies with the cost of money burning.

It seems likely that these results will be useful in a variety of other problems, such as delegation when there are more than two periods, or when there is a more complicated game following the first period of delegation. A key step is to show that the principal has an optimal means of promising a continuation value, and find the explicit form of the function $\gamma$, which maps the agent’s continuation value to the principal’s.

1.8 Acknowledgements

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1.A Proofs of Main Results

Lemma 1.1 \( W \) is absolutely continuous, and where the derivative exists,

\[
W'(\theta) = a(\theta) - \theta
\]  

and

\[
\omega(\theta) = W(\theta) + \frac{1}{2} W'(\theta)^2
\]

Proof Standard from Milgrom and Segal (2002). \( \square \)

Proposition 1.1 Assume that the convexity condition holds. Then a solution to problem (P) exists.

Proof I’ll be using Theorem 4.1.3 from Clarke (1990). Note that we have sufficient assumptions such that \( L \) is measurable and \( L(\theta, \cdot, \cdot) \) is lower semicontinuous. Furthermore, the convexity condition stated above immediately implies the convexity condition required for Theorem 4.1.3.

Define

\[
H(\theta, s, p) = \sup \{pv - L(\theta, s, v) : v \in \mathbb{R} \}
\]

In this case, we know that

\[
H(\theta, s, p) = \sup \{pv - L(\theta, s, v) : v \in \mathbb{R} \}
\]

\[
= \sup \left\{ pv + \left[ u^P(v + \theta, \theta) + \gamma \left( s + \frac{1}{2} v^2 \right) \right] f(\theta) : v \in \mathbb{R} \right\}
\]

\[
\leq \sup \left\{ pv + \left[ u^P(v + \theta, \theta) + \gamma \left( s + \frac{1}{2} v^2 \right) \right] f(\theta) : (s, v) \in \mathbb{R}^2 \right\}
\]

\[
\leq \sup \left\{ pv + u^P(v + \theta, \theta)f(\theta) + \delta \bar{\gamma} f(\theta) : v \in \mathbb{R} \right\}
\]

where \( \bar{\gamma} \) us the upper bound on \( \gamma(\cdot) \) and the last inequality is due to the fact that \( s \) enters the argument of \( \gamma \) linearly. Since the derivative of \( u^P \) with respect to \( a \) is unbounded and \( f(\theta) \) is strictly positive, the last value is bounded by a function of \( p \) and \( \theta \), which is the first “growth
condition”.

The second growth condition is satisfied if it can be shown that the optimal \( W(0) \) lies within some bounded set. If \( \gamma(\cdot) \) reaches its maximum for all \( \omega \) above or below some point, the solution to the problem is immediate (simply choose the \( W' \) which maximizes \( u^P \) for each \( \theta \), and set \( W(0) \) appropriately). Thus, we only need to consider the case in which \( \lim_{\omega \to -\infty} \gamma(\omega) = -\infty \), and show that fixing the endpoint \( W(0) = x_0 \), that the objective function does not continue to fall as \( x_0 \to -\infty \) (the case for \( x_0 \to \infty \) is symmetric). I’ll show that this is impossible: that eventually, lowering \( W(0) \) must make the principal worse off. Define \( \omega_1 \) and \( \omega_2 \), respectively as the minimum and maximum \( \omega \) which maximize \( \gamma(\omega) \), and

\[
\gamma = \int_0^1 \left[ -u^P(\theta, \theta) - \gamma \right] f(\theta) d\theta,
\]

the losses from the principal giving the agent full discretion, and choosing the optimal continuation value. Obviously, an optimal mechanism must improve upon this. From the basic assumptions, there must exist some \( \hat{\nu} > 0 \) such that for all \( \theta \), \( u^P_\theta(\hat{\nu} + \theta, \theta) < 0 \), \( u^P_\theta(-\hat{\nu} + \theta, \theta) > 0 \), and \( -u(-\hat{\nu} + \theta, \theta) \geq u \). Finally, let \( x_0 \) be low enough such that \( -\gamma \left( \frac{1}{2} x_0 + \frac{1}{2} \hat{\nu} \right) > u \) and \( x_0 + \frac{1}{8} x_0^2 > \omega_2 \).

If the optimal mechanism with \( W(0) = x_0 \) improves upon giving the agent full discretion, then there must be some \( \hat{\theta} < 1 \) such that \( W(\hat{\theta}) = \frac{1}{2} x_0 \). Otherwise, for all \( \theta \) the mechanism would either have \( W(\theta) + \frac{1}{2} W'(\theta)^2 < \frac{1}{2} x_0 \), or \( |W'(\theta)| > \hat{\nu} \), giving losses that are higher than \( u \). Since \( W(0) = x_0 \) and \( W(\hat{\theta}) = \frac{1}{2} x_0 \), there must be some mass of \( \theta \) such that \( W'(\theta) \geq \frac{1}{2} x_0 \). Thus, we can marginally decrease any \( W'(\theta) > \sqrt{2\omega_2 - W(\theta)} \) and marginally increase any \( W'(\theta) < -\sqrt{2\omega_2 - W(\theta)} \), increasing the principal’s payoffs with \( W(0) > x_0 \). Thus, for \( x_0 \) low enough, increasing \( W(0) \) above \( x_0 \) increases the principal’s payoffs, so the second growth condition is satisfied. \( \square \)

**Lemma 1A.1** For any \( \varepsilon > 0 \), fix some arc \( W \) on \( [a, b] \) such that \( \forall \theta \in [a, b], W(\theta) \leq \omega - \varepsilon \). Then there exists an \( \hat{\epsilon} > 0 \) and an integrable function \( k(\theta) \) such that for all \( p \), for all \( (\theta, s_1) \) and
(θ, s2) in the tube T(W; ̂ε), one has

\[ |H(θ, s1, p) - H(θ, s2, p)| \leq k(θ)(1 + |p|)|s1 - s2| \]

**Proof** The basic assumptions guarantee that there must exist some \(v\) such that \(pv - L(θ, s1, v)\) attains its supremum. Define one such \(v\) as \(v_1(θ, s1, p)\). Next, define

\[ v_2(θ, s1, p, σ) = v_1(θ, s1, p) - sgn(v_1(θ, s1, p)) \cdot \frac{2}{√ε}|σ| \]

where \(σ = s2 - s1\). Defining

\[ ̂H(θ, s1 + σ, p) = pv_2(θ, s1, p, σ) \]

\[ = [u_2^p(θ, s1, p, σ) + θ, θ] + γ\left(s1 + σ + \frac{1}{2}v_2(θ, s1, p, σ)^2\right)\] \(f(θ)\),

we can take the derivative with respect to \(σ\) at 0 from the right, giving us

\[ -p \cdot sgn(v_1(θ, s1, p)) \cdot \frac{2}{√ε} + u_2^p(θ, s1, p, σ) + θ, θ\left(-sgn(v_1(θ, s1, p))\cdot \frac{2}{√ε}\right) f(θ) \]

\[ + γ\left(s1 + \frac{1}{2}v_1(θ, s1, p)^2\right) \left(1 - v_1(θ, s1, p) \cdot sgn(v_1(θ, s1, p)) \cdot \frac{2}{√ε}\right) f(θ) \]

where \(γ\) indicates the left derivative. With

\[ s = \min_{θ ∈ [a, b]} W(θ) \]

\[ ̄u_a = \sup \left\{ u_a^p(v + θ, θ) | v ∈ \left[-√(-2s + 2̄ω), √(-2s + 2̄ω)\right], θ ∈ [a, b] \right\} \]

\[ ̄γ = \left|γ\left(̄ω - \frac{3}{4}ε\right)\right| \]

\[ γ = |γ(̄ω)| \]

\[ ̄f = \max_{θ ∈ [0, 1]} f(θ) \]
this derivative is bounded below by

\[-\left| \frac{2}{\sqrt{\epsilon}} |p| + \bar{u}_a \frac{2}{\sqrt{\epsilon}} \bar{f} + \max \left( \bar{\gamma}, \gamma \right) \left( 1 + \sqrt{-2s} \right) \bar{f} \right|\]

and the derivative with respect to $\sigma$ at 0 from the left can be bounded by the same value. Thus,

\[
\lim_{s_2 \to s_1} \frac{H(\theta, s_2, p) - H(\theta, s_1, p)}{s_2 - s_1} \geq - \max \left\{ \frac{2}{\sqrt{\epsilon}}, \bar{u}_a \frac{2}{\sqrt{\epsilon}} \bar{f} + \max \left( \bar{\gamma}, \gamma \right) \left( 1 + \sqrt{-2s} \right) \bar{f} \right\} (1 + |p|)
\]

Since this is true for all $s_1$, we get that

\[
|H(\theta, s_2, p) - H(\theta, s_1, p)| \leq \max \left\{ \frac{2}{\sqrt{\epsilon}}, \bar{u}_a \frac{2}{\sqrt{\epsilon}} \bar{f} + \max \left( \bar{\gamma}, \gamma \right) \left( 1 + \sqrt{-2s} \right) \bar{f} \right\} (1 + |p|)|s_1 - s_2|
\]

which is the strong Lipschitz condition. □

Define

\[
V_{a,b,\epsilon}(s) = \min_{W(\theta)} \ell(W(a), W(b) + s) + \int_{a}^{b} L(\theta, W(\theta), W'(\theta)) \, d\theta
\]

where $\epsilon > 0$, $W(\theta)$ is absolutely continuous, $L$ is as before, and

\[
\ell(u, v) = \begin{cases} 0 & \text{if } u = v = -\epsilon \\ -\infty & \text{otherwise} \end{cases}
\]

**Lemma 1.A.2** Assume the convexity condition holds. With the above definitions, we have that if $V_{a,b,\epsilon}(0) > -\infty$

\[
\lim \inf_{s \to 0} \frac{V_{a,b,\epsilon}(s) - V_{a,b,\epsilon}(0)}{|s|} > -\infty
\]
Proof Note that the minimization problem is the problem from before, but with endpoint conditions. Thus, a solution exists for \( s \) in the neighborhood of 0, and \( V_{a,b,e}(s) \) is well defined over this neighborhood. Define a solution to the problem implicitly defined by \( s \) as \( W_s \), and note that \( W_s(a) = W_{-s}(a) \), due to the endpoint condition. Finally, define \( W_\lambda(\theta) = \lambda W_s(\theta) + (1 - \lambda) W_{-s}(\theta) \).

Since \( L(\theta, s, v) \) is convex in \( (s, v) \),

\[
\int_a^b L(\theta, W_\lambda(\theta), W'_\lambda(\theta)) d\theta
\leq \lambda \int_a^b L(\theta, W_s(\theta), W'_s(\theta)) d\theta + (1 - \lambda) \int_a^b L(\theta, W_{-s}(\theta), W'_{-s}(\theta)) d\theta
\]

so \( V_{a,b,e}(\cdot) \) is concave and thus locally Lipschitz. □

Lemma 1.A.3 Assume the convexity condition holds. Then for any function \( W(\theta) \) which solves problem (P), it must be the case that \( \forall \theta \) such that \( W(\theta) < \tilde{\omega} \), there exists an arc \( p(\theta) \) such that

\[
\begin{bmatrix}
-p'(\theta) \\
W'(\theta)
\end{bmatrix} \in \partial H(\theta, W(\theta), p(\theta)) \text{ a.e.}
\]

Furthermore, if \( W(0) < \tilde{\omega} \) then \( p(a) = 0 \) and if \( W(1) < \tilde{\omega} \) then \( p(b) = 0 \).

Proof I will use Theorem 4.2.2 from Clarke (1990). \( L \) satisfies the basic hypotheses and the convexity condition by assumption. For any \( \varepsilon > 0 \), since \( W(\theta) \) is continuous, the set \( \{ \theta \in [0, 1] : \tilde{\omega} \leq W(\theta) \leq -\varepsilon \} \) is compact. For any point \( a \) on the boundary of this set where \( a \) is not equal to 0 or 1, we have \( W(a) = -\varepsilon \). Thus, fixing \( \varepsilon \), for any \( [a, b] \subset \{ \theta \in [0, 1] : \tilde{\omega} \leq W(\theta) \leq -\varepsilon \} \) where \( a \) and \( b \) are in the boundary and \( 0 < a \leq b < 1 \), \( W \) must also maximize the same loss function under the endpoint restrictions of \( W(a) = W(b) = -\varepsilon \). Lemma 1.A.2 shows that this problem with endpoint conditions is calm, and since \( H \) is the same over the interval, Lemma 1.A.1 shows that it satisfies the strong Lipschitz condition near \( W \). Furthermore, if \( a \) is equal to 0, then \( W(\theta) \) must solve the free endpoint problem, implying that \( p(a) = 0 \). Similarly, \( p(b) = 0 \) when \( b = 1 \).
Thus, Theorem 4.2.2 can be applied, giving the result. □

Proposition 1.A.1 Assume that the convexity condition holds. If \( W(\theta) \) solves problem (P), then for \( \theta \in [0,1] \) such that \( W(\theta) < \bar{\omega} \), the following must be true almost everywhere:

1. if \( W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega} \) and \( \gamma''(W(\theta) + \frac{1}{2}W'(\theta)^2) \) exists, then

\[
W''(\theta) = -1 - \frac{\left[u_a^P(W'(\theta) + \theta, \theta) + W'(\theta)\gamma'(W(\theta) + \frac{1}{2}W'(\theta)^2)\right]}{u_a^P(W'(\theta) + \theta, \theta) + W'(\theta)^2\gamma''(W(\theta) + \frac{1}{2}W'(\theta)^2)} + \frac{2\gamma'(W(\theta) + \frac{1}{2}W'(\theta)^2)}{u_a^P(W'(\theta) + \theta, \theta) + W'(\theta)^2\gamma''(W(\theta) + \frac{1}{2}W'(\theta)^2)} - \frac{\gamma'(W(\theta) + \frac{1}{2}W'(\theta)^2)}{u_a^P(W'(\theta) + \theta, \theta)}.
\]

(2) if for some interval \([a,b]\), such that \( 0 < a < b < 1 \), \( \forall \theta \in [a,b] \), \( W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega} \), and \( W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega} \) in the neighborhoods of \( a \) and \( b \), then

\[
\theta \int_a^\theta u_a^P(W'(a) + a, z) f(z) dz \geq W'(a)u_a^P(W'(a) + a, a) f(a) + W'(a)^2\gamma'(\bar{\omega}) f(a)
\]

\[
- W'(\theta)u_a^P(W'(a) + a, \theta) f(\theta) - W'(\theta)^2\gamma'(\bar{\omega}) f(\theta).
\]

and this must hold with equality at \( \theta = b \),

3. if \( \theta \in \{0,1\} \), if \( W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega} \) then

\[
W'(\theta)\gamma'(W(\theta) + \frac{1}{2}W'(\theta)^2) = -u_a^P(W'(\theta) + \theta, \theta),
\]

4. if \( \forall \theta \in [0,b] \), \( W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega} \), with \( W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega} \) for \( \theta \) in the neighborhood of \( b \), then \( \forall \theta \in [0,b] \)

\[
\int_0^\theta u_a^P(W'(0), z) f(z) dz \geq \left[ -W'(\theta)u_a^P(W'(\theta) + \theta, \theta) - W'(\theta)^2\gamma'(\bar{\omega}) \right] f(\theta)
\]

with equality at \( \theta = b \). Similarly, if \( \forall \theta \in [a,1] \), \( W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega} \), with \( W(\theta) +
\[ \frac{1}{2}W'(\theta)^2 < \bar{\omega} \text{ for } \theta \text{ in the neighborhood of } a, \text{ then } \forall \theta \in [a, 1], \]

\[
\int_{\theta} u_a^P(W'(1) + 1, z)f(z)dz \leq \left[W'(\theta)u_a^P(W'(\theta) + \theta, \theta) + W'(\theta)^2\gamma'_{-}(\bar{\omega})\right] f(\theta)
\]

with equality at \( \theta = a \). Finally, if \( \forall \theta \in [0, 1], W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega} \), then \( \forall \theta \in [0, 1] \)

\[
\int_{0}^{\theta} u_a(W'(0), z)f(z)dz \geq W'(\theta) \left[-u_a(W'(\theta) + \theta, \theta) - W'(\theta)\gamma'_{-}(\bar{\omega})\right] g(\theta)
\]

and

\[
\int_{0}^{1} u_a(W'(0), \theta)f(\theta)d\theta = 0.
\]

**Proof** Lemma 1.A.3 shows that under the convexity condition, \( \forall \theta \) such that \( W(\theta) < \bar{\omega} \), there exists an arc \( p(\theta) \) such that

\[
\begin{bmatrix}
-p'(\theta) \\
W'(\theta)
\end{bmatrix} \in \partial H(\theta, W(\theta), p(\theta)) \text{ a.e.}
\]

(1) Using the definitions from Clarke (1990), when \( W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega} \), the envelope theorem gives that

\[
\frac{\partial H}{\partial s}(\theta, W(\theta), p(\theta)) = \gamma' \left(W(\theta) + \frac{1}{2}W'(\theta)^2\right)f(\theta)
\]

and that \( p(\theta) = [-u_a(W'(\theta) + \theta, \theta) - W'(\theta)\gamma' \left(W(\theta) + \frac{1}{2}W'(\theta)^2\right)] f(\theta) \). Where the sec-
Thus, we have that

\[
-p'(\theta) = \left[ u_{uu}^P(W'(\theta) + \theta, \theta) + W'(\theta)^2 \gamma'' \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) \right. \\
+ \left. \gamma' \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) \right] f(\theta) W''(\theta) \\
+ \left[ u_{uu}^P(W'(\theta) + \theta, \theta) + u_{u\theta}^P(W'(\theta) + \theta, \theta) \\
+ W'(\theta)^2 \gamma'' \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) \right] f(\theta) \\
- \left[ u_{\theta u}^P(W'(\theta) + \theta, \theta) + W'(\theta)^2 \gamma' \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) \right] f'(\theta) \\
+ \gamma' \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) f(\theta)
\]

and

\[
W''(\theta) = - \left[ u_{uu}^P(W'(\theta) + \theta, \theta) + u_{u\theta}^P(W'(\theta) + \theta, \theta) + W'(\theta)^2 \gamma'' \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) \right] f(\theta) \\
+ \left[ u_{uu}^P(W'(\theta) + \theta, \theta) + W'(\theta)^2 \gamma' \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) \right] f(\theta) \\
- \left[ u_{uu}^P(W'(\theta) + \theta, \theta) + W'(\theta)^2 \gamma'' \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) \right] f'(\theta) \\
+ \gamma' \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) f(\theta)
\]

\[
= -1 - \frac{u_{uu}^P(W'(\theta) + \theta, \theta) + W'(\theta)^2 \gamma'' \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) f(\theta)}{u_{uu}^P(W'(\theta) + \theta, \theta) + W'(\theta)^2 \gamma' \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) f(\theta)} \\
+ \frac{2 \gamma' \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) - u_{u\theta}^P(W'(\theta) + \theta, \theta)}{u_{uu}^P(W'(\theta) + \theta, \theta) + W'(\theta)^2 \gamma'' \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right) + \gamma' \left( W(\theta) + \frac{1}{2} W'(\theta)^2 \right)}
\]
(2) When $W(\theta) + \frac{1}{2}W'(\theta)^2 = 0$ but $W(\theta) < \bar{\omega}$, we have that

$$\frac{\partial H}{\partial s}(\theta, W(\theta), p(\theta)) = -\frac{1}{W'(\theta)}p(\theta) - \frac{1}{W'(\theta)}u_a^P(W'(\theta) + \theta, \theta) f(\theta).$$

If $W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega}$ for some interval $[a, b]$, then it must be the case that on that interval, $W'(\theta) = W'(a) + a - \theta$. Thus, we get the first order differential equation

$$p'(\theta) = \frac{1}{W'(a) + a - \theta} p(\theta) + \frac{1}{W'(a) + a - \theta} u_a^P(W'(a) + a, \theta) f(\theta)$$

which has the solution

$$p(\theta) = \frac{c_1}{W'(a) + a - \theta} + \frac{\int_a^\theta u_a^P(W'(a) + a, z) f(z) dz}{W'(a) + a - \theta}$$

If $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$ for $\theta$ just below $a$, then the continuity of $p$ implies that

$$p(a) = [ -u_a^P(W'(a) + a, a) - W'(a)\gamma'(\bar{\omega}) ] f(a).$$

Plugging this in we get

$$c_1 = -W'(a)u_a^P(W'(a) + a, a) f(a) - W'(a)^2\gamma'(\bar{\omega}) f(a)$$

$$- \int_a^\theta u_a^P(W'(a) + a, z) f(z) dz$$

Furthermore, with this definition of $p(\theta)$ we have that it must be the case that

$$\int_a^\theta u_a^P(W'(a) + a, z) f(z) dz \geq W'(a)u_a^P(W'(a) + a, a) f(a) + W'(a)^2\gamma'(0) f(a)$$

$$- W'(\theta)u_a^P(W'(a) + a, \theta) f(\theta) - W'(\theta)^2\gamma'(0) f(\theta)$$
and since \( W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega} \) for \( \theta \) just above \( b \),

\[
p(b) = \left[ -u_a^p(W'(b) + b, b) - W'(b)\gamma'(\bar{\omega}) \right] f(b)
\]

which means it must hold with equality at \( \theta = b \).

(3) With the endpoint conditions given, if \( W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega} \) for \( \theta \in \{0, 1\} \),

\[
0 = p(\theta) = \left[ -u_a(W'(\theta) + \theta, \theta) - W'(\theta)\gamma' \left( W(\theta) + \frac{1}{2}W'(\theta)^2 \right) \right] f(\theta)
\]

which simplifies to

\[
W'(\theta)\gamma' \left( W(\theta) + \frac{1}{2}W'(\theta)^2 \right) = -u_a^p(W'(\theta) + \theta, \theta).
\]

(4) Alternatively, if \( W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega} \) for \( \theta \in [0, b] \), then we can use the formula for \( p(\theta) \) from above to show that

\[
c_1 = \int_0^1 u_a^p(W'(0), z) f(z) dz
\]

giving us the inequality and that it must hold with equality at \( \theta = b \). A similar exercise can be done when \( W(\theta) + \frac{1}{2}W'(\theta)^2 \) for \( \theta \in [a, 1] \). When \( W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega} \) for all \( \theta \), then \( c_1 = 0 \), and \( p(0) = p(1) = 0 \) implies the two conditions. \( \Box \)

**Proposition 1.2** Assume that the convexity condition holds, and that \( W(\theta) \) solves problem (P).

Define \( a(\theta) = W'(\theta) + \theta \) and \( \omega(\theta) = W(\theta) + \frac{1}{2}W'(\theta)^2 \). Then for \( \theta \in [0, 1] \) such that \( \omega(\theta) < \bar{\omega} \),

(1) if \( \theta \in \{0, 1\} \),

\[
W'(\theta)\gamma'(\omega(\theta)) = -u_a^p(a(\theta), \theta),
\]

and
(2) if $\gamma''(\omega(\theta))$ exists, then

$$W''(\theta) = -1 + \frac{2\gamma'(\omega(\theta)) - u_{a\theta}(a(\theta), \theta) - [u_{a\theta}^p(a(\theta), \theta) + W'(\theta)\gamma'(\omega(\theta))]}{u_{a\theta}^p(a(\theta), \theta) + W'(\theta)^2\gamma''(\omega(\theta)) + \gamma'(\omega(\theta))}f'(\theta)$$

almost everywhere.

Proof This is an immediate implication of Proposition 1.A.1. □

Proposition 1.A.2 Assume that the convexity condition holds. Suppose that $\forall \theta, W'(\theta) + \frac{1}{2}W'(\theta) \leq \bar{\omega}$ and that there exists an absolutely continuous function $p : [0, 1] \rightarrow \mathbb{R}$ such that

- $W'(\theta) \in \arg \max_v p(\theta)v - L(\theta, W(\theta), v)$
- $p(\theta) = -u_{a\theta}(\theta, \theta) f(\theta)$ if $W(\theta) = \bar{\omega}$.
- $p(0) = 0$ if $W(0) < \bar{\omega}$, and $p(0) \geq 0$ if $W(0) = \bar{\omega}$
- $p(1) = 0$ if $W(1) < \bar{\omega}$, and $p(1) \leq 0$ if $W(1) = \bar{\omega}$

and the following is true almost everywhere:

- $p'(\theta) = -\gamma'(W(\theta) + \frac{1}{2}W'(\theta)) f(\theta)$ if $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$
- $p'(\theta) = -\frac{1}{W'(\theta)} p(\theta) + \frac{1}{W'(\theta)} u_{a\theta}^p(W'(\theta) + \theta, \theta) f(\theta)$ if $W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega}$

and $W(\theta) < \bar{\omega}$, and

- $p'(\theta) \geq -\gamma'(\bar{\omega}) f(\theta)$ if $W(\theta) = \bar{\omega}$.

Then $W(\theta)$ solves problem (P).

Proof The proof will use Theorem 4.3.1 from Clarke (1990) (with $Q(\theta) = 0$ and $\varepsilon = \infty$), showing that $W(\theta)$ solves an adjusted problem, in which

$$\ell(u,v) = \begin{cases} 0 & \text{if } u \leq \bar{\omega} \& v \leq \bar{\omega} \\ \infty & \text{otherwise} \end{cases}.$$  

Obviously, this problem has the same solution as the problem at hand, as the loss function is infinite if $W(\theta) > \bar{\omega}$ for any $\theta$. The first point guarantees that for all $v$,

$$p(\theta)W'(\theta) - L(\theta, W(\theta), W'(\theta)) \geq p(\theta)v - L(\theta, W(\theta), v).$$
As demonstrated above, if $W(\theta) < 0$,

$$\frac{\partial H}{\partial s}(\theta, W(\theta), p(\theta)) = \begin{cases} 
\gamma'(W(\theta) + \frac{1}{2}W'(\theta)) & \text{if } W(\theta) + \frac{1}{2}W'(\theta) < 0 \\
\frac{1}{W'(\theta)} p'(\theta) + \frac{1}{\alpha W'(\theta)} u^p_\alpha (W'(\theta) + \theta, \theta) f(\theta) & \text{otherwise}
\end{cases}$$

Furthermore, in the case when $W(\theta) = 0$ and $p(\theta) = -u^p_\alpha(\theta, \theta) f(\theta)$,

$$\frac{\partial H}{\partial s}(\theta, W(\theta), p(\theta)) = (-\infty, \gamma'(0) f(\theta)).$$

$H(\theta, \cdot, W'(\theta))$ inherits concavity from $\gamma(\cdot)$, so $-p'(\theta) \in \frac{\partial H}{\partial s}(\theta, W(\theta), p(\theta))$ implies that $\forall y$,

$$H(\theta, W(\theta) + y, p(\theta)) - H(\theta, W(\theta), p(\theta)) \leq -p'(\theta)y$$

For the given $W(\theta)$, $\ell(W(0), W(1)) = 0$, so if $p(0) = p(1) = 0$, it must be the case that for all $u$ and $v$,

$$\ell(W(0) + u, W(1) + v) - \ell(W(0), W(1)) \geq -p(1)v + p(0)u.$$ 

If either $W(0) = \bar{\omega}$ and $p(0) > 0$ or $W(1) = \bar{\omega}$ and $p(1) < 0$, then the right hand side is positive only when the left hand side is infinite, so the inequality still holds. Thus, $W(\theta)$ and $p(\theta)$ fulfill all of the requirements from Theorem 4.3.1, and $W(\theta)$ solves the problem. □

**Corollary 1.A.1** Assume that the convexity condition holds. If $\forall \theta, W(\theta) < \bar{\omega}$, $W(\theta)$ is absolutely continuous, and $W(\theta)$ satisfies conditions (1)-(4) from Proposition 1.A.1, then $W(\theta)$ solves problem $(P)$.

**Proof** Define a function $p : [0, 1] \to \mathbb{R}$ such that $p(0) = 0$ and

$$p'(\theta) = \begin{cases} 
-\gamma'(W(\theta) + \frac{1}{2}W'(\theta)) f(\theta) & \text{if } W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega} \\
\frac{1}{W'(\theta)} p(\theta) + \frac{1}{\alpha W'(\theta)} u^p_\alpha (W'(\theta) + \theta, \theta) f(\theta) & \text{otherwise}
\end{cases}.$$
Then the proof of Proposition 1.A.1 shows that conditions (1), (2), and (4) guarantee that when $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$,

$$p(\theta) = \left[ -u_a(W'(\theta) + \theta, \theta) - W'(\theta)\gamma' \left( W(\theta) + \frac{1}{2}W'(\theta)^2 \right) \right] f(\theta),$$

and points (3) and (4) guarantee that $p(1) = 0$. Furthermore, when $W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega}$,

$$p(\theta) \geq W(\theta) \left[ -u_a(W'(\theta) + \theta, \theta) - W'(\theta)\gamma' \left( W(\theta) + \frac{1}{2}W'(\theta)^2 \right) \right] f(\theta),$$

so $W'(\theta) \in \arg\max_v p(\theta)v - L(\theta, W(\theta), v)$. Thus the conditions of Proposition 1.A.2 are fulfilled. □

**Proposition 1.3** Assume that the convexity condition holds. If $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$, $\gamma''(W(\theta) + \frac{1}{2}W'(\theta)^2)$ exists, $W(\theta)$ is absolutely continuous, and $W(\theta)$ satisfies conditions (1) and (2) from Proposition 1.2 almost everywhere, then $W(\theta)$ solves problem (P).

**Proof** All the conditions for application of Corollary 1.A apply, and $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$, so only conditions (1)-(2) from Proposition 1.A.1 need to be satisfied. □

**Theorem 1.1** Suppose that the convexity condition holds, and the following is true:

(1) for $\theta \in \{0, 1\}$,

$$-u_a(a(\theta), \theta) = (a(\theta) - \theta)\gamma'(\omega(\theta));$$

(2) $\gamma''(\omega(\theta))$ exists almost everywhere, with

$$a'(\theta) = \frac{2\gamma'(\omega(\theta)) - u_{a\theta}(a(\theta), \theta) - \left[ u_a^P(a(\theta), \theta) + (a(\theta) - \theta)\gamma'(\omega(\theta)) \right] \frac{f'(\theta)}{f(\theta)}}{u_{a\theta}(a(\theta), \theta) + (a(\theta) - \theta)^2 \gamma''(\omega(\theta)) + \gamma'(\omega(\theta))},$$

and $\omega(\theta)$ is derived from the envelope condition;

(3) $a(\theta)$ is continuous and monotonically increasing.

Then the mechanism defined by $a(\theta)$ and $\omega(\theta)$ is optimal.
Proof This simply combines the conditions of Proposition 1.3, with the requirement that \( W(\theta) \) be absolutely continuous, the envelope condition, and the monotonicity constraint. □

To prove Proposition 1.4, we must solve for the optimal way to promise some amount of continuation utility, \( \omega \), in this setting. The mechanisms that the principal is solving over in problem PDD’’ are deterministic, in that when the agent reports the state, a particular action is taken with probability one. We could instead consider a more general setting, in which the principal commits to a distribution of actions after any given report. In particular, let \( p(\cdot|\theta_1, \theta_2) \) be a probability density function over some set \( A \) for the action taken in the second period when the agent has reported \( \theta_1 \) and \( \theta_2 \). Define

\[
\bar{a}_2(\theta_1, \theta_2) = \int_A a p(a|\theta_1, \theta_2) da
\]

and

\[
\bar{\sigma}^2(\theta_1, \theta_2) = \int_A (a - \bar{a}_2(\theta_1, \theta_2))^2 p(a|\theta_1, \theta_2) da
\]

as the average and variance, respectively, of the actions taken after reports \( \theta_1 \) and \( \theta_2 \). Furthermore, it will be useful to define \( W(\theta_1, \theta_2) \) as the expected utility an agent will receive from the mechanism in the second period, after having reported \( \theta_1 \) in the first period and observing \( \theta_2 \). Thus, for an incentive compatible mechanism,

\[
W(\theta_1, \theta_2) = \int_A -\frac{1}{2} (a - \theta_2)^2 p(a|\theta_1, \theta_2) da.
\]

Lemma 1.A.4 A mechanism is incentive compatible in the second period if and only if

1. \( \bar{a}_2(\theta_1, \cdot) \) is non-decreasing and
2. \( W(\theta_1, \theta_2) = W(\theta_1, 0) + \int_0^{\theta_2} [\bar{a}(\theta_1, z) - z] dz \)

Proof The expected payoff of the agent of type \( \theta_2 \) who reports \( \theta_1 \) in the first period and \( \theta_2' \) in the
second period is

\[
\int_A -\frac{1}{2}(a - \theta_2)^2 p(a|\theta_1, \theta_2^\prime)da = -\frac{1}{2} \tilde{\sigma}^2(\theta_1, \theta_2^\prime) - \frac{1}{2} (\tilde{\alpha}(\theta_1, \theta_2^\prime) - \theta_2)^2.
\]

Thus, part 1 of the “only if” statement follows immediately from IC constraints, and part 2 is the standard envelope condition Milgrom and Segal (2002). To show the “if” statement, it suffices to show that under these conditions, for every \( \theta_2, \theta_2^\prime \in \Theta \),

\[
-\frac{1}{2} \tilde{\sigma}^2(\theta_1, \theta_2) - \frac{1}{2} (\tilde{\alpha}(\theta_1, \theta_2) - \theta_2)^2 \geq -\frac{1}{2} \tilde{\sigma}^2(\theta_1, \theta_2^\prime) - \frac{1}{2} (\tilde{\alpha}(\theta_1, \theta_2^\prime) - \theta_2)^2
\]

We can write

\[
-\frac{1}{2} \tilde{\sigma}^2(\theta_1, \theta_2^\prime) - \frac{1}{2} (\tilde{\alpha}(\theta_1, \theta_2^\prime) - \theta_2)^2
= -\frac{1}{2} \tilde{\sigma}^2(\theta_1, \theta_2^\prime) - \frac{1}{2} (\tilde{\alpha}(\theta_1, \theta_2^\prime) - \theta_2^\prime)^2 + \int_{\theta_2}^{\theta_2^\prime} [\tilde{\alpha}(\theta_1, \theta_2^\prime) - z] dz.
\]

\[
= W(\theta_1, \theta_2^\prime) + \int_{\theta_2}^{\theta_2^\prime} [\tilde{\alpha}(\theta_1, \theta_2^\prime) - z] dz
\]

so

\[
-\frac{1}{2} \tilde{\sigma}^2(\theta_1, \theta_2) - \frac{1}{2} (\tilde{\alpha}(\theta_1, \theta_2) - \theta_2)^2 + \frac{1}{2} \tilde{\sigma}^2(\theta_1, \theta_2^\prime) + \frac{1}{2} (\tilde{\alpha}(\theta_1, \theta_2^\prime) - \theta_2^\prime)^2
= W(\theta_1, \theta_2^\prime) - W(\theta_1, \theta_2^\prime) + \int_{\theta_2}^{\theta_2^\prime} [\tilde{\alpha}(\theta_1, \theta_2^\prime) - z] dz
\]

\[
= \int_{\theta_2}^{\theta_2^\prime} [\tilde{\alpha}(\theta_1, \theta_2^\prime) - \tilde{\alpha}(\theta_1, z)] dz
\]

\[
\geq 0
\]

where the last inequality is due to the fact that \( \tilde{\alpha}_2(\theta_1, \cdot) \) is non-decreasing. Thus, the two conditions are sufficient for incentive compatibility in the second period. □
Lemma 1.A.5 The principal’s expected payoff from the second period can be written as

\[-b_2 [W(\theta_1, 1) - W(\theta_1, 0)] - \frac{1}{2} b_2^2 + \int_0^1 W(\theta_1, \theta_2) d\theta_2.\]

Proof From the definitions above, we have

\[-\frac{1}{2} \sigma^2(\theta_1, \theta_2) - \frac{1}{2} (\tilde{a}(\theta_1, \theta_2) - \theta_2)^2 = W(\theta_1, \theta_2)\]

\[= W(\theta_1, 0) + \int_0^{\theta_2} (\tilde{a}(\theta_1, z) - z) dz\]

The principal’s expected payoff in the second period, using the same decomposition as before, is

\[\int_0^1 \left[ -\frac{1}{2} \sigma^2(\theta_1, \theta_2) - \frac{1}{2} (\tilde{a}(\theta_1, \theta_2) - \theta_2) - b_2^2 \right] d\theta_2.\]

Plugging in the expression for \(\sigma^2\), this can be written as

\[\int_0^1 \left[ W(\theta_1, 0) + \int_0^{\theta_2} (\tilde{a}(\theta_1, z) - z) dz + \frac{1}{2} (\tilde{a}(\theta_1, \theta_2) - \theta_2)^2 - \frac{1}{2} (\tilde{a}(\theta_1, \theta_2) - \theta_2 + b_2)^2 \right] d\theta_2\]

We can switch the order of integration on the second integral to get

\[\int_0^{\theta_2} \int_0^1 (\tilde{a}(\theta_1, z) - z) dz d\theta_2 = \int_0^{\theta_2} \int_0^1 (\tilde{a}(\theta_1, z) - z) d\theta_2 dz\]

\[= \int_0^{\theta_2} (1 - \theta_2)(\tilde{a}(\theta_1, \theta_2) - \theta_2) d\theta_2\]
Thus, the payoff further simplifies to

\[
\int_0^1 \left[ W(\theta_1,0) + (1 - \theta_2)(\tilde{a}(\theta_1, \theta_2) - \theta_2) - \frac{1}{2} b_2 (2\tilde{a}(\theta_1, \theta_2) - 2\theta_2 + b_2) \right] d\theta_2 \\
= W(\theta_1,0) - \frac{1}{2} b_2^2 + \int_0^1 \tilde{a}(\theta_1, \theta_2) (1 - \theta_2 - b_2) d\theta_2 + \int_0^1 \left[ -\theta_2 + \theta_2^2 + \theta_2 b_2 \right] d\theta_2 \\
= W(\theta_1,0) - \frac{1}{2} b_2^2 - \frac{1}{6} + \frac{1}{2} b_2 + \int_0^1 \left( \frac{\partial W(\theta_1, \theta_2)}{\partial \theta_2} + \theta_2 \right) (1 - \theta_2 - b_2) d\theta_2 \\
= W(\theta_1,0) - \frac{1}{2} b_2^2 + \int_0^1 \frac{\partial W(\theta_1, \theta_2)}{\partial \theta_2} (1 - \theta_2 - b_2) d\theta_2 \\
= W(\theta_1,0) - \frac{1}{2} b_2^2 - b_2 W(\theta_1,1) - (1 - b_2) W(\theta_1,0) + \int_0^1 W(\theta_1, \theta_2) d\theta_2 \\
= -b_2 [W(\theta_1,1) - W(\theta_1,0)] - \frac{1}{2} b_2^2 + \int_0^1 W(\theta_1, \theta_2) d\theta_2,
\]

which is the expression in the lemma. □

**Proposition 1.4** In the uniform-quadratic setting, the principal’s problem can be written as

\[
\max_{a_t(\theta_1), \omega(\theta_1)} \mathbb{E} \left[ -\frac{1}{2} (a_t(\theta_1) - \theta_1 + b_1)^2 + \gamma(\omega(\theta_1)) \right] \tag{PDD'}
\]

subject to \( \forall \theta_1, \theta'_1 \in \Theta, \)

\[-\frac{1}{2} (a_t(\theta_1) - \theta_1)^2 + \omega(\theta_1) \geq -\frac{1}{2} (a_t(\theta'_1) - \theta_1)^2 + \omega(\theta'_1) \tag{IC1'}\]

where

\[
\gamma(\omega) = \begin{cases} 
\omega - \frac{\delta}{2} b_2^2 + \frac{6 b_2}{\delta} (-\frac{7}{6} \omega - 3) \frac{1}{2} & \text{if } \omega < -\frac{\delta}{6} \\
\omega - \frac{\delta}{2} b_2^2 + \frac{6 b_2}{\delta} (\frac{6}{3} \omega)^{\frac{1}{2}} & \text{if } -\frac{\delta}{6} \leq \omega \leq 0 \\
-\infty & \text{otherwise}
\end{cases}
\]
I’ll first show that the optimal second period utility function $W(\theta_1, \theta_2)$ conditional on the agent receiving expected utility $\frac{1}{\delta} \omega$ in the second period is

$$W(\theta_1, \theta_2) = -\sqrt{-\frac{2}{\delta} \omega - \frac{1}{12} \theta_2 + \frac{1}{\delta} \omega + 1} - \frac{1}{2} \sqrt{-\frac{2}{\delta} \omega - \frac{1}{12} - \frac{1}{2} \theta_2^2 + \frac{1}{2} \theta_2 - \frac{1}{12}}$$

when $\omega < -\frac{\delta}{6}$ and

$$W(\theta_1, \theta_2) = \begin{cases} 0 & \text{for } \theta_2 \leq 1 + \left(\frac{6}{\delta} \omega\right)^{\frac{1}{3}} \\ -\frac{1}{2} \left(1 + \left(\frac{6}{\delta} \omega\right)^{\frac{1}{3}} - \theta_2^2\right)^2 & \text{otherwise} \end{cases}$$

when $-\frac{\delta}{6} \leq \omega \leq 0.7$ Lemma 1.A.5 shows that after accounting for IC2, the principal’s payoff from the second period takes the form

$$-b_2 [W(\theta_1, 1) - W(\theta_1, 0)] - \frac{1}{2} b_2^2 + \int_0^1 W(\theta_1, \theta_2) d\theta_2$$

Using this formula, and multiplying by the discount factor, it is easily checked that the above $W(\theta_1, \theta_2)$ generates the given function $\gamma(\cdot)$. Suppose that this second period indirect utility function were not optimal. Then there is another indirect utility function derived from an incentive compatible mechanism, $\tilde{W}(\theta_1, \theta_2)$ which gives the principal a higher payoff and the agent the same expected utility, $\omega$. Obviously, this function must be continuous. Furthermore, the monotonicity constraint implies that it cannot be kinked downward (because this would imply a downward jump in the action) and, where twice differentiable, cannot have a second derivative lower than

---

7One can check that this is the indirect second period utility function that arises from the principal placing a cap on the second period action, and the agent choosing optimally given this cap, implying that this utility function is incentive compatible. The cap, as a function of $\omega$, is

$$y(\omega) = \begin{cases} \frac{1}{2} - \sqrt{-\frac{2}{\delta} \omega - \frac{1}{12}} & \text{for } \omega < -\frac{1}{6} \\ 1 + \left(\frac{6}{\delta} \omega\right)^{\frac{1}{3}} & \text{otherwise} \end{cases}$$
−2 (since this would imply a decreasing action). Since
\[
\int_0^1 \tilde{W}(\theta_1, \theta_2) d\theta_2 = \omega = \int_0^1 W(\theta_1, \theta_2) d\theta_2,
\]
the difference between the expected utility for the principal is
\[
-b_2 [\tilde{W}(\theta_1, 1) - \tilde{W}(\theta_1, 0) - W(\theta_1, 1) + W(\theta_1, 0)].
\]
This quantity is positive by assumption, since \(\tilde{W}\) gives the principal a higher payoff, so \(\tilde{W}(\theta_1, 0) - \tilde{W}(\theta_1, 1) > W(\theta_1, 0) - W(\theta_1, 1)\). I’ll show that this is impossible for any incentive compatible second period utility function by considering two cases.

- **Case 1 - \(\tilde{W}(\theta_1, 0) > W(\theta_1, 0)\):** Since \(\int_0^1 \tilde{W}(\theta_1, \theta_2) d\theta_2 = \int_0^1 W(\theta_1, \theta_2) d\theta_2\), there must be a point at which \(\tilde{W}(\theta_1, \theta_2)\) crosses \(W(\theta_1, \theta_2)\). The restriction that \(\frac{\partial^2 \tilde{W}(\theta_1, \theta_2)}{\partial \theta_2^2} \geq -2 = \frac{\partial^2 W(\theta_1, \theta_2)}{\partial \theta_2^2}\), along with the impossibility of downward kinks, implies that \(\frac{\partial \tilde{W}(\theta_1, 0)}{\partial \theta_2} < \frac{\partial W(\theta_1, 0)}{\partial \theta_2}\), so the action associated with \(\tilde{W}(\theta_1, 0)\) is strictly less than that of \(W(\theta_1, 0)\). However, since \(\tilde{W}(\theta_1, 0) > W(\theta_1, 0)\), this implies that the variance associated with \(\tilde{W}(\theta_1, 0)\) is strictly less than that which is associated with \(W(\theta_1, 0)\), an impossibility since the latter is 0.

- **Case 2 - \(\tilde{W}(\theta_1, 0) \leq W(\theta_1, 0)\):** For \(\int_0^1 \tilde{W}(\theta_1, \theta_2) d\theta_2 = \int_0^1 W(\theta_1, \theta_2) d\theta_2\) and \(\tilde{W}(\theta_1, 0) - \tilde{W}(\theta_1, 1) > W(\theta_1, 0) - W(\theta_1, 1)\), \(\tilde{W}(\theta_1, \theta_2)\) must cross \(W(\theta_1, \theta_2)\) at least twice, the first time from below and the second time from above. This would require that \(\frac{\partial \tilde{W}(\theta_1, \theta_2)}{\partial \theta_2} > \frac{\partial W(\theta_1, \theta_2)}{\partial \theta_2}\) at the first intersection, but the opposite at the second, and impossibility due to the restriction on kinks and the fact that \(\frac{\partial^2 \tilde{W}(\theta_1, \theta_2)}{\partial \theta_2^2} \geq -2 = \frac{\partial^2 W(\theta_1, \theta_2)}{\partial \theta_2^2}\) anywhere that they could cross.

Thus, there is no second period utility function arising from an incentive compatible mechanism which gives the principal higher expected utility, and the function \(\gamma(\cdot)\) provides the mapping between the agent’s and principal’s continuation values. □

**Theorem 1.2** In the dynamic delegation game, an optimal mechanism exists and has the following properties:
• the action is strictly increasing and continuous in the first period’s state (no pooling in the first period);
• the maximum action available in the second period is strictly decreasing in the first period’s state;
• the actions taken in the first period are a strict subset of those the principal would take if she were fully informed.

**Proof** Proposition 1.5, proven in Appendix 1.B, shows that there is an indirect utility function $W(\theta)$ which characterizes the solution to problem (PDD'), ignoring the monotonicity constraint. Thus, it will suffice to show that the utility function $a(\theta)$ implied by the solution is monotonic and then identify some of its properties. Since

$$a(\theta) = W'(\theta) + \theta,$$

and the solution to the differential equation will be continuously differentiable, $a(\theta)$ is continuous. To show monotonicity, I’ll show that $\forall \theta \in [0, 1]$ where it exists, $W''(\theta) > -1$.

Consider the region in which $W(\theta) + \frac{1}{2}W'(\theta)^2 < -\frac{\delta}{6}$. Here,

$$W''(\theta) = -1 + \frac{\delta^{-\frac{1}{2}} (-2W(\theta) - W'(\theta)^2 - \frac{1}{12} \delta)^{\frac{3}{2}} + 4b_2W(\theta) + 2b_2W'(\theta)^2 + \frac{1}{6} \delta b_2}{2b_2W(\theta) + \frac{1}{12} \delta b_2}$$

and Lemma 1.B.4 showed that

$$W'(\theta)^2 = -\delta \left[ \frac{2b_2W(\theta) + \frac{1}{12} \delta b_2}{W(\theta) - \lambda} \right]^2 - 2W(\theta) - \frac{1}{12} \delta$$

for some $\lambda \in \mathbb{R}$. Since in this region, $W(\theta) < -\frac{1}{6} \delta$, $W''(\theta) > -1$ if

$$\delta^{-\frac{1}{2}} \left( -2W(\theta) - W'(\theta)^2 - \frac{1}{12} \delta \right)^{\frac{3}{2}} + 4b_2W(\theta) + 2b_2W'(\theta)^2 + \frac{1}{6} \delta b_2$$
is negative. We can substitute to get

\[
\delta \left[ \frac{2b_2W(\theta) + \frac{1}{12} \delta b_2}{W(\theta) - \lambda} \right]^3 - 2b_2\delta \left[ \frac{2b_2W(\theta) + \frac{1}{12} \delta b_2}{W(\theta) - \lambda} \right]^2,
\]

and after factoring we get

\[
\delta b_2 \left[ \frac{2b_2W(\theta) + \frac{1}{12} \delta b_2}{W(\theta) - \lambda} \right]^2 \left[ \frac{2W(\theta) + \frac{1}{12} \delta}{W(\theta) - \lambda} - \frac{1}{2} \right].
\]

Since \( \lambda > 0 \), as shown in the proof of Proposition 1.5, and \( W(\theta) < 0 \), this quantity is negative.

Suppose the solution is instead in the region in which \(-\frac{\delta}{6} \leq W(\theta) + \frac{1}{2}W'(\theta)^2 \leq 0\). Then

\[
W''(\theta) = -1 + \frac{\delta^{-\frac{1}{2}} \left( 6W(\theta) + 3W'(\theta)^2 \right) \frac{\delta^{-\frac{1}{2}} \left( 6W(\theta) + 3W'(\theta)^2 \right)^{\frac{1}{2}} + 4b_2}{12b_2W(\theta) + 2b_2W'(\theta)^2}.
\]

Factoring, we get

\[
W''(\theta) = -1 + \frac{\left( 6W(\theta) + 3W'(\theta)^2 \right) \left[ \delta^{-\frac{1}{2}} \left( 6W(\theta) + 3W'(\theta)^2 \right)^{\frac{1}{2}} + 4b_2 \right]}{12b_2W(\theta) + 2b_2W'(\theta)^2}
\]

and

\[
\left( 6W(\theta) + 3W'(\theta)^2 \right)^{\frac{1}{2}} = \delta^{\frac{1}{2}} \left[ 3b_2W(\theta) - \frac{1}{2} b_2W'(\theta)^2 \right] \frac{W(\theta) - \lambda}{W(\theta) - \lambda}
\]

Thus, this simplifies to

\[
W''(\theta) = -1 + \frac{\left( 6W(\theta) + 3W'(\theta)^2 \right) \left[ \frac{3b_2W(\theta) - \frac{1}{2} b_2W'(\theta)^2}{W(\theta) - \lambda} + 4b_2 \right]}{12b_2W(\theta) + 2b_2W'(\theta)^2}.
\]

Again, since \( \lambda > 0 \), this \( W''(\theta) > -1 \) and the action in the first period is strictly increasing. From before, we know that

\[
\omega(\theta) = W(\theta) + \frac{1}{2}W'(\theta)^2.
\]
Thus, at points of differentiability,

$$\omega'(\theta) = W'(\theta) [1 + W''(\theta)],$$

so this is always negative. Combining this with the fact that $y(\omega)$, the cap on the second period action, is increasing in its argument gives the result that $y(\omega(\theta))$ is strictly decreasing. Finally, in the proofs from Appendix 1.B it is shown that $W'(0) > -b_1$ and $W'(1) < -b_1$. Since

$$a(0) = W'(0)$$

and

$$a(1) = W'(1) + 1,$$

it must be the case that $a(0) > -b_1$, and $a(1) < 1 - b_1$, so the actions taken all lie strictly within the set of actions the principal would take with full information, $[-b_1, 1 - b_1]$. □

**Lemma 1.A.6** In the optimal mechanism, $\gamma(\omega(\theta))$ achieves the maximum value of $\gamma(\cdot)$ for some $\theta \in [0, 1]$.

**Proof** Since the action in the first period is continuous, the maximum action (and thus the continuation value $\omega(\theta)$) must also be continuous. Suppose that $\gamma(\cdot)$ didn’t achieve its maximum. Since $\gamma$ is single peaked and monotonic on either side of the peak, the principal could shift all continuation values either up or down, strictly increasing the principal’s payoff and not affecting incentive compatibility constraints, a contradiction of optimality. □

**Theorem 1.3** In the dynamic delegation game, increasing the agent’s bias in the first period makes the principal strictly worse off.

**Proof** Take some $b_1$ and $\hat{b}_1$, where $b_1 < \hat{b}_1$, and consider the optimal mechanism when the principal is facing an agent with types $\hat{b}_1$ and $b_2$, with $b_2 > 0$. This setting then defines $\hat{a}(\cdot)$ and $\hat{\omega}(\cdot)$. Furthermore, define $\hat{\theta}$ as a value of $\theta$ for which $\gamma(\hat{\omega}(\theta))$ achieves its maximum. Define
\[ a(\theta) = \hat{a}(\theta) + \hat{b}_1 - b_1, \text{ and let} \]
\[ \omega(\theta) = \hat{\omega}(\theta) + (\hat{b}_1 - b_1)(\hat{a}(\theta) - \hat{a}(\hat{\theta})). \]

I’ll show that for \( b_1 \) near \( \hat{b}_1 \), the mechanism defined by \( a \) and \( \omega \) is incentive compatible and strictly improves the principal’s payoffs. Since the original mechanism was incentive compatible, we know that for all \( \theta \) and \( \theta' \),
\[ \frac{1}{2}(\hat{a}(\theta) - \theta)^2 + \hat{\omega}(\theta) \geq \frac{1}{2}(\hat{a}(\theta') - \theta')^2 + \hat{\omega}(\theta') \]

Thus, it must be the case that for all \( \theta \) and \( \theta' \),
\[ \frac{1}{2}(a(\theta) + b_1 - \hat{b}_1 - \theta)^2 + \omega(\theta) - (\hat{b}_1 - b_1)(\hat{a}(\theta) - \hat{a}(\hat{\theta})) \geq \frac{1}{2}(a(\theta') + b_1 - \hat{b}_1 - \theta)^2 + \omega(\theta') - (\hat{b}_1 - b_1)(\hat{a}(\theta') - \hat{a}(\hat{\theta})) \]

This simplifies to
\[ \frac{1}{2}(a(\theta) - \theta)^2 + \omega(\theta) \geq \frac{1}{2}(a(\theta') - \theta')^2 + \omega(\theta'), \]

so the new allocation is incentive compatible. The payoffs from the action taken are the same in both settings. Thus, any difference in payoffs arises completely from the change in continuation values. Payoffs increase weakly if for \( \theta > \hat{\theta} \), \( \hat{\omega}(\theta) < \omega(\theta) \leq \hat{\omega}(\hat{\theta}) \), and for \( \theta < \hat{\theta} \), \( \hat{\omega}(\theta) > \omega(\theta) \geq \hat{\omega}(\hat{\theta}) \), since this moves all continuation payoffs closer to the maximum value of \( \gamma \). Since \( \omega(\hat{\theta}) = \hat{\omega}(\hat{\theta}) \), and it’s clearly the case that \( \omega(\theta) > \hat{\omega}(\theta) \) if and only if \( \theta \geq \hat{\theta} \), then the result holds if \( \omega(\theta) \) is everywhere increasing. Notice that it’s still the case that at points of differentiability,
\[ \omega'(\theta) = W'(\theta)[1 + W''(\theta)] = (\hat{W}'(\theta) + \hat{b}_1 - b_1)(1 + \hat{W}''(\theta)) \]
so since $W'(\theta)$ is bounded away from 0, $\omega$ is increasing for small $b_1$ near $\hat{b}_1$, showing that for any level of bias in the first period, a small decrease in bias weakly increases the principal’s payoffs, so the optimal mechanism in the case of smaller bias must also improve payoffs. □

**Theorem 1.4** In the dynamic delegation game, for high enough second period bias, further increasing this bias improves the principal’s payoffs. As bias in the second period becomes arbitrarily large, the principal’s payoffs approach those she would receive if she had full information in the first period, and had to make the optimal uniformed decision in the second period.

**Proof** First, I’ll show that for $b_2 \geq 0.5$, when $b_2$ increases the principal can use an incentive scheme which implements the same action profile, but achieves higher continuation values for the principal. In particular, I’ll show that as $b_2$ increases, the principal can shift the incentive scheme downward uniformly (i.e. holding $\omega(\theta) - \omega(\theta')$ constant for all $\theta, \theta'$), and that this strictly increases the principal’s payoffs for all $\theta$ such that $\omega(\theta)$ isn’t at the maximum of $\gamma$.

Recall that where it is defined, $\gamma$ is

$$
\gamma(\omega, b_2) = \begin{cases} 
\omega - \frac{\delta}{2} b_2^2 + \frac{\delta b_2}{6} (-\frac{72}{8} \omega - 3)^{\frac{3}{2}} \\
\omega - \frac{\delta}{2} b_2^2 + \frac{\delta b_2}{2} (\frac{6}{\delta} \omega)^{\frac{3}{2}}
\end{cases},
$$

where we add $b_2$ as an argument to the function. Define $\hat{\omega} = \omega + \frac{1}{2} \delta b_2^2 + \frac{1}{24} \delta$, so that $\hat{\omega}$ is just the distance to the maximizer of $\gamma(\cdot, b_2)$ for $b_2 \geq \frac{1}{2}$. Thus, we could redefine $\gamma$ as

$$
\gamma(\hat{\omega}, b_2) = \begin{cases} 
\hat{\omega} - \delta b_2^2 - \frac{1}{24} \delta + \frac{\delta b_2}{6} (- \frac{72}{8} \hat{\omega} + 36 b_2^2)^{\frac{3}{2}} \\
\hat{\omega} - \delta b_2^2 - \frac{1}{24} \delta + \frac{\delta b_2}{2} (\frac{6}{\delta} \hat{\omega} - 3 b_2^2 - \frac{1}{4})^{\frac{3}{2}}
\end{cases}.
$$

If we then take a derivative with respect to $b_2$, we get

$$
\gamma_{b_2}(\hat{\omega}, b_2) = \begin{cases} 
-2 \delta b_2 + \left(-\frac{72}{8} \hat{\omega} + 36 b_2^2\right)^{-\frac{1}{2}} [12 \delta b_2^2 - 12 \hat{\omega}] \\
-2 \delta b_2 + \left(-\frac{7}{2} \delta b_2^2 + 3 \hat{\omega} - \frac{8}{3} \hat{\omega}ight) (\frac{6}{\delta} \hat{\omega} - 3 b_2^2 - \frac{1}{4})^{-\frac{1}{2}}
\end{cases}.
$$
and we can then take a derivative with respect to $\hat{\omega}$ to get

$$\gamma_{b_2, \omega}(\hat{\omega}, b_2) = \begin{cases} 
-\frac{1}{2} \left( \frac{72}{\delta} \hat{\omega} + 36b_2^2 \right)^{-\frac{3}{2}} \left( \frac{72}{\delta} \right) \left[ 12\delta b_2^2 - 12\hat{\omega} \right] - 12 \left( \frac{72}{\delta} \hat{\omega} + 36b_2^2 \right)^{-\frac{3}{2}} \\
-\frac{2\delta(4\delta b_2^2+\delta-24\hat{\omega})(-24b_2^2+4\delta\hat{\omega}-2)^{\frac{3}{2}}}{(12\delta b_2^2+\delta-24\hat{\omega})^2}
\end{cases}$$

In the first region, the derivative with respect to $b_2$ is 0 exactly when $\hat{\omega} = 0$. Furthermore, this derivative is increasing as $\hat{\omega}$ goes away from 0. The derivative is positive whenever $\hat{\omega}$ is in the second region, i.e. $\hat{\omega} \geq \frac{1}{2}\delta b_2^2 - \frac{1}{\delta}$. Thus, when $b_2 \geq \frac{1}{2}$, increasing $b_2$ increases the principal’s continuation value conditional on the agent’s continuation value being the same distance from the maximum of $\gamma$. Because there is no individual rationality constraint, for a higher $b_2$ the principal can simply shift the promised distribution of $\omega$ downward, and this strictly increases her payoffs.

Finally, I’ll show that as $b_2 \to \infty$, the principal’s payoffs are bounded above by $-\frac{1}{24}\delta$, and bounded below by a value that converges to $-\frac{1}{24}\delta$. Notice that for $b_2 \geq 0.5$, the function $\gamma$ has a maximum value of $-\frac{1}{24}\delta$. Thus, the principal’s payoffs are bounded above by what she would receive if she took exactly her preferred action in the first period (receiving 0) and a continuation value of $-\frac{1}{24}\delta$. Next, consider the action profile $a(\theta) = \theta - b_1$, so principal’s preferred action profile is chosen in the first period. This action profile can be supported by $\omega(\theta)$ such that

$$\omega(\theta) = \omega(0) - b_1 \theta$$

In particular, set $\omega(\theta) = -\frac{1}{2}\delta b_2^2 - \frac{1}{24} - b_1 \theta + \frac{1}{2} b_1$. Then for any $\theta$, when $b_2$ is large, the principal has payoffs of 0 in the first period, and continuation value of

$$\gamma(\omega(\theta)) = -\delta b_2^2 - \frac{1}{24}\delta - b_1 \theta + \frac{1}{2} b_1 + \frac{\delta b_2}{6} \left( 36b_2^2 + \frac{72}{\delta} b_1 \theta - \frac{36}{\delta} b_1 \right)^{\frac{1}{2}}.$$
As $b_2 \to \infty$, this approaches

$$-\frac{1}{24} \delta - b_1 \theta + \frac{1}{2} b_1$$

which in expectation is $-\frac{1}{24} \delta$. Since this mechanism is incentive compatible, and approaches the upper bound on the principal’s payoffs, the optimal mechanism’s payoffs must also approach these payoffs, and they are the same as those that the principal would receive if she had full information in the first period, and were completely uninformed with no agent to work with in the second period. □

**Theorem 1.5** The optimal dynamic delegation mechanism is deterministic.

**Proof** The non-optimality of randomizing in the second period was already shown in the proof of Proposition 1.4, and the non-optimality of randomizing over delgation sets offered in the second period follows immediately from the concavity of the function $\gamma(\cdot)$. Thus, it remains to show that it is not optimal to randomize the action after the report in the first period.

Suppose that the principal is randomizing the action taken after the agent’s report in the first period. Then the principal’s and agent’s payoffs can again be decomposed into a function of the average and variance of the action taken in each state. Define these as $\bar{a}_1(\theta)$ and $\sigma^2(\theta)$, respectively and let $\omega(\theta)$ be the continuation value promised to the agent. Suppose that $\sigma^2(\theta) > 0$ for some $\theta$, and define

$$\hat{a}(\theta) = \bar{a}(\theta)$$

$$\hat{\sigma}^2(\theta) = 0$$

$$\hat{\omega}(\theta) = \omega(\theta) - \frac{1}{2} \sigma^2(\theta)$$

Since the original allocation was incentive compatible, we have that

$$-\frac{1}{2} (\hat{a}(\theta) - \theta)^2 - \frac{1}{2} \sigma^2(\theta) + \omega(\theta) \geq -\frac{1}{2} (\hat{a}(\theta') - \theta)^2 - \frac{1}{2} \sigma^2(\theta') + \omega(\theta')$$

$$-\frac{1}{2} (\hat{\omega}(\theta) - \theta)^2 + \hat{\omega}(\theta) \geq -\frac{1}{2} (\hat{\omega}(\theta') - \theta)^2 + \hat{\omega}(\theta')$$
so the new allocation is incentive compatible. Furthermore, since \( \gamma'(\omega) < -1 \), \( \sigma^2(\theta) + \gamma(\omega(\theta) - \sigma^2(\theta)) \), and

\[ -(\bar{a}(\theta) - \theta)^2 - \sigma^2(\theta) + \gamma(\omega(\theta)) < -(\hat{a}(\theta) - \theta)^2 - \hat{\sigma}^2(\theta) + \gamma(\omega(\theta)). \]

This new incentive compatible mechanism improves the principal’s payoffs, so randomizing in the first period can’t be optimal.

**Theorem 1.6** Assume that the convexity condition holds. When money burning is feasible and equally costly to the principal and the agent, the following conditions imply that the optimal mechanism is one in which money burning is everywhere 0, and the actions taken are on an interval \([\theta_L, \theta_H]\).

1. \( \forall \theta \in [0, \theta_L], \int_0^\theta u_n^P(\theta_L, z)f(z)dz \geq [u_n^P(\theta_L, \theta) + (\theta_L - \theta)](\theta - \theta_L)f(\theta), \) with equality at \( \theta_L \).
2. If \( \theta_L = 0 \), \( u_n^P(0, 0) \leq 0 \).
3. If \( \theta_H = 1 \), \( u_n^P(1, 1) \geq 0 \), and
4. \( \forall \theta \in [\theta_H, 1], \int_0^\theta u_n^P(\theta, z)f(z) \leq [u_n^P(\theta_H, \theta) - (\theta_H - \theta)](\theta - \theta_H)f(\theta), \) with equality at \( \theta_H \).

**Proof** The described mechanism has

\[ W(\theta) = \begin{cases} 
\theta_L \theta - \frac{1}{2} \theta^2 - \frac{1}{2} \theta_L^2 & \text{if } \theta < \theta_L \\
0 & \text{if } \theta_L \leq \theta \leq \theta_H \\
\theta_H \theta - \frac{1}{2} \theta^2 - \frac{1}{2} \theta_H^2 & \text{if } \theta_H < \theta 
\end{cases} \]

I’ll define a function \( p(\theta) \) and show that if \( W(\theta) \) has the assumed properties, then \( p(\theta) \) and
$W(\theta)$ satisfy the requirements of Proposition 1.A.2. In particular,

$$p(\theta) = \begin{cases} 
\int_0^\theta u^p_a(\theta, z) f(z) dz / (\theta - \theta) & \text{if } \theta < \theta_L \\
- u^p_a(\theta, \theta) f(\theta) & \text{if } \theta_L \leq \theta \leq \theta_H \\
\int_\theta^{\theta_H} u^p_a(\theta, z) f(z) dz / (\theta - \theta) & \text{if } \theta_H < \theta
\end{cases}$$

Since $\int_0^{\theta_L} u^p_a(\theta_L, z) f(z) dz = 0$,

$$\lim_{\theta \uparrow \theta_L} p(\theta) = \lim_{\theta \uparrow \theta_L} - \frac{\int_0^{\theta_L} u^p_a(\theta, z) f(z) dz - \int_0^{\theta_L} u^p_a(\theta_L, z) f(z) dz}{\theta - \theta_L} = - u^p_a(\theta_L, \theta_L) f(\theta_L).$$

Similarly, $\lim_{\theta \downarrow \theta_H} p(\theta) = - u^p_a(\theta_H, \theta_H) f(\theta_H)$, so $p(\theta)$ is continuous, and fulfills all of the requirements from Proposition 1.A.2 for the level of $p(\theta)$. $p(\theta)$ was defined in such a way that

$$p'(\theta) = \frac{1}{W'(\theta)} p(\theta) + \frac{1}{W'(\theta)} u^p_a(W'(\theta) + \theta, \theta) f(\theta),$$

for $\theta \notin [\theta_L, \theta_H]$, and for $\theta \in [\theta_L, \theta_H]$,

$$p'(\theta) \geq - \gamma'(0) f(\theta)$$

is implied by the derivative of

$$F(\theta) - u^p_a(\theta, \theta) f(\theta)$$

being positive, which is the same as $F(\theta) - u^p_a(\theta, \theta) f(\theta)$ being nondecreasing. □

Remark 1.1 The conditions from Theorem 1.6 imply those in used in Proposition 1(b) by Amador and Bagwell (2013) when applied to this setting.
Proof After translating the setting of this paper into that of Amador and Bagwell, we find that the convexity condition implies that $\kappa = 1$. It’s easy then to see that conditions (c1), (c2'), and (c3') are the same as conditions (2)-(4) above. Suppose that condition (c3) were violated at the point $\theta_2$. Since both sides of the inequality are continuous, then $\exists \hat{\theta}_1$, with $\hat{\theta}_1 < \hat{\theta}_2 \leq \theta_L$ such that

$$\int_0^\theta u_\theta^P(\theta_L,z)f(z)dz < (\theta - \theta_L)F(\theta)$$

for $\theta \in (\hat{\theta}_1, \hat{\theta}_2]$ and

$$\int_0^{\hat{\theta}_1} u_\theta^P(\theta_L,z)f(z)dz = (\hat{\theta}_1 - \theta_L)F(\hat{\theta}_1),$$

implying that

$$\frac{\int_0^{\hat{\theta}_2} u_\theta^P(\theta_L,z)f(z)dz - (\hat{\theta}_2 - \theta_L)F(\hat{\theta}_2) - \int_0^{\hat{\theta}_1} u_\theta^P(\theta_L,z)f(z)dz + (\hat{\theta}_1 - \theta_L)F(\hat{\theta}_1)}{\hat{\theta}_2 - \hat{\theta}_1} < 0.$$  

The mean value theorem then implies that there exists a $\hat{\theta}_3 \in (\hat{\theta}_1, \hat{\theta}_2)$ such that

$$u_\theta^P(\theta_L,\hat{\theta}_3)f(\hat{\theta}_3) - (\hat{\theta}_3 - \theta_L)f(\hat{\theta}_3) - F(\hat{\theta}_3) < 0$$

Thus,

$$\int_0^{\hat{\theta}_3} u_\theta^P(\theta_L,z)f(z)dz < (\hat{\theta}_3 - \theta_L)F(\theta)$$

$$< [u_\theta^P(\theta_L,\hat{\theta}_3) + (\theta_L - \hat{\theta}_3)] f(\hat{\theta}_3)(\hat{\theta}_3 - \theta_L),$$

which violates the condition (1) from above. Similarly, suppose that condition (c2) is violated at
the point \( \hat{\theta}_1 \). Again, there must exist \( \hat{\theta}_2 \), with \( \theta_H < \hat{\theta}_1 < \hat{\theta}_2 \) such that

\[
\int_{\hat{\theta}_1}^{\hat{\theta}_2} u_a^p(\theta_H, z) f(z) dz > (1 - F(\theta))(\theta - \theta_H)
\]

for \( \theta \in [\hat{\theta}_1, \hat{\theta}_2] \) with

\[
\int_{\hat{\theta}_2}^{\hat{\theta}_1} u_a^p(\theta_H, z) f(z) dz = (1 - F(\hat{\theta}_2))(\hat{\theta}_2 - \theta_H).
\]

This gives us that

\[
\frac{\int_{\hat{\theta}_2}^{\hat{\theta}_1} u_a^p(\theta_H, z) f(z) dz - (1 - F(\hat{\theta}_2))(\hat{\theta}_2 - \theta_H) - \int_{\hat{\theta}_1}^{\hat{\theta}_2} u_a^p(\theta_H, z) f(z) dz + (1 - F(\hat{\theta}_1))(\hat{\theta}_1 - \theta_H)}{\hat{\theta}_2 - \hat{\theta}_1} < 0
\]

Then another use of the mean value theorem shows that \( \exists \hat{\theta}_3 \in (\hat{\theta}_1, \hat{\theta}_2) \) such that

\[-u_a^p(\theta_H, \hat{\theta}_3) f(\hat{\theta}_3) - (1 - F(\hat{\theta}_3)) - (\hat{\theta}_2 - \theta_H)(-f(\hat{\theta}_3)) < 0,
\]

so \( (1 - F(\hat{\theta}_3)) > [-u_a^p(\theta_H, \hat{\theta}_3) - (\theta_H - \hat{\theta}_3)] f(\hat{\theta}_3) \), and

\[
\int_{\hat{\theta}_3}^{\hat{\theta}_1} u_a^p(\theta_H, z) f(z) dz > (1 - F(\hat{\theta}_3))(\hat{\theta}_3 - \theta_H)
\]

\[> [-u_a^p(\theta_H, \hat{\theta}_3 - (\theta_H - \hat{\theta}_3))(\hat{\theta}_3 - \theta_H) f(\hat{\theta}_3),
\]

showing that this must violate condition (5) from above. \( \square \)

**Theorem 1.7** In the uniform-quadratic delegation setting with money burning, the optimal mechanism can take one of four forms:

1. Discretion for low actions, with a cap preventing high actions when \( b \leq \frac{1}{2} \) and \( k \geq \frac{1}{2} \)

2. Discretion for low actions, and money burning for high actions when \( b \leq k < \frac{1}{2} \)

3. Constant action for low states and money burning for high states when \( b \geq k \) and \( k \leq \frac{1}{2b+1} \)
4. The optimal uninformed action otherwise

Proof First, notice that if for any combination of $b$ and $k$, if the optimal mechanism doesn’t include any money burning, then in all settings with the same bias and higher $k$, the optimal mechanism is the same. This is because any alternative mechanism must improve payoffs either with money burning or with some other incentive compatible allocation. This implies that there is a contradiction, since the same mechanism must have been incentive compatible for lower $k$, and would have given the principal weakly higher payoffs.

Consider the case with $b \leq k \leq \frac{1}{2}$. The proposed action profile is

$$a(\theta) = \begin{cases} \theta & \text{if } \theta \leq \frac{k-b}{k} \\ \frac{2k-1}{k-1} \theta + \frac{b-k}{k-1} & \text{otherwise} \end{cases}$$

For $\theta > \frac{k-b}{k}$, the envelope condition which gives that

$$\omega'(\theta) = (a(\theta) - \theta) a'(\theta)$$

$$= \frac{2k^2 - k}{(k-1)^2} \theta + \frac{(b-k)(2k-1)}{(k-1)^2}$$

Since $\omega \left( \frac{k-b}{k} \right) = 0$, for $\theta \geq \frac{k-b}{k}$,

$$\omega(\theta) = \frac{2k^2 - k}{2(k-1)^2} \theta^2 + \frac{(b-k)(2k-1)}{(k-1)^2} \theta + \frac{2(k-b)(2k-1) - (2k-1)(k-b)^2}{2k(k-1)}.$$

This gives indirect utility function

$$W(\theta) = \begin{cases} 0 & \text{if } \theta \leq \frac{k-b}{k} \\ \frac{k}{2(k-1)} \theta^2 + \frac{b-k}{k-1} \theta - \frac{(b-k)^2}{2(k-1)^2} + \frac{2(k-b)(2k-1) - (2k-1)(k-b)^2}{2k(k-1)} & \text{otherwise} \end{cases}$$
Finally define

\[ p(\theta) = \begin{cases} 
  b & \text{for } \theta \leq \frac{k-b}{k} \\
  k - k\theta & \text{otherwise}
\end{cases} \]

Notice that \( p(\theta) \) satisfies all of the conditions required in Proposition 1.A.2. The final condition to check is that

\[ W'(\theta) \in \arg\max_v p(\theta)v - \frac{1}{2}(v+b)^2 + \frac{1}{2}kv^2 \]

which implies that \( W'(\theta) = 0 \) when \( \theta < \frac{k-b}{k} \) and \( W'(\theta) = \frac{k}{k-1}\theta + \frac{b-k}{k-1} \) for \( \theta > \frac{k-b}{k} \), which is the case for the indirect utility function given above. When \( k = \frac{1}{2} \) in this solution, the mechanism involves no money burning, and has simple cap on the action at \( 1 - 2b \). Thus, this mechanism is also optima for any \( k \geq \frac{1}{2} \).

Next, consider the case with \( b > k \) and \( k \leq \frac{1}{2b+1} \). Here the proposed action has

\[ a(\theta) = \begin{cases} 
  c & \text{if } \theta \leq d \\
  \frac{2k-1}{k-1}\theta + \frac{b-k}{k-1} & \text{otherwise}
\end{cases}, \]

with \( c \) and \( d \) determined below, \( a(\theta) \) continuous, and money burning for \( \theta > d \). To show that such a mechanism is optimal, I will find a \( p(\theta) \) which satisfies all of the conditions from Proposition 1.A.2, including \( p(0) = p(1) = 0 \). For the strictly increasing portion of the action profile, \( p'(\theta) = -k \), so \( p(\theta) = (1-\theta)k \) on \((d, 1]\). On the flat portion of the profile, it must be the case that

\[ p'(\theta) = \frac{1}{c-\theta} p(\theta) - \frac{c-\theta + b}{c-\theta} \]

which with the initial condition \( p(0) = 0 \) has solution \( p(\theta) = \frac{\theta(\theta - 2b - 2c)}{2(c-\theta)} \). Thus, we are solving a
system of nonlinear equations:

\[
c = \frac{2k - 1}{k - 1}d + \frac{b - k}{k - 1}
\]

\[
(1 - d)k = \frac{d(d - 2b - 2c)}{2(c - d)}.
\]

The equations are simply the requirements that both the action profile and the function \( p(\theta) \) be continuous at the point \( \theta = d \). Solving this system, we get

\[
(2k^2 - 3k + 1)d^2 + (2k - 4k^2)d - 2bk + 2k^2 = 0.
\]

For \( b > k, 0 < k < \frac{1}{1 + 2b} \), this is negative when \( d = 1 \) and positive when \( d = 0 \). Thus, there is a unique value for \( d \) in \([0, 1]\). Thus, with this \( c \) and \( d \) defined, we have that

\[
p(\theta) = \begin{cases} 
\frac{\theta(\theta - 2b - 2c)}{2(c - \theta)} & \text{if } \theta \leq d \\
(1 - \theta)k & \text{otherwise}
\end{cases}
\]

Here, \( p(0) = p(1) = 0 \) and by construction \( p'(\theta) \) satisfies the necessary constraints. The last condition to check is that \( W'(\theta) \) is the solution to the maximization condition in Proposition 1.A.2. For \( \theta \in [0, d] \), naive first order conditions (ignoring the boundary) give that

\[
v^* = \frac{\theta(\theta - 2b - 2c)}{2(c - \theta)(1 - k)} - \frac{b}{1 - k}
\]

Since \( W'(\theta) = c - \theta \) is on the boundary and the problem is convex, we need to check that

\[
\frac{\theta(\theta - 2b - 2c)}{2(c - \theta)(1 - k)} - \frac{b}{1 - k} \leq c - \theta
\]
which simplifies to

\[(1+2k)\theta^2 - (2c+4ck)\theta + 2c^2 - 2c^2k + 2bc \leq 0.\]

The values \(c\) and \(d\) are defined such that this holds with equality at \(\theta = d\) and is not binding at \(\theta = 0\). Since the left hand side of this function is quadratic with a positive coefficient on the quadratic term, the inequality holds everywhere. We also must check that \(W'(\theta)\) is the maximizer of

\[(1-\theta)kv - \frac{1}{2}(v+b)^2 + \frac{1}{2}kv^2,\]

for \(\theta \geq d\), which it is. Thus, the proposed mechanism is optimal. Furthermore, for \(b \geq \frac{1}{2}\), along the boundary \(k = \frac{1}{2b+1}\), the optimal \(c\) and \(d\) are \(\frac{1}{2} - b\) and 1, respectively, which implies that the action is flat at \(\frac{1}{2} - b\). Since this doesn’t use any money burning, for any \(k > \frac{1}{2b+1}\) the same mechanism is optimal. □
1.B  Differential Equation Proofs

This appendix contains the proofs of various lemmas that are used to prove Proposition 1.5. From Proposition 1.2 and the calculations in the main text, we get that the solution to the dynamic delegation problem is characterized by the differential equation

\[ W''(\theta) = \begin{cases} 
-1 + \frac{\delta}{2b_2W(\theta) + 4b_2W'(\theta) + \frac{1}{2}\delta b_2}{2b_2W(\theta) + 4b_2W'(\theta) + \frac{1}{2}\delta b_2} & \text{if } W(\theta) + \frac{1}{2}W'(\theta)^2 < -\frac{\delta}{6} \\
-1 + \frac{\delta}{12b_2W(\theta) + 24b_2W'(\theta)^2} & \text{if } -\frac{\delta}{6} \leq W(\theta) + \frac{1}{2}W'(\theta)^2 \leq 0
\end{cases} \tag{1.B.3} \]

with endpoints \((W(0), W'(0))\) and \((W(1), W'(1))\) given by

\[ W'(\theta)\gamma\left(W(\theta) + \frac{1}{2}W'(\theta)^2\right) = -u_a\left(W'(\theta) + \theta, \theta\right) \tag{1.B.4} \]

for \(\theta \in \{0, 1\}\). Since equation (1.B.3) is defined piecewise, it will be useful in the future to refer to the region \(W(\theta) + \frac{1}{2}W'(\theta)^2 < -\frac{\delta}{6}\) as the first region, and \(-\frac{\delta}{6} \leq W(\theta) + \frac{1}{2}W'(\theta)^2 \leq 0\) as the second region.

**Lemma 1.B.1** If the graph of a solution to equation (1.B.3) lies in the domain

\[ G \equiv \left\{(W(\theta), W'(\theta)) : W(\theta) + \frac{1}{2}W'(\theta)^2 \leq -\epsilon, W(\theta) \geq c\right\} \]

for some \(c < 0\) and \(\epsilon > 0\), then the solution is unique.

**Proof** As in §10 of Filippov and Arscott (1988), define further

\[ G^+ \equiv \left\{(W(\theta), W'(\theta)) : W(\theta) + \frac{1}{2}W'(\theta)^2 < -\frac{\delta}{6}, W(\theta) \geq c\right\} \]

and

\[ G^- \equiv \left\{(W(\theta), W'(\theta)) : -\epsilon \geq W(\theta) + \frac{1}{2}W'(\theta)^2 \geq -\frac{\delta}{6}, W(\theta) \geq c\right\} \]

so that $G^-$ and $G^+$ separate $G$ into two domains by the smooth surface $S_1$, where

$$S_1 = \{(W(\theta), W'(\theta)) : W(\theta) + \frac{1}{2} W'(\theta)^2 = -\frac{\delta}{6}\}.$$ 

Note that the partial derivatives of $W''(W(\theta), W'(\theta))$ are continuous up to the boundaries of $G^-$ and $G^+$. Define $W''_-(W(\theta), W'(\theta))$ and $W''_+(W(\theta), W'(\theta))$ as the limiting values of the function $W''(W(\theta), W'(\theta))$ at the point $W(W(\theta), W'(\theta)) \in S_1$ from the regions $G^-$ and $G^+$ respectively. Define $W^-_N(W(\theta), W'(\theta))$ and $W^+_N(W(\theta), W'(\theta))$ to be the projections of the vectors $[W'(\theta), W''_-(W(\theta), W'(\theta))]$ and $[W'(\theta), W''_+(W(\theta), W'(\theta))]$ onto the normal to $S_1$ directed from $G^-$ to $G^+$ at the point $(W(\theta), W'(\theta))$. That is,

$$W^-_N(W(\theta), W'(\theta)) = \frac{[-1, -W'(\theta)] : [W'(\theta), W''_-(W(\theta), W'(\theta))]}{||[-1, -W'(\theta)]||}$$

We can first find that

$$[-1, -W'(\theta)] : [W'(\theta), W''_-(W(\theta), W'(\theta))]$$

$$= -W'(\theta) + W'(\theta)$$

$$= -W'(\theta) \left[ 3W'(\theta) + \frac{6W'(\theta) + 3W'(\theta)^2}{12b_2W(\theta) + 2b_2W'(\theta)^2} \right]$$

$$= -W'(\theta) \left[ \frac{6W'(\theta) + 3W'(\theta)^2}{12b_2W(\theta) + 2b_2W'(\theta)^2} \right]$$

$$= -\frac{W'(\theta) \left[ \frac{6W'(\theta) + 3W'(\theta)^2}{12b_2W(\theta) + 2b_2W'(\theta)^2} \right]}{b_2(4W'(\theta)^2 + 2\delta)}$$

so

$$W^-_N(W(\theta), W'(\theta)) = \frac{W'(\theta) (1 - 4b_2)}{b_2(4W'(\theta)^2 + 2\delta) \sqrt{1 + W'(\theta)^2}}.$$
Similarly,

\[ [-1, -W'(\theta)] \cdot [W'(\theta), W''(W(\theta), W'(\theta))] \]

\[ = -W'(\theta) \left[ \delta - \frac{1}{4} \left( \frac{b_2}{b_1} \right)^2 \right] \]

\[ = -b_2 \left[ W'(\theta)^2 + \frac{1}{4} \delta \right] \]

\[ = \frac{\delta W'(\theta)}{b_2} \left[ \frac{1}{8} - \frac{1}{4} b_2 \right] \]

\[ = \frac{\delta W'(\theta) [1 - 4b_2]}{b_2 [4W'(\theta)^2 + \delta]} \]

These have the same sign at any point, thus from Theorem 2 of §10 of Filippov and Arscott (1988), uniqueness occurs at all points in the domain \( G \) such that \( W'(\theta) \neq 0 \).

**Lemma 1.B.2** If the graph of a solution to equation (1.B.3) exists in \( G \) for \( \theta \in [0, 1] \), then the solution is continuous in initial conditions for \( \theta \in [0, 1] \).

**Proof** This follows immediately from uniqueness and Theorem 2 of §8 in Filippov and Arscott (1988).

It will be useful to have the point at which both \( W''(\theta) = 0 \) and \( W'(\theta) = 0 \). Plugging in the formula for \( W''(\theta) \), we get \( W(\theta) = -\frac{4}{3} \delta b_2^3 \) or \( W(\theta) = -\frac{1}{2} b_2^2 \delta - \frac{1}{24} \delta \), depending on which of the equation this point is located in.

Define \( S_2 \equiv \{(s, v) : s = q_1(v)\} \) where

\[
q_1(W'(\theta)) = \begin{cases} 
-\frac{\delta}{2} \left( \frac{b_2}{b_1} \right)^2 W'(\theta)^2 - \frac{1}{2} W''(\theta)^2 - \frac{1}{24} \delta & \text{if} \quad \frac{4}{3} \delta \left( \frac{b_2}{b_1} \right)^3 W'(\theta)^3 - \frac{1}{2} W'(\theta)^2 \quad \text{otherwise}
\end{cases}
\]

8The theorem requires that the functions \( W_N^- \) and \( W_N^+ \) be strictly positive or negative. In this case, they can both be 0 for \( b_2 = \frac{1}{4} \), but this value of \( b_2 \) actually makes the second order differential equation continuous, implying uniqueness through standard methods.

9In fact, the theorem gives conditions for right uniqueness, but since this differential equation is autonomous, one can reverse the process (i.e. “run time backwards”) to show left uniqueness.
and

\[ q_2(b_2) = \begin{cases} 
-\frac{4}{3} \delta b_2^3 & \text{for } b_2 \leq \frac{1}{2} \\
-\frac{1}{2} \delta b_2^2 - \frac{1}{4} \delta & \text{otherwise}
\end{cases} \]

**Lemma 1.B.3** There exists a solution to equation (1.B.3) with either (a) \((W(1), W'(1)) \in S_2\) and \(W(\theta) \leq q_1(W(\theta_1))\) for all \(\theta \in [0, 1]\), or (b) \((W(0), W'(0)) \in S_1\) and \(W(\theta) \leq q_1(W'(\theta))\) for all \(\theta \in [0, 1]\).

**Proof** We found above that when \(W(\theta) = q_2(b_2)\) and \(W'(\theta) = 0\), equation (1.B.3) is equal to 0. Thus, \(W(\theta) = q_2(b_2)\) is a solution to equation (1.B.3) for all \(\theta\). For all \(W(\theta) \in G\) such that \(W(\theta) < q_1(b_2)\), \(W''\) is negative, so if \(W(0) = q_2(b_2)\) and \(W'(0) < 0\), then \(W(\theta) < q_2(b_2)\) and \(W'(\theta) < 0\) for all \(\theta > 0\) for which a solution exists. Start at \((W(0), W'(0)) = (q_2(b_2), 0)\).

By decreasing \(W'(0)\) (and holding \(W(0)\) constant), it’s clear from continuity that one of three things must be true: (a) or (b) above, or both \(W(0) < q_1(W'(0))\), \(W(1) < q_1(W'(1))\), and \(W(\theta) \geq q_1(W'(\theta))\) for some set of \(\theta\). Suppose that the last is the case. Then it must be the case that for some \(\theta_a, W(\theta_1) = q_1(W'(\theta_a))\)

\[ W''(\theta_a) = \begin{cases} 
\frac{1}{4\delta(\frac{b_1}{b_2})^3 W'(\theta) - 1} & \text{if } W'(\theta_1) \geq -\frac{b_2}{2b_2} \\
\frac{1}{\delta(\frac{b_1}{b_2})^2 + 1} & \text{otherwise}
\end{cases} \]

The right hand side of the above equation is the second order differential equation for a function whose graph stays along the curve \(q_1\), so to pass through the curve at \(\theta_1\) the second derivative of \(W\) must be less than it. Similarly, there must be some \(\theta_b > \theta_a\) such that \(W(\theta_b) = q_1(W'(\theta_b))\) and

\[ W''(\theta_b) = \begin{cases} 
\frac{1}{4\delta(\frac{b_1}{b_2})^3 W'(\theta) - 1} & \text{if } W'(\theta_1) \geq -\frac{b_2}{2b_2} \\
\frac{1}{\delta(\frac{b_1}{b_2})^2 + 1} & \text{otherwise}
\end{cases} \]

We can plug in \(q_1(W'(\theta))\) to equation (1.B.3) to determine under what conditions these
inequalities hold:

\[
W''(\theta) = -1 + \frac{\delta^{-\frac{1}{3}} \left[ 8\delta \left( \frac{b_2}{b_1} \right)^3 - 3W'(\theta)^2 + 3W'(\theta)^2 \right]^\frac{1}{3} + 24b_2 \left( \frac{4}{3} \delta \left( \frac{b_2}{b_1} \right)^3 W'(\theta)^3 \right)}{12b_2 \left( \frac{4}{3} \delta \left( \frac{b_2}{b_1} \right)^3 W'(\theta)^3 \right) - 6b_2 W'(\theta)^2 + 2b_2 W''(\theta)\}
\]

\[
= \frac{4\delta \left( \frac{b_2}{b_1} \right)^4 W'(\theta)^2 + 4\delta b_2 \left( \frac{b_2}{b_1} \right)^3 W'(\theta) + b_2}{4\delta b_2 \left( \frac{b_2}{b_1} \right)^3 W'(\theta) - b_2}.
\]

Thus, the inequality is fulfilled when

\[
\frac{4\delta \left( \frac{b_2}{b_1} \right)^4 W'(\theta)^2 + 4\delta b_2 \left( \frac{b_2}{b_1} \right)^3 W'(\theta) + b_2}{4\delta b_2 \left( \frac{b_2}{b_1} \right)^3 W'(\theta) - b_2} < \frac{1}{4\delta \left( \frac{b_2}{b_1} \right)^3 W'(\theta) - 1}
\]

\[
4\delta \left( \frac{b_2}{b_1} \right)^4 W'(\theta)^2 + 4\delta b_2 \left( \frac{b_2}{b_1} \right)^3 W'(\theta) > 0
\]

\[
\frac{1}{b_2} W'(\theta) < -1
\]

\[
W'(\theta) < -\frac{1}{b_1}
\]

In the other region, we can also solve for \(W''(\theta)\):

\[
W''(\theta) = -1 + \frac{\delta^{-\frac{1}{3}} \left( \delta \left( \frac{b_2}{b_1} \right)^2 W'(\theta)^2 \right)^\frac{1}{2} - 2\delta b_2 \left( \frac{b_2}{b_1} \right)^2 W'(\theta)^2}{-\delta b_2 \left( \frac{b_2}{b_1} \right)^2 W''(\theta)^2 - b_2 W'(\theta)^2}.
\]
\[(W'(\theta)^2)^3\] is positive, so

\[
W''(\theta) = \frac{\delta b_2 \left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2 + b_2 W'(\theta)^2 - \delta \left(\frac{b_2}{b_1}\right)^3 W'(\theta)^3 - 2\delta b_2 \left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2}{-\delta b_2 \left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2 - b_2 W'(\theta)^2} \\
= -\delta \left(\frac{b_2}{b_1}\right)^3 W'(\theta)^3 - \delta b_2 \left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2 + b_2 W'(\theta)^2 \\
= \frac{-\delta \left(\frac{b_2}{b_1}\right)^3 W'(\theta) - \delta b_2 \left(\frac{b_2}{b_1}\right)^2 + b_2}{-\delta b_2 \left(\frac{b_2}{b_1}\right)^2 - b_2}
\]

Thus, we can again check the inequality,

\[
-\delta \left(\frac{b_2}{b_1}\right)^3 W'(\theta) - \delta b_2 \left(\frac{b_2}{b_1}\right)^2 + b_2 < \frac{1}{-\delta \left(\frac{b_2}{b_1}\right)^2 - 1} \\
-\delta \left(\frac{b_2}{b_1}\right)^3 W'(\theta) > \delta b_2 \left(\frac{b_2}{b_1}\right)^2 \\
W'(\theta) < -b_1
\]

so along the surface, the first inequality holds if and only if \(W'(\theta) < -b_1\). Since \(\theta_b > \theta_a\) implies that \(W'(\theta_b) < W'(\theta_a)\), the above inequalities cannot both hold. □

**Lemma 1.B.4** In the first region, solutions to equation (1.B.3) must satisfy

\[
W'(\theta)^2 = -\delta \left[\frac{2b_2 W'(\theta) + \delta b_2}{W'(\theta) - \lambda}\right]^2 - 2W(\theta) - \delta \frac{1}{12} \tag{1.B.5}
\]

while in the second region, they must satisfy

\[
\frac{\delta^\frac{1}{2} \left[3b_2 W'(\theta) - \frac{1}{2} b_2 W'(\theta)^2\right]}{(6W(\theta) + 3W'(\theta)^2)^\frac{1}{2}} = W(\theta) - \lambda \tag{1.B.6}
\]

for some \(\lambda \in \mathbb{R}\).
Proof From Gelfand and Fomin (1963, p.19), we see that a solution must satisfy
\[ L(\theta, W(\theta), W'(\theta)) - W'(\theta)L_{W(\theta)}(\theta, W(\theta), W'(\theta)) = C. \]

Letting \( C = -\lambda + \frac{1}{2} b_1^2 + \frac{\delta}{2} b_2^2 \), in the first region this gives
\[
\begin{align*}
&b_1 W'(\theta) + \frac{1}{2} b_1^2 - W(\theta) + \frac{\delta}{2} b_2^2 - \frac{\delta b_2}{6} \left( -\frac{72}{\delta} W(\theta) - 36 \frac{\delta}{W'(\theta)^2} - 3 \right)^{\frac{1}{2}} \\
&\quad - b_1 W'(\theta) - 6 b_2 W'(\theta)^2 \left( -\frac{72}{\delta} W(\theta) - 36 \frac{\delta}{W'(\theta)^2} - 3 \right)^{-\frac{1}{2}} = -\lambda + \frac{1}{2} b_1^2 + \frac{\delta}{2} b_2^2
\end{align*}
\]
and this simplifies, to
\[ W'(\theta)^2 = -\delta \left[ \frac{2 b_2 W(\theta) + \delta b_2}{W(\theta) - \lambda} \right]^2 - 2 W(\theta) - \frac{\delta}{12} \]

In the other region, we have
\[
\begin{align*}
&b_1 W'(\theta) - W(\theta) + \frac{1}{2} b_1^2 + \frac{\delta}{2} b_2^2 - \frac{\delta b_2}{2} \left( \frac{6}{\delta} W(\theta) + \frac{3}{\delta} W'(\theta)^2 \right)^{\frac{3}{2}} \\
&\quad - b_1 W'(\theta) + 2 b_2 W'(\theta)^2 \left( \frac{6}{\delta} W(\theta) + \frac{3}{\delta} W'(\theta)^2 \right)^{-\frac{3}{2}} = C
\end{align*}
\]
which simplifies to
\[
\frac{\delta^\frac{1}{2} \left[ \frac{1}{2} b_2 W'(\theta)^2 - 3 b_2 W(\theta) \right]}{(6 W(\theta) + 3 W'(\theta)^2)^{\frac{3}{2}}} = W(\theta) - \lambda,
\]
and these are the equations above. □

Define \( v_a \) and \( v_b \) as the first and second roots of \( 4\delta \left( \frac{b_2}{b_1} \right)^3 v^3 - 3 \left( \frac{1}{2} - 2\delta b_1 \left( \frac{b_2}{b_1} \right)^3 \right) v^2 - 3 b_1 v - 2\delta b_2^3 \), respectively. Also, define \( v_c \) as the second root of
\[
4\delta \left( \frac{b_2}{b_1} \right)^3 v^3 - 3 \left( \frac{1}{2} - 2\delta b_1 \left( \frac{b_2}{b_1} \right)^3 \right) v^2 - 3 b_1 v + \frac{3}{2} b_2 \delta + \frac{1}{2} \delta
\]
and

\[ v_d = \begin{cases} v_a & \text{if } v_a \geq -\frac{b_1}{2b_2} \\ -b_2\left(\frac{b_2}{b_1}\right)^2 - b_2 - \sqrt{\frac{\delta}{b_2^2} + 2\delta b_2^2 \left(\frac{b_2}{b_1}\right)^2 + b_2^2 - 2\delta \left(\frac{b_2}{b_1}\right)^2 + 2\delta b_2^2} \left(\frac{1}{\delta} \left(\frac{b_2}{b_1}\right)^3 + \frac{b_2}{b_1^2} \left(\frac{3}{\delta} b_2\right)\right) \end{cases} \]

otherwise

Lemma 1.B.5 For \( b_2 \leq \frac{1}{2} \), the graph of the solution to equation (1.B.3) passing through \((q_1(v_d), v_d)\) approaches \((q_2(b_2), 0)\).

**Proof** Suppose that a solution to equation 1 has initial conditions \((W(0), W'(0)) = (q_1(v_d), v_d)\).

If this is in the second region, then this implies that

\[
\frac{\delta^{\frac{1}{3}} \left[ \frac{1}{2} b_2 W'(\theta)^2 - 3 b_2 W(\theta) \right]}{(6W(\theta) + 3W'(\theta)^2)^{\frac{1}{3}}} = W(\theta) - \lambda
\]

Using the fact that \( W(0) = q_1(W'(0)) = \frac{4}{3} \delta \left(\frac{b_2}{b_1}\right)^3 W'(0)^3 - \frac{1}{2} W'(0)^2 \), we get

\[
\frac{\delta^{\frac{4}{3}} \left[ \frac{1}{2} b_2 W'(0)^2 - 4\delta b_2 \left(\frac{b_2}{b_1}\right)^3 W'(0)^3 + \frac{1}{2} b_2 W'(0)^2 \right]}{\left(8\delta \left(\frac{b_2}{b_1}\right)^3 W'(0)^3 - 3W'(0)^2 + 3W'(0)^2\right)^{\frac{1}{3}}} = \frac{4}{3} \delta \left(\frac{b_2}{b_1}\right)^3 W'(0)^3 - \frac{1}{2} W'(0)^2 - \lambda
\]

We can be simplified to get

\[ 4\delta \left(\frac{b_2}{b_1}\right)^3 W'(0)^3 - 3 \left(1 - 2\delta b_1 \left(\frac{b_2}{b_1}\right)^3\right) W'(0)^2 - 3b_1 W'(0) - 3\lambda = 0 \]

The only \( \lambda \) for which \( v_d \) satisfies this equation is \( \lambda = \frac{2}{3} \delta b_2^2 \). Alternatively, if \((W(0), W'(0)) = (q_1(v_d), v_d)\) is in the first region, it must be the case that

\[ W'(\theta)^2 = -\delta \left[ \frac{2b_2 W(\theta) + \delta b_2}{W(\theta) - \lambda} \right]^2 - 2W(\theta) - \frac{\delta}{12} \]
We can use the fact that here, \( W(0) = q_1(W'(0)) = -\frac{\delta}{2} \left( \frac{b_2}{b_1} \right)^2 W'(0)^2 - \frac{1}{2} W'(0)^2 - \frac{1}{24} \delta, \) so

\[
W'(0)^2 = -\delta \left[ \frac{-b_2 \delta \left( \frac{b_2}{b_1} \right)^2 W'(0)^2 - b_2 W'(0)^2 - \frac{\delta b_2}{12} + \frac{\delta b_2}{12}}{-\frac{\delta}{2} \left( \frac{b_2}{b_1} \right)^2 W'(0)^2 - \frac{1}{2} W'(0)^2 - \frac{1}{24} \delta - \lambda} \right]^2 \\
+ \delta \left( \frac{b_2}{b_1} \right)^2 \left[ W'(0)^2 + W'(0)^2 + \frac{1}{12} \delta - \frac{1}{12} \delta, \right]
\]

and

\[
\left[ \frac{\delta}{2} \left( \frac{b_2}{b_1} \right)^3 + \frac{1}{2} \frac{b_2}{b_1} \right] W'(0)^2 + \left[ \delta b_2 \left( \frac{b_2}{b_1} \right)^2 + b_2 \right] W'(0) + \frac{1}{24} \delta b_2 + \frac{b_2}{b_1} \lambda = 0.
\]

Again, the only \( \lambda \) for which \( v_d \) satisfies this equation is \( \lambda = \frac{2}{3} \delta b_2^3. \) Since \( \lambda = \frac{2}{3} \delta b_2^3, \) we can show that there is not point at which \( W'(\theta) = 0 \) in the first region.

\[
-\delta \left[ \frac{2b_2 W(\theta) + \frac{\delta b_2}{12}}{W(\theta) - \frac{2}{3} \delta b_2^3} \right]^2 - 2W(\theta) - \frac{\delta}{12} = 0 \\
\frac{\delta b_2^3 \left( 2W(\theta) + \frac{\delta}{12} \right)^2}{(W(\theta) - \frac{2}{3} \delta b_2^3)^2} + 2W(\theta) + \frac{\delta}{12} = 0 \\
\left( W(\theta) - \frac{2}{3} \delta b_2^3 \right)^2 + \delta b_2^3 \left( 2W(\theta) + \frac{\delta}{12} \right) = 0
\]

We can simplify this to get

\[
W(\theta)^2 + \left( 2\delta b_2^2 - \frac{4}{3} \delta b_2^3 \right) W(\theta) + \frac{4}{9} \delta^2 b_2^3 + \frac{\delta^2}{12} b_2^2 = 0
\]

Note that by the quadratic formula, this doesn’t have any solutions if \( 4b_2^2 - \frac{16}{3} b_2^3 - \frac{1}{3} \) is less than zero, which is exactly when \( b_2 \leq \frac{1}{2} \). On the other region, \( W'(\theta) = 0 \) when \( W(\theta) = -\frac{4}{3} \delta b_2^3, \) since
the solution must satisfy

\[-\frac{\delta^\frac{1}{3} (3b_2W(\theta))}{(6W(\theta))^\frac{1}{3}} = W(\theta) - \frac{2}{3}\delta b_2^3\]

\[-\delta^\frac{1}{3} 6^{-\frac{1}{3}} 3b_2W(\theta)^\frac{2}{3} = W(\theta) - \frac{2}{3}\delta b_2^3\]

\[W(\theta) + \delta^\frac{1}{3} 6^{-\frac{1}{3}} 3b_2W(\theta)^\frac{2}{3} - \frac{2}{3}\delta b_2^3 = 0\]

One can take the derivative of this to see that it reaches its local maximum at \(W(\theta) = -\frac{4}{3}\delta b_2^3\), which is \(q_2(b_2)\). Thus, there is only one point less than \(q_1(v_d)\) at which \(W'(\theta) = 0\) when \(\lambda = \frac{2}{3}\delta b_2^3\). Furthermore, there are no values of \(W(\theta)\) between \(q_1(v_d)\) and \(q_2(b_2)\) such that the graph of the solution intersects \(S_2\). Thus, the graph of the solution must approach \((q_2(b_2), 0)\). □

**Lemma 1.B.6** For \(b_2 \leq \frac{1}{2}\), the graph of the solution to equation (1.B.3) that approaches \((q_2(b_2), 0)\) passes through \((q_1(v_b), v_b)\).

**Proof** Since \(b_2 \leq \frac{1}{2}\), \(q_2(b_2)\) is in the second region. Thus, we know that it satisfies the equation

\[\frac{\delta^\frac{1}{3} \left[ \frac{1}{2}b_2W'(\theta)^2 - 3b_2W(\theta) \right]}{(6W(\theta) + 3W'(\theta)^2)^\frac{1}{3}} = W(\theta) - \lambda\]

Since we are considering the graph of the solution that approaches \((q_2(b_2), 0)\), it must be the case that

\[\frac{3b_2\delta^\frac{1}{3}q_2(b_2)}{(6q_2(b_2))^\frac{1}{3}} = q_2(b_2) - \lambda\]

\[\frac{3b_2\delta^\frac{1}{3}(-\frac{4}{3}\delta b_2^3)}{(8\delta b_2^3)^\frac{1}{3}} = -\frac{4}{3}\delta b_2^3 - \lambda\]

\[-\frac{4\delta^\frac{1}{3}b_2^4}{2\delta^\frac{1}{3}b_2} = -\frac{4}{3}\delta b_2^3 - \lambda\]

\[-2\delta b_2^3 = -\frac{4}{3}\delta b_2^3 - \lambda\]

\[\lambda = \frac{2}{3}\delta b_2^3\]
Since $\lambda = \frac{2}{3} \delta b_2^3$, then the second region, the solution must satisfy

$$\frac{\delta \left[ \frac{1}{2} b_2 W'(\theta)^2 - 3 b_2 W(\theta) \right]}{(6W(\theta) + 3W'(\theta)^2)^{\frac{1}{3}}} = W'(\theta) - \frac{2}{3} \delta b_2^3$$

This intersects the curve $q_1(W'(\theta)) = \frac{4}{3} \delta \left( \frac{b_2}{b_1} \right)^3 W'(\theta)^3 - \frac{1}{2} W'(\theta)^2$ when

$$4\delta \left( \frac{b_2}{b_1} \right)^3 W'(\theta)^2 - 3 \left( \frac{1}{2} - 2\delta b_2 \left( \frac{b_2}{b_1} \right)^3 \right) W'(\theta)^2 - 3b_1 W'(\theta) - 2\delta b_2^3 = 0$$

The second root of this equation must be in the second region, so the graph must approach $(q_1(v_b), v_b)$. □

**Lemma 1.B.7** For $b_2 > \frac{1}{2}$, the graph of the solution to equation (1.B.3) passing through

$$\left( -\frac{1}{2} b_1^2 \left( \delta \left( \frac{b_2}{b_1} \right)^2 + 1 \right)^{\frac{1}{2}} + 1 \right)^2 - \frac{1}{24} \delta, -b_1 \left( 1 + \left( \delta \left( \frac{b_2}{b_1} \right)^2 + 1 \right)^{-\frac{1}{2}} \right)$$

approaches $\left( -\frac{1}{2} \delta b_2^2 - \frac{1}{24} \delta, 0 \right)$.

**Proof** Suppose that the graph of the solution passes through

$$\left( -\frac{1}{2} b_1^2 \left( \delta \left( \frac{b_2}{b_1} \right)^2 + 1 \right)^{\frac{1}{2}} + 1 \right)^2 - \frac{1}{24} \delta, -b_1 \left( 1 + \left( \delta \left( \frac{b_2}{b_1} \right)^2 + 1 \right)^{-\frac{1}{2}} \right)$$

For $b_2 > \frac{1}{2}$, this is in the first region, so

$$b_1^2 \left[ 1 + \left( \delta \left( \frac{b_2}{b_1} \right)^2 + 1 \right)^{-\frac{1}{2}} \right]^2 = -\delta \left[ \frac{-b_1^2 b_2 \left( \delta \left( \frac{b_2}{b_1} \right)^2 + 1 \right)^{\frac{1}{2}} + 1 \right]^2 \right]
\left[ -\frac{1}{2} b_1^2 \left( \delta \left( \frac{b_2}{b_1} \right)^2 + 1 \right)^{\frac{1}{2}} + 1 \right]^2 - \frac{1}{24} \delta - \delta
\right]
\left[ b_1^2 \left( \delta \left( \frac{b_2}{b_1} \right)^2 + 1 \right)^{\frac{1}{2}} + 1 \right]^2,$$
and
\[
\pm \frac{1}{1 + \delta \left( \frac{b_2}{b_1} \right)^2} = \frac{b_1^2 \left( \delta \left( \frac{b_2}{b_1} \right)^2 + 1 \right)^{\frac{3}{2}} + 1}{\frac{1}{2} b_1^2 \left( \delta \left( \frac{b_2}{b_1} \right)^2 + 1 \right)^{\frac{3}{2}} + 1 + \frac{1}{24} \delta + \lambda},
\]
which results in \( \lambda = \frac{1}{2} \delta b_2^2 - \frac{1}{24} \) or \( \lambda = -\frac{3}{2} \delta b_2^2 - 2b_1^2 \left( \delta \left( \frac{b_2}{b_1} \right)^2 + 1 \right)^{\frac{1}{2}} - \frac{1}{24} \delta \). The latter of these is associated with positive \( W'(\theta) \), so it must be the case that \( \lambda = \frac{1}{2} \delta b_2^2 - \frac{1}{24} \delta \). Next, note that \( W(\theta) = -\frac{1}{2} \delta b_2^2 - \frac{1}{24} \delta \) is the only solution to
\[
0 = -\delta \left[ \frac{2b_2 W(\theta) + \frac{1}{12} \delta b_2}{W(\theta) - \lambda} \right]^2 - 2W(\theta) - \frac{1}{12} \delta
\]
in this region. The solution also only intersects the curve \( q_1(W'(\theta)) \) at one other point,
\[
-\frac{1}{2} b_1^2 \left[ \left( 1 + \delta \left( \frac{b_2}{b_1} \right)^2 \right)^{\frac{1}{2}} - 1 \right]^2 - \frac{1}{24} \delta > -\frac{1}{2} \delta b_2^2 - \frac{1}{24} \delta.
\]
Finally, I’ll note that the only point at which the graph of the solution could intersect the curve \( W(\theta) + \frac{1}{2} W'(\theta)^2 = -\frac{1}{6} \delta \) is when
\[
W(\theta) = \frac{\frac{1}{2} \delta b_2^2 + \frac{1}{6} \delta b_2 - \frac{1}{24} \delta}{1 - 4b_2^2}
\]
which is greater than \(-\frac{1}{2} \delta b_2^2 - \frac{1}{24} \delta \) for all \( b_2 > \frac{1}{2} \). Thus, for the graph of a solution to equation (1.B.3) to pass through those initial conditions, it must then approach \((-\frac{1}{2} \delta b_2^2 - \frac{1}{24} \delta, 0)\). \( \square \)

**Lemma 1.B.8** For \( b_2 > \frac{1}{2} \), the graph of the solution to equation (1.B.3) passing through \((q_1(v_c), v_c)\) passes through \((-\frac{1}{6} \delta, 0)\).

**Proof** Similar to the proof above, the graph of the solution with these initial conditions is in the
second region, so it will be the case that

\[
4\delta \left( \frac{b_2}{b_1} \right)^3 W'(0)^3 - 3 \left( \frac{1}{2} - 2\delta b_1 \left( \frac{b_2}{b_1} \right)^3 \right) W'(0)^2 - 3b_1W'(0) - 3\lambda = 0.
\]

The only \( \lambda \) for which \( v_c \) satisfies this equation is \( \lambda = \frac{1}{3}b_2\delta - \frac{1}{6}\delta \). Suppose that \( W(\theta) = -\frac{1}{3}\delta \). This implies that

\[
\frac{\delta \left( \frac{1}{2}b_2W'(\theta)^2 + \frac{1}{2}\delta b_2 \right)}{(3W'(\theta)^2 - \delta)^{\frac{3}{2}}} = -\frac{1}{2}\delta b_2, \]

and the only \( W'(\theta) \) for which this is true is \( W'(\theta) = 0 \). With \( \lambda = \frac{1}{2}\delta b_2 - \frac{1}{6}\delta \) and \( W'(\theta) = 0 \), there is no other negative solution for \( W(\theta) \). Finally, if the solution passes through the curve \((q_1(W'(\theta)),W'(\theta))\), then it must be the case that

\[
4\delta \left( \frac{b_2}{b_1} \right)^3 W'(0)^3 - 3 \left( \frac{1}{2} - 2\delta b_1 \left( \frac{b_2}{b_1} \right)^3 \right) W'(0)^2 - 3b_1W'(0) - 3\frac{2}{3}b_2\delta + \frac{1}{6}\delta = 0,
\]

but this has two negative roots, only one of which is greater than \(-2b_1\), so the solution can’t pass through the curve again before passing through \((-\frac{1}{6}\delta,0)\). Thus, the graph of the solution must pass through \((-\frac{1}{6}\delta,0)\). □

**Lemma 1.B.9** If a solution to equation (1.B.3) has \((W(\hat{\theta}),W'(\hat{\theta})) = \left(q_1(-b_1),-b_1\right)\) for some \( \hat{\theta} \), then \( W(\theta) > q_1(W'(\theta)) \) for all \( \theta < \hat{\theta} \) for which the solution exists.

**Proof** For \( b_2 \leq \frac{1}{2} \), \((W(\hat{\theta},W'(\hat{\theta}))) = (q_1(-b_1),-b_1)\) implies that \( \lambda = \frac{3}{2}\delta b_2^3 + \frac{1}{2}b_1^2 \). For \( W'(\theta) > -b_1, (q_1(W'(\theta),W'(\theta))) \) lies completely in the first region, and there is no other point at which the graph of the solution to equation (1.B.3) intersects \( S_1 \). Consider when \( b_2 > \frac{1}{2} \). This implies that \( \lambda = \frac{1}{2}\delta b_2^3 + \frac{1}{2}b_1^2 - \frac{1}{24}\delta \), so there is no other point at which the graph of the solution to equation (1.B.5) intersects \( S_2 \). Suppose that the solution to equation (1.B.3) passed into the second region. The normal to \( S_1 \) in this region is the vector \([-1,4\delta \left( \frac{b_2}{b_1} \right)^3 W'^2 - W']\). Since
along the curve

\[
\begin{bmatrix}
-1,4\delta \left( \frac{b_2}{b_1} \right)^3 W'^2 - W' \\
1 \cdot W', -1 + \frac{\delta^{-\frac{1}{2}} W - \frac{3}{2} W'^2 + 24b_2 W + 12b_2 W'^2}{12b_2 W + 2b_2 W'^2} \\
4\delta \left( \frac{b_2}{b_1} \right)^3 W'^2 + W' \\
4\delta \left( \frac{b_2}{b_1} \right)^3 W'^2 [W' + b_1]
\end{bmatrix}
\]

and this is always positive for \( W' > -b_1 \), the graphs of the solution can only pass through \( S_1 \) from below to above. \( \square \)

**Proposition 1.5** An interior solution to problem (PDD) exists, i.e. there exists a \( W(\theta) \) which solves (PDD) such that \( \forall \theta, W(\theta) + \frac{1}{2} W'(\theta)^2 < \bar{e} < 0. \)

**Proof** Suppose that case (a) from Lemma 1.B.3 were true. The proof actually shows that in this case, there exists a solution to equation (1.B.3) whose graph has an endpoint \( (W(1), W'(1)) \in S_2, W(0) = q_2(b_2), \) and \( W'(0) < 0 \). Consider modifying the initial conditions, increasing \( W(1) \) but keeping \( (W(1), W'(1)) \in S_2 \). Since the solution with \( (W(1), W'(1)) = (q_1(-b_1), -b_1) \) has \( W(\theta) > q_1(W'(\theta)) \) for all \( \theta < 1 \), by continuity in initial conditions (proven in Lemma 1.B.2) there must be a point at which \( W(0) = q_1(W'(0)) \) or a point at which \( (W(\theta), W'(\theta)) \) is on the boundary of \( G \) for \( 0 < \theta < 1 \). Because \( W''(W(\theta), 0) > 0 \) for all \( W(\theta) > q_2(b_2) \), the only way that \( (W(\theta), W'(\theta)) \) could be on the boundary of \( G \) is for \( W(1) = -\bar{e} - \frac{1}{2} W'(1)^2 \). For \( b_2 \leq \frac{1}{2} \), the graph of such a solution would intersect the graph of the solution found in Lemma 1.B.6 and for \( b_2 > \frac{1}{2} \), the graph of such a solution would intersect the graph of the solution in Lemma 1.B.8, violating uniqueness. In this case the solution exists with \( (W(\theta), W'(\theta)) \in G \) for all \( \theta \), and with the given endpoints.

Suppose instead that case (b) from Lemma 1.B.3 is true. Again, shifting the initial conditions along \( S_2 \) (decreasing \( W(0) \)), \( W(1) \) must intersect either the graph of the solution identified in Lemma 1.B.5 (for \( b_2 \leq \frac{1}{2} \)), the graph of the solution identified in Lemma 1.B.7 (for...
$b_2 > \frac{1}{2}$), or $S_2$. Uniqueness guarantees that the former two are impossible, implying that there exists initial conditions $(W(0), W'(0)) \in S_2$ such that $(W(1), W'(1))$ lies in $S_2$. Thus, the solution exists. This also implies that when $b_2 \leq \frac{1}{2}$, the $\lambda$ associated with the solution is strictly greater than $\frac{2}{3} \delta b_2^3$, and when $b_2 > \frac{1}{2}$, the $\lambda$ associated with the solution is strictly greater than $\frac{1}{2} b_2 \delta - \frac{1}{6} \delta$, i.e. $\lambda$ must be strictly positive. □
Chapter 2

Prediction and Model Selection in Experiments

2.1 Introduction

A large body of work in experimental and behavioral economics studies “behavioral anomalies” and the theories that are supposed to explain them, and uses experimental and observational data to evaluate these theories. To a large extent, research in this field has focused on testing models, both in the theoretical and statistical sense. However, such tests don’t always select a single model, and aren’t well designed to select between models when all models are wrong.

This paper uses data from several experiments to argue for an alternative approach to choosing between models. Models are chosen on the basis of the accuracy of their predictions outside of the estimation sample. The analysis focuses on deterministic models of choice, which for a given set of parameters have point predictions in these experiments. Parameters are estimated on one portion of the sample, and used to predict the remainder of the data. This procedure is carried out for a number of random splits of the data, and for various sizes of the split (e.g. parameters are estimated on half the sample and used to predict the other half, then parameters
are estimated on three quarters of the sample, and used to predict the other quarter).

This methodology is applied to one experiment studying risk preferences (Choi, Fisman, Gale, & Kariv, 2007), and two studying time preferences (Andreoni & Sprenger, 2012; Augenblick, Niederle, & Sprenger, 2015). The convex budgets used in these experiments’ designs allow for a clear measurement of the “distance” between a model’s predictions and a given choice.

The results compare the predictions of the constant relative risk aversion (CRRA) and the constant absolute risk aversion (CARA) models, with and without extra “loss aversion” parameters. When comparing the models without these loss aversion parameters, the estimated CARA parameter tends to predict the subject’s next choice better than CRRA for most sample sizes. On the other hand, when we add more flexibility to the model with an extra parameter for loss aversion, CRRA tends to dominate.

Estimating a loss aversion parameter improves prediction in the CRRA model even for small sample sizes. This may be surprising; a more complicated model necessarily improves in-sample fit for all sample sizes, but can often make prediction worse. The fact that adding parameters improves fit here suggests that researchers can obtain meaningful estimates of these parameters even with only a few observations per person, and that it might be valuable to study even more complex models in these settings. This latter conclusion is confirmed here: prediction is improved when we not only estimate the CRRA or CARA parameter for an individual, but also estimate which model they fit into best. In fact, for large sample sizes the model with the highest predictive power is one in which the estimation procedure classifies a subject into which model they fall under, estimates their curvature, and estimates loss aversion parameters.

The same estimation and prediction procedure is applied to assess models of time preferences. A key issue in the literature to this point is the extent to which individuals are present biased, and how well the popular $\beta$-$\delta$ model (Laibson, 1997) captures these factors. Andreoni and Sprenger (2012) found little evidence for present bias over monetary decisions; Augenblick et al. (2015) confirmed this result for monetary decisions, but found that present bias was observed when subjects made choices involving real effort.

The results show that estimating the present bias parameter makes predictions worse
in both experiments when considering all of the data. However, for the data from Augenblick et al. (2015), this masks heterogeneity in predictive power between the two types of decision problems. Estimating the $\beta$ makes predictions clearly worse in monetary decisions, but when estimating on nearly the full sample, models with and without the present bias parameter have nearly equal predictive power. The results also illustrate why researchers have estimated a wide variety of discount rates when subjects choose over time-dated money (Frederick, Loewenstein, & O’donoghue, 2002): the prediction error for real effort decisions is much lower than the prediction error for monetary decisions.

The comparison of predictive powers of models also suggests another potential source of dynamic inconsistency: different discount rates for different sources of utility. Estimating separate discount factors for money and real effort provides improved predictions over the pooled estimate. When an agent has additively separable utility functions with differing discount rates for different goods, the agent is dynamically inconsistent. Thus, this potential source of dynamic inconsistency deserves more interest.

The methods and results presented here should be interpreted as a parallel and complementary approach to traditional methods. They give insights to the predictive power of models, and demonstrate how the models interact with different amounts of data. The empirical measures that are generated are transparent, and immediately interpretable. They can be used by applied modelers who are interested in choosing a model that best captures behavior in a particular situation, and by experimenters determining the necessary quantity and source of data to estimate a particular set of preference parameters.

The paper proceeds as follows: Section 2.2 reviews some of the related literature regarding model selection in economics, evaluating models based on prediction, and testing economic theories. Section 2.3 describes the cross validation procedures I use, and Section 2.4 shows the results when these procedures are applied to experiments studying risk and time preferences. Section 2.5 shows that the results are robust to a number of possible objections. Finally, Section 2.6 is a conclusion.
2.2 Related Literature

Using experimental data to help distinguish between models is not a new idea. Harless and Camerer (1994) compare data from a large number of studies to select between models of risk preferences. They study individual decisions in discrete choice problems from 23 different data sets. Although all models that they study are rejected, they provide guidance about how these models trade off between parsimony and fit, favoring prospect theory, expected utility, or “mixed fanning” (in which indifference curves fan out for unfavorable lotteries and fan in for favorable ones). Hey and Orme (1994) compare 11 different models using both standard statistical tests as well as Akaike’s information criterion, and find that expected utility theory performs well, although several other models fit better (with the caveat that the economic significance of the differences are not large). Camerer and Ho (1994) find that simple models of disappointment aversion and probability weighting fit their data much better than expected utility.

This paper is also not the first to compare models using out of sample prediction. Ericson, White, Laibson, and Cohen (2015) recruit subjects from Amazon.com’s Mechanical Turk to make decisions that differ in framing and timing to better understand time preferences. They cross validate the models with 100 repetitions of estimating parameters on 75% of the data and predicting the remaining 25%, and find that simple, “heuristic” models perform much better than standard discounting and $\beta-\delta$ preferences, which perform similarly to each other. Peysakhovich and Naecker (2017) also recruit subjects from Mechanical Turk, and elicit subjects’ willingness to pay for various risky and ambiguous gambles. They compare the out of sample prediction of several economic models with data-driven machine learning models that are optimized to give better out of sample prediction. They find that for all models, the representative agent assumption is a poor one (even with many fewer data points per estimated parameter, individualized parameter estimates outperform the pooled parameter estimate), and show that for risky gambles, expected utility with probability weighting performs as well out of sample as the Lasso and Ridge regressions. On the other hand, machine learning methods outperform common economic models of ambiguous choice, suggesting that researchers have room to develop better models in this domain.
While not focused directly on comparing models, Halevy, Persitz, and Zrill (in press) use a novel experimental design which estimates preferences and generates choice problems dynamically. After estimating preferences with both nonlinear least squares and what they call a “money metric index,” they compare the predictive power of the parameters estimated using these methods, and find that the latter predicts new decisions better.

2.2.1 Testing and Prediction

In most experimental work, the primary method of evaluating models is based on testing the models. In these cases, the experimenter finds the testable implications of a given theory, suitably adjusted for “noise,” which might be sampling variation or decision error, and carries out a statistical test in the style of Popper (1959).

This model of economic research has been embraced within the microeconomic theory literature as being the proper and scientific way of producing economic research, and decision theorists often focus on the testable implications of their models. However, there seems to be a tension in the methodological discussions about testing models. For instance, Dekel and Lipman (2010) note that a key goal of decision theory is to predict individual choice, and that a model being refuted does not imply that a model should be rejected. On the other hand, they seem to evaluate predictions primarily on the basis of whether or not the model’s predictions are refuted, not how close the data are to the model’s predictions. Similarly, Gilboa (2009) state that “it is important to know that the theories have some empirical content and that given a particular mapping from theoretical objects to actual ones, a theory is not vacuous,” but also that “the question is, therefore, not whether they are right or wrong, but whether they are wrong in a way that invalidates the conclusions drawn from them.” The focus on sharp tests that arise from a model’s empirical content seems to contrast with the belief that a model with strong predictive power is useful even if it is not true.

Indeed, there are numerous theories and models which have been falsified with experimental data but are still in common use today (e.g. expected utility or Nash equilibrium). Practitioners still find these concepts useful, despite the fact that they’ve been rejected by the data. Presumably,
this is because these models are both tractable and “good enough” to capture the phenomenon that the researcher is interested in. This argument is put forth by Simon (1994) in his “principle of continuity of approximation,” which states that “if the conditions of the real world approximate sufficiently well the assumptions of an ideal type, the derivations from these assumptions will be approximately correct.” This article seeks to quantify and compare how closely various models approximate subjects’ behavior in experiments.

Economics have debated on the necessity of testing models’ assumptions for decades. In a well known essay, Friedman (1953) discusses the ways in which economic theories are compared with data, and famously claims that the realism of a theory’s assumptions cannot be used to test that theory. The essay also stresses at length that a model should be evaluated by its accuracy and usefulness relative to other models. Hands (1993) delves further into these topics, suggesting that Popperian methodology fails at providing rules to determine which model is better if both models have been falsified.

This paper proposes and implements an alternative methodology to evaluate economic models. The tools presented here are not demonstrated to be optimal in any statistical sense, but shows that they are simple to implement, easy to interpret, and portable to a variety of settings.

### 2.3 Methodology

The use of cross validation in model selection has a long history, and there are numerous results showing the benefits and drawbacks of various methods (Arlot & Celisse, 2010). The goal of this paper is to develop a methodology that is both easily interpretable and widely applicable, and will sacrifice some efficiency in order to obtain these goals.

A given set of subjects, indexed by $i \in \{1, \ldots, N\}$ made a series of $T$ decisions in an experiment. When faced with a decision problem that has characteristics $x_{i,t}$, the subject chose $y_{i,t}$. Individual $i$’s data set consists of $M_i = \{(x_{i,1}, y_{i,1}), (x_{i,2}, y_{i,2}), \ldots, (x_{i,T}, y_{i,T})\}$. A model maps a decision problem’s characteristics, $x_{i,t}$ to a distribution over $y$.

The particular experiments that are focused on here are those in which subjects make
choices from a convex budget, in which they allocate portions of their budget between two goods or bundles. Once a budget has been described with characteristics $x_{i,t}$, without loss of generality we can label one of the goods as good 1 and the other as good 2, such such that $y_{i,t}$ can be interpreted as the proportion of their budget that they devote towards good one.

Experiments with convex budgets are primary chosen for two reasons. When a researcher only observes a subject making a series of binary decisions, it’s often the case that parameters are only identified to be within some interval. In contrast, when subjects make interior choices from a convex budget problem, it’s usually the case that a set of parameters is point identified. In addition to helping to address identification issues, choices from convex budgets can give a more informative measure of how “far” a choice is from the prediction of a particular model. For instance, in a binary decision problem it may be the case that two models both predict that a subject will prefer good 1 over good 2. If the subject chooses good 2, these models look equally bad. If the subject faced a similar problem with a convex budget, it’s likely that the models would make different predictions (for instance, one model predicts that the subject will choose $y=0.1$, and the other $y=0.4$). If preferences are strictly convex, the difference between the choice that a model predicts and the choice that is taken is increasing in the implied utility cost. Thus, this distance can be thought of as a rough measure of how bad a choice is according to the model.

A model and its parameters give a rule that maps a budget’s characteristics to a choice. Thus, for each vector of parameters $\theta$, such a model predicts that the agent will choose $f(x_{i,j}; \theta) \in [0,1]$, the budget share devoted towards good 1. We can use a subject’s choices and the model’s predictions to get estimates for an individual’s parameters, $\theta$. In many experiments in which the researchers’ goal is to estimate the parameters of a model which best describe a subject’s choices, they use nonlinear least squares (NLLS). To be consistent with this literature, I do the same, which is to say that when estimating a vector of parameters $\theta_{i,j,k}$, they are chosen to solve

$$\min_{\theta} \sum_{(x,y) \in M^{j,k}_i} (f(x; \theta) - y)^2.$$ 

where $M^{j,k}_i$ is the $k$th set of estimation budgets of size $j$ for subject $i$. 
In this paper, each model will be estimated on many different subsets of an individual’s data set. For each \( i, j, \) and \( k \), the subset of budgets that \( \theta_{i,j,k} \) is being estimated on is drawn randomly. Within a set of estimations budgets, budgets are chosen without replacement (i.e. \( M_{i,j}^{j,k} \) has no duplicates), but between repetitions, sets of budgets are drawn with replacement (so it’s possible that \( M_{i,j}^{j,k} = M_{i,j}^{j,k'} \) for \( k \neq k' \)). In all of the results below, models will be estimated 200 times for each estimation sample size.

The main outcome of interest is a model’s predictive capability, measured in predictive mean squared error, on the portion of the sample that the model is not estimated on. Formally, for a given estimation sample size \( j \), this can be defined as

\[
PMSE_j = \sum_{k=1}^{200} \sum_{i=1}^{N} \sum_{(x,y) \in M_i \setminus M_{i,j}^{j,k}} \frac{1}{200(T-j)N} \left( f(x; \hat{\theta}_{i,j,k}^j) - y \right)^2
\]

This will be reported for all the models and experiments considered below. Patterns in the in-sample fit of each of the models are consistent across experiments, and adding parameters improves fit in the way one would expect. Thus, fit is only reported in some cases.

The data used in Sections 2.4 and 2.5 are from several experiments studying risk and time preferences: Choi et al. (2007), Andreoni and Sprenger (2012), and Augenblick et al. (2015). In each of these experiments, subjects make a series of choices, with each choice coming from a convex budget.

I treat these experiments unfairly. The data were not meant to be used in the fashion that I use them below, and I ignore a number of data analysis decisions that the authors use for the sake of being able to compare results across experiments. More information on the differences between the data analysis here and that of the original experiment can be found in Appendix 2.A.
2.4 Results

2.4.1 Risk Preferences

When comparing models of risk preferences, I’ll focus on which curvature parameters capture subjects’ risk behaviors, and whether or not common behavioral parameters are useful for prediction. In particular, when considering curvature, I’ll compare CRRA utility functions, usually parameterized as $u(x) = \frac{1}{1-\rho} x^{1-\rho}$ and CARA utility functions, parameterized as $u(x) = 1 - \exp(-\rho x)$. For each of these cases, I will estimate parameters from the overall expected utility function,

$$U(x, p) = p_0 u(x_0) + p_1 u(x_1).$$

In addition to curvature parameters in an expected utility setting, some experiments allow for the estimation of loss aversion or probability weighting parameters. A simple parameterized version of Gul’s (1991) loss aversion model can be seen as

$$U(x, p) = \{\alpha 1(x_0 \leq x_1)p_0 u(x_0) + p_1 u(x_1), p_0 u(x_0) + \alpha 1(x_0 \geq x_1)p_1 u(x_1)\}.$$

where $\alpha$ is the loss aversion parameter. $\alpha$ is usually expected to be greater than 1, implying that the subject will place a higher weight on the utility coming from the “loss,” or lower payoff.

Results about risk preferences will use data from Choi et al. (2007). The experiment asks subjects to allocate their budget between two Arrow securities, exactly one of which will pay off. The subjects make the allocations under various prices, which allows for the identification of both curvature and loss aversion parameters. The results of the estimation procedure can be seen in Figure 2.1. The figure shows the average estimation mean squared error for exponential utility with CRRA and CARA utility functions. The $x$ axis refers to the size of the estimation sample, while the $y$ axis gives the average squared estimation error, averaged over individuals and repetitions.

This graph shows that for all sample sizes, CARA tends to fit the data better. Figure 2.2
Figure 2.1: Estimation MSE from Choi et al. Parametric models fit smaller samples better than they fit larger samples. For all estimation sample sizes, an expected utility model with a CARA risk parameter fit the data better than an expected utility model with a CRRA parameter.

Figure 2.3 combines these two graphs, and also includes the estimation and prediction mean squared errors for the two models with loss aversion added. As expected, loss aversion always allows the models to fit better (since the simpler model is nested in the loss averse model). Slightly more surprisingly, adding the flexibility of a loss aversion parameter flips the ranking of CRRA and CARA in both estimation and prediction mean squared error: with loss aversion, CRRA fits and predicts better. Furthermore, the predictive mean squared error of CRRA with loss aversion is lower than all of the other models even for small sample sizes, and for high enough sample sizes this model predicts data outside of the estimation sample better than the models without loss aversion fit the data they’re estimated on.
**Figure 2.2:** Prediction MSE from Choi et al. Out of sample prediction improves as the size of the estimation sample size gets larger. Except for very small estimation sample sizes, CARA parameters give better predictions than the CRRA parameters.

**Figure 2.3:** Combined Estimation and Prediction MSE from Choi et al. Estimating loss aversion parameters improves prediction, even for relatively small sample sizes.
2.4.2 Time Preferences

Researchers have found renewed interest in time preferences as they have generated larger data sets which allow them to precisely estimate parameters of interest. In particular, models of dynamic inconsistency and present bias, particularly hyperbolic and quasi-hyperbolic discounting (Strotz, 1955; Laibson, 1997) have been the focus of significant amounts of recent research. Many papers have found evidence of present bias (Frederick et al., 2002), but more recent research has questioned the robustness of this result for time-dated monetary payments (Andreoni & Sprenger, 2012) while still finding evidence for it in other kinds of choices (Augenblick et al., 2015).

A simple and popular way of capturing present bias is the hyperbolic discounting model, in which agent maximizes

\[ U(c_t, c_{t+1}, c_{t+2}, \ldots) = u(c_t) + \sum_{k=1}^{\infty} \beta^k u(c_{t+k}). \]

where \( \beta < 1 \) implies present biased decisions, while \( \beta > 1 \) implies future biased decisions, and \( u(\cdot) \) usually takes the form of CRRA or CARA. A significant body of work has estimated these parameters using laboratory experiments in which subjects chose between earlier, smaller payments and later, larger payments.

Andreoni and Sprenger (2012) estimates these preference parameters from subjects’ choices over dated monetary payments with differing interest rates. Their well known result is that subjects seemed to act in a way that was dynamically consistent (\( \beta \approx 1 \)), in contrast to previous experiments. Using the same data, the prediction mean squared error in Figure 2.4 supports this interpretation: CRRA without quasi-hyperbolic discounting consistently outperforms CRRA with \( \beta \), while CARA with quasi-hyperbolic discounting seems to catch up to the exponential model with sample sizes larger than about 35, but never predicts appreciably better.

The comparison is also present when estimating the “Stone-Geary” parameters (which can be interpreted as background consumption or consumption minima). Despite the potential of higher estimation error, another parameter improves out of sample fit. Even with this additional parameter, the quasi-hyperbolic model doesn’t predict as well as the exponential.
Figure 2.4: Prediction MSE for exponential and quasi-hyperbolic models in Andreoni & Sprenger. Models with CRRA curvature parameters predict better than models with CARA parameters. Adding quasi-hyperbolic discounting makes predictions worse.

A significant difference between the prediction error in Figure 2.3 and Figures 2.4 and 2.5 is the scale of the y axis: mean squared prediction error is much higher when individuals make choices over dated monetary payments than when they make choices over risky prospects. This is not incredibly surprising, given that the discounted utility models described above were originally intended to capture people’s preferences over consumption. Since the experiment is only testing people’s preferences over payments, other factors such as access to credit markets might make the models poorly suited to predicting people’s behavior.

In recognition of this fact, more recent work has estimated time preferences in settings in which subjects are unlikely to have significant access to credit markets (Balakrishnan, Haushofer, & Jakiela, 2017) or over non-fungible goods such as time or effort. Augenblick et al. (2015) implement the same convex time budget design as Andreoni and Sprenger (2012), but over both monetary payments (20 decisions) and over time allocations (40 decisions). When estimating preferences over effort allocations, Augenblick et al. (2015) estimate parameters from cost
functions which take the form

$$c(e_t, e_{t+k}) = (e_t + \omega)^\gamma + \beta^{1-\alpha} \delta^k (e_{t+k} + \omega)^\gamma$$

where $\beta$ and $\delta$ are interpreted as above, and $\gamma$, the curvature parameter on the instantaneous cost function, is expected to be greater than one. Their results confirm Andreoni and Sprenger’s finding of no time inconsistency in preferences over money, but find that in estimating quasi-hyperbolic discounting over effort provision, a significant portion of subjects have $\beta \neq 1$.

Figure 2.6 again compares the exponential discounting model to the quasi-hyperbolic discounting model, allowing different sets of curvature and discounting parameters for money and effort allocations. The horizontal axis indicates the total number of decisions that was included in the estimation, and for each estimation sample size the proportion of “effort” and “money” decisions that were estimated on are kept the same as the overall experiment.\(^1\)

Overall, the exponential model predicts quite well when compared to the quasi-hyperbolic

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\(^1\)Since the overall experiment had 20 money allocations and 40 effort allocations, this implies that for estimation sample sizes of (for instance) 45 used 15 money allocations and 30 effort allocations, randomly selected.
discounting model, dominating it over all estimation sample sizes. However, the results from Augenblick et al. suggest that while individuals seem to act consistently over monetary allocations, they are less consistent over their effort allocations. Since figure 2.6 takes a weighted average of the prediction error in these two settings, it might lead to erroneous conclusions if $\beta - \delta$ predicts poorly over the former but well over the latter. Thus panels A and B of Figure 2.7 show the decomposition of the errors in this case. Perhaps surprisingly, while the prediction MSE of the exponential and quasi-hyperbolic models are closer in effort allocations than in monetary allocations, the exponential model still predicts better on average for all but the largest estimation sample size, in which they have essentially equal predictive power.

In any case, the relative scales of these two panels provide an interesting comparison: these models predict time-dated effort decisions significantly better than time-dated monetary decisions. Indeed, for equivalent sample sizes, the prediction error for effort decisions is similar to the earlier prediction error from risky decisions, and the prediction from the two time-dated money allocations is similar across the two experiments. In light of this, and the previously discussed issues of arbitrage, it seems that the experimental methodology used by Augenblick et
al. will be important going forward in the measurement of time preferences.

Panels C and D of Figure 2.7 also estimate utility functions with an exponential loss function, i.e.

\[ c(e_t, e_{t+k}) = \exp(\rho e_t) + \beta^{1-\delta} \delta^k \exp(\rho e_{t+k}) \]

and CARA utility, for completeness. The two types of curvature have almost equal predictive power over money decisions and effort decisions without quasi-hyperbolic discounting. On the other hand, using a CARA curvature parameter seems to improve predictions over CRRA when the \( \beta-\delta \) model is applied to effort decisions. Indeed, of the four models, CARA with quasi-hyperbolic discounting predicts effort decisions the best for high sample sizes.

A unique feature of Augenblick et al.’s data is that it contains information about subjects’
preferences regarding two different valuable goods. They estimate present bias parameters for effort and money in different regressions, getting separate estimates for the two, and then show that they are uncorrelated. A natural question is how the decision to estimate these parameters separately affects the prediction power of these models. Figure 2.8 shows the results of these regressions.

![Figure 2.8: Prediction MSE, for Augenblick, Niederle, & Sprenger, with and without restrictions on equality of discounting parameters](image)

In the figure, the solid lines are the Prediction MSE of the model in which the effort and money discounting parameters are allowed to be different, while the dotted lines restrict them to be the same. For both the exponential and the $\beta$-$\delta$ model, allowing different discounting factors improves prediction for almost all sample sizes, despite requiring the estimation of more parameters.

The result that different goods might be discounted differently has been discussed in the literature, but these results suggest that the topic deserves more interest. Banerjee and Mullainathan (2010) demonstrate that this sort of discounting leads to time inconsistency (even when $\beta = 1$), and that this sort of preferences can lead to preference reversals which have elsewhere been attributed to present bias. Furthermore, the result here is in accordance with work
by Ubfal (2016) who shows that individuals in a developing economy have good-specific discount rates. Analyzing new sources of data with good specific discount rates is likely to be a fruitful line of research, although it remains to be seen how much is lost by assuming additively separable utility over time.

### 2.5 Robustness

#### 2.5.1 Statistical Significance

The results shown above don’t address the statistical significance of the difference between models’ predictive capabilities. This was deliberate: since a primary goal is to select the set of preferences that will be used by applied modelers, one must choose the “winner,” whether or not the results are significant in the statistical sense.

With this caveat in mind, it will still be useful to those running experiments to have a sense of where more research needs to be done to provide a definitive answer. As a first measure, it’s reassuring that there is substantial consistency within and across experimental data sets. If a model predicts substantially better when estimated on a sample of size 40, it generally also predicts better when estimated on sample sizes of 39 or 41. Furthermore, in the one direct comparison across experiments that can be made here, the money prediction error from Panel A of Figure 2.7 is similar to the prediction error from Figure 2.4 when comparing similar estimation sample sizes. Beyond these points, I also provide two measures of the confidence in these results.

The first measure is a confidence set for each model’s mean squared prediction error. Since for each estimation sample size, a different estimation sample was drawn 200 times, a natural measure of the variance of the prediction error are the 5th and 95th percentiles of the prediction error of these draws.

Figure 2.9 shows these confidence sets for exponential and quasi-hyperbolic discounting under CRRA curvature. In general, these confidence sets have significant overlap. The primary reason for this is that within a given subset of the data, the prediction error of models are highly correlated: some data sets are harder to predict than others.
To account for this correlation, I also calculate t-statistics from matched pair t-test for the difference in means, which accounts for the correlation in difficulty of prediction for a given draw of the data. When this is taken into account, equality of means is rejected at very low p-values. For instance, when comparing the difference in mean Prediction MSE of the expected utility models, as in the first panel of Figure 2.9, the only estimation sample sizes for which equality of means is not rejected is 5 and 6. Equality is always rejected when testing difference in means between the loss aversion models, as in the second panel, and the highest p-value is less than 0.001.

2.5.2 AIC

In addition to predictive measures like those used here, another common method of model selection is the use of information criteria, the most common of which are the Akaike information criterion (AIC) and Bayesian information criterion (BIC) (Akaike, 1998; Schwarz, 1978). Since here we are using non-linear least squares to estimate parameters, the most natural measure is the AIC, which in the case of least squares is usually defined as

\[ AIC = n \ln (MSE) + 2k \]  

(2.1)
where \( n \) is the number of observations, \( MSE \) stands for estimation mean squared error, and \( k \) is the number of parameters. In our case, the total number of observations is the number of subjects in the experiment multiplied by the number of decisions, and the number of parameters is the number of subjects in the experiment multiplied by the number of parameters in the model being studied.

Since the purpose of AIC is to select the best model for the full sample, the most natural comparison between AIC and the methods used in this paper is to consider the rankings provided by AIC versus the rankings given by mean squared prediction error on the largest possible estimation sample size.

Tables 2.1 and 2.2 give the rankings implied by these two measures for two of the experiments that were considered above. These measures only sometimes coincide; in the data from Choi et al. (2007), prediction mean squared error and the AIC would give you the exact same ranking of models. In the data from Augenblick et al. (2015), they substantively differ: for prediction mean squared error, the exponential CARA model is the best and quasi-hyperbolic CRRA with Stone-Geary parameters is the worst, while in AIC the ranking is quite different - the quasi-hyperbolic CARA model is the best and exponential CRRA model with Stone-Geary parameters is the worst.

### Table 2.1: AIC versus Prediction MSE from Choi et al.

<table>
<thead>
<tr>
<th>Prediction MSE Rank</th>
<th>CRRA EU</th>
<th>CRRA LA</th>
<th>CARA EU</th>
<th>CARA LA</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIC Rank</td>
<td>4</td>
<td>1</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

### Table 2.2: AIC versus Prediction MSE from Augenblick, Niederle, & Sprenger

<table>
<thead>
<tr>
<th>Prediction MSE Rank</th>
<th>CRRA Exp SG</th>
<th>CRRA QH SG</th>
<th>CARA Exp</th>
<th>CARA QH</th>
</tr>
</thead>
<tbody>
<tr>
<td>AIC Rank</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

The primary benefit of using AIC (as compared to cross-validation measures) is that it is very easy to calculate - the estimation mean squared error had to have been calculated in the estimation process anyway, and the subsequent calculation of the AIC is trivial. In comparison,
cross validation is much more demanding to compute: it requires estimating each of the models many times, which may involve substantial coding and computational time.

On the other hand, there are several qualities of cross validation which make it superior to the AIC. First is interpretability - the AIC doesn’t have an interpretation itself, but is only useful when comparing models. Mean squared prediction error, on the other hand, is easily interpretable as a measure of the distance between predicted and actual decisions. Related to this point, it may not be easy to compare the results of an AIC calculation in one experiment to that from another; it’s easy to compare the results of prediction error across experiments, and if the experiments use convex budges, they need not even measure preferences in the same choice domain.

Second, the prediction error is more closely tied to the purpose of estimating these preferences. The parameters which are estimated using experimental choices are supposed to capture what subjects will do outside of the lab, and in the past have been evaluated based on their correlation with real life decisions (Meier & Sprenger, 2010; Fisman, Jakiela, Kariv, & Markovits, 2015; Fisman, Jakiela, & Kariv, 2016). Mean squared prediction error is in some sense an intermediate step between the goals of fitting a model to experimental choices and predicting real-world behavior. AIC, on the other hand, can only be interpreted this way insofar as one can argue it reduces estimation error, and the particular formula chosen for AIC is mainly used for historical reasons.\(^2\)

2.5.3 Model Heterogeneity

The estimation procedures used to this point have assumed that each subject’s behavior is explained by the same utility function, but with different parameters. Previous research has demonstrated that there is significant heterogeneity in preference parameters arising in a wide variety of settings. With this in mind, one might ask whether different individuals’ decisions might not only be explained by different parameter values in utility functions, but different families of utility functions themselves.

\(^2\)One might argue that mean squared error is also an arbitrary measure of how well a model fits, and this is true. However, mean squared error is used in both cases, and to use AIC one has to make the extra arbitrary decisions that give the formula in equation (2.1).
To answer this question I treat the model itself as another parameter to be estimated. Thus, I estimate parameters of each model using the data from a subset of individual’s budgets, and choose the model which has the lower mean squared estimation error. The prediction error is then calculated using the parameters from the chosen model from that individual.

![Prediction MSE, including Model Selection, Choi et al.](image)

**Figure 2.10**: Prediction MSE, including Model Selection, Choi et al.

The results of this procedure are shown in figure 2.10 for Choi et al. (2007). Since mean squared estimation error is necessarily lower for the models with loss aversion, I execute the procedure separately for expected utility and loss averse models. Perhaps surprisingly, this heterogenous model procedure without loss aversion outperforms the other expected utility models uniformly, and the heterogenous model procedure with loss aversion improves on the other loss averse models with sufficient estimation sample size. This suggests that allowing for heterogeneity in the family of risk preferences in addition to heterogeneity in parameters might better rationalize individuals’ decisions.

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3 Another interesting result of this procedure is the classification of each subject into one of the two types: CRRA or CARA.
2.5.4 Identification and Experimental Design

The cross validation procedure carried out above ignores important features of experimental design. Researchers design these experiments with a particular mix of choice problems to identify parameters of interest. For instance, in an experiment designed to estimate the present bias parameter $\beta$, subjects face a number of choices in which they trade off between the present and the future, as well as decisions in which they trade off between two future points. When the cross validation procedure splits a subject’s decisions into the estimation and prediction samples, it may do so in a way that doesn’t include enough decisions of a given type. If this is the case, a researcher using cross validation might be overly pessimistic about the more complex model.

It’s likely that this problem will be the worst when the ratio of parameters to identifying points is high. This is because if only a few of the points that can identify a parameter are being used in the estimation procedure, this estimate is likely to be very noisy. Of the models and data sets studied here, the results regarding the prediction error in money decisions for the quasi-hyperbolic model with Stone-Geary background consumption seems to be the most likely case in which it will arise. The “money” portion of the model has 4 parameters: $\beta$, $\delta$, $\gamma$, and $\omega$, and these four parameters are estimated using at most 20 data points. Furthermore, a maximum of 5 of these data points are decisions including only “future” monetary payment, which are critical for the separate identification of $\beta$ and $\delta$.

This issue doesn’t seem to drive the results found above. To show that it is not a major issue, Figure 2.11 disaggregates the prediction MSE by how many “future” decisions are in the estimation sample, and include the prediction MSE from the full sample for comparison. Each of the dashed lines is an average of the prediction MSE, where the number of “only future” choices is held constant.

The main result from Figure 2.11 is that the prediction MSE is not being driven by non-representative estimation samples, in which there are 0, 1, or 2 “only future” future decisions. Instead, the prediction mean squared error the full sample are generally close to those which estimate using 3 to 5 of these “only future” decisions.
2.5.5 Constant Prediction Sample

In all of the analysis above, as one increases the number of budgets used in the estimation, the number of budgets used for prediction gets smaller. When a model is estimated on sample size $j$ and the full experiment generated $T$ observations per subject, the prediction mean squared error is calculated by finding the mean squared prediction error on $T - j$ observations. One might imagine that the changing size of the prediction sample could be influencing the results.

An alternative would be to keep the size of the prediction sample constant. Obviously, if the prediction sample size is held constant at (for instance) 5, the largest possible estimation sample size is $T - 5$. The results of this completing this exercise for the CRRA with and without loss aversion can be seen in Figure 2.6. Here, the mean squared prediction error on a constant sample size of 5 is overlaid on the mean squared prediction error on the complement of the estimation data. The mean squared prediction error on a constant sample size is almost exactly equal to the mean squared prediction error on the changing sample size. In fact, the former seems to be a slightly noisier version of the latter; and this is true for all of the models estimated in this paper. Thus, one can be confident that the changing prediction sample size is not driving any of
the above results.

![Graph](image)

**Figure 2.12:** Prediction MSE, including Constant Comparison in black, Choi et al.

### 2.6 Conclusion

Economic experiments are well posed to allow economists to select between models of behavior. This paper presents a method to select between these models on the basis of out of sample prediction.

The procedure is applied to data from several experiments that elicited risk and time preferences. When studying subjects’ risk preferences, more complicated models that allow for loss aversion and broadly different risk behavior predict better than simpler models. Furthermore, these models tend to predict better even when only estimating on a few data points. When applied to decisions over time preferences, the methodology shows that exponential discounting predicts as well as quasi-hyperbolic discounting, and predicts better for smaller sample sizes. However the models that predict well are also time inconsistent, since they have different discount rates for different goods.

Prediction is presented as a complementary approach to standard methods of testing.
Its outcomes are easily interpretable, and doesn’t suffer from the possibility of rejecting all competing models. In addition to providing guidance to applied modelers, the results are also useful to the empirical and experimental researchers: they implicitly show how much data is needed to convincingly estimate parameters from a given model.

2.7 Acknowledgements

Chapter 2 is being prepared for submission for publication. Breig, Zachary. The dissertation author was the sole author of this paper.
2.A Differences in Data Analysis

2.A.1 Choi et al. (2007)

Choi et al. (2007) estimate curvature and loss aversion parameters with CRRA utility functions. Their estimation procedure minimizes the loss function

\[\sum_{i=1}^{50} \left[ \ln \left( \frac{x_1^i}{x_2^i} \right) - f \left( \ln \left( \frac{x_2^i}{x_1^i} \right); \alpha, \rho, \omega \right) \right]^2,\]

where \(f\) is the ratio of demand that arises from a utility function with \(\alpha\) and \(\rho\) as the parameters, \(\bar{x}_1^i\) and \(\bar{x}_1^i\) the maximum values in the budget, and \(\omega\), an exogenously chosen value that accounts for consumption ratios of 0 or infinity. They use a similar procedure to estimate parameters from the CARA model, but instead of using the log ratio of consumption as the left hand side variable, they use the difference in consumption between the two goods.

Key Differences

1. Instead of using log consumption or consumption differences, the analysis here calculates budget-proportion demands, and minimizes sum of squared differences between these budget-proportion demands and the data.
2. As a result of not relying on demand ratios, the analysis here does not need the extra parameter \(\omega\), which prevents the demand ratio from being 0 or infinite.
3. The analysis from the main text omits a number of subject due to a variety of concerns, such as low CCEI scores or particular choice patterns. This analysis follows the results from the appendix, which includes the full set of subjects.

2.A.2 Andreoni & Sprenger (2012)

(Andreoni & Sprenger, 2012) estimate a number of different utility specifications, with both NLLS and Tobit analysis. In their NLLS section, which is the most closely related to what is done here, they calculate the CRRA demand functions from the agent’s maximization problem,
and estimate the parameters which minimize the sum of square residuals.

**Key Differences**

1. The original analysis uses the level of demand for earlier payment as the left hand side variable, while the analysis here uses the proportion of the budget devoted towards that good. This second approach weights each decision equally, while the former will place higher weight on decisions in which the budget is larger.

2. The analysis here never uses the “reported” level of background consumption, or ω. Instead, it either assumes that the subjects are choosing as if the background consumption 0, or estimates it directly.

3. The original analysis estimates the curvature parameter α rather than the parameter ρ. This is just notation, and one can define α = 1 − ρ.

4. The original analysis doesn’t estimate preference parameters for a small subset of subjects - either because there was not enough variation in their choices to identify a parameter, because the estimation process didn’t converge, or because their behavior exhibited strange choice patterns. The analysis here uses the data from all subjects. If the parameter estimates themselves were a primary objective, the lack of choice variation would be a first order issue. However, since this paper is about comparing models’ predictions, the data can still be used. Regardless, subjects which have little to no variation in their choices are easily predicted by all models, so this concern doesn’t drive the results.

2.A.3 Augenblick, Niederle, & Sprenger (2015)

In the section on individual analysis, Augenblick et al. use NLLS on log consumption ratios to get individual parameter estimates for subjects discounting parameters.

2.A.4 Key Differences

1. Instead of using the log consumption ratio as the left hand side variable, the analysis here uses a NLLS estimation with budget shares as the left hand side variable.
2. The analysis here uses all monetary delay lengths. The main analysis of the original paper focuses on monetary delay lengths of 3 weeks for easier comparison to the nonparametric tests.

3. The original analysis estimates time preferences over effort for 80 subjects, and a subset of 75 for time preferences over money. For easier comparison, the analysis here focuses only on the 75 subjects for which there are both kinds of data.

4. The analysis here never uses the “reported” level of background consumption, or $\omega$. Instead, it either assumes that the subjects are choosing as if the background consumption 0, or estimates it directly.

5. Here, cost and utility function curvature parameters are estimated for each individual
Chapter 3

Repeated Contracting without Commitment

3.1 Introduction

This paper has two primary contributions. First, it formally solves for the equilibrium of the non-commitment or “spot contracting” game of monopolist sales that was first presented by Hart and Tirole (1988). Second, in embeds the “commitment with renegotiation” and “spot contracting” games into a repeated setting, and characterizes the principal’s equilibrium payoffs.

In the stage game I study, a seller interacts with a mass of buyers who have private information about their valuation for a nondurable consumption good. The seller offers contracts, which specify in what periods the buyer will consume, and what price the buyer will pay. The buyer chooses whether or not to accept the contract. If he does, the new contract is carried out. Otherwise, the last contract that the buyer accepted is carried out. The different levels of commitment which I study restrict the set of feasible contracts that the seller can choose from.

A large literature has grown around studying dynamic mechanism design in settings where the principal doesn’t have full commitment power. The model this paper focuses on features a monopolist with constant marginal cost and a buyer with unit demand. Other models studied by
Laffont and Tirole (1987, 1988, 1990) have an increasing cost structure and unbounded demand, but equilibria are complex and difficult to characterize even for two period, two type models.

In general, the optimal structure of a contract depends on the type of commitment the designer is endowed with. In many cases, the contract that the principal will implement if she has full commitment power can’t be implemented through spot contracts, and will not necessarily be renegotiation proof (Baron & Besanko, 1984). Optimal contracts without commitment can look qualitatively different and generate significantly lower payoffs for the principal. Settings without full commitment are thought to be more realistic, because it seems unlikely that all contracts will last the entirety of the relationship, and that courts will prevent any renegotiation.

While the legal structure may not provide a principal with full commitment power, focusing on the one-shot game ignores the potential for reputation effects to make up for a lack of commitment. Previous work has noted that reputation in a repeated game could overcome the principal’s inability commit, but hasn’t shown how the “exogenous” commitment power which is provided by the modeler or the legal system interacts with the “endogenous” commitment power that arises from repeated game incentives. This paper studies how a principal facing a sequence of short-lived agents is affected by this exogenous commitment power.

Empirically, it’s unclear how important principals’ lack of commitment power is in the formation of contracts. Some previous research has shown that insurance contracts are crafted to deal with lack of commitment power on the demand side (i.e. agents’ commitment power) (Hendel & Lizzeri, 2003). Other research has shown that loan renegotiations are a means for lenders to complete contracts, rather than a reaction to past information revelation (Roberts & Sufi, 2009; Roberts, 2015).

Anecdotally, it seems that firms have a strong incentive to adhere to their pricing plans, even if not legally obligated to do so. For instance, roughly two months after the release of the original iPhone in 2007, Apple dropped the price by $200\textsuperscript{1}. Apple’s customers complained of unfair pricing practices, leading the CEO to publicly apologize and offer customers who bought

\textsuperscript{1}An account of consumers’ reaction and Apple’s subsequent actions can be found at http://www.nytimes.com/2007/09/07/technology/07apple.html
it before the price change a $100 store credit. These actions suggest that shareholders were concerned about dropping prices in this way, presumably because doing so would make it more difficult to charge high prices in the future. These considerations are not restricted to technology firms. Publishing houses are known to “strip-and-bind” unsold hard cover books, stripping off the covers and selling them as paperbacks, rather than lowering prices (Clerides, 2002).

This paper draws on the results from Hart and Tirole (1988) in utilizing the uniqueness of the payoffs from the equilibrium in the commitment with renegotiation setting, and solves for the equilibrium of the non-commitment game, which was previously only known for the case in which the buyer and seller don’t discount (Schmidt, 1993). In the equilibrium, the seller makes offers so as to spread out the learning process about the buyer’s type over the relationship. The periods in which this separation can happen are restricted by the prices that the principal has to charge in them: if the principal learns early, the low valuation buyer’s incentive compatibility constraint will be violated. When the principal and agent interact for long enough, the periods in which the principal sells only to high types is determined solely by this incentive compatibility constraint.

When studying the repeated game, I assume that the principal faces a sequence of finitely lived agents (Fudenberg, Kreps, & Maskin, 1990). Thus, the model could be applied to a setting in which customers interact with a monopolist for a few years, after which they will no longer buy the monopolist’s product.

When patient enough, the maximum equilibrium payoffs that the principal can receive in the repeated game are exactly the payoffs from the full commitment game. This provides extra reason to be interested in the full commitment equilibrium, even if it’s unlikely a court would enforce that contract: the payoffs from the full commitment equilibrium provide a bound on the repeated game payoffs.

I also show that, in a sense, the ability of the principal to attain the full commitment payoffs is not necessarily increasing in the exogenous level of commitment. More precisely, there collections of parameters for which the principal is able to obtain the full commitment payoffs in the repeated game when contracts must all be short term, but for which the principal
cannot obtain the full commitment payoffs when the principal can commit to long term contracts which later might be renegotiated. Thus, in the absence of “full commitment” (or in the case that committing fully is otherwise costly), it’s not always obvious that “more” commitment is “better,” or that if given a choice between legal frameworks, an owner of a firm would select the one in which she could sign contracts for longer. These results on the interaction between exogenous and endogenous commitment power in adverse selection settings are related to work by Baker, Gibbons, and Murphy (1994), Schmidt and Schnitzer (1995), and Pearce and Stacchetti (1998), in which implicit and explicit contracts are both used in a hidden-action setting.

The law literature is also somewhat split on whether more commitment power is necessarily better. Jolls (1997) provides the economic arguments for why allowing contract modifications can lead to ex ante inefficiencies, and argues that contract law should facilitate more commitment. Schwartz and Scott (2003) also argue that preventing renegotiation can make all parties to the contract better off, but claim that currently under common law, parties cannot prevent themselves from modifying contracts. In contrast with both of these arguments, Davis (2006) gives evidence that demand for so-called “immutable” contracts is low, and that it’s plausible that the current approach (which doesn’t allow renegotiation to be prevented) is the optimal one.

Section 3.2 of this paper presents the underlying economic framework. Sections 3.3 and 3.4 characterize equilibria in the one-shot and repeated games, respectively. Section 3.5 is a conclusion. The proofs that are not in the main body of text are in Appendix 3.A.

### 3.2 Model

A seller can produce a perishable consumption good at a normalized price of 0. This seller faces a unit mass of buyers, each of which have a unit demand. These buyers have valuation $b \in \{b, \bar{b}\}$, where $0 < b < \bar{b}$. The probability that an agent is of the high type is $P(b = \bar{b}) = \mu$, and to make the problem non-trivial, we’ll always assume that $\mu \bar{b} > b$.

An agent lives for $T < \infty$ periods, has a constant valuation, and wants to consume in each of the periods. Both the principal and the agent have a discount factor $\delta$. 
With some slight changes to notation, we will follow Hart and Tirole (1988) to define a contract as a sequence of message spaces \( \mathcal{M}_t \) for the buyer, a sequence consumption levels \( q_t(m') \in [0, 1] \) where \( q_t \) can depend on all messages up to time \( t \) and \( q_t \) is publicly observed, and a sequence of of prices, \( p_t(m') \), which are again publicly observed and can depend on all messages up to time \( t \). Thus, a contract in period \( t \) is \( C_t = \{(q_t(m^\tau), p_t(m^\tau))_{\tau=t,...,T}\} \). Furthermore, define a continuation contract as

\[
\mathcal{C}(C_{t-1}) = C_{t-1} \setminus \{(q_{t-1}(m^{t-1}), p_{t-1}(m^{t-1}))\},
\]

which is just the contract that was accepted in the previous period without the previous period’s consumption and price.

The buyer’s strategy space in each period depends on the actions taken by the seller. If the seller did not make a contract offer in a given period, then the buyer has no choice to make and any previously agreed upon contract is implemented. If, on the other hand, at least one contract is offered in a given period, the buyer can select among the contracts that are offered to him.

I’ll consider three commitment structures: full commitment (FC), commitment with renegotiation (CR), and spot contracting (SC). The seller’s strategy space is the set of feasible contracts in period \( t \), \( \mathcal{F}_t^j(C_{t-1}) \) which will depend on \( j \), the commitment structure that the seller facts, as well as (perhaps) the contract which was enforced in the previous period. With the definitions below, it will be without loss of generality to assume that before the buyer and seller interact, they are committed to payments and consumption levels of 0 for the entire relationship. Thus, \( C_0 = \{(q_\tau(m^\tau), p_\tau(m^\tau))_{\tau=0,...,T} : q_\tau = p_\tau = 0\} \).

In the full commitment setting, the seller offers an agent a new contract in the first period of their interaction, and is committed to not offering a new contract in future periods. The contract specifies a level of consumption and an amount paid in each period, so we can define

\[
\mathcal{F}_t^{FC}(C_{t-1}) = \begin{cases} 
\{(q_\tau(m^\tau), p_\tau(m^\tau))_{\tau=1,...,T} : q_\tau \in \{0, 1\}, p_\tau \in \mathbb{R}\} & \text{if } t = 1 \\
\mathcal{C}(C_{t-1}) & \text{otherwise}
\end{cases}
\]
This indicates that the seller can offer a new contract to an agent in period 1, but is required to carry out the old contract in all other periods.

When the seller can renegotiate previous contract offers, what she offers is not restricted by the previous offers that she has made, and the set of feasible contracts is

\[ \mathcal{F}^{CR}_t = \{ (q_\tau(m_\tau^\tau), p_\tau(m_\tau^\tau))_{\tau=t, \ldots, T} : q_\tau \in \{0, 1\}, p_\tau \in \mathbb{R} \}. \]

Finally, in the spot contracting setting, the seller can offer an agent a new contract in each period. The offer can specify both quantity and prices for the current period, but cannot promise any consumption level or prices other than 0 for other periods. Thus, the set of feasible contracts is

\[ \mathcal{F}^{SC}_t = \{ (q_\tau, p_\tau)_{\tau=t, \ldots, T} : q_t \in \{0, 1\}, p_t \in \mathbb{R}, \text{ and } q_\tau = p_\tau = 0 \text{ if } \tau > t \}, \]

which again doesn’t depend on any previous contract offer.

In the one-shot game, the principal’s payoffs are the expected discounted sum of payments, or

\[ \sum_{\tau=t}^{T} \delta^{\tau-t} \mathbb{E}^S_t [p_\tau], \]

while an agent with valuation \( b \) maximizes the discounted expected sum of consumption utility minus price paid, or

\[ \sum_{\tau=t}^{T} \delta^{\tau-t} \mathbb{E}^B_t [bq_\tau - p_\tau]. \]

I’ll be looking for these games’ Perfect Bayesian Equilibria, which in this setting is defined as follows. The principal and the agents’ actions are sequentially optimal, which is to say that in any period given their beliefs about the other player’s strategy and (in the case of the seller) the other player’s type, they are maximizing their expected payoffs. The seller’s beliefs
about the valuation of the buyer only change when the buyer takes an action, and are required to satisfy Bayes’ rule whenever possible.

Section 3.4 will consider the repeated version of this game, in which the seller faces a sequence of buyers, each of whom live for $T$ periods. Each buyer’s type is their private information, and the seller cannot contract with the buyer before he observes his type. Both the buyer and seller have the same discount factor $\delta$, which is applied between periods for both parties, and in between the last period one agent is active and the first period another agent is active for the buyer. Furthermore, when computing payoffs for the principal in the repeated game, all payoffs will be normalized by a factor of $1 - \delta$, such that the seller receives a payoff of $b$ for selling to all buyers in all periods at a price equal to the low type’s valuation.

### 3.3 Results - One-Shot Game

The equilibrium of the game with full commitment and the game with commitment and renegotiation is well known.

**Proposition 3.1** *In the equilibrium of the full commitment game, the high type receives a quantity of 1 and pays $\bar{b}$ in each period, and the low type doesn’t consume and pays nothing.*

When priors about the likelihood of the high type are high enough, the optimal full commitment mechanism has the property that the principal sells only to the high type, and extracts full surplus. Notice that the optimal mechanism gives the same allocation to the agent in each period. This is a well known result from Baron and Besanko (1984), and is a function of the fact that the agent’s type is constant. If neither the agent nor the principal knew the agent’s type in future periods, the principal could extract full surplus from the agent by simple offering to give the efficient allocation and charging the expected surplus.

Another feature of the optimal mechanism is its allocative inefficiency: the principal’s marginal cost is zero, and if nothing else changed, increasing the low type’s allocation and the price paid would make the seller strictly better off without affecting the low type’s incentive compatibility constraint. However, this inefficiency allows the seller to extract higher surplus
from the high type. A further consequence of this inefficiency is that the optimal full commitment mechanism is not renegotiation proof. If the seller were attempting to implement this full commitment mechanism in a setting where she was allowed to renegotiate contract offers, she wouldn’t be able to. Entering the second period, the principal has full information about the agents’ types; she’d prefer to offer the low types a new contract with an efficient quantity, and charge $b$. This is would be acceptable to the low type because he isn’t receiving any surplus in either case. Because the high type knows that the low type’s contract will be renegotiated in the second period, he won’t accept the contract in the first period, and the principal can’t implement this allocation.

In a sense, when the contract can be renegotiated, attempting to implement the full commitment allocation causes the seller to learn “too fast.” She can’t help but take advantage of the information she knows in the second period as a result of trying to implement the full commitment allocation. Hart and Tirole (1988) show that if priors are high enough, the optimal mechanism doesn’t learn the agents’ types immediately as in the full commitment case, but instead spreads the learning out over a number of periods so that the principal can extract higher surplus from the high types.

**Proposition 3.2** The equilibrium path of the commitment and renegotiation game is generically unique and takes the following form: there exists a sequence of numbers $0 = \bar{\mu}_1 < \bar{\mu}_2 < \cdots < \bar{\mu}_T < 1$ such that

(i) If current posterior beliefs $\mu_t$ at date $t$ belong to the interval $[\bar{\mu}_i, \bar{\mu}_{i+1})$ for $i \leq T - t + 1$, the principal will sell only to high types for $i$ more periods including the current one. Posterior beliefs are $\bar{\mu}_{i-1}$ at $t + 1$, $\bar{\mu}_{i-2}$ at $t + 2$ and so on. The discounted sum of prices charged in one of these periods is such that the high typed buyer is indifferent between purchasing and waiting for the low type’s contract.

(ii) If current beliefs are such that $\mu_t \geq \bar{\mu}_{T-t+1}$, only high types purchase in ever period, and the discounted sum of prices charged is such that the high type is indifferent between his allocation and not purchasing.

**Proof** Hart and Tirole (1988). □
Proposition 3.2 states that when $T$ is high enough, for a fixed number of periods independent of $T$ the seller makes sales only to buyers with a high valuation, after which she sells to all buyers. The discounted sum of prices charged to the high types makes them indifferent between their allocation and the low types'. The set of agents sold to in a given period is chosen to optimally trade off between selling to a larger group of high valuation buyers earlier, and putting off learning about the agents’ type so that the seller can extract more surplus from the buyers.

The equilibrium described in Proposition 3.2 requires that the buyer and seller be able to at least partially commit for the future. If instead the seller were only able to commit to spot contracts, there are combinations of parameters for which the equilibrium outcome described in Proposition 3.2 would not be incentive compatible. Without any commitment power between periods, the seller will always charge the buyer $\bar{b}$ after learning that a buyer has a high valuation. Thus, to induce such a buyer to reveal their type the seller must charge the buyer a price which leaves the buyer surplus equal to the discounted sum of the low valuation buyer’s future consumption (this is exactly the price which will make him indifferent between accepting his offer and waiting for another offer). However, in some cases this price will be lower than $\bar{b}$, and a buyer with a low valuation would strictly prefer to accept this offer, and no longer participate in the contract.

Thus, the spot contracting setting introduces an additional difficulty to contracting: the ability of the agent to exit the contract after a low price offer, or “take-the-money-and-run” (Laffont & Tirole, 1987). Identifying all of the high types “too soon” requires offering them a contract which the low types would also prefer to take. To prevent this constraint from being violated, the seller might have to wait until later in the relationship to learn the buyers’ types, because otherwise she would have to charge too low of a price.

If only the high valuation buyer is purchasing the good in period $t$, the price being set $t$ is the one which makes this buyer indifferent between his allocation and waiting for the low type’s allocation. The feature which determines this price is the discounted sum of a low valuation buyer’s future consumption according to the seller’s strategy. Thus, selling to only high types in some period is feasible if this discounted sum is less than one.

To solve for the equilibrium of the spot contracting game, it will be useful to define the
following variables:

\[
a_1 \equiv \min \left\{ k : k \in \mathbb{N}^+, \sum_{i=1}^{k} \delta^i > 1 \right\}
\]

and

\[
a_m \equiv \min \left\{ k : k \in \mathbb{N}^+, \sum_{i=1}^{k} \delta^i - \sum_{j=1}^{m-1} \delta^{k-a_j+1} > 1 \right\}.
\]

These variables can be interpreted in the following way: if low valuation buyers are not purchasing in periods \( T - a_i + 1 \) for all \( i \), then the discounted sum of the consumption of low valuation buyers is always less than 1.

A key feature in the description of the equilibrium in the commitment with renegotiation setting was the path that the seller’s posteriors follow. The path that the seller’s posteriors follow plays a similarly important role in the spot contracting setting, and the two are related to each other. Using the definition of \( \bar{\mu}_j \) given by Hart and Tirole, define

\[
\mu_{i,j} = \left( \frac{\bar{b} - b}{b} \right)^i \bar{\mu}_j + 1 - \left( \frac{\bar{b} - b}{b} \right)^i,
\]

and \( M_t = \{ \mu_{i,j} : T - t - a_i + 1 < j \leq T - t - a_i + 1, j \geq 0 \} \). The next result states that if the principal’s beliefs are initially above \( \mu_{i,0} \) for any \( i \), then they won’t fall below that level until a particular period.

**Lemma 3.1** Suppose that \( \delta > \frac{1}{2} \). In any equilibrium of the spot contracting game, if beliefs are initially higher than \( \mu_{i,0} \) then the seller cannot make sales in such a way that beliefs are strictly below the value \( \mu_{i,0} \) in period \( t = T - a_i + 1 \). If \( \mu_t \in [\mu_{i,0}, \mu_{i+1,0}] \) in period \( t = T - a_i + 1 \), the principal sells to only high types in that period, leading to a posterior of \( \mu_{i-1,0} \).

This result is what drives the convergence as \( T \to \infty \) to the “no discrimination” equilibrium described by Hart and Tirole. The proof uses backwards induction, and revolves around the constraint that the seller cannot charge a price below \( b \) and have only high types buy. The intuition is that if beliefs fall to a low level too early, then the seller will be offering a price of \( b \) and selling
to all remaining buyers in most periods in the future. To entice the high types to purchase in an earlier period, the price charged has to be low enough to make up for missing out on future low prices. However, if beliefs are “too low” (and thus the seller will be offering the low price in “too many” future periods), the price that the seller must charge today will be below \( b \), which will violate the low type’s incentive compatibility constraint.

The low valuation buyer’s incentive compatibility constraint will bind exactly when the discounted sum of the low type’s future consumption is strictly greater than one. This shows the importance of the values \( a_1, a_2, \) etc. defined before: as long as the seller “screens” (sells only to high valuation buyers at the price that makes them indifferent all other allocations) in at least periods \( T - a_1 + 1, T - a_2 + 1, \) etc., then the price she charges is always higher than \( b \).

Now that I’ve shown that learning cannot happen “too quickly,” I can characterize the equilibrium of the spot contracting game.

**Proposition 3.3** Suppose that \( \delta > \frac{1}{2} \), and that \( \mu_t \in [\mu_{i,0}, \mu_{i+1,0}) \). The equilibrium of the spot contracting setting is generically unique and takes the following form:

1. If \( t < T - a_i + 1 \), then the principal sells to both the high types and the low types with probability one, the price charged is \( b \), and the posterior \( \mu_{t+1} \) is equal to the current beliefs.
2. If \( t \geq T - a_i + 1 \), the principal sells only to high types, the price charged is the one which makes high types indifferent between buying and waiting for the low type’s allocation, and the posterior \( \mu_{t+1} \in M_{t+1} \).

If priors are high enough, the beginning of the equilibrium of the spot contracting game exhibits “Coasian dynamics” similar to those found in the equilibrium of the commitment with renegotiation game. In each period, the seller identifies and sells to a subset of the high types, updating her beliefs. Eventually, her posteriors fall enough that she can no longer carry out such an equilibrium. After this point, she discriminates only in certain periods, such that the price charged to the high types is never below \( b \).

In an equivalent model, this equilibrium was solved for in the case of \( \delta = 1 \) by Schmidt (1993). Analysis is greatly simplified in that case, because \( a_i = i + 1, \mu_{i,j} = 1 - \left( \frac{b - b}{b} \right)^{i+j} \), and the principal sells to only the high types in periods \( T - n - 1 \) to \( T - 1 \) for a number \( n \) which is
determined by the priors. In fact, as $\delta$ approaches one, the seller makes sales only to high types in the same number of periods in both the renegotiation and spot contracting settings, and the sequence of posteriors the principal has during the separation process of the renegotiation game approaches the sequence she has during the separation process of the spot contracting game. The payoffs in these two settings also become arbitrarily close to each other, since the timing of the screening process doesn’t matter if payoffs aren’t discounted.

3.4 Results - Repeated Game

The previous section focused on how the level of commitment affects the allocation and payment structure that is used in equilibrium. However, a common theme in the study of repeated games is that the incentives that arise in a repeated relationship (or a sequence of relationships) might allow one to credibly commit to taking actions that one wouldn’t usually take. This section investigates the extent to which repetition enables the seller to obtain commitment payoffs even without the ability to make credible commitments.

The model here is one in which a long-lived principal faces a sequence of short-lived agents, each of whom have private information about their own valuations. Because each buyer is only going to interact with the seller for a finite period of time, both the buyer’s and seller’s ability to “punish” or “reward” each other will be limited as compared to a repeated game in which all actors are long lived.

The first result states that $\mu \bar{b}$, the payoff level the seller receives in the game with commitment, acts as an upper bound on payoffs that the seller receives, no matter the discount rate, type of commitment provided, or number of times that the seller and buyer meet.

Lemma 3.2 There is no equilibrium of the repeated game with any commitment structure in which the principal receives payoffs which are higher than $\mu \bar{b}$.

Proof Suppose that there was such an equilibrium in the repeated game. If the seller is receiving discounted expected payoffs which are strictly higher than $\mu \bar{b}$ in the equilibrium, then there must be a particular wave of agents from whom she receives payoffs that are higher than this. These
agents must be best responding to the strategy that the seller is using within the time that they are active.

Consider the following change to the optimal sales mechanism in the one-shot game. The seller commits to using the exact same strategy that she is using on the above wave of agents. The agents in the one shot game have the same incentives as the ones in the repeated game, so the principal must be receiving the same payoffs, but this contradicts the assumption that selling to only high types in all periods is optimal for the principal. Thus, the principal can’t receive payoffs higher than $\mu \bar{b}$, the commitment payoffs, in the repeated game. □

While simple, this result is can be easily applied to other dynamic contracting settings. It shows why one should be interested in the equilibrium of the game with full commitment, even if it’s difficult to imagine a court preventing a contract from being renegotiated. Long-run incentives can overcome an inability to commit to a contract, but asymmetric information still places a limit on the amount of surplus that the principal can extract. This limit is captured by the payoffs the seller receives in the full commitment setting.

The next result is a characterization of the payoffs the seller can receive if she is sufficiently patient. Commitment on the part of the seller can be enforced by the buyers reverting to the equilibrium of the one-shot game after a deviation. The payoffs for the case in which $\delta = 1$ were first computed by Schmidt (1993), and I’ll restate them here. Define

$$v = \bar{b} - \frac{1}{T} (1 - \mu) \bar{b} \sum_{t=1}^{T} \left( \frac{\bar{b}}{b - \bar{b}} \right)^{t-1}$$

if $\exists n \geq T$ s.t. $1 - \left( \frac{\bar{b} - b}{\bar{b}} \right)^n \leq \mu < 1 - \left( \frac{\bar{b} - b}{\bar{b}} \right)^{n+1}$, and

$$v = \frac{T - n}{T} - b + \frac{n}{T} \bar{b} - \frac{1}{T} (1 - \mu) \bar{b} \sum_{t=1}^{n} \left( \frac{\bar{b}}{b - \bar{b}} \right)^{t-1}$$

if $n < T$.

**Proposition 3.4** In both the spot contracting and the renegotiation settings and for any $\pi \in (v, \mu \bar{b})$, for $\delta$ high enough there exists an equilibrium of the repeated game in which the principal
This result can be interpreted as the “folk theorem” for the principal’s payoffs in this setting. The upper bound on the principal’s payoffs, as mentioned before, are the payoffs that she would receive in the full commitment game. Thus, a patient enough seller can attain the full commitment payoffs in the repeated versions of the spot contracting or commitment with renegotiation games. The lower bound on equilibrium payoffs are the same as what the seller gets in the equilibrium of the one shot game, and these payoffs approach $v$ as $\delta$ gets close to 1.

To this point I mostly haven’t discussed the parameter $T$ and its effect on equilibrium outcomes. As described above, $T$ is the number of periods in which the buyer and the seller will interact. When the seller has full commitment power and the buyer’s valuation doesn’t change over time, a change in $T$ has no effect on the principal’s optimal strategy (except that the number of periods specified in the contract she creates varies with $T$). It’s only when the principal is unable to fully commit that $T$ begins to matter.

This paper will adopt the view that $T$ doesn’t represent the length of the contract, but rather number of times a contract can be renegotiated or a new contract can be offered in a given time period. For instance, imagine that the seller will be interested in the contract for two years. Then in the renegotiation setting, $T = 2$ indicates that the buyer and seller meet to renegotiate the contract on January first of the second year. Similarly, $T = 24$ could be interpreted as the buyer and seller meeting monthly, and $T = 730$ implies that they meet daily.

When interested in the interactions between the seller’s patience and commitment power in these settings, this interpretation of $T$ as the number of meetings rather than a length of time suggests studying not the discount rate between periods within a buyer’s “life,” but rather the seller’s discount factor between waves of buyers. In a sense, we can keep this discount rate constant while allowing $\delta$, the rate between periods within a buyer’s lifetime, to reflect the shorter amount of time between periods. Thus, I’ll define $\beta = \delta^T$, the discount factor that is applied to payoffs that arrive one “buyer lifetime” from the current period. I can then study how the set of feasible equilibria varies when $\delta$ and $T$ change, but $\beta$ is held constant.

A natural question is under what conditions the seller is able to attain the payoffs that
she would receive if she had full commitment exogenously provided. Lemma 3.2 shows that no matter what form of commitment the seller has exogenously available, payoffs are bounded above by this value, and Proposition 3.4 demonstrated that if all parties are arbitrarily patient, then these payoffs are attainable in an equilibrium. To study how patient a seller must be to attain this payoff in a given commitment setting, I’ll study the set of \( \beta \)'s for which the principal is able to attain the full commitment payoffs in that commitment setting.

As is usual, such a \( \beta \) is a discount factor which makes equilibrium payoffs better than a short period of “deviation” payoffs followed by “punishment” payoffs. One factor which makes it feasible to compare the sets of \( \beta \) which allow the full commitment payoffs in the spot contracting and renegotiation settings is that the deviation payoff is simple, and is the same for both settings. When carrying out the full commitment allocation, the best time for the seller to deviate is in the second period of a buyer’s lifetime, when she has full information about the buyer’s valuation, and can extract surplus for the remainder of the relationship. For \( T - 1 \) periods from that point, the principal receives \( (1 - \delta)(\mu b + (1 - \mu)b) \). Thus, to check whether attaining the full commitment payoffs is feasible for the seller, one only needs to calculate the payoffs from the punishment equilibrium, which is the same as the payoffs from the equilibrium of the one-shot game.

While this seems like a natural structure to consider, complications arise due to the relationship between the equilibrium of the one-shot game and the discount rate. In particular, the discount rate affects not only how much the punishment is discounted, but also the form of the punishment equilibrium. Thus, it’s not obvious that there exists a unique \( \beta \) which equates the full commitment payoffs to the discounted payoffs from deviating, then entering the punishment equilibrium. Despite this fact, I am able to make a few observations about the set of \( \beta \) for the spot contracting and renegotiation settings.

**Observation 3.1** If contracts are offered and renegotiated quickly enough \( (T \to \infty) \), the full commitment payoffs are attainable in both the spot contracting and renegotiation settings if

\[
\beta > \frac{1 - \mu}{\mu} \cdot \frac{b}{b - \overline{b}}.
\]
**Proof** From the equilibrium described in proposition 3.2, with fixed $\mu$, $\bar{b}$, and $b$, for any discount rate the number of periods in which the seller is only making sales to the high type has a bound which is independent of $T$. Normalizing the payoffs by multiplying them by $(1-\delta)$, the inequality that $\delta$ has to satisfy is

$$
\mu \bar{b} \geq (1-\delta)(\mu \bar{b} + (1-\mu)b)\left(1 + \cdots + \delta^{T-2}\right)
+ \frac{\delta^{T-1}}{1-\delta}(1-\delta)\left[x_1 + \delta x_2 + \cdots + x_n \delta^{n-1} + b\delta^n + \cdots + b\delta^{T-1}\right]
$$

where for all $i$, $b < x_i < \bar{b}$. It’s clear that in the limit as $T \to \infty$, $\delta \to 1$, and the portion of the seller’s payoffs that comes from selling only to high types at price $\bar{b}$ converges to 0. Keeping the terms which don’t approach 0, the expression converges to

$$
(1-\mu)b \leq \delta^{T-1}(\mu \bar{b} + \mu b)
$$

which in the limit is fulfilled when $\beta > \frac{1-\mu}{\mu} \cdot \frac{\bar{b}}{b}$. Essentially the same technique can be applied in the spot contracting case, because as $T$ increases the average discounted payoffs that arise from the equilibrium of the spot contracting game are also $b$. □

Observation 3.1 states that when contract offers and renegotiations happen with high frequency, all sellers whose discount factor is above a particular cutoff will be able to commit to the full commitment allocation in the repeated game. In this particular formulation of the buyer-seller relationship, the cutoff is the same in both the spot contracting and the commitment with renegotiation settings. This is because as the number of periods in a buyer’s lifetime increases in the one-shot version of the game, it becomes increasingly difficult for the seller to extract extra surplus from the high types; the discounted per-period payoffs converge to $b$.

It’s unlikely that the cutoffs under high frequency contract offers in the two commitment settings will be equal to each other with other formulations of the relationship. The intuition behind the form of contracting when $T$ is large is qualitatively different between the two settings. In the commitment with renegotiation setting, contracts must converge to the efficient contract
early in the relationship, making it impossible for the seller to extract additional surplus by screening with inefficient contracts. In the spot contracting setting, the seller pools buyers for most of the relationship, putting off any learning. These lead to equivalent payoffs in this setting, because the pooling allocation is equivalent to the efficient allocation, but this need not be the case in other settings.

The second observation comments on how the ability of a seller to commit varies with the exogenous level of commitment which is provided within the model.

**Observation 3.2** There is a collection of parameters $\mu, b, \bar{b}, T,$ and $\beta$ for which commitment payoffs are attainable in the spot contracting setting but not the renegotiation setting.

**Proof** As shown in the proof of Observation 3.1, the $\delta$ necessary to make commitment payoffs attainable in equilibrium approaches 1 as $T$ get arbitrarily large. Thus, in the punishment equilibrium of the spot contracting game, if

$$1 - \left(\frac{\bar{b} - b}{b}\right)^n \leq \mu < 1 - \left(\frac{\bar{b} - b}{b}\right)^{n+1},$$

then the principal is selling only to high types in periods $T - n$ to $T - 1$. Take the lowest value of $\delta$ such that the commitment payoffs are attainable. At this $\delta$, the punishment payoffs in the spot contracting game are strictly lower than the punishment payoffs in the renegotiation game, and the seller would strictly prefer to deviate. □

Observation 3.2 shows that, in a sense, the *endogenous* commitment power that a long lived agent has is not necessarily “increasing” in the exogenous commitment that is provided by law or within a model. “Exogenous” commitment power “increases” between the spot contracting and renegotiation settings, because the seller can write a contract including everything she could include in the spot contracting setting. However, shifting legal regimes in this way could make it more difficult to commit.

Like most repeated games, the potential punishments from the buyers are what makes commitment feasible. The key feature that drives this observation is that in some cases, punishments are worse in the spot contracting setting than they are in the commitment with renegotiation
setting. The worse punishments, combined with the fact that deviation payoffs are the same, mean that a seller need not be as patient to commit in the spot contracting setting, as compared to the commitment with renegotiation setting.

### 3.5 Conclusion

I consider a classic formulation of a contracting problem in which a seller does not have full commitment power, and show the effects of embedding the one-shot game into a repeated setting.

The equilibrium of the spot contracting game is qualitatively different than that of the commitment with renegotiation game. Hart and Tirole (1988) show that as the length of the spot contracting game converges to infinity, the equilibrium converges to the “no-discrimination” equilibrium, in which the high types don’t purchase at a price greater than the low type’s valuation until a finite and fixed amount of time before the end of the game. This paper showed when the spot contracting game is long, the periods in which the seller is discriminating between types are exactly the ones in which the price she must charge is strictly greater than the low type’s valuation. This paper also shows that for a fixed prior distribution, the equilibrium for intermediate $T$ is in a sense a mixture of the commitment with renegotiation equilibrium and the spot contracting equilibrium with high $T$: the seller discriminates for the first few periods of the relationship and in some later periods that are a fixed distance from $T$. In all other periods, the seller sells to all buyers at a price equal to the low type’s valuation.

Both the commitment with renegotiation and spot contracting games are then embedded into a repeated setting. The seller faces sequential masses of agents, each living for a finite number of periods, and the valuation of the each agent is private.

I first characterize what payments are feasible in these repeated games. No matter the type of commitment game that the agents are playing, the maximum payoffs for the seller are bounded above by the payoffs that she receives in the full commitment setting. This provides extra justification for studying optimal mechanisms under full commitment, even if it’s difficult to
imagine enforcing non-renegotiation clauses. If a long lived mechanism designer faces sequences of short lived agents, the best possible equilibrium outcome is the one which results from full commitment. I also show that the worst punishment payoff in this particular setting is carrying out the one-shot version of the game, and that for a patient enough seller, this punishment is sufficiently bad to ensure that the full commitment outcome is an equilibrium outcome of the repeated game.

I then provide two results that characterize how patient a seller must be to attain the full commitment payoffs in the repeated game. First, I give a simple cutoff (which is the same in both settings) that patience needs to be above in order to obtain the full commitment payoffs as renegotiation happens increasingly often in the commitment with renegotiation game, and as contract offers happen increasingly often in the spot contracting game. I then show that the ability to commit is not necessarily “increasing” in the amount of commitment that the economic setting provides: there are collections of parameters such that the seller can attain the full commitment payoffs in the spot contracting setting, but not the commitment with renegotiation setting.

Future work in this setting should consider the effects of more finely varying the principal’s commitment power. This paper gave results about this limiting behavior of the ability to commit as $T \to \infty$, but one might also study what happens as $T$ varies locally; preliminary numerical analysis suggests that commitment can be non-monotonic in the rate of contractual meetings. A further question of interest is what happens if contracts can be only signed for a portion of the interaction and can be renegotiated.

### 3.6 Acknowledgements

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3.A Proofs

Proposition 3.1 In the equilibrium of the full commitment game, the high type receives a quantity of 1 and pays \( \bar{b} \) in each period, and the low type doesn’t consume and pays nothing.

Proof Notice that the principal only cares about the discounted sum of prices that the agent pays. Since this is a setting with full commitment, the revelation principal holds, and we can focus on direct and truthful mechanisms. Thus, the principal’s problem is

\[
\max \sum_{t=1}^{T} \delta^t (\mu p_t(b) + (1-\mu) p_t(\bar{b}))
\]

subject to

\[
\sum_{t=1}^{T} \delta^t (b q_t(b) - p_t(b)) \geq \sum_{t=1}^{T} \delta^t (\bar{b} q_t(\bar{b}) - p_t(\bar{b}))
\]

and

\[
\sum_{t=1}^{T} \delta^t (b q_t(b) - p_t(b)) \geq 0
\]

It’s obviously optimal to set \( p_t(b) = b q_t(b) \) and \( \bar{b} q_t(\bar{b}) - p_t(\bar{b}) = \bar{b} q_t(b) - p_t(b) \) in each period. Plugging these in, we get that the principal always wants to maximize \( q_t(b) \) and sets \( q_t(b) = 0 \) if \( \mu > \frac{b}{\bar{b}} \). □

To prove Proposition 3.3, I’ll first need several lemmas.

Lemma 3.A.1 In any equilibrium of the spot contracting game, the price charged in each period is at least \( \frac{b}{\bar{b}} \).

Proof Sequential rationality requires that an agent with a low valuation agent always expects to receive weakly positive surplus in the future, and is thus receiving strictly positive surplus if the price is lower than \( \frac{b}{\bar{b}} \) in any period. Then in period \( t \), the principal can again increase all prices without violating individual rationality or incentive compatibility constraints. □

Lemma 3.A.2 In any period, the principal either fully pools the agents, or separates such that
low types do not buy, and high types buy or mix between buying and not buying.

Proof Suppose that the posteriors after period $t$ were such that one group has probability 0 of being a high type and the other had a strictly positive probability that was less than one. Since posteriors about the first group are 0, they will receive the efficient allocation at price $b$ for all future periods. Because the high types aren’t selecting into this contract, the seller can’t be extracting any additional surplus from the high types, so the seller can at most be receiving $b(1 + \delta + \cdots + \delta^{T-t+1})$. The seller can do strictly better than this by pooling in all periods but the last, so this cannot be optimal.

The principal also won’t separate the agents into two groups both of which have posteriors between 0 and 1. Suppose that this were the case. Then if this affects the principal’s payoffs, one group is getting the item in period $t$ and the other is not. Furthermore, since the low type is selecting into both groups, the price charged must be $b$. For the high type to be indifferent between these two allocations, the expected discounted sum of the low type’s consumption must be 1 starting from period $t$. But any such allocation cannot be incentive compatible, because it implies that in the future the high type would have to be charged a price lower than $b$. □

Lemma 3.1 Suppose that $\delta > 0.5$. In any equilibrium of the spot contracting game, the seller cannot make sales in such a way that if beliefs are initially higher than $\mu_{i,0}$, then beliefs are strictly below the value $\mu_{i,0}$ in period $t = T - a_i + 1$. If $\mu_t \in [\mu_{i,0}, \mu_{i+1,0})$ in period $t = T - a_i + 1$, the principal sells to only high types in that period, leading to a posterior of $\mu_{i-1,0}$.

Proof Notice that if $\mu_{i+1} < \mu_{1,0} = 1 - \left(\frac{b-b}{b}\right) = \frac{b}{b}$, the commitment outcome is for the principal to sell to only low types in all future periods. Thus, if the seller attempts to make sales to a portion of the high types at time $t$ in a way which results in posteriors being below $\mu_t$, the price she would have to charge must be less than

$$b - \left(b - \frac{b}{b}\right) \sum_{\tau=1}^{T-t} \delta^\tau.$$
However, $a_1$ is defined such that if $t = T - a_1 + 1$, then

$$\sum_{\tau=1}^{T-t} \delta^\tau > 1,$$

implying that the price charged must be less than $\bar{b}$, which is a contradiction. Furthermore, note that if posteriors are equal to $\bar{b}$ in this period, the principal will separate the agents, selling to all the high types, leading to a posterior of 0 in period $t - a_i + 2$.

Now suppose that in equilibrium, posterior beliefs must be at least

$$1 - \left( \frac{\bar{b} - b}{b} \right)^{i-1}$$

in period $T - a_i - 1 + 1$, and that in that period, if beliefs are between $1 - \left( \frac{\bar{b} - b}{b} \right)^{i-1} \leq \mu_{T-a_i+1} < 1 - \left( \frac{\bar{b} - b}{b} \right)^i$ the seller separates the agents, with the high type randomizing such that beliefs in $t = T - a_i - 1 + 2$ are equal to

$$1 - \left( \frac{\bar{b} - b}{b} \right)^{i-2},$$

and that in some previous period, they sold to only high types. Define $t$ as the last date of separation before period $T - a_i - 1 + 1$. To be worth it to separate and sell in period $t$, beliefs in period $t$ must be at least

$$1 - \left( \frac{\bar{b} - b}{b} \right)^i,$$

so if beliefs are less than this, she doesn’t sell. Thus, if beliefs are below

$$1 - \left( \frac{\bar{b} - b}{b} \right)^i,$$

in period $T - a_i + 1$, and in some period $t < T - a_i + 1$ before this the principal sold only to high types, by the inductive hypothesis the price that the principal charged in period $t$ must be no
higher than
\[
\bar{b} - (\bar{b} - \bar{b}) \left[ \left( \sum_{t=1}^{T-a_i} \delta^t \right) - \left( \sum_{j=1}^{i-1} \delta^{T-a_i+1-j} \right) \right],
\]
otherwise the high type would strictly prefer to pretend to be a low type. But the definition of \(a_i\) is such that if \(t < T - a_i + 1\), this price is less than \(\bar{b}\), which is impossible. Thus, beliefs cannot be strictly \(1 - (\bar{b} - b)^i\) before \(T - a_i + 1\). Furthermore, given that beliefs must be at least
\[
1 - (\frac{\bar{b} - b}{b})^{i-1},
\]
in period \(T - a_i - 1 + 1\), it’s easy to check that in period \(T - a_i + 1\) if beliefs are between
\[
1 - (\frac{\bar{b} - b}{b})^i
\]
and
\[
1 - (\frac{\bar{b} - b}{b})^{i+1}
\]
that it’s optimal to sell to only high types in that period, and then to all buyers at a price of \(b\) between \(T - a_i + 1\) and \(T - a_i - 1 + 2\). □

**Lemma 3.A.3** If beliefs are \(\mu_t = \mu_{i,j}\), the seller is indifferent between making sales to high types in such a way that beliefs follow the pattern \(\mu_{t+1} = \mu_{i,j-1}\), \(\mu_{t+2} = \mu_{i,j-2}\), etc., and \(\mu_{t+1} = \mu_{i,j-2}\), \(\mu_{t+2} = \mu_{i,j-3}\), etc.

**Proof** Notice that beliefs in period \(t + j + 1\) are the same, so profits from these two strategies that come from periods \(t + j + 1\) onwards are the same. Thus the \(\mu_t\) which makes the seller
indifferent is exactly the one which satisfies

\[
\frac{(1 - \mu_i)(\mu_i - \bar{\mu}_{i,j-1})}{(1 - \mu_i)(1 - \bar{\mu}_{i,j-1})} (\bar{b} + \delta \bar{b} + \ldots + \delta^{i-1} \bar{b}) + \frac{(1 - \mu_i)(\mu_{i,j-1} - \bar{\mu}_{i,j-2})}{(1 - \mu_i)(1 - \bar{\mu}_{i,j-2})} (\delta \bar{b} + \ldots + \delta^{i-1} \bar{b})
\]

\[
+ \frac{(1 - \mu_i)(\mu_{i,1} - \bar{\mu}_{i,0})}{(1 - \mu_i)(1 - \bar{\mu}_{i,0})} \delta^{i-1} \bar{b}
\]

\[
= \frac{(1 - \mu_i)(\mu_i - \bar{\mu}_{i,j-2})}{(1 - \mu_i)(1 - \bar{\mu}_{i,j-2})} (\bar{b} + \delta \bar{b} + \ldots + \delta^{i-2} \bar{b} + \delta^{i-1} \bar{b}) + \ldots
\]

\[
+ \frac{(1 - \mu_i)(\mu_{i,1} - \bar{\mu}_{i,0})}{(1 - \mu_i)(1 - \bar{\mu}_{i,j-2})} (\delta^{j-2} \bar{b} + \delta^{j-1} \bar{b}) + \left[ (1 - \mu_i) + \frac{\bar{\mu}_{i,0}(1 - \mu_i)}{1 - \bar{\mu}_{i,0}} \right] \delta^{i-1} \bar{b}.
\]

We can use the definition that

\[
\bar{\mu}_{i,j} = \left( \frac{\bar{b} - b}{b} \right)^{i} \bar{\mu}_{j} - \left( \frac{\bar{b} - b}{b} \right)^{i} + 1,
\]

to substitute, getting

\[
\mu_i - \left( \frac{\bar{b} - b}{b} \right)^{i} \bar{\mu}_{j-1} + \left( \frac{\bar{b} - b}{b} \right)^{i} - 1
\]

\[
\left[ \left( \frac{\bar{b} - b}{b} \right)^{i} - \left( \frac{\bar{b} - b}{b} \right)^{i} \bar{\mu}_{j-1} \right]
\]

\[
(\bar{b} + \delta \bar{b} + \ldots + \delta^{i-1} \bar{b}) + \left[ \left( \frac{\bar{b} - b}{b} \right)^{i} - \left( \frac{\bar{b} - b}{b} \right)^{i} \bar{\mu}_{j-1} \right]
\]

\[
(\bar{b} + \delta \bar{b} + \ldots + \delta^{i-1} \bar{b}) + \left[ \left( \frac{\bar{b} - b}{b} \right)^{i} - \left( \frac{\bar{b} - b}{b} \right)^{i} \bar{\mu}_{j-1} \right]
\]

\[
(\bar{b} + \delta \bar{b} + \ldots + \delta^{i-1} \bar{b}) + \left[ \left( \frac{\bar{b} - b}{b} \right)^{i} - \left( \frac{\bar{b} - b}{b} \right)^{i} \bar{\mu}_{j-1} \right]
\]

\[
\left[ \left( \frac{\bar{b} - b}{b} \right)^{i} - \left( \frac{\bar{b} - b}{b} \right)^{i} \bar{\mu}_{j-1} \right]
\]

\[
(\bar{b} + \delta \bar{b} + \ldots + \delta^{i-1} \bar{b}) + \left[ \left( \frac{\bar{b} - b}{b} \right)^{i} - \left( \frac{\bar{b} - b}{b} \right)^{i} \bar{\mu}_{j-1} \right]
\]

\[
\left[ \left( \frac{\bar{b} - b}{b} \right)^{i} - \left( \frac{\bar{b} - b}{b} \right)^{i} \bar{\mu}_{j-1} \right]
\]

\[
(\bar{b} + \delta \bar{b} + \ldots + \delta^{i-1} \bar{b}) + \left[ \left( \frac{\bar{b} - b}{b} \right)^{i} - \left( \frac{\bar{b} - b}{b} \right)^{i} \bar{\mu}_{j-1} \right]
\]

\[
[1 + \frac{1}{1 - \left( \frac{\bar{b} - b}{b} \right)^{i}} + \left( \frac{\bar{b} - b}{b} \right)^{i} \bar{\mu}_{0}] \delta^{i-1} \bar{b}
\]

\[
[1 + \frac{1}{1 - \left( \frac{\bar{b} - b}{b} \right)^{i}} + \left( \frac{\bar{b} - b}{b} \right)^{i} \bar{\mu}_{0}] \delta^{i-1} \bar{b}
\]
which simplifies to

\[
\begin{align*}
\mu_t - \left( \frac{\bar{b} - b}{b} \right)^i \bar{\mu}_{j-1} + \left( \frac{\bar{b} - b}{b} \right)^i - 1 \\
\frac{(1 - \mu_t)(1 - \bar{\mu}_{j-1})}{(1 - \mu_t)(1 - \bar{\mu}_{j-1})} (\bar{b} + \delta \bar{b} + \cdots + \delta^{j-1} \bar{b})
\end{align*}
\]

\[
\mu_t - \frac{\bar{\mu}_{j-2}}{(1 - \bar{\mu}_{j-1})(1 - \bar{\mu}_{j-2})} (\delta \bar{b} + \cdots + \delta^{j-1} \bar{b}) + \cdots + \frac{\bar{\mu}_1 - \mu_0}{(1 - \bar{\mu}_0)(1 - \mu_0)} \delta \bar{b}
\]

\[
\mu_t - \frac{\bar{\mu}_{j-2}}{1 - \mu_t} (\delta^{j-2} \bar{b} + \delta^{j-1} \bar{b}) + \delta^{j-1} \bar{b}
\]

Notice that from the definitions that if there exists a \( \mu_t \) such that

\[
\begin{align*}
\mu_t - \left( \frac{\bar{b} - b}{b} \right)^i \bar{\mu}_{j-1} + \left( \frac{\bar{b} - b}{b} \right)^i - 1 \\
\frac{(1 - \mu_t)(1 - \bar{\mu}_{j-1})}{(1 - \mu_t)(1 - \bar{\mu}_{j-1})} (\bar{b} + \delta \bar{b} + \cdots + \delta^{j-1} \bar{b})
\end{align*}
\]

\[
\mu_t - \frac{\bar{\mu}_{j-2}}{1 - \mu_t} (\delta^{j-2} \bar{b} + \delta^{j-1} \bar{b}) + \delta^{j-1} \bar{b}
\]

then by the definitions employed in Hart and Tirole (1988), the equality is satisfied. But it’s exactly

\[
\mu_t = 1 - \left( \frac{\bar{b} - b}{b} \right)^i \bar{\mu}_j
\]

which does this, thus \( \mu_{i,j} \) is the belief at which the monopolist will be indifferent between selling for \( j \) or \( j - 1 \) periods before carrying out the remainder of the sequence.

**Proposition 3.3** Suppose that \( \delta > \frac{1}{2} \), and that \( \mu_t \in [\mu_{i,0}, \mu_{i+1,0}) \). The equilibrium of the spot contracting setting is generically unique and takes the following form:

1. If \( t < T - a_i + 1 \), then the principal sells to both the high types and the low types with probability one, the price charged is \( b \) and the posterior \( \mu_{t+1} \) is equal to the current beliefs.
2. If \( t \geq T - a_i + 1 \), the principal sells only to high types, the price charged is the one which
makes high types indifferent between buying and waiting for the low type’s allocation, and the posterior \( \mu_{t+1} \in M_{t+1} \).

**Proof** The first point is an immediate implication of 3.1. The second statement is obviously true in period \( T \): the seller makes sales at a price of \( \bar{b} \) if an only if beliefs are higher than \( \mu_{1,0} = \frac{b}{b} \).

The proof for periods \( t = 2, \ldots, T - 1 \) will use backwards induction, and will be by contradiction in 2 main cases: when \( \mu_{t+1} = \mu_t \) and when \( \mu_{t+1} \neq \mu_t \).

Case 1: Suppose that the inductive hypothesis is true, that \( \mu_t \neq \mu_{t-1} \), and that \( \mu_{t+1} = \mu_t \), but that \( \mu_t \notin M_t \). It must either be the case that \( \mu_t \in M_{t+1} \) if the monopolist is going to sell to only high types at some point in the future, or that \( \mu_t = 0 \), otherwise the principal could sell strictly more in period \( t - 1 \), strictly increasing profits. If \( \mu_{t+1} = \mu_{i,0} \), then from the definition of \( M_t, \mu_t \leq T - a_{t+1} + 1 \) since \( \mu_{t+1} \notin M_t \). However, this violates incentive compatibility because it violates Lemma 3.1. If, on the other hand, \( \mu_{t+1} = \mu_{i,j} \) for \( j > 0 \), then by the inductive hypothesis the monopolist will sell to only high types for the next \( j \) periods, such that posteriors follow the path \( \mu_{t+1} = \mu_{i,j}, \mu_{t+2} = \mu_{i,j-1} \), et cetera. Again, since \( \mu_{t+1} \in M_{t+1} \) but \( \mu_t \notin M_t \), it must be the case that \( T - t - a_{t+1} + 1 = j \). However, this implies that selling only to high types in periods \( t, t+1, \ldots, t+j-1 \) (such that posteriors are \( \mu_t = \mu_{i+1,j-1}, \ldots, \mu_{t+j-1} = \mu_{i+1,0} \)) is feasible. This plan sells only to high types in the same periods, and beliefs are always weakly less in the in the new setting, implying that the sales process is separating out the high types earlier, and earning higher profits. This is a contradiction.

Case 2: Suppose that the inductive hypothesis is true, that \( \mu_t \neq \mu_{t-1} \), and that \( \mu_{t+1} \neq \mu_t \), and is instead some \( \mu_{i,j} \in M_{t+1} \). I’ll show that it must be the case that \( \mu_t = \mu_{i,j+1} \). and from the definition of \( M_t \) and \( M_{t+1} \), it must be the case that \( \mu_{i,j+1} \in M_{t+1} \). Suppose that \( \mu_t < \mu_{i,j+1} \). Then given Lemma 3.A.3, we know the the principal would be made strictly better off by separating the high types for \( j \) periods, rather than \( j + 1 \) periods. Since selling for \( j \) periods starting from period \( t + 1 \) is incentive compatible by assumption, it must then be the case that \( t + j = T - a_{t+1} + 1 \).

But this then means that selling for \( j - 1 \) periods starting from periods \( t + 1 \) is also incentive compatible. Since \( \mu_{i,j-1} < \mu_{i,j} \), carrying this out leads to more sales to the high types earlier, which strictly improves the monopolists payoffs. Finally, suppose that \( \mu_t > \mu_{i,j+1} \). Then the
principal could have sold to strictly more high types in period $t = 1$, again moving payoffs earlier and increasing profits. Thus, it must be the case that $\mu_t \in M_t$. □

**Proposition 3.4** In both the spot contracting and the renegotiation settings and for any $\pi \in (v, \mu \bar{b})$, for $\delta$ high enough there exists an equilibrium of the repeated game in which the principal receives $\pi$.

**Proof** Propositions 3.2 and 3.3 showed that for any collection of parameters, there is only one possible value of the payoffs which arise from the one shot version of either of these games.

Consider the continuation equilibrium starting from period $kT + 1$ (when $k \in \mathbb{N}$) which is worst for the seller. Because this is an equilibrium and the buyers are only active from $kT + 1$ to $(k + 1)T$, they must be best responding during this period. But because it’s the worst continuation equilibrium, there is no way to punish the seller if she deviates so she must be best responding as well. Thus, actions must follow an equilibrium of the one-shot game, of which there is only one. Thus, it suffices to show that as the discount factor approaches 1 that the payoffs from static versions of both the spot contracting and renegotiation games approach $v$.

Notice that as the discount factor approaches 1, $\bar{\mu}_t$ approaches $\mu_{t,0}$. Consider the $n$ such that $1 - \left(\frac{\bar{b} - b}{b}\right)^n \leq \mu < 1 - \left(\frac{\bar{b} - b}{b}\right)^{n+1}$. In the game with renegotiation, when $n < T$, then the principal will sell only to high types for the first $n$ periods, followed by selling to everyone. Some algebra shows that the mass of high types that the principal hasn’t yet identified as high types at the beginning of period $k$ is

$$(1 - \mu) \left[ \left( \frac{\bar{b}}{b - \bar{b}} \right)^{n+1-k} - 1 \right].$$

and the profits that the principal earns from the sales in that period are

$$\bar{b} - \left[ (1 - \mu) + (1 - \mu) \left[ \left( \frac{\bar{b}}{b - \bar{b}} \right)^{n-k} - 1 \right] \right] \bar{b}$$

Thus, adding these profits to the $T - n$ periods of $\bar{b}$ profits and dividing by $T$ to get the per-period
average, one obtains

$$\frac{T-n}{T} \bar{b} + \frac{n}{T} \bar{b} - \frac{1}{T} (1 - \mu) \bar{b} \sum_{t=1}^{n} \left( \frac{\bar{b}}{b-\bar{b}} \right)^{t-1}$$

Next, suppose that $n \geq T$. Again, the mass of high types that the principal hasn’t yet identified as high types at the beginning of period $k$ is

$$(1 - \mu) \left[ \left( \frac{\bar{b}}{b-\bar{b}} \right)^{T-k+1} - 1 \right],$$

so profits are

$$\bar{b} - \frac{1}{T} (1 - \mu) \bar{b} \sum_{t=1}^{T} \left( \frac{\bar{b}}{b-\bar{b}} \right)^{t-1}.$$

Consider instead the spot contracting game. Since the discount factor can be arbitrarily close to 1, $a_i = i + 1$, and $\mu_{i,j} = \mu_{i',j'}$ if and only if $i + j = i' + j'$. Thus, if $1 - \left( \frac{b-b}{b} \right)^n \leq \mu < 1 - \left( \frac{b-b}{b} \right)^{n+1}$ and $n < T$, the seller will sell to only high types in periods $T - n - 1$ to $T - 1$. Furthermore, posteriors will follow the same pattern as the renegotiation game above, only shifted back $T - n - 1$ periods, and the principal be earning $\bar{b}$ profits until she starts separating the agents. As the discount factor gets arbitrarily close to 1, the timing of these periods in which the principal sells only to high types doesn’t matter, and the seller earns the same profits as in the renegotiation case. Furthermore, when $n \geq T$, the profits and allocations from the two settings is exactly the same.

Thus, in both settings the payoffs from the worst equilibrium for the seller are equal to $\nu$, and when the discount factor is high enough any payoffs between $\nu$ and the maximum payoffs, $\mu \bar{b}$, can be sustained by reverting to the static equilibrium if the principal deviates. □
References


