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Structural Optimization of Large-Scale Floating Runways Using a Floating-Mat Hydrodynamic Model

By

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B.S. (Shanghai Jiao Tong University) 1992
M.S. (Shanghai Jiao Tong University) 1995

A dissertation submitted in partial satisfaction of the requirement of the degree of

Doctor of Philosophy

in

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in the

Graduate Division

Of the

University of California, Berkeley

Committee in charge:

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by

Jian Ma
Abstract

Structural Optimization of Large-Scale Floating Runways Using a Floating-Mat Hydrodynamic Model

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Jian Ma

Doctor of Philosophy in Ocean Engineering

University of California, Berkeley

Professor William C. Webster, Chair

The optimization of the strength distribution of the floating runway under wave loads was explored and the techniques needed to make this optimization possible were developed.

The "Hydrodynamic Influence Matrix" was defined to capture the hydrodynamic properties of the floating structure and to incorporate these into the motion equations of the floating structure. Great effort was made to improve the accuracy of the Hydrodynamic Influence Matrices for short waves.

A low-order panel method was developed first to compute the Hydrodynamic Influence Matrices, which were then incorporated into the finite-difference form of the motion equations for the floating-mat model. The spectra of the responses of the floating runway were computed by discretizing the incident wave spectrum into elemental waves and solving the finite-difference equations for each elemental wave. The low-order method required a huge number of variables to treat a realistic floating runway and the resulting numerical accuracy of the computation was insufficient.
A high-order method was then developed to improve the accuracy in the hydrodynamics computation for short waves. Double fifth order interpolation functions were used to represent the source distribution on a high-order panel. Analytically exact formulations and approximate formulations using asymptotic series were developed and combined to compute the singular integrations involving the Rankine part of the Green function and its gradient. Gaussian quadrature was used to compute the integrations involving the remnant part of the Green function. The high-order Hydrodynamic Influence Matrices were then computed with acceptable accuracy for waves with wavelength close to the panel dimension and were incorporated into the high-order motion equations for the floating-mat model. These were solved using a Galerkin method to get the responses of the floating runway in a wave system.

The optimization for the strength distribution of the floating runway was carried out based on consideration of the structural reliability using gradient projection method. Preliminary optimization results for various model structures and the full-size floating runway have been achieved. For the small model structures local optima satisfying the Kuhn-Tucker condition were found. For the large model structure and the full size floating runway preliminary optimized results with reduced total weight were presented as a reference for future work.

August 12, 2003

Professor William C. Webster

Date
Table of Contents

List of Figures iii

List of Tables iv

Acknowledgement vi

1. Introduction 1

2. Low-Order Method 13
   2.1 Background 13
   2.2 Lower-Order Panel Method 17
   2.3 Radiation and Diffraction Problems 22
       2.3.1 Radiation Problem 23
       2.3.2 Diffraction Problem 31
   2.4 Low-Order Method Motion Equation 49
   2.5 Structural Responses in Wave Systems 59

3. High-Order Method 70
   3.1 The Interpolation Functions 70
       3.1.1 The 36-Parameter Flat Bi-Fifth-Order Interpolation Functions 70
   3.2 The High-Order Hydrodynamic Influence Matrices 79
   3.3 The Integration of the Singularities in the Green Function and its Gradient 84
       3.3.1 Analytically Exact Formulations 84
       3.3.2 Asymptotic Approximation Formulations 96
   3.4 The Computation of the High-Order Hydrodynamics Influence Matrices 127
   3.5 The Radiation and Diffraction Problems 130
       3.5.1 Radiation Problem 131
       3.5.2 Diffraction Problem 145
   3.6 The High-Order Motion Equation and Structural Responses 155

4. Structural Optimization 162
   4.1 The Optimization Problem 162
   4.2 The Optimization Process 168
   4.3 The Optimization Results 173

5. Conclusions and Discussions 184
References 188

Appendix I General Higher-Order Interpolation Functions 191

Appendix II Bi-Fifth-Order Basis Functions 194
   II.1 The Expressions of the 36 Panel Basis Functions 194
   II.2 The Shape of the 36 Panel Basis Functions 205
   II.3 The Shape of Some of the Nodal Basis Functions 212
      II.3.1 Nodal Basis Functions at the Center Area of the Mesh 212
      II.3.2 Nodal Basis Functions at Nodes Located on the Edges of the Mesh 217

Appendix III Analytical Formulae for the Integrations of the Potential and its Gradient of Monomial Source Distribution on a Rectangular Panel 222
   III.1 Analytical Formulae for $I'_{i,j,n}, i,j=0,1,...,5$ 223
   III.2 Analytical Formulae for $K'_{x(i,n)}, i,j=0,1,...,5$ 252
   III.3 Analytical Formulae for $K'_{y(i,n)}, i,j=0,1,...,5$ 278
   III.4 Analytical Formulae for $K'_{z(i,n)}, i,j=0,1,...,5$ 304

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List of Figures

Figure 2.4.1 Typical Repeating Unit of the Structure 51  
Figure 2.5.1 ISSC Point Wave Spectrum with 63  
\[ H_{1/3} = 3.99 \text{ m}, T_1 = 10.1 \text{ sec}, \quad 0.3 \leq \omega < 2.0 \text{ (rad/sec)} \]  
Figure 2.5.2 Directional ISSC Wave Spectrum 64  
Figure 3.3.1 Plots of Integrals \{I_1, K_{1x}, K_{1y}, K_{1z}\} verses Distance (log_{10} R) 114  
Computed from Analytically Exact and Asymptotic Approximation Formulae -------Test 1  
Figure 3.3.2 \( \log_{10} - \log_{10} \) Plots of Integrals \{I_1, K_{1x}, K_{1y}, K_{1z}\} verses 118  
Distance R Computed from Analytically Exact and Asymptotic Approximation Formulae -------Test 2  
Figure 3.3.3 Plots of Integrals \{I_1, K_{1x}, K_{1y}, K_{1z}\} verses Distance (log_{10} R) 122  
Computed from Analytically Exact and Asymptotic Approximation Formulae -------Test 1  
Figure 3.3.4 Plots of Integrals \{I_1, K_{1x}, K_{1y}, K_{1z}\} verses Distance (log_{10} R) 126  
Computed from Analytically Exact and Asymptotic Approximation Formulae -------Test 2  
Figure 3.6.1 Deflection of the Full-Size Floating Runway in Incident Regular Waves of wavelength \( \lambda = 415.01 \text{ meters} \) and heading angle \( \alpha = 17.07^\circ \) 161  
Figure 4.3.1 Variation of \( \lambda_{max} \) with \( ar(i,j)'s \) \( (H_{1/3} = 4 \text{ (meter)}, \ T=10 \text{ (sec.)}, \) dominant wave direction 30\(^\circ\) 174  
Figure 4.3.2 Optimum variation of \( ar(i,j)'s \) over the platform \( (H_{1/3} = 4 \text{ (meter)}, \ T=10 \text{ (sec.)}, \) dominant wave direction 30\(^\circ\) 175  
Figure 4.3.3 Optimum variation of \( ar(i,j)'s \) over the platform \( (H_{1/3} = 4 \text{ (meter)}, \ T=10 \text{ (sec.)}, \) dominant wave direction 60\(^\circ\) 176  
Figure 4.3.4 Optimized Distribution of Thickness Parameters \( (2 \times 2 \text{ mesh}, \ 100 \times 100 \times 2 \text{ meters model}) \) 179  
Figure 4.3.5 Optimized Distribution of Thickness Parameters \( (4 \times 4 \text{ mesh}, \ 100 \times 100 \times 2 \text{ meters model}) \) 180  
Figure 4.3.6 Optimized Distribution of Thickness Parameters \( (8 \times 8 \text{ mesh}, \ 100 \times 100 \times 2 \text{ meters model}) \) 180  
Figure 4.3.7 Change of the Total Weight with Panel Numbers of the Mesh \( (1: 2 \times 2 \text{ mesh}, \ 2: 4 \times 4 \text{ mesh}, \ 3: 8 \times 8 \text{ mesh}, \ 100 \times 100 \times 2 \text{ meters model}) \) 181  
Figure 4.3.8 Preliminary Optimized Distribution of Thickness Parameters \( (20 \times 2 \text{ mesh}, \ 2000 \times 200 \times 2 \text{ meters model}) \) 182  
Figure 4.3.9 Preliminary Optimized Distribution of Thickness Parameters \( (40 \times 4 \text{ mesh}, \ 4000 \times 400 \times 2 \text{ meters full-size floating runway}) \) 183
List of Tables

Table 2.2.1  Values of the Sign Coefficients  18
Table 2.3.1  The Symmetry of the Canonical Motions  24
Table 2.3.2  Comparison of the Added-Mass and Damping Coefficients  27

Computed by HYDRO and WAMIT, for 80*40*20m model, wavelength $\lambda_1 = 415.008 (\text{Meter})$

Table 2.3.3  Comparison of the Added-Mass and Damping Coefficients  28

Computed by HYDRO and WAMIT, for 80*40*20m model, wavelength $\lambda_2 = 47.811 (\text{Meter})$

Table 2.3.4  Comparison of the Added-Mass and Damping Coefficients  29

Computed by HYDRO and WAMIT, for 1000*100*2m model, wavelength $\lambda_3 = 1600 (\text{Meter})$

Table 2.3.5  Comparison of the Added-Mass and Damping Coefficients  30

Computed by HYDRO and WAMIT, for 4000*400*2m model, wavelength $\lambda_3 = 1600 (\text{Meter})$

Table 2.3.6  Comparison of the Diffraction Forces Computed by HYDRO and WAMIT (80*40*20m model, wavelength $\lambda_1 = 415.008 (\text{Meter})$, incident angles 0°,19°,46°)  39

Table 2.3.7  Comparison of the Diffraction Forces Computed by HYDRO and WAMIT (80*40*20m model, wavelength $\lambda_2 = 47.811 (\text{Meter})$, incident angles 0°,19°,46°)  42

Table 2.3.8  Comparison of the Diffraction Forces Computed by HYDRO and WAMIT (1000*100*2m model, wavelength $\lambda_3 = 1600 (\text{Meter})$, incident angles 0°,19°,46°)  45

Table 2.3.9  Comparison of the Diffraction Forces Computed by HYDRO and WAMIT (4000*400*2m model, wavelength $\lambda_3 = 1600 (\text{Meter})$, incident angles 0°,19°,46°)  48

Table 2.4.1  Rigidities of the Orthotropic Plate Modeling the Preliminarily Designed

Table 2.5.1  Dividing of the 2-D Wave Spectrum in the Frequency Domain  66
Table 2.5.2  Dividing of the 2-D Wave Spectrum in the Direction Domain  67
Table 3.1.1  Zero and Non-Zero Modes for the Center Node (Intersection of X and Y Symmetry Axes)  75
Table 3.1.2  Zero and Non-Zero Modes for Off-Center Nodes on X Symmetry Axis  76
Table 3.1.3  Zero and Non-Zero Modes for Off-Center Nodes on Y Symmetry Axis  77
Table 3.3.1  Input Data of Test 1  106
Table 3.3.2  Input Data of Test 2  107
Table 3.5.1  Frequencies, Wavelengths, Periods and Panel Size Ratios Used to  134
| Table 3.5.2 | Comparison of the Added-Mass and Damping Coefficients for the model 1000m*100m*2m Computed by HOHYD and HYDRO, for 1000*100*2m model, wavelength \( \lambda = 20.000 \) (Meter) |
| Table 3.5.3 | Comparison of the Added-Mass and Damping Coefficients for the model 1000*100*2m model, wavelength \( \lambda = 22.000 \) (Meter) |
| Table 3.5.4 | Comparison of the Added-Mass and Damping Coefficients for the model 1000*100*2m model, wavelength \( \lambda = 25.000 \) (Meter) |
| Table 3.5.5 | Comparison of the Added-Mass and Damping Coefficients for the model 1000*100*2m model, wavelength \( \lambda = 27.000 \) (Meter) |
| Table 3.5.6 | Comparison of the Added-Mass and Damping Coefficients for the model 1000*100*2m model, wavelength \( \lambda = 30.000 \) (Meter) |
| Table 3.5.7 | Comparison of the Added-Mass and Damping Coefficients for the model 1000*100*2m model, wavelength \( \lambda = 40.000 \) (Meter) |
| Table 3.5.8 | Comparison of the Added-Mass and Damping Coefficients for the model 1000*100*2m model, wavelength \( \lambda = 50.000 \) (Meter) |
| Table 3.5.9 | Comparison of the Added-Mass and Damping Coefficients for the model 1000*100*2m model, wavelength \( \lambda = 75.000 \) (Meter) |
| Table 3.5.10 | Comparison of the Added-Mass and Damping Coefficients for the model 1000*100*2m model, wavelength \( \lambda = 100.000 \) (Meter) |
| Table 3.5.11 | Comparison of the Added-Mass and Damping Coefficients for the model 1000*100*2m model, wavelength \( \lambda = 1600.000 \) (Meter) |
| Table 3.3.12 | Comparison of the Diffraction Forces Computed by HYDRO and HOHYD (1000*100*2m model, wavelength \( \lambda = 1600.000 \) (Meter), incident angles 0°, 19°, 46°) |
| Table 3.3.13 | Comparison of the Diffraction Forces Computed by HYDRO and HOHYD (1000*100*2m model, wavelength \( \lambda = 100.000 \) (Meter), incident angles 0°, 19°, 46°) |
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Chapter 1

Introduction

One important aspect of the so called “Blue Revolution” — the development of the ocean resources — is the utilization of the ocean space, in which various floating structures are involved as in the other aspects of the revolution, especially the Very Large Floating Structures (VLFS’s) (Patrick K. Takahashi and R. Cengiz Ertekin, 1996). VLFS’s are floating structures with a typical dimension at least one order of magnitude larger than the largest traditional floating structures, such as Very Large Crude Carriers (VLCCs), Floating Production, Storage and Offloading (FPSO) system, and so on. These structures have a typical dimension of several hundred meters, so a VLFS will normally have a size of several thousand meters.

In the field of ocean space utilization VLFS’s have many possible applications of either civil or military purposes, like huge oil storage containers, floating artificial islands, floating airports and mobile military bases in the ocean. VLFS’s in the coastal waters are an attractive alternative where land space is not easily available, which is always the case for congested coastal metropolitan areas. They also pose less environmental impact than other ways of utilizing the ocean space, such as land reclamation by using earth or garbage filling (Eiichi Isobe, 1999). VLFS’s in the open ocean, on the other hand, provide mankind with a base station in the remote ocean, either for civil or military purposes. They can be recreational facilities, or industrial manufacturing plants, which, if located outside the Exclusive Economic Zone (EEZ), bypass the domestic law and tax restrictions of any country (Patrick K. Takahashi and R.
Cengiz Ertekin, 1996). When used as military bases, they allow the operating country to deploy its military force all around the world's ocean surface, which gives access to most of the world's strategically important areas and countries (Gene Remmers et al, 1999).

Up to now a number of workshops and conferences have been held on the research of VLFS, among which are the International workshops on Very Large Floating Structures (1st at Hayama, Japan in November 1996, 3rd at Honolulu, USA in September 1999, and 4th at Tokyo, Japan in January 2003). VLFS's also appeared as a topic in other important international conferences on offshore engineering and fluid mechanics. Koichiro Yoshida (1996) and Patrick K. Takahashi and R. Cengiz Ertekin (1996) give a good chronology of these activities before 1996. In Japan, where researches on VLFS's are most intense, the Mega-Float project was initiated in 1995 and completed successfully in 2000 (Eiichi Isobe, 1999, Koichiro Yoshida, 2003). This huge project, including all the aspects about the design, construction, maintenance, operating and environment impact of a VLFS used as a floating runway, has been carried out by a technological research association of 17 companies with support from the government. By 2000, a floating runway, 1000 meters long and 60 meters wide, had been constructed and tested by the actual landing and taking off of small airplanes. This project generated great impact on both the VLFS technology and the public notion about VLFS's. In the USA, extensive research exploring the military usage of the VLFS, i.e. as a Mobile Offshore Base (MOB), was conducted by the Office of Naval Research starting July 1997 and achieved technological advances with both military and civil applications by 2002 (Gene Remmers et al, 1999, R.J. Taylor, 2003).
One important application of VLFS’s is for a runway capable of accommodating the landing and taking off of large commercial airplanes. Its attractiveness is a result of several coincident forces. First most metropolitan areas in the world are located near the ocean or lake coast. Secondly these highly populated cities are most likely to need more airport facilities in the future. Thirdly land space, especially the large amount needed for an airport is not easily available around these cities. Thus a floating airport in the nearby body of water becomes a reasonable alternative. It is often better than the land reclamation alternative for technical, economical and environmental reasons (Eiichi Isobe, 1999).

A VLFS serving as a commercial airplane runway will likely be a flat barge-like structure approximately 4000 meters long, 400 meters wide, and with a relatively very small depth, perhaps 5 meters or even smaller. This is the subject of our research here. Such a barge-like platform would be too large to construct in any current shipyard. As a result, many smaller barge modules will need to be pre-fabricated in shipyards and then towed to the site to be joined together to make a single, continuous unit.

Because of its large scale in the horizontal plane and relatively very small depth, a large-scale floating runway is quite flexible and its response to water waves will be very complicated (Webster 1998). Its own deformation will affect the wave pressure acting on itself and then affect the wave-induced motions of the structure. Actually this situation is considered implicit in the definition of VLFS (Patrick K. Takahashi and R. Cengiz Ertekin, 1996). The traditional methodology of treating a floating structure as a rigid body isn’t applicable for VLFS’s. Instead the methods of hydro-elasticity must be used to analyze the hydrodynamic and structural responses of a VLFS as a deformable body in
the waves. In the past research a floating mat model was used successfully in numerical analysis to find out some of the most important properties of the motions of such structures (Webster and Mamidipudi, 1994, Newman, 1996, Kashiwagi, 1998). Various methods, such as panel method, mode-expansion method and ray-tracing techniques, are used in the numerical analysis to compute the responses of this kind of structures in waves. In our research a 3-Dimension mesh of rectangular panels is used to compute the hydrodynamics properties of the structure.

In the first stage of our research a constant panel method for the hydrodynamics and a finite-difference calculation for the structural responses were used to compute the linear, wave-induced responses of a VLFS used as a floating runway. Hydrodynamic influence matrices were developed to capture the hydrodynamic properties of the structure once for all for a series of input waves of different frequencies (Webster, 1991, Webster and Mamidipudi, 1994, Webster 1998). Using this method, the hydrodynamic properties of a model structure 80 meters long, 40 meters wide, 5 meters deep and with a draft of 2 meters was computed and incorporated into the motion equation of the floating structure by means of the hydrodynamic influence matrices. Then the response spectrum of the structural deformations and stresses were computed from the input wave spectrum (Ma and Webster, 1999). Although this panel method has a clear physical basis and can easily incorporate the 3-D hydrodynamic properties of the floating structure in the motion equation, numerical difficulties prevented application to larger structures, where the wave length of the incoming wave is small compared to the dimensions of the structure itself.

A full size floating runway structure will likely be 4000 meters long, 400 meters wide and 5 meters deep, with a draft of 2 meters. It is very likely such floating runways will be
located in sheltered water close to the shore. It is necessary to compute accurately its
hydrodynamic influence matrices in waves as short as 50 meters, i.e. of waves one
eightieth of the structure length. That is, typical waves encountered by such structures
will be short waves of a length two orders of magnitude smaller than the length of the
structure itself.

In order to accurately compute the hydrodynamic influence matrices of a real-size
floating runway in short waves, a high-order numerical method based on boundary
integration panel method was then developed in our second stage of research to overcome
the computational difficulties. This high-order method is an extension of the first stage
research work using the constant-source panel method and uses a new high-order source
distribution over flat panels for computing the hydrodynamic pressure induced by general
vibration of floating structures. The same idea of a hydrodynamic influence matrix as
presented in (Webster 1979) is adopted. The high-order method uses double fifth-order
polynomials to interpolate the distributions of source density, velocity, displacement,
potential and pressure across the individual panels. In each rectangular panel 36 basis
functions are used to represent the distributions of any physical quantity. The property of
the double fifth-order polynomials ensures that the distribution of physical quantities on
the wetted surface of the runway will be continuous with first and second derivatives
across the panel boundaries.

Computational difficulties encountered in the boundary integration of the 1/R
Rankine singularities are overcome using both analytically exact and asymptotic
approximations. The fifth-order source panel method has proved to be more powerful
than low-order source panel methods in that many fewer panels, and most importantly,
fewer independent variables are needed to get the results with same accuracy as with the traditional constant source panel method. The result is high-order hydrodynamic influence matrices that are more numerically accurate and stable to incorporate the hydrodynamic properties of the structure into its motion equation. This high-order method can effectively capture the hydrodynamic properties of the real-size structure in waves two orders of magnitude shorter than the length of the structure using current computer facilities.

One important purpose of studying the hydrodynamics of the floating runway is to provide load data to be used in analyzing its structural strength. The structural design of the floating runway itself needs to consider two important problems: one is the landing and taking off of heavy commercial airplanes and the other is the structural integrity of the structure itself. In the landing and taking off of airplanes, the force on the landing gears of the airplanes will depend on the slope and curvature of the surface of the floating runway (Webster, 1991). If the deformations of the surface of the floating runway become too large under an incoming wave system, the high force on the landing gear of the airplane will damage the gear structure and cause tragic accidents. The wave-induced stresses in the flexible structure also pose a strength problem of the structure itself. The design of the mooring system also depends on the drift force data of the VLFS in waves, which is a non-linear effect of the incoming waves. So it's necessary to compute the response of the VLFS in wave system. In the research presented here, the optimal structural design of the floating runway itself only is concerned, so we only need to compute the responses in the flat barge-like structure, making use of its hydrodynamics properties computed beforehand.
The flat barge-like structure is considered as a mat floating on the water surface, which can have deflections in the vertical direction and these deflections will generate hydrodynamic, hydrostatic and structural dynamic and static forces on the structure. The hydrodynamics analysis carried out beforehand determines how the hydrodynamic forces will be generated by the deflections. The structural static forces can be determined from plate theory considering the flat floating structure as an equivalent thin plate. If the strength of the floating structure in longitudinal and transverse directions is different, then an orthotropic plate with different equivalent rigidities in the two directions can be used to model the structure.

The equations of motion for an orthotropic floating-mat have been given in many places (see, for instance, Webster and Mamidipudi, 1994, Kashiwagi, 1998). These equations will be discretized to form a linear equation system that can be solved by Gaussian Inversion, LUD or iterative techniques to get the vertical deflections. The method of discretizing depends on how the hydrodynamics properties are computed beforehand, i.e. by a low-order method or a high-order method.

The ultimate purpose of our research is to optimize the structural strength of the flat barge-like VLFS used as a floating runway. Optimization is listed as the last stage work in the flow chart for planning VLFS’s in (Koichiro Yoshida, 1996). But an optimization of the simple preliminary structure design will give an important view to the designer on how the structure should look like before he would make any fundamental design decisions, so it will be also very useful at the early stage of the design work of a VLFS.

Many of the previous studies of barge-kind VLFS platforms have assumed that the runway has structural scantlings that do not vary over the platform. Some of these studies
have been based on modeling the structure as an orthotropic plate with the structural characteristics different in the transverse and longitudinal directions, but with these characteristics invariant over the platform (see: Webster & Mamidipudi, 1994, Kashiwagi, 1998, Newman, 1996). In these studies, it was found that the largest motions and stresses of the platform occur around the edges of the platform.

It would seem plausible, then, to assume that if the magnitude of the motions varies from place to place on the platform, then the structural scantlings should vary in a similar fashion. Through wise design, strength and rigidity requirements might be satisfied using a lighter and more efficient structure tailored to the specific needs of the platform. (Webster, 1998) proposed a framework for the structural optimization problem necessary to accomplish this task and this framework forms the basis of the work presented here.

The cost of constructing such a VLFS may be several billion dollars, so even a one-percent saving through optimization would be very significant. The acquisition cost will mainly be comprised of two parts, the cost of the material (structural steel or concrete) and the cost of manufacture. However, the cost of manufacture is, in part, also related the cost of the material, since it is more expensive to erect thick plates than thin ones. Barge-like platforms are not complicated and thus one would expect that the material-related costs would dominate the overall acquisition cost. So our target of the structural optimization of such a VLFS is set as minimizing the weight of the structure, or the objective function in our optimization problem is the weight of the structure.

The optimization will be under the constraints of maintaining the safety and serviceability of the floating runway. When a VLFS is used as a commercial airport, it must be strong enough to survive the wave environment for a long period (for example
During such a long period, the structure should never be broken by any storm waves that may occur, this is the structural safety requirement. At the same time, the part of the structure used as the airplane runway should be smooth enough for large commercial airplanes to take off and land safely under most of the wave environment conditions. However, as long as the percent of time when airplane operations are not safe is very small, then the runway is still acceptable although at those times it will be out of service temporarily. This is the serviceability requirement of the structure. The constraint on the structural safety is necessarily more severe than that on the structural serviceability.

The structural rigidity and strength are computed on the basis of reliability. So the criteria of the constraints on structural rigidity and strength will be computed from the probability of failure of the structure when exposed to a directional seaway instead of the maximal values of the deformations and stresses (Webster, 1998).

The selection of the design variables will greatly affect or even determine the complexity of the optimization problem. The structural design of the flat barge-like floating runway will take the pattern of a flat long rectangular box, or pontoon, including a strong plane deck plate forming the runway surface, a relatively weaker plane bottom plate, water-tight side plates and webs, girders running between the deck and the bottom and stiffeners on these plates. The structural behavior will like that of an orthotropic combined plate. There are many parameters of the structure that can be selected as the design variable. Since our objective is to find an optimal global design of the structure, we won’t select local details, like the size and shape of stiffeners, lightening holes and brackets on the webs and girders, as the design variables. So we will select only among
parameters like the depth of the structure, the plate thickness of the components in the combined plate and the spacing of the webs and girders.

If we select the depth of the structure as the design variable, the optimization problem will be very complicated because the change in the depth of the structure will affect its hydrodynamic properties and the optimization problem and the hydrodynamic problem will be coupled tightly. Since there may need many steps to reach a local optima and for each step the hydrodynamic influence matrices need to be re-calculated, the computing task will be prohibitively heavy. Also, because a floating runway with a variable draft will be much more expensive to construct, a structure with uniform draft over most of its area will more likely be adopted. Thus, the depth of the structure will be taken to be uniform and pre-determined from design considerations other than structural rigidity and strength. Furthermore, changing the relative proportion of the plate thickness and changing the spacing of the webs and girders will also make the optimization very complicated because they will affect the structural strength in a complicated non-linear way. These parameters are also more likely be determined by design loads such as loads from landing and taking-off of airplanes, the buckling resistance of the webs and girders. That is, the overall geometry of the combined structure will be pre-determined.

Then the structural rigidity and strength will be taken to depend linearly on a single parameter. The initial thickness of all of the structural plates will be multiplied by this parameter to determine the ultimate plate thickness after optimization. This parameter determines the distribution of the rigidities of the combined plate and we will select it as our design variable. At the end of the optimization the optimal distribution of this
thickness parameter, gives an optimal distribution of the rigidities of the combined plate, which will be a very useful input in the design of the floating runway.

The thickness parameter will be a discrete distribution for convenience of fabrication. It’s reasonable to suppose each of the pre-fabricated modules will have uniform thickness. So the number of design variables will be determined by the number of pre-fabricated modules that constitute the whole floating runway.

The minimum scantlings for all of the plates will be taken as those needed to satisfy the requirements like impact loads of taking off and landing airplanes, traffic loads of incoming and outgoing people and cargo, and static loading of the platform in calm water, and so on. Increases in the thickness above this minimum scantling arise from the need to provide the strength to resist the effects of seas that may impinge on the platform.

After all these reasonable simplifications the optimization problem is greatly simplified and can be solved using current available computer facilities to provide useful information for the designers of the structure. In our research optimization is carried out using the so-called “gradient projection” method. This method can reveal a local optimal design but there is no way to assure that this is also a global optimum. It is hoped, however, that it will reveal a better strength distribution for the VLFS under a certain seaway or set of seaways than the uniform distribution of scantlings normally assumed for these platforms.

In carrying out the above optimization with incorporated hydrodynamic effects, heavy computing task and large numerical errors are the two main obstacles encountered. Great efforts have been made to reduce the computing time into tolerable range and to improve accuracy of the numerical computation into acceptable range. In many cases
smaller models are used to verify the correctness of the procedure before the computation on the full-size structure is carried out.
Chapter 2

Low-Order Method

2.1 Background

The hydrodynamics of floating rigid body has been well studied in the framework of linear potential theory (J.V. Wehausen, 1971, J.N. Newman, 1977). The hydrodynamics of a VLFS used as a floating runway is different from that of a floating rigid body because it is deformable itself. This means the VLFS can have more basic modes of motions than the six canonical motions of a rigid body. In fact it can have an infinite number of modes. One method of computing the hydrodynamics of this kind of structure is by using the dry vibration modes of the structure to simulate the wet modes of the structure floating in waves and treat these modes in a similar way as the canonical modes. The research work using this kind of so-called mode-expansion method is introduced in Bishop and Price (1979) and Kashiwagi (2000). Another method involves first discretizing the wetted surface of the body into small panels and letting each panel has only some basic modes of motion. Through the basic modes of motion of each panel the global modes of motion of the VLFS can be simulated. In our research this method is used. In the low-order method the panels can only move in the normal direction as a rigid panel while in the high-order method the much more complex modes of motion of each panel are defined by the basis functions. A hydrodynamic influence matrix can be used to capture the hydrodynamic properties of the structure once for all for a certain wave frequency (Webster, 1991, Webster and Mamidipudi, 1994, Webster 1998). This matrix
can then be incorporated into the motion equation of the structure. So the purpose of hydrodynamics computation is to compute this hydrodynamic influence matrix for every frequency of the input wave.

The method used here is based on linear wave theory. The water depth is assumed to be infinite. A Cartesian coordinate system is set up as follows: the origin is located at the center point of the rectangular horizontal domain of the floating runway on the mean water plane, the x axis is along the longitudinal direction pointing to the bow and the y axis is along the transverse direction pointing to the starboard of the structure, and the z axis is along the vertical direction pointing down. Suppose the motion of the VLFS is small so that the flow around the floating runway can be considered a potential flow and all the incoming, diffracted and radiated waves can be described using Airy linear wave theory. The potential flow of every frequency can then be studied separately and superposed. The potential for the flow about the platform resulting from an incoming wave of frequency $\omega$ can be expressed as:

$$\varphi(x, y, z; \omega, t) = \varphi_0(x, y, z)e^{-j\omega t}$$  \hspace{1cm} (2.1.1)

where $j = \sqrt{-1}$, $t$ is the time and $\varphi_0(x, y, z)$ is the initial value. The potential satisfies

$$\nabla^2 \varphi(x, y, z; \omega, t) = 0$$

and the water particle velocity is given by

$$u(x, y, z; \omega, t) = \nabla \cdot \varphi(x, y, z; \omega, t).$$

Similarly other physical quantities, like the displacement of the body surface or deflection of the platform $d(x, y, z; \omega, t)$, the velocity of the platform surface $v(x, y, z; \omega, t)$, the source intensity $Q(x, y, z; \omega, t)$ and the pressure $p(x, y, z; \omega, t)$, will depend on the time $t$ through the $e^{-j\omega t}$ term. Since the motion is small, the oscillating
water particle is considered approximately located at its mean position. Then the relation between the displacement and velocity can be written as:

\[ v(x, y, z; \omega, t) = \frac{\partial d(x, y, z; \omega, t)}{\partial t} = -j \omega d(x, y, z; \omega, t) \quad (2.1.2) \]

The pressure caused by the waves can be determined from the potential through the Lagrange-Cauchy Integral:

\[ \frac{p}{\rho} + \frac{v^2}{2} + gz + \frac{\partial \phi}{\partial t} = f(t) \quad (2.1.3) \]

where \( g \) is the gravity acceleration, \( z \) is the elevation, \( \rho \) is the mass density of water, \( v \) is the water velocity, \( p \) is the pressure, \( f(t) \) is a function of time and is constant over the domain. Without loss of generality this constant can be set to zero. The term \( gz \) represents the static pressure. Under Airy linear wave assumption \( \frac{v^2}{2} \) is of higher order and can be neglected. The dynamic pressure caused by the wave can be computed from the wave potential as:

\[ p = -\rho \frac{\partial \phi}{\partial t} \quad (2.1.4) \]

The wave potential in (2.1.1) will be chosen so that it satisfies the Laplace Equation in the domain and the boundary conditions at the free surface already. It needs to satisfy the following boundary condition on the body surface:

\[ \vec{n} \cdot \nabla \phi = v_n \quad (2.1.5) \]

where \( \vec{n} \) is the normal vector of the body surface and \( v_n \) is the normal velocity of the body surface.
A source distribution on the body surface is used to simulate the effect of the presence of the body. The complex Green Function $G(x, y, z; \xi, \eta, \zeta; \omega)$ is defined as the potential at $\bar{x} = (x, y, z)$ due to a unit source which is located at position $\bar{\xi} = (\xi, \eta, \zeta)$ and vibrating with frequency $\omega$. This potential, or the Green Function $G(x, y, z; \xi, \eta, \zeta; \omega)$, satisfies the Laplace Equation in the domain, the boundary conditions on the free surface and the appropriate radiation conditions in the far field. Then the potential of a surface $S$ with a complex source distribution can be written as the integral:

$$
\varphi(x, y, z) = \int_{S} Q(\xi, \eta, \zeta)G(x, y, z; \xi, \eta, \zeta; \omega) dS
$$

(2.1.6)

This potential must satisfy the boundary condition (2.1.5) on the surface so we get another equation as follows:

$$
\nu_n(x, y, z) = \bar{n} \cdot \nabla \int_{S} Q(\xi, \eta, \zeta)G(x, y, z; \xi, \eta, \zeta; \omega) dS
$$

(2.1.7)

By using some discretizing method equation (2.1.7) gives a relation between the source distribution and the normal velocity of the body surface, and equation (2.1.6) gives a relation between the source distribution $Q(\xi, \eta, \zeta)$ and the potential generated by the body surface. From them, a relation between the normal velocity $\nu_n(x, y, z)$ and the potential $\varphi(x, y, z)$ can be found by canceling the source distribution $Q(\xi, \eta, \zeta)$ through substitution. Using (2.1.2) and (2.1.4) a direct relation between the pressure and the displacement of the body surface can be found, which can be expressed as the Hydrodynamic Influence Matrix. Different ways of discretizing the distributions on the surface characterize the different methods introduced in the following sections.
2.2 Low-Order Panel Method

The low-order panel method is described in (Webster, 1998) but more details are added here.

First since the geometry of the structure is symmetric about x and y axis, we can decompose the motion of the structure into a 4-fold symmetry (bow-stern, port-starboard). This allows us to decompose the hydrodynamic problem for the whole platform into four separate hydrodynamic problems for platforms one fourth of the size. Under each symmetry only one quadrant, here the first quadrant, of the structure needs to be discretized. The values of any physical quantities are defined independently only on this quadrant. On the other quadrants the values are determined according to the symmetry. In addition, for simplicity we assume that the structure strength also has this same four-fold symmetry (although there is nothing in the method that necessitates this assumption). As a result, we just need to solve the motion equations for one quadrant of the plate but we need to solve them for four times, each time for one of the four symmetries.

Suppose the length of the floating runway is L, its width is B and its draft is D and use the Cartesian coordinate system in section 2.1. We discretize the surface of the quadrant in the domain \([0, L/2] \times [0, B/2] \times [0, D]\), or the first quadrant, into a 3-dimensional mesh of rectangular panels. Suppose the number of panels in the x direction is \(nl\), in the y direction \(nb\), in the z direction \(nt\). Then on the bottom plate of the quadrant there are \(num = nl \times nb\) panels, on the starboard plate there are \(nl \times nt\) panels, on the bow plate there are \(nb \times nt\) panels. Totally there are \(node = nl \times nb + nl \times nt + nb \times nt\) panels.
in the quadrant. And for each panel in this reference quadrant, there are three corresponding panels in the symmetric position of the other three quadrants.

All the distributions of physical quantities over each panel $i$ in the first quadrant are approximated with constant distributions over each panel with values equal to original distributions at the center point of the panel. The corresponding values in the symmetric panels of other three quadrants have the same magnitude as that in the first quadrant but their sign will depend on the sign coefficient of the symmetry considered. The sign coefficient for each quadrant $iquad$ under each symmetry $isym$ is shown in Table 2.2.1.

<table>
<thead>
<tr>
<th>Sign Coefficient $\text{sign(isym,quadrant)}$</th>
<th>$isym = 1$</th>
<th>$isym = 2$</th>
<th>$isym = 3$</th>
<th>$isym = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$iquad = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$[0, L/2] \times [0, B/2] \times [0, D]$</td>
<td>+1</td>
<td>+1</td>
<td>+1</td>
<td>+1</td>
</tr>
<tr>
<td>$iquad = 2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$[0, L/2] \times [-B/2, 0] \times [0, D]$</td>
<td>+1</td>
<td>-1</td>
<td>+1</td>
<td>-1</td>
</tr>
<tr>
<td>$iquad = 3$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$[-L/2, 0] \times [0, B/2] \times [0, D]$</td>
<td>+1</td>
<td>+1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$iquad = 4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$[-L/2, 0] \times [-B/2, 0] \times [0, D]$</td>
<td>+1</td>
<td>-1</td>
<td>-1</td>
<td>+1</td>
</tr>
</tbody>
</table>

Table 2.2.1 Values of the Sign Coefficients
For each panel \( i \) in the first quadrant, together with the other three symmetric panels in the other three quadrants the source distribution is constant with an intensity 
\[ \text{sign}(\text{isym}, \text{iquad}) \cdot Q_{i, \text{isym}} e^{-j\alpha t}, \] 
which is different for different symmetries. The source distribution over the whole body surface will generate continuous velocity potential throughout the fluid domain including the body surface. For each symmetry, \( \text{isym} \), the velocity potential distribution over each panel \( m \) in the first quadrant is approximated by a constant with the value \( \varphi_{m, \text{isym}} \), which is equal to the potential at the center point of the panel. Then from (2.1.6) we have:

\[
\varphi_{m, \text{isym}} = \sum_{i=1}^{\text{node}} \sum_{\text{iquad}=1}^{4} \text{sign}(\text{isym}, \text{iquad}) \cdot Q_{i, \text{isym}} e^{-j\alpha t} \int_{S_{i, \text{iquad}}} G(\bar{x}_m; \bar{z}_i, \text{iquad}; \omega) dS_{i, \text{iquad}} \\
= e^{-j\alpha t} \sum_{i=1}^{\text{node}} Q_{i, \text{isym}} U_{m, \text{isym}} \quad m=1,2,...,\text{node}, \quad \text{isym}=1,2,3,4
\]  

(2.2.1)

where \( S_{i, \text{iquad}} \) is the area of the symmetric image of panel \( i \) in quadrant \( \text{iquad} \), and

\[
U_{m, \text{isym}} = \sum_{\text{iquad}=1}^{4} \text{sign}(\text{isym}, \text{iquad}) \int_{S_{i, \text{iquad}}} G(\bar{x}_m; \bar{z}_i, \text{iquad}; \omega) dS_{i, \text{iquad}}
\]  

(2.2.2)

Then from (2.1.4) the constant pressure on panel \( m \) in the first quadrant for symmetry \( \text{isym} \), which is equal to the value at the panel center, can be written as:

\[
p_{m, \text{isym}} = -\rho \frac{\partial \varphi_{m, \text{isym}}}{\partial t} = j\rho \omega e^{-j\alpha t} \sum_{i=1}^{\text{node}} Q_{i, \text{isym}} U_{m, \text{isym}} m=1,2,...,\text{node}, \quad \text{isym}=1,2,3,4
\]  

(2.2.3)

The normal velocity over panel \( m \) in the first quadrant is approximated by a constant value which is equal to the value at the panel center. It can be computed as in (2.1.7) as:
\[ v_{m, isym} = \bar{n} \cdot \nabla \varphi_{m, isym} \]
\[ = \sum_{i=1}^{\text{node}} \sum_{\text{quad}=1}^{4} \text{sign}(isym, iquad) \cdot Q_{i, isym} e^{-j\alpha} (\bar{n} \cdot \nabla \int_{S_{i, quad}} G(\bar{x}_m; \bar{x}_i, quad; \omega) dS_{i, quad}) \]
\[ = e^{-j\alpha} \sum_{i=1}^{\text{node}} Q_{i, isym} B_{m, isym}, m = 1, 2, ..., \text{node}, \ isym = 1, 2, 3, 4, \]
(2.2.4)

where

\[ B_{m, isym} = \sum_{\text{quad}=1}^{4} \text{sign}(isym, iquad) \cdot (\bar{n} \cdot \nabla \int_{S_{i, quad}} G(\bar{x}_m; \bar{x}_i, quad; \omega) dS_{i, quad}) \]
(2.2.5)

\[ m, i = 1, 2, ..., \text{node}, \ isym = 1, 2, 3, 4 \]

The source-induced quantities \( U_{m, isym} \) and \( B_{m, isym} \) are integrations that can be computed using Gaussian Quadratures. The free-surface Green Function and its gradient are computed using the standard subroutine FINGREEN (Newman, 1985).

In (2.2.1), (2.2.3) and (2.2.4) the term \( e^{-j\alpha} \) appears on both sides of the equation either explicitly or implicitly, from now on this term will be omitted in the expressions to save space and considered implicit for variables changing with time. From (2.2.4) we can solve for the source distribution \( Q_{i, isym}, i = 1, 2, ..., \text{node} \) of each symmetry. After plugging in (2.1.2) we can get:

\[ Q_{i, isym} = \sum_{j=1}^{\text{node}} B^{-1} i_{j, isym} v_{j, isym} = -j\omega \sum_{j=1}^{\text{node}} B^{-1} i_{j, isym} d_{j, isym}, i = 1, 2, ..., \text{node}, \ isym = 1, 2, 3, 4 \]
(2.2.6)
where \( d_{j,\text{sym}} \) is the normal displacement at the panel center and is considered constant over the panel. Substitute (2.2.6) in (2.2.3) we can get:

\[
p_{m,\text{sym}} = \rho \omega^2 \sum_{j=1}^{\text{node}} \sum_{i=1}^{\text{node}} U_{mi,\text{sym}} B^{-1}_{ij,\text{sym}} d_{j,\text{sym}}
\]

\[
= \rho \omega^2 \sum_{j=1}^{\text{node}} H_{mj,\text{sym}} d_{j,\text{sym}} \quad m = 1,2,\ldots,\text{node}, \quad \text{sym} = 1,2,3,4 \tag{2.2.7}
\]

where \( H_{mj,\text{sym}} \) is the hydrodynamic influence matrix for this low-order panel method. This matrix relates the pressures to the displacements of panels and is defined as:

\[
H_{mj,\text{sym}} = \sum_{i=1}^{\text{node}} U_{mi,\text{sym}} B^{-1}_{ij,\text{sym}} m, i, j = 1,2,\ldots,\text{node}, \quad \text{sym} = 1,2,3,4 \tag{2.2.8}
\]

These hydrodynamic influence matrices encapsulate all the hydrodynamic properties of the 3-Dimension mesh of the body surface. They can be computed beforehand once and for all for the four symmetries and for all the frequencies considered.

The direction of the displacements \( d_{j,\text{sym}} \) in (2.2.8) is determined by the normal vector \( \vec{n} \) in (2.2.4) and (2.2.6). For the hydrodynamic problem the normal vector is defined as pointing out of the fluid domain according to traditional convention, so the positive displacements are in this same direction.
2.3 Radiation and Diffraction Problems

The pressure distributions on the body surface due to the radiation and diffraction problems of floating runways can be computed for every frequency of the input waves from the displacement vectors using the Hydrodynamic Influence Matrices as in (2.2.7). The resulted diffraction pressure vector will be used in the Motion Equation of the floating-mat model as the loads input. Also these pressure distributions can be integrated over the body surface to get the radiation and diffraction forces and compared the results with those from WAMIT, an offshore industry standard motions program for rigid bodies (WAMIT Version 5.4, 1998), to verify that the Hydrodynamic Influence Matrices are defined and computed correctly and with adequate accuracy. For the purpose of verification, small model structures with similar shape as the real size floating runway are used.
2.3.1 Radiation Problem

In the radiation problem the radiation forces are computed from the integration of pressure on the body surface under the six canonical motions. Suppose the normal vector of panel $i$ in the first quadrant is $\vec{n}_i = (n_{i1}, n_{i2}, n_{i3})$ and the position vector of the center of the panel is $\vec{x}_i = (x_i, y_i, z_i)$. Then the normal velocity at the center of each panel in the first quadrant for the six canonical motions can be defined by the components of the so-called extended normal vector. By using (2.1.2) we can get the corresponding vectors of displacements at the panels centers in the first quadrant as follows:

$$
\begin{align*}
  d_{ij} &= \frac{j}{\omega} n_{ij}, & j &= 1,2,3 \\
  d_{ij} &= \frac{j}{\omega} (\vec{x}_i \times \vec{n}_i)_{j-3}, & j &= 4,5,6
\end{align*}
$$

(2.3.1)

The displacement vectors for the canonical motions can be used as input data to compute the pressure distribution on body surface using the Hydrodynamic Influence Matrices as in (2.2.7). Each of the canonical motions has a specific symmetry so only the displacement in the first quadrant and the Hydrodynamic Influence Matrices of that symmetry are needed to compute the vector of pressure on the panels in the first quadrant. The pressure on the whole body surface can be computed from this vector of pressure for the panels in the first quadrant according to the symmetry as the canonical motion. Table 2.3.1 shows the symmetry of the canonical motions.
<table>
<thead>
<tr>
<th>Canonical Motions</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Surge</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sway</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Heave</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Roll</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Pitch</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Yaw</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Symmetry</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>(i_{sym} = )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 2.3.1 The Symmetry of the Canonical Motions

A program called HYDRO was developed to determine the Hydrodynamic Influence Matrices and the hydrodynamic forces on a platform in waves. The radiation forces computed using HYDRO are compared with those computed from WAMIT. First, a small floating structure model with deep draft is used to verify that the results from HYDRO are close enough to the results from WAMIT. The model is 80 meters long, 40 meters wide and has a draft of 20 meters. The results for a lower wave frequency and a higher wave frequency are compared. The lower wave frequency is \(\omega_1 = 0.38519 \text{ (Rad/Sec)}\) and the corresponding period is \(T_1 = 16.3119 \text{ (Sec)}\) and the wavelength is \(\lambda_1 = 415.008 \text{ (Meter)}\). This wave is very long compared with the size of the model structure. The higher wave frequency is \(\omega_2 = 1.13485 \text{ (Rad/Sec)}\) and the corresponding period is \(T_2 = 5.53658 \text{ (Sec)}\) and the wave length is \(\lambda_2 = 47.811 \text{ (Meter)}\). This wavelength is comparable with the size of the model structure. In both WAMIT and HYDRO the first quadrant of the wetted surface is discretized into 32 panels in the longitudinal direction, 20 panels in the transverse direction and 10 panels in the vertical
direction. The relatively long wavelength compared with the panel size and mild ratios of the dimensions of the model ensure that the results from WAMIT would be very accurate and are suitable to verify the soundness of the programming and computation of HYDRO.

Table 2.3.2 and Table 2.3.3 compare the results of the radiation forces from both programs and their relative differences under the two wave frequencies. The results from WAMIT are the non-dimensionalized added-mass coefficient \( \overline{A}_j \) and damping coefficient \( \overline{B}_j \) as defined in section (4.2) of the WAMIT version 5.4 manual (page 4-3). The direct results from HYDRO are the integration of the radiation pressure on the first quadrant surface and are also non-dimensionalized into the same added-mass and damping coefficients by timing by 4 (account for the 4 quadrants) and dividing by \( i\omega \cdot \rho l^4 \), where \( L=40 \text{m} \) is half the length, \( k=3 \) for surge, sway and heave motions and \( k=5 \) for roll, pitch and yaw motions.

From Table 2.3.2 and Table 2.3.3 it can be seen that in the radiation problem HYDRO gives results very close to those from WAMIT. Although there are always some differences depending on the wave frequency and the dimensions of the structure, it can be concluded that the computation and programming in HYDRO are correct.

Second, a small floating structure model with small draft and with shape similar to the real-size floating runway and the real-size structure is used to compare the results from HYDRO the results from WAMIT. The small model is 1000 meters long, 100 meters wide and has a draft of 2 meters and the real structure is 4000 meters long, 400 meters wide and has a draft of 2 meters. For these two models WAMIT may be unable to converge for waves with period smaller than 18 seconds (with wavelength 500 meters).
So the results for a very low wave frequency are compared. Now the wave frequency is $\omega_3 = 0.19617 \text{(Rad/Sec)}$, the corresponding period $T_3 = 32.029 \text{(Sec)}$ and the wavelength $\lambda_3 = 1600 \text{(Meter)}$. This wavelength is of the same order of magnitude as the length of the structure but is much longer than the width of the structure. And the draft of the structure is close to zero when compared to this wavelength. In both WAMIT and HYDRO the first quadrant of the wetted surface is discretized into 150 panels in the longitudinal direction, 15 panels in the transverse direction and 1 panels in the vertical direction. The wavelength is very large compared with the panel size so that the error in the results won’t come from a coarse mesh.

Table 2.3.4 and Table 2.3.5 compare the results of the radiation forces from both programs and their relative differences for the two models. The results from WAMIT and HYDRO are also non-dimensionalized into the added-mass and damping coefficients as before with $L$ equal to half the lengths of the models.

From Table 2.3.4 and Table 2.3.5 it can be seen that for these two models HYDRO gives results very close to those from WAMIT for the heave, roll and pitch motions which are not significantly affected by the small draft. But for the surge, sway and yaw motions which are significantly affected by the small draft the results from HYDRO are quite different to those from WAMIT. The differences for all the motions are much larger for the real-size model for which the draft is relatively much smaller compared with the length and width. This phenomenon indicates either HYDRO or WAMIT or both of them can’t compute the effects of the very narrow side and bow plates accurately. This defect is not un-expected. Fortunately the effects of the very narrow side and bow plates on the hydrodynamics properties of the whole structure are very small and can be neglected.
Although it takes HYDRO much longer time to compute the results than WAMIT because HYDRO uses Gauss-Jordan method to invert the matrices and WAMIT uses iteration method instead, HYDRO works well for much higher wave frequencies when WAMIT will exceed its maximum number of iteration.

<table>
<thead>
<tr>
<th>Canonical Motions</th>
<th>WAMIT</th>
<th>HYDRO</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\bar{A}_{\eta \eta_1}$</td>
<td>$\bar{B}_{\eta \eta_1}$</td>
<td>$\bar{A}_{\eta \eta_2}$</td>
</tr>
<tr>
<td>1</td>
<td>4.387E-01</td>
<td>6.813E-02</td>
<td>4.360E-01</td>
</tr>
<tr>
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<td>1.183E+00</td>
<td>1.795E-01</td>
<td>1.169E+00</td>
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<td>3</td>
<td>8.712E-01</td>
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<td>8.761E-01</td>
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<td>3.380E-03</td>
<td>5.917E-02</td>
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<tr>
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<td>2.276E-03</td>
<td>1.674E-01</td>
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<tr>
<td>6</td>
<td>1.819E-01</td>
<td>6.580E-04</td>
<td>1.792E-01</td>
</tr>
</tbody>
</table>

Table 2.3.2  Comparison of the Added-Mass and Damping Coefficients Computed by HYDRO and WAMIT, for 80*40*20m model, wavelength $\lambda_i = 415.008$ (Meter)
<table>
<thead>
<tr>
<th>Canonical Motions</th>
<th>WAMIT</th>
<th>HYDRO</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{y_1}$</td>
<td>$B_{y_1}$</td>
<td>$A_{y_2}$</td>
<td>$B_{y_2}$</td>
</tr>
<tr>
<td>1</td>
<td>5.343E-02</td>
<td>1.269E-01</td>
<td>5.286E-02</td>
</tr>
<tr>
<td>2</td>
<td>1.095E-01</td>
<td>2.534E-01</td>
<td>1.086E-01</td>
</tr>
<tr>
<td>3</td>
<td>8.006E-01</td>
<td>1.301E-03</td>
<td>8.028E-01</td>
</tr>
<tr>
<td>4</td>
<td>3.750E-02</td>
<td>4.724E-03</td>
<td>3.652E-02</td>
</tr>
<tr>
<td>5</td>
<td>1.627E-01</td>
<td>1.781E-03</td>
<td>1.619E-01</td>
</tr>
<tr>
<td>6</td>
<td>6.964E-02</td>
<td>7.314E-02</td>
<td>6.842E-02</td>
</tr>
</tbody>
</table>

Table 2.3.3  Comparison of the Added-Mass and Damping Coefficients Computed by HYDRO and WAMIT, for 80*40*20m model, wavelength $\lambda_2 = 47.811 \text{ (Meter)}$
<table>
<thead>
<tr>
<th>Canonical Motions</th>
<th>WAMIT</th>
<th>HYDRO</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A_{y_1}$</td>
<td>$B_{y_1}$</td>
<td>$A_{y_2}$</td>
</tr>
<tr>
<td>1</td>
<td>1.37E-05</td>
<td>1.16E-06</td>
<td>1.467E-05</td>
</tr>
<tr>
<td>2</td>
<td>1.49E-04</td>
<td>3.82E-06</td>
<td>1.604E-04</td>
</tr>
<tr>
<td>3</td>
<td>5.65E-02</td>
<td>4.80E-02</td>
<td>5.596E-02</td>
</tr>
<tr>
<td>4</td>
<td>6.59E-05</td>
<td>2.15E-06</td>
<td>6.533E-05</td>
</tr>
<tr>
<td>5</td>
<td>2.22E-02</td>
<td>8.36E-03</td>
<td>2.208E-02</td>
</tr>
<tr>
<td>6</td>
<td>4.76E-05</td>
<td>3.70E-07</td>
<td>5.096E-05</td>
</tr>
</tbody>
</table>

Table 2.3.4  Comparison of the Added-Mass and Damping Coefficients Computed by HYDRO and WAMIT, for 1000*100*2m model, wavelength $\lambda = 1600$ (Meter)
<table>
<thead>
<tr>
<th>Canonical Motions</th>
<th>WAMIT</th>
<th>HYDRO</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A_{\gamma 1}$</td>
<td>$B_{\gamma 1}$</td>
<td>$A_{\gamma 2}$</td>
</tr>
<tr>
<td>1</td>
<td>1.111E-06</td>
<td>1.626E-07</td>
<td>1.420E-06</td>
</tr>
<tr>
<td>2</td>
<td>1.218E-05</td>
<td>2.160E-06</td>
<td>1.588E-05</td>
</tr>
<tr>
<td>3</td>
<td>3.247E-02</td>
<td>2.814E-02</td>
<td>3.197E-02</td>
</tr>
<tr>
<td>4</td>
<td>6.627E-05</td>
<td>1.726E-05</td>
<td>6.523E-05</td>
</tr>
<tr>
<td>5</td>
<td>1.089E-02</td>
<td>8.460E-03</td>
<td>1.073E-02</td>
</tr>
<tr>
<td>6</td>
<td>4.011E-06</td>
<td>6.306E-07</td>
<td>5.216E-06</td>
</tr>
</tbody>
</table>

Table 2.3.5 Comparison of the Added-Mass and Damping Coefficients Computed by
HYDRO and WAMIT, for 4000*400*2m model, wavelength $\lambda_2 = 1600$ (Meter)
2.3.2 Diffraction Problem

In the diffraction problem the diffraction pressure under each input waves are computed using the Hydrodynamic Influence Matrices and stored to be used later as the loads input of the Motion Equation. Then the diffraction forces are computed by integrating the pressure distributions on the body surface and also compared with those computed using WAMIT to verify the correctness and accuracy of our programs.

An incoming wave with frequency $\omega$ and incident angle $\theta$ can be described using the complex velocity potential as follows:

$$\varphi_I(x, y, z; \omega, \theta) = \frac{G}{\omega} e^{-kz} e^{j(kx \cos \theta + ky \sin \theta)} (e^{-j\omega t}) \quad (2.3.2)$$

This distribution of the velocity potential on the body surface is also approximated by panel-wise constants equal to the values at the panel centers. As before the velocity potential distribution and other distributions like the displacement, normal velocity, pressure on the whole body surface are all decomposed into four distributions, each have one of the four symmetries. Suppose $\varphi_{ii,iquad}$ is the velocity potential at the center of panel $i$ in the first quadrant when $iquad=1$, or the symmetric panel in other quadrants when $iquad=2,3,4$. Then the velocity potential, $\varphi_{ii, isym}$, of symmetry $isym=1,2,3,4$ on the panel $i$ of the first quadrant can be computed as:

$$\varphi_{ii, isym} = \sum_{iquad=1}^{4} \text{sign}((isym,iquad)\varphi_{ii,iquad}, i=1,2,\ldots, node, \quad isym=1,2,3,4 \quad (2.3.3)$$
The corresponding wave elevation (positive downward as defined by the z axis) on the panel $i$ of the first quadrant of symmetry $isym$ can be computed from the velocity potential as:

$$
\eta_{li,isym} = \frac{1}{g} \frac{\partial \varphi_{li,isym}(x, y, z; \omega, \theta)}{\partial t} \\
= \sum_{iquad=1}^{4} -\text{sign}(isym, iquad) je^{-k_{ix,iquad}z} e^{j(k_{ix,iquad} \cos \theta + k_{iy,iquad} \sin \theta)},
$$

(2.3.4) $i = 1, 2, ..., node, \quad isym = 1, 2, 3, 4$

The pressure of symmetry $isym$ on panel $i$ in the first quadrant caused by the incident wave, which is equal to the value at the center of the panel, is then:

$$
\rho_{li,isym} = -\rho \frac{\partial \varphi_{li,isym}(x_i, y_i, z_i; \omega, \theta)}{\partial t} \\
= j \omega \varphi_{li,isym}(x_i, y_i, z_i; \omega, \theta) = -\rho g \eta_{li,isym}(x_i, y_i, z_i; \omega, \theta),
$$

(2.3.5) $i = 1, 2, ..., node, \quad isym = 1, 2, 3, 4$

Similarly the incident wave will cause an oscillating velocity in the normal direction, which is determined from the normal derivative of the velocity potential and is related to the displacement as in (2.1.2). The normal velocity on panel $i$ in the first quadrant and having symmetry $isym$ can be computed from the velocity potential by the boundary condition on the body surface as follows:

$$
\nu_{nl,i,isym} \approx \frac{\partial (d_{li,isym}(x_i, y_i, z_i))}{\partial t} = -j \alpha d_{li,isym}(x_i, y_i, z_i) \\
= \bar{n} \cdot \nabla \varphi_{li,isym}(x_i, y_i, z_i; \omega, \theta) \\
= (n_{i1} j k \cos \theta + n_{i2} j k \sin \theta - kn_{i3}) \varphi_{li,isym}(x_i, y_i, z_i; \omega, \theta),
$$

(2.3.6) $i = 1, 2, ..., node, \quad isym = 1, 2, 3, 4$
The oscillating displacements of panel \(i\) in the first quadrant with symmetry \(isym\) corresponding to the scattered waves, \(d_{Si, isym}\), will have equal magnitude but opposite direction to those of the incident waves and can then be computed as:

\[
d_{Si, isym} = -d_{Li, isym}(x_i, y_i, z_i) = -\frac{j}{\omega} (n_{i1} jk \cos \theta + n_{i2} jk \sin \theta - kn_{i3}) \varphi_{Li, isym}(x_i, y_i, z_i; \omega, \theta) = (jn_{i1} \cos \theta + jn_{i2} \sin \theta - n_{i3}) \eta_{Li, isym}(x_i, y_i, z_i; \omega, \theta),
\]

\(i = 1, 2, ..., \text{node, } isym = 1, 2, 3, 4\) (2.3.7)

Using the Hydrodynamic Influence Matrices, the related pressure from the scattered waves, on panel \(i\) in the first quadrant and with symmetry \(isym\), can be computed as:

\[
P_{Si, isym} = \rho \omega^2 [H_{ij, isym}]^{d_{Si, isym}}_{i, j = 1, 2, ..., \text{node, } isym = 1, 2, 3, 4} (2.3.8)
\]

The diffraction pressure is the summing of the pressure of the incident waves and the pressure of the scattered waves:

\[
P_{Di, isym} = P_{Li, isym} + P_{Si, isym} = j \omega \varphi_{Li, isym}(x_i, y_i, z_i; \omega, \theta) + \rho \omega^2 [H_{ij, isym}]^{d_{Si, isym}}_{i, j = 1, 2, ..., \text{node, } isym = 1, 2, 3, 4} (2.3.9)
\]

This vector of diffraction pressure will be used as the loads input for the Motion Equation of the floating-mat model and will be computed and stored for each incident wave of different frequency and incidental angle.

Also the diffraction pressure is integrated over the whole body surface to get the diffraction forces. These forces (computed by the program HYDRO using low-order method) are compared with those computed using WAMIT in order to verify that the Hydrodynamics Influence Matrix and the vector of diffraction pressure are computed correctly.
Same as in the radiation results comparison in subsection (3.5.1), first a small floating structure model 80 meters long, 40 meters wide and with a deep draft of 20 meters is used. The results for a lower wave frequency $\omega_1 = 0.38519 \frac{Rad}{Sec}$ (with corresponding period $T_1 = 16.3119 \ (Sec)$ and wavelength $\lambda_1 = 415.008 \ (Meter)$) and a higher wave frequency $\omega_2 = 1.13485 \frac{Rad}{Sec}$ (with corresponding period $T_2 = 5.53658 \ (Sec)$ and wavelength $\lambda_2 = 47.811 \ (Meter)$) are compared. In both WAMIT and HYDRO a quadrant of the wetted surface is discretized into 32 panels in the longitudinal direction, 20 panels in the transverse direction and 10 panels in the vertical direction. The relatively long wavelength compared with the panel size and mild ratios of the dimensions of the model ensure that the results from WAMIT would be very accurate so that they are suitable to be used to verify the soundness of the programming and computation in HYDRO. For each frequency the diffraction forces are compared for waves of three incident angles, i.e. $\theta = 0^\circ$, $\theta = 19^\circ$ and $\theta = 46^\circ$.

Tables 2.3.6(a,b,c) and Tables 2.3.7(a,b,c) show the results of the diffraction forces from both programs and the differences of their modules under the two wave frequencies. The results from WAMIT are the non-dimensionalized modules of the exciting forces as defined in section (4.3) of the WAMIT version 5.4 manual (page 4-3) together with the phase angles. The direct results from HYDRO are the integration of the diffraction pressure on the first quadrant surface and are transformed by multiplying by 4 (account for the 4 quadrants) and dividing by $-i \cdot \rho g L^m$, where L=40m is half the length, $m=2$ for surge, sway and heave motions and $m=3$ for roll, pitch and yaw motions, before the modules and phase angles are computed.
From Tables 2.3.6(a,b,c) and Tables 2.3.7(a,b,c) it can be seen that in the diffraction problem HYDRO gives results very close to those from WAMIT. Although there are always some errors depending on the wave frequency and the dimensions of the structure, it can be concluded that the computation and programming in HYDRO are correct and that HYDRO computes correctly the Hydrodynamic Influence Matrices and the vectors of diffraction pressure which are the input data to the Motion Equation. But one obvious difference is that for the forces of mode 4, 5, and 6 the phase angles given by HYDRO and WAMIT differ by 180°. This appears to result from a different definition of the extended normal vector in WAMIT.

Then a small floating structure model with very small draft and with shape similar to the real-size floating runway and the real-size structure is used to compare the diffraction results from HYDRO and WAMIT, same as for the radiation problem in subsection (2.3.1). The small model is 1000 meters long, 100 meters wide and has a draft of 2 meters and the real structure is 4000 meters long, 400 meters wide and has a draft of 2 meters. The results for a very low wave frequency are compared in case WAMIT won’t converge. Same as in the radiation problem the wave frequency is \( \omega_3 = 0.19617 \frac{\text{Rad}}{{\text{Sec}}} \), the corresponding period \( T_3 = 32.029 \text{ (Sec)} \) and the wavelength \( \lambda_3 = 1600 \text{ (Meter)} \). In both WAMIT and HYDRO one quadrant of the wetted surface is discretized into 150 panels in the longitudinal direction, 15 panels in the transverse direction and 1 panels in the vertical direction. The wavelength is very large compared with the panel size so that the error in the results won’t come from a coarse mesh.

Tables 2.3.8(a,b,c) and Tables 2.3.9(a,b,c) compares the results of the diffraction forces from WAMIT and HYDRO for these two models. From Tables 2.3.8(a,b,c) it can
be seen that for this model HYDRO gives diffraction force modules very close to those from WAMIT except for modes 2, 4 and 6 at $\theta = 19^\circ$ when the relative differences will be large. Since for modes 2, 4 and 6 at incident angle $\theta = 19^\circ$ the modules from HYDRO and WAMIT are all very small as seen in Table 2.3.8(b), these large differences are not significant and can be neglected. The phase angles for forces of modes 1, 2, 3 all conform very well. The phase angles for forces of modes 4, 5, 6 still have a difference of $180^\circ$. Tables 2.3.9(a,b,c) shows similar results for the real structure except that the differences are relatively larger and that the differences in modules of modes 2, 4 and 6 at incident angle $\theta = 19^\circ$ are still small.

From these comparisons it can be verified again that the programming and computing in HYDRO are correct and it computes the Hydrodynamics Influence Matrix and the vector of diffraction pressure correctly.
<table>
<thead>
<tr>
<th>Force Mode</th>
<th>WAMIT</th>
<th>HYDRO</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Module $F_1$</td>
<td>Phase Angle $\alpha_1$ °</td>
<td>Module $F_2$</td>
</tr>
<tr>
<td>1</td>
<td>6.582E-01</td>
<td>88.0</td>
<td>6.561E-01</td>
</tr>
<tr>
<td>2</td>
<td>0.000E+00</td>
<td>90.0</td>
<td>0.000E+00</td>
</tr>
<tr>
<td>3</td>
<td>1.007E+00</td>
<td>11.0</td>
<td>1.004E+00</td>
</tr>
<tr>
<td>4</td>
<td>0.000E+00</td>
<td>90.0</td>
<td>0.000E+00</td>
</tr>
<tr>
<td>5</td>
<td>1.239E-01</td>
<td>88.0</td>
<td>1.245E-01</td>
</tr>
<tr>
<td>6</td>
<td>0.000E+00</td>
<td>90.0</td>
<td>0.000E+00</td>
</tr>
</tbody>
</table>

(a) Wave Incident Angle $\theta = 0°$
<table>
<thead>
<tr>
<th>Force Mode</th>
<th>WAMIT</th>
<th>HYDRO</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Module $F_1$</td>
<td>Phase Angle $\alpha_1$</td>
<td>Module $F_2$</td>
</tr>
<tr>
<td>1</td>
<td>6.278E-01</td>
<td>88.0</td>
<td>6.254E-01</td>
</tr>
<tr>
<td>2</td>
<td>3.449E-01</td>
<td>85.0</td>
<td>3.450E-01</td>
</tr>
<tr>
<td>3</td>
<td>1.010E+00</td>
<td>11.0</td>
<td>1.008E+00</td>
</tr>
<tr>
<td>4</td>
<td>4.746E-02</td>
<td>86.0</td>
<td>4.719E-02</td>
</tr>
<tr>
<td>5</td>
<td>1.169E-01</td>
<td>88.0</td>
<td>1.173E-01</td>
</tr>
<tr>
<td>6</td>
<td>4.030E-02</td>
<td>0.0</td>
<td>4.044E-02</td>
</tr>
</tbody>
</table>

(b) Wave Incident Angle $\theta = 19^\circ$
<table>
<thead>
<tr>
<th>Force Mode</th>
<th>WAMIT</th>
<th>HYDRO</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Module</td>
<td>Phase Angle</td>
<td>Module</td>
</tr>
<tr>
<td>1</td>
<td>4.768E-01</td>
<td>88.0</td>
<td>4.737E-01</td>
</tr>
<tr>
<td>2</td>
<td>7.750E-01</td>
<td>85.0</td>
<td>7.719E-01</td>
</tr>
<tr>
<td>3</td>
<td>1.025E+00</td>
<td>11.0</td>
<td>1.023E+00</td>
</tr>
<tr>
<td>4</td>
<td>1.063E-01</td>
<td>85.0</td>
<td>1.053E-01</td>
</tr>
<tr>
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<td>8.506E-02</td>
<td>88.0</td>
<td>8.517E-02</td>
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<tr>
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<td>6.589E-02</td>
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<td>6.567E-02</td>
</tr>
</tbody>
</table>

(c) Wave Incident Angle $\theta = 46^\circ$

Table 2.3.6  Comparison of the Diffraction Forces Computed by HYDRO and WAMIT

(80*40*20m model, wavelength $\lambda_i = 415.008\, (Meter)$, incident angles $0^\circ, 19^\circ, 46^\circ$)
<table>
<thead>
<tr>
<th>Force Mode</th>
<th>WAMIT</th>
<th></th>
<th>HYDRO</th>
<th></th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>Module</td>
<td>Phase Angle</td>
<td>Module</td>
<td>Phase Angle</td>
<td>$\frac{F_2 - F_1}{F_1}$</td>
<td></td>
</tr>
<tr>
<td>$F_1$</td>
<td>$\alpha_1$</td>
<td>$F_2$</td>
<td>$\alpha_2$</td>
<td>N/A</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>3.403E-01</td>
<td>-57.0</td>
<td>3.399E-01</td>
<td>-56.8</td>
<td>-0.132%</td>
</tr>
<tr>
<td>2</td>
<td>0.000E+00</td>
<td>90.0</td>
<td>0.000E+00</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>3</td>
<td>1.721E-02</td>
<td>-77.0</td>
<td>1.687E-02</td>
<td>-75.8</td>
<td>-1.988%</td>
</tr>
<tr>
<td>4</td>
<td>0.000E+00</td>
<td>90.0</td>
<td>0.000E+00</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>5</td>
<td>3.713E-02</td>
<td>124.0</td>
<td>3.705E-02</td>
<td>-55.7</td>
<td>-0.229%</td>
</tr>
<tr>
<td>6</td>
<td>0.000E+00</td>
<td>90.0</td>
<td>0.000E+00</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>

(a) Wave Incident Angle $\theta = 0^\circ$
<table>
<thead>
<tr>
<th>Force Mode</th>
<th>WAMIT</th>
<th>HYDRO</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$F_1$</td>
<td>$\alpha_1^\circ$</td>
<td>$F_2$</td>
</tr>
<tr>
<td>1</td>
<td>3.143E-01</td>
<td>-75.0</td>
<td>3.131E-01</td>
</tr>
<tr>
<td>2</td>
<td>1.517E-01</td>
<td>-113.0</td>
<td>1.519E-01</td>
</tr>
<tr>
<td>3</td>
<td>1.861E-02</td>
<td>-85.0</td>
<td>1.814E-02</td>
</tr>
<tr>
<td>4</td>
<td>2.265E-02</td>
<td>-115.0</td>
<td>2.259E-02</td>
</tr>
<tr>
<td>5</td>
<td>3.669E-02</td>
<td>106.0</td>
<td>3.652E-02</td>
</tr>
<tr>
<td>6</td>
<td>6.150E-02</td>
<td>-174.0</td>
<td>6.063E-02</td>
</tr>
</tbody>
</table>

(b) Wave Incident Angle $\theta = 19^\circ$
<table>
<thead>
<tr>
<th>Force Mode</th>
<th>WAMIT Module $F_1$</th>
<th>Phase Angle $\alpha_1$ °</th>
<th>HYDRO Module $F_2$</th>
<th>Phase Angle $\alpha_2$ °</th>
<th>Relative Differences $\frac{F_2 - F_1}{F_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.732E-01</td>
<td>-143.0</td>
<td>1.713E-01</td>
<td>-143.5</td>
<td>-1.100%</td>
</tr>
<tr>
<td>2</td>
<td>9.415E-02</td>
<td>-71.0</td>
<td>9.185E-02</td>
<td>-69.9</td>
<td>-2.439%</td>
</tr>
<tr>
<td>3</td>
<td>8.354E-03</td>
<td>-138.0</td>
<td>7.909E-03</td>
<td>-141.0</td>
<td>-5.333%</td>
</tr>
<tr>
<td>4</td>
<td>1.202E-02</td>
<td>-72.0</td>
<td>1.166E-02</td>
<td>108.9</td>
<td>-3.004%</td>
</tr>
<tr>
<td>5</td>
<td>1.302E-02</td>
<td>49.0</td>
<td>1.250E-02</td>
<td>-132.1</td>
<td>-3.981%</td>
</tr>
<tr>
<td>6</td>
<td>2.543E-01</td>
<td>5.0</td>
<td>2.541E-01</td>
<td>-174.5</td>
<td>-0.084%</td>
</tr>
</tbody>
</table>

(c) Wave Incident Angle $\theta = 46^\circ$

Table 2.3.7  Comparison of the Diffraction Forces Computed by HYDRO and WAMIT

(80*40*20m model, wavelength $\lambda_2 = 47.811$ (Meter), incident angles $0^\circ,19^\circ,46^\circ$)
<table>
<thead>
<tr>
<th>Force Mode</th>
<th>WAMIT Module $F_1$</th>
<th>Phase Angle $\alpha_1$</th>
<th>HYDRO Module $F_2$</th>
<th>Phase Angle $\alpha_2$</th>
<th>Relative Differences $\frac{F_2 - F_1}{F_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.174E-03</td>
<td>102.0</td>
<td>1.175E-03</td>
<td>102.3</td>
<td>0.090%</td>
</tr>
<tr>
<td>2</td>
<td>0.000E+00</td>
<td>90.0</td>
<td>0.000E+00</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>3</td>
<td>1.362E-01</td>
<td>22.0</td>
<td>1.364E-01</td>
<td>22.1</td>
<td>0.119%</td>
</tr>
<tr>
<td>4</td>
<td>0.000E+00</td>
<td>90.0</td>
<td>0.000E+00</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>5</td>
<td>1.157E-01</td>
<td>101.0</td>
<td>1.161E-01</td>
<td>-79.3</td>
<td>0.300%</td>
</tr>
<tr>
<td>6</td>
<td>0.000E+00</td>
<td>90.0</td>
<td>0.000E+00</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>

(a) Wave Incident Angle $\theta = 0^\circ$
<table>
<thead>
<tr>
<th>Force Mode</th>
<th>WAMIT Module $F_1$</th>
<th>Phase Angle $\alpha_1^\circ$</th>
<th>HYDRO Module $F_2$</th>
<th>Phase Angle $\alpha_2^\circ$</th>
<th>Relative Differences $\frac{F_2 - F_1}{F_1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.228E-03</td>
<td>102.0</td>
<td>1.196E-03</td>
<td>102.1</td>
<td>-2.618%</td>
</tr>
<tr>
<td>2</td>
<td>5.381E-04</td>
<td>90.0</td>
<td>3.059E-04</td>
<td>90.0</td>
<td>-43.154%</td>
</tr>
<tr>
<td>3</td>
<td>1.502E-01</td>
<td>22.0</td>
<td>1.414E-01</td>
<td>21.9</td>
<td>-5.910%</td>
</tr>
<tr>
<td>4</td>
<td>4.043E-04</td>
<td>-90.0</td>
<td>2.305E-04</td>
<td>90.0</td>
<td>-42.988%</td>
</tr>
<tr>
<td>5</td>
<td>1.147E-01</td>
<td>101.0</td>
<td>1.158E-01</td>
<td>-79.3</td>
<td>0.927%</td>
</tr>
<tr>
<td>6</td>
<td>4.460E-04</td>
<td>0.0</td>
<td>2.710E-04</td>
<td>-180.0</td>
<td>-39.243%</td>
</tr>
</tbody>
</table>

(b) Wave Incident Angle $\theta = 19^\circ$
<table>
<thead>
<tr>
<th>Force Mode</th>
<th>WAMIT</th>
<th></th>
<th>HYDRO</th>
<th></th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Module</td>
<td>Phase Angle $\alpha_1$</td>
<td>Module</td>
<td>Phase Angle $\alpha_2$</td>
<td>$\frac{F_2 - F_1}{F_1}$</td>
</tr>
<tr>
<td>1</td>
<td>1.284E-03</td>
<td>100.0</td>
<td>1.290E-03</td>
<td>99.7</td>
<td>0.513%</td>
</tr>
<tr>
<td>2</td>
<td>1.651E-03</td>
<td>90.0</td>
<td>1.623E-03</td>
<td>90.0</td>
<td>-1.697%</td>
</tr>
<tr>
<td>3</td>
<td>2.117E-01</td>
<td>20.0</td>
<td>2.098E-01</td>
<td>19.6</td>
<td>-0.909%</td>
</tr>
<tr>
<td>4</td>
<td>1.243E-03</td>
<td>-90.0</td>
<td>1.225E-03</td>
<td>90.0</td>
<td>-1.386%</td>
</tr>
<tr>
<td>5</td>
<td>1.008E-01</td>
<td>101.0</td>
<td>1.020E-01</td>
<td>-79.5</td>
<td>1.158%</td>
</tr>
<tr>
<td>6</td>
<td>8.619E-04</td>
<td>0.0</td>
<td>8.636E-04</td>
<td>-180.0</td>
<td>0.196%</td>
</tr>
</tbody>
</table>

(c) Wave Incident Angle $\theta = 46^\circ$

Table 2.3.8  Comparison of the Diffraction Forces Computed by HYDRO and WAMIT

(1000*100*2m model, wavelength $\lambda_3 = 1600$ (Meter), incident angles 0°, 19°, 46°)
<table>
<thead>
<tr>
<th>Force Mode</th>
<th>WAMIT</th>
<th></th>
<th>HYDRO</th>
<th></th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Module</td>
<td>Phase Angle</td>
<td>Module</td>
<td>Phase Angle</td>
<td>$\frac{F_2 - F_1}{F_1}$</td>
</tr>
<tr>
<td>1</td>
<td>2.170E-04</td>
<td>111.0</td>
<td>2.211E-04</td>
<td>110.6</td>
<td>1.888%</td>
</tr>
<tr>
<td>2</td>
<td>0.000E+00</td>
<td>90.0</td>
<td>0.000E+00</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>3</td>
<td>2.217E-02</td>
<td>49.0</td>
<td>2.217E-02</td>
<td>49.4</td>
<td>0.007%</td>
</tr>
<tr>
<td>4</td>
<td>0.000E+00</td>
<td>90.0</td>
<td>0.000E+00</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>5</td>
<td>1.560E-02</td>
<td>56.0</td>
<td>1.559E-02</td>
<td>-124.1</td>
<td>-0.049%</td>
</tr>
<tr>
<td>6</td>
<td>0.000E+00</td>
<td>90.0</td>
<td>0.000E+00</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>

(a) Wave Incident Angle $\theta = 0^\circ$
<table>
<thead>
<tr>
<th>Force Mode</th>
<th>WAMIT</th>
<th>HYDRO</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Module $F_1$</td>
<td>Phase Angle $\alpha_1$ $^\circ$</td>
<td>Module $F_2$</td>
</tr>
<tr>
<td>1</td>
<td>1.964E-04</td>
<td>98.0</td>
<td>2.003E-04</td>
</tr>
<tr>
<td>2</td>
<td>9.428E-05</td>
<td>94.0</td>
<td>9.784E-05</td>
</tr>
<tr>
<td>3</td>
<td>2.099E-02</td>
<td>37.0</td>
<td>2.095E-02</td>
</tr>
<tr>
<td>4</td>
<td>2.595E-04</td>
<td>-85.0</td>
<td>2.682E-04</td>
</tr>
<tr>
<td>5</td>
<td>1.806E-02</td>
<td>18.0</td>
<td>1.809E-02</td>
</tr>
<tr>
<td>6</td>
<td>2.719E-05</td>
<td>-160.0</td>
<td>2.877E-05</td>
</tr>
</tbody>
</table>

(b) Wave Incident Angle $\theta = 19^\circ$
<table>
<thead>
<tr>
<th>Force Mode</th>
<th>WAMIT</th>
<th>HYDRO</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Module $F_1$</td>
<td>Phase Angle $\alpha_1^\circ$</td>
<td>Module $F_2$</td>
</tr>
<tr>
<td>1</td>
<td>2.469E-04</td>
<td>-47.0</td>
<td>2.521E-04</td>
</tr>
<tr>
<td>2</td>
<td>2.075E-04</td>
<td>-79.0</td>
<td>2.200E-04</td>
</tr>
<tr>
<td>3</td>
<td>3.655E-02</td>
<td>-108.0</td>
<td>3.704E-02</td>
</tr>
<tr>
<td>4</td>
<td>5.682E-04</td>
<td>102.0</td>
<td>6.012E-04</td>
</tr>
<tr>
<td>5</td>
<td>2.706E-02</td>
<td>-41.0</td>
<td>2.692E-02</td>
</tr>
<tr>
<td>6</td>
<td>2.405E-04</td>
<td>-174.0</td>
<td>2.470E-04</td>
</tr>
</tbody>
</table>

(c) Wave Incident Angle $\theta = 46^\circ$

Table 2.3.9  Comparison of the Diffraction Forces Computed by HYDRO and WAMIT

(4000*400*2m model, wavelength $\lambda_1 = 1600 (\text{Meter})$, incident angles $0^\circ,19^\circ,46^\circ$)
2.4 Low-Order Method Motion Equation

Into order to compute the responses of the floating runway in waves, the Motion Equation is established using the so-called floating-mat model that has been used by many peoples in the previous research (Webster and Mamidipudi, 1994, Newman, 1996, Kashiwagi, 1998). In this floating-mat model, the structure can be regarded as a thin plate floating on the water surface because the depth and draft of the floating runway are much smaller than its length and width. The structure is assumed to have different strength in the longitudinal and transverse directions, so it’s modelized as an orthotropic plate using the method shown in the appendix of (H.A.Schade,1938). Although the depth of the structure is neglected so that the plate is considered a 2-dimensional sheet, the structural rigidities, mass density and hydrodynamic pressure are all computed from the 3-Dimentional structure.

The rigidities of the equivalent orthotropic plate are computed from the combined plate using orthotropic plate theory and based on a simplified preliminary design which is also the initial design of the optimization process.

A simple preliminary design of the floating runway is first discussed beforehand to give a starting point or baseline for the optimization procedure, which is called initial or baseline design. The initial design gives the initial value of the structural strength for the optimization which will be changed in the optimizing process. The preliminary design provides an image of the probable design of the floating structure. Through studying the preliminary design we can decide on which minor details can be omitted and can concentrate on the main structural properties in our optimization work.
It is assumed that the structure is a steel plate fabrication in the shape of a flat box of 4000 meters long, 400 meters wide and 5 meters deep. The draft is taken to be 2 meters and will be kept at this level by adding or removing ballast weight. It is supposed that the buoyancy and gravity of the structure is balanced everywhere in the horizontal domain by ballasting so that there is no hydrostatic static loads in the structure when floating in the calm water.

The structure will have single deck and single bottom plate with longitudinal girders and transverse webs running between. Some of the girders and webs will serve as watertight bulkheads to ensure the safety of the structure and others will have lightening holes in them. However, girders and webs with lightening holes are normally strengthened so that they are equally efficient as the watertight ones in carrying bending and shear load. That is, all the girders and webs can be considered as having equal strength. They can be simulated with solid plates of same equivalent plate thickness. The spacing of the transverse webs and longitudinal girders are normally different because the different length and width of the structure. The transverse webs are spaced to be 25 meters apart and the longitudinal girders are spaced 10 meters apart. There may also be combined webs and girders which serve to increase to the effectiveness of deck and bottom plate. Thus, the deck and bottom plate will be considered fully effective while these combined webs and girders will be treated same as the stiffeners.

The deck plate is used as the runway and will be very strong itself. Thus, it also needs to be fitted with strong stiffeners. In the initial design its thickness is set to be 30mm. All other plates have a thickness of 10mm. In reality, the deck structure will be a complex cross-stiffened structure composed of many beams and girders to provide the deck with
the strength necessary to support the landing and taking off of airplanes. The simple pate
that we are using to model the deck should be thought of as just a representation with
equivalent strength of this complex structure. On the longitudinal girders and transverse
webs there will also be stiffeners and other strengthening details according to normal
structure design rules. But in a similar vein, these minor longitudinal and transverse
structure items are omitted. Their effect on global strength can be supposed to be
represented by an increase in the equivalent plate thickness of the main structure. After
simplification, a typical repeating unit of the structure is as shown in Fig. 2.4.1.

![Diagram of a typical repeating unit of the structure]

Figure 2.4.1 Typical Repeating Unit of the Structure

Consider the floating runway with the repeating unit as shown in Figure 2.4.1, the
basic idea is to replace it with a homogeneous orthotropic plate. As in classic plate theory
the structure needs to satisfy some limitations. First the thickness of the plate must be
small when compared to its other two dimensions, normally less than one-fifth of them.
This is true for the structure discussed here since the thickness to width ratio is
approximately one-eightieth and the thickness to length ratio is even 10 times smaller.
Secondly the deflection must also be small when compared to the thickness, normally less than one-fifth of the latter. This may be not true for the bottom plate under water pressure which will act partially as a membrane with middle plane loaded. But for the whole structure modeled as a plate this is normally true since the deflection is small and the boundary conditions don't enforce membrane loads. Thirdly the effects of vertical shearing and normal stresses on deflection must be neglectable. The vertical shear is resisted mainly by the widely spaced webs and may be of certain magnitude. But compared to the bending deflection it is relatively small. Also it is assumed that there are enough webs to keep the effects of vertical shearing small and neglectable. This limitation will let the Kirchhoff's assumption on straight normal line valid.

In this initial design of the runway, the scantlings are constant over the whole platform. In an ultimate design the structure, the scantlings, and thus the strength distribution, will vary throughout the platform as a result of optimization.

From the theory of orthotropic plates, the relation between the moments in the plate and the deflection of the plate, can be written as:

\[
M_x = -(D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2})
\]

\[
M_y = -(D_y \frac{\partial^2 w}{\partial y^2} + D_1 \frac{\partial^2 w}{\partial x^2})
\]

\[
M_{xy} = -2D_{xy} \frac{\partial^2 w}{\partial x \partial y}
\]

(2.4.1)

where \(D_x\), \(D_y\), \(D_1\) and \(D_{xy}\) are the rigidities of the orthotropic plate, so is \(H = D_1 + 2D_{xy}\). This relation comes from the integration of sectional stresses on the rectangular difference element of the plate and the stress-deflection relations. It is valid
for thin plate model with constant equivalent thickness even when the rigidities change with the location \( \bar{x} = (x, y) \). For the orthotropic plate here, which represents a flat combined structure, the thickness is the depth of the structure. The change of thickness of the plates in the structure won't affect the depth of the structure but will change the rigidities of the equivalent orthotropic plate which are then functions of the location \( \bar{x} = (x, y) \).

The initial rigidities of the equivalent homogeneous orthotropic plate can be easily computed from the structural scantlings using the formulae given in the appendix of (H.A. Schade, 1938). The results are shown in Table 2.4.1. It is also noticed that the neutral axes of the whole X-section and whole Y-section, and that of the deck and bottom plate only, are very close to each other (within 0.20 meter or 1/25 of the depth from each other).

<table>
<thead>
<tr>
<th>( D_x )</th>
<th>( D_y )</th>
<th>( D_t )</th>
<th>( H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^6 N.m)</td>
<td>(10^6 N.m)</td>
<td>(10^6 N.m)</td>
<td>(10^6 N.m)</td>
</tr>
<tr>
<td>46723.</td>
<td>44580.</td>
<td>12958.</td>
<td>43193.</td>
</tr>
</tbody>
</table>

Table 2.4.1 Rigidities of the Orthotropic Plate Modeling the Preliminarily Designed

Runway From the equilibrium of forces and moments, we can get the following differential Equation of Equilibrium, which also holds for the plate with non-uniform rigidities:

\[
\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -q(x, y)
\] (2.4.2)
Substitute (2.4.1) in (2.4.2), and notice that the rigidities $D_x$, $D_y$, $D_1$ and $D_{xy}$ are also functions of location $\bar{x} = (x, y)$, we can get the basic differential equation for the bending of plates with non-uniform rigidities:

\[
D_x \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial D_x}{\partial x} \frac{\partial^3 w}{\partial x^3} + (\frac{\partial^2 D_x}{\partial x^2} + \frac{\partial^2 D_1}{\partial y^2}) \frac{\partial^2 w}{\partial x^2} + (4 \frac{\partial D_{xy}}{\partial y} + 2 \frac{\partial D_1}{\partial y}) \frac{\partial^3 w}{\partial x \partial y^2} + (2D_1 + 4D_{xy}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + 4 \frac{\partial^2 D_{xy}}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} + (4 \frac{\partial D_{xy}}{\partial x} + 2 \frac{\partial D_1}{\partial x}) \frac{\partial^3 w}{\partial x^2 \partial y^2} + \frac{\partial^2 D_1}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + 2 \frac{\partial D_1}{\partial y} \frac{\partial^3 w}{\partial y^3} + D_y \frac{\partial^4 w}{\partial y^4} = q(x, y) \tag{2.4.3}
\]

The corresponding boundary conditions for free edges parallel to the $x$ or $y$ axis can be derived similarly:

\[
\left\{ \begin{aligned}
M_x|_{x=x_0} &= D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2} \\
V_x|_{x=x_0} &= (Q_x + \frac{\partial M_{xy}}{\partial y})|_{x=x_0} = (\frac{\partial M_x}{\partial x} + 2 \frac{\partial M_{xy}}{\partial y})|_{x=x_0} \\
&= \frac{\partial D_x}{\partial x} \frac{\partial^2 w}{\partial x^2} + D_x \frac{\partial^3 w}{\partial x^3} + \frac{\partial D_1}{\partial y} \frac{\partial^2 w}{\partial y^2} + D_1 \frac{\partial^3 w}{\partial x\partial y^2} + 4 \frac{\partial D_{xy}}{\partial y} \frac{\partial^2 w}{\partial x \partial y} + 4D_{xy} \frac{\partial^3 w}{\partial x^2 \partial y} |_{x=x_0} = 0
\end{aligned} \right. \tag{2.4.4a}
\]

\[
\left\{ \begin{aligned}
M_y|_{y=y_0} &= D_y \frac{\partial^2 w}{\partial y^2} + D_1 \frac{\partial^2 w}{\partial x^2} \\
V_y|_{y=y_0} &= (Q_y + \frac{\partial M_{xy}}{\partial x})|_{y=y_0} = (\frac{\partial M_y}{\partial y} + 2 \frac{\partial M_{xy}}{\partial x})|_{y=y_0} \\
&= \frac{\partial D_y}{\partial y} \frac{\partial^2 w}{\partial y^2} + D_y \frac{\partial^3 w}{\partial y^3} + \frac{\partial D_1}{\partial x} \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^3 w}{\partial x^2 \partial y} + 4 \frac{\partial D_{xy}}{\partial x} \frac{\partial^2 w}{\partial x \partial y} + 4D_{xy} \frac{\partial^3 w}{\partial x^2 \partial y} |_{y=y_0} = 0
\end{aligned} \right. \tag{2.4.4b}
\]

And at the corner with two neighboring free edges, there is an additional boundary condition requiring the concentrated force to be zero:

\[
F_z = 2M_{xy} = -4D_{xy} \frac{\partial^2 w}{\partial x \partial y} |_{x=x_0, y=y_0} = 0 \tag{2.4.4c}
\]
For the structure considered here, the rigidities don’t vary continuously. Since the structure will be constructed from smaller barge-like modules which are fabricated separately, the thickness inside each module is supposed to be uniform. For the purpose of computing the hydrodynamics, the bottom (and if it is an edge module, the side too) will be discretized into several low-order panels. Inside each module the structure is identical and the rigidities are constants. But from module to module the plate thickness changes and so do the rigidities. At each location inside the module the differential equation for plate bending (2.4.3) will take the simple classic form as:

\[ D_x \frac{\partial^4 w}{\partial x^4} + (2D_1 + 4D_{xy}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = q(x, y) \]  

(2.4.5)

Similarly the boundary conditions (2.4.4a,b) will have the simple forms respectively as:

\[
\begin{align*}
M_x \bigg|_{x=x_0} &= D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2} \bigg|_{x=x_0} = 0 \\
V_x \bigg|_{x=x_0} &= D_x \frac{\partial^3 w}{\partial x^3} + D_1 \frac{\partial^3 w}{\partial x \partial y^2} + 4D_{xy} \frac{\partial^3 w}{\partial x \partial y} \bigg|_{x=x_0} = 0
\end{align*}
\]  

(2.4.6a)

\[
\begin{align*}
M_y \bigg|_{y=y_0} &= D_y \frac{\partial^2 w}{\partial y^2} + D_1 \frac{\partial^2 w}{\partial x^2} \bigg|_{y=y_0} = 0 \\
V_y \bigg|_{y=y_0} &= D_y \frac{\partial^3 w}{\partial y^3} + D_1 \frac{\partial^3 w}{\partial x^2 \partial y} + 4D_{xy} \frac{\partial^3 w}{\partial x^2} \bigg|_{y=y_0} = 0
\end{align*}
\]  

(2.4.6b)

Now consider this orthotropic plate as floating on water surface, the floating-mat. The equilibrium of a differential piece of the floating-mat under lateral pressure gives the motion equation:

\[
\left( D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} \right) + T \rho_b \ddot{w} + \rho g w = -q
\]  

(2.4.7)
where \( w \) is the deflection satisfying the boundary conditions (2.4.4c, 2.4.6a,b) and \( q \) is the lateral water pressure.

In the low-order method the continuous deflection distribution on the floating-mat is represented by a panel-wise constant distribution. For each panel the deflection equal to the value at the panel center. Also the deflection is decomposed into the four symmetries as in section (2.2) so that only the deflection in the first quadrant, i.e. \( w_{i,\text{sym}}(x_i,y_i,z_i;\omega), i = 1,2,...,\text{node}; isym = 1,2,3,4 \), is specific and needs to be solved for.

The deflection in other quadrants can be determined by the sign coefficients as \( \text{sign}(isym, i\text{quad})w_{i,\text{sym}}, i = 1,2,...,\text{node}; \text{isym} = 1,2,3,4; i\text{quad} = 1,2,3,4 \). The right hand side lateral pressure is discretized in a similar way into \( q_{i,\text{sym}}(x_i,y_i,z_i;\omega), i = 1,2,...,\text{node}; \text{isym} = 1,2,3,4 \).

Since we consider the problem in frequency domain, the deflection oscillates with time by the same term \( e^{-j\omega} \) as other variables like the pressure. Then we can get:

\[
\ddot{w}_{i,\text{sym}}(x_i,y_i,z_i;\omega) = -\omega^2 w_{i,\text{sym}}(x_i,y_i,z_i)
\]  
(2.4.8)

In the low-order method the finite difference method is used to discretize the first term relating to the bending of an orthotropic plate (Webster, 1998). For panels in the central area, normal center difference formulae for the derivatives are used. For panels in the edge and corner area polynomial surfaces are fit to get formulae for the derivatives. In order to avoid increasing the size of the matrix, the boundary conditions at the free edges of the plate are embedded in the structural finite difference matrix \([D_{ij}]\) by fitting polynomial surfaces that satisfy the boundary conditions at the edge and corner area of the plate.
Then a structural finite difference matrix \([D_{ij}]\) can be assembled so that the structural term can be written as:

\[
(D_x \frac{\partial^4 w}{\partial x^4} + 2H \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4})_i = [D_{ij}] [w_j]
\]  
(2.4.9)

The right hand side lateral pressure \(-q_{i,isym}\) consists of hydrodynamic pressure only. The hydrodynamic pressure consists of three parts:

\[
q_{i,isym} = P_{li,isym} + P_{si,isym} + P_{ri,isym},
\]  
(2.4.10)

which is the pressure due to incident waves, scattered waves and radiated waves respectively. The pressure due to incident waves and scattered waves can be computed as in (2.3.5) and (2.3.8), or together as diffracted wave pressure as in (2.3.9). The pressure due to radiated waves can be computed using the Hydrodynamic Influence Matrices and the deflections \(\{w_{i,isym}\}\) of the floating-mat.

Notice that the Hydrodynamic Influence Matrices are prepared with the normal vector \(\vec{n}\) pointing to the outside of the fluid domain according to popular conventions in hydrodynamics literature and in our problem the deflection \(\{w_{i,isym}\}\) of the floating-mat are defined positive downward as the Z-axis, opposite to the normal vector on the bottom plate which points upward. So (2.2.7) should be applied with \(d_{i,isym} = -w_{i,isym}\) in order to get the radiation pressure on panel \(i\) under symmetry \(isym\) as:

\[
P_{ri,isym} = -\rho \omega^2 \left[H_{ij,isym} \{w_{j,isym}\}_i, j = 1,2,\ldots, num; isym = 1,2,3,4\right.
\]  
(2.4.11)

Substitute (2.4.8), (2.4.9), (2.4.10), (2.3.9) and (2.4.11) into (2.4.7) and rearrange the unknown and knows parts at the two sides of the equation, we can write the Motion Equation in matrix form:

\[57\]
\[
D_j - \rho \ddot{H}_{j,\text{sym}} + (\rho g - T_j) \ddot{\theta} \delta_j \begin{pmatrix}
  w_{j,\text{sym}}^i \\
\end{pmatrix} = \begin{pmatrix}
  -j \omega \rho \sigma_{j,\text{sym}}(\phi, \theta) - \rho \ddot{H}_{k,\text{sym}} d_{k,\text{sym}}^i \\
\end{pmatrix} \quad (2.4.12)
\]

\(i, j = 1, 2, \ldots; \text{num} k = 1, 2, \ldots; \text{nodesym} = 1, 2, 3, 4\)

This is the Motion Equation of the floating-mat using low-order method. It can be solved using traditional Gaussian Reduction or LUD method, or using iteration methods for every wave frequency, direction and symmetry to get the deflection responses \(\{w_{i,\text{sym}}\}\) of each panel in the first quadrant, which in turn can be used to compute all the deformation and stress responses in the whole structure. We should note that although we have exploited the symmetries, the deflections themselves and the stresses in the floating-mat that resulted thereafter do not exhibit a four-fold symmetry.
2.5 Structural Responses in Wave Systems

Seven kinds of responses are calculated for the structure. Two of them are the first and second order derivatives of the plate deflection (slope and curvature of the plate). These two together with the airplane velocity determine the vertical velocity and acceleration encountered by the plane, which then affect the safety for airplane to land and take off. If these platform shape parameters exceed predetermined maximum values then the platform is not operable. These responses are computed only for the strip near the platform centerline along which planes will take off and land.

The next five responses are: the maximum normal stresses in the $x$ and $y$ cross section, $\sigma_x$ and $\sigma_y$; the horizontal in-plane shear stress, $\tau_{xy}$; and the vertical shear stresses in the web plate of $x$ and $y$ direction, $\tau_x$ and $\tau_y$. These stresses determine whether or not the strength of the structure is sufficient. If a certain pre-determined value for each of these quantities is exceeded, the structure is supposed to be broken. Stresses are related to the derivatives of the deflection of the plate, which are computed by finite difference method. Formulae for the evaluation of these can be found in books on plates and shells and will not be repeated here.

The constraints on the flatness and strength of the floating runway are set based on a probabilistic point of view since the seaway that causes these is random in nature. The surface smoothness and structural stresses of the equivalent plate are random time series as responses of a linear system to the input time series, i.e. the sea waves. Then the probabilistic properties of these quantities can be computed from the wave spectrum by solving the equations of motion.
It is supposed that the structure during its lifetime will be exposed to a variety of different storm systems, each of a different intensity and direction. As a simplification for the current study, we will consider the platform exposed to only one storm, a directional wave system described by the product of a two-parameter ISSC wave spectrum and a 4th-power-cosine spreading function suggested by ISSC, as given by:

$$S_x(\omega, \theta) = \left[ AB \omega^{-5} e^{-B\omega^4} \right] \left( \frac{8}{3\pi} \cos^4(\theta - \theta_0) \right)$$  \hspace{1cm} (2.5.1)

where

$$A = 0.25(H_{1/3})^2 \hspace{1cm} B = (0.817 \frac{2\pi}{T_1})^4$$

$H_{1/3}$ = the significant wave height of the wave system

$T_1$ = the mean wave period corresponding to average frequency of component waves

$\theta_0$ = the prominent wave direction of the wave system

$\theta$ = the angle relative to the principal wind direction of the wave system

The significant wave height and mean wave period are selected for the spectrum to represent a short-term wave system that would be significant for the floating runway in this study. In this context they are selected as:

$$H_{1/3} = 4 \text{ m}, \hspace{1cm} T_1 = 10 \text{ sec.}$$ \hspace{1cm} (2.5.2)

The zeroth moment (area beneath) $m_0$ and second moment $m_2$ of the selected ISSC point spectrum are finite while its fourth moment $m_4$ is infinite, as pointed out by St. Denis (St. Denis, 1980), which makes it wide-banded with band width parameter

$$\varepsilon = \sqrt{1 - \frac{m_2^2}{m_0 m_4}} = 1.0,$$ \hspace{1cm} (2.5.3)
and the mean period between peaks (Lewis, 1989)

\[ T_c = 2\pi \sqrt{\frac{m_2}{m_4}} = 0.0 \quad (2.5.4) \]

while an alternate definition of the band width parameter shows it's narrow-banded:

\[ q = \sqrt{1 - \frac{m_1^2}{m_0 m_2}} = 0.391 \quad (2.5.5) \]

This is because the long tail of the spectrum. It is recommended to truncate the spectrum at \( \omega = 5\omega_m \) in (Lewis, 1989), where \( \omega_m \) is its modal frequency. But here the specified spectrum becomes very small after \( \omega = 1.5 \text{Rad/sec} \), so we truncated it at \( \omega = 4.12\omega_m = 2.0 \text{Rad/sec} \) where the spectrum \( S(\omega = 2.0) = 0.0086m^2 \cdot \text{sec} \) is small enough and the wave length \( \lambda = 15.4 \text{meters} \) is also small enough. The point spectrum is also truncated at low frequency \( \omega = 0.3 \text{Rad/sec} \) where the spectrum \( S(\omega) = 0.0216m^2 \cdot \text{sec} \) is small enough and the wave length \( \lambda = 684.2 \text{meters} \) is already very long. In this truncated frequency domain there is 99.5% of the total wave energy.

For the truncated spectrum the actual mean period corresponding to average frequency of component waves becomes (Lewis, 1989)

\[ T_{\text{1}} = 2\pi \frac{m_0}{m_1} \approx 10.1 \text{sec}(\approx 10 \text{sec}) \quad (2.5.6) \]

The actual mean period between peaks now becomes

\[ T_c = 2\pi \sqrt{\frac{m_2}{m_4}} \approx 7.1 \text{sec} \quad (2.5.7) \]

The actual significant wave height now becomes

\[ H_{\frac{1}{3}} = 4\sqrt{m_0} \approx 3.99 \text{meters}(\approx 4.00 \text{meters}) \quad (2.5.8) \]
A plot of the adopted point wave spectrum is shown in Figure 2.5.1. The actual bandwidth parameters become $\varepsilon = 0.665$ and $q = 0.338$. Because the wave energy is actually concentrated in a narrow range of frequency as shown from Figure 2.5.1 and by the latter bandwidth parameter we can consider the wave system is narrow banded. The large value of $\varepsilon$ is caused by the long tail which contains little energy and is not important to structural responses.

For the spreading function we select the prominent wave direction to be $\theta_0 = 60^\circ$ and the scope of wave directions are determined from $-90^\circ < \theta - \theta_0 < 90^\circ$ as follows:

$$-30^\circ < \theta < 150^\circ, \theta_0 = 60^\circ$$

(2.5.9)
$S(\omega), m^2 \cdot \text{sec},$

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{ISSC Point Wave Spectrum with $H_{1/3} = 3.99 \, m, T_1 = 10.1 \, \text{sec}, \quad 0.3 < \omega < 2.0 \,(\text{rad} / \text{sec})$}
\end{figure}
The adopted 2-D directional spectrum is shown in Figure 2.5.2.

The structural responses are studied for a period of *half an hour* when this wave system can be considered approximately stationary. To study the structural responses in this wave system, the directional wave spectrum is then discretized both in the frequency domain and in the direction domain into a number of component waves of unit amplitude, each represents a same amount of energy of the 2-D directional spectrum. First the volume under the 2-D spectrum surface (which is proportional to the wave energy) is divided into blocks of variable height by 9 frequencies between 0.3 Rad/sec and 2.0
Rad/sec and by 8 directions between $-90^\circ$ and $90^\circ$ ($\theta_0$ will be added later). These dividing frequencies and directions are selected so that they divide the total volume (wave energy) equally in the frequency and direction domain. The resulted blocks will have equal volume (wave energy). Then each block is represented by a component wave having the same amount of energy. The frequency and direction of the component waves are set at the centroid of the block volume. The dividing of the 2-D wave spectrum in the frequency and angle domains, as well as the frequencies and directions of the component waves are shown in Table 2.5.1 and Table 2.5.2.
<table>
<thead>
<tr>
<th>Division No.</th>
<th>Frequency Interval (Rad/sec)</th>
<th>Frequencies of Component waves (Rad/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(0.3000000, 0.4166045)</td>
<td>0.3851902</td>
</tr>
<tr>
<td>2</td>
<td>(0.4166045, 0.4555039)</td>
<td>0.4368349</td>
</tr>
<tr>
<td>3</td>
<td>(0.4555039, 0.4896626)</td>
<td>0.4727040</td>
</tr>
<tr>
<td>4</td>
<td>(0.4896626, 0.5241000)</td>
<td>0.5067298</td>
</tr>
<tr>
<td>5</td>
<td>(0.5241000, 0.5617556)</td>
<td>0.5425384</td>
</tr>
<tr>
<td>6</td>
<td>(0.5617556, 0.6059571)</td>
<td>0.5831325</td>
</tr>
<tr>
<td>7</td>
<td>(0.6059571, 0.6622853)</td>
<td>0.6327593</td>
</tr>
<tr>
<td>8</td>
<td>(0.6622853, 0.7433393)</td>
<td>0.6998214</td>
</tr>
<tr>
<td>9</td>
<td>(0.7433393, 0.8920120)</td>
<td>0.8079115</td>
</tr>
<tr>
<td>10</td>
<td>(0.8920120, 2.0000000)</td>
<td>1.1348488</td>
</tr>
</tbody>
</table>

Table 2.5.1  Dividing of the 2-D Wave Spectrum in the Frequency Domain
<table>
<thead>
<tr>
<th>Division No.</th>
<th>Direction Interval (Rad)</th>
<th>Directions of Component waves (Rad)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(-\frac{\pi}{2}, -0.55732)$</td>
<td>-0.74921</td>
</tr>
<tr>
<td>2</td>
<td>$(-0.55732, -0.35535)$</td>
<td>-0.44971</td>
</tr>
<tr>
<td>3</td>
<td>$(-0.35535, -0.20171)$</td>
<td>-0.27629</td>
</tr>
<tr>
<td>4</td>
<td>$(-0.20171, -0.06564)$</td>
<td>-0.13285</td>
</tr>
<tr>
<td>5</td>
<td>$(-0.06564, 0.06564)$</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>$(0.06564, 0.20171)$</td>
<td>0.13285</td>
</tr>
<tr>
<td>7</td>
<td>$(0.20171, 0.35535)$</td>
<td>0.27629</td>
</tr>
<tr>
<td>8</td>
<td>$(0.35535, 0.55732)$</td>
<td>0.44971</td>
</tr>
<tr>
<td>9</td>
<td>$(0.55732, \frac{\pi}{2})$</td>
<td>0.74921</td>
</tr>
</tbody>
</table>

Table 2.5.2 Dividing of the 2-D Wave Spectrum in the Direction Domain

The responses of the structure are calculated for each of the component waves by solving the Motion Equation (2.4.10) and then calculating the deformations and stresses using finite difference method. The structural responses for each panel $(i, j)$ in the whole structure under component waves of the same frequency but different directions are summed to get the Transfer Functions of the 7 structural responses, noted as $H(i,j,k; \omega)$,
\[ i=1,2,\ldots,2nl, \ j=1,2,\ldots,2nb, \ k=1,2,\ldots,7 \]. Then the spectra of the structural responses of panel \((i, j)\) can be computed from the input wave spectrum as:

\[
S_y(i, j, k; \omega) = \left| H(i, j, k; \omega) \right|^2 S_X(\omega), \quad i=1,2,\ldots,2nl, \ j=1,2,\ldots,2nb, \ k=1,2,\ldots,7 \quad (2.5.10)
\]

The mean (zero up-crossing) period of structural responses can be determined by:

\[
T_y(i, j, k) = \frac{1}{V_0^+(i, j, k)} = 2\pi \sqrt{\frac{m_0(i, j, k)}{m_2(i, j, k)}}, \quad (2.5.11)
\]

where

\[
m_0(i, j, k) = \int_0^\infty \int_0^\tau S_y(i, j, k; \omega) d\omega d\theta
\]

\[
m_2(i, j, k) = \int_0^\infty \int_0^\tau \omega^2 S_y(i, j, k; \omega) d\omega d\theta
\]

Since the input wave spectrum is narrow-banded, we also suppose the structural responses are narrow-banded. Then the peaks of the responses in panel \((i, j)\), with mean period \(T_y(i, j, k)\) follow the Rayleigh distribution. The probability for response peak \(\sigma_k\) to be below value \(x\) can be expressed as:

\[
P_{i, j, k}(\sigma_k < x) = 1 - e^{-\frac{x^2}{2m_0(i, j, k)}} \quad (2.5.12)
\]

\[ i=1,2,\ldots,2nl, \ j=1,2,\ldots,2nb, \ k=1,2,\ldots,7 \]

By order statistics, the probability of failure, i.e. the probability for the response peak \(\sigma_k\) to exceed a pre-determined value \(a_k\) in a certain period (here half an hour), can be computed for panel \((i, j)\) by:

\[
Pf(i, j, k; a_k) = P_{i, j, k}(\sigma_k > a_k) = 1 - P_{i, j, k}(\sigma_k < a_k) = 1 - \left[ 1 - e^{-\frac{a_k^2}{2m_0(i, j, k)}} \right]^\frac{1800}{T_y(i, j, k)} \quad (2.5.13)
\]

68
The probability of failure is computed for all the seven structural responses of all the panels in the whole floating-mat. For each kind of responses the probability of failure is tolerated when it is below a certain level. The tolerated levels for the slope and curvature are higher because these two only affect the availability of the floating runway while those for the stresses determine the safety of the structure. Then a single measure of the structure strength can be computed as (Webster, 1998):

\[
\lambda_{\text{max}} = \text{Max}_{i=1}^{7} \left\{ \frac{2n_l \cdot 2n_b \text{ Max } \left[ Pf(i, j, k) \right] - u_k}{u_k} \right\} \quad (2.5.14)
\]

where \( u_k \) is the tolerated probability of failure for response \( k=1,2,\ldots,7 \), and \( \lambda_{\text{max}} \) is the combined single measure of the structural strength applying the concept of reliability. The \( u_k \)'s for slope and curvature are set to 0.01 and those for stresses are set to 0.001.

Values of \( \lambda_{\text{max}} \) are in the range \( \lambda_{\text{max}} \geq -1 \). If this measure is greater than zero, the structure is too weak and at least one criterion is violated. If it is less than zero, the structure is too strong. And \( \lambda_{\text{max}} = 0 \) denotes the limit state surface where the structure is just feasible. The result of the optimization should stay on this surface.

The optimization process is further discussed in Chapter 4.
Chapter 3

High-Order Method

3.1 The Interpolation Functions

In order to perform the hydrodynamics and structural computations for the motions of a barge-like structure, it will be necessary to introduce interpolation functions for various physical quantities. In Appendix I, the use of general higher-order interpolation functions are discussed. For the research presented here, the bi-fifth-order interpolation functions are chosen, as described below.

3.1.1 The 36-Parameter Flat Bi-Fifth-Order Interpolation Functions

A 36-parameter flat bi-fifth-order interpolation function is chosen to represent the distribution of any physical quantity, such as deflections, pressures and source density, on a flat rectangular panel of the floating-mat.

At each corner node \((x_i, y_i), i=1,2,3,4\) of the panel, 9 parameters, i.e. the value and 8 derivatives up to second order in both \(x\) and \(y\), of the interpolation function are prescribed. From the total 36 parameters in a panel, a bi-fifth-order interpolation function,

\[
f(x, y) = \sum_{i=0}^{5} \sum_{j=0}^{5} a_{ij} x^i y^j,
\]

(3.1.1)
can be determined uniquely. Let the 36 parameters be denoted as $f_{ij}$, where $i=1,2,3,4$ refer to the corner nodes and $j=1,2,...,9$ refer to the value and derivatives (i.e. $f(x_i,y_i)$, $f_x(x_i,y_i)$, $f_y(x_i,y_i)$, $f_{xy}(x_i,y_i)$, $f_{xx}(x_i,y_i)$, $f_{yy}(x_i,y_i)$, $f_{xyy}(x_i,y_i)$, $f_{xxy}(x_i,y_i)$, $f_{xyp}(x_i,y_i)$) respectively.

When these bi-fifth-order panels are joined together in a flat array such that the parameters corresponding to corner nodes are identical, then the resulting function and all its derivatives up to second order, i.e. $f(x,y)$, $f_x(x,y)$, $f_y(x,y)$, $f_{xy}(x,y)$, $f_{xx}(x,y)$ and $f_{yy}(x,y)$, are continuous across all the four edges. High-order derivatives of the above six functions in the direction of the edges are also always continuous, i.e. $f_{xxx}(x,y)$, $f_{xxxx}(x,y)$, ..., $f_{x...x}(x,y)$, $f_{xyy}(x,y)$, $f_{xxy}(x,y)$, ..., $f_{x...y}(x,y)$ and $f_{xyy}(x,y)$, $f_{xyyy}(x,y)$, ..., $f_{y...y}(x,y)$, $f_{xyy}(x,y)$, $f_{xxy}(x,y)$, ..., $f_{x...y}(x,y)$ and $f_{xyy}(x,y)$, $f_{xxy}(x,y)$, ..., $f_{x...y}(x,y)$ are continuous across edges parallel to x-axis and $f_{xyy}(x,y)$, $f_{xyyy}(x,y)$, ..., $f_{y...y}(x,y)$, $f_{xyy}(x,y)$, $f_{xxy}(x,y)$, ..., $f_{x...y}(x,y)$ and $f_{xyy}(x,y)$, $f_{xxy}(x,y)$, ..., $f_{x...y}(x,y)$ are continuous across edges parallel to y-axis.

The bi-fifth-order interpolation function is then decomposed into the form of a sum of 36 products of the basis functions, $b_{ij}(x,y)$, with their corresponding parameters, $f_{ij}$:

$$f(x,y) = \sum_{i=1}^{4} \sum_{j=1}^{9} f_{ij} b_{ij}(x,y), \quad (3.1.2)$$

Let us define 9 differential operators, $D_{kl}$, $k=1,2,...; l=1,2,...,9$, in the following way:

$$D_{k1}g(x,y) = g(x,y)|_{node1},$$

$$D_{k2}g(x,y) = \partial_x g(x,y)|_{node1},$$

$$D_{k3}g(x,y) = \partial_y g(x,y)|_{node1},$$
\[ D_{k4} g(x, y) = \partial_{xy} g(x, y) \left|_{node 1} \right., \]
\[ D_{k5} g(x, y) = \partial_{xx} g(x, y) \left|_{node 1} \right., \]
\[ D_{k6} g(x, y) = \partial_{yy} g(x, y) \left|_{node 1} \right., \]
\[ D_{k7} g(x, y) = \partial_{xx} g(x, y) \left|_{node 1} \right., \]
\[ D_{k8} g(x, y) = \partial_{xy} g(x, y) \left|_{node 1} \right., \]
\[ D_{k9} g(x, y) = \partial_{xyy} g(x, y) \left|_{node 1} \right.. \]

Then \( b_{ij}(x, y) \) are chosen so that:

\[ D_{kl} b_{ij}(x, y) = \delta(k, i) \delta(l, j). \] (3.1.3)

The polynomials representing the 36 basis functions for a panel occupying the area \([-\frac{a}{2} \leq x \leq \frac{a}{2}] \times [-\frac{b}{2} \leq y \leq \frac{b}{2}] \) and the shapes of the 36 basis functions in a square panel with unity lateral length are shown in Appendix II.

In order to evaluate the 36 parameters \( f_{ij}, i=1,2,3,4; j=1,2,...,9 \) we need to know the various derivatives at the node \( i \) of the function to be interpolated. The analytical form of the function to be interpolated is unknown. We only know the value of the interpolated function for a finite set of points on the panel. One such set could be, for instance, a set of 2-D grid points for Gaussian Quadrature, or a set of evenly distributed grid points. A relationship must be established between the function values at the grid points, \( d_k, k=1, 2, ..., n \) and the 36 parameters, \( f_{ij}, i=1,2,3,4; j=1,2,...,9 \). The relations between them can be represented by a linear transform matrix:
\{ f_y \} = [A] \{ d_k \} \hspace{1cm} (3.1.4)

To determine the 36 parameters we need at least 36 function values at 36 grid points on the panel. If there are 36 function values at 36 grid points, we can find an 'exact' relationship to compute the 36 parameters. The term 'exact' just means that the interpolation surface determined by the 36 parameters has exactly the same value as prescribed at the 36 grid points while it may have value different from the interpolated function elsewhere. If there are more than 36 function values at more grid points, we can find 'least-square-approximate' relations to compute the 36 parameters. Although the interpolation surface determined by the 36 parameters may have value different from the interpolated function at the grid points and elsewhere, the sum of squared differences between them at the grid points is least among any other choice of the 36 parameters. In our research not-evenly distributed nodes, like the Gaussian Quadrature grid points of $6 \times 6$, $7 \times 7$ and $8 \times 8$ points and evenly distributed grid points of $6 \times 6$, $7 \times 7$, $8 \times 8$ and $16 \times 16$ points have been tried. The not-evenly distributed nodes lose the regularity of the evenly distributed grid points, a regularity that can be taken advantage of in order to avoid repeated integration for same relative location of the field point to the source distribution. As shown in section 3.4 below, only evenly distributed grid points are selected. In order to capture as much information about the distribution inside the panels as possible, $16 \times 16$ evenly distributed grid points are ultimately used.

Consider a node in the bottom plate mesh of the floating runway. If it is in the inner part of the mesh, then it will be common to four panels. For any one of the 9 parameters defined at that node there is a corresponding basis function in each of the four connected panels. These four basis functions will form a nodal basis function.
$n_{cenb}_j(x,y), j=1,2,...,9$, where $i$ is the node index in the bottom plate mesh and $j$ is the
parameter index. The center nodal basis function $n_{cenb}_j(x,y)$ is defined by the
connected four panel basis functions and satisfies $D_{kl}n_{cenb}_j(x,y) = \delta(k,i)\delta(l,j)$ for the
center node $i$ and the surrounding 8 nodes. Similarly, a node on the edge of the bottom
plate (but not a corner) mesh connects two panels. The nodal basis functions at these edge
nodes, $n_{edgb}_j(x,y)'s$, are defined on the two connected panels and have the similar
parameter values as the center nodal basis functions. But the corner nodes involve only
one panel, so their nodal basis functions, $n_{corb}_j(x,y)'s$, are just the panel basis
functions at the corresponding corners. The shape of some of the nodal basis functions,
$n_{cenb}_j(x,y)$ and $n_{edgb}_j(x,y)$ are shown in Appendix II.

Similarly, to evaluate the parameters at a node in the bottom plate mesh, we would
use the function values at the grid points in all the connected panels around the node.
Similar transformation rules as (3.1.4) are obtained between the nodal parameters and
functions values at surrounding grid points for nodes in the center area and four edges of
the mesh using least-square method. A transformation rule for nodal parameters using
$16 \times 16$ grid points around the node is developed.

Because the interpolated functions are decomposed into 4 symmetries about the X
and Y axes and only the distributions in the first quadrant are modeled, symmetric
conditions must be imposed on the symmetry axes of the bottom plate mesh, i.e. edges at
$x=0$ and $y=0$. So for nodes on these edges only those parameters that don’t violate the
symmetry conditions can be non-zero. So we don’t need to evaluate those zero
parameters that violate the symmetry conditions using the transformation rule. Instead,
those zero modes are deleted in the process of computing the Hydrodynamic Influence Matrices and also when the Motion Equations are solved. The zero modes (represented by 0) and non-zero modes (represented by 1) for the 4 symmetries and for different areas on the symmetry axes are listed in Table 3.1.1, Table 3.1.2 and Table 3.1.3.

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<th>No. of Symmetry</th>
<th>isym=1</th>
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Table 3.1.1  Zero and Non-Zero Modes for the Center Node (Intersection of X and Y Symmetry Axes)
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Table 3.1.2  Zero and Non-Zero Modes for Off-Center Nodes on X Symmetry Axis
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Table 3.1.3  Zero and Non-Zero Modes for Off-Center Nodes on Y Symmetry Axis

This family of bi-fifth-order functions is capable of representing one wave within a panel. One drawback of them is that their continuity is still not enough for the forth-order differentiation that appears in the plate bending term of the Motion Equation of the floating-mat. When they are used to represent the bending deformation of a plate, the resulted structural pressure will be not-continuous across the panel boundary and will have multi-values at a node connected to 2 or 4 panels. Although this family of interpolation functions can be used directly when computing the hydrodynamic properties (the Hydrodynamic Influence Matrices), an approximate method need to be used to
determine the deformation effects. In our research a weak-form of the Motion Equation is derived using Galerkin Method to overcome this difficulty.
3.1 The High-Order Hydrodynamic Influence Matrices

If we use high-order interpolation functions to represent the physical quantities distributed over the mesh of high-order panels on the body surface, the represented quantities will have certain degree of smoothness guaranteed by the property of the high-order interpolation functions as stated in section 3.1. In our research the 5th order interpolation functions are finally chosen so the represented distributions will have continuous second-order derivatives across the panel boundary. The physical quantities represented this way include plate deflection, source distribution, pressure distribution, normal vector distribution, and so on.

First the body surface is discretized as for the low-order method. The four-fold symmetry is also used to reduce the scale of computation so only the first quadrant of the wetted body surface of \([0, L/2] \times [0, B/2] \times [0, D]\) needs to be discretized and the motion equation needs to be solved for each one of the four symmetries.

The bottom plate is discretized into \(num = nl \times nb\) panels as for the low-order method except that fewer panels are needed for the same accuracy of the computation. Thus, the panel size on the bottom plate is much larger than for the low-order method. But the draft of the floating runway is very small (2 meters). This situation will generate very narrow panels if we discretize the side plate and the bow plate into same number of panels as on the bottom plate. Also the high-order interpolation along the short vertical side will waste computation time and storage space. So in the high-order method, the side plate and bow plate are discretized into more panels in the longitudinal or transverse direction to make their aspect ratio closer to 1. Further only low-order panels are used. The low-order side
and bow panel numbers are increased by \( k \) times the high-order panels on the bottom plate. In the high-order method these low-order panels are treated similarly as the high-order panels with only one node, i.e. the node at the panel center, and only one basis function, i.e. the unity distribution over that panel. \( k \) is taken to be an integer so that the side and bow panels will maintain the same regularity as the grid points. This allows avoiding repeated integration for same relative location of the field point to the source distribution, as shown in section 3.4. Several values of \( k \) were tried, but finally \( k=15 \) is selected to match the 16×16 grid points on the bottom plate.

So the total number of the nodes in the mesh of the first quadrant for \( nt=1 \) and \( k=15 \) is

\[
\text{node} = (nl + 1) \times (nb + 1) + k \times nl \times nt + k \times nb \times nt = (nl + 1) \times (nb + 1) + 15 \times nl + 15 \times nb.
\]

The total number of coefficients that define a distribution in the first quadrant is

\[
\text{nunkn} = 9 \times (nl + 1) \times (nb + 1) + k \times nl \times nt + k \times nb \times nt.
\]

The mesh for the other quadrants is just symmetric images of that in the first quadrant. Similar to low-order method, the distributions in the other quadrants are related to those in the first quadrant by the sign coefficients for the symmetries as given in Table (2.2.1).

Suppose the deflection for symmetry \( isym \) in the first quadrant is \( w_{isym,1}(x, y) \) and it is decomposed according to the basis functions as (the time dependent term \( e^{-j\omega t} \) is dropped hereafter):

\[
w_{isym,1}(x, y) = \sum_{i=1}^{nunkn} a_{i,isym} B_i(x, y), \quad isym = 1, 2, 3, 4 \tag{3.2.1}
\]

Similarly the source distribution is decomposed for the symmetry, \( isym \), in all the four quadrants according to the basis functions as follows:
\[
O_{i\text{sym},i\text{quad}}(x, y) = \text{sign}(i\text{sym}, i\text{quad}) \sum_{j=1}^{n\text{unkn}} \tilde{O}_{j,i\text{sym}} B_j(x, y),
\]
where \( i\text{sym} = 1,2,3,4; \ i\text{quad} = 1,2,3,4 \) (3.2.2)

The coefficients \( \tilde{O}_{j,i\text{sym}} \)'s are the value and derivatives of the source distribution, and can be related to the coefficients \( a_{i,i\text{sym}} \)'s through matrices \( q_{i,i\text{sym}} \)'s as follows:

\[
\tilde{O}_{j,i\text{sym}} = \sum_{i=1}^{n\text{unkn}} a_{i,i\text{sym}} q_{ij,i\text{sym}}
\]
(3.2.3)

Adopting Einstein’s convention from now on in this section, the velocity potential generated at a point \((x, y, z)\) in the first quadrant mesh of the wetted body surface, by the source distribution in all the four quadrants under symmetry \( i\text{sym} \) (always not summed), can now be written as:

\[
\phi_{i\text{sym}}(x, y, z; \omega) = a_{i,i\text{sym}} q_{ij,i\text{sym}} \sum_{i\text{quad}=1}^{4} \text{sign}(i\text{sym}, i\text{quad}) \left[ \int \int B_j(\xi, \zeta) G(x, y, z; \xi, \zeta; \omega) d\xi d\zeta \right] (3.2.4)
\]

Substitute (3.2.1) and (3.2.4) into the rigid body B.C.:

\[
n \cdot \nabla \phi(x, y, z; \omega) = -j \alpha_n(x, y, z) \quad (3.2.5)
\]
and we can get:

\[
a_{i,i\text{sym}} q_{ij,i\text{sym}} n \cdot \nabla \sum_{i\text{quad}=1}^{4} \text{sign}(i\text{sym}, i\text{quad}) \left[ \int \int B_j(\xi, \zeta) G(x, y, z; \xi, \zeta; \omega) d\xi d\zeta \right] S_{i\text{quad}} = -j \alpha n_{i,i\text{sym}} B_i(x, y, z) \quad (3.2.6)
\]

In (3.2.4) and (3.2.6) the free-surface Green Function \( G(x, y, z; \xi, \zeta; \omega) \) and its gradient can be computed using standard subroutine FINGREEN (Newman, 1985) as in low-order method.

Carrying out the same decomposition according to the basis functions on the last part at the left hand side we get:

81
\[ C_{j,\text{sym}}(x, y, z; \omega) = n \cdot \sum_{i\text{quad}=1}^{4} \text{sign}(i\text{sym}, i\text{quad}) \left[ \int_{S_{\text{quad}}} B_j(\xi, \zeta) G(x, y, z; \xi, \zeta; \omega) d\xi d\zeta \right] \]

\[ = n \cdot \sum_{i\text{quad}=1}^{4} \text{sign}(i\text{sym}, i\text{quad}) \left[ \int_{S_{\text{quad}}} B_j(\xi, \zeta) \nabla G(x, y, z; \xi, \zeta; \omega) d\xi d\zeta \right] \]

\[ = c_{j,\text{sym}}(\omega) B_k(x, y, z) \quad (3.2.7) \]

Substituting (3.2.7) in (3.2.6) we get:

\[ a_{i,\text{sym}} q_{ij,\text{sym}} c_{jk,\text{sym}} B_k(x, y, z) = -j \alpha a_{i,\text{sym}} B_i(x, y, z) \quad (3.2.8) \]

Since the basis functions are independent, their coefficients at the two sides of (3.2.8) must be equal:

\[ a_{i,\text{sym}} q_{ij,\text{sym}} c_{jk,\text{sym}} = -j \alpha a_{k,\text{sym}} \quad (3.2.9) \]

To make (3.2.9) an identity we can just let

\[ q_{ij,\text{sym}} = -j \alpha c_{ij,\text{sym}}^{-1} \quad (3.2.10) \]

Similar to (2.2.3) we can get the pressure distribution corresponding to the deflection distribution as follows:

\[ p_{\text{sym}}(x, y, z; \omega) = j \rho \omega \varphi_{\text{sym}}(x, y, z; \omega) \]

\[ = j \rho \omega a_{i,\text{sym}} q_{ij,\text{sym}} \sum_{i\text{quad}=1}^{4} \text{sign}(i\text{sym}, i\text{quad}) \left[ \int_{S_{\text{quad}}} B_j(\xi, \zeta) G(x, y, z; \xi, \zeta; \omega) d\xi d\zeta \right] \]

\[ = \rho \omega^2 a_{i,\text{sym}} c_{ij,\text{sym}}^{-1} \sum_{i\text{quad}=1}^{4} \text{sign}(i\text{sym}, i\text{quad}) \left[ \int_{S_{\text{quad}}} B_j(\xi, \zeta) G(x, y, z; \xi, \zeta; \omega) d\xi d\zeta \right] \]

\[ = \rho \omega^2 a_{i,\text{sym}} c_{ij,\text{sym}}^{-1} U_{j,\text{sym}}(x, y, z; \omega) = (\rho \omega^2 a_{i,\text{sym}} c_{ij,\text{sym}}^{-1} U_{j,\text{sym}}) B_i(x, y, z; \omega) \]

\[ = p_{k,\text{sym}} B_k(x, y, z; \omega) \quad (3.2.11) \]

where another decomposition according to basis functions is carried out on the integrated distribution in (3.2.11) as:
\[ U_{j, isym}(x, y, z; \omega) = \sum_{i,quad=1}^{4} \text{sign}(isym, iquad) \int_{S_{quad}} B_j(\xi, \zeta) G(x, y, z; \xi, \zeta; \omega) d\xi d\zeta \]

\[ = u_{jk, isym} B_k(x, y, z; \omega) \tag{3.2.12} \]

where \( u_{jk, isym} \) is the \( k \)th of the nine value and derivatives of the distribution at node \( j \), corresponding to the \( k \)th nodal basis function \( B_k \).

The Hydrodynamic Influence Matrix for symmetry \( isym \) is defined as:

\[ [h_{ik, isym}] = \left[ c_{ij, isym}^{-1} u_{jk, isym} \right] \quad i, j, k = 1, 2, \ldots, nunkn; \quad isym = 1, 2, 3, 4 \tag{3.2.13} \]

Compared with the Hydrodynamic Influence Matrices for the low-order method defined by (2.2.8), the matrices here involve the coefficients of the high-order interpolation. But similarly the hydrodynamic influence matrices defined by (3.2.13) also encapsulate all the hydrodynamic properties of the 3-Dimension mesh of the body surface. They also can be computed once for all beforehand for the four symmetries and for all the frequencies considered.

Then the coefficients \( p_{k, isym} \) in the decomposition of the pressure distribution in the last part of (3.2.11) can be written as:

\[ p_{k, isym} = \rho \omega^2 a_{i, isym} h_{ik, isym} \tag{3.2.14} \]

Computing the matrices \( \{c_{ij, isym}\} \) and \( \{u_{ij, isym}\} \) is very time-consuming. This is done by first evaluating the corresponding surface integrations at each of the grid points (i.e. point \((x, y, z)\)) in the first quadrant of the mesh and then transforming them into nodal parameters using a linear transformation rule as shown in (3.1.4). For the reasons stated in section 3.1 right after (3.1.4), 16x16 evenly distributed grid points are ultimately used.
3.3 The Integration of the Singularities in the Green Function and its Gradient

3.3.1 Analytically Exact Formulations

In the high-order method, when we compute the velocity potential and the velocity on the body surface, we need to integrate the product of the high-order source distribution and the free-surface Green Function or its gradient over rectangular panels, as in $U_{j,isym}(x, y, z; \omega)$ and $C_{j,isym}(x, y, z; \omega)$. Normally this kind of integration can be evaluated using the Gaussian Quadrature. But Gaussian Quadrature is not sufficiently accurate for all situations encountered here where the kernel is singular. When the field point is inside or close to the source panel, the result of the Gaussian Quadrature usually contains large numeric errors and is not usable. This phenomenon is caused by the fact that a part of the free-surface Green Function, e.g. $1/R$, with $R$ being the distance between the field point and the source point, is singular or nearly so when the field point is inside or close to the source panel. When the field point is inside the source panel, $1/R$ is infinite at the field point because $R=0$. In the close vicinity of the field point, $1/R$ increases so fast that the polynomial basis of Gaussian Quadrature can’t represent the actual function well. The singular behavior is highly localized around the field point so increasing the order of the Gaussian Quadrature does not help much. For typical functions a $4 \times 4$ point Gaussian Quadrature is quite accurate, but in this case even a $40 \times 40$ points Gaussian Quadrature yields considerable error. The situation with Grad($1/R$), which appears as a part of the gradient of the free-surface Green Function, is even worse because of its higher order of singularity.
One common method to deal with this kind of localized singularity is to subdivide the panel into many small patches. Inside each of the small patches the integrand will be relatively of same character for a low order Gaussian Quadrature to get acceptable results. Many small patches would be needed to get accurate results and thus the computation will turn out to be quiet burdensome.

Another method is to use the analytically exact formulae obtained from carrying out the integration analytically. Once we have analytically exact formulae, we can use these to get an exact result under any relative position of the field point and the source panel. It is simple in principle. However actually many factors will adversely affect the computation and wise manipulations of the analytically exact formulae and the computing process are definitely necessary. Some approximate formulae are also needed to reduce the complexity in the numerical computing of the formulae. The analytically exact formulae for the integrations are first prepared here.

First consider the integrations in $U_{j,\text{sym}}(x, y, z; \omega)$ and $C_{j,\text{sym}}(x, y, z; \omega)$ over $S_{\text{quad}}$. Their integrands involve the product of nodal basis function $B_j(\xi, \zeta)$ and the Green Function $G(x, y, z; \xi, \zeta; \omega)$ or its gradient $\nabla G(x, y, z; \xi, \zeta; \omega)$. The nodal basis function is non-zero only in the panels connected to the node, which may be four, two or one panels as shown in section 3.1. So the integrations in $U_{j,\text{sym}}(x, y, z; \omega)$ and $C_{j,\text{sym}}(x, y, z; \omega)$ over $S_{\text{quad}}$ can be computed as sums of the integrations of the products of the simple basis function and the Green Function or its gradient over all the single panels. Consider the integration over a panel in the domain of $(bb, aa) \times (dd, cc)$ on the $\xi - \zeta$ plane, there are totally 36 different forms involving the 36 basis functions defined on the panel:

85
\[ \int_{\delta \eta \delta \xi \delta \zeta} B_j(\xi, \zeta) G(x, y, z; \xi, \zeta; \omega) \, d\xi d\zeta \text{ or } \int_{\delta \eta \delta \xi \delta \zeta} B_j(\xi, \zeta) \nabla G(x, y, z; \xi, \zeta; \omega) \, d\xi d\zeta, \]  

(3.3.1)

where \( j = 1, 2, \ldots, 36 \). (For simplicity in notation we will replace the double index \( i = 1, 2, 3, 4, \) and \( j = 1, 2, \ldots, 9 \) with one index \( j = 1, 2, \ldots, 36 \).)

The coordinates \((\xi, \zeta)\) refer to the position of the source and will be denoted as \((x_s, y_s, z_s)\) hereafter while the coordinates \((x, y, z)\) refer to the position of the field point and will be denoted as \((x_f, y_f, z_f)\) hereafter. The singular part of the Green Function \( G(x_f, y_f, z_f; x_s, y_s, \omega) \) which can not be handled with Gaussian Quadrature is point source potential:

\[ \psi(x_f, y_f, z_f; x_s, y_s, z_s) = \frac{1}{R} = \frac{1}{\sqrt{(x_s - x_f)^2 + (y_s - y_f)^2 + (z_s - z_f)^2}} \]  

(3.3.2a)

Because the draft of the floating runway is very small (2 meters), the potential of the image of the above point source about the free surface is nearly singular and can also cause numerical errors in the Gaussian Quadrature. So the potential of the image of the point source, which is also a part of \( G(x_f, y_f, z_f; x_s, y_s, \omega) \), needs to be dealt with too.

The image of the point source at \((x_s, y_s, z_s)\) is located at \((x_s, y_s, -z_s)\) so its potential can be written as:

\[ \psi'(x_f, y_f, z_f; x_s, y_s, -z_s) = \frac{1}{R} = \frac{1}{\sqrt{(x_s - x_f)^2 + (y_s - y_f)^2 + (-z_s - z_f)^2}} \]  

(3.3.2b)

The corresponding gradient of the potential of the point source, which is a part of the gradient of the Green Function \( \nabla G(x_f, y_f, z_f; x_s, y_s, \omega) \) that needs to be dealt with outside Gaussian Quadrature, can be written as:

86
\[ \nabla_f \psi(x_f, y_f, z_f; x_s, y_s, z_s) = \nabla_{x_f} \psi \mathbf{i} + \nabla_{y_f} \psi \mathbf{j} + \nabla_{z_f} \psi \mathbf{k} \]

\[ = \frac{x_s - x_f}{(x_s - x_f)^2 + (y_s - y_f)^2 + (z_s - z_f)^2} \mathbf{i} + \frac{y_s - y_f}{(x_s - x_f)^2 + (y_s - y_f)^2 + (z_s - z_f)^2} \mathbf{j} + \frac{z_s - z_f}{(x_s - x_f)^2 + (y_s - y_f)^2 + (z_s - z_f)^2} \mathbf{k} \]  

(3.3.3a)

Similarly the potential gradient of the image point source also needs to be dealt with outside Gaussian Quadrature, which can be written as:

\[ \nabla_f \psi'(x_f, y_f, z_f; x_s, y_s, -z_s) = \frac{x_s - x_f}{(x_s - x_f)^2 + (y_s - y_f)^2 + (-z_s - z_f)^2} \mathbf{i} + \frac{y_s - y_f}{(x_s - x_f)^2 + (y_s - y_f)^2 + (-z_s - z_f)^2} \mathbf{j} + \frac{-z_s - z_f}{(x_s - x_f)^2 + (y_s - y_f)^2 + (-z_s - z_f)^2} \mathbf{k} \]  

(3.3.3b)

The potential caused by the source panel \( n \) with a distribution defined by a basis function \( B_k(x_s, y_s), k=1,2,...,36 \), at the field point \( (x_f, y_f, z_f) \) is then as follows:

\[ I_{k,n}(x_f, y_f, z_f) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x_f, y_f, z_f; x_s, y_s, z_s) B_k(x_s, y_s) \, dx_s \, dy_s \]  

(3.3.4)

Similarly the gradient of the potential caused by the source panel \( n \) with a distribution defined by a basis function \( B_k(x_s, y_s), k=1,2,...,36 \), at the field point \( (x_f, y_f, z_f) \) is as follows:
\[ K_{k,n}(x_f, y_f, z_f) = \int_{dd} \int_{bb} \nabla^2 \psi(x_f, y_f, z_f, x_s, y_s, z_s) B_k(x_s, y_s) \, dx_s dy_s \]

\[ = \int_{dd} \int_{bb} \nabla^2 x \psi B_k \, dx_s dy_s, \quad \int_{dd} \int_{bb} \nabla y \psi B_k \, dx_s dy_s, \quad \int_{dd} \int_{bb} \nabla z \psi B_k \, dx_s dy_s \}

\[ = \{K_{k,n}(x_f, y_f, z_f), K_{k,n}(x_f, y_f, z_f), K_{k,n}(x_f, y_f, z_f)\} \]

The goal here is to find the analytically exact formulae of the integrations

\[ I_{k,n}(x_f, y_f, z_f) \quad \text{and} \quad \{K_{k,n}(x_f, y_f, z_f), K_{k,n}(x_f, y_f, z_f), K_{k,n}(x_f, y_f, z_f)\}, \quad k=1,2,\ldots,36. \]

The input data for these formulae are the domain of the \( i \)th source panel \((bb, aa) \times (dd, cc)\), the coordinates of the field points \((x_f, y_f, z_f)\), and the form of the basis functions \( B_k(x, y) \), which is defined by the 36 coefficients \( a_{kj}, \quad i=0, 5; j=0, 5 \)

of the monomial terms \( x^i y^j, \quad i=0, 5; j=0, 5 \). Here \( x \) and \( y \) are the two panel Cartesian coordinates in the plane of the source panel and are originated at the center of each panel. On the other hand \( x_s \) and \( y_s \) are global coordinates in the plane of the source panel, originated at the center of the floating runway on the free surface.

But these formulae are very difficult to get from analytical integration because of the complexity of the integrand which is the product of a 5th-order, 2-variable polynomial and a singular function. Some wise manipulations of the integral are necessary and they are described as follows. First, since the basis functions are polynomials which are linear combinations of monomials \( x^i y^j, i=0, 5; j=0, 5 \) as follows:

\[ B_k(x, y) = \sum_{i=0}^{5} \sum_{j=0}^{5} a_{kij} x^i y^j, \quad k=1, 2, \ldots, 36, \quad (3.3.6) \]
where $a_{s_{ij}}$'s are coefficients corresponding to coordinates $x_s$ and $y_s$ and are functions of integral bounds $aa$, $bb$, $cc$, $dd$ and $a_{k_{ij}}$'s which correspond to panel coordinates $x$ and $y$.

The $I_{k,n}$'s and $\{K_{k_{x,n}}, K_{k_{y,n}}, K_{k_{z,n}}\}$'s can also be computed as linear combinations with coefficients $a_{s_{ij}}$'s of elementary integration items, i.e. $I_{ij,n}$'s and $\{K_{s_{ij,n}}, K_{y_{ij,n}}, K_{z_{ij,n}}\}$'s defined as follows:

$$I_{ij,n} = \int \int_{dd bb} \frac{x_s^i y_s^j}{\sqrt{(x_s - x_f)^2 + (y_s - y_f)^2 + (z_s - z_f)^2}} \, dx_s \, dy_s$$

$$K_{s_{ij,n}} = \int \int_{dd bb} \frac{x_s - x_f}{[(x_s - x_f)^2 + (y_s - y_f)^2 + (z_s - z_f)^2]^{3/2}} \, dx_s \, dy_s$$

$$K_{y_{ij,n}} = \int \int_{dd bb} \frac{y_s - y_f}{[(x_s - x_f)^2 + (y_s - y_f)^2 + (z_s - z_f)^2]^{3/2}} \, dx_s \, dy_s$$

$$K_{z_{ij,n}} = \int \int_{dd bb} \frac{z_s - z_f}{[(x_s - x_f)^2 + (y_s - y_f)^2 + (z_s - z_f)^2]^{3/2}} \, dx_s \, dy_s$$

(3.3.7a)

(The above terms are not subscripted with $k$ because they are general among integrations relating to different basis functions $B_k(x_s, y_s), \ k=1,2,nunk$. On the other hand terms $I_{k,n}$ and $\{K_{k_{x,n}}, K_{k_{y,n}}, K_{k_{z,n}}\}$ and coefficients $a_{s_{ij}}$'s are subscripted with $k$ because they are specific for the $k$th basis function. Similarly all the integration terms are subscripted with $n$ because they are related to the specific panel $n$.)

Obviously the linear combinations for $I_{k,n}$ and $\{K_{k_{x,n}}, K_{k_{y,n}}, K_{k_{z,n}}\}$, which are specific for the $k$th basis function, are as follows:
\[ I_{k,n} = \sum_{i=0}^{5} \sum_{j=0}^{5} a_{sk,j} I_{ij,n}, \]
\[ K_{k,n} = \sum_{i=0}^{5} \sum_{j=0}^{5} a_{sk,j} K_{ij,n}, \]
\[ K_{ky,n} = \sum_{i=0}^{5} \sum_{j=0}^{5} a_{sk,j} K_{ij,n}, \]
\[ K_{kz,n} = \sum_{i=0}^{5} \sum_{j=0}^{5} a_{sk,j} K_{ij,n}, \quad k = 1, 2, \ldots, 36 \]  

(3.3.8)

But the elementary formulae for the integrals in (3.3.7a) are still hard to obtain through analytical integration even with the help of powerful symbolic analysis software like the Mathematica. Thus, further simplifications are needed. One can see that it will help if we replace the square root in the denominator \[ \sqrt{(x_i - x_f)^2 + (y_i - y_f)^2 + (z_i - z_f)^2} \] by \[ \sqrt{x^2 + y^2 + (z_i - z_f)^2} \] through a translation of the coordinate system that changes the variables of integration \( x_i \) and \( y_i \) into new variables \( x = x_i - x_f \) and \( y = y_i - y_f \). The coordinates \( x \) and \( y \) correspond to a translated coordinate system with the origin at the field point. This will change the limits of the domain of integration \( aa, bb, cc, dd \) into \( a = aa - x_f, b = bb - x_f, c = cc - y_f, d = dd - y_f \) respectively. And the monomials \( x_i^i y_i^j, i, j = 0, 1, 2, \ldots, 5 \), will be changed into polynomials \( (x + x_f)^i (y + y_f)^j \), which are again a linear combination of the monomials \( x^i y^j \) of the new coordinates. So the new constructive elementary integrals that need to be computed beforehand are as follow:
\[
I'_{ij,n} = \int_{d}^{c} \int_{b}^{a} \frac{x'y'}{x^2 + y^2 + (z_s - z_f)^2} \, dx \, dy
\]

\[
K'_{xij,n} = \int_{d}^{c} \int_{b}^{a} \frac{x}{[x^2 + y^2 + (z_s - z_f)^2]^{3/2}} \, dx \, dy
\]

\[
K'_{yij,n} = \int_{d}^{c} \int_{b}^{a} \frac{y}{[x^2 + y^2 + (z_s - z_f)^2]^{3/2}} \, dx \, dy
\]

\[
K'_{zij,n} = \int_{d}^{c} \int_{b}^{a} \frac{z_s - z_f}{[x^2 + y^2 + (z_s - z_f)^2]^{3/2}} \, dx \, dy
\]

(3.3.9a) \[i, j = 0,1,...,5\]

All the above integrals can be integrated analytically using Mathematica for \(i, j=0,1,2,...,5\) to get \(36 \times 4 = 144\) analytically exact formulae. For a certain source panel \(n\) and a certain field point, the input data \((bb, aa), (dd, cc)\) and \((x_f, y_f, z_f)\) are determined so the integrals \(I'_{ij,n}\) and \{\(K'_{xij,n}\), \(K'_{yij,n}\), \(K'_{zij,n}\)\} can be computed through these analytically exact formulae. Then integrals in (3.3.7a) can be computed from linear combinations of the integrals in (3.3.9a) as follows:
\[
\begin{align*}
\left\{ \begin{array}{l}
I_{ij,n} \\
K_{xij,n} \\
K_{yij,n} \\
K_{zij,n}
\end{array} \right\} &= \iint_n (x + x_f)^i (y + y_f)^j \left( \frac{1}{r} \right) \frac{1}{\nabla} \left( \frac{1}{r} \right) dx\,dy \\
&= \sum_{m=0}^{i} \sum_{l=0}^{j} \binom{i}{m} \binom{j}{l} x_f^{i-m} y_f^{j-l} \iint_n x^m y^l \left( \frac{1}{r} \right) \frac{1}{\nabla} \left( \frac{1}{r} \right) dx\,dy \\
&= \sum_{m=0}^{i} \sum_{l=0}^{j} \binom{i}{m} \binom{j}{l} x_f^{i-m} y_f^{j-l} \left\{ \begin{array}{l}
I'_{ml,n} \\
K'_{xml,n} \\
K'_{yml,n} \\
K'_{zml,n}
\end{array} \right\}
\end{align*}
\]

where \( \frac{1}{r} = \frac{1}{\sqrt{x^2 + y^2 + (z_s - z_f)^2}} \) and

\[
\nabla \left( \frac{1}{r} \right) = \begin{cases} 
\frac{x}{[x^2 + y^2 + (z_s - z_f)^2]^{3/2}} \\
\frac{y}{[x^2 + y^2 + (z_s - z_f)^2]^{3/2}} \\
\frac{z_s - z_f}{[x^2 + y^2 + (z_s - z_f)^2]^{3/2}} \\
\frac{z_f}{[x^2 + y^2 + (z_s - z_f)^2]^{3/2}}
\end{cases}
\]

(3.3.10)

When the source panel is on the horizontal bottom plate, the \( x \) and \( y \) directions in the integrations in (3.3.9a) coincide with the global coordinate directions defined in section 2.1. But when the source panel is located on vertical plates of the floating runway, i.e. the side plate and the bow plate, we always select the horizontal direction as the \( x \) direction and the vertical direction as the \( y \) direction. This treatment will not cause any difference from the case when the source panel is located in the bottom plate except when we are integrating for the image of the source panel.

The image source about the free surface of point source at \( (x_s, y_s, z_s) \) in the source panel is located at \( (x_s, y_s, -z_s) \). Note that \( z \) is the coordinate direction pointing down with
its origin on the free surface. The potential and gradient of the potential are as shown in (3.3.2b) and (3.3.3b) respectively. When the source panel is located in a horizontal plane, we just need to replace \( z_s \) with \(-z_s\) in (3.3.7a) and (3.3.9a) to compute the integrals of the image source as shown below in (3.3.7b) and (3.3.9b) because the \( z_s \) coordinate is not integrated.

\[
I_{ij,n} = \int \int_{ab} \frac{x_s^i y_s^j}{\sqrt{(x_s-x_f)^2 + (y_s-y_f)^2 + (-z_s-z_f)^2}} \, dx_s \, dy_s
\]

\[
K_{xij,n} = \int \int_{ab} \frac{x_s-x_f}{[(x_s-x_f)^2 + (y_s-y_f)^2 + (-z_s-z_f)^2]^{3/2}} \, dx_s \, dy_s
\]

\[
K_{yij,n} = \int \int_{ab} \frac{y_s-y_f}{[(x_s-x_f)^2 + (y_s-y_f)^2 + (-z_s-z_f)^2]^{3/2}} \, dx_s \, dy_s
\]

\[
K_{zij,n} = \int \int_{ab} \frac{-z_s-z_f}{[(x_s-x_f)^2 + (y_s-y_f)^2 + (-z_s-z_f)^2]^{3/2}} \, dx_s \, dy_s
\]

\[
I'_{ij,n} = \int \int_{ab} \frac{x^i y^j}{\sqrt{x^2 + y^2 + (-z_s-z_f)^2}} \, dx \, dy
\]

\[
K'_{xij,n} = \int \int_{ab} \frac{x}{[x^2 + y^2 + (-z_s-z_f)^2]^{3/2}} \, dx \, dy
\]

\[
K'_{yij,n} = \int \int_{ab} \frac{y}{[x^2 + y^2 + (-z_s-z_f)^2]^{3/2}} \, dx \, dy
\]

\[
K'_{zij,n} = \int \int_{ab} \frac{-z_s-z_f}{[x^2 + y^2 + (-z_s-z_f)^2]^{3/2}} \, dx \, dy
\]

But when the source panel is located in vertical plates, the vertical coordinates \( z_s \) and \( z \) are integrated. The image of the source from \((x_s,y_s,z_s)\) to \((x_s,y_s,-z_s)\) will affect the coordinate translation applied in the process of simplifying the original integrals in (3.3.7) into the integrals in (3.3.9). For these source panels, we always select the
horizontal direction in the panel plane as the \(x\) coordinate direction and the vertical direction as the \(y\) coordinate direction for the integration. So the \(x\) direction in the integrations can be the \(x\) or \(y\) direction in the original global coordinate system, depending on whether the source panel is on the side plate or on the bow plate. The \(y\) direction in the integrations is actually the \(z\) direction in the original global coordinate system. And the domain of integral for the image of the vertical source panel will also be translated differently, i.e. \(c = -dd - z_f\), \(d = -cc - z_f\) now. The integrals in (3.3.7b) and (3.3.9b) will then become as follows (Assume the horizontal direction in the panel plane is \(x\) direction, which is the case for source panels in the side plate):

\[
I_{ij,n} = \int_{a}^{b} \int_{d}^{b} \frac{x_z z_y}{\sqrt{(x_z - x_f)^2 + (z_z - z_f)^2 + (y_z - y_f)^2}}\, dx_z dz_z
\]

\[
K_{xij,n} = \int_{a}^{b} \int_{d}^{b} \frac{x_z z_y}{[(x_z - x_f)^2 + (z_z - z_f)^2 + (y_z - y_f)^2]^{3/2}}\, dx_z dz_z
\]

\[
K_{yij,n} = \int_{a}^{b} \int_{d}^{b} \frac{y_z}{[(x_z - x_f)^2 + (z_z - z_f)^2 + (y_z - y_f)^2]^{3/2}}\, dx_z dz_z
\]

\[
K_{zij,n} = \int_{a}^{b} \int_{d}^{b} \frac{-z_z}{[(x_z - x_f)^2 + (z_z - z_f)^2 + (y_z - y_f)^2]^{3/2}}\, dx_z dz_z
\]

\[
I'_{ij,n} = \int_{d}^{b} \int_{d}^{b} \frac{x' y' f}{\sqrt{x'^2 + y'^2 + (y_z - y_f)^2}}\, dx' dy'
\]

\[
K'_{xij,n} = \int_{d}^{b} \int_{d}^{b} \frac{x}{[x'^2 + y'^2 + (y_z - y_f)^2]^{3/2}}\, dx' dy'
\]

\[
K'_{yij,n} = \int_{d}^{b} \int_{d}^{b} \frac{y_z}{[x'^2 + y'^2 + (y_z - y_f)^2]^{3/2}}\, dx' dy'
\]

\[
K'_{zij,n} = \int_{d}^{b} \int_{d}^{b} \frac{y'}{[x'^2 + y'^2 + (y_z - y_f)^2]^{3/2}}\, dx' dy'
\]
After the integrals in (3.3.7) are computed using the analytically exact formulae for the integrals in (3.3.9) and the linear combination in (3.3.10), they can in turn be used to compute the integrals $I_{k,n}$ and $K_{k,n}$ in (3.3.4) and (3.3.5) through the linear combination in (3.3.8).

The analytically exact formulae for the integrals in (3.3.9a), (3.3.9b) and (3.3.9c) are same. But the input data about the domain of the source panel, i.e. $a$, $b$, $c$, $d$, are different and determine whether the analytically exact formulae are computing the integrals for a source panel on the bottom plate, or for a source panel on the side plate, or for a source panel on the bow plate, or for the image of these source panels. The 512 analytically exact formulae for $I_{ij,n}$ and $\{K'_{iij,n}, K'_{ij,n}, K'_{iij,n}\}$ are presented in Appendix III.
3.3.2 Asymptotic Approximation Formulations

When the input data are substituted into the analytically exact formulae for (3.3.9a,b,c) and then into the linear combination formulae of (3.3.10) and (3.3.8) in section 3.3.1 to compute the integrals $I_{k,n}$ and $K_{k,n}$ in (3.3.4) and (3.3.5), serious numerical error may occur because of the characteristics of these formulae when the field point is far away from the source panel. When the coordinates of the field point $x_f$ and $y_f$ are much larger than any panel dimension, the limits of the integrations, $a$, $b$, $c$ and $d$, are also large. But in the analytically exact formulae for these integrations in (3.3.9a,b,c), as listed in Appendix III, very high order powers of these limits, such as $a^{11}$, $b^{11}$, $c^{10}$ and $d^{10}$, are involved. So very large numbers will be generated and differentiated yielding relatively very small numbers. This results in large errors and when the field point goes far away from the source panel the results from these analytically exact formulae blow up. What’s more, in the subsequent step, when the results from the analytically exact formulae for (3.3.9a,b,c) are combined linearly to compute the integrals in (3.3.7a,b,c), very large numbers will arise and contaminate in the computation again. Consider the case of

$$(x + x_f)^5(y + y_f)^5 = x_f^5 y_f + 5 x_f^4 y_f^4 x + 5 x_f^5 y_f^4 y + 25 x_f^4 y_f^4 xy + ... + x^5 y^5.$$ 

The integral involving this term, e.g. $I_{55,n}$, will be combined from $I_{y,n}$'s as

$$I_{55,n} = x_f^5 y_f^5 + 5 x_f^4 y_f^4 I_{10,n} + 5 x_f^5 y_f^4 I_{10,n} + 25 x_f^4 y_f^4 I_{11,n} + ... + I_{55,n},$$

in which $x_f^5 y_f^5$ will be a very large number when $x_f$ and $y_f$ are large numbers, while $I_{55,n}$ is usually
small. So the summing of these numbers of very different scale causes serious errors again.

In order to overcome this kind of computational difficulty, approximate formulae using asymptotic series to replace \( \psi = \frac{1}{R} \) and \( \nabla_{f} \psi = \{ \nabla_{x_{f}} \psi, \nabla_{y_{f}} \psi, \nabla_{z_{f}} \psi \} \) in the integrands of the integrations for \( I_{k,n} \)'s and \( \{ K_{ix,n}, K_{iy,n}, K_{iz,n} \} \)'s are applied and then integrated to get robust formulae for these integrations that have robust performance even when the field point is far away from the source panel.

Consider the \( \psi = \frac{1}{R} \) in the integrands of \( I_{k,n} \)'s. Through Taylor Expansion at the center of the source panel \((x_{0}, y_{0})\), an asymptotic series can be obtained as follows:

\[
\left( \frac{1}{R} \right)_{3} = \frac{1}{\sqrt{(x_{f} - x_{0})^{2} + (y_{f} - y_{0})^{2} + (z_{f} - z_{0})^{2}}}
\approx \frac{1}{R} + \frac{1}{2} \left[ (x_{f} - x_{0})(x_{f} - x_{0}) + (y_{f} - y_{0})(y_{f} - y_{0}) + \frac{(x_{f} - x_{0})^{2}}{2} + \frac{(y_{f} - y_{0})^{2}}{2} \right] \frac{1}{R^{3}} \hspace{1cm} \text{(3.3.11a)}
\]

\[
+ \frac{3}{2} \left[ (x_{f} - x_{0})(x_{f} - x_{0}) + (y_{f} - y_{0})(y_{f} - y_{0}) \right] \frac{1}{R^{3}} \left( + o\left( \frac{1}{R^{3}} \right) \right)
\]

Similarly, an asymptotic series of \( \nabla_{f} \psi = \{ \nabla_{x_{f}} \psi, \nabla_{y_{f}} \psi, \nabla_{z_{f}} \psi \} \) can be obtained as follows:
\[
\left( \nabla_{x_f} \frac{1}{R} \right)_4 = \frac{x_s - x_f}{[(x_f - x_i)^2 + (y_f - y_i)^2 + (z_f - z_i)^2]^{3/2}} \\
= (x_i - x_f) \frac{1}{R^3} + \left[ -3(x_i - x_o)(x_f - x_o)^2 + \frac{9}{2}(x_i - x_o)^2(x_f - x_o) \\
+ \frac{3}{2}(x_f - x_o)(y_s - y_o)^3 + \frac{3}{2}(x_i - x_o)(y_s - y_o)(y_f - y_o) \\
- 3(y_s - y_o)(x_f - x_o)(y_f - y_o) \right] \frac{1}{R^5} \\
- \frac{15}{2} \left[ (x_s - x_o)^2(y_f - y_o) + (x_s - x_o)(y_f - y_o) \right] \frac{1}{R^7} \left( + o \left( \frac{1}{R^9} \right) \right)
\] (3.3.12a)

\[
\left( \nabla_{y_f} \frac{1}{R} \right)_4 = \frac{y_s - y_f}{[(x_f - x_i)^2 + (y_f - y_i)^2 + (z_f - z_i)^2]^{3/2}} \\
= (y_i - y_f) \frac{1}{R^3} + \left[ -3(y_i - y_o)(y_f - y_o)^3 - (x_i - x_o)^2(y_f - y_o) \\
+ (x_i - x_o)(x_f - x_o)(y_f - y_o) - 3(y_i - y_o)^3(y_f - y_o) \right] \frac{1}{R^5} \\
- \frac{15}{2} \left[ (x_s - x_o)^2(y_f - y_o) + 2(x_s - x_o)(y_s - y_o)(y_f - y_o) \right] \frac{1}{R^7} \left( + o \left( \frac{1}{R^9} \right) \right)
\] (3.3.13a)

\[
\left( \nabla_{z_f} \frac{1}{R} \right)_5 = \frac{z_s - z_f}{[(x_f - x_i)^2 + (y_f - y_i)^2 + (z_f - z_i)^2]^{3/2}} \\
= \frac{z_i - z_f}{R^3} - \frac{3}{2} \left[ (x_i - x_o)^2 + 3(x_i - x_o)(x_f - x_o) \\
+ (y_i - y_o)^3 - 6(y_i - y_o)(y_f - y_o) \right] \frac{z_s - z_f}{R^5} \\
+ \frac{15}{2} \left[ (x_s - x_o)(x_f - x_o) + (y_s - y_o)(y_f - y_o) \right] \frac{z_s - z_f}{R^7} \left( + o \left( \frac{1}{R^9} \right) \right)
\] (3.3.14a)
In the above asymptotic series, the coordinates of the field point \((x_f, y_f)\) are of the same order of magnitude as the distance \(R\) between the field point \((x_f, y_f)\) and the center of source panel. But integration variables \((x_i, y_i)\) and coordinates of the center of source panel \((x_0, y_0)\) are of the same order of magnitude as dimensions of the source panel which are much smaller than \(R\) when the field point is far away from the source panel. (This is the result of our selection of a corner of the source panel as the origin of the coordinate system for the input data, which include integration bounds \(aa, bb, cc, dd\) and field point \((x_f, y_f)\)). This assumption guides which terms will be retained in the series when a certain order of approximation is kept. The height difference \((z_i - z_f)\) of the field and source point that appears in \(\nabla_z \frac{1}{R}\) is always small compared to the panel dimensions and will not affect which terms will be retained. The last term in these asymptotic expressions, such as \(o\left(\frac{1}{R^3}\right)\), shows the order of small of the omitted residuals. For example \(o\left(\frac{1}{R^3}\right)\) means the omitted residuals is a small of order higher than \(\frac{1}{R^3}\).

After applying these asymptotic series in (3.3.7a,b,c), the integrands become polynomials in the integration variables and analytically exact formulae can be obtained for these integrals. In these formulae polynomials in \(x_f\) and \(y_f\) are divided by powers of \(R\), which are large enough so that the result is small. Thus, the previous computational difficulties are bypassed. When the field point is close to the source panel, the distance \(R\) is small and these formulae have larger errors than the analytically exact formulae. But when the distance \(R\) increases the errors of these formulae will go to zero asymptotically.
As R increases, the accuracy of the asymptotic formulae increases. However as R increases, the accuracy of the exact formulation decreases and eventually blows up. In order to decide when to use which formulae, we first must assure that there is an overlap region where both formulations give accurate results. A combination of these two methods can then give satisfactory results under all the situations.

The residual errors of the asymptotic series can be reduced to higher order small if we include higher order of (1/R) in the series. But this will let higher order powers of large numbers (R, x_f and y_f) to arise and will cause computational errors again. So the orders of (1/R) are kept moderate to maintain the balance.

When the field point is relatively close to the source panel, the distance from the field point to the source panel center R is not much larger than the coordinates of the center of source panel (x_o, y_o) and the integration variables (x_s, y_s). The asymptotic approximations in (3.3.11a), (3.3.12a), (3.3.13a) and (3.3.14a) will show the disadvantage of being incomplete because some terms involving (x_o, y_o) and (x_s, y_s) have been omitted based on the assumption that R and (x_f, y_f) is much larger than (x_o, y_o), (x_s, y_s) and R. So a complete form, retaining all the terms in the numerator over the denominator of the same order powers of R, are developed and applied in developing the formulae, which are as shown in (3.3.11b), (3.3.12b) (3.3.13b) and (3.3.14b).
\[
\left(\frac{1}{R}\right)_{3\text{complete}} = \frac{1}{\sqrt{(x_f - x_s)^2 + (y_f - y_s)^2 + (z_f - z_s)^2}} \\
\approx \frac{1}{R} \left[ -x_f^2 + 2x_f(x_s - x_0) + x_0^2 - y_s^2 + 2y_f(y_s - y_0) + y_0^2 \frac{1}{R^3} \right] + \frac{3}{2} \left[ -x_f^2 + 2x_f(x_s - x_0) + x_0^2 - y_s^2 + 2y_f(y_s - y_0) + y_0^2 \right] \frac{1}{R^5} + o\left(\frac{1}{R^7}\right)
\]

\[
\left(\nabla_{x_f} \frac{1}{R}\right)_{4\text{complete}} = \frac{x_s - x_f}{[(x_f - x_s)^2 + (y_f - y_s)^2 + (z_f - z_s)^2]^{3/2}} \\
\approx \frac{x_s - x_f}{R^3} \left[ -x_f^2 + 2x_f(x_s - x_0) + x_0^2 - y_s^2 + 2y_f(y_s - y_0) + y_0^2 \right] \frac{x_s - x_f}{R^5} + \frac{15}{8} \left[ -x_f^2 + 2x_f(x_s - x_0) + x_0^2 - y_s^2 + 2y_f(y_s - y_0) + y_0^2 \right] \frac{x_s - x_f}{R^7} + o\left(\frac{1}{R^9}\right)
\]

\[
\left(\nabla_{y_f} \frac{1}{R}\right)_{4\text{complete}} = \frac{y_s - y_f}{[(x_f - x_s)^2 + (y_f - y_s)^2 + (z_f - z_s)^2]^{3/2}} \\
\approx \frac{y_s - y_f}{R^3} \left[ -x_f^2 + 2x_f(x_s - x_0) + x_0^2 - y_s^2 + 2y_f(y_s - y_0) + y_0^2 \right] \frac{y_s - y_f}{R^5} + \frac{15}{8} \left[ -x_f^2 + 2x_f(x_s - x_0) + x_0^2 - y_s^2 + 2y_f(y_s - y_0) + y_0^2 \right] \frac{y_s - y_f}{R^7} + o\left(\frac{1}{R^9}\right)
\]
\[
\left( \nabla_x \frac{1}{R} \right)_{\text{complete}} = \frac{z_s - z_f}{\left[ (x_f - x_s)^2 + (y_f - y_s)^2 + (z_f - z_s)^2 \right]^{\frac{3}{2}}} \\
= \frac{z_s - z_f}{R^3} \\
\quad + \frac{3}{2} \left[ -x_s^2 + 2x_f(x_s - x_0) + x_0^2 - y_s^2 + 2y_f(y_s - y_0) + y_0^2 \right] \frac{z_s - z_f}{R^5} \\
\quad + \frac{15}{8} \left[ -x_s^2 + 2x_f(x_s - x_0) + x_0^2 - y_s^2 + 2y_f(y_s - y_0) + y_0^2 \right] \frac{z_s - z_f}{R^7} \\
\quad \left( + o\left( \frac{1}{R^r} \right) \right)
\]

On the other hand lower order asymptotic approximation formulae may be useful when \( R \) becomes very large. Since these approximations involve fewer terms, they can save computation time and avoid high order powers of \( R \) while maintaining the accuracy when \( R \) is large enough. These formulae can be obtained by integrating (3.3.7a,b,c) using lower order asymptotic approximations of \( \psi = \frac{1}{R} \) and \( \nabla_f \psi = \nabla_f \frac{1}{R} \) than those in (3.3.11a), (3.3.12a), (3.3.13a) and (3.3.14a). A set of lower order asymptotic approximations are shown in (3.3.11c), (3.3.12c), (3.3.13c) and (3.3.14c).

\[
\left( \frac{1}{R} \right)_2 = \frac{1}{\sqrt{(x_f - x_s)^2 + (y_f - y_s)^2 + (z_f - z_s)^2}} \\
\approx \frac{1}{R} \left[ (x_f - x_0)(x_s - x_0) + (y_f - y_0)(y_s - y_0) \right] \frac{1}{R^3} \left( + o\left( \frac{1}{R^2} \right) \right)
\]

\[
\left( \nabla_x \frac{1}{R} \right)_3 = \frac{x_s - x_f}{\left[ (x_f - x_s)^2 + (y_f - y_s)^2 + (z_f - z_s)^2 \right]^{\frac{3}{2}}} \\
\approx (x_s - x_f) \frac{1}{R^3} \left[ -3(x_s - x_0)(x_f - x_0)^2 \\
\quad + 3(y_s - y_0)(x_f - x_0)(y_f - y_0) \right] \frac{1}{R^5} \left( + o\left( \frac{1}{R^3} \right) \right)
\]
\[
\left( \nabla_y \frac{1}{R} \right)_3 = \frac{y_y - y_f}{\left[ (x_f - x_y)^2 + (y_f - y_y)^2 + (z_f - z_y)^2 \right]^{3/2}} \\
\approx \left( y_y - y_f \right) \frac{1}{R^3} - 3 \left[ (x_y - x_0)(x_f - x_0)(y_f - y_0) + (y_y - y_0)(y_f - y_0)^2 \right] \frac{1}{R^5} \left( + o \left( \frac{1}{R^3} \right) \right)
\]

\[
\left( \nabla_y \frac{1}{R} \right)_4 = \frac{z_y - z_f}{\left[ (x_f - x_y)^2 + (y_f - y_y)^2 + (z_f - z_y)^2 \right]^{3/2}} \\
\approx \frac{z_y - z_f}{R^3} + 3 \left[ (x_y - x_0)(x_f - x_0) + (y_y - y_0)(y_f - y_0) \right] \frac{z_y - z_f}{R^5} \left( + o \left( \frac{1}{R^4} \right) \right)
\]

The behavior of even lower (lowest) order asymptotic approximations is tested. The lowest order asymptotic approximations are shown in (3.3.11d), (3.3.12d), (3.3.13d) and (3.3.14d). They require least computation time but are useful only when R is extremely large.

\[
\left( \frac{1}{R} \right)_1 = \frac{1}{\sqrt{(x_f - x_0)^2 + (y_f - y_0)^2 + (z_f - z_0)^2}} \approx \frac{1}{R} \left( + o \left( \frac{1}{R} \right) \right) \tag{3.3.11d}
\]

\[
\left( \nabla_x \frac{1}{R} \right)_2 = \frac{x_x - x_f}{\left[ (x_f - x_y)^2 + (y_f - y_y)^2 + (z_f - z_y)^2 \right]^{3/2}} \approx \frac{1}{R^3} \left( + o \left( \frac{1}{R^2} \right) \right) \tag{3.3.12d}
\]

\[
\left( \nabla_y \frac{1}{R} \right)_2 = \frac{y_y - y_f}{\left[ (x_f - x_y)^2 + (y_f - y_y)^2 + (z_f - z_y)^2 \right]^{3/2}} \approx \frac{1}{R^3} \left( + o \left( \frac{1}{R^2} \right) \right) \tag{3.3.13d}
\]

\[
\left( \nabla_z \frac{1}{R} \right)_3 = \frac{z_z - z_f}{\left[ (x_f - x_y)^2 + (y_f - y_y)^2 + (z_f - z_y)^2 \right]^{3/2}} \approx \frac{1}{R^3} \left( + o \left( \frac{1}{R^3} \right) \right) \tag{3.3.14d}
\]

To find out the behavior of these formulae derived from the above approximation, as well as the scopes in which each of them will be satisfactory, integrals involving various
locations of the field point and different sizes of the source panel are evaluated using these formulae. The first group of results are computed by first using the analytically exact formulae for integrals in (3.3.9a, b, c) to compute \( \{ I_{ij,n}, K_{xij,n}, K_{yij,n}, K_{zij,n} \} \), then using combining formulae in (3.3.10) to compute \( \{ I_{ij,n}, K_{xij,n}, K_{yij,n}, K_{zij,n} \} \), and finally using formulae in (3.3.8) to compute \( \{ I_{k,n}, K_{kx,n}, K_{ky,n}, K_{kz,n} \} \). The second group of results are computed by using the highest order and complete asymptotic approximations in (3.3.11b), (3.3.12b), (3.3.13b) and (3.3.14b) to compute \( \{ I_{ij,n}, K_{xij,n}, K_{yij,n}, K_{zij,n} \} \) directly and then using formulae in (3.3.8) to compute \( \{ I_{k,n}, K_{kx,n}, K_{ky,n}, K_{kz,n} \} \). The third, forth and fifth groups of results are computed the same way as the second group except that they use the incomplete highest order asymptotic approximations in (3.3.11a), (3.3.12a), (3.3.13a) and (3.3.14a), or lower order asymptotic approximations in (3.3.11c), (3.3.12c), (3.3.13c) and (3.3.14c), or the lowest order asymptotic approximations in (3.3.11d), (3.3.12d), (3.3.13d) and (3.3.14d), respectively, to compute \( \{ I_{ij,n}, K_{xij,n}, K_{yij,n}, K_{zij,n} \} \) directly.

Here the results of two tests are presented. The tests are designed to find out the behavior of the analytically exact and asymptotic approximation formulae under the worst situations for numerical computation. The panel is selected to be 50 meters by 50 meters square panel which will be used in the computation of the hydrodynamics of the real size floating runway. In the first test the field point stays on a line parallel and also close to one side of the panel. In the second test the field point stays on the diagonal line and its extension of the panel. In both tests the field point will move from inside the panel to thousands of meters away from the panel center. The great difference in the values of

104
the input data for the analytically exact and asymptotic approximation formulae is designed to reveal computational errors that may occur in the practical computations. The input data of the first test are shown in Table 3.3.1 and the input data of the second test are shown in Table 3.3.2.
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<th>Input Data (meters)</th>
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<td></td>
<td>$z_f$</td>
<td>2.0D0</td>
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Table 3.3.1 Input Data of Test 1
<table>
<thead>
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<th>Variables</th>
<th>Input Data (meters)</th>
<th>Variables</th>
<th>Input Data (meters)</th>
</tr>
</thead>
<tbody>
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<td>( y_f )</td>
<td>{23.5, 18.75, 12.5, 6.25, -6.25, -12.5, -18.75, -25.0, -37.5, -50.0, -75.0, -100.0, -150.0, -200.0, -300.0, -400.0, -500.0, -600.0, -700.0, -800.0, -1000.0, -2000.0, -4000.0}</td>
</tr>
<tr>
<td>cc</td>
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<td>( z_f )</td>
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</table>

Table 3.3.2  Input Data of Test 2

Figures (3.3.1a-d) and (3.3.2a-d) show the \( \log_{10}|Y| - \log_{10}|X| \) plots on the behavior of the five groups of results for \( \{I_{k.n}, K_{kx,n}, K_{ky,n}, K_{kz,n}\}, k = 1 \) in Test 1 and Test 2, when the distance R between the field point and the center of the source panel increases from close to 0 to beyond 4000 (meters). In these figures different line types and dot types are used to plot the results of \( \{I_1, K_{1x}, K_{1y}, K_{1z}\} \) using the analytically exact formulae and the
approximate formulae derived from different asymptotic approximation formulae, i.e. the 1st approximation formulae (3.3.11d, 3.3.12d, 3.3.13d, 3.3.14d), the 2nd approximation formulae (3.3.11c, 3.3.12c, 3.3.13c, 3.3.14c), the 3rd approximation formulae (3.3.11a, 3.3.12a, 3.3.13a, 3.3.14a), and the 3rd complete approximation formulae (3.3.11b, 3.3.12b, 3.3.13b, 3.3.14b).

From the whole-field plots in figures (3.3.1a-d) and (3.3.2a-d) it can be seen that the approximate formulae do converge to the analytically exact formulae when R increases until a certain value. The results from the 3rd complete, 3rd, 2nd and 1st approximation formulae will become respectively the most, less and least close to the results from the analytically exact formulae when the distance R is not very small. But when R continues to increase beyond a certain value, the results from the analytically exact formulae blow up suddenly. This phenomenon is caused by the computational errors arising from the high-order powers of large numbers in the analytically exact formulae. The results from the groups of asymptotic approximation formulae do not deteriorate with large distance R and continue converging together to the correct result.

Similarly figures (3.3.3a-d) and (3.3.4a-d) show plots of \( \{I_{1,n}, K_{1x,n}, K_{1y,n}, K_{1z,n}\} \) against \( \log_{10}R \) in the near field in Test 1 and Test 2, when the distance R between the field point and the center of the source panel is within a moderate value. These plots are prepared to show the near field behavior of the analytically exact and different asymptotic approximation formulae for the integral.

From plots in figures (3.3.3a-d) and (3.3.4a-d) it can be seen that in the near field when R is small the differences between the results from the analytically exact formulae and the different approximate formulae are quite large and the analytically exact formulae
gives the correct results under all the situations. So the analytically exact formulae should be used in the computation of the integrals for the near field. When R increases beyond a certain point, and before the results of the analytically exact formulae blow up, the results from the highest asymptotic approximation formulae become close enough to those from the analytically exact formulae. It is at that point that we transfer from the analytically exact formulae to the highest order asymptotic approximation formulae. For different panel size s and different route by which the field point goes away from the source panel, this point is in a narrow interval around \( R/s \approx 3 \sim 4 \). The point at which we transfer from the highest order asymptotic approximation formulae to the lower order asymptotic approximation formulae is not so critical because there is no risk of having the results blow up. A good selection can be \( R/s \geq 32 \) to save some computing time. And the lowest order asymptotic approximation formulae may be not used at all.

It can also be seen that in the near field the results from the complete asymptotic approximation formulae in (3.3.11b, 3.3.12b, 3.3.13b, 3.3.14b) are not necessarily better than the results from the same order incomplete asymptotic approximation formulae in (3.3.11a, 3.3.12a, 3.3.13a, 3.3.14a). Similarly in the near field the results from the higher order asymptotic approximation formulae are also not necessarily better than the results from the lower order asymptotic approximation formulae. In some cases the results from these approximate formulae blow up in the near field and the lowest order approximate formulae give closer results to the analytically exact formulae. But when the field point moves out of the near field, normally when \( R/s \geq 1.5 \sim 2 \), the complete and higher order approximate formulae always give better result. In the practical computation the
analytically exact formulae are used for the near field and then when \( R/s \geq 2 \) the complete 3\(^{rd} \) approximate formulae will be used to achieve highest accuracy.
Test 1 of Different Formulae
Integral $I_1$

(a)
Test 1 of Different Formulae
Integral $I_{lx}$

Magnitude of Integral $I_{lx}$ (meter sec.)

Distance $R$ (meter)

(b)
Test 1 of Different Formulae
Integral $I_y$

(c)
Figure 3.3.1 Plots of Integrals \( \{I_1, K_1, K_{1,1}, K_{1,2}\} \) versus Distance (\( \log_{10} R \)) Computed from Analytically Exact and Asymptotic Approximation Formulae -------- Test 1
Test 2 of Different Formulae
Integral $I_1$

- 3rd Complete Approx.
- 3rd Approx.
- 2nd Approx.
- 1st Approx.
- Exact

Magnitude of Integral $I_1$ (meter/sec.)

Distance $R$ (meter)

(a)
Test 2 of Different Formulae
Integral $I_x$

(b)
Figure 3.3.2 \( \log_{10} - \log_{10} \) Plots of Integrals \( \{I_1, K_{1x}, K_{1y}, K_{1z}\} \) verses Distance \( R \)

Computed from Analytically Exact and Asymptotic Approximation Formulae ------ Test 2
Test 1 of Different Formulae
Integral $I_{1x}$

Value of Integral $I_{1x}$ (meter/sec.)

Distance $R$ (meter)

- 3rd Complete Approx.
- 3rd Approx.
- 2nd Approx.
- 1st Approx.
- Exact

(b)
Test 1 of Different Formulae
Integral I_{ty}

Value of Integral I_{ty} (meter/sec.)

Distance R (meter)

- 3rd Complete Approx.
- 3rd Approx.
- 2nd Approx.
- 1st Approx.
- Exact

(c)
Figure 3.3.3 Plots of Integrals \( \{I_1, K_{1z}, K_{1y}, K_{1z}\} \) versus Distance \((\log_{10} R)\) Computed from Analytically Exact and Asymptotic Approximation Formulae ------ Test 1
Test 2 of Different Formulae

Integral $I_1$

- 3rd Complete Approx.
- 3rd Approx.
- 2nd Approx.
- 1st Approx.
- Exact

Value of Integral $I_1$ (meter/sec.)

Distance R (meter)

(a)
Test 2 of Different Formulae
Integral $I_{1s}$
Test 2 of Different Formulae
Integral $I_{1y}$

Value of Integral $I_{1y}$ (meter/sec.)

Distance R (meter)

(c)
Test 2 of Different Formulae
Integral $I_{1z}$

Value of Integral $I_{1z}$ (meter/sec.)

Distance $R$ (meter)

---

Figure 3.3.4 Plots of Integrals $\{I_1, K_{1x}, K_{1y}, K_{1z}\}$ verses Distance ($\log_{10} R$) Computed from Analytically Exact and Asymptotic Approximation Formulae-------Test 2
3.4 The Computation of the High-Order Hydrodynamics Influence Matrices

After the integration relating to the singular or nearly singular part of the free-surface Green Function has been separated and computed using the analytically exact and asymptotic approximation formulae introduced in section 3.3, the remnant of the free-surface Green Function is computed by the standard subroutine FINGREEN and then integrated using Gaussian Quadrature. Then these two parts are added together to get the results for the integrals $C_{j,\text{sym}}(x, y, z; \omega)$ in (3.2.7) and $U_{j,\text{sym}}(x, y, z; \omega)$ in (3.2.12).

The computation of these integrations is very time-consuming and great effort has been committed to reduce the computation task.

First according to distance from the field point to the source panel Gaussian Quadrature of different orders can be used to compute the integration relating to the non-singular part of the Green Function. When the distance is small the Green Function has significant value and also changes rapidly. So Gaussian Quadrature of higher order should be used. When the distance is large the Green Function has very small value and also doesn’t change much. So Gaussian Quadrature of lower order can be used to save computation time.

Secondly in the various analytically exact and asymptotic approximation formulae shown in section 3.3 there are many expressions which appear frequently. For example the powers of $a, b, c, d, z$ and expressions like $\sqrt{a^2 + c^2 + z^2}$ appear repeatedly in most of the analytically exact formulae for integrals in (3.3.9) as shown in Appendix II. These expressions can be computed beforehand and stored once for all to avoid the repeated computation. The lower powers of $a, b, c, d, z$ will be computed first. Then the higher
powers of $a, b, c, d, z$ will be computed using the results of the lower powers. Then the more and more complicated expressions like $\sqrt{a^2 + c^2 + z^2}$ and $LN(a + \sqrt{a^2 + c^2 + z^2})$ will be computed from the results of the simpler expressions. At last the complicated formulae will be computed by combining the results of their constituting expressions. Since these analytically exact and asymptotic approximation formulae are very long and complicated, this way of avoiding the repeated computation will reduce the computation task significantly.

Thirdly according to the “relative similarity relation” (Kashiwagi, 1998a), the Green Function depends on the relative location of the field point and the source point only. For the same source distributions in the panels of same size, the integrals $C_{j,\text{sym}}(x, y, z; \omega)$ and $U_{j,\text{sym}}(x, y, z; \omega)$ have same values for field points of same relative location.

In the mesh of the wetted surface of the floating runway the panels on the bottom plate have identical size. Similarly the panels on the side plate and the panels on the bow plate also have their unique identical sizes. The source distributions in the bottom panels are described by the 36 high-order basis functions which are identical under identical panel size. The source distributions in the side panels and the bow panels are uniform and are also identical under identical panel size. So totally there are only three different kinds of source panels in the mesh, i.e. the bottom source panel, the side source panel and the bow source panel.

The field points on the bottom plate are evenly distributed grid nodes in the panels of identical size and are evenly distributed in the whole plate. The field points on the side and bow plate are the centers of the panels so they are also evenly distributed in the
whole side or bow plate. So there are also three kinds of field points, each evenly distributed in a plate.

Three source panels, located at determined places in the mesh, are selected to represent the three kinds of source panels, one for each kind. For each of the three source panels, the scopes of the distributions of the three kinds of field points are determined to cover all the possible relative locations of the source panels and the field points. Then for these three source panels and the relative field points, the integrals $C_{j,\text{sym}}(x, y, z; \omega)$ and $U_{j,\text{sym}}(x, y, z; \omega)$ are computed and the results are stored in matrices beforehand. Finally when the values of the integrals $C_{j,\text{sym}}(x, y, z; \omega)$ and $U_{j,\text{sym}}(x, y, z; \omega)$ are needed for any source panel and any field point, the corresponding elements of these matrices are chosen, according to the kind of the source panel and the relative location of the field point to the source panel. This way, the repeated computation are avoided significantly.

After these efforts to reduce the computational task, the computation of the high-order Hydrodynamics Influence Matrices for 10 wave frequencies is possible for a full size runway, which is 4000 meters long, 400 meters wide and having 2 meters draft. The shortest wave have a wavelength of 47.811 meters and the mesh in the first quadrant has 40 panels in the longitudinal direction, 4 panels in the transverse direction and 1 panel in the vertical direction. The panels on the bottom plate are high-order panels and the panels on the side and bow plate are constant low-order panels which are subdivided by a factor of 15 as explained in section (3.2).
3.5 The Radiation and Diffraction Problems

Similar as in the low-order method, the pressure distributions on the body surface in the radiation and diffraction problems of floating runways can be computed for every frequency of the input waves from the displacement vectors using the high-order Hydrodynamic Influence Matrices as in (3.2.11). The difference is that in the high-order method the displacement vectors and the pressure vectors all consist of the coefficients of the high-order decomposition of these distributions. The resulted diffraction pressure vector will be used in the high-order Motion Equation of the floating-mat model as the loads input. These pressure distributions can also be integrated over the body surface to get the radiation and diffraction forces. These are compared the results with those from WAMIT and those from the low-order method to verify that the high-order Hydrodynamic Influence Matrices are defined and computed correctly and accurately. For the purpose of verification, small model structures with similar shape as the real size floating runway are used. The details of the radiation and diffraction problems are discussed separately below.
3.5.1 Radiation Problem

Similarly as in the low-order method the radiation forces are computed from the integration of the pressure on the body surface undergoing the six canonical motions. The displacement of the canonical motions at the side and bow plate is determined similarly as in the low-order method. But in the high-order method we need to decompose the displacements of the bottom plate according to the basis functions to get the displacement coefficients.

The extended normal vector at point \( \bar{x} = (x, y, z) \) on the bottom plate in the first quadrant is \( \bar{n} = (n_1, n_2, n_3, n_4, n_5, n_6) = (0, 0, -1, -y, x, 0) \). From (2.1.2) the corresponding displacement of canonical motion \( i \) is as follows:

\[
d_i(x, y, z) = \frac{j}{\partial} n_i, \quad i = 1, 2, 3, 4, 5, 6. \tag{3.5.1}
\]

At each node in the bottom plate mesh only the value and first order derivatives of the canonical displacement can be non-zero. The high-order displacement vector is computed easily from the position of the node in the first quadrant of the mesh. The displacement vectors for the canonical motions are used as input data to compute the coefficient vectors of the pressure distributions on the first quadrant body surface using the high-order Hydrodynamic Influence Matrices as in (3.2.14). Similar to the low-order method, the canonical motions have symmetries as shown in Table 2.3.1 so only the displacements in the first quadrant and the Hydrodynamic Influence Matrices of the right symmetry are needed to compute the pressure vectors for the first quadrant. The pressure distributions in the first quadrant are computed from these vectors of coefficients through interpolation using the basis functions. Pressure distributions in the other quadrants are
determined by the same symmetry as the corresponding canonical motion. Finally the pressure distributions can be integrated over the wetted body surface to get the radiation forces.

The radiation forces computed using high-order Hydrodynamic Influence Matrices (from the program HOHYD) are compared with those computed using the low-order method (from program HYDRO) for a model floating structure of 1000 meters long, 100 meters wide and with a draft of 2 meters under different waves frequencies which are listed in Table 3.5.1 together with the corresponding wavelengths, periods and the panel size ratios for the low-order method and the high-order method. For the low-order method (in HYDRO) the first quadrant of the wetted surface is discretized into 150 panels in the longitudinal direction, 15 panels in the transverse direction and 1 panels in the vertical direction. For the high-order method (in HOHYD) the first quadrant of the bottom plate is discretized into 10 panels in the longitudinal direction, 1 panels in the transverse direction and the side and bow plates are discretized into 1 panels in the vertical direction and subdivided by a factor of 15 in the horizontal directions as explained in section (3.2). The panel size on the bottom plate used in HOHYD is 50 meters by 50 meters and that used in HYDRO is 3.333 meters by 3.333 meters. The results from HYDRO are expected to have enough accuracy since there are at least 6 panels within a wavelength.

Similarly as in section (2.3), the results from HYDRO and HOHYD are also non-dimensionalized into added-mass coefficient \( \bar{A}_g \) and damping coefficient \( \bar{B}_g \) as defined in section (4.2) of the WAMIT version 5.4 manual (page 4-3). The results from HOHYD
and HYDRO together with their relative differences are compared in Tables from (3.5.2) to (3.5.11) for the wavelengths listed in Table (3.5.1).

From Tables from (3.5.2) to (3.5.11) it can be seen that when the wavelength is larger than 40 meters (panel size ratio for HOHYD less than 1.25) the results from the high-order method program HOHYD are close to the results from the low-order method program HYDRO for the three important motions, i.e. heave, roll and pitch. The relative differences for the other three motions are still tolerable although much higher, same as the case when HYDRO is compared with WAMIT. This shows the programming and computation in the high-order method program HOHYD are correct and that it computes the hydrodynamics properties of the model structure correctly for waves longer than 40 meters.

For waves shorter than 40 meters the results from high-order method program HOHYD have great differences when compared to HODRO even for the three important motions. This indicates HOHYD can not compute the hydrodynamics properties of the model structure correctly for waves shorter than 40 meters. In practice the panel size ratio is kept to be $\frac{\text{Panel Size}}{\lambda} \leq 1.1$ to ensure the accuracy in the results.

So the high-order method program HOHYD is very powerful in computing the hydrodynamics of the floating runway structure. By using 40 panels in the longitudinal direction, 4 panels in the transverse direction for one quadrant of the real size floating runway we can compute the hydrodynamics correctly when ratio of the structure length to the wavelength reaches 80.
| Frequency $\omega$ (Rad/Sec) | Wavelength $\lambda$ (Meter) | Period $T$ (Second) | Panel Size Ratio | Panel Size/$\lambda$
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>High-Order Method</td>
<td>Low-Order Method</td>
</tr>
<tr>
<td>1.755</td>
<td>20.00</td>
<td>3.581</td>
<td>2.500</td>
<td>0.167</td>
</tr>
<tr>
<td>1.673</td>
<td>22.00</td>
<td>3.756</td>
<td>2.273</td>
<td>0.152</td>
</tr>
<tr>
<td>1.569</td>
<td>25.00</td>
<td>4.004</td>
<td>2.000</td>
<td>0.133</td>
</tr>
<tr>
<td>1.510</td>
<td>27.00</td>
<td>4.161</td>
<td>1.852</td>
<td>0.123</td>
</tr>
<tr>
<td>1.433</td>
<td>30.00</td>
<td>4.386</td>
<td>1.667</td>
<td>0.111</td>
</tr>
<tr>
<td>1.241</td>
<td>40.00</td>
<td>5.064</td>
<td>1.250</td>
<td>0.083</td>
</tr>
<tr>
<td>1.110</td>
<td>50.00</td>
<td>5.662</td>
<td>1.000</td>
<td>0.067</td>
</tr>
<tr>
<td>0.906</td>
<td>75.000</td>
<td>6.934</td>
<td>0.667</td>
<td>0.044</td>
</tr>
<tr>
<td>0.785</td>
<td>100.00</td>
<td>8.007</td>
<td>0.500</td>
<td>0.033</td>
</tr>
<tr>
<td>0.196</td>
<td>1600.00</td>
<td>32.029</td>
<td>0.031</td>
<td>0.002</td>
</tr>
</tbody>
</table>

Table 3.5.1  Frequencies, Wavelengths, Periods and Panel Size Ratios Used to Compare HOHYD and HYDRO for the model 1000m*100m*2m
<table>
<thead>
<tr>
<th>Canonical Mots</th>
<th>HYDRO</th>
<th>HOHYD</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A_{y_1}$</td>
<td>$B_{y_1}$</td>
<td>$A_{y_2}$</td>
</tr>
<tr>
<td>1</td>
<td>2.679E-06</td>
<td>7.601E-06</td>
<td>2.289E-06</td>
</tr>
<tr>
<td>2</td>
<td>2.493E-05</td>
<td>7.581E-05</td>
<td>2.556E-05</td>
</tr>
<tr>
<td>3</td>
<td>2.895E-02</td>
<td>8.187E-04</td>
<td>2.667E-02</td>
</tr>
<tr>
<td>4</td>
<td>3.774E-05</td>
<td>1.792E-06</td>
<td>3.635E-05</td>
</tr>
<tr>
<td>5</td>
<td>8.706E-03</td>
<td>2.563E-04</td>
<td>8.051E-03</td>
</tr>
<tr>
<td>6</td>
<td>8.500E-06</td>
<td>2.532E-05</td>
<td>9.017E-06</td>
</tr>
</tbody>
</table>

Table 3.5.2  Comparison of the Added-Mass and Damping Coefficients Computed by HOHYD and HYDRO, for 1000*100*2m model, wavelength $\lambda = 20.000$ (Meter)
<table>
<thead>
<tr>
<th>Canonical Motions</th>
<th>HYDRO</th>
<th>HOHYD</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( A_{y_1} )</td>
<td>( B_{y_1} )</td>
<td>( A_{y_2} )</td>
</tr>
<tr>
<td>1</td>
<td>3.165E-06</td>
<td>7.965E-06</td>
<td>1.670E-06</td>
</tr>
<tr>
<td>2</td>
<td>2.909E-05</td>
<td>7.959E-05</td>
<td>2.914E-05</td>
</tr>
<tr>
<td>3</td>
<td>2.864E-02</td>
<td>1.051E-03</td>
<td>2.541E-02</td>
</tr>
<tr>
<td>4</td>
<td>3.735E-05</td>
<td>2.281E-06</td>
<td>3.646E-05</td>
</tr>
<tr>
<td>6</td>
<td>9.919E-06</td>
<td>2.658E-05</td>
<td>9.703E-06</td>
</tr>
</tbody>
</table>

Table 3.5.3  Comparison of the Added-Mass and Damping Coefficients Computed by HOHYD and HYDRO, for 1000*100*2m model, wavelength \( \lambda = 22.000 \) (Meter)
<table>
<thead>
<tr>
<th>Canonical Motions</th>
<th>HYDRO</th>
<th>HOHYD</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A_{\parallel 1}$</td>
<td>$B_{\parallel 1}$</td>
<td>$A_{\parallel 2}$</td>
</tr>
<tr>
<td>1</td>
<td>3.797E-06</td>
<td>8.415E-06</td>
<td>2.674E-06</td>
</tr>
<tr>
<td>2</td>
<td>3.520E-05</td>
<td>8.422E-05</td>
<td>3.603E-05</td>
</tr>
<tr>
<td>3</td>
<td>2.821E-02</td>
<td>1.424E-03</td>
<td>2.756E-02</td>
</tr>
<tr>
<td>4</td>
<td>3.686E-05</td>
<td>3.055E-06</td>
<td>3.398E-05</td>
</tr>
<tr>
<td>5</td>
<td>8.496E-03</td>
<td>4.456E-04</td>
<td>7.286E-03</td>
</tr>
<tr>
<td>6</td>
<td>1.202E-05</td>
<td>2.811E-05</td>
<td>1.278E-05</td>
</tr>
</tbody>
</table>

Table 3.5.4  Comparison of the Added-Mass and Damping Coefficients Computed by HOHYD and HYDRO, for 1000*100*2m model, wavelength $\lambda = 25.000$ (Meter)
<table>
<thead>
<tr>
<th>Canonical Motions</th>
<th>HYDRO</th>
<th>HOHYD</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\bar{A}_{y_1}$</td>
<td>$\bar{B}_{y_1}$</td>
<td>$\bar{A}_{y_2}$</td>
</tr>
<tr>
<td>1</td>
<td>4.259E-06</td>
<td>8.655E-06</td>
<td>7.004E-06</td>
</tr>
<tr>
<td>2</td>
<td>3.911E-05</td>
<td>8.675E-05</td>
<td>3.884E-05</td>
</tr>
<tr>
<td>3</td>
<td>2.796E-02</td>
<td>1.684E-03</td>
<td>3.421E-02</td>
</tr>
<tr>
<td>4</td>
<td>3.659E-05</td>
<td>3.589E-06</td>
<td>3.586E-05</td>
</tr>
<tr>
<td>5</td>
<td>8.423E-03</td>
<td>5.268E-04</td>
<td>1.378E-02</td>
</tr>
<tr>
<td>6</td>
<td>1.336E-05</td>
<td>2.894E-05</td>
<td>1.312E-05</td>
</tr>
</tbody>
</table>

Table 3.5.5  Comparison of the Added-Mass and Damping Coefficients Computed by HOHYD and HYDRO, for 1000*100*2m model, wavelength $\lambda = 27.000 (\text{Meter})$
<table>
<thead>
<tr>
<th>Canonical Motions</th>
<th>HYDRO</th>
<th>HOHYD</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A_{y_1}$</td>
<td>$B_{y_1}$</td>
<td>$A_{y_2}$</td>
</tr>
<tr>
<td>1</td>
<td>4.858E-06</td>
<td>8.888E-06</td>
<td>2.898E-07</td>
</tr>
<tr>
<td>2</td>
<td>4.474E-05</td>
<td>8.986E-05</td>
<td>4.427E-05</td>
</tr>
<tr>
<td>3</td>
<td>2.761E-02</td>
<td>2.087E-03</td>
<td>2.839E-02</td>
</tr>
<tr>
<td>4</td>
<td>3.626E-05</td>
<td>4.402E-06</td>
<td>3.707E-05</td>
</tr>
<tr>
<td>5</td>
<td>8.324E-03</td>
<td>6.524E-04</td>
<td>8.327E-03</td>
</tr>
<tr>
<td>6</td>
<td>1.529E-05</td>
<td>2.995E-05</td>
<td>1.525E-05</td>
</tr>
</tbody>
</table>

Table 3.5.6  Comparison of the Added-Mass and Damping Coefficients Computed by HOHYD and HYDRO, for 1000*100*2m model, wavelength $\lambda = 30.000 (\text{Meter})$
<table>
<thead>
<tr>
<th>Canonical Motions</th>
<th>HYDRO</th>
<th>HOHYD</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A_{y1}$</td>
<td>$B_{y1}$</td>
<td>$A_{y2}$</td>
</tr>
<tr>
<td>1</td>
<td>6.662E-06</td>
<td>9.344E-06</td>
<td>7.765E-06</td>
</tr>
<tr>
<td>2</td>
<td>6.114E-05</td>
<td>9.607E-05</td>
<td>6.064E-05</td>
</tr>
<tr>
<td>3</td>
<td>2.671E-02</td>
<td>3.472E-03</td>
<td>2.655E-02</td>
</tr>
<tr>
<td>4</td>
<td>3.572E-05</td>
<td>7.092E-06</td>
<td>3.550E-05</td>
</tr>
<tr>
<td>5</td>
<td>8.075E-03</td>
<td>1.084E-03</td>
<td>8.172E-03</td>
</tr>
<tr>
<td>6</td>
<td>2.091E-05</td>
<td>3.188E-05</td>
<td>2.075E-05</td>
</tr>
</tbody>
</table>

Table 3.5.7  Comparison of the Added-Mass and Damping Coefficients Computed by HOHYD and HYDRO, for 1000*100*2m model, wavelength $\lambda = 40.000$ (Meter)
<table>
<thead>
<tr>
<th>Canonical Motions</th>
<th>HYDRO</th>
<th>HOHYD</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A_{\psi_1}$</td>
<td>$B_{\psi_1}$</td>
<td>$A_{\psi_2}$</td>
</tr>
<tr>
<td>2</td>
<td>7.434E-05</td>
<td>9.860E-05</td>
<td>7.419E-05</td>
</tr>
<tr>
<td>3</td>
<td>2.611E-02</td>
<td>4.841E-03</td>
<td>2.594E-02</td>
</tr>
<tr>
<td>4</td>
<td>3.582E-05</td>
<td>9.585E-06</td>
<td>3.551E-05</td>
</tr>
<tr>
<td>5</td>
<td>7.917E-03</td>
<td>1.509E-03</td>
<td>7.952E-03</td>
</tr>
<tr>
<td>6</td>
<td>2.540E-05</td>
<td>3.253E-05</td>
<td>2.535E-05</td>
</tr>
</tbody>
</table>

Table 3.5.8  Comparison of the Added-Mass and Damping Coefficients Computed by HOHYD and HYDRO, for 1000*100*2m model, wavelength $\lambda = 50.000$ (Meter)
<table>
<thead>
<tr>
<th>Canonical Motions</th>
<th>HYDRO</th>
<th>HOHYD</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{y_1} )</td>
<td>1.036E-05</td>
<td>1.104E-05</td>
<td>( \frac{A_{y_1} - A_{y_2}}{A_{y_2}} )</td>
</tr>
<tr>
<td>( B_{y_1} )</td>
<td>8.915E-06</td>
<td>8.327E-06</td>
<td>( \frac{B_{y_1} - B_{y_2}}{B_{y_2}} )</td>
</tr>
<tr>
<td>1</td>
<td>-6.203%</td>
<td>7.060%</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>9.811E-05</td>
<td>9.951E-05</td>
<td>-1.407%</td>
</tr>
<tr>
<td>3</td>
<td>2.538E-02</td>
<td>2.518E-02</td>
<td>0.796%</td>
</tr>
<tr>
<td>4</td>
<td>7.763E-03</td>
<td>7.752E-03</td>
<td>0.139%</td>
</tr>
<tr>
<td>5</td>
<td>3.340E-05</td>
<td>3.393E-05</td>
<td>-1.577%</td>
</tr>
<tr>
<td>6</td>
<td>3.220E-05</td>
<td>3.464E-05</td>
<td>-7.036%</td>
</tr>
</tbody>
</table>

Table 3.5.9  Comparison of the Added-Mass and Damping Coefficients Computed by HOHYD and HYDRO, for 1000*100*2m model, wavelength \( \lambda = 75.000 \) (Meter)
<table>
<thead>
<tr>
<th>Canonical Motions</th>
<th>HYDRO</th>
<th>HOHYD</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A_{y1}$</td>
<td>$B_{y1}$</td>
<td>$A_{y2}$</td>
</tr>
<tr>
<td>1</td>
<td>1.180E-05</td>
<td>8.481E-06</td>
<td>1.220E-05</td>
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<td>2</td>
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<td>1.174E-04</td>
</tr>
<tr>
<td>3</td>
<td>2.525E-02</td>
<td>1.065E-02</td>
<td>2.504E-02</td>
</tr>
<tr>
<td>4</td>
<td>4.030E-05</td>
<td>1.818E-05</td>
<td>3.980E-05</td>
</tr>
<tr>
<td>5</td>
<td>7.802E-03</td>
<td>3.302E-03</td>
<td>7.776E-03</td>
</tr>
<tr>
<td>6</td>
<td>3.866E-05</td>
<td>3.089E-05</td>
<td>3.977E-05</td>
</tr>
</tbody>
</table>

Table 3.5.10  Comparison of the Added-Mass and Damping Coefficients Computed by HOHYD and HYDRO, for 1000*100*2m model, wavelength $\lambda = 100.000$ (Meter)
<table>
<thead>
<tr>
<th>Canonical Motions</th>
<th>HYDRO</th>
<th>HOHYD</th>
<th>Relative Differences</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\bar{A}_\parallel$</td>
<td>$\bar{B}_\parallel$</td>
<td>$\bar{A}_\parallel$</td>
</tr>
<tr>
<td>1</td>
<td>1.467E-05</td>
<td>1.373E-06</td>
<td>1.463E-05</td>
</tr>
<tr>
<td>2</td>
<td>1.604E-04</td>
<td>4.490E-06</td>
<td>1.707E-04</td>
</tr>
<tr>
<td>3</td>
<td>5.596E-02</td>
<td>4.776E-02</td>
<td>5.557E-02</td>
</tr>
<tr>
<td>4</td>
<td>6.533E-05</td>
<td>2.153E-06</td>
<td>6.486E-05</td>
</tr>
<tr>
<td>5</td>
<td>2.208E-02</td>
<td>8.326E-03</td>
<td>2.200E-02</td>
</tr>
<tr>
<td>6</td>
<td>5.096E-05</td>
<td>4.352E-07</td>
<td>5.414E-05</td>
</tr>
</tbody>
</table>

Table 3.5.11  Comparison of the Added-Mass and Damping Coefficients Computed by HOHYD and HYDRO, for 1000*100*2m model, wavelength $\lambda = 1600.000 \text{ (} \text{Meter)}$
3.5.2 Diffraction Problem

Similarly as in the low-order method the diffraction pressure vectors under each input waves are computed using the high-order Hydrodynamic Influence Matrices and stored to be used later as the loads input of the high-order Motion Equation. Then the diffraction forces are computed by integrating the pressure distributions on the body surface and also compared with those computed using WAMIT and low-order method to verify the correctness and accuracy of our programs.

The velocity potential of an incoming wave with frequency $\omega$ and incident angle $\theta$ is shown in (2.3.2). It is also decomposed according to the four symmetries as in (2.3.3). For panels on the side and bow plates the coefficients in the vector representing this velocity potential can be computed in the same way as in the low-order method, which is shown in (2.3.2) and (2.3.3). But for the high-order panels on the bottom plate, the potential distribution is represented by the coefficients of the decomposition according to the basis functions which are normally non-zero. These coefficients are the values and derivatives of the potential distribution. At node $i$ the 9 derivatives of the potential of an incoming wave with frequency $\omega$ and incident angle $\theta$ can be computed as:

$$
\varphi_{i,\text{sym}}^{(m,n)} = \frac{\partial^{m,n} \varphi_{i,\text{sym}}}{\partial x^m \partial y^n} = \sum_{i_{\text{quad}}=1}^{4} \frac{\mathbf{G}}{\omega} \text{sign}(i_{\text{sym}},i_{\text{quad}}) e^{-k_{x_{\text{quad}}} \omega t} e^{j(k_{x_{\text{quad}}} \cos \phi + k_{y_{\text{quad}}} \sin \phi)} (jk \cos \phi)^m (jk \sin \phi)^n \quad (3.5.2)
$$

$$m, n = 0,1,2; \quad \text{sym} = 1,2,3,4$$

The coefficients vector of the pressure of symmetry $i_{\text{sym}}$ caused by the incident wave can be computed similarly as in (2.3.5):
\begin{equation}
\left\{ p_{i,k,\text{sym}} \right\} = -\rho \frac{\partial}{\partial t} \{ \varphi_{i,k,\text{sym}}(x_i, y_i, z_i; \omega, \theta) \} = j \omega \rho \{ \varphi_{i,k,\text{sym}}(x_i, y_i, z_i; \omega, \theta) \}, \quad k = 1, 2, \ldots, n \text{unkn}, \quad isym = 1, 2, 3, 4
\end{equation}

Similarly the normal velocity on the body surface caused by the incident wave, which is related to the displacement as in (2.1.2), can be computed from the wave potential as in (2.3.6). The oscillating displacements corresponding to the scattered waves, \( d_{i,k,\text{sym}} \), can also be computed as in (2.3.7). But for the high-order bottom panels we also need to compute the derivatives of \( d_{i,k,\text{sym}} \) to form the coefficients vector. Similarly as in (3.5.2) these derivatives of \( d_{i,k,\text{sym}} \) can be computed according to the formula of the incident wave potential as follows:

\begin{equation}
\begin{aligned}
d^{(m,n)}_{i,k,\text{sym}} &= -\frac{j}{\omega} \left( n_{jk} \cos \theta + n_{jk} \sin \theta - kn_{\text{sym}} \right) \varphi_{i,k,\text{sym}}(x, y, z; \omega, \theta) \\
&\quad \cdot \left( jk \cos \theta \right)^m \left( jk \sin \theta \right)^n \\
&\quad m, n = 0, 1, 2, \quad isym = 1, 2, 3, 4
\end{aligned}
\end{equation}

Using the high-order Hydrodynamic Influence Matrices, the coefficients vectors of the pressure from the scattered waves can be computed similarly as in (2.3.8) for the low-order method:

\begin{equation}
\left\{ p_{i,k,\text{sym}} \right\} = \rho \omega^2 \left\{ d_{i,k,\text{sym}} \right\} \left[ H_{i,k,\text{sym}} \right], \quad i, k = 1, 2, \ldots, n \text{unkn}; \quad isym = 1, 2, 3, 4
\end{equation}

The coefficients vectors of the diffraction pressure are the sums of the coefficients vectors of the incident wave pressure and the scattered wave pressure similarly as in (2.3.9):

\begin{equation}
\left\{ p_{i,k,\text{sym}} \right\} = \left\{ p_{i,k,\text{sym}} \right\} + \left\{ p_{i,k,\text{sym}} \right\} \\
= j \omega \rho \left\{ \varphi_{i,k,\text{sym}}(x, y, z; \omega, \theta) \right\} + \rho \omega^2 \left\{ d_{i,k,\text{sym}} \right\} \left[ H_{i,k,\text{sym}} \right], \quad i, k = 1, 2, \ldots, n \text{unkn}; \quad isym = 1, 2, 3, 4
\end{equation}
These vectors for the diffraction pressure will be used as the loads input in the high-order Motion Equation of the floating-mat model and will be computed and stored for each incident wave of different frequency and incidental angle.

Also the diffraction pressure distribution can be interpolated from the coefficients vector and integrated over the body surface to get the diffraction forces. The diffraction forces computed using high-order method (from the program HOHYD) are compared with those computed using the low-order method (from program HYDRO) for a model floating structure of 1000 meters long, 100 meters wide and with a draft of 2 meters under different waves frequencies as listed in Table 3.5.1. The mesh dividing and the panel size for HOHYD and HYDRO are the same as for the radiation computation in subsection (3.5.1) and the results from HYDRO are expected to have enough accuracy since there are at least 6 panels within a wavelength.

Similarly as in subsection (2.3.2), the results from HYDRO and HOHYD are transformed into non-dimensionalized modules and phase angles of the diffraction forces. The results from HOHYD and HYDRO for the wavelength $\lambda_3 = 1600.000$ (Meter) and incident angles (0°, 19°, 46°) are compared in Tables 3.5.12(a, b, c). The results for the wavelength $\lambda_4 = 100.000$ (Meter) and incident angles (0°, 19°, 46°) are compared in Tables 3.5.13(a, b, c).

From Tables 3.5.12(a, b, c) it can be seen that when the wavelength is large the results from the high-order method program HOHYD are very close to the results from the low-order method program HYDRO for all the 6 modes of diffraction forces. This shows the programming and computation in the high-order method program HOHYD are
correct and it computes the Hydrodynamics Influence Matrix and the vector of diffraction pressure correctly for long waves.

From Tables 3.5.13(a, b, c) it can be seen that when the wavelength is small compared with the length of the structure, the results from the high-order method program HOHYD are still similar to the results from the low-order method program HYDRO for the 6 modes of diffraction forces. But in this case the relative difference between results from HOHYD and HYDRO can sometimes be quite large. This shows although the programming and computation in the high-order method program HOHYD are correct and its accuracy in computing diffraction forces decreases quickly when the incident waves become shorter.
<table>
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<th>Force Mode</th>
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<th>HYDRO</th>
<th>Relative Differences</th>
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(a) Wave Incident Angle $\theta = 0^\circ$
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<th>HYDRO</th>
<th>Relative Differences</th>
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<td>Module</td>
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<td>$\alpha_1$</td>
<td>$F_2$</td>
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(b) Wave Incident Angle $\theta = 19^\circ$
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<th>HYDRO</th>
<th>Relative Differences</th>
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(c) Wave Incident Angle $\theta = 46^\circ$

Table 3.3.12 Comparison of the Diffraction Forces Computed by HYDRO and HOHYD

(1000*100*2m model, wavelength $\lambda_3 = 1600.000$ (Meter), incident angles $0^\circ, 19^\circ, 46^\circ$)
<table>
<thead>
<tr>
<th>Force Mode</th>
<th>HOHYD</th>
<th>HYDRO</th>
<th>Relative Differences</th>
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<td>Phase Angle $\alpha_2 , ^\circ$</td>
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(a) Wave Incident Angle $\theta = 0^\circ$
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<th>HYDRO</th>
<th>Relative Differences</th>
</tr>
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<td>Module $F_1$</td>
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<td>157.1</td>
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(b) Wave Incident Angle $\theta = 19^\circ$
<table>
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<th>HOHYD</th>
<th>HYDRO</th>
<th>Relative Differences</th>
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(c) Wave Incident Angle $\theta = 46^\circ$

Table 3.3.13  Comparison of the Diffraction Forces Computed by HYDRO and HOHYD

(1000*100*2m model, wavelength $\lambda_4 = 100.000$ (Meter), incident angles $0^\circ, 19^\circ, 46^\circ$)
3.6 The High-Order Motion Equation and Structural Responses

In the high-order method the floating-mat model of the floating runway is also applied. The high-order Motion Equation can be derived from (2.4.7) using high-order interpolations of the deflection and pressure distributions. Since the interpolation function is double-5th order, less unknown deflection coefficients need to be solved for. The high-order Motion Equation is also decomposed into four symmetries as in the low-order method.

Substitute (3.2.1) into (2.4.7) and considering (3.2.14), (3.5.6), with similar steps as in the low-order method as shown in (2.4.8) and (2.4.10), the following high-order Motion Equation can be derived:

\[
\begin{align*}
        a_{i,\text{sym}}(D_y \frac{\partial^4}{\partial x^4} & + 2H \frac{\partial^4}{\partial x^2 \partial y^2} + D_y \frac{\partial^4}{\partial y^4})B_i(x, y) + a_{i,\text{sym}} \delta_{ij}(\rho g - T \rho_0 \omega^2)B_j(x, y) \\
        - \rho \omega^2 a_{i,\text{sym}} h_i B_j(x, y) = - p_{D_j,\text{sym}} B_j(x, y)
\end{align*}
\]  \quad (3.6.1)

where the subscripts \(i\) and \(j\) are summed for the first quadrant bottom plate only, i.e. from 1 to \(9 \cdot (nl + 1) \cdot (nb + 1)\), according to the Einstein Convention.

Because the interpolation functions have continuous derivatives up to second order only, the fourth order derivatives in the structural part of (3.6.1) will be non-continuous and can not be decomposed according to the basis functions. So the Galerkin method has been applied to discretize the Motion Equation. Multiplying both sides of (3.6.1) with the nodal basis functions \(B_k(x, y), k = 1,2,\ldots,9(nl + 1)(nb + 1)\) and integrate over the bottom plate, we can get linear equation systems of the discretized Motion Equation:
\[
\begin{align*}
\begin{bmatrix}
D_{ik}' + [\delta^j_i (\rho g - T \rho_b \omega^2) + \rho \omega^2 h_{ij, sym}] b_{jk}'
\end{bmatrix}^T \begin{bmatrix}
a_{i, sym}
\end{bmatrix} = \begin{bmatrix}
- p_{Dj, sym} b_{jk}'
\end{bmatrix} \quad (3.6.2)
\end{align*}
\]

where

\[
D_{ik}' = \int_{\text{bottom plate}} \left[ (D_1(x,y) \frac{\partial \delta^4}{\partial x^4} + 2H(x,y) \frac{\partial \delta^4}{\partial x \partial y^2} + D_2(x,y) \frac{\partial \delta^4}{\partial y^4}) B_i(x,y) \right] B_k(x,y) \, dx \, dy \quad (3.6.3)
\]

\[
b_{ik}' = \int_{\text{bottom plate}} B_i(x,y) B_k(x,y) \, dx \, dy \quad (3.6.4)
\]

The rigidities \( D_1(x,y) \), \( H(x,y) \) and \( D_2(x,y) \) are functions of \((x, y)\) because the plate thickness changes with \((x, y)\). Because the basis functions are determined when the high-order mesh is determined, the matrix \( b_{ik}' \) just needs to be computed once and the matrix \( D_{ik}' \) needs to be computed again only when the thickness distribution is changed. Because the nodal basis functions are defined over the panels connected to the nodes only, both the matrices \( b_{ik}' \) and \( D_{ik}' \) are narrow-banded.

The Motion Equation in (3.6.2) demands more computation time to prepare because the high-order Hydrodynamic Influence Matrix \( h_{ij, sym} \) and the diffraction loads vector \( p_{Dj, sym} \) will be multiplied by the matrix \( b_{ik}' \) to get the left-hand-side matrix of the linear system. But since \( b_{ik}' \) and \( D_{ik}' \) are narrow-banded, the increase in computation time can be reduced by applying this property in the evaluation of the matrix product.

The Galerkin method used in the Motion Equation (3.6.2) also has the advantage of being able to incorporate the free-end boundary conditions of the floating-mat into the Motion Equation, as shown by (Kashiwagi, 1998b). This is done by applying partial
integration in the weighted residual form of the structural motion equation and forcing the boundary conditions to be satisfied as natural boundary conditions, as shown below.

Multiply both sides of (2.4.2) by a nodal basis function \( B_k(x, y) \) and then integrate over the bottom plate, we get a weighted residual form of the original partial differential equation:

\[
\int B_k(x, y) \left[ \frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} \right] dx dy = - \int B_k(x, y) q(x, y) dx dy \quad (3.6.5)
\]

By partially integrating the left side, we can get:

\[
\int B_k(x, y) \left( \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right) \bigg|_{a}^{b} dx dy - \int \left( \frac{\partial B_k(x, y)}{\partial x} M_x \right) \bigg|_{-a}^{a} dy \\
+ \int \frac{\partial^2 B_k(x, y)}{\partial x^2} M_x dx dy - \int \frac{\partial B_k(x, y)}{\partial x} \frac{\partial M_{xy}}{\partial y} dx dy \\
+ \int B_k(x, y) \left( \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \right) \bigg|_{b}^{a} dx dy - \int \left( \frac{\partial B_k(x, y)}{\partial y} M_y \right) \bigg|_{-b}^{b} dx \\
+ \int \frac{\partial^2 B_k(x, y)}{\partial y^2} M_y dx dy - \int \frac{\partial B_k(x, y)}{\partial y} \frac{\partial M_{xy}}{\partial x} dx dy = - \int B_k(x, y) q(x, y) dx dy \quad (3.6.6)
\]

Modifying the free-edge boundary condition (2.4.4a) and (2.4.4b) as follows:

\[
\begin{align*}
M_x \bigg|_{x=\pm a} &= 0 \\
\left( \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \right) \bigg|_{x=\pm a} &= \frac{\partial M_y}{\partial y} \bigg|_{x=\pm a} 
\end{align*} \quad (3.6.7a)
\]

\[
\begin{align*}
M_y \bigg|_{y=\pm b} &= 0 \\
\left( \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} \right) \bigg|_{y=\pm b} &= - \frac{\partial M_x}{\partial x} \bigg|_{y=\pm b} 
\end{align*} \quad (3.6.7b)
\]

Substitute them into (3.6.6) we get:
\[
\int_{a}^{b} (-B_k(x,y) \frac{\partial M_{xy}}{\partial y}) \, dy + \int_{a}^{b} (-B_k(x,y) \frac{\partial M_{xy}}{\partial x}) \, dx \\
+ \iint \frac{\partial^2 B_k(x,y)}{\partial x^2} M_{xy} \, dx dy - \iint \frac{\partial B_k(x,y)}{\partial x} \frac{\partial M_{xy}}{\partial y} \, dx dy \\
+ \iint \frac{\partial^2 B_k(x,y)}{\partial y^2} M_{xy} \, dx dy - \iint \frac{\partial B_k(x,y)}{\partial y} \frac{\partial M_{xy}}{\partial x} \, dx dy = -\iint B_k(x,y)q(x,y) \, dx dy 
\] (3.6.8)

Partially integrate the first two terms in (3.6.8) and apply the boundary conditions at corners (2.4.4c) as follows:

\[
\int_{a}^{b} (-B_k(x,y) \frac{\partial M_{xy}}{\partial y}) \, dy + \int_{a}^{b} (-B_k(x,y) \frac{\partial M_{xy}}{\partial x}) \, dx \\
= (-B_k(x,y)M_{xy})_{a}^{b} - (B_k(x,y)M_{xy})_{-a}^{-b} + \int_{a}^{b} \left( \frac{\partial B_k(x,y)}{\partial y} M_{xy} \right) \, dy + \int_{a}^{b} \left( \frac{\partial B_k(x,y)}{\partial x} M_{xy} \right) \, dx \\
= -2B_k(x,y)[M_{xy}(a,b) - M_{xy}(-a,b) - M_{xy}(a,-b) + M_{xy}(-a,-b)] \\
+ \int_{a}^{b} \left( \frac{\partial B_k(x,y)}{\partial y} M_{xy} \right) \, dy + \int_{a}^{b} \left( \frac{\partial B_k(x,y)}{\partial x} M_{xy} \right) \, dx \\
= \int_{a}^{b} \left( \frac{\partial B_k(x,y)}{\partial y} M_{xy} \right) \, dy + \int_{a}^{b} \left( \frac{\partial B_k(x,y)}{\partial x} M_{xy} \right) \, dx
\]

Substitute it into (3.6.8) and we get:

\[
\int_{a}^{b} \left( \frac{\partial B_k(x,y)}{\partial y} M_{xy} \right) \, dy + \int_{a}^{b} \left( \frac{\partial B_k(x,y)}{\partial x} M_{xy} \right) \, dx \\
+ \iint \frac{\partial^2 B_k(x,y)}{\partial x^2} M_{xy} \, dx dy - \iint \frac{\partial B_k(x,y)}{\partial x} \frac{\partial M_{xy}}{\partial y} \, dx dy \\
+ \iint \frac{\partial^2 B_k(x,y)}{\partial y^2} M_{xy} \, dx dy - \iint \frac{\partial B_k(x,y)}{\partial y} \frac{\partial M_{xy}}{\partial x} \, dx dy = -\iint B_k(x,y)q(x,y) \, dx dy 
\] (3.6.9)

For the convenience of numerical computation, rewrite the line integrals in (3.6.9) into surface integrals by differentiating the integrand by x or y as follows:
\[
\int_a^b \left( \frac{\partial B_k(x,y)}{\partial y} M_{xy} \right) dy + \int_b^a \left( \frac{\partial B_k(x,y)}{\partial x} M_{xy} \right) dx = \iint \left( \frac{\partial B_k(x,y)}{\partial x} \frac{\partial M_{xy}}{\partial y} + \frac{\partial B_k(x,y)}{\partial y} \frac{\partial M_{xy}}{\partial x} \right) dxdy
\]

Substitute it into (3.6.9) and we get a weak form of the original partial differential equation on the balance of the force resultants in a plate with the 4-edge–free boundary conditions incorporated:

\[
\iint \left( \frac{\partial^2 B_k(x,y)}{\partial x^2} M_x + 2 \frac{\partial^2 B_k(x,y)}{\partial x \partial y} M_{xy} + \frac{\partial^2 B_k(x,y)}{\partial y^2} M_y \right) dxdy = - \iint B_k(x,y) q(x,y) dxdy \tag{3.6.10}
\]

Substitute (2.4.1) into it and we get a weak form of the original partial differential equation on the deflection of a plate with the 4-edge–free boundary conditions incorporated:

\[
\iint \left[ D_x \frac{\partial^2 B_k(x,y)}{\partial x^2} + D_y \frac{\partial^2 B_k(x,y)}{\partial y^2} + 4D_{xy} \frac{\partial^2 B_k(x,y)}{\partial x \partial y} \right] \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} dxdy = \iint B_k(x,y) q(x,y) dxdy \tag{3.6.11}
\]

When the plate is isotropic and has uniform rigidities, the right-hand-side of the above weak form is exactly the same as that derived by (Kashiwagi, 1998b).

Substitute (3.2.1) into (3.6.11) we can get a new form of the matrix \( D_{ik} \), noted as \( D_{ik}^* \) here, with the 4-edge–free boundary conditions incorporated as follows:
\[ D_{ik} = \iint_{\text{bottom plate}} \left\{ D_x \frac{\partial^2 B_k(x,y)}{\partial x^2} + D_y \frac{\partial^2 B_k(x,y)}{\partial y^2} + D_z \frac{\partial^2 B_k(x,y)}{\partial z^2} \right\} \, dx \, dy \] (3.6.12)

Using \( D_{ik} \) instead of \( D_k \) in (3.6.2), we get a discrete high-order Motion Equation for the floating-mat model which can be solved for the coefficients vector of the deflection \( \{ a_{i,i_{\text{sym}}} \}, i = 1, 2, \ldots, n\, \text{unkn} ; \, i_{\text{sym}} = 1, 2, 3, 4 \). The great advantage of the high-order method is that under same accuracy requirement the total number of unknown variables, \( n\, \text{unkn} \), is much smaller than that of the low-order method. On the other words, by solving linear systems of the same scale, the high-order method enable us to get accurate results for much shorter waves than the low-order method.

From the coefficients vectors \( \{ a_{i,i_{\text{sym}}} \} \), the deflection everywhere on the floating-mat (equal to the deflection on the bottom plate) can be interpolated using the basis functions. Then same as in the low-order method the various structural responses can be computed from the deflection. Finally the directional spectra of the various structural responses under an input wave spectrum can be computed by solving the Motion Equation for incident waves of different frequencies and heading angles, same as in the low-order method.

Figure 3.6.1 shows the deflection of the full-size floating runway which is 4000 meters long, 400 meters wide, 5 meters deep and with a draft of 2 meter, in incident regular waves of wavelength \( \lambda = 415.01 \, \text{meters} \) and heading angle \( \alpha = 17.07^\circ \). It can be seen that the structure is quite flexible. The structural deflection shows longer wavelength...
than the water waves. The deflection is largest at the edge encountering the waves and also larger at the leeward edge than in the middle area.

Figure 3.6.1 Deflection of the Full-Size Floating Runway in Incident Regular Waves of wavelength \( \lambda = 415.01 \) meters and heading angle \( \alpha = 17.07^\circ \)
Chapter 4

Structural Optimization

4.1 The Optimization Problem

The statement of the optimization problem involves the definition of the design variables, the objective function and the constraints. Here the design variables are the thickness parameters for the modules, i.e.

\[ ar(i,j), \ i=1,2,...,nlm; \ j=1,2,...,nbm. \]  

(4.1.1)

where \( nlm \) and \( nbm \) are the numbers of modules divided in the longitudinal and transverse directions respectively, in the first quadrant of the structure. There are totally \( numm = nlm \times nbm \) such modules. The thickness distribution of the structure is supposed to be symmetric about the longitudinal and transverse centerlines so only the thickness in one quadrant needs to be considered in terms of the design variables. The optimization process introduced here is almost identical for both low-order and high-order computation of the hydrodynamics. Thus, both will be treated simultaneously here.

All the modules have the same initial structural geometry and will keep this geometry in the process of the optimization, which means they will maintain the same height, same transverse web spacing, same longitudinal girder spacing and the same ratio between their web thickness, girder thickness, deck plate thickness and bottom plate thickness. This makes the rigidities \( (D_x, D_y, D_1, D_o) \) of the equivalent orthotropic plate have constant relationship to each other. That is, we constrain changes in strength for a single module \((i,j)\) to be restricted to changes in the thickness of all its plates in proportion to a
single parameter, which is noted as $ar(i,j)$. Suppose the deck, bottom, web and girder plates in the module have initial thickness $(t_{deck0}, t_{bottom0}, t_{web0}, t_{girder0})$ and the equivalent orthotropic plate has initial rigidities $(D_{x0}, D_{y0}, D_{01}, D_{xy0})$, then at a point in the optimization process with parameters $ar(i,j)$, $i=1,2,...,nlm$; $j=1,2,...,nbm$ the thickness of the plates in the modules $(i, j), i=1,2,...,nlm; j=1,2,...,nbm$ will be $(ar(i,j) \cdot t_{deck0}, ar(i,j) \cdot t_{bottom0}, ar(i,j) \cdot t_{web0}, ar(i,j) \cdot t_{girder0})$ and the corresponding rigidities of the equivalent orthotropic plate will be $(ar(i,j) \cdot D_{x0}, ar(i,j) \cdot D_{y0}, ar(i,j) \cdot D_{01}, ar(i,j) \cdot D_{xy0})$. So at any point in the optimization process the design variable $ar(i,j)$ determines the strength of the module $(i,j), i=1,2,...,nlm; j=1,2,...,nbm$ in a linear way.

The objective function in our optimization problem is the total weight of the structure, i.e. $4 \times \sum_{i=1}^{nlm} \sum_{j=1}^{nbm} ar(i,j) \cdot weight0(i,j)$, where $weight0(i,j)$ is the initial weight of the module $(i,j)$. Since we suppose all the modules are initially identical, the $weight0(i,j)$ is a constant over the structure. The total weight of the structure is equal to the product of a constant and the sum of $ar(i,j)$'s. So we can chose the objective function simply to be the sum of $ar(i,j)$'s:

$$weight\{\{ar(i,j)\}\} = \sum_{i=1}^{nlm} \sum_{j=1}^{nbm} ar(i,j)$$

(4.1.2)

This objective function is a linear function of the design variables and is very easy to evaluate. The objective of the optimization process is to minimize the value of this function by move the design variables $\{ar(i,j)\}$ step by step toward an optimal point.
\{ar_{opt}(i,j)\}$. There may be many optimal points of design variables where the objective function reaches a local minimum.

The optimization will be under constraints on the rigidity and strength of the structure. In our research only the responses from the wave loading are considered even though the real floating runway will be subject to other operating and static loadings. Two responses are considered for the rigidity of the structure. One is the first order longitudinal derivative of the deflection of the equivalent orthotropic plate, \( \frac{\partial w(x,y)}{\partial x} \), which is the longitudinal slope of the runway. Another rigidity response is the second order longitudinal derivative of the deflection of the equivalent orthotropic plate, \( \frac{\partial^2 w(x,y)}{\partial x^2} \), which determines the magnitude of the curvature in the longitudinal direction.

These two responses will affect the safety of taking off and landing of the airplane. Five responses are considered for the strength of the structure: the maximum normal stresses in the \( x \) direction and \( y \) direction cross sections, \( \sigma_x \) and \( \sigma_y \); the horizontal in-plane shear stress, \( \tau_{xy} \); and the vertical shear stresses in the web plate in the \( x \) and \( y \) directions, \( \tau_x \) and \( \tau_y \). These stresses are related to the derivatives of the deflection of the plate that are computed by finite difference method. Formulas for the evaluation of these responses can be found in books on plates and shells and will not be repeated here. According to the condition of the floating structure, some of these seven responses are used to evaluate the probability of failure of the structure. For convenience these responses are noted as \( \sigma_i, i = 1,...,nstr \), where \( nstr \) is the number of responses selected.
In structural reliability analysis, the resistance of the structure is normally considered as a random variable with certain probability distribution to take care of the uncertainties involved in the material, fabrication and other aspects. But since little data exist on these uncertainties and we are interested only in a primary strength distribution that can be a reference to the initial design of the floating runway considering the probabilistic wave loadings, we will suppose the structural resistances for each of the responses are constants determined by reasonable design considerations. So if a certain pre-determined value for each of these resistances is exceeded by the corresponding probabilistic wave responses, the structure is supposed to be broken. For convenience these pre-determined permitted resistance values are noted as $R_i, i = 1, \ldots, 7$.

Then the probability of failure of the structure for response $k$ at any grid point $(i,j)$ of the structure is as follows:

$$pf(i, j, k) = \text{Prob}(\sigma_k > R_k), i = 1, 2, \ldots, nlw2; j = 1, 2, \ldots, nbw2; k = 1, 2, \ldots \text{nstr}$$  \hspace{1cm} (4.1.3)

where $nlw2, nbw2$ are numbers of grid points in the longitudinal and transverse directions in all the four quadrants of the structure. In the low-order method there are $nlw2 = nl \cdot 2$, and $nbw2 = nb \cdot 2$ because only the centers of the low-order panels constitute the grid points. In the high-order method there are $nlw2 = 8 \cdot nl \cdot 2$ and $nbw2 = 8 \cdot nb \cdot 2$ because $8 \times 8$ evenly distributed grid points in each high-order panel are selected where the probability of failure is evaluated.

There are all totaled $\text{nstr} \times nlw2 \times nbw2$ values of probability of failure. This is a very large number, especially for the high-order method. To limit the number of constraints at a convenient level, we can choose to evaluate the maximum of the probability of failure among all modules and impose constraints on these maximum values as follows:
\[ pfm(i, j, k) = \max_{m=8(i-1)+1; n=8(j-1)+1} \left[ pf(m, n, k) \right] \leq u_k, \quad (4.1.4) \]

\[ i = 1, 2, \ldots; nlm \cdot 2; j = 1, 2, \ldots; nbm \cdot 2; k = 1, 2, \ldots, nstr \]

So there will be totally \( nstr \times nlm \times nbm \) constraints. The number of constraints can be further reduced to one by imposing the limit on the maximum probability of failure among all the grid points in the whole structure from each kind of responses as follows:

\[ \max_{k=1}^{nstr} \left( \max_{nwl2; nbw2} \max_{i=1; j=1}^{nlm \cdot 2} \left[ pf(i, j, k) \right] - u_k \right) \leq 0 \quad (4.1.5) \]

The permitted maximum probability of failure, \( u_k, k = 1, 2, \ldots, nstr \), are selected for each of the responses according to the significance of their failure. Since the slope and curvature responses affect the serviceability and the stresses responses affect the structural safety, the permitted maximum probabilities of failure of the former two responses can be much larger than the latter five responses. The \( u_k \)'s for slope and curvature are set to be 0.01 and those for stresses are set to be 0.001.

To regulate the multiple constraints shown in (4.1.4) the following strength criteria are computed from the local maximum probability of failure among modules:

\[ \lambda(i, j, k) = \frac{pfm(i, j, k) - u_k}{u_k} \quad (4.1.6) \]

And for the mono-constraint shown in (4.1.5) a single strength criterion is computed from the maximum probability of failure for each responses as shown in (2.5.14):
Values of $\lambda(i,j,k)$'s are in the range $[-1.0, 99.0]$ for slope and curvature and $[-1.0, 999.0]$ for the stresses. If any one of these measures is greater than zero, the structure is too weak and at least one constraint is violated. If all these measures are less than zero, the structure is too strong. And the critical surface which divides the feasible and non-feasible domain can be defined as:

$$\lambda_{max} = 0,$$

for mono-constraint method

or

$$\begin{cases} 
\lambda(i,j,k) = 0 & \text{for active constraints} \\
\lambda(i,j,k) < 0 & \text{for non-active constraints}
\end{cases}$$

for multi-constraints method \hspace{1cm} (4.1.8)

$for i = 1,2,\ldots,nlm \cdot 2; j = 1,2,\ldots,nbm \cdot 2; k = 1,2,\ldots,nstr;$

For the multiple constraints method the active constraints are determined by the standard:

$$\lambda(i,j,k) \text{ is active if } \lambda(i,j,k) \geq -\epsilon$$

(4.1.9)

where $\epsilon$ is a small positive number, e.g. $\epsilon = 0.05$. Suppose there will be totally $nact$ active constraints.

The result of the optimization should stay on this surface. So the constraint of our optimization problem is:

$$\lambda_{max} \leq 0,$$

for mono-constraint method

or

$$\lambda(i,j,k) \leq 0,$$

for multi-constraints method \hspace{1cm} (4.1.10)

$for i = 1,2,\ldots,nlm \cdot 2; j = 1,2,\ldots,nbm \cdot 2; k = 1,2,\ldots,nstr;$
4.2 The Optimization Process

Initially all the design variables (thickness parameters) $ar(i,j), i=1,2,...,nlm; j=1,2,...,nbm$ are set to be a certain constant and the criterion $\lambda_{max}$ is computed. If the design variables are not on the limit-state surface defined in (4.1.8) the so-called zooming-in steps will be started and a feasible solution is searched by changing all the design variables $ar(i,j)$ simultaneously proportional to a power of $(1+\lambda_{max})$. Thus, the values of $ar(i,j)$ at the $(n+1)$th step are related to those at the $n$th step by

$$ar(i,j)_{n+1} = ar(i,j)_n \times (1 + \lambda_{max})^x$$  \hspace{1cm} (4.2.1)

The order of the power $x$ is selected between 0.001 and 0.05 because the probability of failure is very sensitive to the strength factors. This zooming-in step iterates until $\lambda_{max}$ is so close to zero that the design variables are considered to be on the limit-state surface and the solution is feasible. In addition to the initial zooming-in steps needed to get the first feasible solution, zooming-in steps may also be introduced after an optimization step to bring $\lambda_{max}$ to nearly zero. The first feasible solution gives a better starting point for the actual optimization.

The optimization procedure applies a version of the gradient projection method that has been generalized to nonlinear constraints (see: Rosen (1961), Haftka & Gurdal (1992). The procedure used here is the one suggested by Haug and Arora (1979) and presented in (Haftka & Gurdal (1992)). At an optimization step $n$ the value of $weight(ar(i,j)), \lambda(i,j,k)$ and $\lambda_{max}$ are evaluated for the current feasible design variables vector $\{ar(i,j)\}_n$. 

168
Then the gradient of the criteria of the active constraints, which is a second order tensor, is evaluated:

\[ N_n = \begin{bmatrix} \frac{\partial \lambda(i_1, j_1, k_1)}{\partial r_n(l,1)} & \frac{\partial \lambda(i_2, j_2, k_2)}{\partial r_n(l,1)} & \ldots & \frac{\partial \lambda(i_{nact}, j_{nact}, k_{nact})}{\partial r_n(l,1)} \\ \frac{\partial \lambda(i_1, j_1, k_1)}{\partial r_n(l,2)} & \frac{\partial \lambda(i_2, j_2, k_2)}{\partial r_n(l,2)} & \ldots & \frac{\partial \lambda(i_{nact}, j_{nact}, k_{nact})}{\partial r_n(l,2)} \\ \ldots & \ldots & \ldots & \ldots \\ \frac{\partial \lambda(i_1, j_1, k_1)}{\partial r_n(nl, nb)} & \frac{\partial \lambda(i_2, j_2, k_2)}{\partial r_n(nl, nb)} & \ldots & \frac{\partial \lambda(i_{nact}, j_{nact}, k_{nact})}{\partial r_n(nl, nb)} \end{bmatrix}_{num+nact} \]  

\hspace{1cm} (4.2.3)

The projection of the steepest descent direction (the minus gradient of the weight function) on the hyper-plane tangent to the critical surface defined by (4.1.8) is computed as:

\[ S = P \nabla W = -\left[ I - N_n (N_n^T N_n)^{-1} N_n^T \right]_{num+num} \nabla W \]

\[ = -\left[ I - N_n (N_n^T N_n)^{-1} N_n^T \right]_{num+num} [1,0,1,0,\ldots,1,0]^T_{num} \]  

\hspace{1cm} (4.2.4)

The step length, \( \alpha \), that the design variables vector will move along this direction at the optimization step \( n \) is determined by specifying a desired reduction \( \gamma \) in the weight function by:

\[ \alpha = -\frac{\gamma}{S^T \cdot \nabla W (\{ar(i,j)\}_{n})} \]  

\hspace{1cm} (4.2.5)

The restoration move is along the subspace normal to the critical surface and will reduce the non-zero active constraints to zero:

\[ \{ar(i,j)\}_{n+1} - \{ar(i,j)\}_{n} = -N_n (N_n^T N_n)^{-1}\{\lambda(i,j,k)\}_{nactn} \]  

\hspace{1cm} (4.2.6)

Following Haug and Arora (1979) and (Haftka & Gurdal (1992)) the projection and restoration moves can be combined in one optimization step as:

\[ \{ar(i,j)\}_{n+1} = \{ar(i,j)\}_{n} + \alpha S - N_n (N_n^T N_n)^{-1}\{\lambda(i,j,k)\}_{nactn} \]  

\hspace{1cm} (4.2.7)
The restoration in this step may not be accurate enough if the constraint surface is highly nonlinear. Also after the above combined step, the thickness parameters may be out of reasonable range, becoming too large or too small or even negative. So adjusting steps are needed in which the thickness parameter vector \( \{ar(i, j)\}_{n+1} \) goes back along the same direction as in the step in (4.27). This process is repeated until after several adjusting steps, each with half the step length of the former one, a thickness parameter vector \( \{ar(i, j)\}_{n+1} \) within reasonable range is achieved.

If under the new design variables the structure is still feasible, then another optimization step will be carried out. But if the structure becomes infeasible, an adjusting step will be carried out to let the thickness parameters vector \( \{ar(i, j)\}_{n+1} \) go back half way. If the design variables are still not feasible after several adjusting and re-evaluating steps, zooming-in steps may be necessary to find a feasible solution to start the optimization step.

The above procedure is iterated until the projection of the negative gradient of the weight function on the hyper-plane tangent to the critical surface, \( S \) as computed from (4.2.4), is zero. \( S = 0 \) is the so-called Kuhn-Tucker condition for a local optimal or stationary solution. In practice it takes a great many steps to reach a point where the Kuhn-Tucker condition is satisfied. As result, the optimization is usually stopped when the decrease of the weight function from one step to the next is small.

The gradient of the strength criteria \( \lambda(i,j,k) \) with respect to the design variables is computed by finite differences. This requires evaluating \( \lambda(i,j,k) \) at least \( (numm+1) \) times (for forward finite differences). Since evaluating \( \lambda(i,j,k) \) requires solution of the Motion Equation using as input the wave spectrum, this will be overwhelmingly time-consuming.
Thus, a fast reanalysis method must be used, which takes advantage of the fact that the finite difference of the design variables are very small.

To compute the derivative \( \frac{\partial \lambda(i,j,k)}{\partial r_n(k,l)} \) we increase \( r_n(k,l) \) by a very small amount \( \nabla r_n(k,l) \). In the low-order method \( \nabla r_n(k,l) \) will change the structural matrix \( \{ D_j \} \) by \( \{ \nabla D_j \} \), which is sparse consisting of a few non-zero elements. Suppose the deflection \( w_{j,isym}, j=1,2,...,num; isym=1,2,3,4 \) will change by \( \nabla w_{j,isym} \). After plugging these into the low-order Motion Equation (2.4.12) and omitting the second order terms, the equation which controls \( \nabla w_{j,isym} \) can be written as:

\[
\left[ D_j - \rho \bar{\gamma}^2 H_{ij,isym} + (\rho g - T \rho \bar{\gamma}^2) \delta^i \right] \{ \nabla w_{j,isym} \} = \{ \nabla D_j \} \left[ w_{j,isym} \right] 
\]

\[ i,j=1,2,...,num; k=1,2,...,node_{isym}=1,2,3,4. \] (4.2.8)

Similarly in the high-order method, \( \nabla r_n(k,l) \) will change the structural matrix \( \{ D_{ik} \} \) by \( \{ \nabla D_{ik} \} \), which also is also sparse consisting of a few non-zero elements. Suppose the deflection parameters \( a_{i,isym}, i=1,2,...,nunkn; isym=1,2,3,4 \) will change by \( \nabla a_{i,isym} \). After plugging these into the low-order Motion Equation (3.6.2) and omitting the second order terms, the equation which controls \( \nabla a_{i,isym} \) can be written as:

\[
D_{ik}^{'} + [\delta^i_j (\rho g - T \rho \bar{\gamma}^2) + \rho \bar{\gamma}^2 h_{ij,isym}] b_{ik}^{'} \left[ \nabla a_{i,isym} \right]^T = \left[ \nabla D_{ik}^{'} \right]^T \left[ a_{i,isym} \right] 
\]

(4.2.9)
After the right-hand-side vector has been computed from the original deflection and
the sparse rigidity difference matrix, the deflection difference $\nabla w_{j, isym}$ or the deflection
parameter difference $\nabla a_{i, isym}$ is computed by a direct and inexpensive substitution
because the left-hand-side matrix has already been inverted. The new deflection can then
be used to compute the new criteria $\lambda'(i, j, k)$'s, where the prime indicates the value of
$\lambda(i, j, k)$ computed using $a_{i, isym} + \nabla a_{i, isym}$. The derivative $\frac{\partial \lambda(i, j, k)}{\partial a_{n}(k, l)}$ can then be
computed from the forward finite difference formula as $\frac{\lambda'(i, j, k) - \lambda(i, j, k)}{\nabla a_{n}(k, l)}$. After only
one Gaussian Inversion all the derivatives for an optimization step can be computed this
way. This method dramatically reduces the computation task while maintaining the
necessary accuracy.

The accuracy can be improved by using central finite difference formula instead of
the forward finite difference formula. Let $a_{n}(k, l)$ decrease by the very small amount
$\nabla a_{n}(k, l)$ and compute the new criteria criteria $\lambda''(i, j, k)$. Then the derivative
$\frac{\partial \lambda(i, j, k)}{\partial a_{n}(k, l)}$ can be computed from the central finite difference formula as
$\frac{\lambda'(i, j, k) - \lambda''(i, j, k)}{2\nabla a_{n}(k, l)}$, where the double-prime indicates the value of $\lambda(i, j, k)$ computed
using $a_{i, isym} - \nabla a_{i, isym}$. 

172
4.3 The Optimization Results

Because solving the Motion Equation is very time-consuming, the optimization process is still a very heavy computing task even after many efforts to simplify and reduce the computational task. Model structures are then used to test the procedures and reveal elemental information on the optimization of the full-size structure.

4.3.1 The Optimization Results Using Low-Order Method on a Small Model

The test structure considered here is 80 meters long, 40 meters wide, 5 meters deep and has a draft of 2 meters. The low-order panel size is 2.5 meters by 2.5 meters, which is sufficiently smaller than the considered wavelengths to assure good computation of the hydrodynamic effects. The directional seaway is supposed to have a significant wave height of 4m and a mean period of 10s. Two predominant directions are considered 30° and 60° and a cosine-squared spreading is used. The frequency domain is discretized into ten frequency segments and the direction into five azimuth segments. There are 16*8 panels in each quarter of the structure, for a total of 512 panels for the whole structure.

The limits on slope and curvature are set to 0.5° and 0.001 respectively. Limits on normal stress and shear stress are set to 235Mpa and 136Mpa, which are the yield stresses for mild structural steel. One constraint on \( \lambda_{max} \) is considered in the optimization problem.
Figure 4.3.1 shows the results of the “zooming” steps for the seas at a predominant direction of 30°. It was found that the initial design was not strong enough and the thickness parameters \( \lambda_{i,j} \)'s have all been increased. At first the value of \( \lambda_{max} \) decreases very slowly. After some point it decreases much more rapid and passes through zero (a feasible solution) at \( \lambda_{i,j} = 1.65 \) for all of the \( i \) and \( j \). This defines the scantlings of the platform if one chooses to make the scantlings of all of the sections identical. After several iterations of the optimization steps, the distribution of the \( \lambda_{i,j} \) become stationary and these results are shown in Figure 4.3.2.

It can be seen that there are regions near the bow of the platform that have strength factors considerably in excess of 1.65. However, the majority of the platform has optimum thickness parameters equal to 1.0, which is set as the minimum value feasible. The result is a structure that is about 30% lighter and presumably less expensive than the uniform strength design.

![Image of graph showing variation of \( \lambda_{max} \) with \( \lambda_{i,j} \)'](image)

**Fig.4.3.1** Variation of \( \lambda_{max} \) with \( \lambda_{i,j} \)’s

\[ H_{1/3} = 4 \text{ (meter)}, \ T=10 \text{ (sec.)}, \ \text{dominant wave direction 30°} \]
Figure 4.3.2 Optimum variation of $ar(i,j)$'s over the platform

$(H_{1/3} = 4\, \text{meter}, T=10\, \text{sec.},$ dominant wave direction $30^\circ)$

Figure 4.3.3 shows similar results for the same platform exposed to directional seas from $60^\circ$. The results are not much different from those for $30^\circ$ but the values of $ar(i,j)$'s near the bow are generally lower. In this case the structural weight is about 28% lower than the minimum structure with uniform scantlings.
Figure 4.3.3 Optimum variation of $a_r(i,j)$'s over the platform

($H_{1/3} = 4 \text{ (meter)}, T=10 \text{ (sec.)}, \text{dominant wave direction } 60^\circ$)
4.3.2 The Optimization Results Using High-Order Method

First a small square test structure which is 100 meters long, 100 meters wide, 5 meters deep and has a draft of 2 meters is considered. It is supposed that the structure will encounter only two incident waves of frequency $\omega_1 = 1.1348 \frac{Rad}{Sec}$ (wavelength $\lambda_1 = 47.815$ (meter)) and $\omega_2 = 1.56940 \frac{Rad}{Sec}$ (wavelength $\lambda_2 = 25.000$ (meter)), each with two incident angles of 0° and 90°, instead of a real wave system.

For the incident waves considered, a $2 \times 2$ mesh for the bottom plate already has sufficiently small panels to assure good computation of the hydrodynamic effects. Five structural responses are considered with the limits on slope and curvature set to be 0.001 and 0.0001 respectively, limits on normal stresses $\sigma_x$ and $\sigma_y$ set to be 90Mpa and limit on plane shear stress $\tau_{xy}$ set to be 60Mpa. Multiple $(nstr \times nlw2 \times nbw2)$ constraints on $\lambda(i, j, k)'s$ are considered in the optimization problem.

Figure 4.3.4 shows the distribution of thickness parameters after the Kuhn-Tucker condition has been approximately met in the optimization. The result shows that the center area of the structure will be stronger than the side area of the structure.

For the purpose of comparison $4 \times 4$ and $8 \times 8$ mesh for the bottom plate are also used in the computation. Figure 4.3.5 and Figure 4.3.6 shows the distribution of thickness parameters when optimized using these finer meshes. These distributions are very close to each other and the distribution for a $2 \times 2$ mesh, with the central area stronger than side areas. Figure 4.3.7 shows how the total weight of the achieved optimum changes
when finer meshes are used. The optimal total weight decreases at first when a finer mesh is used until a point after which a finer mesh will not reduce the optimal weight.

Then a model resembling the shape of a full-size floating runway is optimized under a wave system. The wave system is chosen to have significant wave height $H_{1/3} = 4$ (meter) and mean period $T=6$ (sec.) with cosine-squared form spreading function. The dominant incident angle is $0^\circ$. The spectrum is represented by waves of five frequencies and 5 heading angles. The Motion Equation is based on a $20 \times 2$ high-order panel mesh of the floating-mat model. Five structural responses (curvature, normal stresses $\sigma_x$ and $\sigma_y$, plane shear stress $\tau_{xy}$ and vertical shear stress $\tau_x$) are considered with the limits on curvature set to be 0.0001, limits on normal stresses $\sigma_x$ and $\sigma_y$ set to be 90Mpa and limit on shear stresses $\tau_{xy}$ and $\tau_x$ set to be 60Mpa. Multiple $(nstr \times nlw2 \times nbw2)$ constraints on $\lambda(i, j, k)$'s are considered in the optimization problem.

For this model structure it is hard to reach the local optimal point which satisfies the Kuhn-Tucker condition. But after several optimization steps the total weight can be reduced significantly. Figure 4.3.8 shows the feasible distribution of thickness parameters after 5 optimization steps.

Finally the full-size floating runway is optimized under the wave system introduced in section (2.5). As described in section (2.5), the wave system has significant wave height $H_{1/3} = 4$ (meter) and mean period $T=10$ (sec.) with ISSC recommended cosine-$4^{th}$ power form spreading function. The dominant incident angle is $60^\circ$. The spectrum is represented by waves of 10 frequencies and 9 heading angles as shown in Table (2.5.1) and Table (2.5.2). The Motion Equation is based on a $40 \times 4$ high-order mesh of the
floating-mat model. Five structural responses (curvature, normal stresses $\sigma_x$ and $\sigma_y$, plane shear stress $\tau_{xy}$, and vertical shear stress $\tau_x$) are considered with the limits on curvature set to be 0.0001, limits on normal stresses $\sigma_x$ and $\sigma_y$ set to be 90Mpa and limit on shear stresses $\tau_{xy}$ and $\tau_x$ set to be 60Mpa. Multiple ($nstr \times nlw2 \times nbw2$) constraints on $\lambda(i, j, k)$'s are considered in the optimization problem.

For this full-size structure the optimization is still very time-consuming even after all the simplifications. it is hard to reach the local optimal point which satisfies the Kuhn-Tucker condition. Figure 4.3.9 shows the feasible distribution of thickness parameters after one optimization step.

![Diagram](image.png)

Figure 4.3.4 Optimized Distribution of Thickness Parameters

$$(2 \times 2 \text{ mesh, } 100 \times 100 \times 2 \text{ meters model})$$
Figure 4.3.5 Optimized Distribution of Thickness Parameters

(4 × 4 mesh, 100 × 100 × 2 meters model)

Figure 4.3.6 Optimized Distribution of Thickness Parameters

(8 × 8 mesh, 100 × 100 × 2 meters model)
Figure 4.3.7 Change of the Total Weight with Panel Numbers of the Mesh

(1: 2×2 mesh, 2: 4×4 mesh, 3: 8×8 mesh, 100×100×2 meters model)
Figure 4.3.8 Preliminary Optimized Distribution of Thickness Parameters

(20×2 mesh, 2000×200×2 meters model)
Figure 4.3.9 Preliminary Optimized Distribution of Thickness Parameters

(40 x 4 mesh, 4000 x 400 x 2 meters full-size floating runway)
Chapter 5

CONCLUSIONS AND DISCUSSIONS

The optimization of the strength distribution of the floating runway is critically important for the feasibility of practical large-scale floating platforms. These platforms are different from current offshore platforms in that they are so large that they are quite flexible and can not be treated as a rigid body. In the research here, the optimization of such structures under wave loads is explored and the techniques needed to make this kind of optimization possible are developed.

The hydrodynamics computation is the most important and also most difficult aspect of the process. For the large-scale platforms considered here, the ratio of wave length to platform length is in the order of 1:100 and this makes the computation difficult. The concept of a “Hydrodynamic Influence Matrix” which captures all the hydrodynamic properties of the floating structure and which incorporates them into the motion equation of the floating structure is introduced and exploited. The purpose of the detailed hydrodynamics computation presented here is to prepare the Hydrodynamic Influence Matrices of enough accuracy for various wave frequencies of interest for large-scale platforms.

In Chapter 2 a low-order panel method is developed to compute Hydrodynamic Influence Matrices in which the source distribution over a panel is constant. The accuracy of this method was verified by comparing its radiation and diffraction force results for rigid platforms with those from WAMIT, an industry standard for computing motions and loads for typical offshore platforms. These low-order Hydrodynamic Influence Matrices are then incorporated into the finite-difference form of the motion equation

184
resulting in a floating-mat model. The spectra of the responses of the floating runway are then computed by solving the finite-difference equation using a discretized incident wave spectrum.

Because of the very large scale of the floating runway the incident waves can be very short compared with the length of the structure. Under these situations it was found that the low-order panel method required the use of an extremely large number of panels to be able to capture the effects of the short waves. Although storage of the correspondingly large matrices is difficult, the overwhelming problem is one of the numerical accuracy involved in inverting the required matrices. In the end, this loss of accuracy prevents the use of low-order panel methods for this problem.

In Chapter 3 a high-order panel method is then developed to overcome this shortcoming of the low-order method. Double fifth-order interpolation functions are used to represent the source distribution as well as the displacement and pressure distribution within a flat high-order panel. In order for the high-order method to achieve high accuracy for short waves, analytically-exact integrations of the singular Rankine part of the Green function and its gradient are developed through the use of a computer-based symbolic manipulation program (MATHEMATICA). The analytical formulae are then used to compute these integrations accurately when the field point is close to the source panel. Since these exact results blow up if the field point is far from the panel, approximate formulations of these integrations are then developed for these situations by using asymptotic series for the Rankine source potential and its gradient. These approximate formulae can compute the involved integrations accurately when the field point is far away from the source panel. By combining the analytical formulae and the
approximate formulae for the integrations involving the Rankine source, as well as the Gaussian Quadrature for the integration of the remnant part of the Green function, high-order Hydrodynamic Influence Matrices are computed with acceptable accuracy for waves with wavelength $\frac{1}{80}$ of the length of the floating runway. The high-order Hydrodynamic Influence Matrices are incorporated into the corresponding high-order motion equations of the floating-mat model. These equations are solved through using the Galerkin method (a weak formulation) to get the responses of the floating runway in a wave system.

In Chapter 4 the optimization procedure for the strength distribution of the floating runway is designed based on the structural reliability concepts using gradient projection method. The objective function is the total weight of the structure and the design variables are the thickness parameters. Constraints are imposed on the probability of failure of the structure exposed in narrow-band wave systems. One or multiple constraints are established by combining the probability of failure of each module of the structure for different responses into one or multiple criteria.

Preliminary optimizations are carried out for various model structures and the full-size floating runway. For the small model structures local optima satisfying the Kuhn-Tucker condition are found. For the large model structure and the full size floating runway preliminary optimized results with reduced total weight are presented as a reference for future work.

At present the optimization procedure works well and some optimal design point is reached. But more work need to be done to compute the hydrodynamics more accurately and to reduce the huge computation task involved in the optimization process. The high-
order method has been proved to be a powerful way to compute the hydrodynamics of the floating runway. To improve its accuracy, a program to compute the free-surface Green function with higher precision than the subroutine ‘FINGREEN’ used here needs to be developed first. Then the performance of higher order or more powerful interpolation functions needs to be explored. If in the future the Hydrodynamic Influence Matrix can be computed accurately for waves of wavelength $\frac{1}{200}$ of the length of the full size floating runway, the hydrodynamics problem can be considered solved perfectly for the hydro-elasticity problem involved in the optimization.

In the optimization part more work needs to be done to reduce the times that the motion equations in the hydroelastic problem need to be solved. Because of the very large number of degrees of freedom, this consumes most of the calculation time. A re-design of the optimization problem may be needed to make the critical surface smoother and to make the optimization steps come out of the feasible zone less frequently. More special techniques to reduce the size of the motion equations and to solve it more efficiently also need to be explored.

Only the local optimum is searched for in the research presented above. The methods to search for the global optimum like the “Simulated Annealing Algorithm” and the “Genetic Algorithms” (Haftka and Gurdal, 1992), which need a large number of evaluations of the object function and the constraints, are not practical until the involved computation task has been reduced dramatically.
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Appendix I

General Higher-Order Interpolation Functions

Higher-order polynomial interpolation functions can be constructed in a rectangular panel. The general case is studied and listed below.

First consider polynomials complete up to a certain order of power of \( x \) and \( y \). For any bi-(2n-I)th-order polynomial function of the form \( f(x, y) = \sum_{i=0}^{2n-12n-1} \sum_{j=0}^{1} a_{ij} x^i y^j \), there are totally \( 2n \times 2n = 4n^2 \) unknown coefficient \( a_{ij} \)'s. At each corner \( k \) of the panel prescribe all the derivatives up to the \( m \)th order of \( x \) and \( y \), i.e. let \( f_{ij}(x_k, y_k) = \frac{\partial^{i+j} f(x_k, y_k)}{\partial x^i \partial y^j} \), \( 0 \leq i, j \leq m; k = 1, 2, 3, 4 \) be known, so that there are totally \( 4 \times \sum_{i=0}^{m} (2l + 1) = 4(m + 1)^2 \) known parameters. Let the number of unknown coefficients equal to the number of the known parameters so that they can be solved for and have deterministic solution, we get the following relation between the order of the polynomial function and the order of prescribed derivatives:

\[
    n = m + 1
\]

(3.1.5)

For the polynomial function determined as above, all the derivatives prescribed at corner nodes are continuous across the four edges connecting to other panels. This property can be explained as follows: The curve of the derivative \( \frac{\partial^i f(x, y)}{\partial x^i} \), \( 0 \leq i \leq m \)

191

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(or \( \frac{\partial^j f(x,y)}{\partial y^j} \), \( 0 \leq j \leq m \)) along an edge parallel to Y (or X) axis where \( x=constant \) (\( y=constant \)) is a polynomial of \( y \) (or \( x \)) up to \( (2n-1) \)th order power with \( 2n \) unknown coefficients and we have \( 2(m+1) \) prescribed parameters of the form

\[
\frac{\partial^{i+j} f(x,y)}{\partial x^i \partial y^j}, \quad 0 \leq j \leq m \quad (or \quad 0 \leq i \leq m) \]

at the two nodes of that edge, which control the concerned derivative along the edge. Since \( 2n=2(m+1) \), the prescribed parameters determine the curve of the derivative uniquely along the edge. All the derivatives of the form \( \frac{\partial^{i+j} f(x,y)}{\partial x^i \partial y^j}, \quad 0 \leq i \leq m \quad j \geq 0 \quad (or \quad 0 \leq i \quad 0 \leq j \leq m) \) are continuous across edges parallel to Y (or X) axis. Across all the panel boundaries derivatives up to \( m \)th order of \( x \) and \( y \) will be continuous.

For the bi-fifth-order interpolation functions used in our research \( n=3 \) and \( m=2 \), so the derivatives \( \frac{\partial^{i+j} f(x,y)}{\partial x^i \partial y^j}, 0 \leq i \leq 2, j \geq 0 \) are continuous across panel edges parallel to Y axis where \( x=constant \) and the derivatives \( \frac{\partial^{i+j} f(x,y)}{\partial x^i \partial y^j}, i \geq 0, 0 \leq j \leq 2 \), are continuous across panel edges parallel to X axis where \( y=constant \), and across all panel boundaries derivatives up to second order of \( x \) and \( y \) are continuous, as shown in the former section for the bi-fifth-order interpolation functions.

When \( n=2 \) and \( m=1 \), the interpolation functions will be bi-3rd-order

\[
f(x,y) = \sum_{i=0}^{3} \sum_{j=0}^{3} a_{ij} x^i y^j\]

with 16 unknown coefficients and at each corner \( i \) there are 4 parameters prescribed, i.e. \( (f(x_i,y_i), f_x(x_i,y_i), f_y(x_i,y_i), f_{xy}(x_i,y_i)) \). These 16-
parameter bi-third-order interpolation functions have continuous first order derivatives across all panel boundaries.

The simplest case is $n=1$ and $m=0$, when the interpolation functions will be bi-linear $f(x,y) = \sum_{i=0}^{1} \sum_{j=0}^{1} a_{ij} x^i y^j$ with 4 unknown coefficients and at each corner $i$ only the function value will be prescribed. These 4-parameter bi-linear interpolation functions are continuous across all panel boundaries.

In order to guarantee the continuity of derivatives up to fourth-order which will appear in the plate-bending term of the Motion Equation, $m \geq 4$ is required. The simplest case is when $m=4$ and $n=5$ which gives the bi-nineth-order polynomials $f(x,y) = \sum_{i=0}^{9} \sum_{j=0}^{9} a_{ij} x^i y^j$ with 100 unknown coefficients and at each corner $i$ there are 25 parameters prescribed. This kind of interpolation functions is more powerful but also much more complicated.

Other incomplete forms of higher-order interpolation functions are tried, e.g. those with 24 parameters ($f(x_i,y_i)$, $f_x(x_i,y_i)$, $f_y(x_i,y_i)$, $f_{xy}(x_i,y_i)$, $f_{xx}(x_i,y_i)$, $f_{yy}(x_i,y_i)$ prescribed at corner $i$). Only a certain form of higher-order functions can accommodate arbitrary values of the 24 parameters. But their continuity is not satisfactory. Even the first order derivatives ($f_x(x_i,y_i)$ and $f_y(x_i,y_i)$) are not necessarily continuous let alone the second order derivatives. If the interpolation functions are selected wisely the first order continuity can be guaranteed but the second order derivatives are still not continuous even with the significant increase in the complexity.
Appendix II

Bi-Fifth-Order Basis Functions

The basis Functions are written as $b_i(x,y)$, where $i=1,2,3,4$ are the indices of panel corners; $j=1,2,...,9$ are the indices of the nine parameters (value and derivatives of the interpolated function) $(f(x_i,y_j), f_x(x_i,y_j), f_y(x_i,y_j), f_{xy}(x_i,y_j), f_{xx}(x_i,y_j), f_{yy}(x_i,y_j), f_{xy}(x_i,y_j), f_{xxy}(x_i,y_j), f_{xyy}(x_i,y_j))$. The bi-fifth-order basis functions are polynomials of $x$ and $y$, each up to the 5th power. The basis functions are defined by (3.1.3), i.e.

$$f_{ijkl} = \begin{cases} 1, & \text{if } i=k \text{ and } j=l \\ 0, & \text{otherwise} \end{cases} (3.1.3)$$

if we evaluate the parameters $f_{ijkl}, k=1,2,3,4; l=1,2,...9$ of the bi-fifth-order basis function $b_i(x,y), i=1,2,3,4; j=1,2,...9$ at the panel corners.

II.1 The Expressions of the 36 Panel Basis Functions

The panel occupying the area: $\left(-\frac{a}{2} \leq x \leq \frac{a}{2}\right) \times \left(-\frac{b}{2} \leq y \leq \frac{b}{2}\right)$

Location of Corner 1: $\left(-\frac{a}{2}, -\frac{b}{2}\right)$; Location of Corner 2: $\left(\frac{a}{2}, -\frac{b}{2}\right)$

Location of Corner 3: $\left(\frac{a}{2}, \frac{b}{2}\right)$; Location of Corner 4: $\left(-\frac{a}{2}, \frac{b}{2}\right)$
\[ b_{11}(x, y) = \]
\[
\frac{1}{4} \left( \frac{15}{16a} + \frac{5}{2a^3} - \frac{3}{a^5} + \frac{15}{16b} + \frac{225}{64ab} - \frac{75}{8a^3} + \frac{y}{b^3} \right) - \frac{5}{2b^3} - \frac{36x^5 y^5}{a^2 b^5} 
\]

\[ b_{21}(x, y) = \]
\[
\frac{1}{4} \left( \frac{15}{16a} + \frac{5}{2a^3} - \frac{3}{a^5} + \frac{15}{16b} + \frac{225}{64ab} - \frac{75}{8a^3} + \frac{y}{b^3} \right) - \frac{5}{2b^3} + \frac{36x^5 y^5}{a^2 b^5} 
\]

\[ b_{31}(x, y) = \]
\[
\frac{1}{4} \left( \frac{15}{16a} + \frac{5}{2a^3} - \frac{3}{a^5} + \frac{15}{16b} + \frac{225}{64ab} - \frac{75}{8a^3} + \frac{y}{b^3} \right) - \frac{5}{2b^3} + \frac{36x^5 y^5}{a^2 b^5} 
\]

\[ b_{41}(x, y) = \]
\[
\frac{1}{4} \left( \frac{15}{16a} + \frac{5}{2a^3} - \frac{3}{a^5} + \frac{15}{16b} + \frac{225}{64ab} - \frac{75}{8a^3} + \frac{y}{b^3} \right) - \frac{5}{2b^3} + \frac{36x^5 y^5}{a^2 b^5} 
\]

\[ b_{12}(x, y) = \]
\[
\frac{5a}{64} - \frac{2a^3}{32} - \frac{3x^5}{8a} + \frac{5x^3}{4a^2} + \frac{x^4}{4a^3} - \frac{3x^5}{2b^4} - \frac{75a y}{256b} + \frac{105x y}{128b} + \frac{45x^2 y}{32b} 
\]

\[ \frac{1}{16a b} - \frac{25}{16a b} + \frac{25a y^3}{32b^3} - \frac{75x y}{16a b} + \frac{15x y}{16b} + \frac{21x y^3}{16b} - \frac{9x^2 y^3}{16b} + \frac{15x^2 y}{16b} + \frac{18x^5 y^5}{2a^2 b^5} 
\]
\[ b_{22}(x, y) = \]
\[
\frac{5a}{64} - \frac{7x}{32} + \frac{3x^2}{8a} + \frac{5x^3}{4a^2} - \frac{x^4}{4a^3} + \frac{3x^5}{2a^4} + \frac{75ay}{256b} + \frac{105xy}{128b} - \frac{45x^2y}{32ab} - \frac{75x^3y}{16a^2b} + \frac{15x^4y}{16a^3b} + \frac{45x^5y}{8a^4b} + \frac{25ay^3}{32b^2} - \frac{25x^2y^3}{16b^3} + \frac{15x^3y^3}{4ab^3} + \frac{25x^4y^3}{2a^2b^3} - \frac{5x^4y^3}{2a^3b^3} + \frac{15x^5y^3}{16b^5} + \frac{15a^2y^5}{8b^5} + \frac{2a^5b^5}{a^5b^5} + \frac{18x^8y^5}{a^5b^5} + \frac{2a^6b^5}{a^6b^5} + \frac{18x^8y^5}{a^6b^5} + \frac{3x^6y^5}{a^6b^5}
\]

\[ b_{32}(x, y) = \]
\[
\frac{5a}{64} - \frac{7x}{32} + \frac{3x^2}{8a} + \frac{5x^3}{4a^2} - \frac{x^4}{4a^3} + \frac{3x^5}{2a^4} + \frac{75ay}{256b} + \frac{105xy}{128b} + \frac{45x^2y}{32ab} + \frac{75x^3y}{16a^2b} + \frac{15x^4y}{16a^3b} + \frac{45x^5y}{8a^4b} + \frac{25ay^3}{32b^2} + \frac{25x^2y^3}{16b^3} + \frac{15x^3y^3}{4ab^3} + \frac{25x^4y^3}{2a^2b^3} + \frac{5x^4y^3}{2a^3b^3} + \frac{15x^5y^3}{16b^5} + \frac{15a^2y^5}{8b^5} + \frac{2a^5b^5}{a^5b^5} + \frac{18x^8y^5}{a^5b^5} + \frac{3x^6y^5}{a^6b^5}
\]

\[ b_{42}(x, y) = \]
\[
\frac{5a}{64} - \frac{7x}{32} + \frac{3x^2}{8a} + \frac{5x^3}{4a^2} - \frac{x^4}{4a^3} + \frac{3x^5}{2a^4} + \frac{75ay}{256b} + \frac{105xy}{128b} + \frac{45x^2y}{32ab} + \frac{75x^3y}{16a^2b} + \frac{15x^4y}{16a^3b} + \frac{45x^5y}{8a^4b} + \frac{25ay^3}{32b^2} + \frac{25x^2y^3}{16b^3} + \frac{15x^3y^3}{4ab^3} + \frac{25x^4y^3}{2a^2b^3} + \frac{5x^4y^3}{2a^3b^3} + \frac{15x^5y^3}{16b^5} + \frac{15a^2y^5}{8b^5} + \frac{2a^5b^5}{a^5b^5} + \frac{18x^8y^5}{a^5b^5} + \frac{3x^6y^5}{a^6b^5}
\]

\[ b_{13}(x, y) = \]
\[
\frac{5b}{64} - \frac{75bx}{256a} + \frac{25bx^3}{32a^3} - \frac{15bx^5}{16a^5} + \frac{7}{32} + \frac{105xy}{128a} - \frac{35x^2y}{16a^3} + \frac{21x^3y}{8a^5} - \frac{3}{8a} + \frac{45x^2y^2}{32ab} + \frac{15x^3y^2}{4a^2b} + \frac{9x^3y^2}{4a^3b} + \frac{5y^3}{4b^2} + \frac{75x^2y^3}{16ab^2} + \frac{25x^3y^3}{2a^2b^2} + \frac{15x^4y^3}{a^3b^2} + \frac{18x^8y^3}{a^3b^4} + \frac{3x^6y^5}{2a^3b^4} + \frac{18x^8y^3}{a^3b^4} + \frac{3x^6y^5}{a^3b^4}
\]
\[ b_{32}(x, y) = \frac{5b}{64} + \frac{75bx}{256} - \frac{25bx^3}{32a^3} + \frac{15bx^5}{16a^5} - \frac{7y}{32} - \frac{105xy}{128a} + \frac{35x^3y}{16a^3} - \frac{21x^5y}{8a^5} - \frac{3y^2}{8b} - \frac{45xy^2}{32ab} + \frac{15x^3y^2}{4a^3b} + \frac{9x^5y^2}{2a^5b} - \frac{5y^3}{4b^2} + \frac{75x^3y^3}{16ab^2} - \frac{25x^5y^3}{2a^3b^2} + \frac{15x^5y^5}{a^5b^5} + \frac{y^4}{2b^4} - \frac{15xy^4}{16ab^3} - \frac{5x^3y^4}{2a^3b^3} + \frac{3x^5y^4}{a^5b^3} - \frac{3y^6}{2b^6} - \frac{45x^3y^6}{8ab^4} + \frac{15x^5y^6}{a^3b^4} - \frac{18x^5y^8}{a^5b^4} \]

\[ b_{33}(x, y) = \frac{5b}{64} - \frac{75bx}{256a} - \frac{25bx^3}{32a^3} - \frac{15bx^5}{16a^5} - \frac{7y}{32} + \frac{105xy}{128a} - \frac{35x^3y}{16a^3} - \frac{21x^5y}{8a^5} - \frac{3y^2}{8b} + \frac{45xy^2}{32ab} - \frac{15x^3y^2}{4a^3b} - \frac{9x^5y^2}{2a^5b} + \frac{5y^3}{4b^2} - \frac{75x^3y^3}{16ab^2} + \frac{25x^5y^3}{2a^3b^2} - \frac{15x^5y^5}{a^5b^5} - \frac{y^4}{2b^4} + \frac{15xy^4}{16ab^3} + \frac{5x^3y^4}{2a^3b^3} - \frac{3x^5y^4}{a^5b^3} + \frac{3y^6}{2b^6} - \frac{45x^3y^6}{8ab^4} - \frac{15x^5y^6}{a^3b^4} + \frac{18x^5y^8}{a^5b^4} \]

\[ b_{34}(x, y) = \frac{5b}{64} - \frac{75bx}{256a} - \frac{25bx^3}{32a^3} + \frac{15bx^5}{16a^5} - \frac{7y}{32} + \frac{105xy}{128a} - \frac{35x^3y}{16a^3} + \frac{21x^5y}{8a^5} + \frac{3y^2}{8b} - \frac{45xy^2}{32ab} - \frac{15x^3y^2}{4a^3b} - \frac{9x^5y^2}{2a^5b} + \frac{5y^3}{4b^2} - \frac{75x^3y^3}{16ab^2} + \frac{25x^5y^3}{2a^3b^2} - \frac{15x^5y^5}{a^5b^5} + \frac{y^4}{2b^4} + \frac{15xy^4}{16ab^3} + \frac{5x^3y^4}{2a^3b^3} - \frac{3x^5y^4}{a^5b^3} + \frac{3y^6}{2b^6} + \frac{45x^3y^6}{8ab^4} + \frac{15x^5y^6}{a^3b^4} + \frac{18x^5y^8}{a^5b^4} \]

\[ b_{44}(x, y) = \frac{25ab}{1024} - \frac{35bx}{512} - \frac{15bx^3}{128a} - \frac{25bx^5}{64a^3} - \frac{25bx^7}{64a^5} - \frac{35ay}{128a} - \frac{49xy}{256} + \frac{21x^2y}{64a} - \frac{35x^3y}{32a^3} - \frac{21x^5y}{16ab} + \frac{21x^7y}{16a^3b} - \frac{3x^3y^2}{16b^2} - \frac{21x^5y^2}{8a^3b} - \frac{15x^7y^2}{8a^5b} + \frac{3y^3}{16a^3} - \frac{9x^3y^3}{4a^5b} - \frac{25x^5y^3}{8ab^2} - \frac{25x^7y^3}{8a^3b^2} + \frac{15x^9y^3}{2a^5b^2} + \frac{5a^3y^4}{16b^3} - \frac{7xy^4}{8a^3b^3} + \frac{3y^6}{2b^5} - \frac{21xy^6}{16a^3b^3} - \frac{9x^3y^6}{8a^5b^3} + \frac{15x^5y^6}{2a^3b^5} - \frac{3x^7y^6}{2a^5b^5} + \frac{9xy^8}{4a^5b^5} + \frac{5x^3y^8}{4a^7b^5} + \frac{21x^5y^8}{2a^3b^7} + \frac{9x^7y^8}{2a^5b^7} + \frac{3x^9y^8}{a^7b^7} + \frac{xy^{10}}{2a^9b^7} - \frac{3x^3y^{10}}{2a^11b^7} + \frac{15xy^{10}}{2a^9b^9} - \frac{x^{12}}{2a^{11}b^9} + \frac{9x^4y^{12}}{2a^{13}b^9} - \frac{21x^6y^{12}}{2a^9b^{11}} - \frac{9x^8y^{12}}{2a^5b^{11}} + \frac{15x^{10}y^{12}}{2a^3b^{13}} - \frac{3x^{12}y^{12}}{a^5b^{13}} + \frac{9x^{14}y^{12}}{a^7b^{13}} + \frac{x^{16}}{2a^{13}b^{13}} - \frac{9x^{18}y^{12}}{2a^{15}b^{13}} + \frac{3x^{20}y^{12}}{a^{17}b^{13}} + \frac{9x^{22}y^{12}}{a^{19}b^{13}} + \frac{x^{24}}{2a^{17}b^{13}} - \frac{9x^{26}y^{12}}{2a^{19}b^{13}} + \frac{3x^{28}y^{12}}{a^{21}b^{13}} + \frac{9x^{30}y^{12}}{a^{23}b^{13}} + \frac{x^{32}}{2a^{23}b^{13}}
\[ b_{24}(x, y) = \]
\[
\frac{25ab}{1024} - \frac{35bx}{512} + \frac{15bx^2}{128a} - \frac{25bx^3}{64a^2} + \frac{5bx^4}{64a^3} - \frac{15bx^5}{32a^4} - \frac{35ay}{512} + \frac{49xy}{256} - \frac{21x^2y}{64a}
\]

\[ b_{34}(x, y) = \]
\[
\frac{25ab}{1024} + \frac{35bx}{512} + \frac{15bx^2}{128a} + \frac{25bx^3}{64a^2} + \frac{5bx^4}{64a^3} + \frac{15bx^5}{32a^4} + \frac{35ay}{512} + \frac{49xy}{256} + \frac{21x^2y}{64a} + \frac{35x^3y}{32a^2} + \frac{7x^4y}{16a^3} + \frac{21x^5y}{128b} + \frac{21x^6y}{64b} + \frac{9x^7y}{16ab} + \frac{9x^8y}{8a^2b} + \frac{9x^9y}{8ab^2} + \frac{21x^{10}y}{4a^3b} + \frac{9x^{11}y}{2a^4b}
\]

\[ b_{44}(x, y) = \]
\[
\frac{25ab}{1024} + \frac{35bx}{512} + \frac{15bx^2}{128a} - \frac{25bx^3}{64a^2} - \frac{5bx^4}{64a^3} - \frac{15bx^5}{32a^4} - \frac{35ay}{512} + \frac{49xy}{256} + \frac{21x^2y}{64a} + \frac{35x^3y}{32a^2} - \frac{7x^4y}{16a^3} - \frac{21x^5y}{128b} - \frac{21x^6y}{64b} - \frac{9x^7y}{16ab} - \frac{9x^8y}{8a^2b} - \frac{9x^9y}{8ab^2} - \frac{21x^{10}y}{4a^3b} - \frac{9x^{11}y}{2a^4b}
\]

\[ b_{15}(x, y) = \]
\[
\frac{a^2}{128} - \frac{x^2}{64} - \frac{x^3}{16} + \frac{x^4}{8a} - \frac{x^5}{8a^2} - \frac{15a^2y}{4a^3} - \frac{15axy}{512b} + \frac{256b}{256b}
\]

\[ 198 \]
\[ b_{25}(x, y) = \frac{a^2}{128} + \frac{ax}{64} - \frac{x^2}{16} - \frac{x^3}{8a} - \frac{x^4}{8a^2} + \frac{x^5}{4a^3} - \frac{15a^2 y}{512b} - \frac{15ax y}{256b} + \frac{15x^2 y}{64b} + \frac{32ab}{32a^2 b} + \frac{16a^3 b}{64b^3} - \frac{64b^3}{32b^3} - \frac{32b^3}{8b^3} - \frac{4ab^3}{a^3 b^3} + \frac{5x^3 y}{32b^3} - \frac{5x^4 y}{32a^2 b^3} + \frac{3x^5 y}{32a b^3} - \frac{3x^6 y}{16b^3} - \frac{4b^3}{2ab^3} - \frac{2a^2 b^3}{a^3 b^3} \]

\[ b_{35}(x, y) = \frac{a^2}{128} + \frac{ax}{64} - \frac{x^2}{16} - \frac{x^3}{8a} - \frac{x^4}{8a^2} + \frac{x^5}{4a^3} - \frac{15a^2 y}{512b} + \frac{15ax y}{256b} + \frac{15x^2 y}{64b} - \frac{32ab}{32a^2 b} + \frac{16a^3 b}{64b^3} - \frac{64b^3}{32b^3} + \frac{32b^3}{8b^3} + \frac{4ab^3}{a^3 b^3} - \frac{5x^3 y}{32b^3} + \frac{5x^4 y}{32a^2 b^3} - \frac{3x^5 y}{32a b^3} - \frac{3x^6 y}{16b^3} - \frac{4b^3}{2ab^3} + \frac{2a^2 b^3}{a^3 b^3} \]

\[ b_{45}(x, y) = \frac{a^2}{128} + \frac{ax}{64} - \frac{x^2}{16} - \frac{x^3}{8a} - \frac{x^4}{8a^2} - \frac{x^5}{4a^3} + \frac{15a^2 y}{512b} + \frac{15ax y}{256b} + \frac{15x^2 y}{64b} - \frac{32ab}{32a^2 b} + \frac{16a^3 b}{64b^3} + \frac{64b^3}{32b^3} + \frac{32b^3}{8b^3} - \frac{4ab^3}{a^3 b^3} - \frac{5x^3 y}{32b^3} + \frac{5x^4 y}{32a^2 b^3} - \frac{3x^5 y}{32a b^3} - \frac{3x^6 y}{16b^3} + \frac{4b^3}{2ab^3} + \frac{2a^2 b^3}{a^3 b^3} + \frac{3x^6 y}{a^3 b^3} \]

\[ b_{16}(x, y) = \frac{b^2}{128} + \frac{15b^2 x}{512a} + \frac{5b^2 x^3}{64a^3} - \frac{3b^2 x^5}{32a^5} - \frac{b y}{64} + \frac{15bx y}{256a} - \frac{5bx^3 y}{32a^3} + \frac{3bx^5 y}{16a^5} - \frac{y^2}{16} + \frac{15x y^2}{64a} - \frac{5x y^4}{8a^3} + \frac{3x y^6}{4a^5} + \frac{y^8}{8b} - \frac{15x y^8}{32ab} + \frac{5x y^{10}}{4a^3 b} - \frac{3x y^{12}}{2a^2 b^2} - \frac{8b^2}{32a^2 b^2} + \frac{4a^2 b^2}{4a^2 b^2} - \frac{16ab^3}{2a^2 b^3} - \frac{16a^3 b^3}{a^3 b^3} \]
\[ b_{26}(x, y) = \]
\[
\frac{b^2}{128} + \frac{15b^2 x}{512a} + \frac{5b^2 x^3}{64a^3} - \frac{3b^2 x^5}{32a^4} - \frac{b y}{64} - \frac{15b x y}{256a} + \frac{5b x^3 y}{32a^3} - \frac{3b x^5 y}{16a^4} - \\
\frac{y^2}{16} - \frac{15x y^2}{64a} + \frac{5x^3 y^2}{8a^3} - \frac{3x^5 y^2}{4a^4} + \frac{y y^3}{8b} - \frac{15x y^3}{32a b} + \frac{5x^3 y^3}{4a^4 b} - \frac{3x^5 y^3}{2a^5 b} + \\
\frac{y^4}{8b^2} - \frac{15x y^4}{32a b^2} - \frac{5x^3 y^4}{4a^3 b^2} + \frac{3x^5 y^4}{2a^4 b^2} - \frac{y^5}{4b^3} - \frac{15x y^5}{16a b^3} + \frac{5x^3 y^5}{2a^3 b^3} - \frac{3x^5 y^5}{a^4 b^3}
\]

\[ b_{36}(x, y) = \]
\[
\frac{b^2}{128} + \frac{15b^2 x}{512a} + \frac{5b^2 x^3}{64a^3} - \frac{3b^2 x^5}{32a^4} - \frac{b y}{64} + \frac{15b x y}{256a} - \frac{5b x^3 y}{32a^3} + \frac{3b x^5 y}{16a^4} - \\
\frac{y^2}{16} - \frac{15x y^2}{64a} + \frac{5x^3 y^2}{8a^3} - \frac{3x^5 y^2}{4a^4} + \frac{y y^3}{8b} - \frac{15x y^3}{32a b} + \frac{5x^3 y^3}{4a^4 b} - \frac{3x^5 y^3}{2a^5 b} + \\
\frac{y^4}{8b^2} - \frac{15x y^4}{32a b^2} + \frac{5x^3 y^4}{4a^3 b^2} - \frac{3x^5 y^4}{2a^4 b^2} + \frac{y^5}{4b^3} + \frac{15x y^5}{16a b^3} - \frac{5x^3 y^5}{2a^3 b^3} + \frac{3x^5 y^5}{a^4 b^3}
\]

\[ b_{46}(x, y) = \]
\[
\frac{b^2}{128} - \frac{15b^2 x}{512a} + \frac{5b^2 x^3}{64a^3} - \frac{3b^2 x^5}{32a^4} - \frac{b y}{64} - \frac{15b x y}{256a} - \frac{5b x^3 y}{32a^3} + \frac{3b x^5 y}{16a^4} - \\
\frac{y^2}{16} + \frac{15x y^2}{64a} - \frac{5x^3 y^2}{8a^3} + \frac{3x^5 y^2}{4a^4} - \frac{y y^3}{8b} + \frac{15x y^3}{32a b} - \frac{5x^3 y^3}{4a^4 b} + \frac{3x^5 y^3}{2a^5 b} + \\
\frac{y^4}{8b^2} - \frac{15x y^4}{32a b^2} - \frac{5x^3 y^4}{4a^3 b^2} + \frac{3x^5 y^4}{2a^4 b^2} - \frac{y^5}{4b^3} + \frac{15x y^5}{16a b^3} - \frac{5x^3 y^5}{2a^3 b^3} + \frac{3x^5 y^5}{a^4 b^3}
\]

\[ b_{17}(x, y) = \]
\[
\frac{5a^2 b}{2048} - \frac{5ab x}{1024} + \frac{5b x^2}{256} - \frac{5b x^3}{128a} + \frac{5b x^4}{128a^2} - \frac{5b x^5}{64a^3} + \frac{7a^2 y}{1024} - \frac{7ax y}{512} + \frac{7x^2 y}{128} - \\
\frac{7x^3 y}{64a} - \frac{7x^4 y}{64a^2} + \frac{3a^2 y^2}{32a^3} - \frac{3a x y^2}{256b} + \frac{3x^2 y^2}{32b} + \frac{3a^2 y^2}{128b} - \frac{3x^2 y^2}{16a b} - \frac{3x^3 y^2}{16a^2 b} + \frac{3x^4 y}{8a^3 b} + \\
\frac{5a^2 y}{128b^2} - \frac{5ax y}{64b^2} + \frac{5x y^3}{16b^2} + \frac{5x^3 y^3}{8a^2 b^2} - \frac{5x^5 y^3}{4a^3 b^2} + \frac{a^2 y^4}{128b^3} - \frac{ax y^4}{64b^3} + \frac{x^2 y^4}{8a^2 b^3} - \\
\frac{x^3 y^4}{8a b^3} - \frac{x^4 y^4}{8a^2 b^3} - \frac{3a^2 y^3}{32a b^3} + \frac{3a x y^3}{64b^3} + \frac{3x^2 y^3}{32b^3} + \frac{3a^2 y^3}{8b^4} - \frac{3x^2 y^3}{4a b^4} - \frac{3a^2 y^3}{4a^2 b^4} + \frac{2a b^4}{2a^3 b^4}
\]
\[ \begin{align*}
\mathbf{b}_{18}(x, y) &= \frac{5ab^2}{2048} - \frac{7b^2x}{1024} + \frac{3b^2x^2}{256a} + \frac{5b^2x^3}{128a^2} + \frac{b^2x^4}{128a^3} - \frac{3b^2x^5}{64a^4} - \frac{5ab}{1024} + \frac{7bx}{512} - \frac{3bx^2}{128a} - \\
&\quad - \frac{5bx^3}{64a^2} + \frac{bx^4}{64a^3} - \frac{3bx^5}{32a^4} + \frac{5ax^2}{256} + \frac{7x}{128} - \frac{3x^2y^2}{16a} - \frac{5x^3y^2}{16a^2} + \frac{x^4}{16a^3} + \frac{3x^5y^2}{8a^4} + \\
&\quad \frac{5ax^3}{128b} - \frac{x^4}{64b} + \frac{3x^5}{16ab} + \frac{5ay^2}{8a^2b} + \frac{7xy^2}{64b} + \frac{3x^2y^2}{32b} + \frac{8ab}{8a^3b} - \frac{4a^2b}{4a^3b} + \frac{2a^3b}{2a^3b}.
\end{align*} \]

\[ \begin{align*}
\mathbf{b}_{28}(x, y) &= \frac{5ab^2}{2048} - \frac{7b^2x}{1024} + \frac{3b^2x^2}{256a} + \frac{5b^2x^3}{128a^2} + \frac{b^2x^4}{128a^3} - \frac{3b^2x^5}{64a^4} - \frac{5ab}{1024} + \frac{7bx}{512} - \frac{3bx^2}{128a} - \\
&\quad - \frac{5bx^3}{64a^2} + \frac{bx^4}{64a^3} - \frac{3bx^5}{32a^4} + \frac{5ax^2}{256} + \frac{7x}{128} - \frac{3x^2y^2}{16a} - \frac{5x^3y^2}{16a^2} + \frac{x^4}{16a^3} + \frac{3x^5y^2}{8a^4} + \\
&\quad \frac{5ax^3}{128b} - \frac{x^4}{64b} + \frac{3x^5}{16ab} + \frac{5ay^2}{8a^2b} + \frac{7xy^2}{64b} + \frac{3x^2y^2}{32b} + \frac{8ab}{8a^3b} - \frac{4a^2b}{4a^3b} + \frac{2a^3b}{2a^3b} + \\
&\quad - \frac{8a^2b}{8a^3b} - \frac{4a^3b}{4a^3b} + \frac{2a^3b}{2a^3b}.
\end{align*} \]

\[ \begin{align*}
\mathbf{b}_{38}(x, y) &= \frac{5ab^2}{2048} - \frac{7b^2x}{1024} + \frac{3b^2x^2}{256a} + \frac{5b^2x^3}{128a^2} + \frac{b^2x^4}{128a^3} - \frac{3b^2x^5}{64a^4} - \frac{5ab}{1024} + \frac{7bx}{512} - \frac{3bx^2}{128a} - \\
&\quad - \frac{5bx^3}{64a^2} + \frac{bx^4}{64a^3} - \frac{3bx^5}{32a^4} + \frac{5ax^2}{256} + \frac{7x}{128} - \frac{3x^2y^2}{16a} - \frac{5x^3y^2}{16a^2} + \frac{x^4}{16a^3} + \frac{3x^5y^2}{8a^4} + \\
&\quad \frac{5ax^3}{128b} - \frac{x^4}{64b} + \frac{3x^5}{16ab} + \frac{5ay^2}{8a^2b} + \frac{7xy^2}{64b} + \frac{3x^2y^2}{32b} + \frac{8ab}{8a^3b} - \frac{4a^2b}{4a^3b} + \frac{2a^3b}{2a^3b} + \\
&\quad - \frac{8a^2b}{8a^3b} - \frac{4a^3b}{4a^3b} + \frac{2a^3b}{2a^3b}.
\end{align*} \]
\[ b_{48}(x, y) = \]
\[
\begin{array}{c}
\frac{5a b^2}{2048} - \frac{7b^2 x}{1024} + \frac{3b^2 x^2}{256a} + \frac{5b^2 x^3}{128a^2} + \frac{b^2 x^4}{128a^3} - \frac{3b^2 x^5}{64a^4} + \frac{5ab y}{1024} - \frac{7b x y}{512} + \frac{3bx^2 y}{128a} + \\
\frac{5b x^3 y}{64a^2} + \frac{bx^4 y}{64a^3} - \frac{3bx^5 y}{32a^4} + \frac{5a y^2}{256} + \frac{7xy^2}{128} + \frac{3x^2 y^2}{32a} - \frac{5x^3 y^2}{16a^2} - \frac{x^4 y^2}{16a^3} + \frac{3x^5 y^2}{8a^4} + \\
\frac{5a y^3}{128b} + \frac{7xy^3}{64b} + \frac{3x^2 y^3}{16ab} - \frac{5x^3 y^3}{8a^2 b} + \frac{7x y^5}{8a^3 b} - \frac{3x^2 y^5}{4a^4 b} + \frac{5a y^5}{128b^2} + \frac{7x y^5}{64b^2} - \frac{3x^2 y^5}{16a b^2} + \\
\frac{5x^3 y^5}{8a^2 b^2} - \frac{x^4 y^5}{8a^3 b^2} + \frac{3x^5 y^5}{4a^4 b^2} + \frac{3x^6 y^5}{2a^5 b^3} \\
\end{array}
\]

\[ b_{49}(x, y) = \]
\[
\begin{array}{c}
\frac{a^2 b^2}{4096} - \frac{ab^2 x}{2048} + \frac{-b^2 x^2}{512} - \frac{-b^2 x^3}{256a} - \frac{-b^2 x^4}{256a^2} + \frac{-b^2 x^5}{128a^3} - \frac{ab y}{2048} + \frac{-ab x y}{1024} + \frac{1}{256} b x^2 y - \\
\frac{b x^3 y}{128a} - \frac{-b x^4 y}{128a^2} + \frac{-b x^5 y}{64a^3} + \frac{a^2 y^2}{512} - \frac{-1}{256} ax y^2 + \frac{x^2 y^2}{64} - \frac{-x^3 y^2}{32a} + \frac{-x^4 y^2}{32a^2} - \frac{-x^5 y^2}{16a^3} + \\
\frac{a^2 y^3}{256b} - \frac{ax y^3}{128b^2} + \frac{-x^2 y^3}{32b} + \frac{-x^3 y^3}{16ab} - \frac{-x^4 y^3}{16a^2 b} + \frac{-x^5 y^3}{8a^3 b} - \frac{x^6 y^3}{256b^2} + \frac{-128b^2}{-128b^2} - \frac{-32b^2}{32b^2} + \\
\frac{x^3 y^4}{16ab^2} + \frac{x^4 y^4}{16a^2 b^2} + \frac{-x^5 y^4}{8a^3 b^2} + \frac{-128b^3}{128b^3} - \frac{-64b^3}{64b^3} + \frac{-16b^3}{16b^3} - \frac{-8a b^3}{8a b^3} - \frac{-8a^2 b^3}{8a^2 b^3} - \frac{-4a^3 b^3}{4a^3 b^3} + \\
\frac{4a^4 b^3}{4a^4 b^3} + \frac{4a^3 b^3}{4a^3 b^3} + \frac{4a^4 b^3}{4a^4 b^3} + \frac{4a^5 b^3}{4a^5 b^3} + \frac{4a^6 b^3}{4a^6 b^3} \\
\end{array}
\]

\[ b_{29}(x, y) = \]
\[
\begin{array}{c}
\frac{a^2 b^2}{4096} + \frac{ab^2 x}{2048} - \frac{b^2 x^2}{512} - \frac{-b^2 x^3}{256a} + \frac{b^2 x^4}{256a^2} + \frac{-b^2 x^5}{128a^3} - \frac{ab y}{2048} + \frac{-ab x y}{1024} + \frac{1}{256} b x^2 y + \\
\frac{b x^3 y}{128a} - \frac{bx^4 y}{128a^2} + \frac{-bx^5 y}{64a^3} - \frac{a^2 y^2}{512} + \frac{1}{256} ax y^2 + \frac{x^2 y^2}{64} + \frac{x^3 y^2}{32a} - \frac{x^4 y^2}{32a^2} + \frac{x^5 y^2}{16a^3} + \\
\frac{a^2 y^3}{256b} + \frac{-ax y^3}{128b^2} + \frac{x^2 y^3}{32b} + \frac{x^3 y^3}{16ab} + \frac{x^4 y^3}{16a^2 b} + \frac{x^5 y^3}{8a^3 b} + \frac{x^6 y^3}{256b^2} + \frac{128b^2}{-128b^2} + \frac{32b^2}{32b^2} - \\
\frac{x^3 y^4}{16ab^2} - \frac{x^4 y^4}{16a^2 b^2} + \frac{-x^5 y^4}{8a^3 b^2} + \frac{128b^3}{128b^3} + \frac{64b^3}{64b^3} + \frac{16b^3}{16b^3} + \frac{8a b^3}{8a b^3} + \frac{8a^2 b^3}{8a^2 b^3} - \frac{4a^3 b^3}{4a^3 b^3} + \\
\frac{4a^4 b^3}{4a^4 b^3} + \frac{4a^3 b^3}{4a^3 b^3} + \frac{4a^4 b^3}{4a^4 b^3} + \frac{4a^5 b^3}{4a^5 b^3} + \frac{4a^6 b^3}{4a^6 b^3} \\
\end{array}
\]
\[
\begin{align*}
&b_{35}(x, y) = \\
&\quad \frac{a^2 b^2}{4096} + \frac{a b^2 x}{2048} + \frac{b^2 x^2}{512} - \frac{b^2 x^3}{256 a} + \frac{b^3 x^4}{256 a^2} + \frac{b^2 x^5}{128 a^3} + \frac{a^2 b y}{2048} + \frac{a b x y}{1024} - \frac{1}{256 b x^2 y} - \\
&\quad \frac{b x^3 y}{128 a} + \frac{b x^4 y}{128 a^2} + \frac{b x^5 y}{64 a^3} - \frac{a^2 y^2}{512} + \frac{1}{256} a x y^2 + \frac{x^2 y^2}{64} + \frac{x^3 y^2}{32 a} - \frac{x^4 y^2}{32 a^2} - \frac{x^5 y^2}{16 a^3} - \\
&\quad \frac{a^2 y^3}{256 b} + \frac{a x y^3}{128 b^2} + \frac{x^2 y^3}{32 b} + \frac{x^3 y^3}{16 a b} - \frac{x^4 y^3}{16 a^2 b} + \frac{a^2 y^4}{8 a^3 b} + \frac{a x y^4}{256 b^2} + \frac{x^2 y^4}{128 b^2} + \frac{x^3 y^4}{32 b^2} - \\
&\quad \frac{x^4 y^4}{16 a b^2} + \frac{x^5 y^4}{16 a^2 b^2} + \frac{x^6 y^4}{8 a^3 b^2} + \frac{x^7 y^4}{128 b^3} + \frac{x^8 y^4}{64 b^3} - \frac{x^9 y^4}{16 b^3} - \frac{x^{10} y^4}{8 a b^3} + \frac{x^{11} y^4}{8 a^2 b^3} + \frac{x^{12} y^4}{4 a^3 b^3}
\end{align*}
\]

\[
\begin{align*}
&b_{49}(x, y) = \\
&\quad \frac{a^2 b^2}{4096} - \frac{a b^2 x}{2048} - \frac{b^2 x^2}{512} - \frac{b^3 x^3}{256 a} + \frac{b^2 x^4}{256 a^2} - \frac{b^2 x^5}{128 a^3} + \frac{a^2 b y}{2048} - \frac{a b x y}{1024} - \frac{1}{256 b x^2 y} + \\
&\quad \frac{b x^3 y}{128 a} + \frac{b x^4 y}{128 a^2} - \frac{b x^5 y}{64 a^3} + \frac{a^2 y^2}{512} + \frac{1}{256} a x y^2 + \frac{x^2 y^2}{64} - \frac{32 a}{32 a^2} + \frac{x^4 y^2}{16 a^3} - \\
&\quad \frac{a^2 y^3}{256 b} + \frac{a x y^3}{128 b^2} - \frac{x^2 y^3}{32 b} - \frac{x^3 y^3}{16 a b} + \frac{x^4 y^3}{16 a^2 b} + \frac{a^2 y^4}{8 a^3 b} + \frac{a x y^4}{256 b^2} - \frac{x^2 y^4}{128 b^2} - \frac{x^3 y^4}{32 b^2} - \\
&\quad \frac{x^4 y^4}{16 a b^2} - \frac{x^5 y^4}{16 a^2 b^2} - \frac{x^6 y^4}{8 a^3 b^2} - \frac{x^7 y^4}{128 b^3} - \frac{x^8 y^4}{64 b^3} + \frac{x^9 y^4}{16 b^3} + \frac{x^{10} y^4}{8 a b^3} - \frac{x^{11} y^4}{8 a^2 b^3} - \frac{x^{12} y^4}{4 a^3 b^3}
\end{align*}
\]
II.2 The Shape of the 36 Panel Basis Functions

The panel occupying the area: \((-\frac{1}{2} \leq x \leq \frac{1}{2}) \times (-\frac{1}{2} \leq y \leq \frac{1}{2})\)

Location of Corner 1: \((-\frac{1}{2}, -\frac{1}{2})\);

Location of Corner 2: \((\frac{1}{2}, -\frac{1}{2})\);

Location of Corner 3: \((\frac{1}{2}, \frac{1}{2})\);

Location of Corner 4: \((-\frac{1}{2}, \frac{1}{2})\)

Fig. II.2.1 Basis Function \(b_{11}(x, y)\)

Fig. II.2.2 Basis Function \(b_{21}(x, y)\)

Fig. II.2.3 Basis Function \(b_{31}(x, y)\)

Fig. II.2.4 Basis Function \(b_{41}(x, y)\)
Fig. II.2.5  Basis Function $b_{12}(x, y)$

Fig. II.2.6  Basis Function $b_{22}(x, y)$

Fig. II.2.7  Basis Function $b_{32}(x, y)$

Fig. II.2.8  Basis Function $b_{42}(x, y)$

Fig. II.2.9  Basis Function $b_{13}(x, y)$

Fig. II.2.10  Basis Function $b_{23}(x, y)$
Fig. II.2.17  Basis Function $b_{15}(x,y)$

Fig. II.2.18  Basis Function $b_{25}(x,y)$

Fig. II.2.19  Basis Function $b_{35}(x,y)$

Fig. II.2.20  Basis Function $b_{45}(x,y)$

Fig. II.2.21  Basis Function $b_{16}(x,y)$

Fig. II.2.22  Basis Function $b_{26}(x,y)$
Fig. II.2.35  Basis Function $b_{39}(x,y)$  

Fig. II.2.36  Basis Function $b_{49}(x,y)$
II.3 The Shape of Some of the Nodal Basis Functions

II.3.1 Nodal Basis Functions at the Center Area of the Mesh

Node \( i \) is located at \((0,0)\). The nodal basis function is distributed on four panels connected at node \( i \) \((0,0)\): \((0 \leq x \leq 1) \times (0 \leq y \leq 1)\), \((0 \leq x \leq 1) \times (-1 \leq y \leq 0)\), \((-1 \leq x \leq 0) \times (0 \leq y \leq 1)\), \((-1 \leq x \leq 0) \times (-1 \leq y \leq 0)\). The whole area of distribution is \((-1 \leq x \leq 1) \times (-1 \leq y \leq 1)\) and includes nine nodes, i.e. \((-1, -1)\), \((-1, 0)\), \((-1, 1)\), \((0, -1)\), \((0, 0)\), \((0, 1)\), \((1, -1)\), \((1, 0)\), \((1, 1)\).

1. \( ncenb_{ij}(x,y) \): All the parameters at all the nine nodes are zero except that \( f(x,y) = 1 \) at node \( i \) \((0,0)\)
2. \( ncenb_{12}(x, y) \): All the parameters at all the nine nodes are zero except that \( \frac{\partial f(x, y)}{\partial x} = 1 \) at node \( i (0, 0) \)

3. \( ncenb_{13}(x, y) \): All the parameters at all the nine nodes are zero except that \( \frac{\partial f(x, y)}{\partial y} = 1 \) at node \( i (0, 0) \)
4. \( ncenb_{14}(x, y) \): All the parameters at all the nine nodes are zero except that

\[
\frac{\partial^2 f(x, y)}{\partial x \partial y} = 1 \text{ at node } i (0, 0)
\]

5. \( ncenb_{15}(x, y) \): All the parameters at all the nine nodes are zero except that

\[
\frac{\partial^2 f(x, y)}{\partial x^2} = 1 \text{ at node } i (0, 0)
\]
6. $ncenb_{i0}(x, y)$: All the parameters at all the nine nodes are zero except that

$$\frac{\partial^2 f(x, y)}{\partial y^2} = 1 \text{ at node } i (0, 0)$$

7. $ncenb_{i2}(x, y)$: All the parameters at all the nine nodes are zero except that

$$\frac{\partial^3 f(x, y)}{\partial x^2 \partial y} = 1 \text{ at node } i (0, 0)$$
8. \( ncenb_{i9}(x, y) \): All the parameters at all the nine nodes are zero except that

\[
\frac{\partial^3 f(x, y)}{\partial x \partial y^2} = 1 \text{ at node } i (0, 0)
\]

9. \( ncenb_{j9}(x, y) \): All the parameters at all the nine nodes are zero except that

\[
\frac{\partial^4 f(x, y)}{\partial x^2 \partial y^2} = 1 \text{ at node } i (0, 0)
\]
II.3.2 Nodal Basis Functions at Nodes Located on the Edges of the mesh

Node \( i \) is located at \((0,0)\) and on the edge of the mesh parallel to X-axis. The nodal basis function is distributed on two panels along the edge which are connected at node \( i \) \((0, 0)\): \((0 \leq x \leq 1) \times (0 \leq y \leq 1)\), \((-1 \leq x \leq 0) \times (0 \leq y \leq 1)\). The whole area of distribution is \((-1 \leq x \leq 1) \times (0 \leq y \leq 1)\) and includes six nodes, i.e. \((-1, 0)\), \((-1, 1)\), \((0, 0)\), \((0, 1)\), \((1, 0)\), \((1, 1)\).

1. \( nedgb_n (x, y) \): All the parameters at all the six nodes are zero except that \( f(x, y) = 1 \) at node \( i \) \((0, 0)\)
2. $nedgb_2(x, y)$: All the parameters at all the nodes are zero except that $\frac{\partial f(x, y)}{\partial x} = 1$ at node $i(0, 0)$.

3. $nedgb_3(x, y)$: All the parameters at all the nodes are zero except that $\frac{\partial f(x, y)}{\partial y} = 1$ at node $i(0, 0)$. 
4. $\text{nedgb}_4(x, y)$: All the parameters at all the nodes are zero except that $\frac{\partial^2 f(x, y)}{\partial x \partial y} = 1$ at node $i (0, 0)$

5. $\text{nedgb}_5(x, y)$: All the parameters at all the nodes are zero except that $\frac{\partial^2 f(x, y)}{\partial x^2} = 1$ at node $i (0, 0)$
6. \textit{nedgb}_{06}(x, y): All the parameters at all the nodes are zero except that \( \frac{\partial^2 f(x, y)}{\partial y^2} = 1 \) at node \( i (0, 0) \)

7. \textit{nedgb}_{17}(x, y): All the parameters at all the nodes are zero except that \( \frac{\partial^3 f(x, y)}{\partial x^2 \partial y} = 1 \) at node \( i (0, 0) \)
8. \( \text{nedgb}_{18}(x, y) \): All the parameters at all the nodes are zero except that \( \frac{\partial^3 f(x, y)}{\partial x \partial y^2} = 1 \) at node \( i(0, 0) \)

![Diagram](image1)

9. \( \text{nedgb}_{9}(x, y) \): All the parameters at all the nodes are zero except that \( \frac{\partial^4 f(x, y)}{\partial x^2 \partial y^2} = 1 \) at node \( i(0, 0) \)

![Diagram](image2)
Appendix III

Analytical Formulae for the Integrations of the Potential and its Gradient of Monomial Source Distribution on a Rectangular Panel

The analytical formulae for the integrations of the potential and its Gradient of monomial source distribution on a rectangular panel, i.e. $I'_{ij,n}$ and $\{K'_{xij,n}, K'_{yij,n}, K'_{zij,n}\}$ as defined in (3.3.9) below, are listed here. The $z$ in (3.3.9) corresponds to $z_s - z_f$ in (3.3.9a), $-z_s - z_f$ in (3.3.9b) and to $y_s - y_f$ in (3.3.9c), which is the height of the source panel relative to the field point in the $z$ direction.

\[
I'_{ij,n} = \int_{d}^{c} \int_{b}^{a} \frac{x^i y^j}{\sqrt{x^2 + y^2 + z^2}} \, dx \, dy
\]

\[
K'_{xij,n} = \int_{d}^{c} \int_{b}^{a} \frac{x^i y^j}{\sqrt{x^2 + y^2 + z^2}} \, dx \, dy
\]

\[
K'_{yij,n} = \int_{d}^{c} \int_{b}^{a} \frac{y^i x^j}{\sqrt{x^2 + y^2 + z^2}} \, dx \, dy
\]

\[
K'_{zij,n} = \int_{d}^{c} \int_{b}^{a} \frac{z^i y^j}{\sqrt{x^2 + y^2 + z^2}} \, dx \, dy
\]

(3.3.9)

The source panel is located in the area $[b, a] \times [d, c]$, where $[b, a]$ is the interval of integration in the $x$ direction and $[d, c]$ is the interval of integration in the $y$ direction. Here $x$ and $y$ are coordinate directions parallel to the sides of the rectangular source panel and originated at the field point. Normally $x$ direction is chosen to be longitudinal or at least horizontal and $y$ chosen to be transverse or vertical but this is not required.

By determining the input data $a$, $b$, $c$, $d$ and $z$ correspondingly, these analytical formulae are used to compute the integrations in (3.3.9) for source panels on the bottom.
plate, the side plate and bow plate of the floating runway and also for their image source panels about the free surface. But if the field point is located on the four lines defining the four sides of the source panel, when \( z \) and at least one of \( a, b, c, d \) are equal to zero, some of the analytical formulae may be not defined.

For each one of \( I_{ij,n}^{'} \) and \( \{K_{xij,n}^{'}, K_{yij,n}^{'}, K_{zij,n}^{'} \} \) there are 36 integrals corresponding to \( i,j=0,1,\ldots,5 \).

### III.1 Analytical Formulae for \( I_{ij,n}^{'} \), \( i,j=0,1,\ldots,5 \)

\[
I_{00,n}^{'} = \\
\frac{1}{2} \left( -2 z \left( \arctan \left( c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, -az \right) \right. \right. \\
\left. \left. + \arctan \left( c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, bz \right) \right) \right. \\
\left. \left. + \arctan \left( d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, az \right) \right) \right. \\
\left. \left. + \arctan \left( d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, -bz \right) \right) \right. \\
\left. + 2 c \log \left( a + \sqrt{a^2 + c^2 + z^2} \right) + 2 a \log \left( c + \sqrt{a^2 + c^2 + z^2} \right) \right) \\
\left. - 2 c \log \left( b + \sqrt{b^2 + c^2 + z^2} \right) - 2 b \log \left( c + \sqrt{b^2 + c^2 + z^2} \right) \right) \\
\left. - 2 d \log \left( a + \sqrt{a^2 + d^2 + z^2} \right) - 2 a \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) \right) \\
\left. + 2 d \log \left( b + \sqrt{b^2 + d^2 + z^2} \right) + 2 b \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) \right)
\]

\[
I_{10,n}^{'} = \\
\frac{1}{2} c \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) + \frac{1}{2} d \left( -\sqrt{a^2 + d^2 + z^2} + \sqrt{b^2 + d^2 + z^2} \right) + \\
\frac{1}{2} a^2 \log \left( \frac{c + \sqrt{a^2 + c^2 + z^2}}{d + \sqrt{a^2 + d^2 + z^2}} \right) + \frac{1}{2} b^2 \log \left( \frac{d + \sqrt{b^2 + d^2 + z^2}}{c + \sqrt{b^2 + c^2 + z^2}} \right) + \\
\frac{1}{2} z^2 \log \left( \frac{c + \sqrt{a^2 + c^2 + z^2}}{c + \sqrt{b^2 + c^2 + z^2}} \right) \left( d + \sqrt{b^2 + d^2 + z^2} \right)
\]
\[ I'_{20, a} = \frac{1}{3} \left( \arctan \left[ a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, cz \right] + \arctan \left[ b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, -cz \right] + \arctan \left[ a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2}, -dz \right] + \arctan \left[ b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, dz \right] \right) + \]
\[ \frac{1}{6} \left( ac \sqrt{a^2 + c^2 + z^2} - bc \sqrt{b^2 + c^2 + z^2} \right) - ad \sqrt{a^2 + d^2 + z^2} + bd \sqrt{b^2 + d^2 + z^2} -
\]
\[ c^3 \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] - 3cz^2 \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] +
\]
\[ 2a^3 \log \left[ c + \sqrt{a^2 + c^2 + z^2} \right] + c^3 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] +
\]
\[ 3cz^2 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] - 2b^3 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] +
\]
\[ d^3 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 3dz^2 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] -
\]
\[ 2a^3 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] - d^3 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] -
\]
\[ 3dz^2 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] + 2b^3 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] \]

\[ I'_{30, a} = \frac{1}{12} \left( a^2 c \sqrt{a^2 + c^2 + z^2} - 2c^3 \sqrt{a^2 + c^2 + z^2} - 5cz^2 \sqrt{a^2 + c^2 + z^2} -
\right.
\]
\[ b^2 c \sqrt{b^2 + c^2 + z^2} + 2c^3 \sqrt{b^2 + c^2 + z^2} + 5cz^2 \sqrt{b^2 + c^2 + z^2} -
\]
\[ a^2 d \sqrt{a^2 + d^2 + z^2} + 2d^3 \sqrt{a^2 + d^2 + z^2} + 5dz^2 \sqrt{a^2 + d^2 + z^2} +
\]
\[ b^2 d \sqrt{b^2 + d^2 + z^2} - 2d^3 \sqrt{b^2 + d^2 + z^2} - 5dz^2 \sqrt{b^2 + d^2 + z^2} +
\]
\[ 3(a^4 - z^4) \log \left[ c + \sqrt{a^2 + c^2 + z^2} \right] - 3(b^4 - z^4) \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] -
\]
\[ 3a^4 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] + 3z^4 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] +
\]
\[ 3b^4 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] - 3z^4 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] \]
\[ I'_{40,n} = \]
\[- \frac{1}{5} z^5 \left( \text{ArcTan} \left[ a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, -cz \right] + \right. \]
\[
\text{ArcTan} \left[ b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, cz \right] + \text{ArcTan} \left[ a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2}, dz \right] + \\
\left. \text{ArcTan} \left[ b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, -dz \right] \right) + \\
\frac{1}{40} \left( 2a^3 c \sqrt{a^2 + c^2 + z^2} - 3ac^3 \sqrt{a^2 + c^2 + z^2} - 7ac^2 \sqrt{a^2 + c^2 + z^2} - \\
2b^3 c \sqrt{b^2 + c^2 + z^2} + 3bc^3 \sqrt{b^2 + c^2 + z^2} + 7bcz^2 \sqrt{b^2 + c^2 + z^2} - \\
2a^3 d \sqrt{a^2 + d^2 + z^2} + 3ad^3 \sqrt{a^2 + d^2 + z^2} + 7adz^2 \sqrt{a^2 + d^2 + z^2} + \\
2b^3 d \sqrt{b^2 + d^2 + z^2} - 3bd^3 \sqrt{b^2 + d^2 + z^2} - 7bdz^2 \sqrt{b^2 + d^2 + z^2} + \\
(3c^5 + 10c^3 z^2 + 15cz^4) \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] + 8a^5 \log \left[ c + \sqrt{a^2 + c^2 + z^2} \right] - \\
3c^5 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] - 10c^3 z^2 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] - \\
15cz^4 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] - 8b^5 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] - \\
3d^5 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - 10d^3 z^2 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - \\
15dz^4 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - 8a^5 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] + \\
3d^5 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] + 10d^3 z^2 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] + \\
15dz^4 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] + 8b^5 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] \right] \]
$I_{01, a} = \frac{1}{90} \left( \frac{3a^4 c \sqrt{a^2 + c^2 + z^2} - 4a^2 c^3 \sqrt{a^2 + c^2 + z^2} + 8c^5 \sqrt{a^2 + c^2 + z^2} - 
abla a^2 c^2 \sqrt{a^2 + c^2 + z^2} + 26c^3 \sqrt{a^2 + c^2 + z^2} + 33cz^4 \sqrt{a^2 + c^2 + z^2} - 
abla b^4 c \sqrt{b^2 + c^2 + z^2} + 4b^2 c^3 \sqrt{b^2 + c^2 + z^2} - 8c^5 \sqrt{b^2 + c^2 + z^2} + 
abla b^2 c \sqrt{b^2 + c^2 + z^2} - 26c^3 \sqrt{b^2 + c^2 + z^2} - 33cz^4 \sqrt{b^2 + c^2 + z^2} - 
abla a^4 d \sqrt{a^2 + d^2 + z^2} + 4a^2 d^3 \sqrt{a^2 + d^2 + z^2} - 8d^5 \sqrt{a^2 + d^2 + z^2} + 
abla a^2 d \sqrt{a^2 + d^2 + z^2} - 26d^3 \sqrt{a^2 + d^2 + z^2} - 33dz^4 \sqrt{a^2 + d^2 + z^2} + 
abla b^4 d \sqrt{b^2 + d^2 + z^2} - 4b^2 d^3 \sqrt{b^2 + d^2 + z^2} + 8d^5 \sqrt{b^2 + d^2 + z^2} - 
abla b^2 d \sqrt{b^2 + d^2 + z^2} + 26d^3 \sqrt{b^2 + d^2 + z^2} + 33dz^4 \sqrt{b^2 + d^2 + z^2} + 15(a^2 + z^2) \log[c + \sqrt{a^2 + c^2 + z^2} - 15(b^2 + z^2) \log[c + \sqrt{b^2 + c^2 + z^2} - 15a^2 \log[d + \sqrt{a^2 + d^2 + z^2} + 15b^2 \log[d + \sqrt{b^2 + d^2 + z^2}] ]} ]$

$\frac{1}{2} a \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) + \frac{1}{2} b \left( -\sqrt{b^2 + c^2 + z^2} + \sqrt{b^2 + d^2 + z^2} \right) + \frac{1}{2} c^2 \log \left( \frac{a + \sqrt{a^2 + c^2 + z^2}}{b + \sqrt{b^2 + c^2 + z^2}} \right) + \frac{1}{2} d^2 \log \left( \frac{b + \sqrt{b^2 + d^2 + z^2}}{a + \sqrt{a^2 + d^2 + z^2}} \right) + \frac{1}{2} z^2 \log \left( \frac{a + \sqrt{a^2 + c^2 + z^2}}{b + \sqrt{b^2 + c^2 + z^2}} \right) \left( \frac{b + \sqrt{b^2 + d^2 + z^2}}{a + \sqrt{a^2 + d^2 + z^2}} \right) }$

$I_{11, a} = \frac{1}{3} c^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) + \frac{1}{3} a^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) + \frac{1}{3} b^2 \left( -\sqrt{b^2 + c^2 + z^2} + \sqrt{b^2 + d^2 + z^2} \right) + \frac{1}{3} d^2 \left( -\sqrt{a^2 + d^2 + z^2} + \sqrt{b^2 + d^2 + z^2} \right) + \frac{1}{3} z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} + \sqrt{b^2 + d^2 + z^2} \right) $
\[ I_{21,n} = \frac{1}{8} \left( 2a^3 \sqrt{a^2 + c^2 + z^2} + ac^2 \sqrt{a^2 + c^2 + z^2} + az^2 \sqrt{a^2 + c^2 + z^2} - \right. \\
2b^3 \sqrt{b^2 + c^2 + z^2} - bc^2 \sqrt{b^2 + c^2 + z^2} - bz^2 \sqrt{b^2 + c^2 + z^2} - \\
2a^3 \sqrt{a^2 + d^2 + z^2} - ad^2 \sqrt{a^2 + d^2 + z^2} - az^2 \sqrt{a^2 + d^2 + z^2} + \\
2b^3 \sqrt{b^2 + d^2 + z^2} + bd^2 \sqrt{b^2 + d^2 + z^2} + bz^2 \sqrt{b^2 + d^2 + z^2} - \\
(c^2 + z^2)^2 \log[a + \sqrt{a^2 + c^2 + z^2}] + (c^2 + z^2)^2 \log[b + \sqrt{b^2 + c^2 + z^2}] + \\
d^4 \log[a + \sqrt{a^2 + d^2 + z^2}] + 2d^2 z^2 \log[a + \sqrt{a^2 + d^2 + z^2}] + \\
z^4 \log[a + \sqrt{a^2 + d^2 + z^2}] - d^4 \log[b + \sqrt{b^2 + d^2 + z^2}] - \\
2d^2 z^2 \log[b + \sqrt{b^2 + d^2 + z^2}] - z^4 \log[b + \sqrt{b^2 + d^2 + z^2}] \right) \\
\]

\[ I'_{31,n} = \frac{1}{15} \left( -2z^4 \sqrt{a^2 + c^2 + z^2} - \right. \\
3b^3 \sqrt{b^2 + c^2 + z^2} - b^2 z^2 \sqrt{b^2 + c^2 + z^2} + 2z^4 \sqrt{b^2 + c^2 + z^2} + \\
2d^3 \sqrt{a^2 + d^2 + z^2} + 4d^2 z^2 \sqrt{a^2 + d^2 + z^2} + 2z^4 \sqrt{a^2 + d^2 + z^2} + \\
3b^3 \sqrt{b^2 + d^2 + z^2} - b^2 d^2 \sqrt{b^2 + d^2 + z^2} - 2z^4 \sqrt{b^2 + d^2 + z^2} + \\
b^2 z^2 \sqrt{b^2 + d^2 + z^2} - 4d^2 z^2 \sqrt{b^2 + d^2 + z^2} - 2z^4 \sqrt{b^2 + d^2 + z^2} - \\
2c^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + c^2 + z^2} \right) + 3a^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) - \\
c^2 \left( \sqrt{b^2 + c^2 + z^2} + 4z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) \right) + \\
a^2 \left( c^2 \sqrt{a^2 + c^2 + z^2} - d^2 \sqrt{a^2 + d^2 + z^2} + z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) \right) \\
\]
\[ I_{41,s} = \frac{1}{48} \left( 8a^5 \sqrt{a^2 + c^2 + z^2} + 2a^3 c^2 \sqrt{a^2 + c^2 + z^2} - 3ac^4 \sqrt{a^2 + c^2 + z^2} + \\
2a^3 z^2 \sqrt{a^2 + c^2 + z^2} - 6ac^2 \sqrt{a^2 + c^2 + z^2} - 3az^4 \sqrt{a^2 + c^2 + z^2} - \\
8b^5 \sqrt{b^2 + c^2 + z^2} - 2b^3 c^2 \sqrt{b^2 + c^2 + z^2} + 3bc^4 \sqrt{b^2 + c^2 + z^2} - \\
2b^3 z^2 \sqrt{b^2 + c^2 + z^2} + 6bc^2 \sqrt{b^2 + c^2 + z^2} + 3bz^4 \sqrt{b^2 + c^2 + z^2} - \\
8a^5 \sqrt{a^2 + d^2 + z^2} - 2a^3 d^2 \sqrt{a^2 + d^2 + z^2} + 3ad^4 \sqrt{a^2 + d^2 + z^2} + \\
2a^3 z^2 \sqrt{a^2 + d^2 + z^2} + 6ad^2 \sqrt{a^2 + d^2 + z^2} + 3az^4 \sqrt{a^2 + d^2 + z^2} + \\
8b^5 \sqrt{b^2 + d^2 + z^2} + 2b^3 d^2 \sqrt{b^2 + d^2 + z^2} - 3bd^4 \sqrt{b^2 + d^2 + z^2} + \\
2b^3 z^2 \sqrt{b^2 + d^2 + z^2} - 6bd^2 \sqrt{b^2 + d^2 + z^2} - 3bzd^4 \sqrt{b^2 + d^2 + z^2} + \\
3(c^2 + z^2)^3 \log(a + \sqrt{a^2 + c^2 + z^2}) - 3(c^2 + z^2)^3 \log(b + \sqrt{b^2 + c^2 + z^2}) - \\
3d^6 \log(a + \sqrt{a^2 + d^2 + z^2}) - 9d^4 z^2 \log(a + \sqrt{a^2 + d^2 + z^2}) - \\
9d^2 z^4 \log(a + \sqrt{a^2 + d^2 + z^2}) - 3z^6 \log(a + \sqrt{a^2 + d^2 + z^2}) + \\
3d^6 \log(b + \sqrt{b^2 + d^2 + z^2}) + 9d^4 z^2 \log(b + \sqrt{b^2 + d^2 + z^2}) + \\
9d^2 z^4 \log(b + \sqrt{b^2 + d^2 + z^2}) + 3z^6 \log(b + \sqrt{b^2 + d^2 + z^2}) \right) \]
\[ I_{51,n}^{'} = \frac{1}{105} \left( 8 z^6 \sqrt{a^2 + c^2 + z^2} - 15 b^6 \sqrt{b^2 + c^2 + z^2} - 3 b^4 z^2 \sqrt{b^2 + c^2 + z^2} + 4 b^2 z^4 \sqrt{b^2 + c^2 + z^2} - 8 z^6 \sqrt{b^2 + c^2 + z^2} - 8 d^6 \sqrt{a^2 + d^2 + z^2} - 24 d^4 z^2 \sqrt{a^2 + d^2 + z^2} - 24 d^2 z^4 \sqrt{a^2 + d^2 + z^2} - 8 z^6 \sqrt{a^2 + d^2 + z^2} + 15 b^6 \sqrt{b^2 + d^2 + z^2} + 3 b^4 d^2 \sqrt{b^2 + d^2 + z^2} - 4 b^2 d^4 \sqrt{b^2 + d^2 + z^2} + 8 d^6 \sqrt{b^2 + d^2 + z^2} + 3 b^4 z^2 \sqrt{b^2 + d^2 + z^2} - 8 b^2 d^2 z^2 \sqrt{b^2 + d^2 + z^2} + 24 d^4 z^2 \sqrt{b^2 + d^2 + z^2} - 4 b^2 z^4 \sqrt{b^2 + d^2 + z^2} + 24 d^2 z^4 \sqrt{b^2 + d^2 + z^2} + 8 z^6 \sqrt{b^2 + d^2 + z^2} + 8 c^6 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) + 15 a^6 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) + 4 c^4 \left( b^2 \sqrt{b^2 + c^2 + z^2} + 6 z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) \right) + c^2 \left( -3 b^4 \sqrt{b^2 + c^2 + z^2} + 8 b^2 z^2 \sqrt{b^2 + c^2 + z^2} + 24 z^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) \right) + 3 a^4 \left( c^2 \sqrt{a^2 + c^2 + z^2} - d^2 \sqrt{a^2 + d^2 + z^2} + z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) - 4 a^2 \left( c^4 \sqrt{a^2 + c^2 + z^2} + 2 c^2 z^2 \sqrt{a^2 + c^2 + z^2} - d^4 \sqrt{a^2 + d^2 + z^2} - 2 d^2 z^2 \sqrt{a^2 + d^2 + z^2} + z^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) \right) \]
\[ P_{02,n} = \frac{1}{3} z^3 (\text{ArcTan}[c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, az] + \text{ArcTan}[c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, -bz] + \text{ArcTan}[d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, -az] + \text{ArcTan}[d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, bz]) + \]
\[ \frac{1}{6} (a c \sqrt{a^2 + c^2 + z^2} - b c \sqrt{b^2 + c^2 + z^2} - a d \sqrt{a^2 + d^2 + z^2} + b d \sqrt{b^2 + d^2 + z^2} + 2 c^3 \log[a + \sqrt{a^2 + c^2 + z^2}] - a^2 \log[c + \sqrt{a^2 + c^2 + z^2}] - 3 a z^2 \log[c + \sqrt{a^2 + c^2 + z^2}] - 2 c^3 \log[b + \sqrt{b^2 + c^2 + z^2}] + b^3 \log[c + \sqrt{b^2 + c^2 + z^2}] + 3 b z^2 \log[c + \sqrt{b^2 + c^2 + z^2}] - 2 a^3 \log[d + \sqrt{a^2 + d^2 + z^2}] + a^3 \log[d + \sqrt{a^2 + d^2 + z^2}] + 3 a z^2 \log[d + \sqrt{a^2 + d^2 + z^2}] + 2 d^3 \log[b + \sqrt{b^2 + d^2 + z^2}] - b^3 \log[d + \sqrt{b^2 + d^2 + z^2}] - 3 b z^2 \log[d + \sqrt{b^2 + d^2 + z^2}]) \]

\[ P_{12,n} = \frac{1}{8} (2 d^2 b^2 c \sqrt{a^2 + c^2 + z^2} + 2 c^3 \sqrt{a^2 + c^2 + z^2} + c z^2 \sqrt{a^2 + c^2 + z^2} - b^2 c \sqrt{b^2 + c^2 + z^2} - 2 c^3 \sqrt{b^2 + c^2 + z^2} - c z^2 \sqrt{b^2 + c^2 + z^2} - a^2 d \sqrt{a^2 + d^2 + z^2} - 2 d^3 \sqrt{a^2 + d^2 + z^2} - d z^2 \sqrt{a^2 + d^2 + z^2} + b^2 d \sqrt{b^2 + d^2 + z^2} + 2 d^3 \sqrt{b^2 + d^2 + z^2} + dz^2 \sqrt{b^2 + d^2 + z^2} - (a^2 + z^2)^2 \log[c + \sqrt{a^2 + c^2 + z^2}] + b^4 \log[c + \sqrt{b^2 + c^2 + z^2}] + 2 b^2 z^2 \log[c + \sqrt{b^2 + c^2 + z^2}] + (a^2 + z^2)^2 \log[d + \sqrt{a^2 + d^2 + z^2}] - b^4 \log[d + \sqrt{b^2 + d^2 + z^2}] - 2 b^2 z^2 \log[d + \sqrt{b^2 + d^2 + z^2}]) \]
\[
I'_{22,a} = \\
- \frac{1}{15} z^5 \left( \text{ArcTan}\left[ z^2 + c \sqrt{a^2 + c^2 + z^2}, -az \right] + \text{ArcTan}\left[ z^2 + c \sqrt{b^2 + c^2 + z^2}, bz \right] + \text{ArcTan}\left[ d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, az \right] + \text{ArcTan}\left[ d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, -bz \right] \right) + \\
\frac{1}{30} \left( 3a^3 c \sqrt{a^2 + c^2 + z^2} + 3ac^3 \sqrt{a^2 + c^2 + z^2} + 2acz^2 \sqrt{a^2 + c^2 + z^2} - \\
3b^3 c \sqrt{b^2 + c^2 + z^2} - 3bc^3 \sqrt{b^2 + c^2 + z^2} - 2bcz^2 \sqrt{b^2 + c^2 + z^2} - \\
3a^3 d \sqrt{a^2 + d^2 + z^2} - 3ad^3 \sqrt{a^2 + d^2 + z^2} - 2adz^2 \sqrt{a^2 + d^2 + z^2} + \\
3b^3 d \sqrt{b^2 + d^2 + z^2} + 3bd^3 \sqrt{b^2 + d^2 + z^2} + 2bdz^2 \sqrt{b^2 + d^2 + z^2} - \\
(3c^5 + 5c^3 z^2) \log[a + \sqrt{a^2 + c^2 + z^2}] - (3a^5 + 5a^3 z^2) \log[c + \sqrt{a^2 + c^2 + z^2}] + \\
3c^5 \log[b + \sqrt{b^2 + c^2 + z^2}] + 5c^3 z^2 \log[b + \sqrt{b^2 + c^2 + z^2}] + \\
3b^5 \log[c + \sqrt{b^2 + c^2 + z^2}] + 5b^3 z^2 \log[c + \sqrt{b^2 + c^2 + z^2}] + \\
3d^5 \log[a + \sqrt{a^2 + d^2 + z^2}] + 5d^3 z^2 \log[a + \sqrt{a^2 + d^2 + z^2}] + \\
3a^5 \log[d + \sqrt{a^2 + d^2 + z^2}] + 5a^3 z^2 \log[d + \sqrt{a^2 + d^2 + z^2}] - \\
3d^5 \log[b + \sqrt{b^2 + d^2 + z^2}] - 5d^3 z^2 \log[b + \sqrt{b^2 + d^2 + z^2}] - \\
3b^5 \log[d + \sqrt{b^2 + d^2 + z^2}] - 5b^3 z^2 \log[d + \sqrt{b^2 + d^2 + z^2}] \right)
\]
\[ I_{32,a} = \frac{1}{72} \left( 6 a^4 c \sqrt{a^2 + c^2 + z^2} + 4 a^2 c^3 \sqrt{a^2 + c^2 + z^2} - \\
8 c^5 \sqrt{a^2 + c^2 + z^2} + 3 a^2 c z^2 \sqrt{a^2 + c^2 + z^2} - 14 c^3 z^2 \sqrt{a^2 + c^2 + z^2} - \\
3 c z^4 \sqrt{a^2 + c^2 + z^2} - 6 b^4 c \sqrt{b^2 + c^2 + z^2} - 4 b^2 c^3 \sqrt{b^2 + c^2 + z^2} + \\
8 c^5 \sqrt{b^2 + c^2 + z^2} - 3 b^2 c z^2 \sqrt{b^2 + c^2 + z^2} + 14 c^3 z^2 \sqrt{b^2 + c^2 + z^2} + \\
3 c z^4 \sqrt{b^2 + c^2 + z^2} - 6 a^4 d \sqrt{a^2 + d^2 + z^2} - 4 a^2 d^3 \sqrt{a^2 + d^2 + z^2} + \\
8 d^5 \sqrt{a^2 + d^2 + z^2} - 3 a^2 d z^2 \sqrt{a^2 + d^2 + z^2} + 14 d^3 z^2 \sqrt{a^2 + d^2 + z^2} + \\
3 d z^4 \sqrt{a^2 + d^2 + z^2} + 6 b^4 d \sqrt{b^2 + d^2 + z^2} + 4 b^2 d^3 \sqrt{b^2 + d^2 + z^2} - \\
8 d^5 \sqrt{b^2 + d^2 + z^2} + 3 b^2 d z^2 \sqrt{b^2 + d^2 + z^2} - 14 d^3 z^2 \sqrt{b^2 + d^2 + z^2} - \\
3 d z^4 \sqrt{b^2 + d^2 + z^2} - 3 \left( 2 a^6 + 3 a^4 z^2 - 3 z^6 \right) \log[c + \sqrt{a^2 + c^2 + z^2}] + \\
3 \left( 2 b^6 + 3 b^4 z^2 - 3 z^6 \right) \log[c + \sqrt{b^2 + c^2 + z^2}] + \\
6 a^6 \log[d + \sqrt{a^2 + d^2 + z^2}] + 9 a^4 z^2 \log[d + \sqrt{a^2 + d^2 + z^2}] - \\
3 z^6 \log[d + \sqrt{a^2 + d^2 + z^2}] - 6 b^6 \log[d + \sqrt{b^2 + d^2 + z^2}] - \\
9 b^4 z^2 \log[d + \sqrt{b^2 + d^2 + z^2}] + 3 z^6 \log[d + \sqrt{b^2 + d^2 + z^2}] \right) \]
\[ I'_{42,n} = \]
\[ \frac{1}{35} z^7 \]
\[ \left( \arctan \left[ a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, cz \right] + \arctan \left[ b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, -cz \right] + \right. \]
\[ \left. \arctan \left[ a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2}, -dz \right] + \arctan \left[ b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, dz \right] + \right. \]
\[ \frac{1}{280} \left( 20a^5 c \sqrt{a^2 + c^2 + z^2} + 10a^3 c^3 \sqrt{a^2 + c^2 + z^2} - 15a^5 c \sqrt{a^2 + c^2 + z^2} + \right. \]
\[ 8a^3 c z^2 \sqrt{a^2 + c^2 + z^2} - 27a^3 c^3 \sqrt{a^2 + c^2 + z^2} - 8ac^4 \sqrt{a^2 + c^2 + z^2} - \right. \]
\[ 20b^5 c \sqrt{b^2 + c^2 + z^2} - 10b^3 c^3 \sqrt{b^2 + c^2 + z^2} + 15bc^5 \sqrt{b^2 + c^2 + z^2} - \right. \]
\[ 8b^3 c z^2 \sqrt{b^2 + c^2 + z^2} + 27bc^5 \sqrt{b^2 + c^2 + z^2} + 8bc^4 \sqrt{b^2 + c^2 + z^2} - \right. \]
\[ 20a^5 d \sqrt{a^2 + d^2 + z^2} - 10a^3 d^3 \sqrt{a^2 + d^2 + z^2} + 15ad^5 \sqrt{a^2 + d^2 + z^2} - \right. \]
\[ 8a^3 dz^2 \sqrt{a^2 + d^2 + z^2} + 27ad^5 \sqrt{a^2 + d^2 + z^2} + 8adz^4 \sqrt{a^2 + d^2 + z^2} + \right. \]
\[ 20b^5 d \sqrt{b^2 + d^2 + z^2} - 10b^3 d^3 \sqrt{b^2 + d^2 + z^2} - 15bd^5 \sqrt{b^2 + d^2 + z^2} + \right. \]
\[ 8b^3 dz^2 \sqrt{b^2 + d^2 + z^2} - 27bd^5 \sqrt{b^2 + d^2 + z^2} - \right. \]
\[ 8bdz^4 \sqrt{b^2 + d^2 + z^2} + (15c^7 + 42c^5 z^2 + 35c^3 z^4) \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] - \right. \]
\[ 4(5a^7 + 7a^5 z^2) \log \left[ c + \sqrt{a^2 + c^2 + z^2} \right] - 15c^7 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + \right. \]
\[ 42c^5 z^2 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] - 35c^3 z^4 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + \right. \]
\[ 20b^7 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] + 28b^5 z^2 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] - \right. \]
\[ 15d^7 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - 42d^5 z^2 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - \right. \]
\[ 35d^3 z^4 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 20a^7 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] + \right. \]
\[ 28a^5 z^2 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] + 15d^7 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] + \right. \]
\[ 42d^5 z^2 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] + 35d^3 z^4 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - \right. \]
\[ 20b^7 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] - 28b^5 z^2 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] \]
\[
I'_{52,a} = \frac{1}{720} \left( 45 a^6 c \sqrt{a^2 + c^2 + z^2} + 18 a^4 c^3 \sqrt{a^2 + c^2 + z^2} - 24 a^2 c^5 \sqrt{a^2 + c^2 + z^2} + 
48 c^7 \sqrt{a^2 + c^2 + z^2} + 15 a^4 c^2 \sqrt{a^2 + c^2 + z^2} - 44 a^2 c^3 \sqrt{a^2 + c^2 + z^2} + 
136 c^5 z^2 \sqrt{a^2 + c^2 + z^2} - 15 a^2 c^4 \sqrt{a^2 + c^2 + z^2} + 118 c^3 z^4 \sqrt{a^2 + c^2 + z^2} + 
15 c z^6 \sqrt{a^2 + c^2 + z^2} - 45 b^6 c \sqrt{b^2 + c^2 + z^2} - 18 b^4 c^3 \sqrt{b^2 + c^2 + z^2} + 
24 b^2 c^5 \sqrt{b^2 + c^2 + z^2} - 48 c^7 \sqrt{b^2 + c^2 + z^2} - 15 b^4 c^2 \sqrt{b^2 + c^2 + z^2} + 
44 b^2 c^3 z^2 \sqrt{b^2 + c^2 + z^2} - 136 c^5 z^2 \sqrt{b^2 + c^2 + z^2} + 15 b^2 c^4 \sqrt{b^2 + c^2 + z^2} - 
118 c^3 z^4 \sqrt{b^2 + c^2 + z^2} - 15 c z^6 \sqrt{b^2 + c^2 + z^2} - 45 a^6 d \sqrt{a^2 + d^2 + z^2} - 
18 a^4 d^3 \sqrt{a^2 + d^2 + z^2} + 24 a^2 d^5 \sqrt{a^2 + d^2 + z^2} - 48 d^7 \sqrt{a^2 + d^2 + z^2} - 
15 a^4 d z^2 \sqrt{a^2 + d^2 + z^2} + 44 a^2 d^3 z^2 \sqrt{a^2 + d^2 + z^2} - 136 a^4 d^5 z^2 \sqrt{a^2 + d^2 + z^2} + 
15 a^2 d z^4 \sqrt{a^2 + d^2 + z^2} - 118 a^4 d^3 z^2 \sqrt{a^2 + d^2 + z^2} - 15 d z^6 \sqrt{a^2 + d^2 + z^2} + 
45 b^6 d \sqrt{b^2 + d^2 + z^2} + 18 b^4 d^3 \sqrt{b^2 + d^2 + z^2} - 24 b^2 d^5 \sqrt{b^2 + d^2 + z^2} + 
48 d^7 \sqrt{b^2 + d^2 + z^2} + 15 b^4 d z^2 \sqrt{b^2 + d^2 + z^2} - 44 b^2 d^3 z^2 \sqrt{b^2 + d^2 + z^2} + 
136 d^5 z^2 \sqrt{b^2 + d^2 + z^2} - 15 b^2 d z^4 \sqrt{b^2 + d^2 + z^2} + 118 d^3 z^4 \sqrt{b^2 + d^2 + z^2} + 
15 d z^6 \sqrt{b^2 + d^2 + z^2} - 15 (3 a^8 + 4 a^6 z^2 + z^4) \log \left( c + \sqrt{a^2 + c^2 + z^2} \right) + 
15 (3 b^8 + 4 b^6 z^2 + z^4) \log \left( c + \sqrt{b^2 + c^2 + z^2} \right) + 
45 a^8 \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) + 60 a^6 z^2 \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) + 
15 z^8 \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) - 45 b^8 \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) - 
60 b^6 z^2 \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) - 15 z^8 \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) \right)
\]
\[ I'_{03,a} = \frac{1}{12} (-2a^3 \sqrt{a^2 + c^2 + z^2} + ac^2 \sqrt{a^2 + c^2 + z^2} - 5az^2 \sqrt{a^2 + c^2 + z^2} + \\
2b^3 \sqrt{b^2 + c^2 + z^2} - bc^2 \sqrt{b^2 + c^2 + z^2} + 5bz^2 \sqrt{b^2 + c^2 + z^2} + \\
2a^3 \sqrt{a^2 + d^2 + z^2} - ad^2 \sqrt{a^2 + d^2 + z^2} + 5az^2 \sqrt{a^2 + d^2 + z^2} - \\
2b^3 \sqrt{b^2 + d^2 + z^2} + bd^2 \sqrt{b^2 + d^2 + z^2} - 5bz^2 \sqrt{b^2 + d^2 + z^2} + \\
3(c^4 - z^4) \log[a + \sqrt{a^2 + c^2 + z^2}] - 3c^4 \log[b + \sqrt{b^2 + c^2 + z^2}] + \\
3z^4 \log[b + \sqrt{b^2 + c^2 + z^2}] - 3(d^4 - z^4) \log[a + \sqrt{a^2 + d^2 + z^2}] + \\
3d^4 \log[b + \sqrt{b^2 + d^2 + z^2}] - 3z^4 \log[b + \sqrt{b^2 + d^2 + z^2}]) \]

\[ I'_{13,a} = \frac{1}{15} (-2z^4 \sqrt{a^2 + c^2 + z^2} + \\
2b^4 \sqrt{b^2 + c^2 + z^2} + 4b^2z^2 \sqrt{b^2 + c^2 + z^2} + 2z^4 \sqrt{b^2 + c^2 + z^2} - \\
3d^4 \sqrt{a^2 + d^2 + z^2} - d^2z^2 \sqrt{a^2 + d^2 + z^2} + 2z^4 \sqrt{a^2 + d^2 + z^2} - \\
2b^4 \sqrt{b^2 + d^2 + z^2} + b^2d^2 \sqrt{b^2 + d^2 + z^2} + 3d^4 \sqrt{b^2 + d^2 + z^2} - \\
4b^2z^2 \sqrt{b^2 + d^2 + z^2} + d^2z^2 \sqrt{b^2 + d^2 + z^2} + 4z^4 \sqrt{b^2 + d^2 + z^2} + \\
3c^4 (\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2}) - 2a^4 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2}) + \\
c^2 (a^2 \sqrt{a^2 + c^2 + z^2} - b^2 \sqrt{b^2 + c^2 + z^2} + z^2 (\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2})) - \\
a^2 (d^2 \sqrt{a^2 + d^2 + z^2} + 4z^2 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2}))) \]
$$I'_{23,n} = \frac{1}{72} (-8a^5 \sqrt{a^2 + c^2 + z^2} + 4a^3 c^2 \sqrt{a^2 + c^2 + z^2} + 6ac^4 \sqrt{a^2 + c^2 + z^2} -$$
$$14a^3 z^2 \sqrt{a^2 + c^2 + z^2} + 3ac^2 z^2 \sqrt{a^2 + c^2 + z^2} - 3az^4 \sqrt{a^2 + c^2 + z^2} +$$
$$8b^5 \sqrt{b^2 + c^2 + z^2} - 4b^3 c^2 \sqrt{b^2 + c^2 + z^2} - 6b c^4 \sqrt{b^2 + c^2 + z^2} +$$
$$14b^3 z^2 \sqrt{b^2 + c^2 + z^2} - 3b c^2 z^2 \sqrt{b^2 + c^2 + z^2} + 3bz^4 \sqrt{b^2 + c^2 + z^2} +$$
$$8a^5 \sqrt{a^2 + d^2 + z^2} - 4a^3 d^2 \sqrt{a^2 + d^2 + z^2} - 6ad^4 \sqrt{a^2 + d^2 + z^2} +$$
$$14a^3 z^2 \sqrt{a^2 + d^2 + z^2} - 3ad^2 z^2 \sqrt{a^2 + d^2 + z^2} + 3az^4 \sqrt{a^2 + d^2 + z^2} -$$
$$8b^5 \sqrt{b^2 + d^2 + z^2} + 4b^3 d^2 \sqrt{b^2 + d^2 + z^2} + 6bd^4 \sqrt{b^2 + d^2 + z^2} -$$
$$14b^3 z^2 \sqrt{b^2 + d^2 + z^2} + 3bd^2 z^2 \sqrt{b^2 + d^2 + z^2} - 3bz^4 \sqrt{b^2 + d^2 + z^2} -$$
$$3(2c^6 + 3c^4 z^2 - z^6) \log[a + \sqrt{a^2 + c^2 + z^2}] + 6c^6 \log[b + \sqrt{b^2 + c^2 + z^2}] +$$
$$9c^4 z^2 \log[b + \sqrt{b^2 + c^2 + z^2}] - 3z^6 \log[b + \sqrt{b^2 + c^2 + z^2}] +$$
$$3(2d^6 + 3d^4 z^2 - z^6) \log[a + \sqrt{a^2 + d^2 + z^2}] - 6d^6 \log[b + \sqrt{b^2 + d^2 + z^2}] -$$
$$9d^4 z^2 \log[b + \sqrt{b^2 + d^2 + z^2}] + 3z^6 \log[b + \sqrt{b^2 + d^2 + z^2}]$$
\[
I'_{33, n} = \frac{1}{105} \left( 4z^6 \sqrt{a^2 + c^2 + z^2} + 10 b^6 \sqrt{b^2 + c^2 + z^2} + \\
16 b^4 z^2 \sqrt{b^2 + c^2 + z^2} + 2 b^2 z^4 \sqrt{b^2 + c^2 + z^2} - 4 z^6 \sqrt{b^2 + c^2 + z^2} + \\
10 d^6 \sqrt{a^2 + d^2 + z^2} + 16 d^4 z^2 \sqrt{a^2 + d^2 + z^2} + 2 d^2 z^4 \sqrt{a^2 + d^2 + z^2} - \\
4 z^6 \sqrt{a^2 + d^2 + z^2} - 10 b^6 \sqrt{b^2 + d^2 + z^2} + 5 b^4 d^2 \sqrt{b^2 + d^2 + z^2} + \\
5 b^2 d^4 \sqrt{b^2 + d^2 + z^2} - 10 d^6 \sqrt{b^2 + d^2 + z^2} - 16 b^4 z^2 \sqrt{b^2 + d^2 + z^2} + \\
3 b^2 d^2 z^2 \sqrt{b^2 + d^2 + z^2} - 16 d^4 z^2 \sqrt{b^2 + d^2 + z^2} - \\
2 b^2 z^4 \sqrt{b^2 + d^2 + z^2} - 2 d^2 z^4 \sqrt{b^2 + d^2 + z^2} + 4 z^6 \sqrt{b^2 + d^2 + z^2} - \\
10 c^6 (\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2}) - 10 a^6 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2}) + \\
c^4 (-5 b^2 \sqrt{b^2 + c^2 + z^2} - 16 z^2 (\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2})) + \\
c^2 (5 b^4 \sqrt{b^2 + c^2 + z^2} + 3 b^2 z^2 \sqrt{b^2 + c^2 + z^2} + \\
2 z^4 (\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2})) + a^4 (5 c^2 \sqrt{a^2 + c^2 + z^2} - \\
5 d^2 \sqrt{a^2 + d^2 + z^2} - 16 z^2 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2})) + \\
a^2 (5 c^4 \sqrt{a^2 + c^2 + z^2} + 3 c^2 z^2 \sqrt{a^2 + c^2 + z^2} - 5 d^4 \sqrt{a^2 + d^2 + z^2} - \\
3 d^2 z^2 \sqrt{a^2 + d^2 + z^2} - 2 z^4 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2})) \right)
\]
\[ I_{43,a} = \frac{1}{192} \left( -16a^7 \sqrt{a^2 + c^2 + z^2} + 8a^5c^2 \sqrt{a^2 + c^2 + z^2} + 6a^3c^4 \sqrt{a^2 + c^2 + z^2} - 
\right.
\left. 9ac^6 \sqrt{a^2 + c^2 + z^2} - 24a^5z^2 \sqrt{a^2 + c^2 + z^2} + 4a^3c^2z^2 \sqrt{a^2 + c^2 + z^2} - 
\right.
\left. 15ac^4z^2 \sqrt{a^2 + c^2 + z^2} - 2a^3z^4 \sqrt{a^2 + c^2 + z^2} - 3ac^2z^4 \sqrt{a^2 + c^2 + z^2} + 
\right.
\left. 3az^6 \sqrt{a^2 + c^2 + z^2} + 16b^7 \sqrt{b^2 + c^2 + z^2} - 8b^5c^2 \sqrt{b^2 + c^2 + z^2} - 
\right.
\left. 6b^3c^4 \sqrt{b^2 + c^2 + z^2} + 9bc^6 \sqrt{b^2 + c^2 + z^2} + 24b^5z^2 \sqrt{b^2 + c^2 + z^2} - 
\right.
\left. 4b^3c^2z^2 \sqrt{b^2 + c^2 + z^2} + 15bc^4z^2 \sqrt{b^2 + c^2 + z^2} + 2b^3z^4 \sqrt{b^2 + c^2 + z^2} + 
\right.
\left. 16a^7 \sqrt{a^2 + d^2 + z^2} - 6a^3d^4 \sqrt{a^2 + d^2 + z^2} + 9ad^6 \sqrt{a^2 + d^2 + z^2} + 
\right.
\left. 24a^5z^2 \sqrt{a^2 + d^2 + z^2} - 4a^3d^2z^2 \sqrt{a^2 + d^2 + z^2} + 15ad^4z^2 \sqrt{a^2 + d^2 + z^2} + 
\right.
\left. 2a^3z^4 \sqrt{a^2 + d^2 + z^2} + 3ad^2z^4 \sqrt{a^2 + d^2 + z^2} - 3az^6 \sqrt{a^2 + d^2 + z^2} - 
\right.
\left. 16b^7 \sqrt{b^2 + d^2 + z^2} + 8b^5d^2 \sqrt{b^2 + d^2 + z^2} + 6b^3d^4 \sqrt{b^2 + d^2 + z^2} - 
\right.
\left. 9bd^6 \sqrt{b^2 + d^2 + z^2} - 24b^5z^2 \sqrt{b^2 + d^2 + z^2} + 4b^3d^2z^2 \sqrt{b^2 + d^2 + z^2} - 
\right.
\left. 15bd^4z^2 \sqrt{b^2 + d^2 + z^2} - 2b^3z^4 \sqrt{b^2 + d^2 + z^2} - 3bd^2z^4 \sqrt{b^2 + d^2 + z^2} + 
\right.
\left. 3b^3d^2 \sqrt{b^2 + d^2 + z^2} + 3(3c^2 - z^2)(c^2 + z^2)^3 \log[a + (c^2 + z^2)^3] - 
\right.
\left. 3(3c^2 - z^2)(c^2 + z^2)^3 \log[b + \sqrt{b^2 + c^2 + z^2}] - 
\right.
\left. 9d^8 \log[a + \sqrt{a^2 + d^2 + z^2}] - 24d^6z^2 \log[a + \sqrt{a^2 + d^2 + z^2}] - 
\right.
\left. 18d^4z^4 \log[a + \sqrt{a^2 + d^2 + z^2}] + 3z^8 \log[a + \sqrt{a^2 + d^2 + z^2}] + 
\right.
\left. 9d^8 \log[b + \sqrt{b^2 + d^2 + z^2}] + 24d^6z^2 \log[b + \sqrt{b^2 + d^2 + z^2}] + 
\right.
\left. 18d^4z^4 \log[b + \sqrt{b^2 + d^2 + z^2}] - 3z^8 \log[b + \sqrt{b^2 + d^2 + z^2}] \right) \]
\[ I_{33,n} = \frac{1}{945} \left( -16 z^8 \sqrt{a^2 + c^2 + z^2} + 70 b^8 \sqrt{b^2 + c^2 + z^2} + \\
100 b^6 z^2 \sqrt{b^2 + c^2 + z^2} + 6 b^4 z^4 \sqrt{b^2 + c^2 + z^2} - 8 b^2 z^6 \sqrt{b^2 + c^2 + z^2} + \\
16 z^8 \sqrt{a^2 + d^2 + z^2} - 56 d^8 \sqrt{a^2 + d^2 + z^2} - 152 d^6 z^2 \sqrt{a^2 + d^2 + z^2} - \\
120 d^4 z^4 \sqrt{a^2 + d^2 + z^2} - 8 d^2 z^6 \sqrt{a^2 + d^2 + z^2} + 16 z^8 \sqrt{a^2 + d^2 + z^2} - \\
70 b^8 \sqrt{b^2 + d^2 + z^2} + 35 b^6 d^2 \sqrt{b^2 + d^2 + z^2} + 21 b^4 d^4 \sqrt{b^2 + d^2 + z^2} - \\
28 b^2 d^6 \sqrt{b^2 + d^2 + z^2} + 56 d^8 \sqrt{b^2 + d^2 + z^2} - 100 b^6 z^2 \sqrt{b^2 + d^2 + z^2} + \\
15 b^4 d^4 z^2 \sqrt{b^2 + d^2 + z^2} - 48 b^2 d^4 z^4 \sqrt{b^2 + d^2 + z^2} + 152 d^6 z^2 \sqrt{b^2 + d^2 + z^2} - \\
6 b^4 z^4 \sqrt{b^2 + d^2 + z^2} - 12 b^2 d^2 z^4 \sqrt{b^2 + d^2 + z^2} + 120 d^4 z^4 \sqrt{b^2 + d^2 + z^2} + \\
8 b^2 d^6 \sqrt{b^2 + d^2 + z^2} + 8 d^2 z^6 \sqrt{b^2 + d^2 + z^2} - 16 z^8 \sqrt{b^2 + d^2 + z^2} - \\
56 c^8 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) - 70 a^8 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) + \\
4 c^6 \left( 7 b^2 \sqrt{b^2 + c^2 + z^2} + 38 z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) \right) + \\
3 c^4 \left( -7 b^4 \sqrt{b^2 + c^2 + z^2} + 16 b^2 d^2 \sqrt{b^2 + c^2 + z^2} + \\
40 z^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) \right) + \\
c^2 \left( -35 b^6 \sqrt{b^2 + c^2 + z^2} - 15 b^4 z^2 \sqrt{b^2 + c^2 + z^2} + 12 b^2 z^4 \sqrt{b^2 + c^2 + z^2} + \\
8 z^6 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) \right) + 5 a^6 \left( 7 c^2 \sqrt{a^2 + c^2 + z^2} - \\
7 d^2 \sqrt{a^2 + d^2 + z^2} - 20 z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) + \\
3 a^4 \left( 7 c^4 \sqrt{a^2 + c^2 + z^2} + 5 c^2 z^2 \sqrt{a^2 + c^2 + z^2} - 7 d^4 \sqrt{a^2 + d^2 + z^2} - \\
5 d^2 z^4 \sqrt{a^2 + d^2 + z^2} - 2 z^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) - \\
4 a^2 \left( 7 c^6 \sqrt{a^2 + c^2 + z^2} + 12 c^4 z^2 \sqrt{a^2 + c^2 + z^2} + 3 c^2 z^4 \sqrt{a^2 + c^2 + z^2} - \\
7 d^6 \sqrt{a^2 + d^2 + z^2} - 12 d^4 z^2 \sqrt{a^2 + d^2 + z^2} - \\
3 d^2 z^4 \sqrt{a^2 + d^2 + z^2} - 2 z^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \right) \r
\[ I_{04,5} = \]
\[- \frac{1}{5} z^5 \left( \arctan \left[ c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, -az \right] + \right. \]
\[ \arctan \left[ c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, bz \right] + \arctan \left[ d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, az \right] + \]
\[ \arctan \left[ d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, -bz \right] \right) \]
\[ \frac{1}{40} \left( -3 a^3 c \sqrt{a^2 + c^2 + z^2} + 2 a c^3 \sqrt{a^2 + c^2 + z^2} - 7 a c z^2 \sqrt{a^2 + c^2 + z^2} + \right. \]
\[ 3 b^3 c \sqrt{b^2 + c^2 + z^2} - 2 b c^3 \sqrt{b^2 + c^2 + z^2} + 7 b c z^2 \sqrt{b^2 + c^2 + z^2} + \]
\[ 3 a^3 d \sqrt{a^2 + d^2 + z^2} - 2 a d^3 \sqrt{a^2 + d^2 + z^2} + 7 a d z^2 \sqrt{a^2 + d^2 + z^2} - \]
\[ 3 b^3 d \sqrt{b^2 + d^2 + z^2} + 2 b d^3 \sqrt{b^2 + d^2 + z^2} - 7 b d z^2 \sqrt{b^2 + d^2 + z^2} + \]
\[ 8 c^5 \log \left( a^2 + c^2 + z^2 \right) + (3 a^3 + 10 a^3 z^2 + 15 a z^4) \log \left( c + \sqrt{a^2 + c^2 + z^2} \right) - \]
\[ 8 c^5 \log \left( b + \sqrt{b^2 + c^2 + z^2} \right) - 3 b^5 \log \left( c + \sqrt{b^2 + c^2 + z^2} \right) - \]
\[ 10 b^3 z^2 \log \left( c + \sqrt{b^2 + c^2 + z^2} \right) - 15 b z^4 \log \left( c + \sqrt{b^2 + c^2 + z^2} \right) - \]
\[ 8 d^5 \log \left( a + \sqrt{a^2 + d^2 + z^2} \right) - 3 a^5 \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) - \]
\[ 10 a^3 z^2 \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) - 15 a z^4 \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) + \]
\[ 8 d^5 \log \left( b + \sqrt{b^2 + d^2 + z^2} \right) + 3 b^5 \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) + \]
\[ 10 b^3 z^2 \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) + 15 b z^4 \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) \) \]
\[ I_{14,n} = \frac{1}{48} \left( -3a^4 c \sqrt{a^2 + c^2 + z^2} + 2a^2 c^3 \sqrt{a^2 + c^2 + z^2} + 8c^5 \sqrt{a^2 + c^2 + z^2} - \
6a^2 c^2 \sqrt{a^2 + c^2 + z^2} + 2c^3 z^2 \sqrt{a^2 + c^2 + z^2} - 3cz^4 \sqrt{a^2 + c^2 + z^2} + \
3b^4 c \sqrt{b^2 + c^2 + z^2} - 2b^2 c^3 \sqrt{b^2 + c^2 + z^2} - 8c^5 \sqrt{b^2 + c^2 + z^2} + \
6b^2 c^2 \sqrt{b^2 + c^2 + z^2} - 2c^3 z^2 \sqrt{b^2 + c^2 + z^2} + 3cz^4 \sqrt{b^2 + c^2 + z^2} + \
3a^4 d \sqrt{a^2 + d^2 + z^2} - 2a^2 d^3 \sqrt{a^2 + d^2 + z^2} - 8d^5 \sqrt{a^2 + d^2 + z^2} + \
6a^2 d^2 \sqrt{a^2 + d^2 + z^2} - 2d^3 z^2 \sqrt{a^2 + d^2 + z^2} + 3dz^4 \sqrt{a^2 + d^2 + z^2} - \
3b^4 d \sqrt{b^2 + d^2 + z^2} + 2b^2 d^3 \sqrt{b^2 + d^2 + z^2} + 8d^5 \sqrt{b^2 + d^2 + z^2} - \
6b^2 d^2 \sqrt{b^2 + d^2 + z^2} + 2d^3 z^2 \sqrt{b^2 + d^2 + z^2} - 3dz^4 \sqrt{b^2 + d^2 + z^2} + \
3(a^2 + z^2)^3 \log(c + \sqrt{a^2 + c^2 + z^2}) - 3b^6 \log(c + \sqrt{b^2 + c^2 + z^2}) - \
9b^4 z^2 \log(c + \sqrt{b^2 + c^2 + z^2}) - 9b^2 z^4 \log(c + \sqrt{b^2 + c^2 + z^2}) - \
3z^6 \log(c + \sqrt{b^2 + c^2 + z^2}) - 3(a^2 + z^2)^3 \log(d + \sqrt{a^2 + d^2 + z^2}) + \
3b^6 \log(d + \sqrt{b^2 + d^2 + z^2}) + 9b^4 z^2 \log(d + \sqrt{b^2 + d^2 + z^2}) + \
9b^2 z^4 \log(d + \sqrt{b^2 + d^2 + z^2}) + 3z^6 \log(d + \sqrt{b^2 + d^2 + z^2}) \right) \]
$$I'_{24,n} =$$

$$-\frac{1}{35} z^7$$

$$\left( \text{ArcTan}\left[ c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, az \right] + \text{ArcTan}\left[ c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, -bz \right] +$$

$$\text{ArcTan}\left[ d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, -az \right] +$$

$$\text{ArcTan}\left[ d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, bz \right] \right)$$

$$\frac{1}{280} \left(-15 a^5 c \sqrt{a^2 + c^2 + z^2} + 10 a^3 c^3 \sqrt{a^2 + c^2 + z^2} + 20 ac^5 \sqrt{a^2 + c^2 + z^2} -$$

$$27 a^3 c^2 \sqrt{a^2 + c^2 + z^2} + 8 ac^3 z^2 \sqrt{a^2 + c^2 + z^2} - 8 ac^4 \sqrt{a^2 + c^2 + z^2} +$$

$$15 b^5 c \sqrt{b^2 + c^2 + z^2} - 10 b^3 c^3 \sqrt{b^2 + c^2 + z^2} - 20 bc^5 \sqrt{b^2 + c^2 + z^2} +$$

$$27 b^3 c^2 \sqrt{b^2 + c^2 + z^2} - 8 bc^3 z^2 \sqrt{b^2 + c^2 + z^2} + 8 bc^4 \sqrt{b^2 + c^2 + z^2} +$$

$$15 a^5 d \sqrt{a^2 + d^2 + z^2} - 10 a^3 d^3 \sqrt{a^2 + d^2 + z^2} - 20 ad^5 \sqrt{a^2 + d^2 + z^2} +$$

$$27 a^3 d^2 \sqrt{a^2 + d^2 + z^2} - 8 ad^3 z^2 \sqrt{a^2 + d^2 + z^2} + 8 ad^4 \sqrt{a^2 + d^2 + z^2} -$$

$$15 b^5 d \sqrt{b^2 + d^2 + z^2} + 10 b^3 d^3 \sqrt{b^2 + d^2 + z^2} + 20 bd^5 \sqrt{b^2 + d^2 + z^2} -$$

$$27 b^3 d^2 \sqrt{b^2 + d^2 + z^2} + 8 bd^3 z^2 \sqrt{b^2 + d^2 + z^2} -$$

$$8 bd^4 \sqrt{b^2 + d^2 + z^2} - 4 (5 c^7 + 7 c^5 z^2) \log [a + \sqrt{a^2 + c^2 + z^2}] +$$

$$(15 a^7 + 42 a^5 z^2 + 35 a^3 z^4) \log [c + \sqrt{a^2 + c^2 + z^2}] + 20 c^7 \log [b + \sqrt{b^2 + c^2 + z^2}] +$$

$$28 c^5 z^2 \log [b + \sqrt{b^2 + c^2 + z^2}] - 15 b^7 \log [c + \sqrt{b^2 + c^2 + z^2}] -$$

$$42 b^5 z^2 \log [c + \sqrt{b^2 + c^2 + z^2}] - 35 b^3 z^4 \log [c + \sqrt{b^2 + c^2 + z^2}] +$$

$$20 d^7 \log [a + \sqrt{a^2 + d^2 + z^2}] + 28 d^5 z^2 \log [a + \sqrt{a^2 + d^2 + z^2}] -$$

$$15 a^7 \log [d + \sqrt{a^2 + d^2 + z^2}] - 42 a^5 z^2 \log [d + \sqrt{a^2 + d^2 + z^2}] -$$

$$35 a^3 z^4 \log [d + \sqrt{a^2 + d^2 + z^2}] - 20 d^7 \log [b + \sqrt{b^2 + d^2 + z^2}] -$$

$$28 d^5 z^2 \log [b + \sqrt{b^2 + d^2 + z^2}] + 15 b^7 \log [d + \sqrt{b^2 + d^2 + z^2}] +$$

$$42 b^5 z^2 \log [d + \sqrt{b^2 + d^2 + z^2}] + 35 b^3 z^4 \log [d + \sqrt{b^2 + d^2 + z^2}]$$
\[ I'_{34,a} = \frac{1}{192} \left( -9a^6 c \sqrt{a^2 + c^2 + z^2} + 6a^4 c^3 \sqrt{a^2 + c^2 + z^2} + 8a^2 c^5 \sqrt{a^2 + c^2 + z^2} - 16c^7 \sqrt{a^2 + c^2 + z^2} - 15a^4 c z^2 \sqrt{a^2 + c^2 + z^2} - 4a^2 c^3 z^2 \sqrt{a^2 + c^2 + z^2} - 24c^5 z^2 \sqrt{a^2 + c^2 + z^2} - 3c^6 z^2 \sqrt{a^2 + c^2 + z^2} + 9b^6 c \sqrt{b^2 + c^2 + z^2} - 6b^4 c^3 \sqrt{b^2 + c^2 + z^2} - 8b^2 c^5 \sqrt{b^2 + c^2 + z^2} + 16c^7 \sqrt{b^2 + c^2 + z^2} + 15b^4 c z^2 \sqrt{b^2 + c^2 + z^2} - 4b^2 c^3 z^2 \sqrt{b^2 + c^2 + z^2} + 24c^5 z^2 \sqrt{b^2 + c^2 + z^2} - 3b^2 c z^4 \sqrt{b^2 + c^2 + z^2} + 2c^3 z^4 \sqrt{b^2 + c^2 + z^2} - 3c z^6 \sqrt{b^2 + c^2 + z^2} + 9a^6 d \sqrt{a^2 + d^2 + z^2} - 6a^4 d^3 \sqrt{a^2 + d^2 + z^2} + 8a^2 d^5 \sqrt{a^2 + d^2 + z^2} + 16d^7 \sqrt{a^2 + d^2 + z^2} + 15a^4 d z^2 \sqrt{a^2 + d^2 + z^2} - 4a^2 d^3 z^2 \sqrt{a^2 + d^2 + z^2} + 24d^5 z^2 \sqrt{a^2 + d^2 + z^2} - 3a^2 d z^4 \sqrt{a^2 + d^2 + z^2} - 2d^3 z^4 \sqrt{a^2 + d^2 + z^2} - 3d z^6 \sqrt{a^2 + d^2 + z^2} - 9b^6 d \sqrt{b^2 + d^2 + z^2} + 6b^4 d^3 \sqrt{b^2 + d^2 + z^2} - 8b^2 d^5 \sqrt{b^2 + d^2 + z^2} - 16d^7 \sqrt{b^2 + d^2 + z^2} - 15b^4 d z^2 \sqrt{b^2 + d^2 + z^2} - 4b^2 d^3 z^2 \sqrt{b^2 + d^2 + z^2} - 24d^5 z^2 \sqrt{b^2 + d^2 + z^2} - 3b^2 d z^4 \sqrt{b^2 + d^2 + z^2} - 2d^3 z^4 \sqrt{b^2 + d^2 + z^2} + 3d z^6 \sqrt{b^2 + d^2 + z^2} - 3(3a^2 - z^2)(a^2 + z^2)^3 \log [c + \sqrt{a^2 + c^2 + z^2}] - 9b^8 \log [c + \sqrt{b^2 + c^2 + z^2}] - 24b^6 z^2 \log [c + \sqrt{b^2 + c^2 + z^2}] - 18b^4 z^4 \log [c + \sqrt{b^2 + c^2 + z^2}] + 3z^8 \log [c + \sqrt{b^2 + c^2 + z^2}] - 3(3a^2 - z^2)(a^2 + z^2)^3 \log [d + \sqrt{a^2 + d^2 + z^2}] + 9b^8 \log [d + \sqrt{b^2 + d^2 + z^2}] + 24b^6 z^2 \log [d + \sqrt{b^2 + d^2 + z^2}] + 18b^4 z^4 \log [d + \sqrt{b^2 + d^2 + z^2}] - 3z^8 \log [d + \sqrt{b^2 + d^2 + z^2}] \right) \]
\[ I_{4r,a} = -\frac{1}{105} z^2 \left[ \arctan \left( c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, -a z \right) + \arctan \left( b^2 + c^2 + z^2, b z \right) + \frac{1}{2520} \right] \]

\[ = \frac{1}{105} z^2 \left[ \arctan \left( a^2 + c^2 + z^2, b^2 + c^2 + z^2 \right) + \arctan \left( d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, -d z \right) \right] \]

\[ = \frac{1}{105} \left[ 70 a^5 c^8 \sqrt{a^2 + c^2 + z^2} - 165 a^5 c^7 z \sqrt{a^2 + c^2 + z^2} + 70 a^3 c^5 \sqrt{a^2 + c^2 + z^2} - 105 a^7 c \sqrt{a^2 + d^2 + z^2} + 105 a^7 c \sqrt{a^2 + d^2 + z^2} \right] \]

\[ = \frac{1}{105} \left[ 70 a^5 c^8 \sqrt{a^2 + c^2 + z^2} - 165 a^5 c^7 z \sqrt{a^2 + c^2 + z^2} + 70 a^3 c^5 \sqrt{a^2 + c^2 + z^2} - 105 a^7 c \sqrt{a^2 + d^2 + z^2} + 105 a^7 c \sqrt{a^2 + d^2 + z^2} \right] \]

\[ = \frac{1}{105} \left[ 70 a^5 c^8 \sqrt{a^2 + c^2 + z^2} - 165 a^5 c^7 z \sqrt{a^2 + c^2 + z^2} + 70 a^3 c^5 \sqrt{a^2 + c^2 + z^2} - 105 a^7 c \sqrt{a^2 + d^2 + z^2} + 105 a^7 c \sqrt{a^2 + d^2 + z^2} \right] \]

\[ = \frac{1}{105} \left[ 70 a^5 c^8 \sqrt{a^2 + c^2 + z^2} - 165 a^5 c^7 z \sqrt{a^2 + c^2 + z^2} + 70 a^3 c^5 \sqrt{a^2 + c^2 + z^2} - 105 a^7 c \sqrt{a^2 + d^2 + z^2} + 105 a^7 c \sqrt{a^2 + d^2 + z^2} \right] \]

\[ = \frac{1}{105} \left[ 70 a^5 c^8 \sqrt{a^2 + c^2 + z^2} - 165 a^5 c^7 z \sqrt{a^2 + c^2 + z^2} + 70 a^3 c^5 \sqrt{a^2 + c^2 + z^2} - 105 a^7 c \sqrt{a^2 + d^2 + z^2} + 105 a^7 c \sqrt{a^2 + d^2 + z^2} \right] \]

\[ = \frac{1}{105} \left[ 70 a^5 c^8 \sqrt{a^2 + c^2 + z^2} - 165 a^5 c^7 z \sqrt{a^2 + c^2 + z^2} + 70 a^3 c^5 \sqrt{a^2 + c^2 + z^2} - 105 a^7 c \sqrt{a^2 + d^2 + z^2} + 105 a^7 c \sqrt{a^2 + d^2 + z^2} \right] \]

\[ = \frac{1}{105} \left[ 70 a^5 c^8 \sqrt{a^2 + c^2 + z^2} - 165 a^5 c^7 z \sqrt{a^2 + c^2 + z^2} + 70 a^3 c^5 \sqrt{a^2 + c^2 + z^2} - 105 a^7 c \sqrt{a^2 + d^2 + z^2} + 105 a^7 c \sqrt{a^2 + d^2 + z^2} \right] \]

\[ = \frac{1}{105} \left[ 70 a^5 c^8 \sqrt{a^2 + c^2 + z^2} - 165 a^5 c^7 z \sqrt{a^2 + c^2 + z^2} + 70 a^3 c^5 \sqrt{a^2 + c^2 + z^2} - 105 a^7 c \sqrt{a^2 + d^2 + z^2} + 105 a^7 c \sqrt{a^2 + d^2 + z^2} \right] \]

\[ = \frac{1}{105} \left[ 70 a^5 c^8 \sqrt{a^2 + c^2 + z^2} - 165 a^5 c^7 z \sqrt{a^2 + c^2 + z^2} + 70 a^3 c^5 \sqrt{a^2 + c^2 + z^2} - 105 a^7 c \sqrt{a^2 + d^2 + z^2} + 105 a^7 c \sqrt{a^2 + d^2 + z^2} \right] \]

\[ = \frac{1}{105} \left[ 70 a^5 c^8 \sqrt{a^2 + c^2 + z^2} - 165 a^5 c^7 z \sqrt{a^2 + c^2 + z^2} + 70 a^3 c^5 \sqrt{a^2 + c^2 + z^2} - 105 a^7 c \sqrt{a^2 + d^2 + z^2} + 105 a^7 c \sqrt{a^2 + d^2 + z^2} \right] \]

\[ = \frac{1}{105} \left[ 70 a^5 c^8 \sqrt{a^2 + c^2 + z^2} - 165 a^5 c^7 z \sqrt{a^2 + c^2 + z^2} + 70 a^3 c^5 \sqrt{a^2 + c^2 + z^2} - 105 a^7 c \sqrt{a^2 + d^2 + z^2} + 105 a^7 c \sqrt{a^2 + d^2 + z^2} \right] \]

\[ = \frac{1}{105} \left[ 70 a^5 c^8 \sqrt{a^2 + c^2 + z^2} - 165 a^5 c^7 z \sqrt{a^2 + c^2 + z^2} + 70 a^3 c^5 \sqrt{a^2 + c^2 + z^2} - 105 a^7 c \sqrt{a^2 + d^2 + z^2} + 105 a^7 c \sqrt{a^2 + d^2 + z^2} \right] \]
\[ I'_{54,a} = \frac{1}{2400} \left( -90a^8 c \sqrt{a^2 + c^2 + z^2} + 60a^6 c^3 \sqrt{a^2 + c^2 + z^2} + 48a^4 c^5 \sqrt{a^2 + c^2 + z^2} - \\
64a^2 c^7 \sqrt{a^2 + c^2 + z^2} + 128c^9 \sqrt{a^2 + c^2 + z^2} - 135a^6 c^2 \sqrt{a^2 + c^2 + z^2} + \\
30a^4 c^3 z^2 \sqrt{a^2 + c^2 + z^2} - 104a^2 c^5 z^2 \sqrt{a^2 + c^2 + z^2} + 336c^7 z^2 \sqrt{a^2 + c^2 + z^2} - \\
15a^2 c^7 \sqrt{a^2 + c^2 + z^2} + 10c^3 z^4 \sqrt{a^2 + c^2 + z^2} - 15c^8 z^4 \sqrt{a^2 + c^2 + z^2} + \\
90b^8 c \sqrt{b^2 + c^2 + z^2} - 60b^6 c^3 \sqrt{b^2 + c^2 + z^2} - 48b^4 c^5 \sqrt{b^2 + c^2 + z^2} + \\
64b^2 c^7 \sqrt{b^2 + c^2 + z^2} - 128c^9 \sqrt{b^2 + c^2 + z^2} + 135b^6 c^2 \sqrt{b^2 + c^2 + z^2} - \\
30b^4 c^3 z^2 \sqrt{b^2 + c^2 + z^2} + 104b^2 c^5 z^2 \sqrt{b^2 + c^2 + z^2} - 336c^7 z^2 \sqrt{b^2 + c^2 + z^2} + \\
15b^4 c^7 \sqrt{b^2 + c^2 + z^2} + 20b^2 c^3 z^4 \sqrt{b^2 + c^2 + z^2} - 248c^5 z^4 \sqrt{b^2 + c^2 + z^2} - \\
15b^2 c^7 \sqrt{b^2 + c^2 + z^2} - 10c^3 z^4 \sqrt{b^2 + c^2 + z^2} + 15cz^8 \sqrt{b^2 + c^2 + z^2} + \\
90a^8 d \sqrt{a^2 + d^2 + z^2} - 60a^6 d^3 \sqrt{a^2 + d^2 + z^2} - 48a^4 d^5 \sqrt{a^2 + d^2 + z^2} + \\
64a^2 d^7 \sqrt{a^2 + d^2 + z^2} - 128d^9 \sqrt{a^2 + d^2 + z^2} + 135a^6 d z^2 \sqrt{a^2 + d^2 + z^2} - \\
30a^4 d^3 z^2 \sqrt{a^2 + d^2 + z^2} + 104a^2 d^5 z^2 \sqrt{a^2 + d^2 + z^2} - 336d^7 z^2 \sqrt{a^2 + d^2 + z^2} + \\
15a^2 d^7 \sqrt{a^2 + d^2 + z^2} + 20a^2 d^3 z^4 \sqrt{a^2 + d^2 + z^2} - 248d^5 z^4 \sqrt{a^2 + d^2 + z^2} - \\
15a^2 d^7 \sqrt{a^2 + d^2 + z^2} - 10d^3 z^6 \sqrt{a^2 + d^2 + z^2} + 15dz^8 \sqrt{a^2 + d^2 + z^2} - \\
90b^8 d \sqrt{b^2 + d^2 + z^2} + 60b^6 d^3 \sqrt{b^2 + d^2 + z^2} + 48b^4 d^5 \sqrt{b^2 + d^2 + z^2} - \\
64b^2 d^7 \sqrt{b^2 + d^2 + z^2} + 128d^9 \sqrt{b^2 + d^2 + z^2} - 135b^6 dz^2 \sqrt{b^2 + d^2 + z^2} + \\
30b^4 d^3 z^2 \sqrt{b^2 + d^2 + z^2} - 104b^2 d^5 z^2 \sqrt{b^2 + d^2 + z^2} + \\
336d^7 z^2 \sqrt{b^2 + d^2 + z^2} - 15b^4 d^3 z^4 \sqrt{b^2 + d^2 + z^2} - 20b^2 d^3 z^4 \sqrt{b^2 + d^2 + z^2} + \\
248d^5 z^4 \sqrt{b^2 + d^2 + z^2} + 15b^2 d^5 z^2 \sqrt{b^2 + d^2 + z^2} + 10d^3 z^6 \sqrt{b^2 + d^2 + z^2} - \\
15dz^8 \sqrt{b^2 + d^2 + z^2} + 15(a^2 + z^2)^3 (6a^4 - 3a^2 z^2 + z^4) \log(c + \sqrt{a^2 + c^2 + z^2}) - \\
15(b^2 + z^2)^3 (6b^4 - 3b^2 z^2 + z^4) \log(c + \sqrt{b^2 + c^2 + z^2}) - \\
90a^{10} \log(d + \sqrt{a^2 + d^2 + z^2}) - 225a^8 z^2 \log(d + \sqrt{a^2 + d^2 + z^2}) - \\
150a^6 z^4 \log(d + \sqrt{a^2 + d^2 + z^2}) - 15z^{10} \log(d + \sqrt{a^2 + d^2 + z^2}) + \\
90b^{10} \log(d + \sqrt{b^2 + d^2 + z^2}) + 225b^8 z^2 \log(d + \sqrt{b^2 + d^2 + z^2}) + \\
150b^6 z^4 \log(d + \sqrt{b^2 + d^2 + z^2}) + 15z^{10} \log(d + \sqrt{b^2 + d^2 + z^2}) \right) \]
\[ I_{05,a} = \frac{1}{90} \left( 8 a^5 \sqrt{a^2 + c^2 + z^2} - 4 a^3 c^2 \sqrt{a^2 + c^2 + z^2} + 3 a c^4 \sqrt{a^2 + c^2 + z^2} + \\ 26 a^3 z^2 \sqrt{a^2 + c^2 + z^2} - 9 a c^2 z^2 \sqrt{a^2 + c^2 + z^2} + 33 a z^4 \sqrt{a^2 + c^2 + z^2} - \\ 8 b^5 \sqrt{b^2 + c^2 + z^2} + 4 b^3 c^2 \sqrt{b^2 + c^2 + z^2} - 3 b c^4 \sqrt{b^2 + c^2 + z^2} - \\ 26 b^3 z^2 \sqrt{b^2 + c^2 + z^2} + 9 b c^2 z^2 \sqrt{b^2 + c^2 + z^2} - 33 b z^4 \sqrt{b^2 + c^2 + z^2} - \\ 8 a^3 z^2 \sqrt{a^2 + d^2 + z^2} + 4 a^3 d^2 \sqrt{a^2 + d^2 + z^2} - 3 a d^4 \sqrt{a^2 + d^2 + z^2} - \\ 26 a^3 z^2 \sqrt{a^2 + d^2 + z^2} + 9 a d^2 z^2 \sqrt{a^2 + d^2 + z^2} - 33 a z^4 \sqrt{a^2 + d^2 + z^2} + \\ 8 b^5 \sqrt{b^2 + d^2 + z^2} - 4 b^3 d^2 \sqrt{b^2 + d^2 + z^2} + 3 b d^4 \sqrt{b^2 + d^2 + z^2} + \\ 26 b^3 z^2 \sqrt{b^2 + d^2 + z^2} - 9 b d^2 z^2 \sqrt{b^2 + d^2 + z^2} + 33 b z^4 \sqrt{b^2 + d^2 + z^2} + \\ 15 (c^6 + z^6) \log [a + \sqrt{a^2 + c^2 + z^2}] - 15 c^6 \log [b + \sqrt{b^2 + c^2 + z^2}] - \\ 15 z^6 \log [b + \sqrt{b^2 + c^2 + z^2}] - 15 (d^6 + z^6) \log [a + \sqrt{a^2 + d^2 + z^2}] + \\ 15 d^6 \log [b + \sqrt{b^2 + d^2 + z^2}] + 15 z^6 \log [b + \sqrt{b^2 + d^2 + z^2}] \right) \]
$$I_{15,n} = \frac{1}{105} \left( 8z^6 \sqrt{a^2 + c^2 + z^2} - 8b^6 \sqrt{b^2 + c^2 + z^2} - \\
24b^4z^2 \sqrt{b^2 + c^2 + z^2} - 24b^2z^4 \sqrt{b^2 + c^2 + z^2} - 8z^6 \sqrt{b^2 + c^2 + z^2} - \\
15d^6 \sqrt{a^2 + d^2 + z^2} - 3d^4z^2 \sqrt{a^2 + d^2 + z^2} + 4d^2z^4 \sqrt{a^2 + d^2 + z^2} - \\
8z^6 \sqrt{a^2 + d^2 + z^2} + 8b^4 \sqrt{b^2 + d^2 + z^2} - 4b^4d^2 \sqrt{b^2 + d^2 + z^2} - \\
3b^2d^4 \sqrt{b^2 + d^2 + z^2} + 15d^6 \sqrt{b^2 + d^2 + z^2} + 24b^4z^2 \sqrt{b^2 + d^2 + z^2} - \\
8b^2d^2z^2 \sqrt{b^2 + d^2 + z^2} + 3d^4z^2 \sqrt{b^2 + d^2 + z^2} + \\
24b^2z^4 \sqrt{b^2 + d^2 + z^2} - 4d^2z^4 \sqrt{b^2 + d^2 + z^2} + 8z^6 \sqrt{b^2 + d^2 + z^2} - \\
15c^6 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) + 8a^6 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) + \\
3c^4 \left( a^2 \sqrt{a^2 + c^2 + z^2} - b^2 \sqrt{b^2 + c^2 + z^2} + z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) \right) - \\
4c^2 \left( a^2 \sqrt{a^2 + c^2 + z^2} + 2a^2z^2 \sqrt{a^2 + c^2 + z^2} - b^4 \sqrt{b^2 + c^2 + z^2} - \\
2b^2z^2 \sqrt{b^2 + c^2 + z^2} + 2a^2z^2 \sqrt{a^2 + c^2 + z^2} - b^4 \sqrt{b^2 + c^2 + z^2} \right) + \\
4a^4 \left( d^2 \sqrt{a^2 + d^2 + z^2} + 6z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) + \\
a^2 \left( -3d^4 \sqrt{a^2 + d^2 + z^2} + 8d^2z^2 \sqrt{a^2 + d^2 + z^2} + \\
24z^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) \right)$$
$$I_{25,a} = \frac{1}{720} \left( 48a^7\sqrt{a^2 + c^2 + z^2} - 24a^5c^2\sqrt{a^2 + c^2 + z^2} + \\
18a^3c^4\sqrt{a^2 + c^2 + z^2} + 45ac^6\sqrt{a^2 + c^2 + z^2} + 136a^2z^3\sqrt{a^2 + c^2 + z^2} - \\
44a^3c^2z^2\sqrt{a^2 + c^2 + z^2} + 15ac^4z^2\sqrt{a^2 + c^2 + z^2} + 118a^3z^4\sqrt{a^2 + c^2 + z^2} - \\
15a^2z^4\sqrt{a^2 + c^2 + z^2} + 15az^6\sqrt{a^2 + c^2 + z^2} - 48b^7\sqrt{b^2 + c^2 + z^2} + \\
24b^5c^2\sqrt{b^2 + c^2 + z^2} - 18b^3c^4\sqrt{b^2 + c^2 + z^2} - 45bc^6\sqrt{b^2 + c^2 + z^2} - \\
136b^5z^2\sqrt{b^2 + c^2 + z^2} + 44b^3c^2z^2\sqrt{b^2 + c^2 + z^2} - 15b^4c^2z^2\sqrt{b^2 + c^2 + z^2} - \\
118b^3z^4\sqrt{b^2 + c^2 + z^2} + 15b^2c^4z^2\sqrt{b^2 + c^2 + z^2} - 15b^2z^6\sqrt{b^2 + c^2 + z^2} - \\
48a^7\sqrt{a^2 + d^2 + z^2} + 24a^5d^2\sqrt{a^2 + d^2 + z^2} - 18a^3d^4\sqrt{a^2 + d^2 + z^2} - \\
45ad^6\sqrt{a^2 + d^2 + z^2} - 136a^5z^2\sqrt{a^2 + d^2 + z^2} + 44a^3d^2z^2\sqrt{a^2 + d^2 + z^2} - \\
15adb^2z^2\sqrt{a^2 + d^2 + z^2} - 118a^3z^4\sqrt{a^2 + d^2 + z^2} + 15ad^2z^4\sqrt{a^2 + d^2 + z^2} - \\
15az^6\sqrt{a^2 + d^2 + z^2} + 48b^7\sqrt{b^2 + d^2 + z^2} - 24b^5d^2\sqrt{b^2 + d^2 + z^2} - \\
18b^3d^4\sqrt{b^2 + d^2 + z^2} + 45bd^6\sqrt{b^2 + d^2 + z^2} + 136b^5z^2\sqrt{b^2 + d^2 + z^2} - \\
44b^3d^2z^2\sqrt{b^2 + d^2 + z^2} + 15bd^4z^2\sqrt{b^2 + d^2 + z^2} + \\
118b^3z^4\sqrt{b^2 + d^2 + z^2} - 15bd^2z^4\sqrt{b^2 + d^2 + z^2} + 15b^2z^6\sqrt{b^2 + d^2 + z^2} - \\
15(3c^8 + 4c^6z^2 + z^8)\log\left[a + \sqrt{a^2 + c^2 + z^2}\right] + 45c^8\log\left[b + \sqrt{b^2 + c^2 + z^2}\right] + \\
60c^6z^2\log\left[b + \sqrt{b^2 + c^2 + z^2}\right] + 15z^8\log\left[b + \sqrt{b^2 + c^2 + z^2}\right] + \\
15(3d^8 + 4d^6z^2 + z^8)\log\left[a + \sqrt{a^2 + d^2 + z^2}\right] - 45d^8\log\left[b + \sqrt{b^2 + d^2 + z^2}\right] - \\
60d^6z^2\log\left[b + \sqrt{b^2 + d^2 + z^2}\right] - 15z^8\log\left[b + \sqrt{b^2 + d^2 + z^2}\right] \right)
I'_{35,a} = 
\frac{1}{945} \left( -16 z^8 \sqrt{a^2 + c^2 + z^2} - 56 b^8 \sqrt{b^2 + c^2 + z^2} - 
152 b^6 z^2 \sqrt{b^2 + c^2 + z^2} - 120 b^4 z^4 \sqrt{b^2 + c^2 + z^2} - 8 b^2 z^6 \sqrt{b^2 + c^2 + z^2} + 
16 z^8 \sqrt{b^2 + c^2 + z^2} + 70 d^8 \sqrt{a^2 + d^2 + z^2} + 100 d^6 z^2 \sqrt{a^2 + d^2 + z^2} + 
6 d^4 z^4 \sqrt{a^2 + d^2 + z^2} - 8 d^2 z^6 \sqrt{a^2 + d^2 + z^2} + 16 z^8 \sqrt{a^2 + d^2 + z^2} + 
56 b^8 \sqrt{b^2 + d^2 + z^2} - 28 b^6 d^2 \sqrt{b^2 + d^2 + z^2} + 21 b^4 d^4 \sqrt{b^2 + d^2 + z^2} + 
35 b^2 d^6 \sqrt{b^2 + d^2 + z^2} - 70 d^8 \sqrt{b^2 + d^2 + z^2} + 152 b^6 z^2 \sqrt{b^2 + d^2 + z^2} - 
48 b^4 d^2 z^2 \sqrt{b^2 + d^2 + z^2} + 15 b^2 d^4 z^2 \sqrt{b^2 + d^2 + z^2} - 
100 d^6 z^2 \sqrt{b^2 + d^2 + z^2} + 120 b^4 z^4 \sqrt{b^2 + d^2 + z^2} - 12 b^2 d^2 z^2 \sqrt{b^2 + d^2 + z^2} - 
6 d^4 z^4 \sqrt{b^2 + d^2 + z^2} + 8 b^2 z^6 \sqrt{b^2 + d^2 + z^2} + 8 d^2 z^6 \sqrt{b^2 + d^2 + z^2} - 
16 z^8 \sqrt{b^2 + d^2 + z^2} - 70 c^8 (\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2}) + 
56 a^8 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2}) + 5 c^6 (7 a^2 \sqrt{a^2 + c^2 + z^2} - 
7 b^2 \sqrt{b^2 + c^2 + z^2} - 20 z^2 (\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2})) + 
3 c^4 (7 a^4 \sqrt{a^2 + c^2 + z^2} + 5 a^2 z^2 \sqrt{a^2 + c^2 + z^2} - 7 b^4 \sqrt{b^2 + c^2 + z^2} - 
5 b^2 z^2 \sqrt{b^2 + c^2 + z^2} - 2 z^4 (\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2})) - 
4 c^2 (7 a^6 \sqrt{a^2 + c^2 + z^2} + 12 a^4 z^2 \sqrt{a^2 + c^2 + z^2} + 3 a^2 z^4 \sqrt{a^2 + c^2 + z^2} - 
7 b^6 \sqrt{b^2 + c^2 + z^2} - 12 b^4 z^2 \sqrt{b^2 + c^2 + z^2} - 
3 b^2 z^4 \sqrt{b^2 + c^2 + z^2} - 2 z^6 (\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2})) + 
4 a^6 (7 d^2 \sqrt{a^2 + d^2 + z^2} + 38 z^2 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2})) + 
3 a^4 (-7 d^4 \sqrt{a^2 + d^2 + z^2} + 16 d^2 z^2 \sqrt{a^2 + d^2 + z^2} + 
40 z^4 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2}) + 
3 a^2 (-35 d^6 \sqrt{a^2 + d^2 + z^2} - 15 d^4 z^2 \sqrt{a^2 + d^2 + z^2} + 12 d^2 z^4 \sqrt{a^2 + d^2 + z^2} + 
8 z^6 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2}))
\right)
\[
I_{45,a} = \frac{1}{2400}
\left(128a^9 \sqrt{a^2 + c^2 + z^2} - 64a^7 c^2 \sqrt{a^2 + c^2 + z^2} + 48a^5 c^4 \sqrt{a^2 + c^2 + z^2} +
60a^3 c^6 \sqrt{a^2 + c^2 + z^2} - 90a^8 c^2 \sqrt{a^2 + c^2 + z^2} + 336a^7 z^2 \sqrt{a^2 + c^2 + z^2} -
104a^5 c^2 z^2 \sqrt{a^2 + c^2 + z^2} + 30a^3 c^4 z^2 \sqrt{a^2 + c^2 + z^2} - 135a^6 z^2 \sqrt{a^2 + c^2 + z^2} -
248a^6 z^2 \sqrt{a^2 + c^2 + z^2} - 20a^3 c^2 z^4 \sqrt{a^2 + c^2 + z^2} = 15ac^2 z^4 \sqrt{a^2 + c^2 + z^2} +
10a^3 z^6 \sqrt{a^2 + c^2 + z^2} + 15ac^2 z^6 \sqrt{a^2 + c^2 + z^2} - 15az^8 \sqrt{a^2 + c^2 + z^2} -
128b^7 \sqrt{b^2 + c^2 + z^2} + 64b^7 c^2 \sqrt{b^2 + c^2 + z^2} - 48b^5 c^4 \sqrt{b^2 + c^2 + z^2} -
60b^3 c^6 \sqrt{b^2 + c^2 + z^2} + 90bc^8 \sqrt{b^2 + c^2 + z^2} - 336b^7 z^2 \sqrt{b^2 + c^2 + z^2} +
104b^5 c^2 z^2 \sqrt{b^2 + c^2 + z^2} - 30b^3 c^4 z^2 \sqrt{b^2 + c^2 + z^2} + 135bc^6 z^2 \sqrt{b^2 + c^2 + z^2} -
248b^5 z^4 \sqrt{b^2 + c^2 + z^2} + 20b^3 c^2 z^4 \sqrt{b^2 + c^2 + z^2} + 15bc^4 z^4 \sqrt{b^2 + c^2 + z^2} -
10b^3 z^6 \sqrt{b^2 + c^2 + z^2} - 15bc^2 z^6 \sqrt{b^2 + c^2 + z^2} + 15b^2 z^8 \sqrt{b^2 + c^2 + z^2} -
128a^9 \sqrt{a^2 + d^2 + z^2} + 64a^7 d^2 \sqrt{a^2 + d^2 + z^2} - 48a^5 d^4 \sqrt{a^2 + d^2 + z^2} -
60a^3 d^6 \sqrt{a^2 + d^2 + z^2} + 90ad^8 \sqrt{a^2 + d^2 + z^2} - 336a^7 z^2 \sqrt{a^2 + d^2 + z^2} +
104a^5 d^2 z^2 \sqrt{a^2 + d^2 + z^2} - 30a^3 d^4 z^2 \sqrt{a^2 + d^2 + z^2} + 135ad^6 z^2 \sqrt{a^2 + d^2 + z^2} -
248a^5 z^4 \sqrt{a^2 + d^2 + z^2} + 20a^3 d^2 z^4 \sqrt{a^2 + d^2 + z^2} + 15ad^4 z^4 \sqrt{a^2 + d^2 + z^2} -
10a^3 z^6 \sqrt{a^2 + d^2 + z^2} - 15ad^2 z^6 \sqrt{a^2 + d^2 + z^2} + 15az^8 \sqrt{a^2 + d^2 + z^2} +
128b^9 \sqrt{b^2 + d^2 + z^2} - 64b^7 d^2 \sqrt{b^2 + d^2 + z^2} + 48b^5 d^4 \sqrt{b^2 + d^2 + z^2} +
60b^3 d^6 \sqrt{b^2 + d^2 + z^2} - 90bd^8 \sqrt{b^2 + d^2 + z^2} - 336b^7 z^2 \sqrt{b^2 + d^2 + z^2} -
104b^5 d^2 z^2 \sqrt{b^2 + d^2 + z^2} + 30b^3 d^4 z^2 \sqrt{b^2 + d^2 + z^2} -
135bd^6 z^2 \sqrt{b^2 + d^2 + z^2} + 248bd^5 z^4 \sqrt{b^2 + d^2 + z^2} - 20bd^3 z^4 \sqrt{b^2 + d^2 + z^2} -
15bd^4 z^4 \sqrt{b^2 + d^2 + z^2} + 10b^3 d^2 z^6 \sqrt{b^2 + d^2 + z^2} + 15bd^2 z^6 \sqrt{b^2 + d^2 + z^2} -
15bz^8 \sqrt{b^2 + d^2 + z^2} + 15(c^2 + z^2)^3 (6c^4 - 3c^2 z^2 + z^4) \log[a + \sqrt{a^2 + c^2 + z^2}] -
90c^{10} \log[b + \sqrt{b^2 + c^2 + z^2}] - 225c^8 z^2 \log[b + \sqrt{b^2 + c^2 + z^2}] -
150c^6 z^4 \log[b + \sqrt{b^2 + c^2 + z^2}] - 15z^{10} \log[b + \sqrt{b^2 + c^2 + z^2}] -
15(d^2 + z^2)^3 (6d^4 - 3d^2 z^2 + z^4) \log[a + \sqrt{a^2 + d^2 + z^2}] +
90d^{10} \log[b + \sqrt{b^2 + d^2 + z^2}] + 225d^8 z^2 \log[b + \sqrt{b^2 + d^2 + z^2}] +
150d^6 z^4 \log[b + \sqrt{b^2 + d^2 + z^2}] + 15z^{10} \log[b + \sqrt{b^2 + d^2 + z^2}] +
\right).
\]

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\[ I_{ss,n} = \frac{1}{10395} \left( 64 z^{10} \sqrt{a^2 + c^2 + z^2} - 504 b^{10} \sqrt{b^2 + c^2 + z^2} - 1288 b^8 z^2 \sqrt{b^2 + c^2 + z^2} - 904 b^6 z^4 \sqrt{b^2 + c^2 + z^2} - 24 b^4 z^6 \sqrt{b^2 + c^2 + z^2} + 32 b^2 z^8 \sqrt{b^2 + c^2 + z^2} - 64 z^{10} \sqrt{a^2 + d^2 + z^2} - 504 d^{10} \sqrt{a^2 + d^2 + z^2} - 1288 d^8 z^2 \sqrt{a^2 + d^2 + z^2} - 904 d^6 z^4 \sqrt{a^2 + d^2 + z^2} - 24 d^4 z^6 \sqrt{a^2 + d^2 + z^2} + 32 d^2 z^8 \sqrt{a^2 + d^2 + z^2} - 64 z^{10} \sqrt{a^2 + d^2 + z^2} + 504 b^{10} \sqrt{b^2 + d^2 + z^2} - 252 b^8 d^2 \sqrt{b^2 + d^2 + z^2} + 189 b^6 d^4 \sqrt{b^2 + d^2 + z^2} + 189 b^4 d^6 \sqrt{b^2 + d^2 + z^2} - 252 b^2 d^8 \sqrt{b^2 + d^2 + z^2} + 504 d^{10} \sqrt{b^2 + d^2 + z^2} + 1288 b^8 z^2 \sqrt{b^2 + d^2 + z^2} - 392 b^6 d^2 z^2 \sqrt{b^2 + d^2 + z^2} + 105 b^4 d^4 z^2 \sqrt{b^2 + d^2 + z^2} - 392 b^2 d^6 z^2 \sqrt{b^2 + d^2 + z^2} + 1288 d^8 z^2 \sqrt{b^2 + d^2 + z^2} - 904 b^6 z^4 \sqrt{b^2 + d^2 + z^2} - 60 b^4 d^2 z^4 \sqrt{b^2 + d^2 + z^2} - 60 b^2 d^4 z^4 \sqrt{b^2 + d^2 + z^2} + 904 d^6 z^4 \sqrt{b^2 + d^2 + z^2} + 24 b^4 z^6 \sqrt{b^2 + d^2 + z^2} + 48 b^2 d^2 z^6 \sqrt{b^2 + d^2 + z^2} + 24 d^4 z^6 \sqrt{b^2 + d^2 + z^2} - 32 b^2 d^8 \sqrt{b^2 + d^2 + z^2} - 32 d^2 z^8 \sqrt{b^2 + d^2 + z^2} + 64 z^{10} \sqrt{b^2 + d^2 + z^2} + 504 c^{10} (\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2}) + 504 a^{10} (\sqrt{a^2 + d^2 + z^2} - \sqrt{a^2 + d^2 + z^2}) + 28 a^8 (9 b^2 \sqrt{b^2 + c^2 + z^2} + 46 z^2 \sqrt{a^2 + c^2 + z^2} + 2b^2 + c^2 + z^2) + 904 z^4 (\sqrt{a^2 + d^2 + z^2} - \sqrt{b^2 + c^2 + z^2}) + \]

\[ 3 c^4 (18 b^2 \sqrt{b^2 + c^2 + z^2} - 35 b^2 z^2 \sqrt{b^2 + c^2 + z^2} + 20 b^2 z^2 \sqrt{a^2 + c^2 + z^2} + 8 z^6 (\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2})) + c^2 (252 b^8 \sqrt{b^2 + c^2 + z^2} + 392 b^6 z^2 \sqrt{b^2 + c^2 + z^2} + 60 b^4 z^4 \sqrt{b^2 + c^2 + z^2} + 48 b^2 z^6 \sqrt{b^2 + c^2 + z^2} - 32 z^8 (\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2})) - 28 a^8 (9 c^2 \sqrt{a^2 + c^2 + z^2} - 9 d^2 \sqrt{a^2 + d^2 + z^2} - 46 z^2 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2})) + a^6 (189 c^4 \sqrt{a^2 + c^2 + z^2} - 392 c^2 z^2 \sqrt{a^2 + c^2 + z^2} - 189 d^4 \sqrt{a^2 + d^2 + z^2} + 3 a^4 (63 c^6 \sqrt{a^2 + c^2 + z^2} + 20 c^2 z^2 \sqrt{a^2 + c^2 + z^2} - 63 d^6 \sqrt{a^2 + d^2 + z^2} - 35 d^2 z^2 \sqrt{a^2 + d^2 + z^2} + 20 d^2 z^2 \sqrt{a^2 + d^2 + z^2} + 8 z^6 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2})) - 4 a^2 (63 c^8 \sqrt{a^2 + c^2 + z^2} + 98 c^6 z^2 \sqrt{a^2 + c^2 + z^2} + 15 c^4 z^4 \sqrt{a^2 + c^2 + z^2} + 12 c^2 z^6 \sqrt{a^2 + d^2 + z^2} - 63 d^8 \sqrt{a^2 + d^2 + z^2} - 98 d^6 z^2 \sqrt{a^2 + d^2 + z^2} - 15 d^4 z^4 \sqrt{a^2 + d^2 + z^2} + 12 d^2 z^6 \sqrt{a^2 + d^2 + z^2} + 8 z^6 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2})))) \]
III.2 Analytical Formulae for $K'_{x_{ij},n}$, $i,j=0,1,...,5$

\[ K'_{x_{100},n} = \]
\[ \quad - \log \left[ c + \sqrt{a^2 + c^2 + z^2} \right] + \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] + \]
\[ \quad \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] - \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] \]

\[ K'_{x_{110},n} = \]
\[ \quad -z \left( \arctan \left[ a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, -cz \right] + \arctan \left[ b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, cz \right] + \right. \]
\[ \quad \left. \arctan \left[ a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2}, dz \right] + \arctan \left[ b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, -d \right] \right) + \]
\[ \quad c \log \frac{a + \sqrt{a^2 + c^2 + z^2}}{b + \sqrt{b^2 + c^2 + z^2}} + d \log \frac{b + \sqrt{b^2 + d^2 + z^2}}{a + \sqrt{a^2 + d^2 + z^2}} \]

\[ K'_{x_{200},n} = \]
\[ \quad c \sqrt{a^2 + c^2 + z^2} - c \sqrt{b^2 + c^2 + z^2} - d \sqrt{a^2 + d^2 + z^2} + \]
\[ \quad d \sqrt{b^2 + d^2 + z^2} + z^2 \log \left[ c + \sqrt{a^2 + c^2 + z^2} \right] - z^2 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] - \]
\[ \quad z^2 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] + z^2 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] \]

\[ K'_{x_{300},n} = \]
\[ \quad -z^3 \left( \arctan \left[ a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, cz \right] + \arctan \left[ b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, -cz \right] + \right. \]
\[ \quad \left. \arctan \left[ a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2}, -dz \right] + \arctan \left[ b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, dz \right] \right) + \]
\[ \quad \frac{1}{2} \left( ac \sqrt{a^2 + c^2 + z^2} - bc \sqrt{b^2 + c^2 + z^2} - ad \sqrt{a^2 + d^2 + z^2} + \right. \]
\[ \quad \left. bd \sqrt{b^2 + d^2 + z^2} \right) \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] + \]
\[ \quad c^3 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + 3 cz^2 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + \]
\[ \quad d^3 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 3 dz^2 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - \]
\[ \quad d^3 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - 3 dz^2 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] \]
\[ K'_{x40,n} = \frac{1}{3} \left( a^2 c \sqrt{a^2 + c^2 + z^2} - 2c^3 \sqrt{a^2 + c^2 + z^2} - 5c^2 z^2 \sqrt{a^2 + c^2 + z^2} - b^2 c \sqrt{b^2 + c^2 + z^2} + \\
2c^3 \sqrt{b^2 + c^2 + z^2} + 5c^2 z^2 \sqrt{b^2 + c^2 + z^2} - a^2 d \sqrt{a^2 + d^2 + z^2} + 2d^3 \sqrt{a^2 + d^2 + z^2} + \\
5dz^2 \sqrt{a^2 + d^2 + z^2} + b^2 d \sqrt{b^2 + d^2 + z^2} - 2d^3 \sqrt{b^2 + d^2 + z^2} - \\
5dz^2 \sqrt{b^2 + d^2 + z^2} - 3z^4 \log(c + \sqrt{a^2 + c^2 + z^2}) + 3z^4 \log(c + \sqrt{b^2 + c^2 + z^2}) + \\
3z^4 \log(d + \sqrt{a^2 + d^2 + z^2}) - 3z^4 \log(d + \sqrt{b^2 + d^2 + z^2}) \right) \]

\[ K'_{x50,n} = -z^5 \left( \arctan[c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, -az] + \arctan[c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, bz] + \\
\arctan[d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, az] + \arctan[d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, -bz] \right) + \\
\frac{1}{8} \left( 2a^3 c \sqrt{a^2 + c^2 + z^2} - 3ac^3 \sqrt{a^2 + c^2 + z^2} - 7ac^2 z^2 \sqrt{a^2 + c^2 + z^2} + \\
2b^3 c \sqrt{b^2 + c^2 + z^2} + 3bc^3 \sqrt{b^2 + c^2 + z^2} + 7bcz^2 \sqrt{b^2 + c^2 + z^2} - \\
2a^3 d \sqrt{a^2 + d^2 + z^2} + 3ad^3 \sqrt{a^2 + d^2 + z^2} + 7ad^2 z^2 \sqrt{a^2 + d^2 + z^2} + \\
2b^3 d \sqrt{b^2 + d^2 + z^2} - 3bd^3 \sqrt{b^2 + d^2 + z^2} - 7bdz^2 \sqrt{b^2 + d^2 + z^2} + \\
(3c^5 + 10c^3 z^2 + 15cz^4) \log(a + \sqrt{a^2 + c^2 + z^2}) - 3c^5 \log(b + \sqrt{b^2 + c^2 + z^2}) - \\
10c^3 z^2 \log(b + \sqrt{b^2 + c^2 + z^2}) - 15cz^4 \log(b + \sqrt{b^2 + c^2 + z^2}) - \\
3d^5 \log(a + \sqrt{a^2 + d^2 + z^2}) - 10d^3 z^2 \log(a + \sqrt{a^2 + d^2 + z^2}) - \\
15dz^4 \log(a + \sqrt{a^2 + d^2 + z^2}) + 3d^5 \log(b + \sqrt{b^2 + d^2 + z^2}) + \\
10d^3 z^2 \log(b + \sqrt{b^2 + d^2 + z^2}) + 15dz^4 \log(b + \sqrt{b^2 + d^2 + z^2}) \right) \]
\[ K'_{x01, n} = -\sqrt{a^2 + c^2 + z^2} + \sqrt{b^2 + c^2 + z^2} + \sqrt{a^2 + d^2 + z^2} - \sqrt{b^2 + d^2 + z^2} \]

\[ K'_{x11, n} = \frac{1}{2} \left( -a \sqrt{a^2 + c^2 + z^2} + b \sqrt{b^2 + c^2 + z^2} + a \sqrt{a^2 + d^2 + z^2} - b \sqrt{b^2 + d^2 + z^2} + \right) \]

\[ (c^2 + z^2) \log \left[ \frac{a + \sqrt{a^2 + c^2 + z^2}}{b + \sqrt{b^2 + c^2 + z^2}} \right] + (d^2 + z^2) \log \left[ \frac{b + \sqrt{b^2 + d^2 + z^2}}{a + \sqrt{a^2 + d^2 + z^2}} \right] \]

\[ K'_{x21, n} = \frac{1}{3} \left( 2z^2 \sqrt{a^2 + c^2 + z^2} + b^2 \sqrt{b^2 + c^2 + z^2} - 2z^2 \sqrt{b^2 + c^2 + z^2} - 2d^2 \sqrt{a^2 + d^2 + z^2} - \right) \]

\[ 2z^2 \sqrt{a^2 + d^2 + z^2} - b^2 \sqrt{b^2 + d^2 + z^2} + 2d^2 \sqrt{b^2 + d^2 + z^2} + 2z^2 \sqrt{b^2 + d^2 + z^2} + \]

\[ 2c^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + c^2 + z^2} \right) + a^2 \left( -\sqrt{a^2 + c^2 + z^2} + \sqrt{a^2 + d^2 + z^2} \right) \]
$$K'_{z_{31,n}} = \frac{1}{8} \left( \frac{2a^5}{\sqrt{a^2 + b^2 + z^2}} + \frac{a^3 c^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{3a c^4}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^3 z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{6a c^2 z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{3az^4}{\sqrt{a^2 + c^2 + z^2}} + \frac{2b^5}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^3 c^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{3b c^4}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^3 z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{6bc^2 z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{3b^4 z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{2a^5}{\sqrt{a^2 + d^2 + z^2}} - \frac{a^3 d^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{3a d^4}{\sqrt{a^2 + d^2 + z^2}} - \frac{a^3 z^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{6ad^2 z^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{3az^4}{\sqrt{a^2 + d^2 + z^2}} - \frac{2b^5}{\sqrt{b^2 + d^2 + z^2}} + \frac{b^3 d^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{3b d^4}{\sqrt{b^2 + d^2 + z^2}} + \frac{b^3 z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{6bd^2 z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{3b^4 z^2}{\sqrt{b^2 + d^2 + z^2}} - 3(c^2 + z^2)^2 \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] + 3(c^2 + z^2)^2 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + 3d^4 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 6d^2 z^2 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 3z^4 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - 3d^4 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - 6d^2 z^2 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - 3z^4 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] \right)$$
\[
K'_{x_{41,n}} = \frac{1}{15} \left( -\frac{8z^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{3b^6}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^4z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{4b^2y^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{8y^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{8d^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{24d^4z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{24d^2y^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{8y^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{3b^6}{\sqrt{a^2 + d^2 + z^2}} - \frac{b^4d^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{4b^2d^4}{\sqrt{b^2 + d^2 + z^2}} - \frac{8d^6}{\sqrt{b^2 + d^2 + z^2}} + \frac{b^4y^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{8b^2d^2y^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{24d^4y^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{4b^2y^4}{\sqrt{b^2 + d^2 + z^2}} - \frac{24d^2y^4}{\sqrt{b^2 + d^2 + z^2}} - \frac{8z^6}{\sqrt{b^2 + d^2 + z^2}} + 8e^6 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{b^2 + c^2 + z^2}} \right) \right) + \frac{a^6 \left( \frac{3}{\sqrt{a^2 + c^2 + z^2}} + \frac{3}{\sqrt{a^2 + d^2 + z^2}} \right) + c^2 \left( \frac{b^4}{\sqrt{b^2 + c^2 + z^2}} \right)}{\sqrt{b^2 + c^2 + z^2}} + \frac{8b^2z^2}{\sqrt{b^2 + c^2 + z^2}} + 24z^4 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{b^2 + c^2 + z^2}} \right) + \frac{4c^4 \left( \frac{b^2}{\sqrt{b^2 + c^2 + z^2}} + z^2 \left( \frac{6}{\sqrt{a^2 + c^2 + z^2}} + \frac{6}{\sqrt{b^2 + c^2 + z^2}} \right) \right) }{\sqrt{b^2 + c^2 + z^2}} + \frac{a^4 \left( \frac{c^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{d^2}{\sqrt{a^2 + d^2 + z^2}} + z^2 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} - \frac{1}{\sqrt{a^2 + d^2 + z^2}} \right) \right) }{\sqrt{a^2 + c^2 + z^2}} + \frac{4a^2 \left( \frac{c^4}{\sqrt{a^2 + c^2 + z^2}} - \frac{2c^2z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{d^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{2d^2z^2}{\sqrt{a^2 + d^2 + z^2}} + z^4 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{a^2 + d^2 + z^2}} \right) \right) }{\sqrt{a^2 + d^2 + z^2}} + \frac{256}{\sqrt{a^2 + d^2 + z^2}} \right)
\]
\[ K_{x_{31,n}}' = \frac{1}{48} \left( \frac{8a^7}{\sqrt{a^2 + c^2 + z^2}} + \frac{2a^5c^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{5a^3c^4}{\sqrt{a^2 + c^2 + z^2}} - \frac{15ac^6}{\sqrt{a^2 + c^2 + z^2}} \right) + \frac{2a^5z^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{10a^3c^2z^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{45ac^4z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{5a^3z^4}{\sqrt{a^2 + c^2 + z^2}} \]

\[ - \frac{\sqrt{a^2 + c^2 + z^2}}{2b^5c^2} + \frac{8b^7}{\sqrt{a^2 + c^2 + z^2}} - \frac{2b^5c^2}{\sqrt{a^2 + c^2 + z^2}} \]

\[ + \frac{\sqrt{b^2 + c^2 + z^2}}{5b^3c^4} - \frac{15bc^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{2b^5z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{10b^3c^2z^2}{\sqrt{b^2 + c^2 + z^2}} \]

\[ + \frac{\sqrt{b^2 + c^2 + z^2}}{45b^4c^2z^2} - \frac{5b^3z^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{45b^4c^2z^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{15b^6z^2}{\sqrt{b^2 + c^2 + z^2}} \]

\[ + \frac{\sqrt{a^2 + d^2 + z^2}}{8a^7} - \frac{2a^5d^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{5a^3d^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{15ad^6}{\sqrt{a^2 + d^2 + z^2}} \]

\[ + \frac{2a^5z^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{10a^3d^2z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{45ad^4z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{5a^3d^4z^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{15az^6}{\sqrt{a^2 + d^2 + z^2}} \]

\[ - \frac{8b^7}{\sqrt{a^2 + d^2 + z^2}} + \frac{2b^5d^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{15bd^6}{\sqrt{a^2 + d^2 + z^2}} + \frac{2b^5z^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{10b^3d^2z^2}{\sqrt{a^2 + d^2 + z^2}} \]

\[ + \frac{\sqrt{b^2 + d^2 + z^2}}{5b^3d^4} - \frac{15bd^6}{\sqrt{b^2 + d^2 + z^2}} + \frac{2b^5z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{10b^3d^2z^2}{\sqrt{b^2 + d^2 + z^2}} \]

\[ + \frac{\sqrt{b^2 + d^2 + z^2}}{45bd^4z^2} - \frac{5b^3z^4}{\sqrt{b^2 + d^2 + z^2}} + \frac{45bd^2z^4}{\sqrt{b^2 + d^2 + z^2}} + \frac{15bz^6}{\sqrt{b^2 + d^2 + z^2}} \]

\[ + 15(c^2 + z^2)^3 \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] - 15(c^2 + z^2)^3 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] - 15d^6 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - 45d^4z^2 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - 45d^2z^4 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - 15z^6 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 15d^6 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] + 45d^4z^2 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] + 45d^2z^4 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] + 15z^6 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] \right) \]

257
\[ K'_{x_{12},n} = \]
\[
\frac{1}{2} \left( -c \sqrt{a^2 + c^2 + z^2} + c \sqrt{b^2 + c^2 + z^2} + d \sqrt{a^2 + d^2 + z^2} - d \sqrt{b^2 + d^2 + z^2} +
(a^2 + z^2) \log \left( \frac{c + \sqrt{a^2 + c^2 + z^2}}{d + \sqrt{a^2 + d^2 + z^2}} \right) + (b^2 + z^2) \log \left( \frac{d + \sqrt{b^2 + d^2 + z^2}}{c + \sqrt{b^2 + c^2 + z^2}} \right) \right)
\]

\[ - \frac{1}{3} \left( \text{ArcTan} \left[ a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, cz \right] + \text{ArcTan} \left[ b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, -cz \right] -
\text{ArcTan} \left[ a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2}, -dz \right] + \text{ArcTan} \left[ b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, dz \right] +
\frac{1}{6} \left( -2ac \sqrt{a^2 + c^2 + z^2} + 2bc \sqrt{b^2 + c^2 + z^2} + 2ad \sqrt{a^2 + d^2 + z^2} -
2bd \sqrt{b^2 + d^2 + z^2} + 2c^3 \log \left( a + \sqrt{a^2 + c^2 + z^2} \right) + 2a^3 \log \left( c + \sqrt{a^2 + c^2 + z^2} \right) -
2c^3 \log \left( b + \sqrt{b^2 + c^2 + z^2} \right) - 2b^3 \log \left( c + \sqrt{b^2 + c^2 + z^2} \right) -
2d^3 \log \left( a + \sqrt{a^2 + d^2 + z^2} \right) - 2a^3 \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) +
2d^3 \log \left( b + \sqrt{b^2 + d^2 + z^2} \right) + 2b^3 \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) \right)\]

\[ K'_{x_{22},n} = \]
\[
\frac{1}{4} \left( -a^2 c \sqrt{a^2 + c^2 + z^2} + 2c^3 \sqrt{a^2 + c^2 + z^2} + cz^2 \sqrt{a^2 + c^2 + z^2} + b^2 c \sqrt{b^2 + c^2 + z^2} -
2c^3 \sqrt{b^2 + c^2 + z^2} - cz^2 \sqrt{b^2 + c^2 + z^2} + a^2 d \sqrt{a^2 + d^2 + z^2} -
2d^3 \sqrt{a^2 + d^2 + z^2} - dz^2 \sqrt{a^2 + d^2 + z^2} - b^2 d \sqrt{b^2 + d^2 + z^2} +
2d^3 \sqrt{b^2 + d^2 + z^2} + dz^2 \sqrt{b^2 + d^2 + z^2} + (a^3 - z^4) \log \left( c + \sqrt{a^2 + c^2 + z^2} \right) +
(-b^4 + z^4) \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) - a^4 \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) +
z^4 \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) + b^4 \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) - z^4 \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) \right) \]
\[ K'_{32, \alpha} = \]
\[-\frac{1}{5} z^5 \left( \text{ArcTan} \left[ a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, -cz \right] + \text{ArcTan} \left[ b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, cz \right] + \text{ArcTan} \left[ a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2}, dz \right] + \text{ArcTan} \left[ b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, -dz \right] \right) + \]
\[ \frac{1}{10} \left( -2a^3 c \sqrt{a^2 + c^2 + z^2} + 3ac^3 \sqrt{a^2 + c^2 + z^2} + 2acz^2 \sqrt{a^2 + c^2 + z^2} + 2b^3 c \sqrt{b^2 + c^2 + z^2} - 3bc^3 \sqrt{b^2 + c^2 + z^2} - 2bcz^2 \sqrt{b^2 + c^2 + z^2} + 2a^3 d \sqrt{a^2 + d^2 + z^2} - 3ad^3 \sqrt{a^2 + d^2 + z^2} - 2adz^2 \sqrt{a^2 + d^2 + z^2} - 2b^3 d \sqrt{b^2 + d^2 + z^2} + 3bd^3 \sqrt{b^2 + d^2 + z^2} + 2bdz^2 \sqrt{b^2 + d^2 + z^2} - (3c^5 + 5c^3 z^2) \text{Log} \left[ a + \sqrt{a^2 + c^2 + z^2} \right] + 2a^5 \text{Log} \left[ c + \sqrt{a^2 + c^2 + z^2} \right] + 3c^5 \text{Log} \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + 5c^3 z^2 \text{Log} \left[ b + \sqrt{b^2 + c^2 + z^2} \right] - 2b^5 \text{Log} \left[ c + \sqrt{b^2 + c^2 + z^2} \right] + 3d^5 \text{Log} \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 5d^3 z^2 \text{Log} \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - 2a^5 \text{Log} \left[ d + \sqrt{a^2 + d^2 + z^2} \right] - 3d^5 \text{Log} \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - 5d^3 z^2 \text{Log} \left[ b + \sqrt{b^2 + d^2 + z^2} \right] + 2b^5 \text{Log} \left[ d + \sqrt{b^2 + d^2 + z^2} \right] \right) \]
\[
K'_{x_{42},r} = \frac{1}{18} \left( \frac{3a^6c}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^4c^3}{\sqrt{a^2 + c^2 + z^2}} - \frac{4a^2c^5}{\sqrt{a^2 + c^2 + z^2}} - \frac{8c^7}{\sqrt{a^2 + c^2 + z^2}} - \frac{7a^2c^3z^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{22c^5z^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{17c^3z^4}{\sqrt{a^2 + c^2 + z^2}} - \frac{3cz^6}{\sqrt{a^2 + c^2 + z^2}} + \frac{3b^6c}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^4c^3}{\sqrt{b^2 + c^2 + z^2}} + \frac{4b^2c^5}{\sqrt{b^2 + c^2 + z^2}} + \frac{8c^7}{\sqrt{b^2 + c^2 + z^2}} + \frac{7b^2c^3z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{22c^5z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{17c^3z^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{3cz^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{3a^6d}{\sqrt{a^2 + d^2 + z^2}} - \frac{a^4d^3}{\sqrt{a^2 + d^2 + z^2}} + \frac{4a^2d^5}{\sqrt{a^2 + d^2 + z^2}} + \frac{8d^7}{\sqrt{a^2 + d^2 + z^2}} + \frac{7a^2d^3z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{22d^5z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{17d^3z^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{3dz^6}{\sqrt{a^2 + d^2 + z^2}} - \frac{3b^6d}{\sqrt{b^2 + d^2 + z^2}} + \frac{b^4d^3}{\sqrt{b^2 + d^2 + z^2}} - \frac{4b^2d^5}{\sqrt{b^2 + d^2 + z^2}} - \frac{8d^7}{\sqrt{b^2 + d^2 + z^2}} + \frac{7b^2d^3z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{22d^5z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{17d^3z^4}{\sqrt{b^2 + d^2 + z^2}} + \frac{3dz^6}{\sqrt{b^2 + d^2 + z^2}} - 3(a^6 + z^6) \log[c + \sqrt{a^2 + c^2 + z^2}] - 3(b^6 + z^6) \log[c + \sqrt{b^2 + c^2 + z^2}] - 3a^6 \log[d + \sqrt{a^2 + d^2 + z^2}] - 3z^6 \log[d + \sqrt{a^2 + d^2 + z^2}] \\
+ 3b^6 \log[d + \sqrt{b^2 + d^2 + z^2}] + 3z^6 \log[d + \sqrt{b^2 + d^2 + z^2}]} \right)
\]
\[ K'_{x_{52,n}} = \]
\[-\frac{1}{7} z^7 \left( \frac{\text{ArcTan}\left( c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, az \right) + \text{ArcTan}\left( c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, -bz \right) + \text{ArcTan}\left( d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, -az \right) + \text{ArcTan}\left( d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, bz \right)}{56} \right) \]
\[= \frac{1}{56}(-8a^5c\sqrt{a^2+c^2+z^2} + 10a^3c^3\sqrt{a^2+c^2+z^2} - 15ac^5\sqrt{a^2+c^2+z^2} + 8a^3c^2\sqrt{a^2+c^2+z^2} - 27ac^3z^2\sqrt{a^2+c^2+z^2} - 8ac^2z^3\sqrt{a^2+c^2+z^2} + 8a^3c^2\sqrt{b^2+c^2+z^2} - 10b^3c^3\sqrt{b^2+c^2+z^2} + 15bc^5\sqrt{b^2+c^2+z^2} - 8b^3c^2\sqrt{b^2+c^2+z^2} + 27bc^3z^2\sqrt{b^2+c^2+z^2} + 8bc^2z^3\sqrt{b^2+c^2+z^2} + 8a^5d\sqrt{a^2+d^2+z^2} - 10a^3d^3\sqrt{a^2+d^2+z^2} + 15ad^5\sqrt{a^2+d^2+z^2} - 8a^3dz^2\sqrt{a^2+d^2+z^2} + 27ad^3z^2\sqrt{a^2+d^2+z^2} + 8ad^2z^3\sqrt{a^2+d^2+z^2} - 8b^5d\sqrt{b^2+d^2+z^2} + 10b^3d^3\sqrt{b^2+d^2+z^2} - 15bd^5\sqrt{b^2+d^2+z^2} - 8b^3dz^2\sqrt{b^2+d^2+z^2} - 27bd^3z^2\sqrt{b^2+d^2+z^2} - 8bd^2z^3\sqrt{b^2+d^2+z^2} + (15c^7 + 42c^5z^2 + 35c^3z^4)\text{Log}[a + \sqrt{a^2+c^2+z^2}] + 8a^7\text{Log}[c + \sqrt{a^2+c^2+z^2}] - 15c^7\text{Log}[b + \sqrt{b^2+c^2+z^2}] - 42c^5z^2\text{Log}[b + \sqrt{b^2+c^2+z^2}] - 35c^3z^2\text{Log}[b + \sqrt{b^2+c^2+z^2}] - 8b^7\text{Log}[c + \sqrt{b^2+c^2+z^2}] - 15d^7\text{Log}[a + \sqrt{a^2+d^2+z^2}] - 42d^5z^2\text{Log}[a + \sqrt{a^2+d^2+z^2}] - 35d^3z^2\text{Log}[a + \sqrt{a^2+d^2+z^2}] - 8a^7\text{Log}[d + \sqrt{a^2+d^2+z^2}] + 15d^7\text{Log}[b + \sqrt{b^2+d^2+z^2}] + 42d^5z^2\text{Log}[b + \sqrt{b^2+d^2+z^2}] + 35d^3z^2\text{Log}[b + \sqrt{b^2+d^2+z^2}] + 8b^7\text{Log}[d + \sqrt{b^2+d^2+z^2}] + 35d^3z^2\text{Log}[b + \sqrt{b^2+d^2+z^2}] + 8b^7\text{Log}[d + \sqrt{b^2+d^2+z^2}]) \]

\[ K'_{x_{103,n}} = \]
\[-\frac{1}{3} (2z^2\sqrt{a^2+c^2+z^2} - 2b^2\sqrt{b^2+c^2+z^2} - 2z^2\sqrt{b^2+c^2+z^2} + 2d^2\sqrt{a^2+d^2+z^2} - 2z^2\sqrt{a^2+d^2+z^2} - 2b^2\sqrt{b^2+d^2+z^2} - d^2\sqrt{b^2+d^2+z^2} + 2z^2\sqrt{b^2+d^2+z^2} + c^2(-\sqrt{a^2+c^2+z^2} + \sqrt{b^2+c^2+z^2} + 2a^2(\sqrt{a^2+c^2+z^2} - \sqrt{a^2+d^2+z^2})))) \]
\[ K'_{x_{13, \pi}} = \frac{1}{4} \left( 2a^3 \sqrt{a^2 + c^2 + z^2} - ac^2 \sqrt{a^2 + c^2 + z^2} + az^2 \sqrt{a^2 + c^2 + z^2} - 2b^3 \sqrt{b^2 + c^2 + z^2} + \\
bc^2 \sqrt{b^2 + c^2 + z^2} - bz^2 \sqrt{b^2 + c^2 + z^2} - 2a^3 \sqrt{a^2 + d^2 + z^2} + ad^2 \sqrt{a^2 + d^2 + z^2} - \\
az^2 \sqrt{a^2 + d^2 + z^2} + 2b^3 \sqrt{b^2 + d^2 + z^2} - bd^2 \sqrt{b^2 + d^2 + z^2} + \\
bz^2 \sqrt{b^2 + d^2 + z^2} + (c^4 - z^4) \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] - c^4 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + \\
z^4 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] - \left( -d^4 + z^4 \right) \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + \\
d^4 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - z^4 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] \right) \]

\[ K'_{x_{23, \pi}} = \frac{1}{15} \left( -4z^4 \sqrt{a^2 + c^2 + z^2} - \\
6b^4 \sqrt{b^2 + c^2 + z^2} - 2b^2z^2 \sqrt{b^2 + c^2 + z^2} + 4z^4 \sqrt{b^2 + c^2 + z^2} - \\
6d^4 \sqrt{a^2 + d^2 + z^2} - 2d^2z^2 \sqrt{a^2 + d^2 + z^2} + 4z^4 \sqrt{a^2 + d^2 + z^2} + \\
6b^4 \sqrt{b^2 + d^2 + z^2} - 3b^2d^2 \sqrt{b^2 + d^2 + z^2} + 6d^4 \sqrt{b^2 + d^2 + z^2} + \\
2b^2z^2 \sqrt{b^2 + d^2 + z^2} + 2d^2z^2 \sqrt{b^2 + d^2 + z^2} - 4z^4 \sqrt{b^2 + d^2 + z^2} + \\
6c^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) + 6a^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) + \\
c^2 \left( 3b^2 \sqrt{b^2 + c^2 + z^2} + 2z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) \right) + \\
a^2 \left( -3c^2 \sqrt{a^2 + c^2 + z^2} + 3d^2 \sqrt{a^2 + d^2 + z^2} + 2z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) \right) \]
\[ K'_{33,n} = \frac{1}{24} \left( 8a^5 \sqrt{a^2 + c^2 + z^2} - 4a^3 c^2 \sqrt{a^2 + c^2 + z^2} + 
\right. \\
6ac^4 \sqrt{a^2 + c^2 + z^2} + 2a^3 z^2 \sqrt{a^2 + c^2 + z^2} + 3ac^2 z^2 \sqrt{a^2 + c^2 + z^2} - 
3az^4 \sqrt{a^2 + c^2 + z^2} - 8b^5 \sqrt{b^2 + c^2 + z^2} + 4b^3 c^2 \sqrt{b^2 + c^2 + z^2} - 
6bc^4 \sqrt{b^2 + c^2 + z^2} - 2b^3 z^2 \sqrt{b^2 + c^2 + z^2} - 3bc^2 z^2 \sqrt{b^2 + c^2 + z^2} + 
3bz^4 \sqrt{b^2 + c^2 + z^2} - 8a^5 \sqrt{a^2 + d^2 + z^2} + 4a^3 d^2 \sqrt{a^2 + d^2 + z^2} - 
6ad^4 \sqrt{a^2 + d^2 + z^2} - 2a^3 d^2 \sqrt{a^2 + d^2 + z^2} - 3ad^2 z^2 \sqrt{a^2 + d^2 + z^2} + 
3az^4 \sqrt{a^2 + c^2 + z^2} + 8b^5 \sqrt{b^2 + d^2 + z^2} - 4b^3 d^2 \sqrt{b^2 + d^2 + z^2} + 
6bd^4 \sqrt{b^2 + d^2 + z^2} + 2b^3 z^2 \sqrt{b^2 + d^2 + z^2} + 3bd^2 z^2 \sqrt{b^2 + d^2 + z^2} - 
3bz^4 \sqrt{b^2 + d^2 + z^2} - 3(2c^6 + 3c^4 z^2 - z^6) \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] + 
3(2c^6 + 3c^4 z^2 - z^6) \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + 
6d^6 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 9d^4 z^2 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - 
3z^6 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - 6d^6 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - 
9d^4 z^2 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] + 3z^6 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] \right) \]
\[ K'_{x_{43,a}} = \] 
\[
\frac{1}{105} \left( 16 z^6 \sqrt{a^2 + c^2 + z^2} - 30 b^6 \sqrt{b^2 + c^2 + z^2} + 6 b^4 z^2 \sqrt{b^2 + c^2 + z^2} + 8 b^2 z^4 \sqrt{b^2 + c^2 + z^2} - 16 z^6 \sqrt{b^2 + c^2 + z^2} + 40 d^6 \sqrt{a^2 + c^2 + z^2} + 64 d^4 z^2 \sqrt{a^2 + d^2 + z^2} + 8 d^2 z^4 \sqrt{a^2 + d^2 + z^2} - 16 z^6 \sqrt{a^2 + d^2 + z^2} + 30 b^6 \sqrt{b^2 + d^2 + z^2} - 15 b^4 d^2 \sqrt{b^2 + d^2 + z^2} + 20 b^2 d^4 \sqrt{b^2 + d^2 + z^2} - 40 d^6 \sqrt{b^2 + d^2 + z^2} + 6 b^4 z^2 \sqrt{b^2 + d^2 + z^2} + 12 b^2 d^2 z^2 \sqrt{b^2 + d^2 + z^2} - 64 d^4 z^2 \sqrt{b^2 + d^2 + z^2} - 8 b^2 z^4 \sqrt{b^2 + d^2 + z^2} - 8 d^2 z^4 \sqrt{b^2 + d^2 + z^2} + 16 z^6 \sqrt{b^2 + d^2 + z^2} - 40 c^6 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) + 30 a^6 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) - 4 c^4 \left( 5 b^2 \sqrt{b^2 + c^2 + z^2} + 16 z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) \right) + c^2 \left( 15 b^4 \sqrt{b^2 + c^2 + z^2} - 12 b^2 z^2 \sqrt{b^2 + c^2 + z^2} - 8 z^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) \right) - 3 a^4 \left( 5 c^2 \sqrt{a^2 + c^2 + z^2} - 5 d^2 \sqrt{a^2 + d^2 + z^2} - 2 z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) + 4 a^2 \left( 5 c^4 \sqrt{a^2 + c^2 + z^2} + 3 c^2 z^2 \sqrt{a^2 + c^2 + z^2} - 5 d^4 \sqrt{a^2 + d^2 + z^2} - 3 d^2 z^2 \sqrt{a^2 + d^2 + z^2} - 2 z^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) \right) \]
\[ K'_{x33,n} = \frac{1}{192} \left( 48 a^7 \sqrt{a^2 + c^2 + z^2} - 24 a^5 c^2 \sqrt{a^2 + c^2 + z^2} + 30 a^3 c^4 \sqrt{a^2 + c^2 + z^2} - \\
45 a c^6 \sqrt{a^2 + c^2 + z^2} + 8 a^5 z^2 \sqrt{a^2 + c^2 + z^2} + 20 a^3 c^2 z^2 \sqrt{a^2 + c^2 + z^2} - \\
75 a c^4 z^2 \sqrt{a^2 + c^2 + z^2} - 10 a^3 z^4 \sqrt{a^2 + c^2 + z^2} - 15 a c^2 z^4 \sqrt{a^2 + c^2 + z^2} + \\
15 a z^6 \sqrt{a^2 + c^2 + z^2} - 48 b^7 \sqrt{b^2 + c^2 + z^2} + 24 b^5 c^2 \sqrt{b^2 + c^2 + z^2} - \\
30 b^3 c^4 \sqrt{b^2 + c^2 + z^2} + 45 b c^6 \sqrt{b^2 + c^2 + z^2} - 8 b^5 z^2 \sqrt{b^2 + c^2 + z^2} - \\
20 b^3 c^2 z^2 \sqrt{b^2 + c^2 + z^2} + 75 b c^4 z^2 \sqrt{b^2 + c^2 + z^2} + 10 b^3 z^4 \sqrt{b^2 + c^2 + z^2} + \\
15 b c^2 z^4 \sqrt{b^2 + c^2 + z^2} - 15 b z^6 \sqrt{b^2 + c^2 + z^2} - 48 a^7 \sqrt{a^2 + d^2 + z^2} + \\
24 a^5 d^2 \sqrt{a^2 + d^2 + z^2} - 30 a^3 d^4 \sqrt{a^2 + d^2 + z^2} + 45 a d^6 \sqrt{a^2 + d^2 + z^2} - \\
8 a^5 z^2 \sqrt{a^2 + d^2 + z^2} - 20 a^3 d^2 z^2 \sqrt{a^2 + d^2 + z^2} + 75 a d^4 z^2 \sqrt{a^2 + d^2 + z^2} + \\
10 a^3 z^4 \sqrt{a^2 + d^2 + z^2} + 15 a d^2 z^4 \sqrt{a^2 + d^2 + z^2} - 15 a z^6 \sqrt{a^2 + d^2 + z^2} + \\
48 b^7 \sqrt{b^2 + d^2 + z^2} - 24 b^5 d^2 \sqrt{b^2 + d^2 + z^2} + 30 b^3 d^4 \sqrt{b^2 + d^2 + z^2} - \\
45 b d^6 \sqrt{b^2 + d^2 + z^2} + 8 b^5 z^2 \sqrt{b^2 + d^2 + z^2} + 20 b^3 d^2 z^2 \sqrt{b^2 + d^2 + z^2} - \\
75 b d^4 z^2 \sqrt{b^2 + d^2 + z^2} - 10 b^3 z^4 \sqrt{b^2 + d^2 + z^2} - 15 b d^2 z^4 \sqrt{b^2 + d^2 + z^2} + \\
15 b z^6 \sqrt{b^2 + d^2 + z^2} + 15 (3 c^2 - z^2)(c^2 + z^2)^3 \log[a + \sqrt{a^2 + c^2 + z^2}] - \\
(3 c^2 - z^2)(c^2 + z^2)^3 \log[b + \sqrt{b^2 + c^2 + z^2}] - \\
45 d^8 \log[a + \sqrt{a^2 + d^2 + z^2}] - 120 d^6 z^2 \log[a + \sqrt{a^2 + d^2 + z^2}] - \\
90 d^4 z^4 \log[a + \sqrt{a^2 + d^2 + z^2}] + 15 z^8 \log[a + \sqrt{a^2 + d^2 + z^2}] + \\
45 d^8 \log[b + \sqrt{b^2 + d^2 + z^2}] + 120 d^6 z^2 \log[b + \sqrt{b^2 + d^2 + z^2}] + \\
90 d^4 z^4 \log[b + \sqrt{b^2 + d^2 + z^2}] - 15 z^8 \log[b + \sqrt{b^2 + d^2 + z^2}] \right) \]
\[ K'_{x04,n} = \frac{1}{8} \left( \frac{3a^4c}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^2c^3}{\sqrt{a^2 + c^2 + z^2}} - \frac{2c^5}{\sqrt{a^2 + c^2 + z^2}} + \frac{6a^2cz^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{c^3z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{3cz^4}{\sqrt{a^2 + c^2 + z^2}} + \frac{3b^4c}{\sqrt{a^2 + c^2 + z^2}} - \frac{b^2c^3}{\sqrt{b^2 + c^2 + z^2}} + \frac{2c^5}{\sqrt{b^2 + c^2 + z^2}} + \frac{6b^2cz^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{3cz^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{3a^4d}{\sqrt{a^2 + d^2 + z^2}} - \frac{a^2d^3}{\sqrt{a^2 + d^2 + z^2}} + \frac{2d^5}{\sqrt{a^2 + d^2 + z^2}} - \frac{6a^2dz^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{d^3z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{3dz^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{3b^4d}{\sqrt{a^2 + d^2 + z^2}} + \frac{b^2d^3}{\sqrt{a^2 + d^2 + z^2}} - \frac{2d^5}{\sqrt{b^2 + d^2 + z^2}} + \frac{6b^2dz^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{d^3z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{3dz^4}{\sqrt{b^2 + d^2 + z^2}} + \frac{3(a^2 + z^2)^2 \log[c + \sqrt{a^2 + c^2 + z^2}] + 3b^4 \log[c + \sqrt{b^2 + c^2 + z^2}] + 6b^2z^2 \log[c + \sqrt{b^2 + c^2 + z^2}] + 3z^4 \log[c + \sqrt{b^2 + c^2 + z^2}] + 3(a^2 + z^2)^2 \log[d + \sqrt{a^2 + d^2 + z^2}] - 3b^4 \log[d + \sqrt{b^2 + d^2 + z^2}] - 6b^2z^2 \log[d + \sqrt{b^2 + d^2 + z^2}] - 3z^4 \log[d + \sqrt{b^2 + d^2 + z^2}] - 266}{\text{Reproduced with permission of the copyright owner. Further reproduction prohibited without permission.}} \]
\[ K'_{14, n} = \]
\[- \frac{1}{5} z^5 \left( \text{ArcTan}[c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, -az] + \text{ArcTan}[e^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, -bz] + \right. \]
\[ \left. \text{ArcTan}[d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, az] + \text{ArcTan}[d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, -bz] + \right. \]
\[ \frac{1}{10} \left( 3 a^3 c \sqrt{a^2 + c^2 + z^2} - 2 ac^3 \sqrt{a^2 + c^2 + z^2} + 2 acz^2 \sqrt{a^2 + c^2 + z^2} - \right. \]
\[ \left. 3 b^3 c \sqrt{b^2 + c^2 + z^2} + 2 bc^3 \sqrt{b^2 + c^2 + z^2} - 2 bcz^2 \sqrt{b^2 + c^2 + z^2} - \right. \]
\[ \left. 3 a^3 d \sqrt{a^2 + d^2 + z^2} + 2 ad^3 \sqrt{a^2 + d^2 + z^2} - 2 adz^2 \sqrt{a^2 + d^2 + z^2} - \right. \]
\[ \left. 3 b^3 d \sqrt{b^2 + d^2 + z^2} - 2 bd^3 \sqrt{b^2 + d^2 + z^2} + \right. \]
\[ 2 b dz^2 \sqrt{b^2 + d^2 + z^2} + \right. \]
\[ \left. (3 a^5 + 5 a^3 z^2) \text{Log}[c + \sqrt{a^2 + c^2 + z^2}] - 2 c^5 \text{Log}[b + \sqrt{b^2 + c^2 + z^2}] + \right. \]
\[ 3 b^5 \text{Log}[c + \sqrt{b^2 + c^2 + z^2}] + 5 b^3 z^2 \text{Log}[c + \sqrt{b^2 + c^2 + z^2}] - \right. \]
\[ \left. 2 d^5 \text{Log}[a + \sqrt{a^2 + d^2 + z^2}] + 3 a^5 \text{Log}[d + \sqrt{a^2 + d^2 + z^2}] + \right. \]
\[ \left. 5 a^3 z^2 \text{Log}[d + \sqrt{a^2 + d^2 + z^2}] + 2 d^5 \text{Log}[b + \sqrt{b^2 + d^2 + z^2}] - \right. \]
\[ \left. 3 b^5 \text{Log}[d + \sqrt{b^2 + d^2 + z^2}] - 5 b^3 z^2 \text{Log}[d + \sqrt{b^2 + d^2 + z^2}] \right) \]
\[
K'_{x24,n} = \frac{1}{24} \left( 6 a^4 c \sqrt{a^2 + c^2 + z^2} - 4 a^2 c^3 \sqrt{a^2 + c^2 + z^2} + 8 c^5 \sqrt{a^2 + c^2 + z^2} + \\
3 a^2 c^2 \sqrt{a^2 + c^2 + z^2} + 2 c^3 z^2 \sqrt{a^2 + c^2 + z^2} - 3 c z^4 \sqrt{a^2 + c^2 + z^2} - \\
6 b^4 c \sqrt{b^2 + c^2 + z^2} + 4 b^2 c^3 \sqrt{b^2 + c^2 + z^2} - 8 c^5 \sqrt{b^2 + c^2 + z^2} - \\
3 b^2 c z^2 \sqrt{b^2 + c^2 + z^2} - 2 c^3 z^2 \sqrt{b^2 + c^2 + z^2} + 3 c z^4 \sqrt{b^2 + c^2 + z^2} - \\
6 a^4 d \sqrt{a^2 + d^2 + z^2} + 4 a^2 d^3 \sqrt{a^2 + d^2 + z^2} - 8 d^5 \sqrt{a^2 + d^2 + z^2} - \\
3 a^2 d z^2 \sqrt{a^2 + d^2 + z^2} - 2 d^3 z^2 \sqrt{a^2 + d^2 + z^2} + 3 d z^4 \sqrt{a^2 + d^2 + z^2} + \\
6 b^4 d \sqrt{b^2 + d^2 + z^2} - 4 b^2 d^3 \sqrt{b^2 + d^2 + z^2} + 8 d^5 \sqrt{b^2 + d^2 + z^2} + \\
3 b^2 d z^2 \sqrt{b^2 + d^2 + z^2} + 2 b^3 d^2 \sqrt{b^2 + d^2 + z^2} - 3 d z^4 \sqrt{b^2 + d^2 + z^2} - \\
3 (2 a^6 + 3 a^4 z^2 - z^6) \left( \log(c + \sqrt{a^2 + c^2 + z^2}) + 6 b^6 \log(c + \sqrt{b^2 + c^2 + z^2}) + \\
9 b^4 z^2 \log(c + \sqrt{b^2 + c^2 + z^2}) - 3 z^6 \log(c + \sqrt{b^2 + c^2 + z^2}) + \\
3 (2 a^6 + 3 a^4 z^2 - z^6) \left( \log(d + \sqrt{a^2 + d^2 + z^2}) - 6 b^6 \log(d + \sqrt{b^2 + d^2 + z^2}) - \\
9 b^4 z^2 \log(d + \sqrt{b^2 + d^2 + z^2}) + 3 z^6 \log(d + \sqrt{b^2 + d^2 + z^2}) \right) \right) \right)
\]
\[ K'_{34,4} = -\frac{3}{35} z^7 \left( \text{ArcTan} \left[ a^2 + c^2 + z^2, az \right] + \text{ArcTan} \left[ c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, -b z \right] + \text{ArcTan} \left[ d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, -a z \right] + \text{ArcTan} \left[ d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, b z \right] \right) + \frac{1}{70} \left( 15 a^5 c \sqrt{a^2 + c^2 + z^2} - 10 a^3 c^3 \sqrt{a^2 + c^2 + z^2} + 15 a c^5 \sqrt{a^2 + c^2 + z^2} + 6 a^3 c^2 \sqrt{a^2 + c^2 + z^2} + 6 a c^3 z^2 \sqrt{a^2 + c^2 + z^2} - 6 ac z^4 \sqrt{a^2 + c^2 + z^2} - 15 b^5 c \sqrt{b^2 + c^2 + z^2} + 10 b^3 c^3 \sqrt{b^2 + c^2 + z^2} - 15 b c^5 \sqrt{b^2 + c^2 + z^2} - 6 b^3 c^2 \sqrt{b^2 + c^2 + z^2} - 6 b c^3 z^2 \sqrt{b^2 + c^2 + z^2} + 6 b cz^4 \sqrt{b^2 + c^2 + z^2} - 15 a^5 d \sqrt{a^2 + d^2 + z^2} + 10 a^3 d^3 \sqrt{a^2 + d^2 + z^2} - 15 a d^5 \sqrt{a^2 + d^2 + z^2} - 6 a^3 d^2 \sqrt{a^2 + d^2 + z^2} - 6 a d^3 z^2 \sqrt{a^2 + d^2 + z^2} + 6 ad^4 \sqrt{a^2 + d^2 + z^2} + 15 b^5 d \sqrt{b^2 + d^2 + z^2} - 10 b^3 d^3 \sqrt{b^2 + d^2 + z^2} + 15 b d^5 \sqrt{b^2 + d^2 + z^2} + 6 b^3 d^2 \sqrt{b^2 + d^2 + z^2} + 6 b d^3 z^2 \sqrt{b^2 + d^2 + z^2} - 6 b d z^4 \sqrt{b^2 + d^2 + z^2} - 3 \left( 5 c^7 + 7 c^5 z^2 \right) \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] - 3 \left( 5 a^7 + 7 a^5 z^2 \right) \log \left[ c + \sqrt{a^2 + c^2 + z^2} \right] + 15 c^7 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + 21 c^5 z^2 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + 15 b^7 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] + 21 b^5 z^2 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] + 15 d^7 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 21 d^5 z^2 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 15 a^7 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] + 21 a^5 z^2 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] - 15 d^7 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - 21 d^5 z^2 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - 15 b^7 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] - 21 b^5 z^2 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] \right) \]
\[ K'_{44,n} = \]
\[
\frac{1}{48} \left( \begin{array}{ccccccc}
9a^6c & 3a^6c^3 & 2a^4c^5 & 8a^2c^7 & 16c^9 \\
\sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} \\
12a^6cz^2 & a^4cz^2 & 12a^2c^5z^2 & 40cz^2 & a^2c^3z^4 \\
\sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} \\
26c^3z^4 & c^3z^6 & 3cz^8 & 9b^8c & 3b^6c^3 \\
\sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} \\
2b^4c^5 & 8b^2c^7 & 16c^9 & 12b^6cz^2 & b^4c^3z^2 \\
\sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} \\
12b^2c^5z^2 & 40c^7z^2 & b^2c^3z^4 & 26c^5z^4 & c^3z^6 \\
\sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} \\
3cz^8 & 9a^8d & 3a^6d^3 & 2a^4d^5 & 8a^2d^7 \\
\sqrt{b^2 + c^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} \\
16d^9 & 12a^6dz^2 & a^4d^3z^2 & 12a^2d^5z^2 & 40d^7z^2 \\
\sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} \\
a^2d^3z^4 & 26d^5z^4 & d^3z^6 & 3dz^8 & 9b^8d \\
\sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} \\
3b^6d^3 & 2b^4d^5 & 8b^2d^7 & 16d^9 & \sqrt{b^2 + d^2 + z^2} \\
\sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} \\
12b^6dz^2 & b^4d^3z^2 & 12b^2d^5z^2 & 40d^7z^2 & \sqrt{b^2 + d^2 + z^2} \\
\sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} \\
b^2d^3z^4 & 26d^5z^4 & d^3z^6 & 3dz^8 & \sqrt{b^2 + d^2 + z^2} \\
\sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} \\
3(3a^8 + 4a^6z^2 + z^8)\log[c + \sqrt{a^2 + c^2 + z^2}] + 3(3b^8 + 4b^6z^2 + z^8)\log[c + \sqrt{b^2 + c^2 + z^2}] + \\
9a^8\log[d + \sqrt{a^2 + d^2 + z^2}] + 12a^6z^2\log[d + \sqrt{a^2 + d^2 + z^2}] + \\
3z^8\log[d + \sqrt{a^2 + d^2 + z^2}] - 9b^8\log[d + \sqrt{b^2 + d^2 + z^2}] - \\
12b^6z^2\log[d + \sqrt{b^2 + d^2 + z^2}] - 3z^8\log[d + \sqrt{b^2 + d^2 + z^2}] \right) 
\right)
\]
\[ K'_{x54,a} = \]
\[ -\frac{1}{21} \left( \text{ArcTan}[c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, -az] + \text{ArcTan}[c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, bz] + \right. \]
\[ \left. \text{ArcTan}[d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, az] + \text{ArcTan}[d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, -bz] \right) + \]
\[ \frac{1}{504} \left( 84a^7c \sqrt{a^2 + c^2 + z^2} - 56a^5c^3 \sqrt{a^2 + c^2 + z^2} + 70a^3c^5 \sqrt{a^2 + c^2 + z^2} - \right. \]
\[ 105ac^7 \sqrt{a^2 + c^2 + z^2} + 24ac^5z^2 \sqrt{a^2 + c^2 + z^2} + 40a^3c^3z^2 \sqrt{a^2 + c^2 + z^2} - \]
\[ 165ac^5z^2 \sqrt{a^2 + c^2 + z^2} - 24a^3z^4 \sqrt{a^2 + c^2 + z^2} - 24ac^3z^4 \sqrt{a^2 + c^2 + z^2} + \]
\[ 24ac^5z^2 \sqrt{a^2 + c^2 + z^2} - 84b^7c \sqrt{b^2 + c^2 + z^2} + 56b^5c^3 \sqrt{b^2 + c^2 + z^2} - \]
\[ 70b^3c^5 \sqrt{b^2 + c^2 + z^2} + 105bc^7 \sqrt{b^2 + c^2 + z^2} - 24b^5cz^2 \sqrt{b^2 + c^2 + z^2} - \]
\[ 40b^3c^3z^2 \sqrt{b^2 + c^2 + z^2} + 165b^5c^3z^2 \sqrt{b^2 + c^2 + z^2} + 24b^3c^3z^4 \sqrt{b^2 + c^2 + z^2} + \]
\[ 24b^3c^3z^2 \sqrt{b^2 + c^2 + z^2} - 24bc^7z^2 \sqrt{b^2 + c^2 + z^2} - 84ad^7 \sqrt{a^2 + d^2 + z^2} + \]
\[ 56a^3d^3 \sqrt{a^2 + d^2 + z^2} - 70a^3d^5 \sqrt{a^2 + d^2 + z^2} + 105ad^7 \sqrt{a^2 + d^2 + z^2} - \]
\[ 24a^3dz^2 \sqrt{a^2 + d^2 + z^2} - 40a^3dz^3 \sqrt{a^2 + d^2 + z^2} + 165ad^5z^2 \sqrt{a^2 + d^2 + z^2} + \]
\[ 24a^3dz^3 \sqrt{a^2 + d^2 + z^2} + 24ad^3z^3 \sqrt{a^2 + d^2 + z^2} - 24adz^6 \sqrt{a^2 + d^2 + z^2} + \]
\[ 84b^7d \sqrt{b^2 + d^2 + z^2} - 56b^5d^3 \sqrt{b^2 + d^2 + z^2} + 70b^3d^5 \sqrt{b^2 + d^2 + z^2} - \]
\[ 105bd^7 \sqrt{b^2 + d^2 + z^2} + 24b^3dz^2 \sqrt{b^2 + d^2 + z^2} + 40b^3dz^2 \sqrt{b^2 + d^2 + z^2} - \]
\[ 165bd^5z^2 \sqrt{b^2 + d^2 + z^2} - 24b^3dz^4 \sqrt{b^2 + d^2 + z^2} - 24bd^3z^4 \sqrt{b^2 + d^2 + z^2} + \]
\[ 24bd^5z^2 \sqrt{b^2 + d^2 + z^2} + 3c^5(35c^4 + 90c^2z^2 + 63z^4) \log[a + \sqrt{a^2 + c^2 + z^2}] - \]
\[ 12a^7(7a^2 + 9z^2) \log[c + \sqrt{a^2 + c^2 + z^2}] - 105c^9 \log[b + \sqrt{b^2 + c^2 + z^2}] - \]
\[ 270c^7z^2 \log[b + \sqrt{b^2 + c^2 + z^2}] - 189c^5z^4 \log[b + \sqrt{b^2 + c^2 + z^2}] + \]
\[ 84b^9 \log[c + \sqrt{b^2 + c^2 + z^2}] + 108b^7z^2 \log[c + \sqrt{b^2 + c^2 + z^2}] - \]
\[ 105d^9 \log[a + \sqrt{a^2 + d^2 + z^2}] - 270d^7z^2 \log[a + \sqrt{a^2 + d^2 + z^2}] - \]
\[ 189d^5z^4 \log[a + \sqrt{a^2 + d^2 + z^2}] + 84a^9 \log[d + \sqrt{a^2 + d^2 + z^2}] + \]
\[ 108a^7z^2 \log[d + \sqrt{a^2 + d^2 + z^2}] + 105d^9 \log[b + \sqrt{b^2 + d^2 + z^2}] + \]
\[ 270d^7z^2 \log[b + \sqrt{b^2 + d^2 + z^2}] + 189d^5z^4 \log[b + \sqrt{b^2 + d^2 + z^2}] - \]
\[ 84b^9 \log[d + \sqrt{b^2 + d^2 + z^2}] - 108b^7z^2 \log[d + \sqrt{b^2 + d^2 + z^2}] \]
\[
K'_{x_{05,n}} = \frac{1}{15} \left( \frac{8z^6}{\sqrt{a^2 + c^2 + z^2}} + \frac{8b^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{24b^4z^2}{\sqrt{b^2 + c^2 + z^2}} + \right.
\frac{24b^2z^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{8z^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{3d^6}{\sqrt{a^2 + d^2 + z^2}} - \frac{d^4z^2}{\sqrt{a^2 + d^2 + z^2}} + \\
\frac{4d^2z^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{8z^6}{\sqrt{a^2 + d^2 + z^2}} - \frac{8b^6}{\sqrt{a^2 + d^2 + z^2}} - \frac{4b^4d^2}{\sqrt{a^2 + d^2 + z^2}} + \\
\frac{\sqrt{a^2 + d^2 + z^2}}{\sqrt{b^2 + d^2 + z^2}} - \frac{3d^6}{\sqrt{a^2 + d^2 + z^2}} - \frac{24b^4z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{8b^2d^2z^2}{\sqrt{b^2 + d^2 + z^2}} + \\
\frac{\sqrt{b^2 + d^2 + z^2}}{\sqrt{b^2 + d^2 + z^2}} - \frac{24b^2z^4}{\sqrt{b^2 + d^2 + z^2}} - \frac{4d^2z^4}{\sqrt{b^2 + d^2 + z^2}} - \frac{8z^6}{\sqrt{b^2 + d^2 + z^2}} + \\
\frac{c^6}{\sqrt{a^2 + c^2 + z^2}} + \frac{3}{\sqrt{b^2 + c^2 + z^2}} + 8a^6 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{a^2 + d^2 + z^2}} \right) + \\
\frac{c^4}{\sqrt{a^2 + c^2 + z^2}} - \frac{b^2}{\sqrt{b^2 + c^2 + z^2}} + z^2 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} - \frac{1}{\sqrt{b^2 + c^2 + z^2}} \right) + \\
\frac{4c^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{a^4}{\sqrt{a^2 + c^2 + z^2}} - \frac{2a^2z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{b^4}{\sqrt{b^2 + c^2 + z^2}} + \\
\frac{2b^2z^2}{\sqrt{b^2 + c^2 + z^2}} + z^4 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{b^2 + c^2 + z^2}} \right) + \\
\frac{a^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{d^4}{\sqrt{a^2 + d^2 + z^2}} + 24z^4 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{a^2 + d^2 + z^2}} \right) + \\
\frac{4a^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{d^2}{\sqrt{a^2 + d^2 + z^2}} + z^2 \left( \frac{6}{\sqrt{a^2 + c^2 + z^2}} + \frac{6}{\sqrt{a^2 + d^2 + z^2}} \right) \right).
\[
K'_{\alpha_{15,n}} = \frac{1}{18} \left( \frac{8a^7}{\sqrt{a^2 + c^2 + z^2}} - \frac{4a^5c^2}{\sqrt{a^2 + c^2 + z^2}^2} + \frac{a^3c^4}{\sqrt{a^2 + c^2 + z^2}^3} - \frac{3ac^6}{\sqrt{a^2 + c^2 + z^2}^4} - \frac{22a^5z^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{7a^3c^2z^2}{\sqrt{a^2 + c^2 + z^2}^2} - \frac{17a^3z^4}{\sqrt{a^2 + c^2 + z^2}^3} - \frac{3az^6}{\sqrt{a^2 + c^2 + z^2}^4} + \frac{8b^7}{\sqrt{b^2 + c^2 + z^2}} + \frac{4b^5c^2}{\sqrt{b^2 + c^2 + z^2}^2} - \frac{b^3c^4}{\sqrt{b^2 + c^2 + z^2}^3} + \frac{3bc^6}{\sqrt{b^2 + c^2 + z^2}^4} + \frac{22b^5z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{7b^3c^2z^2}{\sqrt{b^2 + c^2 + z^2}^2} + \frac{17b^3z^4}{\sqrt{b^2 + c^2 + z^2}^3} + \frac{3bz^6}{\sqrt{b^2 + c^2 + z^2}^4} + \frac{8a^7}{\sqrt{a^2 + d^2 + z^2}} - \frac{4a^5d^2}{\sqrt{a^2 + d^2 + z^2}^2} - \frac{a^3d^4}{\sqrt{a^2 + d^2 + z^2}^3} + \frac{3ad^6}{\sqrt{a^2 + d^2 + z^2}^4} + \frac{22a^5z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{7a^3d^2z^2}{\sqrt{a^2 + d^2 + z^2}^2} + \frac{17a^3z^4}{\sqrt{a^2 + d^2 + z^2}^3} + \frac{3az^6}{\sqrt{a^2 + d^2 + z^2}^4} - \frac{8b^7}{\sqrt{b^2 + d^2 + z^2}} + \frac{4b^5d^2}{\sqrt{b^2 + d^2 + z^2}^2} + \frac{b^3d^4}{\sqrt{b^2 + d^2 + z^2}^3} - \frac{3bd^6}{\sqrt{b^2 + d^2 + z^2}^4} + \frac{22b^5z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{7b^3d^2z^2}{\sqrt{b^2 + d^2 + z^2}^2} - \frac{17b^3z^4}{\sqrt{b^2 + d^2 + z^2}^3} - \frac{3bz^6}{\sqrt{b^2 + d^2 + z^2}^4} + \frac{3(c^6 + z^6) \log[a + \sqrt{a^2 + c^2 + z^2}]}{3c^6 \log[b + \sqrt{b^2 + c^2 + z^2}]} - 3c^6 \log[b + \sqrt{b^2 + c^2 + z^2}] - 3z^6 \log[b + \sqrt{b^2 + c^2 + z^2}] + 3(d^6 + z^6) \log[a + \sqrt{a^2 + d^2 + z^2}] + 3d^6 \log[b + \sqrt{b^2 + d^2 + z^2}] + 3z^6 \log[b + \sqrt{b^2 + d^2 + z^2}] \right) \right)
\[
K'_{x_{25,n}} = \frac{1}{105} \left( 16 z^6 \sqrt{a^2 + c^2 + z^2} + 40 b^6 \sqrt{b^2 + c^2 + z^2} + 
64 b^4 z^2 \sqrt{b^2 + c^2 + z^2} + 8 b^2 z^4 \sqrt{b^2 + c^2 + z^2} - 16 z^6 \sqrt{b^2 + c^2 + z^2} - 
30 d^6 \sqrt{a^2 + d^2 + z^2} - 6 d^4 z^2 \sqrt{a^2 + d^2 + z^2} + 8 d^2 z^4 \sqrt{a^2 + d^2 + z^2} - 
16 z^6 \sqrt{a^2 + d^2 + z^2} - 40 b^6 \sqrt{b^2 + d^2 + z^2} + 20 b^4 d^2 \sqrt{b^2 + d^2 + z^2} - 
15 b^2 d^4 \sqrt{b^2 + d^2 + z^2} + 30 d^6 \sqrt{b^2 + d^2 + z^2} - 64 b^4 z^2 \sqrt{b^2 + d^2 + z^2} + 
12 b^2 d^2 z^2 \sqrt{b^2 + d^2 + z^2} + 6 d^4 z^2 \sqrt{b^2 + d^2 + z^2} - 
8 b^2 z^4 \sqrt{b^2 + d^2 + z^2} - 8 d^2 z^4 \sqrt{b^2 + d^2 + z^2} + 16 z^6 \sqrt{b^2 + d^2 + z^2} + 
30 c^6 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) - 40 a^6 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) - 3 c^4 
\right)
\]
\[
K_{x35,n} = \frac{1}{48} \left( \begin{array}{cccc}
16a^9 & 8a^7c^2 & 2a^5c^4 & 3a^3c^6 \\
\sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} \\
40a^7z^2 & 12a^5c^2z^2 & 3a^3c^4z^2 & 12a^5c^6z^2 \\
\sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} \\
a^3c^2z^4 & + & a^3z^6 & + \\
\sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} \\
2b^5c^4 & + & 3b^3c^6 & + \\
\sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} \\
12b^5c^2z^2 & + & b^3c^4z^2 & + \\
\sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} \\
b^3z^4 & + & b^3z^6 & + \\
\sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} \\
8a^7d^2 & + & 2a^5d^4 & + \\
\sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} \\
12a^9d^2z^2 & + & a^3d^4z^2 & + \\
\sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} \\
a^3d^2z^4 & + & a^3z^6 & + \\
\sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} \\
8b^7d^2 & + & 2b^5d^4 & + \\
\sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} \\
40b^7z^2 & + & 12b^5d^2z^2 & + \\
\sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} \\
26b^5z^4 & + & b^3d^2z^4 & + \\
\sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} \\
3(3c^8 + 4c^6z^2 + z^8)\log[a + \sqrt{a^2 + c^2 + z^2}] + 9c^8\log[b + \sqrt{b^2 + c^2 + z^2}] + \\
12c^6z^2\log[b + \sqrt{b^2 + c^2 + z^2}] + 3z^8\log[b + \sqrt{b^2 + c^2 + z^2}] + \\
3(3d^8 + 4d^6z^2 + z^8)\log[a + \sqrt{a^2 + d^2 + z^2}] - 9d^8\log[b + \sqrt{b^2 + d^2 + z^2}] - \\
12d^6z^2\log[b + \sqrt{b^2 + d^2 + z^2}] - 3z^8\log[b + \sqrt{b^2 + d^2 + z^2}]
\end{array} \right)
\]
\[ K_{45,a}^r = \frac{1}{1890} \left( \begin{array}{c} \sqrt{a^2 + c^2 + z^2} - \left( \frac{560 a^{10}}{\sqrt{a^2 + z^2}} + \frac{280 a^8 c^2}{\sqrt{a^2 + z^2}} - \frac{210 a^6 c^4}{\sqrt{a^2 + z^2}} - \frac{280 a^4 c^6}{\sqrt{a^2 + z^2}} - \frac{560 a^2 c^8}{\sqrt{a^2 + z^2}} + \frac{1360 a^8 z^2}{\sqrt{a^2 + z^2}} + \frac{400 a^6 c^2 z^2}{\sqrt{a^2 + z^2}} - \frac{90 a^4 c^4 z^2}{\sqrt{a^2 + z^2}} + \frac{520 a^2 c^6 z^2}{\sqrt{a^2 + z^2}} + \frac{560 a^8 z^2}{\sqrt{a^2 + z^2}} - \frac{848 a^6 z^4}{\sqrt{a^2 + z^2}} \right) \right) \right) + 1 \]

\[ \left( \frac{\sqrt{b^2 + c^2 + z^2}}{\sqrt{b^2 + z^2}} - \left( \frac{560 b^{10}}{\sqrt{b^2 + z^2}} - \frac{280 b^8 c^2}{\sqrt{b^2 + z^2}} + \frac{210 b^6 c^4}{\sqrt{b^2 + z^2}} - \frac{280 b^4 c^6}{\sqrt{b^2 + z^2}} + \frac{560 b^2 c^8}{\sqrt{b^2 + z^2}} + \frac{1360 b^8 z^2}{\sqrt{b^2 + z^2}} - \frac{400 b^6 c^2 z^2}{\sqrt{b^2 + z^2}} + \frac{90 b^4 c^4 z^2}{\sqrt{b^2 + z^2}} + \frac{520 b^2 c^6 z^2}{\sqrt{b^2 + z^2}} + \frac{560 b^8 z^2}{\sqrt{b^2 + z^2}} - \frac{848 b^6 z^4}{\sqrt{b^2 + z^2}} \right) \right) \right) + 1 \]

\[ \left( \frac{\sqrt{a^2 + d^2 + z^2}}{\sqrt{a^2 + d^2 + z^2}} - \left( \frac{560 d^{10}}{a^2 + d^2 + z^2} - \frac{1360 d^8 z^2}{a^2 + d^2 + z^2} + \frac{400 d^6 d^2 z^2}{a^2 + d^2 + z^2} + \frac{30 d^4 d^4 z^2}{a^2 + d^2 + z^2} - \frac{400 a^2 d^6 z^2}{a^2 + d^2 + z^2} + \frac{1360 d^8 z^2}{a^2 + d^2 + z^2} + \frac{848 d^6 z^4}{a^2 + d^2 + z^2} \right) \right) \right) + 1 \]
III.3 Analytical Formulae for $K'_{ij,n}$, $i,j=0,1,...,5$

\[
K'_{y00,n} = -\log\left[ a + \sqrt{a^2 + c^2 + z^2} \right] + \log\left[ b + \sqrt{b^2 + c^2 + z^2} \right] + \\
\log\left[ a + \sqrt{a^2 + d^2 + z^2} \right] - \log\left[ b + \sqrt{b^2 + d^2 + z^2} \right]
\]

\[
K'_{y10,n} = -\sqrt{a^2 + c^2 + z^2} + \sqrt{b^2 + c^2 + z^2} + \sqrt{a^2 + d^2 + z^2} - \sqrt{b^2 + d^2 + z^2}
\]

\[
K'_{y20,n} = \frac{1}{2}\left( -a\sqrt{a^2 + c^2 + z^2} + b\sqrt{b^2 + c^2 + z^2} + a\sqrt{a^2 + d^2 + z^2} - b\sqrt{b^2 + d^2 + z^2} + \\
(c^2 + z^2)\log\left[ \frac{a + \sqrt{a^2 + c^2 + z^2}}{b + \sqrt{b^2 + c^2 + z^2}} \right] + (d^2 + z^2)\log\left[ \frac{b + \sqrt{b^2 + d^2 + z^2}}{a + \sqrt{a^2 + d^2 + z^2}} \right] \right)
\]

\[
K'_{y30,n} = \frac{1}{3}\left[ 2z^2\sqrt{a^2 + c^2 + z^2} + b^2\sqrt{b^2 + c^2 + z^2} - \\
2z^2\sqrt{b^2 + c^2 + z^2} - 2d^2\sqrt{a^2 + d^2 + z^2} - 2z^2\sqrt{a^2 + d^2 + z^2} - \\
b^2\sqrt{b^2 + d^2 + z^2} + 2d^2\sqrt{b^2 + d^2 + z^2} + 2z^2\sqrt{b^2 + d^2 + z^2} + \\
2c^2\left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) + a^2\left( -\sqrt{a^2 + c^2 + z^2} + \sqrt{a^2 + d^2 + z^2} \right) \right]
\]
\[ K'_{y40,n} = \frac{1}{8} \left( \frac{2a^5}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^3c^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{3ac^4}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^3z^2}{\sqrt{a^2 + c^2 + z^2}} \right) + \frac{6ac^2z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{3az^4}{\sqrt{a^2 + c^2 + z^2}} + \frac{2b^5}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^3c^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{3bc^4}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^3z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{6bc^2z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{3bz^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{2a^5}{\sqrt{a^2 + d^2 + z^2}} + \frac{a^3d^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{3ad^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{a^3z^2}{\sqrt{a^2 + d^2 + z^2}} \right) + \frac{6ad^2z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{3az^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{2b^5}{\sqrt{b^2 + d^2 + z^2}} + \frac{b^3d^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{3bd^4}{\sqrt{b^2 + d^2 + z^2}} - \frac{b^3z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{6bd^2z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{3bz^4}{\sqrt{b^2 + d^2 + z^2}} \right) + \frac{3(c^2 + z^2)^2 \log(a + \sqrt{a^2 + c^2 + z^2}) + 3(c^2 + z^2)^2 \log(b + \sqrt{b^2 + c^2 + z^2}) + 3d^4 \log(a + \sqrt{a^2 + d^2 + z^2}) + 6d^2z^2 \log(a + \sqrt{a^2 + d^2 + z^2}) + 3z^4 \log(a + \sqrt{a^2 + d^2 + z^2}) - 3d^4 \log(b + \sqrt{b^2 + d^2 + z^2}) - 6d^2z^2 \log(b + \sqrt{b^2 + d^2 + z^2}) - 3z^4 \log(b + \sqrt{b^2 + d^2 + z^2}) \right) \]
\[ K'_{y_{50, x}} = \]
\[
\frac{1}{15} \left( -\frac{8}{\sqrt{a^2 + c^2 + z^2}} + \frac{3}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^4 z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{4 b^2 z^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{8 z^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{8 d^6}{\sqrt{a^2 + d^2 + z^2}} + \frac{24 d^4 z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{24 d^2 z^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{8 z^6}{\sqrt{a^2 + d^2 + z^2}} + \frac{2 d^2 z^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{8 b^2 d^2 z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{24 d^4 z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{4 b^2 z^4}{\sqrt{b^2 + d^2 + z^2}} - \frac{b^4 z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{8 d^6}{\sqrt{b^2 + d^2 + z^2}} + \frac{b^4 z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{8 b^2 d^2 z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{24 d^4 z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{8 b^2 d^2 z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{24 d^4 z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{4 b^2 z^4}{\sqrt{b^2 + d^2 + z^2}} + \frac{24 d^4 z^4}{\sqrt{b^2 + d^2 + z^2}} + \frac{8 z^6}{\sqrt{b^2 + d^2 + z^2}} + 8 c^6 \left( -\frac{1}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{b^2 + c^2 + z^2}} \right) + \frac{3}{\sqrt{a^2 + c^2 + z^2}} \right) + \frac{3}{\sqrt{a^2 + d^2 + z^2}} + \frac{b^4}{\sqrt{b^2 + c^2 + z^2}} + c^2 \left( -\frac{8 b^2 z^2}{\sqrt{b^2 + c^2 + z^2}} + 24 z^4 \left( -\frac{1}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{b^2 + c^2 + z^2}} \right) \right) + 4 c^4 \left( -\frac{b^2}{\sqrt{b^2 + c^2 + z^2}} + z^2 \left( -\frac{6}{\sqrt{a^2 + c^2 + z^2}} + \frac{6}{\sqrt{b^2 + c^2 + z^2}} \right) \right) + a^4 \left( -\frac{c^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{d^2}{\sqrt{a^2 + d^2 + z^2}} + z^2 \left( -\frac{1}{\sqrt{a^2 + c^2 + z^2}} - \frac{1}{\sqrt{a^2 + d^2 + z^2}} \right) \right) + 4 a^2 \left( -\frac{c^4}{\sqrt{a^2 + c^2 + z^2}} - \frac{2 c^2 z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{d^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{2 d^2 z^4}{\sqrt{a^2 + d^2 + z^2}} + z^4 \left( -\frac{1}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{a^2 + d^2 + z^2}} \right) \right) \right)} \]
\[ K'_{y_{01,n}} = -z \left( \text{ArcTan}\left[ c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, -az \right] + \right. \]

\[ \left. \text{ArcTan}\left[ c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, bz \right] + \text{ArcTan}\left[ d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, az \right] + \right. \]

\[ \left. \text{ArcTan}\left[ d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, -bz \right] + \right) \]

\[ a \log\left( \frac{c + \sqrt{a^2 + c^2 + z^2}}{d + \sqrt{a^2 + d^2 + z^2}} \right) + b \log\left( \frac{d + \sqrt{b^2 + d^2 + z^2}}{c + \sqrt{b^2 + c^2 + z^2}} \right) \]

\[ K'_{y_{11,n}} = \frac{1}{2} c \left( -\sqrt{a^2 + c^2 + z^2} + \sqrt{b^2 + c^2 + z^2} \right) + \frac{1}{2} d \left( \sqrt{a^2 + d^2 + z^2} - \sqrt{b^2 + d^2 + z^2} \right) + \]

\[ \frac{1}{2} (a^2 + z^2) \log\left( \frac{c + \sqrt{a^2 + c^2 + z^2}}{d + \sqrt{a^2 + d^2 + z^2}} \right) + \frac{1}{2} (b^2 + z^2) \log\left( \frac{d + \sqrt{b^2 + d^2 + z^2}}{c + \sqrt{b^2 + c^2 + z^2}} \right) \]

\[ K'_{y_{21,n}} = -\frac{1}{3} z^3 \]

\[ \left( \text{ArcTan}\left[ a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, cz \right] + \text{ArcTan}\left[ b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, -cz \right] + \right. \]

\[ \left. \text{ArcTan}\left[ a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2}, -dz \right] + \right. \]

\[ \left. \text{ArcTan}\left[ b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, dz \right] + \right) + \]

\[ \frac{1}{6} (-2 ac \sqrt{a^2 + c^2 + z^2} + 2 bc \sqrt{b^2 + c^2 + z^2} + 2 ad \sqrt{a^2 + d^2 + z^2} - \]

\[ 2 bd \sqrt{b^2 + d^2 + z^2} + 2 c^3 \log(a + \sqrt{a^2 + c^2 + z^2}) + 2 a^3 \log(c + \sqrt{a^2 + c^2 + z^2}) - \]

\[ 2 c^3 \log(b + \sqrt{b^2 + c^2 + z^2}) - 2 b^3 \log(c + \sqrt{b^2 + c^2 + z^2}) - \]

\[ 2 d^3 \log(a + \sqrt{a^2 + d^2 + z^2}) - 2 a^3 \log(d + \sqrt{a^2 + d^2 + z^2}) + \]

\[ 2 d^3 \log(b + \sqrt{b^2 + d^2 + z^2}) + 2 b^3 \log(d + \sqrt{b^2 + d^2 + z^2}) \) \]
\[ K'_{y_{31,a}} = \]
\[
\frac{1}{4} \left( -a^2 c \sqrt{a^2 + c^2 + z^2} + 2c^3 \sqrt{a^2 + c^2 + z^2} + c^2 \sqrt{a^2 + c^2 + z^2} + \right.
\]
\[
b^2 c \sqrt{b^2 + c^2 + z^2} - 2c^3 \sqrt{b^2 + c^2 + z^2} - c^2 \sqrt{b^2 + c^2 + z^2} + \right.
\]
\[
a^2 d \sqrt{a^2 + d^2 + z^2} - 2d^3 \sqrt{a^2 + d^2 + z^2} - d^2 \sqrt{a^2 + d^2 + z^2} - \right.
\]
\[
b^2 d \sqrt{b^2 + d^2 + z^2} + 2d^3 \sqrt{b^2 + d^2 + z^2} + d^2 \sqrt{b^2 + d^2 + z^2} + \right.
\]
\[
(a^4 - z^4) \log\left[ c + \sqrt{a^2 + c^2 + z^2} \right] + (-b^4 + z^4) \log\left[ c + \sqrt{b^2 + c^2 + z^2} \right] - \right.
\]
\[
a^4 \log\left[ d + \sqrt{a^2 + d^2 + z^2} \right] + z^4 \log\left[ d + \sqrt{a^2 + d^2 + z^2} \right] + \right.
\]
\[
b^4 \log\left[ d + \sqrt{b^2 + d^2 + z^2} \right] - z^4 \log\left[ d + \sqrt{b^2 + d^2 + z^2} \right] \right) \]

\[ K'_{y_{31,a}} = \]
\[
-\frac{1}{5} z^5 \left( \arctan\left[ a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, -cz \right] + \right.
\]
\[
\arctan\left[ b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, cz \right] + \arctan\left[ a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2}, dz \right] + \right.
\]
\[
\arctan\left[ b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, -dz \right] \right) + \right.
\]
\[
\frac{1}{10} \left( -2a^3 c \sqrt{a^2 + c^2 + z^2} + 3ac^3 \sqrt{a^2 + c^2 + z^2} + 2ac^2 \sqrt{a^2 + c^2 + z^2} + \right.
\]
\[
2b^3 c \sqrt{b^2 + c^2 + z^2} - 3bc^3 \sqrt{b^2 + c^2 + z^2} - 2bc^2 \sqrt{b^2 + c^2 + z^2} + \right.
\]
\[
2a^3 d \sqrt{a^2 + d^2 + z^2} - 3ad^3 \sqrt{a^2 + d^2 + z^2} - 2ad^2 \sqrt{a^2 + d^2 + z^2} - \right.
\]
\[
2b^3 d \sqrt{b^2 + d^2 + z^2} + 3bd^3 \sqrt{b^2 + d^2 + z^2} + \right.
\]
\[
2bdz^2 \sqrt{b^2 + d^2 + z^2} - (3c^5 + 5c^3 z^2) \log\left[ a + \sqrt{a^2 + c^2 + z^2} \right] + \right.
\]
\[
2a^5 \log\left[ c + \sqrt{a^2 + c^2 + z^2} \right] + 3c^5 \log\left[ b + \sqrt{b^2 + c^2 + z^2} \right] + \right.
\]
\[
5c^3 z^2 \log\left[ b + \sqrt{b^2 + c^2 + z^2} \right] - 2b^5 \log\left[ c + \sqrt{b^2 + c^2 + z^2} \right] + \right.
\]
\[
3d^5 \log\left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 5d^3 z^2 \log\left[ a + \sqrt{a^2 + d^2 + z^2} \right] - \right.
\]
\[
2a^5 \log\left[ d + \sqrt{a^2 + d^2 + z^2} \right] - 3d^5 \log\left[ b + \sqrt{b^2 + d^2 + z^2} \right] - \right.
\]
\[
5d^3 z^2 \log\left[ b + \sqrt{b^2 + d^2 + z^2} \right] + 2b^5 \log\left[ d + \sqrt{b^2 + d^2 + z^2} \right] \right) \]
\[ K'_{y,51,n} = \]
\[
\frac{1}{18} \left( -\frac{3 \, \text{a}^6 \, \text{c}}{\sqrt{\text{a}^2 + \text{c}^2 + \text{z}^2}} + \frac{\text{a}^4 \, \text{c}^3}{\sqrt{\text{a}^2 + \text{c}^2 + \text{z}^2}} - \frac{4 \, \text{a}^2 \, \text{c}^5}{\sqrt{\text{a}^2 + \text{c}^2 + \text{z}^2}} - \frac{8 \, \text{c}^7}{\sqrt{\text{a}^2 + \text{c}^2 + \text{z}^2}} \right) - \frac{7 \, \text{b}^6 \, \text{c}^2}{\sqrt{\text{a}^2 + \text{c}^2 + \text{z}^2}} + \frac{22 \, \text{c}^5 \, \text{z}^2}{\sqrt{\text{a}^2 + \text{c}^2 + \text{z}^2}} - \frac{17 \, \text{c}^3 \, \text{z}^4}{\sqrt{\text{a}^2 + \text{c}^2 + \text{z}^2}} - \frac{3 \, \text{c} \, \text{z}^6}{\sqrt{\text{a}^2 + \text{c}^2 + \text{z}^2}} + \frac{3 \, \text{b}^6 \, \text{c}}{\sqrt{\text{b}^2 + \text{c}^2 + \text{z}^2}} - \frac{\text{b}^4 \, \text{c}^3}{\sqrt{\text{b}^2 + \text{c}^2 + \text{z}^2}} + \frac{4 \, \text{b}^2 \, \text{c}^5}{\sqrt{\text{b}^2 + \text{c}^2 + \text{z}^2}} - \frac{8 \, \text{c}^7}{\sqrt{\text{b}^2 + \text{c}^2 + \text{z}^2}} \right) - \frac{7 \, \text{b}^2 \, \text{c}^3 \, \text{z}^2}{\sqrt{\text{b}^2 + \text{c}^2 + \text{z}^2}} + \frac{22 \, \text{c}^5 \, \text{z}^2}{\sqrt{\text{b}^2 + \text{c}^2 + \text{z}^2}} - \frac{17 \, \text{c}^3 \, \text{z}^4}{\sqrt{\text{b}^2 + \text{c}^2 + \text{z}^2}} - \frac{3 \, \text{c} \, \text{z}^6}{\sqrt{\text{b}^2 + \text{c}^2 + \text{z}^2}} + \frac{3 \, \text{a}^6 \, \text{d}}{\sqrt{\text{a}^2 + \text{d}^2 + \text{z}^2}} - \frac{\text{a}^4 \, \text{d}^3}{\sqrt{\text{a}^2 + \text{d}^2 + \text{z}^2}} + \frac{4 \, \text{a}^2 \, \text{d}^5}{\sqrt{\text{a}^2 + \text{d}^2 + \text{z}^2}} - \frac{8 \, \text{d}^7}{\sqrt{\text{a}^2 + \text{d}^2 + \text{z}^2}} \right) - \frac{7 \, \text{a}^2 \, \text{d}^3 \, \text{z}^2}{\sqrt{\text{a}^2 + \text{d}^2 + \text{z}^2}} + \frac{22 \, \text{d}^5 \, \text{z}^2}{\sqrt{\text{a}^2 + \text{d}^2 + \text{z}^2}} - \frac{17 \, \text{d}^3 \, \text{z}^4}{\sqrt{\text{a}^2 + \text{d}^2 + \text{z}^2}} - \frac{3 \, \text{d} \, \text{z}^6}{\sqrt{\text{a}^2 + \text{d}^2 + \text{z}^2}} + \frac{3 \, \text{b}^6 \, \text{d}}{\sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2}} + \frac{\text{b}^4 \, \text{d}^3}{\sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2}} - \frac{4 \, \text{b}^2 \, \text{d}^5}{\sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2}} - \frac{8 \, \text{d}^7}{\sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2}} \right) - \frac{7 \, \text{b}^2 \, \text{d}^3 \, \text{z}^2}{\sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2}} + \frac{22 \, \text{d}^5 \, \text{z}^2}{\sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2}} - \frac{17 \, \text{d}^3 \, \text{z}^4}{\sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2}} - \frac{3 \, \text{d} \, \text{z}^6}{\sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2}} + \frac{3 \, (\text{a}^6 + \text{z}^6) \, \text{Log}[\text{c} + \sqrt{\text{a}^2 + \text{c}^2 + \text{z}^2}] - 3 \, (\text{b}^6 + \text{z}^6) \, \text{Log}[\text{c} + \sqrt{\text{b}^2 + \text{c}^2 + \text{z}^2}] - 3 \, \text{a}^6 \, \text{Log}[\text{d} + \sqrt{\text{a}^2 + \text{d}^2 + \text{z}^2}] - 3 \, \text{b}^6 \, \text{Log}[\text{d} + \sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2}] + 3 \, \text{d}^6 \, \text{Log}[\text{d} + \sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2}] + 3 \, \text{b}^6 \, \text{Log}[\text{d} + \sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2}] + 3 \, \text{b}^6 \, \text{Log}[\text{d} + \sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2}] + 3 \, \text{z}^6 \, \text{Log}[\text{d} + \sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2}]
\right)\]

\[ K'_{y,02,n} = \]
\[
\text{a} \sqrt{\text{a}^2 + \text{c}^2 + \text{z}^2} - \text{b} \sqrt{\text{b}^2 + \text{c}^2 + \text{z}^2} - \text{a} \sqrt{\text{a}^2 + \text{d}^2 + \text{z}^2} + \text{b} \sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2} + \text{z}^2 \text{Log}[\text{a} + \sqrt{\text{a}^2 + \text{d}^2 + \text{z}^2}] + \text{z}^2 \text{Log}[\text{b} + \sqrt{\text{b}^2 + \text{d}^2 + \text{z}^2}]
\]
\[ K'_{y_{12,n}} = \frac{1}{3} \left( 2 z^2 \sqrt{a^2 + c^2 + z^2} - 2 b^2 \sqrt{b^2 + c^2 + z^2} - \right. \\
\left. 2 z^2 \sqrt{b^2 + c^2 + z^2} + d^2 \sqrt{a^2 + d^2 + z^2} - 2 z^2 \sqrt{a^2 + d^2 + z^2} + \\
2 b^2 \sqrt{b^2 + d^2 + z^2} - d^2 \sqrt{b^2 + d^2 + z^2} + 2 z^2 \sqrt{b^2 + d^2 + z^2} + \\
c^2 \left( -\sqrt{a^2 + c^2 + z^2} + \sqrt{b^2 + c^2 + z^2} \right) + 2 a^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) \]

\[ K'_{y_{22,n}} = \frac{1}{4} \left( 2 a^3 \sqrt{a^2 + c^2 + z^2} - a^2 \sqrt{a^2 + c^2 + z^2} + a z^2 \sqrt{a^2 + c^2 + z^2} - 2 b^3 \sqrt{b^2 + c^2 + z^2} + \\
b c^2 \sqrt{b^2 + c^2 + z^2} - b z^2 \sqrt{b^2 + c^2 + z^2} - 2 a^3 \sqrt{a^2 + d^2 + z^2} + a d^2 \sqrt{a^2 + d^2 + z^2} - \\
a z^2 \sqrt{a^2 + d^2 + z^2} + 2 b^3 \sqrt{b^2 + d^2 + z^2} - b d^2 \sqrt{b^2 + d^2 + z^2} + \\
b z^2 \sqrt{b^2 + d^2 + z^2} + \left( c^4 - z^4 \right) \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] - c^4 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + \\
z^4 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + \left( - d^4 + z^4 \right) \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + \\
d^4 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - z^4 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] \right) \]

\[ K'_{y_{32,n}} = \frac{1}{15} \left( -4 z^4 \sqrt{a^2 + c^2 + z^2} - 6 b^4 \sqrt{b^2 + c^2 + z^2} - \\
2 b^2 z^2 \sqrt{b^2 + c^2 + z^2} + 4 z^2 \sqrt{b^2 + c^2 + z^2} - 6 d^4 \sqrt{a^2 + d^2 + z^2} - \\
2 d^2 z^2 \sqrt{b^2 + d^2 + z^2} + 4 z^2 \sqrt{b^2 + d^2 + z^2} + 6 b^4 \sqrt{b^2 + d^2 + z^2} - \\
3 b^2 d^2 \sqrt{b^2 + d^2 + z^2} + 6 d^4 \sqrt{b^2 + d^2 + z^2} + \\
2 b^2 z^2 \sqrt{b^2 + d^2 + z^2} + 2 d^2 z^2 \sqrt{b^2 + d^2 + z^2} - 4 z^4 \sqrt{b^2 + d^2 + z^2} + \\
6 c^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) + 6 a^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) + \\
c^2 \left( 3 b^2 \sqrt{b^2 + c^2 + z^2} + 2 z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) \right) + \\
a^2 \left( -3 c^2 \sqrt{a^2 + c^2 + z^2} + 3 d^2 \sqrt{a^2 + d^2 + z^2} + \\
2 z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) \right) \]
\[ K'_{y,z,x} = \frac{1}{24} \left( 8 a^5 \sqrt{a^2 + c^2 + z^2} - 4 a^3 c^2 \sqrt{a^2 + c^2 + z^2} + 
\right. \\
6 ac^4 \sqrt{a^2 + c^2 + z^2} + 2 a^3 z^2 \sqrt{a^2 + c^2 + z^2} + 3 ac^2 z^2 \sqrt{a^2 + c^2 + z^2} - \\
3 az^4 \sqrt{a^2 + c^2 + z^2} - 8 b^5 \sqrt{b^2 + c^2 + z^2} + 4 b^3 c^2 \sqrt{b^2 + c^2 + z^2} - \\
6 b c^4 \sqrt{b^2 + c^2 + z^2} - 2 b^3 z^2 \sqrt{b^2 + c^2 + z^2} - 3 b c^2 z^2 \sqrt{b^2 + c^2 + z^2} + \\
3 b z^4 \sqrt{b^2 + c^2 + z^2} - 8 a^5 \sqrt{a^2 + d^2 + z^2} + 4 a^3 d^2 \sqrt{a^2 + d^2 + z^2} - \\
6 ad^4 \sqrt{a^2 + d^2 + z^2} - 2 a^3 z^2 \sqrt{a^2 + d^2 + z^2} - 3 ad^2 z^2 \sqrt{a^2 + d^2 + z^2} + \\
3 az^4 \sqrt{a^2 + d^2 + z^2} + 8 b^5 \sqrt{b^2 + d^2 + z^2} - 4 b^3 d^2 \sqrt{b^2 + d^2 + z^2} - \\
6 b d^4 \sqrt{b^2 + d^2 + z^2} + 2 b^3 z^2 \sqrt{b^2 + d^2 + z^2} + 3 b d^2 z^2 \sqrt{b^2 + d^2 + z^2} - \\
3 b z^4 \sqrt{b^2 + d^2 + z^2} - 3 (2 c^6 + 3 c^4 z^2 - z^6) \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] + \\
3 (2 c^6 + 3 c^4 z^2 - z^6) \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + \\
6 d^6 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 9 d^4 z^2 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - \\
3 z^6 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - 6 d^6 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - \\
9 d^4 z^2 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] + 3 z^6 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] \right) \]
\[ K'_{y_{32,n}} = \frac{1}{105} \left( 16 z^6 \sqrt{a^2 + c^2 + z^2} - 30 b^6 \sqrt{b^2 + c^2 + z^2} - 6 b^4 z^2 \sqrt{b^2 + c^2 + z^2} + 8 b^2 z^4 \sqrt{b^2 + c^2 + z^2} - 16 z^6 \sqrt{b^2 + c^2 + z^2} + 40 d^6 \sqrt{a^2 + d^2 + z^2} + 64 d^4 z^2 \sqrt{a^2 + d^2 + z^2} + 8 d^2 z^4 \sqrt{a^2 + d^2 + z^2} - 16 z^6 \sqrt{a^2 + d^2 + z^2} + 30 b^6 \sqrt{b^2 + d^2 + z^2} - 15 b^4 d^2 \sqrt{b^2 + d^2 + z^2} + 20 b^2 d^4 \sqrt{b^2 + d^2 + z^2} + 6 b^4 z^2 \sqrt{b^2 + d^2 + z^2} + 12 b^2 d^2 \sqrt{b^2 + d^2 + z^2} - 8 b^2 z^4 \sqrt{b^2 + d^2 + z^2} - 8 d^2 z^4 \sqrt{b^2 + d^2 + z^2} + 16 z^6 \sqrt{b^2 + d^2 + z^2} - 40 d^6 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) + 30 a^6 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) - 4 c^4 \left( 5 b^2 \sqrt{b^2 + c^2 + z^2} + 16 z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) \right) + c^2 \left( 15 b^4 \sqrt{b^2 + c^2 + z^2} - 12 b^2 z^2 \sqrt{b^2 + c^2 + z^2} - 8 z^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) \right) - 3 a^4 \left( 5 c^2 \sqrt{a^2 + c^2 + z^2} - 5 d^2 \sqrt{a^2 + d^2 + z^2} - 2 z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) + 4 a^2 \left( 5 c^4 \sqrt{a^2 + c^2 + z^2} + 3 c^2 z^2 \sqrt{a^2 + c^2 + z^2} - 5 d^4 \sqrt{a^2 + d^2 + z^2} - 3 d^2 z^2 \sqrt{a^2 + d^2 + z^2} - 2 z^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) \right) \right) \]

\[ K'_{y_{03,n}} = -z^3 \left( \text{ArcTan} \left[ c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, a z \right] + \text{ArcTan} \left[ c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, -b z \right] + \text{ArcTan} \left[ d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, -a z \right] + \text{ArcTan} \left[ d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, b z \right] \right) + \frac{1}{2} \left( a c \sqrt{a^2 + c^2 + z^2} - b c \sqrt{b^2 + c^2 + z^2} - a d \sqrt{a^2 + d^2 + z^2} + b d \sqrt{b^2 + d^2 + z^2} - (a^2 + 3 a z^2) \log \left( c + \sqrt{a^2 + c^2 + z^2} \right) + b^2 \log \left( c + \sqrt{b^2 + c^2 + z^2} \right) + 3 b z^2 \log \left( c + \sqrt{b^2 + c^2 + z^2} \right) + a^3 \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) + 3 a z^2 \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) - b^3 \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) - 3 b z^2 \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) \right) \]

286
\[ K_{y13,\pi} ' = \frac{1}{8} \left( \frac{3 a^4 c}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^2 c^3}{\sqrt{a^2 + c^2 + z^2}} - \frac{2 c^5}{\sqrt{a^2 + c^2 + z^2}} + \frac{6 a^2 c z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{c^3 z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{3 c z^4}{\sqrt{a^2 + c^2 + z^2}} - \frac{3 b^4 c}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^2 c^3}{\sqrt{b^2 + c^2 + z^2}} + \frac{2 c^5}{\sqrt{b^2 + c^2 + z^2}} - \frac{6 b^2 c z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{c^3 z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{3 c z^4}{\sqrt{b^2 + c^2 + z^2}} \right) 
\frac{3 a^4 d}{\sqrt{a^2 + d^2 + z^2}} - \frac{a^2 d^3}{\sqrt{a^2 + d^2 + z^2}} + \frac{2 d^5}{\sqrt{a^2 + d^2 + z^2}} - \frac{6 a^2 d z^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{d^3 z^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{3 d z^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{3 b^4 d}{\sqrt{b^2 + d^2 + z^2}} + \frac{b^2 d^3}{\sqrt{b^2 + d^2 + z^2}} - \frac{2 d^5}{\sqrt{b^2 + d^2 + z^2}} - \frac{6 b^2 d z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{d^3 z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{3 d z^4}{\sqrt{b^2 + d^2 + z^2}} \right) 
\left( 3 \left( a^2 + z^2 \right)^2 \log \left[ c + \sqrt{a^2 + c^2 + z^2} \right] + 3 b^4 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] + 6 b^2 z^2 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] \right) 
\left( 3 \left( a^2 + z^2 \right)^2 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] - 3 b^4 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] - 6 b^2 z^2 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] \right) \right) \]
\[ K'_{y,23,n} = \\
- \frac{1}{5} z^5 \left( \text{ArcTan} \left[ c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, -az \right] + \right. \\
\text{ArcTan} \left[ c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, bz \right] + \text{ArcTan} \left[ d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, az \right] + \\
\text{ArcTan} \left[ d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, -bz \right] + \right. \\
\left. \frac{1}{10} \left( 3a^3 c \sqrt{a^2 + c^2 + z^2} - 2ac^3 \sqrt{a^2 + c^2 + z^2} + 2ac^2 \sqrt{a^2 + c^2 + z^2} - \\
3b^3 c \sqrt{b^2 + c^2 + z^2} + 2bc^3 \sqrt{b^2 + c^2 + z^2} - 2bcz^2 \sqrt{b^2 + c^2 + z^2} - \\
3a^3 d \sqrt{a^2 + d^2 + z^2} + 2ad^3 \sqrt{a^2 + d^2 + z^2} - 2adz^2 \sqrt{a^2 + d^2 + z^2} + \\
3b^3 d \sqrt{b^2 + d^2 + z^2} - 2bd^3 \sqrt{b^2 + d^2 + z^2} + \\
2bdz^2 \sqrt{b^2 + d^2 + z^2} + 2c^5 \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] - \\
(3a^5 + 5a^3 z^2) \log \left[ c + \sqrt{a^2 + c^2 + z^2} \right] - 2c^5 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + \\
3b^5 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] + 5b^3 z^2 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] - \\
2d^5 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 3a^5 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] + \\
5a^3 z^2 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] + 2d^5 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - \\
3b^5 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] - 5b^3 z^2 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] \right) \]
\[ K'_{y,33,n} = \frac{1}{24} \left( 6 a^4 c \sqrt{a^2 + c^2 + z^2} - 4 a^2 c^3 \sqrt{a^2 + c^2 + z^2} + 8 c^5 \sqrt{a^2 + c^2 + z^2} + \
3 a^2 c z^2 \sqrt{a^2 + c^2 + z^2} + 2 a^2 \sqrt{a^2 + c^2 + z^2} - 3 c z^4 \sqrt{a^2 + c^2 + z^2} - \
6 b^4 c \sqrt{b^2 + c^2 + z^2} + 4 b^2 c^3 \sqrt{b^2 + c^2 + z^2} - 8 c^5 \sqrt{b^2 + c^2 + z^2} - \
3 b^2 c z^2 \sqrt{b^2 + c^2 + z^2} + 2 c^4 \sqrt{b^2 + c^2 + z^2} + 3 c z^4 \sqrt{b^2 + c^2 + z^2} - \
6 a^4 d \sqrt{a^2 + d^2 + z^2} + 4 a^2 d^3 \sqrt{a^2 + d^2 + z^2} - 8 d^5 \sqrt{a^2 + d^2 + z^2} - \
3 a^2 d z^2 \sqrt{a^2 + d^2 + z^2} - 2 d^3 z^2 \sqrt{a^2 + d^2 + z^2} + 3 d z^4 \sqrt{a^2 + d^2 + z^2} + \
6 b^4 d \sqrt{b^2 + d^2 + z^2} - 4 b^2 d^3 \sqrt{b^2 + d^2 + z^2} + 8 d^5 \sqrt{b^2 + d^2 + z^2} - \
3 b^2 d z^2 \sqrt{b^2 + d^2 + z^2} - 2 d^3 z^2 \sqrt{b^2 + d^2 + z^2} + 8 d^5 \sqrt{b^2 + d^2 + z^2} - \
3 (2 a^6 + 3 a^4 z^2 - z^6) \log[c + \sqrt{a^2 + c^2 + z^2}] + 6 b^6 \log[c + \sqrt{b^2 + c^2 + z^2}] + \
9 b^4 z^2 \log[c + \sqrt{b^2 + c^2 + z^2}] - 3 z^6 \log[c + \sqrt{b^2 + c^2 + z^2}] + \
3 (2 a^6 + 3 a^4 z^2 - z^6) \log[d + \sqrt{a^2 + d^2 + z^2}] - 6 b^6 \log[d + \sqrt{b^2 + d^2 + z^2}] - \
9 b^4 z^2 \log[d + \sqrt{b^2 + d^2 + z^2}] + 3 z^6 \log[d + \sqrt{b^2 + d^2 + z^2}]] \]
\[ K'_{y_{43,n}} = \frac{3}{35} z^7 \]

\[
\left( \text{ArcTan}\left[ b^2 + c^2 + z^2 , a z \right] + \text{ArcTan}\left[ c^2 + b^2 + c^2 + z^2 , -b z \right] + \text{ArcTan}\left[ d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2} , -a z \right] + \right.
\]

\[
\frac{1}{70} \left( 15 a^5 c \sqrt{a^2 + c^2 + z^2} - 10 a^3 c^3 \sqrt{a^2 + c^2 + z^2} + 15 a c^5 \sqrt{a^2 + c^2 + z^2} + 6 a^3 c^2 \sqrt{a^2 + c^2 + z^2} + 6 a c^3 z^2 \sqrt{a^2 + c^2 + z^2} - 6 a c z^4 \sqrt{a^2 + c^2 + z^2} - 15 b^5 c \sqrt{b^2 + c^2 + z^2} + 10 b^3 c^3 \sqrt{b^2 + c^2 + z^2} - 15 b c^5 \sqrt{b^2 + c^2 + z^2} - 6 b^3 c z^2 \sqrt{b^2 + c^2 + z^2} - 6 b c^3 z^2 \sqrt{b^2 + c^2 + z^2} + 6 b c z^4 \sqrt{b^2 + c^2 + z^2} - 15 a^5 d \sqrt{a^2 + d^2 + z^2} + 10 a^3 d^3 \sqrt{a^2 + d^2 + z^2} - 15 a d^5 \sqrt{a^2 + d^2 + z^2} - 6 a^3 d z^2 \sqrt{a^2 + d^2 + z^2} - 6 a d^3 z^2 \sqrt{a^2 + d^2 + z^2} + 6 a d z^4 \sqrt{a^2 + d^2 + z^2} + 15 b^5 d \sqrt{b^2 + d^2 + z^2} - 10 b^3 d^3 \sqrt{b^2 + d^2 + z^2} + 15 b d^5 \sqrt{b^2 + d^2 + z^2} - 6 b^3 d z^2 \sqrt{b^2 + d^2 + z^2} + 6 b d^3 z^2 \sqrt{b^2 + d^2 + z^2} - 6 b d z^4 \sqrt{b^2 + d^2 + z^2} - 3(5 c^2 + 7 c^2 z^2) \log(a + \sqrt{a^2 + c^2 + z^2}) - 3(5 a^2 + 7 a^2 z^2) \log(c + \sqrt{a^2 + c^2 + z^2}) + 15 c^7 \log(\sqrt{b^2 + c^2 + z^2}) + 21 c^5 z^2 \log(\sqrt{b^2 + c^2 + z^2}) + 15 b^7 \log(\sqrt{c + b^2 + c^2 + z^2}) + 21 b^5 z^2 \log(\sqrt{c + b^2 + c^2 + z^2}) + 15 d^7 \log(\sqrt{a + d^2 + a^2 + z^2}) + 21 d^5 z^2 \log(\sqrt{a + d^2 + a^2 + z^2}) + 15 a^7 \log(\sqrt{d + a^2 + d^2 + z^2}) + 21 a^5 z^2 \log(\sqrt{d + a^2 + d^2 + z^2}) - 15 d^7 \log(\sqrt{b + b^2 + d^2 + z^2}) - 21 b^7 \log(\sqrt{b + b^2 + d^2 + z^2}) - 15 b^7 \log(\sqrt{d + b^2 + d^2 + z^2}) - 21 d^7 \log(\sqrt{d + b^2 + d^2 + z^2}) \right)
\]
\[ K'_{y_{53,n}} = \]
\[
\begin{align*}
1 & \left( \frac{9a^8c}{\sqrt{a^2+c^2+z^2}} + \frac{3a^6c^3}{\sqrt{a^2+c^2+z^2}} + \frac{2a^4c^5}{\sqrt{a^2+c^2+z^2}} + \frac{8a^2c^7}{\sqrt{a^2+c^2+z^2}} + \frac{16c^9}{\sqrt{a^2+c^2+z^2}} \right) \\
& \left( \frac{12a^6cz^2}{\sqrt{a^2+c^2+z^2}} + \frac{a^4c^3z^2}{\sqrt{a^2+c^2+z^2}} + \frac{a^2c^5z^2}{\sqrt{a^2+c^2+z^2}} + \frac{40c^7z^2}{\sqrt{a^2+c^2+z^2}} + \frac{a^2c^3z^4}{\sqrt{a^2+c^2+z^2}} \right) \\
& \left( \frac{26c^5z^4}{\sqrt{a^2+c^2+z^2}} + \frac{c^3z^6}{\sqrt{a^2+c^2+z^2}} + \frac{3cz^8}{\sqrt{a^2+c^2+z^2}} + \frac{9b^8c}{\sqrt{a^2+c^2+z^2}} + \frac{3b^6c^3}{\sqrt{a^2+c^2+z^2}} \right) \\
& \left( \frac{2b^4c^5}{\sqrt{a^2+c^2+z^2}} + \frac{8b^2c^7}{\sqrt{a^2+c^2+z^2}} + \frac{16c^9}{\sqrt{a^2+c^2+z^2}} + \frac{12b^6cz^2}{\sqrt{a^2+c^2+z^2}} + \frac{b^4c^3z^2}{\sqrt{a^2+c^2+z^2}} \right) \\
& \left( \frac{12b^2c^5z^2}{\sqrt{a^2+c^2+z^2}} + \frac{40c^7z^2}{\sqrt{a^2+c^2+z^2}} + \frac{b^2c^3z^4}{\sqrt{a^2+c^2+z^2}} + \frac{26c^5z^4}{\sqrt{a^2+c^2+z^2}} + \frac{c^3z^6}{\sqrt{a^2+c^2+z^2}} \right) \\
& \left( \frac{3cz^8}{\sqrt{a^2+c^2+z^2}} + \frac{9a^8d}{\sqrt{a^2+d^2+z^2}} + \frac{3a^6d^3}{\sqrt{a^2+d^2+z^2}} + \frac{2a^4d^5}{\sqrt{a^2+d^2+z^2}} + \frac{8a^2d^7}{\sqrt{a^2+d^2+z^2}} \right) \\
& \left( \frac{16d^9}{\sqrt{a^2+d^2+z^2}} + \frac{12a^6dz^2}{\sqrt{a^2+d^2+z^2}} + \frac{a^4d^3z^2}{\sqrt{a^2+d^2+z^2}} + \frac{12a^2d^5z^2}{\sqrt{a^2+d^2+z^2}} + \frac{40d^7z^2}{\sqrt{a^2+d^2+z^2}} \right) \\
& \left( \frac{a^2d^3z^4}{\sqrt{a^2+d^2+z^2}} + \frac{26d^5z^4}{\sqrt{a^2+d^2+z^2}} + \frac{d^3z^6}{\sqrt{a^2+d^2+z^2}} + \frac{3dz^8}{\sqrt{a^2+d^2+z^2}} + \frac{9b^8d}{\sqrt{a^2+d^2+z^2}} \right) \\
& \left( \frac{3b^6d^3}{\sqrt{a^2+d^2+z^2}} + \frac{2b^4d^5}{\sqrt{a^2+d^2+z^2}} + \frac{8b^2d^7}{\sqrt{a^2+d^2+z^2}} + \frac{16d^9}{\sqrt{a^2+d^2+z^2}} \right) \\
& \left( \frac{12b^6dz^2}{\sqrt{a^2+d^2+z^2}} + \frac{b^4d^3z^2}{\sqrt{a^2+d^2+z^2}} + \frac{12b^2d^5z^2}{\sqrt{a^2+d^2+z^2}} + \frac{40d^7z^2}{\sqrt{a^2+d^2+z^2}} \right) \\
& \left( \frac{b^2d^3z^4}{\sqrt{a^2+d^2+z^2}} + \frac{26d^5z^4}{\sqrt{a^2+d^2+z^2}} + \frac{d^3z^6}{\sqrt{a^2+d^2+z^2}} + \frac{3dz^8}{\sqrt{a^2+d^2+z^2}} \right) \\
& \left( \frac{3(3a^8+4a^6z^2+z^4)\log[c+\sqrt{a^2+c^2+z^2}]+3(3b^8+4b^6z^2+z^4)\log[c+\sqrt{b^2+c^2+z^2}]}{2} + \ight. \\
& \left. \frac{9a^8\log[d+\sqrt{a^2+d^2+z^2}]+12a^6z^2\log[d+\sqrt{a^2+d^2+z^2}]+3z^8\log[d+\sqrt{a^2+d^2+z^2}]}{2} \right) \\
& \left( \frac{9b^8\log[d+\sqrt{b^2+d^2+z^2}]+12b^6z^2\log[d+\sqrt{b^2+d^2+z^2}]+3z^8\log[d+\sqrt{b^2+d^2+z^2}]}{2} \right)
\end{align*}
\]
\[
K'_{y04, r} = \frac{1}{3} \left( -2 a^3 \sqrt{a^2 + c^2 + z^2} + a c^2 \sqrt{a^2 + c^2 + z^2} - 5 a z^2 \sqrt{a^2 + c^2 + z^2} + \\
2 b^3 \sqrt{b^2 + c^2 + z^2} - b c^2 \sqrt{b^2 + c^2 + z^2} + 5 b z^2 \sqrt{b^2 + c^2 + z^2} + \\
2 a^3 \sqrt{a^2 + d^2 + z^2} - a d^2 \sqrt{a^2 + d^2 + z^2} + 5 a z^2 \sqrt{a^2 + d^2 + z^2} - \\
2 b^3 \sqrt{b^2 + d^2 + z^2} + b d^2 \sqrt{b^2 + d^2 + z^2} - 5 b z^2 \sqrt{b^2 + d^2 + z^2} - \\
3 z^4 \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] + 3 z^4 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + \\
3 z^4 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - 3 z^4 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] \right)
\]
\[ K'_{y14, \pi} = \frac{1}{15} \left( \frac{8z^6}{\sqrt{a^2 + c^2 + z^2}} + \frac{8b^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{24b^4z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{24b^2z^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{8z^6}{\sqrt{a^2 + c^2 + z^2}} + \frac{3d^6}{\sqrt{a^2 + d^2 + z^2}} - \frac{d^4z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{4d^2z^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{8z^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{8b^6}{\sqrt{b^2 + d^2 + z^2}} - \frac{4b^4z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{b^2d^4}{\sqrt{b^2 + d^2 + z^2}} - \frac{3d^6}{\sqrt{b^2 + d^2 + z^2}} - \frac{24b^4z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{8b^2d^2z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{d^4z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{4d^2z^4}{\sqrt{b^2 + d^2 + z^2}} \right) \]
\[ K'_{y_{24,n}} = \begin{align*}
\frac{1}{18} \left( & - \frac{8a^7}{\sqrt{a^2 + c^2 + z^2}} - \frac{4a^5c^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^3c^4}{\sqrt{a^2 + c^2 + z^2}} - \frac{3ac^6}{\sqrt{a^2 + c^2 + z^2}} - \\
& 22a^5z^2 - \frac{7a^3c^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{17a^3z^4}{\sqrt{a^2 + c^2 + z^2}} - 3az^6 + \\
& \sqrt{b^2 + c^2 + z^2} + \frac{4b^5c^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^3c^4}{\sqrt{b^2 + c^2 + z^2}} + 3bc^6 + \\
& 22b^5z^2 + \frac{7b^3c^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{17b^3z^4}{\sqrt{b^2 + c^2 + z^2}} + 3bz^6 + \\
& \sqrt{b^2 + c^2 + z^2} + \frac{8a^7}{\sqrt{a^2 + d^2 + z^2}} + \frac{4a^5d^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{a^3d^4}{\sqrt{a^2 + d^2 + z^2}} + 3ad^6 + \\
& 22a^5z^2 + \frac{7a^3d^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{17a^3z^4}{\sqrt{a^2 + d^2 + z^2}} + 3az^6 + \\
& \sqrt{b^2 + d^2 + z^2} + \frac{8b^7}{\sqrt{b^2 + d^2 + z^2}} + \frac{4b^5d^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{b^3d^4}{\sqrt{b^2 + d^2 + z^2}} + 3bd^6 + \\
& 22b^5z^2 + \frac{7b^3d^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{17b^3z^4}{\sqrt{b^2 + d^2 + z^2}} + 3bz^6 + \\
& 3(c^6 + z^6) \log\left[ a + \sqrt{a^2 + c^2 + z^2} \right] - 3c^6 \log\left[ b + \sqrt{b^2 + c^2 + z^2} \right] - \\
& 3z^6 \log\left[ b + \sqrt{b^2 + c^2 + z^2} \right] - (d^6 + z^6) \log\left[ a + \sqrt{a^2 + d^2 + z^2} \right] + \\
& 3d^6 \log\left[ b + \sqrt{b^2 + d^2 + z^2} \right] + 3z^6 \log\left[ b + \sqrt{b^2 + d^2 + z^2} \right] \right) \end{align*} \]
$K'_{r, 34, n} = \frac{1}{105} \left( 16z^6 \sqrt{a^2 + c^2 + z^2} + 40b^6 \sqrt{b^2 + c^2 + z^2} + 64b^4z^2 \sqrt{b^2 + c^2 + z^2} + 8b^2z^4 \sqrt{b^2 + c^2 + z^2} - 16z^6 \sqrt{b^2 + c^2 + z^2} - 30d^6 \sqrt{a^2 + d^2 + z^2} - 6d^4z^2 \sqrt{a^2 + d^2 + z^2} + 8d^2z^4 \sqrt{a^2 + d^2 + z^2} - 16z^6 \sqrt{a^2 + d^2 + z^2} - 40b^6 \sqrt{b^2 + d^2 + z^2} + 20b^4d^2 \sqrt{b^2 + d^2 + z^2} - 15b^2d^4 \sqrt{b^2 + d^2 + z^2} + 30d^6 \sqrt{b^2 + d^2 + z^2} - 64b^4d^2 \sqrt{b^2 + d^2 + z^2} + 12b^2d^2z^2 \sqrt{b^2 + d^2 + z^2} + 6d^4z^2 \sqrt{b^2 + d^2 + z^2} - 8b^2d^2z^4 \sqrt{b^2 + d^2 + z^2} - 8d^2z^4 \sqrt{b^2 + d^2 + z^2} + 16z^6 \sqrt{b^2 + d^2 + z^2} + 30c^6(\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2}) - 40a^6(\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2}) - 3c^4(5a^2 \sqrt{a^2 + c^2 + z^2} - 5b^2 \sqrt{b^2 + c^2 + z^2} - 2z^2(\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2}) + 4c^2(5a^4 \sqrt{a^2 + c^2 + z^2} + 3a^2z^2 \sqrt{a^2 + c^2 + z^2} - 5b^4 \sqrt{b^2 + c^2 + z^2} - 3b^2z^2 \sqrt{b^2 + c^2 + z^2} - 2z^4(\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2}) - 4a^4(5d^2 \sqrt{a^2 + d^2 + z^2} + 16z^2(\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2}) + a^2(15d^4 \sqrt{a^2 + d^2 + z^2} - 12d^2z^2 \sqrt{a^2 + d^2 + z^2} - 8z^4(\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2}) ) \right)$
\[ K'_{y_{44,n}} = \left\{ \begin{array}{l}
\frac{16a^9}{\sqrt{a^2 + c^2 + z^2}} - \frac{8a^7c^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{2a^5c^4}{\sqrt{a^2 + c^2 + z^2}} + \frac{3a^3c^6}{\sqrt{a^2 + c^2 + z^2}} - \frac{9ac^8}{\sqrt{a^2 + c^2 + z^2}} \\
- \frac{40a^7z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{12a^5c^4z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^3c^6z^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{12ac^8z^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{26a^5z^4}{\sqrt{a^2 + c^2 + z^2}} \\
\frac{a^3c^2z^4}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^3z^6}{\sqrt{a^2 + c^2 + z^2}} + \frac{3az^8}{\sqrt{a^2 + c^2 + z^2}} + \frac{16b^6}{\sqrt{a^2 + c^2 + z^2}} - \frac{8a^7d^2}{\sqrt{a^2 + c^2 + z^2}} \\
\frac{2b^5c^4}{\sqrt{b^2 + c^2 + z^2}} - \frac{3b^3c^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{9b^8}{\sqrt{b^2 + c^2 + z^2}} - \frac{40b^7z^2}{\sqrt{b^2 + c^2 + z^2}} \\
\frac{12b^5c^2z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^3c^4z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{12bc^6z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{26b^5z^4}{\sqrt{b^2 + c^2 + z^2}} \\
\frac{b^3c^2z^4}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^3z^6}{\sqrt{b^2 + c^2 + z^2}} - \frac{3b^2z^8}{\sqrt{b^2 + c^2 + z^2}} + \frac{16b^9}{\sqrt{b^2 + c^2 + z^2}} - \frac{8b^7d^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{2b^5d^4}{\sqrt{b^2 + c^2 + z^2}} \\
\frac{3b^3d^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{9bd^8}{\sqrt{b^2 + c^2 + z^2}} - \frac{40b^7z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{12b^5d^2z^2}{\sqrt{b^2 + c^2 + z^2}} \\
\frac{b^3d^4z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{12bd^6z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{26b^5z^4}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^3d^2z^4}{\sqrt{b^2 + c^2 + z^2}} \\
\frac{b^6z^6}{\sqrt{b^2 + d^2 + z^2}} + \frac{3b^2z^8}{\sqrt{b^2 + d^2 + z^2}} - 3(3c^8+4c^6z^2+z^8)\log\left[a+\sqrt{a^2+c^2+z^2}\right] + \\
9c^8\log\left[b+\sqrt{b^2+c^2+z^2}\right] + 12c^6z^2\log\left[b+\sqrt{b^2+c^2+z^2}\right] + \\
3z^8\log\left[b+\sqrt{b^2+c^2+z^2}\right] + 3(3d^8+4d^6z^2+z^8)\log\left[a+\sqrt{a^2+d^2+z^2}\right] - \\
9d^8\log\left[b+\sqrt{b^2+d^2+z^2}\right] - 12d^6z^2\log\left[b+\sqrt{b^2+d^2+z^2}\right] - 3z^8\log\left[b+\sqrt{b^2+d^2+z^2}\right]
\end{array} \right\} \right.$
\[
K_{r_{54,n}}' = \frac{1}{1890} \left( \sqrt{a^2 + c^2 + z^2} - \frac{560a^{10}}{\sqrt{a^2 + z^2}} + \frac{280a^8c^2}{\sqrt{a^2 + z^2}} - \frac{210a^6c^4}{\sqrt{a^2 + z^2}} + \frac{280a^4c^6}{\sqrt{a^2 + z^2}} - \right.
\frac{560a^2c^8}{\sqrt{a^2 + z^2}} - \frac{1360a^8z^2}{\sqrt{a^2 + z^2}} + \frac{400a^6c^2z^2}{\sqrt{a^2 + z^2}} - \frac{90a^4c^4z^2}{\sqrt{a^2 + z^2}} + \frac{520a^2c^6z^2}{\sqrt{a^2 + z^2}} - \right.
\frac{560c^8z^2}{\sqrt{a^2 + z^2}} - \frac{848a^6z^4}{\sqrt{a^2 + z^2}} + \frac{24a^4c^2z^4}{\sqrt{a^2 + z^2}} - \frac{72a^2c^4z^4}{\sqrt{a^2 + z^2}} + \frac{800c^6z^4}{\sqrt{a^2 + z^2}} + \frac{16a^4z^6}{\sqrt{a^2 + z^2}} - \right.
\frac{32a^2c^2z^6}{\sqrt{a^2 + z^2}} - \frac{48c^4z^6}{\sqrt{a^2 + z^2}} + \frac{64a^2z^8}{\sqrt{a^2 + z^2}} - \frac{64c^2z^8}{\sqrt{a^2 + z^2}} - \frac{128z^{10}}{\sqrt{a^2 + z^2}} \bigg) + \right.
\frac{1}{1890} \left( \sqrt{b^2 + c^2 + z^2} - \frac{560b^{10}}{\sqrt{b^2 + z^2}} + \frac{280b^8c^2}{\sqrt{b^2 + z^2}} - \frac{210b^6c^4}{\sqrt{b^2 + z^2}} + \frac{280b^4c^6}{\sqrt{b^2 + z^2}} - \right.
\frac{560b^2c^8}{\sqrt{b^2 + z^2}} - \frac{1360b^8z^2}{\sqrt{b^2 + z^2}} + \frac{400b^6c^2z^2}{\sqrt{b^2 + z^2}} - \frac{90b^4c^4z^2}{\sqrt{b^2 + z^2}} + \frac{520b^2c^6z^2}{\sqrt{b^2 + z^2}} - \right.
\frac{560c^8z^2}{\sqrt{b^2 + z^2}} - \frac{848b^6z^4}{\sqrt{b^2 + z^2}} + \frac{24b^4c^2z^4}{\sqrt{b^2 + z^2}} - \frac{72b^2c^4z^4}{\sqrt{b^2 + z^2}} + \frac{800c^6z^4}{\sqrt{b^2 + z^2}} + \frac{16b^4z^6}{\sqrt{b^2 + z^2}} - \right.
\frac{32b^2c^2z^6}{\sqrt{b^2 + z^2}} - \frac{48c^4z^6}{\sqrt{b^2 + z^2}} + \frac{64b^2z^8}{\sqrt{b^2 + z^2}} - \frac{64c^2z^8}{\sqrt{b^2 + z^2}} - \frac{128z^{10}}{\sqrt{b^2 + z^2}} \bigg) + \right.
\frac{1}{1890} \left( \sqrt{a^2 + d^2 + z^2} - \frac{560a^{10}}{a^2 + d^2 + z^2} + \frac{280a^8d^2}{a^2 + d^2 + z^2} - \frac{70a^6d^4}{a^2 + d^2 + z^2} - \frac{70a^4d^6}{a^2 + d^2 + z^2} + \right.
\frac{280a^2d^8}{a^2 + d^2 + z^2} - \frac{560d^{10}}{a^2 + d^2 + z^2} + \frac{1360a^8z^2}{a^2 + d^2 + z^2} + \frac{400a^6d^2z^2}{a^2 + d^2 + z^2} - \frac{30a^4d^4z^2}{a^2 + d^2 + z^2} + \frac{400a^2d^6z^2}{a^2 + d^2 + z^2} + \right.
\frac{1360d^8z^2}{a^2 + d^2 + z^2} - \frac{848a^6d^4}{a^2 + d^2 + z^2} - \frac{24a^4d^2z^4}{a^2 + d^2 + z^2} + \frac{24a^2d^4z^4}{a^2 + d^2 + z^2} + \frac{848d^6z^4}{a^2 + d^2 + z^2} - \frac{16a^4z^6}{a^2 + d^2 + z^2} - \right.
\frac{32a^2d^2z^6}{a^2 + d^2 + z^2} - \frac{16d^4z^6}{a^2 + d^2 + z^2} + \frac{64a^2z^8}{a^2 + d^2 + z^2} - \frac{64d^2z^8}{a^2 + d^2 + z^2} - \frac{128z^{10}}{a^2 + d^2 + z^2} \bigg) + \right.
\frac{1}{1890} \left( \sqrt{b^2 + d^2 + z^2} - \frac{560b^{10}}{b^2 + d^2 + z^2} + \frac{280b^8d^2}{b^2 + d^2 + z^2} - \frac{70b^6d^4}{b^2 + d^2 + z^2} - \frac{70b^4d^6}{b^2 + d^2 + z^2} + \right.
\frac{280b^2d^8}{b^2 + d^2 + z^2} - \frac{560d^{10}}{b^2 + d^2 + z^2} + \frac{1360b^8z^2}{b^2 + d^2 + z^2} + \frac{400b^6d^2z^2}{b^2 + d^2 + z^2} - \frac{30b^4d^4z^2}{b^2 + d^2 + z^2} + \frac{400b^2d^6z^2}{b^2 + d^2 + z^2} + \right.
\frac{1360d^8z^2}{b^2 + d^2 + z^2} - \frac{848b^6d^4}{b^2 + d^2 + z^2} - \frac{24b^4d^2z^4}{b^2 + d^2 + z^2} + \frac{24b^2d^4z^4}{b^2 + d^2 + z^2} + \frac{848d^6z^4}{b^2 + d^2 + z^2} - \frac{16b^4z^6}{b^2 + d^2 + z^2} - \right.
\frac{32b^2d^2z^6}{b^2 + d^2 + z^2} - \frac{16d^4z^6}{b^2 + d^2 + z^2} + \frac{64b^2z^8}{b^2 + d^2 + z^2} - \frac{64d^2z^8}{b^2 + d^2 + z^2} - \frac{128z^{10}}{b^2 + d^2 + z^2} \bigg) \right)}
\[ K'_{y_{05,n}} = \]
\[-z^5 \left( \text{ArcTan} [a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, -cz] + \text{ArcTan} [b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, cz] + \right. \]
\[ \left. \text{ArcTan} [a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, dz] + \text{ArcTan} [b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, -dz] \right) + \]
\[ \frac{1}{8} \left( -3a^3 c \sqrt{a^2 + c^2 + z^2} + 2ac^2 \sqrt{a^2 + c^2 + z^2} - 7acz^2 \sqrt{a^2 + c^2 + z^2} + \right. \]
\[ \left. 3b^3 c \sqrt{b^2 + c^2 + z^2} - 2bc^3 \sqrt{b^2 + c^2 + z^2} + 7bcz^2 \sqrt{b^2 + c^2 + z^2} + \right. \]
\[ \left. 3a^3 d \sqrt{a^2 + d^2 + z^2} - 2ad^3 \sqrt{a^2 + d^2 + z^2} + 7adz^2 \sqrt{a^2 + d^2 + z^2} - \right. \]
\[ \left. 3b^3 d \sqrt{b^2 + d^2 + z^2} + 2bd^3 \sqrt{b^2 + d^2 + z^2} - 7bdz^2 \sqrt{b^2 + d^2 + z^2} + \right. \]
\[ \left. (3a^5 + 10a^3 z^2 + 15az^4) \log [c + \sqrt{a^2 + c^2 + z^2}] - 3b^5 \log [c + \sqrt{b^2 + c^2 + z^2}] - \right. \]
\[ \left. 10b^3 z^2 \log [c + \sqrt{b^2 + c^2 + z^2}] - 15bz^4 \log [c + \sqrt{b^2 + c^2 + z^2}] - \right. \]
\[ \left. 3a^5 \log [d + \sqrt{a^2 + d^2 + z^2}] - 10a^3 z^2 \log [d + \sqrt{a^2 + d^2 + z^2}] - \right. \]
\[ \left. 15az^4 \log [d + \sqrt{a^2 + d^2 + z^2}] + 3b^5 \log [d + \sqrt{b^2 + d^2 + z^2}] + \right. \]
\[ \left. 10b^3 z^2 \log [d + \sqrt{b^2 + d^2 + z^2}] + 15bz^4 \log [d + \sqrt{b^2 + d^2 + z^2}] \right) \]
\[ K'_{y15,n} = \frac{1}{48} \left( \begin{array}{cccc}
15a^6c & 5a^4c^3 & 2a^2c^5 & 8c^7 \\
\sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} \\
45a^4cz^2 & 10a^2c^3z^2 & 2c^5z^2 & 45a^2cz^4 \\
\sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} \\
5c^3z^4 & 15cz^6 & 15b^6c & 5b^4c^3 \\
\sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} \\
2b^2c^5 & 8c^7 & 45b^4cz^2 & 10b^2c^3z^2 \\
\sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} \\
2c^5z^2 & 45b^2cz^4 & 5c^3z^4 & 15cz^6 \\
\sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} & \sqrt{b^2 + c^2 + z^2} \\
15a^6d & 5a^4d^3 & 2a^2d^5 & 8d^7 \\
\sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} \\
45a^4dz^2 & 10a^2d^3z^2 & 2d^5z^2 & 45a^2dz^4 \\
\sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} \\
5d^3z^4 & 15dz^6 & 15b^6d & 5b^4d^3 \\
\sqrt{a^2 + d^2 + z^2} & \sqrt{a^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} \\
2b^2d^5 & 8d^7 & 45b^4dz^2 & 10b^2d^3z^2 \\
\sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} \\
2d^5z^2 & 45b^2dz^4 & 5d^3z^4 & 15dz^6 \\
\sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} & \sqrt{b^2 + d^2 + z^2} \\
15(a^2 + z^3)^3 \log(c + \sqrt{a^2 + c^2 + z^2}) - 15b^6 \log(c + \sqrt{b^2 + c^2 + z^2}) \\
45b^4z^2 \log(c + \sqrt{b^2 + c^2 + z^2}) - 45b^2z^2 \log(c + \sqrt{b^2 + c^2 + z^2}) \\
15z^6 \log(c + \sqrt{b^2 + c^2 + z^2}) + 15(a^2 + z^3)^3 \log(d + \sqrt{a^2 + d^2 + z^2}) + 15b^6 \log(d + \sqrt{b^2 + d^2 + z^2}) + 45b^4z^2 \log(d + \sqrt{b^2 + d^2 + z^2}) + 15z^6 \log(d + \sqrt{b^2 + d^2 + z^2}) \right) \right) 
\]
\[
K'_{y_{25,n}} = \frac{1}{7} z^7 \left[ \text{ArcTan} \left( a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, cz \right) + \text{ArcTan} \left( b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, -cz \right) + \text{ArcTan} \left( a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2}, -dz \right) + \text{ArcTan} \left( b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, d \right) \right] + \frac{1}{56} \left( -15a^5 c^4 \sqrt{a^2 + c^2 + z^2} + 10a^3 c^3 \sqrt{a^2 + c^2 + z^2} - 8ac^5 \sqrt{a^2 + c^2 + z^2} - 27a^3 cz^2 \sqrt{a^2 + c^2 + z^2} + 8ac^3 z^2 \sqrt{a^2 + c^2 + z^2} - 8ac^2 z^2 \sqrt{a^2 + c^2 + z^2} + 15b^5 \sqrt{a^2 + c^2 + z^2} - 10b^3 c^3 \sqrt{b^2 + c^2 + z^2} + 8bc^5 \sqrt{b^2 + c^2 + z^2} + 27b^3 cz^2 \sqrt{b^2 + c^2 + z^2} - 8bc^3 z^2 \sqrt{b^2 + c^2 + z^2} + 8bc^2 z^2 \sqrt{b^2 + c^2 + z^2} + 15a^5 d \sqrt{a^2 + d^2 + z^2} - 10a^3 d^3 \sqrt{a^2 + d^2 + z^2} + 8ad^5 \sqrt{a^2 + d^2 + z^2} - 27a^3 dz^2 \sqrt{a^2 + d^2 + z^2} - 8ad^3 z^2 \sqrt{a^2 + d^2 + z^2} + 8ad^2 z^2 \sqrt{a^2 + d^2 + z^2} - 15b^5 d \sqrt{b^2 + d^2 + z^2} + 10b^3 d^3 \sqrt{b^2 + d^2 + z^2} - 8bd^5 \sqrt{b^2 + d^2 + z^2} + 27b^3 dz^2 \sqrt{b^2 + d^2 + z^2} + 8bd^3 z^2 \sqrt{b^2 + d^2 + z^2} - 8bd^2 z^2 \sqrt{b^2 + d^2 + z^2} + 8c^7 \text{Log} [a + \sqrt{a^2 + z^2}] + (15a^7 + 42a^5z^2 + 35a^3z^4) \text{Log} [c + \sqrt{a^2 + c^2 + z^2}] - 8c^7 \text{Log} [b + \sqrt{b^2 + c^2 + z^2}] - 15b^7 \text{Log} [c + \sqrt{b^2 + c^2 + z^2}] - 42b^5 z^2 \text{Log} [c + \sqrt{b^2 + c^2 + z^2}] - 35b^3 z^4 \text{Log} [c + \sqrt{b^2 + c^2 + z^2}] - 8d^7 \text{Log} [a + \sqrt{a^2 + d^2 + z^2}] - 15a^7 \text{Log} [d + \sqrt{a^2 + d^2 + z^2}] - 42a^5 z^2 \text{Log} [d + \sqrt{a^2 + d^2 + z^2}] - 35a^3 z^4 \text{Log} [d + \sqrt{a^2 + d^2 + z^2}] + 8d^7 \text{Log} [b + \sqrt{b^2 + d^2 + z^2}] + 15b^7 \text{Log} [d + \sqrt{b^2 + d^2 + z^2}] + 42b^5 z^2 \text{Log} [d + \sqrt{b^2 + d^2 + z^2}] + 35b^3 z^4 \text{Log} [d + \sqrt{b^2 + d^2 + z^2}] \right) \]
\]
\[ K'_{y35,n} = \frac{1}{192} \left( -45a^6c \sqrt{a^2 + c^2 + z^2} + 30a^4c^3 \sqrt{a^2 + c^2 + z^2} - 24a^2c^5 \sqrt{a^2 + c^2 + z^2} + 48c^7 \sqrt{a^2 + c^2 + z^2} - 75a^4cz^2 \sqrt{a^2 + c^2 + z^2} + 20a^2c^3z^2 \sqrt{a^2 + c^2 + z^2} + 8c^5z^2 \sqrt{a^2 + c^2 + z^2} - 75a^4cz^2 \sqrt{a^2 + c^2 + z^2} + 15a^2cz^4 \sqrt{a^2 + c^2 + z^2} - 10c^3z^4 \sqrt{a^2 + c^2 + z^2} + 45b^6c \sqrt{b^2 + c^2 + z^2} - 30b^4c^3 \sqrt{b^2 + c^2 + z^2} - 48c^7 \sqrt{b^2 + c^2 + z^2} - 75b^4cz^2 \sqrt{b^2 + c^2 + z^2} - 20b^2c^3z^2 \sqrt{b^2 + c^2 + z^2} - 8c^5z^2 \sqrt{b^2 + c^2 + z^2} + 15b^2cz^4 \sqrt{b^2 + c^2 + z^2} + 10c^3z^4 \sqrt{b^2 + c^2 + z^2} - 15cz^6 \sqrt{a^2 + d^2 + z^2} + 45a^6d \sqrt{a^2 + d^2 + z^2} - 30a^4d^3 \sqrt{a^2 + d^2 + z^2} + 24a^2d^5 \sqrt{a^2 + d^2 + z^2} - 48d^7 \sqrt{a^2 + d^2 + z^2} + 75a^4dz^2 \sqrt{a^2 + d^2 + z^2} - 20a^2d^3z^2 \sqrt{a^2 + d^2 + z^2} - 8d^5z^2 \sqrt{a^2 + d^2 + z^2} + 15a^2dz^4 \sqrt{a^2 + d^2 + z^2} + 10d^3z^4 \sqrt{a^2 + d^2 + z^2} - 15dz^6 \sqrt{a^2 + d^2 + z^2} - 45b^6d \sqrt{b^2 + d^2 + z^2} + 30b^4d^3 \sqrt{b^2 + d^2 + z^2} - 48d^7 \sqrt{b^2 + d^2 + z^2} - 75b^4dz^2 \sqrt{b^2 + d^2 + z^2} + 20b^2d^3z^2 \sqrt{b^2 + d^2 + z^2} + 8d^5z^2 \sqrt{b^2 + d^2 + z^2} + 15b^2dz^4 \sqrt{b^2 + d^2 + z^2} + 15(3a^2 - z^2)(a^2 + z^2)^3 \log[c + \sqrt{a^2 + c^2 + z^2}] - 45b^8 \log[c + \sqrt{b^2 + c^2 + z^2}] - 120b^6z^2 \log[c + \sqrt{b^2 + c^2 + z^2}] - 90b^4z^4 \log[c + \sqrt{b^2 + c^2 + z^2}] + 15z^8 \log[c + \sqrt{b^2 + c^2 + z^2}] - 15(3a^2 - z^2)(a^2 + z^2)^3 \log[d + \sqrt{a^2 + d^2 + z^2}] + 45b^8 \log[d + \sqrt{b^2 + d^2 + z^2}] + 120b^6z^2 \log[d + \sqrt{b^2 + d^2 + z^2}] + 90b^4z^4 \log[d + \sqrt{b^2 + d^2 + z^2}] - 15z^8 \log[d + \sqrt{b^2 + d^2 + z^2}] \right) \]
\[ K'_{y,s_5,n} = \]
\[ -\frac{1}{21} z^9 \left( \text{ArcTan} \left[ a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, -c z \right] + \text{ArcTan} \left[ b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, c z \right] + \right. \]
\[ \left. \text{ArcTan} \left[ a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2}, d z \right] + \text{ArcTan} \left[ b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, -d z \right] \right) + \]
\[ \frac{1}{504} \left( -105 a^7 c \sqrt{a^2 + c^2 + z^2} + 70 a^5 c^3 \sqrt{a^2 + c^2 + z^2} - 56 a^3 c^5 \sqrt{a^2 + c^2 + z^2} + ight. \]
\[ 84 a^5 c^3 \sqrt{a^2 + c^2 + z^2} - 165 a^5 c z^2 \sqrt{a^2 + c^2 + z^2} + 40 a^3 c^3 z^2 \sqrt{a^2 + c^2 + z^2} + ight. \]
\[ 24 a c^5 z^2 \sqrt{a^2 + c^2 + z^2} - 24 a^3 c z^4 \sqrt{a^2 + c^2 + z^2} - 24 a c^3 z^4 \sqrt{a^2 + c^2 + z^2} + ight. \]
\[ 24 a c z^6 \sqrt{a^2 + c^2 + z^2} + 105 b^7 c \sqrt{b^2 + c^2 + z^2} - 70 b^5 c^3 \sqrt{b^2 + c^2 + z^2} - ight. \]
\[ 40 b^3 c^3 z^2 \sqrt{b^2 + c^2 + z^2} - 24 b c^5 z^2 \sqrt{b^2 + c^2 + z^2} + 24 b^3 c z^4 \sqrt{b^2 + c^2 + z^2} + ight. \]
\[ 24 b c^3 z^4 \sqrt{b^2 + c^2 + z^2} - 24 b c z^6 \sqrt{b^2 + c^2 + z^2} + 70 a^7 d \sqrt{a^2 + d^2 + z^2} - ight. \]
\[ 70 a^5 d^3 \sqrt{a^2 + d^2 + z^2} - 56 a^3 d^5 \sqrt{a^2 + d^2 + z^2} - 84 a d^7 \sqrt{a^2 + d^2 + z^2} + ight. \]
\[ 165 a^5 d z^2 \sqrt{a^2 + d^2 + z^2} - 40 a^3 d^3 z^2 \sqrt{a^2 + d^2 + z^2} - 24 a d^5 z^2 \sqrt{a^2 + d^2 + z^2} + ight. \]
\[ 24 a^3 d z^4 \sqrt{a^2 + d^2 + z^2} + 24 a d^3 z^4 \sqrt{a^2 + d^2 + z^2} - 24 a d z^6 \sqrt{a^2 + d^2 + z^2} - ight. \]
\[ 105 b^7 d \sqrt{b^2 + d^2 + z^2} + 70 b^5 d^3 \sqrt{b^2 + d^2 + z^2} - 56 b^3 d^5 \sqrt{b^2 + d^2 + z^2} + ight. \]
\[ 84 b d^7 \sqrt{b^2 + d^2 + z^2} - 165 b^5 d z^2 \sqrt{b^2 + d^2 + z^2} + 40 b d^3 z^2 \sqrt{b^2 + d^2 + z^2} + ight. \]
\[ 24 b d z^4 \sqrt{b^2 + d^2 + z^2} - 24 b d z^6 \sqrt{b^2 + d^2 + z^2} - 24 b d^3 z^4 \sqrt{b^2 + d^2 + z^2} + ight. \]
\[ 24 b d^5 z^2 - 12 c^7 \left( 7 c^2 + 9 z^2 \right) \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] + ight. \]
\[ 3 a^5 \left( 35 a^4 + 90 a^2 z^2 + 63 z^4 \right) \log \left[ c + \sqrt{a^2 + c^2 + z^2} \right] + 84 c^9 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + ight. \]
\[ 108 c^7 z^2 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] - 105 b^9 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] - ight. \]
\[ 270 b^7 z^2 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] + 189 b^5 z^4 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] + ight. \]
\[ 84 d^9 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 108 d^7 z^2 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] - ight. \]
\[ 105 a^9 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] - 270 a^7 z^2 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] - ight. \]
\[ 189 a^5 z^4 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] - 84 d^9 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - ight. \]
\[ 108 d^7 z^2 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] + 105 b^9 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] + ight. \]
\[ 270 b^7 z^2 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] + 189 b^5 z^4 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] + \]
\[ K_{y,s,n} = \]
\[
\begin{array}{cccccccc}
\sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} & \sqrt{a^2 + c^2 + z^2} \\
225 a^6 c z^2 & 45 a^6 c z^2 & 6 a^6 c z^2 & 88 a^6 c z^2 & 304 a^6 c z^2 & 150 a^6 c z^4 & \\
5 a^6 z^4 & 4 a^6 z^4 & 184 a^6 z^4 & 5 a^6 z^6 & 2 a^6 z^6 & 5 a^6 z^8 & \\
15 c z^{10} & 90 b^{10} c & 30 b^{10} c & 12 b^8 c^7 & 16 b^8 c^7 & \\
64 b^2 c^9 & 128 c^{11} & 225 b^2 c z^2 & 45 b^6 c z^2 & 6 b^4 c z^2 & \\
88 b^2 c z^2 & 304 b^6 c z^2 & 150 b^6 c z^4 & 5 b^6 c z^4 & 4 b^4 c z^4 & \\
184 c^4 z^4 & 5 b^2 c z^6 & 2 c^5 z^6 & 5 c^3 z^8 & 15 c z^{10} & \\
15 b^6 d^9 & 30 a^6 d^3 & 12 a^6 d^3 & 16 a^6 d^7 & 64 a^2 d^9 & 128 d^{11} & \\
225 a^6 d^2 z^2 & 45 a^6 d^2 z^2 & 6 a^6 d^2 z^2 & 88 a^6 d^2 z^2 & 304 a^6 d^2 z^2 & 150 a^6 d^2 z^4 & \\
5 a^6 z^4 & 4 a^6 z^4 & 184 a^6 z^4 & 5 a^6 z^6 & 2 a^6 z^6 & \\
5 a^6 z^8 & 15 d z^{10} & 90 b^{10} d & 30 b^8 d^3 & 12 b^8 d^3 & \\
16 b^8 d^9 & 64 b^2 d^9 & 128 d^{11} & 225 b^2 d z^2 & 45 b^6 d z^2 & \\
88 b^2 d z^2 & 304 b^6 d z^2 & 150 b^6 d z^4 & 5 b^6 d z^4 & \\
184 d^4 z^4 & 5 b^2 d z^6 & 2 d^5 z^6 & 5 d^3 z^8 & \\
15 d z^{10} & + 15 (a^2 + z^2)^3 (6 a^4 - 3 a^2 z^2 + z^4) \log \left[ c + \sqrt{a^2 + c^2 + z^2} \right] - 90 b^{10} \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] - \\
225 b^8 z^2 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] - 150 b^6 z^4 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] - 15 z^{10} \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] - \\
15 (a^2 + z^2)^3 (6 a^4 - 3 a^2 z^2 + z^4) \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] + 90 b^{10} \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] + \\
225 b^8 z^2 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] + 150 b^6 z^4 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] + 15 z^{10} \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right]
\end{array}
\]
III.4 Analytical Formulae for $K'_{2i,j,n}$, $i,j=0,1,...,5$

\[ K'_{200,n} = \]
\[ \arctan \left( c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, -az \right) + \arctan \left( c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, bz \right) + \]
\[ \arctan \left( d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, az \right) + \arctan \left( d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, -bz \right) \]

\[ K'_{210,n} = \]
\[ z \log \left( \frac{c + \sqrt{b^2 + c^2 + z^2}}{c + \sqrt{a^2 + c^2 + z^2}} \right) \left( d + \sqrt{a^2 + d^2 + z^2} \right) \left( d + \sqrt{b^2 + d^2 + z^2} \right) \]

\[ K'_{220,n} = \]
\[ \left\{ \begin{array}{l}
|z| - z \left( \arctan \left( a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, -cz \right) + \right.
\arctan \left( b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, cz \right) + \\
\arctan \left( a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2}, dz \right) + \\
\arctan \left( b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, -dz \right) \biggr) + \\
c \log \left( \frac{a + \sqrt{a^2 + c^2 + z^2}}{b + \sqrt{b^2 + c^2 + z^2}} \right) + d \log \left( \frac{b + \sqrt{b^2 + d^2 + z^2}}{a + \sqrt{a^2 + d^2 + z^2}} \right) \biggr) \end{array} \right. \]

\[ K'_{230,n} = \]
\[ z \left( c \sqrt{a^2 + c^2 + z^2} - c \sqrt{b^2 + c^2 + z^2} - d \sqrt{a^2 + d^2 + z^2} + d \sqrt{b^2 + d^2 + z^2} \right) \]
\[ z^2 \log \left( c + \sqrt{a^2 + c^2 + z^2} \right) - z^2 \log \left( c + \sqrt{b^2 + c^2 + z^2} \right) + \\
z^2 \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) + z^2 \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) \]
\[ K'_{z_{40,n}} = \]
\[
z \left( -z^3 \left( \text{ArcTan} \left[ a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, cz \right] + \text{ArcTan} \left[ b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, -cz \right] + \text{ArcTan} \left[ a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2}, -dz \right] + \text{ArcTan} \left[ b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, dz \right] \right) \right) + \]
\[
\frac{1}{2} \left( ac \sqrt{a^2 + c^2 + z^2} - bc \sqrt{b^2 + c^2 + z^2} - ad \sqrt{a^2 + d^2 + z^2} + bd \sqrt{b^2 + d^2 + z^2} - (c^3 + 3cz^2) \text{Log}[a + \sqrt{a^2 + c^2 + z^2}] + c^3 \text{Log}[b + \sqrt{b^2 + c^2 + z^2}] + 3cz^2 \text{Log}[b + \sqrt{b^2 + c^2 + z^2}] + d^3 \text{Log}[a + \sqrt{a^2 + d^2 + z^2}] + 3dz^2 \text{Log}[a + \sqrt{a^2 + d^2 + z^2}] - d^3 \text{Log}[b + \sqrt{b^2 + d^2 + z^2}] - 3dz^2 \text{Log}[b + \sqrt{b^2 + d^2 + z^2}] \right) \]

\[ K'_{z_{50,n}} = \]
\[
\frac{1}{3} z \left( a^2 c \sqrt{a^2 + c^2 + z^2} - 2c^3 \sqrt{a^2 + c^2 + z^2} - 5cz^2 \sqrt{a^2 + c^2 + z^2} - b^2 c \sqrt{b^2 + c^2 + z^2} + 2c^3 \sqrt{b^2 + c^2 + z^2} + 5cz^2 \sqrt{b^2 + c^2 + z^2} - a^2 d \sqrt{a^2 + d^2 + z^2} + 2d^3 \sqrt{a^2 + d^2 + z^2} + 5dz^2 \sqrt{a^2 + d^2 + z^2} + b^2 d \sqrt{b^2 + d^2 + z^2} - 2d^3 \sqrt{b^2 + d^2 + z^2} - 5dz^2 \sqrt{b^2 + d^2 + z^2} - 3z^4 \text{Log}[c + \sqrt{a^2 + c^2 + z^2}] + 3z^4 \text{Log}[c + \sqrt{b^2 + c^2 + z^2}] + 3z^4 \text{Log}[d + \sqrt{a^2 + d^2 + z^2}] - 3z^4 \text{Log}[d + \sqrt{b^2 + d^2 + z^2}] \right) \]

\[ K'_{z_{101,n}} = \]
\[
z \text{Log} \left( \frac{b + \sqrt{b^2 + c^2 + z^2}}{a + \sqrt{a^2 + d^2 + z^2}} \right) \left( a + \sqrt{a^2 + c^2 + z^2} \right) \left( b + \sqrt{b^2 + d^2 + z^2} \right) \]

\[ K'_{z_{11,n}} = \]
\[
z \left( -\sqrt{a^2 + c^2 + z^2} + \sqrt{b^2 + c^2 + z^2} + \sqrt{a^2 + d^2 + z^2} - \sqrt{b^2 + d^2 + z^2} \right) \]
\[ K'_{221,n} = \]
\[
\frac{1}{2} \left\{ \frac{z}{-a \sqrt{a^2 + c^2 + z^2} + b \sqrt{b^2 + c^2 + z^2} + a \sqrt{a^2 + d^2 + z^2} - b \sqrt{b^2 + d^2 + z^2}} \right. \\
\left. (c^2 + z^2) \log \left[ \frac{a + \sqrt{a^2 + c^2 + z^2}}{b + \sqrt{b^2 + c^2 + z^2}} \right] + (d^2 + z^2) \log \left( \frac{b + \sqrt{b^2 + d^2 + z^2}}{a + \sqrt{a^2 + d^2 + z^2}} \right) \right\} \]

\[ K'_{331,n} = \]
\[
\frac{1}{3} z \left( 2 z^2 \sqrt{a^2 + c^2 + z^2} + b^2 \sqrt{b^2 + c^2 + z^2} - 2 z^2 \sqrt{b^2 + c^2 + z^2} - 2 d^2 \sqrt{a^2 + d^2 + z^2} - \\
2 z^2 \sqrt{a^2 + d^2 + z^2} - b^2 \sqrt{b^2 + d^2 + z^2} + 2 d^2 \sqrt{b^2 + d^2 + z^2} + 2 z^2 \sqrt{b^2 + d^2 + z^2} + \\
2 c^2 (\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2}) + a^2 (\sqrt{a^2 + c^2 + z^2} + \sqrt{a^2 + d^2 + z^2}) \right) \]
\[
K_{z41,n} = \\
\left( \frac{2a^5}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^3 c^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{3ac^4}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^3 z^2}{\sqrt{a^2 + c^2 + z^2}} + \\
\frac{6ac^2 z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{3az^4}{\sqrt{a^2 + c^2 + z^2}} + \frac{2b^5}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^3 c^2}{\sqrt{b^2 + c^2 + z^2}} - \\
\frac{3bc^4}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^3 z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{6bc^2 z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{3b^4 z^2}{\sqrt{b^2 + c^2 + z^2}} + \\
\frac{2a^5 z^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{a^3 d^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{3ad^4}{\sqrt{a^2 + d^2 + z^2}} - \frac{a^3 z^2}{\sqrt{a^2 + d^2 + z^2}} - \\
\frac{6ad^2 z^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{3az^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{2b^5}{\sqrt{b^2 + d^2 + z^2}} + \frac{b^3 d^2}{\sqrt{b^2 + d^2 + z^2}} + \\
\frac{3bd^4}{\sqrt{b^2 + d^2 + z^2}} + \frac{b^3 z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{6bd^2 z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{3b^4 z^2}{\sqrt{b^2 + d^2 + z^2}} - \\
3(c^2 + z^2)^2 \log\left( a + \sqrt{a^2 + c^2 + z^2} \right) + 3(c^2 + z^2)^2 \log\left( b + \sqrt{b^2 + c^2 + z^2} \right) + \\
3d^4 \log\left( a + \sqrt{a^2 + d^2 + z^2} \right) - 3d^4 \log\left( b + \sqrt{b^2 + d^2 + z^2} \right) - \\
6d^2 z^2 \log\left( b + \sqrt{b^2 + d^2 + z^2} \right) - 3z^4 \log\left( b + \sqrt{b^2 + d^2 + z^2} \right) \right)
\]
\[
K'_{z,1,n} = \frac{1}{15} \left( \frac{8z^6}{\sqrt{a^2 + c^2 + z^2}} + \frac{3b^6}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^4z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{4b^2z^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{8z^6}{\sqrt{a^2 + d^2 + z^2}} + \frac{8d^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{24d^4z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{24d^2z^4}{\sqrt{a^2 + d^2 + z^2}} - \frac{3b^6}{\sqrt{b^2 + d^2 + z^2}} + \frac{8b^4d^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{8d^6}{\sqrt{b^2 + d^2 + z^2}} + \frac{b^4z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{8b^2d^2z^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{24d^4z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{4b^2z^4}{\sqrt{b^2 + d^2 + z^2}} - \frac{24d^2z^4}{\sqrt{b^2 + d^2 + z^2}} \right)
\]
\[
+ \frac{8z^6}{\sqrt{b^2 + d^2 + z^2}} + 8e^6 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{b^2 + c^2 + z^2}} \right) + \frac{3}{\sqrt{a^2 + c^2 + z^2}} - \frac{3}{\sqrt{a^2 + d^2 + z^2}} + c^2 \left( \frac{b^4}{\sqrt{b^2 + c^2 + z^2}} \right) + \frac{8b^2z^2}{\sqrt{b^2 + c^2 + z^2}} + 24z^4 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{b^2 + c^2 + z^2}} \right) + \frac{b^2}{\sqrt{b^2 + c^2 + z^2}} + z^2 \left( \frac{6}{\sqrt{a^2 + c^2 + z^2}} + \frac{6}{\sqrt{b^2 + c^2 + z^2}} \right) + \frac{c^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{d^2}{\sqrt{a^2 + d^2 + z^2}} + z^2 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} - \frac{1}{\sqrt{a^2 + d^2 + z^2}} \right) + 4a^2 \left( \frac{-c^4}{\sqrt{a^2 + c^2 + z^2}} - \frac{2c^2z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{d^4}{\sqrt{a^2 + d^2 + z^2}} - \frac{2d^2z^2}{\sqrt{a^2 + d^2 + z^2}} + z^2 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{a^2 + d^2 + z^2}} \right) \right) \right)
\]
\[ K'_{x02,n} = \]
\[ = \frac{z}{-z} \left( \text{ArcTan}\left[ c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, -az \right] + \text{ArcTan}\left[ c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, bz \right] + \right. \]
\[ \text{ArcTan}\left[ d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, az \right] + \]
\[ \text{ArcTan}\left[ d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, -bz \right] + \]
\[ a \log \left( \frac{c + \sqrt{a^2 + c^2 + z^2}}{d + \sqrt{a^2 + d^2 + z^2}} \right) + b \log \left( \frac{d + \sqrt{b^2 + d^2 + z^2}}{c + \sqrt{b^2 + c^2 + z^2}} \right) \]
\[ K'_{z12,n} = \]
\[ = \frac{1}{2} z \left( -c \sqrt{a^2 + c^2 + z^2} + c \sqrt{b^2 + c^2 + z^2} + d \sqrt{a^2 + d^2 + z^2} - d \sqrt{b^2 + d^2 + z^2} + \right. \]
\[ (a^2 + z^2) \log \left( \frac{c + \sqrt{a^2 + c^2 + z^2}}{d + \sqrt{a^2 + d^2 + z^2}} \right) + (b^2 + z^2) \log \left( \frac{d + \sqrt{b^2 + d^2 + z^2}}{c + \sqrt{b^2 + c^2 + z^2}} \right) \]
\[ K'_{z22,n} = \]
\[ = z \left( -\frac{1}{3} z^3 \left( \text{ArcTan}\left[ a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, cz \right] + \right. \right. \]
\[ \text{ArcTan}\left[ b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, -cz \right] + \text{ArcTan}\left[ a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2}, \right. \]
\[ -dz \right) + \text{ArcTan}\left[ b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, dz \right] + \right. \]
\[ \frac{1}{6} \left( -2 ac \sqrt{a^2 + c^2 + z^2} + 2 bc \sqrt{b^2 + c^2 + z^2} + 2 ad \sqrt{a^2 + d^2 + z^2} - 2 bd \right. \]
\[ \sqrt{b^2 + d^2 + z^2} + 2 c^3 \log \left( a + \sqrt{a^2 + c^2 + z^2} \right) + 2 a^3 \log \left( c + \sqrt{a^2 + c^2 + z^2} \right) - \]
\[ 2 c^3 \log \left( b + \sqrt{b^2 + c^2 + z^2} \right) - 2 b^3 \log \left( c + \sqrt{b^2 + c^2 + z^2} \right) - \]
\[ 2 a^3 \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) + 2 c^3 \log \left( b + \sqrt{b^2 + d^2 + z^2} \right) + 2 a^3 \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) + \]
\[ 2 a^3 \log \left( d + \sqrt{a^2 + d^2 + z^2} \right) + 2 b^3 \log \left( d + \sqrt{b^2 + d^2 + z^2} \right) \right) \]
\[ K'_{z_{32}, n} = \]
\[
\frac{1}{4} z \left( -a^2 c \sqrt{a^2 + c^2 + z^2} + 2 c^3 \sqrt{a^2 + c^2 + z^2} + cz^2 \sqrt{a^2 + c^2 + z^2} + b^2 c \sqrt{b^2 + c^2 + z^2} - 2 c^3 \sqrt{b^2 + c^2 + z^2} - cz^2 \sqrt{b^2 + c^2 + z^2} + a^2 d \sqrt{a^2 + d^2 + z^2} - 2 d^3 \sqrt{a^2 + d^2 + z^2} =
\right)
\]
\[
dz^2 \sqrt{a^2 + d^2 + z^2} + (a^4 - z^4) \log[c + \sqrt{a^2 + c^2 + z^2}] +
\]
\[
(-b^4 + z^4) \log[c + \sqrt{b^2 + c^2 + z^2}] - a^4 \log[d + \sqrt{a^2 + d^2 + z^2}] +
\]
\[
z^4 \log[d + \sqrt{a^2 + d^2 + z^2}] + b^4 \log[d + \sqrt{b^2 + d^2 + z^2}] - z^4 \log[d + \sqrt{b^2 + d^2 + z^2}] \right)
\]

\[ K'_{z_{42}, n} = \]
\[
z \left( -\frac{1}{5} z^5 \left( \arctan[a^2 + z^2 + a \sqrt{a^2 + c^2 + z^2}, -cz] +
\right)
\]
\[
\arctan[b^2 + z^2 + b \sqrt{b^2 + c^2 + z^2}, cz] + \arctan[a^2 + z^2 + a \sqrt{a^2 + d^2 + z^2},
\right]
\]
\[
dz] + \arctan[b^2 + z^2 + b \sqrt{b^2 + d^2 + z^2}, -dz] +
\]
\[
\frac{1}{10} \left( -2a^3 c \sqrt{a^2 + c^2 + z^2} + 3ac^3 \sqrt{a^2 + c^2 + z^2} + 2acz^2 \sqrt{a^2 + c^2 + z^2} +
\right)
\]
\[
2bc^3 \sqrt{b^2 + c^2 + z^2} - 3bc^3 \sqrt{b^2 + c^2 + z^2} - 2bcz^2 \sqrt{b^2 + c^2 + z^2} +
\]
\[
2a^3 d \sqrt{a^2 + d^2 + z^2} - 3ad^3 \sqrt{a^2 + d^2 + z^2} - 2adz^2 \sqrt{a^2 + d^2 + z^2} -
\]
\[
2b^3 d \sqrt{b^2 + d^2 + z^2} + 3bd^3 \sqrt{b^2 + d^2 + z^2} +
\]
\[
2b dz^2 \sqrt{b^2 + d^2 + z^2} - (3c^5 + 5c^3 z^2) \log[a + \sqrt{a^2 + c^2 + z^2}] +
\]
\[
2a^5 \log[c + \sqrt{a^2 + c^2 + z^2}] + 3c^5 \log[b + \sqrt{b^2 + c^2 + z^2}] +
\]
\[
5c^3 z^2 \log[b + \sqrt{b^2 + c^2 + z^2}] - 2b^5 \log[c + \sqrt{b^2 + c^2 + z^2}] +
\]
\[
3d^5 \log[a + \sqrt{a^2 + d^2 + z^2}] + 5d^3 z^2 \log[a + \sqrt{a^2 + d^2 + z^2}] -
\]
\[
2a^5 \log[d + \sqrt{a^2 + d^2 + z^2}] - 3d^5 \log[b + \sqrt{b^2 + d^2 + z^2}] -
\]
\[
5d^3 z^2 \log[b + \sqrt{b^2 + d^2 + z^2}] + 2b^5 \log[d + \sqrt{b^2 + d^2 + z^2}] \right) \right).
\]
\[ K'_{252,n} = \]
\[ \frac{1}{18} \left( \frac{3a^6c}{\sqrt{a^2 + c^2 + z^2}} - \frac{a^4c^3}{\sqrt{a^2 + c^2 + z^2}} - \frac{4a^2c^5}{\sqrt{a^2 + c^2 + z^2}} - \frac{8c^7}{\sqrt{a^2 + c^2 + z^2}} - \frac{7a^2c^3z^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{22c^5z^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{17c^3z^4}{\sqrt{a^2 + c^2 + z^2}} - \frac{3cz^6}{\sqrt{a^2 + c^2 + z^2}} - \frac{3b^6c}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^4c^3}{\sqrt{b^2 + c^2 + z^2}} - \frac{4b^2c^5}{\sqrt{b^2 + c^2 + z^2}} - \frac{8c^7}{\sqrt{b^2 + c^2 + z^2}} - \frac{7b^2c^3z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{22c^5z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{17c^3z^4}{\sqrt{b^2 + c^2 + z^2}} - \frac{3cz^6}{\sqrt{b^2 + c^2 + z^2}} - \frac{3a^6d}{\sqrt{a^2 + d^2 + z^2}} - \frac{a^4d^3}{\sqrt{a^2 + d^2 + z^2}} - \frac{4a^2d^5}{\sqrt{a^2 + d^2 + z^2}} - \frac{8d^7}{\sqrt{a^2 + d^2 + z^2}} - \frac{7a^2d^3z^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{22d^5z^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{17d^3z^4}{\sqrt{a^2 + d^2 + z^2}} - \frac{3d^7z^6}{\sqrt{a^2 + d^2 + z^2}} - \frac{3(b^6 + z^6) \log[c + \sqrt{a^2 + c^2 + z^2}]}{3a^6 \log[d + \sqrt{a^2 + d^2 + z^2}]} - \frac{3z^6 \log[d + \sqrt{b^2 + d^2 + z^2}]}{3b^6 \log[d + \sqrt{b^2 + d^2 + z^2}]} + \right) \]

\[ K'_{203,n} = \]
\[ z(a \sqrt{a^2 + c^2 + z^2} - b \sqrt{b^2 + c^2 + z^2} - a \sqrt{a^2 + d^2 + z^2} + b \sqrt{b^2 + d^2 + z^2} + z^2 \log[a + \sqrt{a^2 + c^2 + z^2}] - z^2 \log[b + \sqrt{b^2 + c^2 + z^2}] - z^2 \log[a + \sqrt{a^2 + d^2 + z^2}] + z^2 \log[b + \sqrt{b^2 + d^2 + z^2}]) \]
\begin{align*}
K'_{z_{13},n} &= \frac{1}{3} z \left( 2z^2 \sqrt{a^2 + c^2 + z^2} - 2b^2 \sqrt{b^2 + c^2 + z^2} - 2z^2 \sqrt{b^2 + c^2 + z^2} - d^2 \sqrt{a^2 + d^2 + z^2} - 2z^2 \sqrt{a^2 + d^2 + z^2} + 2b^2 \sqrt{b^2 + d^2 + z^2} - d^2 \sqrt{b^2 + d^2 + z^2} + 2z^2 \sqrt{b^2 + d^2 + z^2} + c^2 (-\sqrt{a^2 + c^2 + z^2} + \sqrt{b^2 + c^2 + z^2}) + 2a^2 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2}) \right) \\
K'_{z_{23},n} &= \frac{1}{4} z \left( 2a^3 \sqrt{a^2 + c^2 + z^2} - 2b^3 \sqrt{b^2 + c^2 + z^2} + b c^2 \sqrt{b^2 + c^2 + z^2} - b z^2 \sqrt{b^2 + c^2 + z^2} - 2a^2 \sqrt{a^2 + d^2 + z^2} + a d^2 \sqrt{a^2 + d^2 + z^2} - a z^2 \sqrt{a^2 + d^2 + z^2} + 2b^2 \sqrt{b^2 + d^2 + z^2} - b d^2 \sqrt{b^2 + d^2 + z^2} + b z^2 \sqrt{b^2 + d^2 + z^2} + (c^4 - z^4) \log [a + \sqrt{a^2 + c^2 + z^2}] - c^4 \log [b + \sqrt{b^2 + c^2 + z^2}] + z^4 \log [b + \sqrt{b^2 + c^2 + z^2}] + (-d^4 + z^4) \log [a + \sqrt{a^2 + d^2 + z^2}] + d^4 \log [b + \sqrt{b^2 + d^2 + z^2}] - z^4 \log [b + \sqrt{b^2 + d^2 + z^2}] \right) \\
K'_{z_{33},n} &= \frac{1}{15} z \left( -4z^4 \sqrt{a^2 + c^2 + z^2} - 6 b^4 \sqrt{b^2 + c^2 + z^2} - 2 b^2 z^2 \sqrt{b^2 + c^2 + z^2} + 4z^4 \sqrt{b^2 + c^2 + z^2} - 6d^4 \sqrt{a^2 + d^2 + z^2} - 2d^2 z^2 \sqrt{a^2 + d^2 + z^2} + 4z^4 \sqrt{a^2 + d^2 + z^2} + 6b^4 \sqrt{b^2 + d^2 + z^2} - 3b^2 d^2 \sqrt{b^2 + d^2 + z^2} + 6d^4 \sqrt{b^2 + d^2 + z^2} + 2b^2 z^2 \sqrt{b^2 + d^2 + z^2} + 2d^2 z^2 \sqrt{b^2 + d^2 + z^2} - 4z^4 \sqrt{b^2 + d^2 + z^2} + 6c^4 (\sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2}) + 6a^4 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2}) + c^2 (3b^2 \sqrt{b^2 + c^2 + z^2} + 2z^2 \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2}) + a^2 (3b^2 \sqrt{a^2 + c^2 + z^2} + 2z^2 \sqrt{a^2 + d^2 + z^2} + 2z^2 (\sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2})) \right) 
\end{align*}
\[
K'_{z_{43,n}} = \\
\frac{1}{24} z \left(8 a^5 \sqrt{a^2 + c^2 + z^2} - 4 a^3 c^2 \sqrt{a^2 + c^2 + z^2} + \\
6 a c^4 \sqrt{a^2 + c^2 + z^2} + 2 a^3 z^2 \sqrt{a^2 + c^2 + z^2} + 3 a c^2 z^2 \sqrt{a^2 + c^2 + z^2} - \\
3 a z^4 \sqrt{a^2 + c^2 + z^2} - 8 b^5 \sqrt{b^2 + c^2 + z^2} + 4 b^3 c^2 \sqrt{b^2 + c^2 + z^2} - \\
6 b c^4 \sqrt{b^2 + c^2 + z^2} - 2 b^3 z^2 \sqrt{b^2 + c^2 + z^2} - 3 b c^2 z^2 \sqrt{b^2 + c^2 + z^2} + \\
3 b z^4 \sqrt{b^2 + c^2 + z^2} - 8 a^5 \sqrt{a^2 + d^2 + z^2} + 4 a^3 d^2 \sqrt{a^2 + d^2 + z^2} - \\
6 a d^4 \sqrt{a^2 + d^2 + z^2} - 2 a^3 z^2 \sqrt{a^2 + d^2 + z^2} - 3 a d^2 z^2 \sqrt{a^2 + d^2 + z^2} + \\
3 a z^4 \sqrt{a^2 + d^2 + z^2} + 8 b^5 \sqrt{b^2 + d^2 + z^2} - 4 b^3 d^2 \sqrt{b^2 + d^2 + z^2} - \\
6 b d^4 \sqrt{b^2 + d^2 + z^2} + 2 b^3 z^2 \sqrt{b^2 + d^2 + z^2} + 3 b d^2 z^2 \sqrt{b^2 + d^2 + z^2} - \\
3 b z^4 \sqrt{b^2 + d^2 + z^2} - 3(2 c^6 + 3 c^4 z^2 - z^6) \log(a + \sqrt{a^2 + c^2 + z^2}) + \\
3(2 c^6 + 3 c^4 z^2 - z^6) \log(b + \sqrt{b^2 + c^2 + z^2}) + \\
6 a^6 \log(a + \sqrt{a^2 + d^2 + z^2}) + 9 d^4 z^2 \log(a + \sqrt{a^2 + d^2 + z^2}) - \\
3 z^6 \log(a + \sqrt{a^2 + d^2 + z^2}) - 6 d^6 \log(b + \sqrt{b^2 + d^2 + z^2}) - \\
9 d^4 z^2 \log(b + \sqrt{b^2 + d^2 + z^2}) + 3 z^6 \log(b + \sqrt{b^2 + d^2 + z^2}) \right) 
\]
\[ K'_{253,n} = \frac{1}{105} z \left( 16z^6 \sqrt{a^2 + c^2 + z^2} - 30b^6 \sqrt{b^2 + c^2 + z^2} - 
\right. \\
6b^4z^2 \sqrt{b^2 + c^2 + z^2} + 8b^2z^4 \sqrt{b^2 + c^2 + z^2} - 16z^6 \sqrt{b^2 + c^2 + z^2} + 
40d^6 \sqrt{a^2 + d^2 + z^2} + 64d^2z^2 \sqrt{a^2 + d^2 + z^2} + 8d^2z^4 \sqrt{a^2 + d^2 + z^2} - 
16z^6 \sqrt{a^2 + d^2 + z^2} + 30b^6 \sqrt{b^2 + d^2 + z^2} - 15b^4d^2 \sqrt{b^2 + d^2 + z^2} + 
20b^2d^4 \sqrt{b^2 + d^2 + z^2} - 40d^6 \sqrt{b^2 + d^2 + z^2} + 6b^4z^2 \sqrt{b^2 + d^2 + z^2} + 
12b^2d^2z^2 \sqrt{b^2 + d^2 + z^2} - 64d^4z^2 \sqrt{b^2 + d^2 + z^2} - 
8b^2z^4 \sqrt{b^2 + d^2 + z^2} - 8d^2z^4 \sqrt{b^2 + d^2 + z^2} + 16z^6 \sqrt{b^2 + d^2 + z^2} - 
40e^6 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) + 30a^6 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) - 
4e^4 \left( 5b^2 \sqrt{b^2 + c^2 + z^2} + 16z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) \right) + 
\left. \frac{c^2(15b^4 \sqrt{b^2 + c^2 + z^2} - 12b^2z^2 \sqrt{b^2 + c^2 + z^2} - 
8z^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) - \sqrt{b^2 + c^2 + z^2} \right)}{1} \right) 
\] 

\[ K'_{204,n} = z \left( -z^3 \left( \text{ArcTan}[c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2} , az] + \right. \\
\text{ArcTan}[c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2} , -bz] + \text{ArcTan}[d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2} , \\
-az] + \text{ArcTan}[d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2} , bz] \right) + 
\right) \\
\frac{1}{2} \left( ac \sqrt{a^2 + c^2 + z^2} - bc \sqrt{b^2 + c^2 + z^2} - ad \sqrt{a^2 + d^2 + z^2} + 
bd \sqrt{b^2 + d^2 + z^2} - (a^3 + 3az^2) \text{Log}[c + \sqrt{a^2 + c^2 + z^2}] + 
b^3 \text{Log}[c + \sqrt{b^2 + c^2 + z^2}] + 3bz^2 \text{Log}[c + \sqrt{b^2 + c^2 + z^2}] + 
a^3 \text{Log}[d + \sqrt{a^2 + d^2 + z^2}] + 3az^2 \text{Log}[d + \sqrt{a^2 + d^2 + z^2}] - 
b^3 \text{Log}[d + \sqrt{b^2 + d^2 + z^2}] - 3bz^2 \text{Log}[d + \sqrt{b^2 + d^2 + z^2}] \right) 
\]
\[ K'_{z_{14,\alpha}} = \frac{1}{8} \left( \frac{3a^4c}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^2c^3}{\sqrt{a^2 + c^2 + z^2}} - \frac{2c^5}{\sqrt{a^2 + c^2 + z^2}} + \frac{6a^2cz^2}{\sqrt{a^2 + c^2 + z^2}} \right. \]
\[ \frac{c^3z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{3cz^4}{\sqrt{a^2 + c^2 + z^2}} - \frac{3b^4c}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^2c^3}{\sqrt{b^2 + c^2 + z^2}} \]
\[ \frac{2c^5}{\sqrt{b^2 + c^2 + z^2}} - \frac{6b^2cz^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{c^3z^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{3cz^4}{\sqrt{b^2 + c^2 + z^2}} \]
\[ \frac{3a^4d}{\sqrt{a^2 + d^2 + z^2}} - \frac{a^2d^3}{\sqrt{a^2 + d^2 + z^2}} + \frac{2d^5}{\sqrt{a^2 + d^2 + z^2}} - \frac{6a^2dz^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{d^3z^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{3dz^4}{\sqrt{a^2 + d^2 + z^2}} \]
\[ \frac{3b^4d}{\sqrt{a^2 + d^2 + z^2}} + \frac{b^2d^3}{\sqrt{a^2 + d^2 + z^2}} - \frac{2d^5}{\sqrt{b^2 + d^2 + z^2}} + \frac{6b^2dz^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{d^3z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{3dz^4}{\sqrt{b^2 + d^2 + z^2}} \]
\[ 3(a^2 + z^2)^2 \log[c + \sqrt{a^2 + c^2 + z^2}] + 3b^4 \log[c + \sqrt{b^2 + c^2 + z^2}] + 6b^2z^2 \log[c + \sqrt{b^2 + c^2 + z^2}] + 3z^4 \log[c + \sqrt{b^2 + c^2 + z^2}] + 3(a^2 + z^2)^2 \log[d + \sqrt{a^2 + d^2 + z^2}] - 3b^4 \log[d + \sqrt{b^2 + d^2 + z^2}] - 6b^2z^2 \log[d + \sqrt{b^2 + d^2 + z^2}] - 3z^4 \log[d + \sqrt{b^2 + d^2 + z^2}] \]
\[ K'_{2z24,n} = \]
\[ z \left( -\frac{1}{5} z^5 \left( \text{ArcTan}[c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, -az] + \right. \right. \]
\[ \text{ArcTan}[c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, bz] + \text{ArcTan}[d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, \]
\[ \left. \left. az] + \text{ArcTan}[d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, -bz] \right) + \right) \]
\[ \frac{1}{10} \left( 3 a^3 c \sqrt{a^2 + c^2 + z^2} - 2 ac^3 \sqrt{a^2 + c^2 + z^2} + 2 acz^2 \sqrt{a^2 + c^2 + z^2} - \right. \]
\[ 3 b^3 c \sqrt{b^2 + c^2 + z^2} + 2 bc^3 \sqrt{b^2 + c^2 + z^2} - 2 bcz^2 \sqrt{b^2 + c^2 + z^2} - \right. \]
\[ 3 a^3 d \sqrt{a^2 + d^2 + z^2} + 2 ad^3 \sqrt{a^2 + d^2 + z^2} - 2 adz^2 \sqrt{a^2 + d^2 + z^2} + \right. \]
\[ 3 b^3 d \sqrt{b^2 + d^2 + z^2} - 2 b dz^2 \sqrt{b^2 + d^2 + z^2} + \right. \]
\[ 2 b dz^2 \sqrt{b^2 + d^2 + z^2} + 2 c^5 \log[a + \sqrt{a^2 + c^2 + z^2}] - \right. \]
\[ (3 a^5 + 5 a^3 z^2) \log[c + \sqrt{a^2 + c^2 + z^2}] - 2 c^5 \log[b + \sqrt{b^2 + c^2 + z^2}] + \right. \]
\[ 3 b^5 \log[c + \sqrt{b^2 + c^2 + z^2}] + 5 b^3 z^2 \log[c + \sqrt{b^2 + c^2 + z^2}] - \right. \]
\[ 2 d^5 \log[a + \sqrt{a^2 + d^2 + z^2}] + 3 a^5 \log[d + \sqrt{a^2 + d^2 + z^2}] + \right. \]
\[ 5 a^3 z^2 \log[d + \sqrt{a^2 + d^2 + z^2}] + 2 d^5 \log[b + \sqrt{b^2 + d^2 + z^2}] - \right. \]
\[ 3 b^5 \log[d + \sqrt{b^2 + d^2 + z^2}] - 5 b^3 z^2 \log[d + \sqrt{b^2 + d^2 + z^2}] \right) \]
\[ K'_{z34,n} = \frac{1}{24} z \left( 6a^4 c \sqrt{a^2 + c^2 + z^2} - 4a^2 c^3 \sqrt{a^2 + c^2 + z^2} + 8c^5 \sqrt{a^2 + c^2 + z^2} + 
right. \\
left. 3a^2 cz^2 \sqrt{a^2 + c^2 + z^2} - 2c^3 z^2 \sqrt{a^2 + c^2 + z^2} - 3cz^4 \sqrt{a^2 + c^2 + z^2} - 
right. \\
left. 6b^4 c \sqrt{b^2 + c^2 + z^2} + 4b^2 c^3 \sqrt{b^2 + c^2 + z^2} - 8c^5 \sqrt{b^2 + c^2 + z^2} - 
right. \\
left. 3b^2 cz^2 \sqrt{b^2 + c^2 + z^2} - 2c^3 z^2 \sqrt{b^2 + c^2 + z^2} + 3cz^4 \sqrt{b^2 + c^2 + z^2} - 
right. \\
left. 6a^4 d \sqrt{a^2 + d^2 + z^2} + 4a^2 d^3 \sqrt{a^2 + d^2 + z^2} - 8d^5 \sqrt{a^2 + d^2 + z^2} - 
right. \\
left. 3a^2 dz^2 \sqrt{a^2 + d^2 + z^2} - 2d^3 z^2 \sqrt{a^2 + d^2 + z^2} + 3dz^4 \sqrt{a^2 + d^2 + z^2} + 
right. \\
left. 6b^4 d \sqrt{b^2 + d^2 + z^2} - 4b^2 d^3 \sqrt{b^2 + d^2 + z^2} + 8d^5 \sqrt{b^2 + d^2 + z^2} + 
right. \\
left. 3b^2 dz^2 \sqrt{b^2 + d^2 + z^2} + 2d^3 z^2 \sqrt{b^2 + d^2 + z^2} - 3dz^4 \sqrt{b^2 + d^2 + z^2} - 
right. \\
left. 3(2a^6 + 3a^4 z^2 - 2c^6) \log[c + \sqrt{a^2 + c^2 + z^2}] + 6b^6 \log[c + \sqrt{b^2 + c^2 + z^2}] + 
right. \\
left. 9b^4 z^2 \log[c + \sqrt{b^2 + c^2 + z^2}] - 3z^6 \log[c + \sqrt{b^2 + c^2 + z^2}] + 
right. \\
left. 3(2a^6 + 3a^4 z^2 - 2c^6) \log[d + \sqrt{a^2 + d^2 + z^2}] - 6b^6 \log[d + \sqrt{b^2 + d^2 + z^2}] - 
right. \\
left. 9b^4 z^2 \log[d + \sqrt{b^2 + d^2 + z^2}] + 3z^6 \log[d + \sqrt{b^2 + d^2 + z^2}] \right) \]
\[ K'_{244,n} = \frac{a}{z} \left( -\frac{3}{35} z^7 \left( \text{ArcTan} \left[ c^2 + z^2 + c \sqrt{a^2 + c^2 + z^2}, az \right] + \text{ArcTan} \left[ c^2 + z^2 + c \sqrt{b^2 + c^2 + z^2}, -bz \right] + \text{ArcTan} \left[ d^2 + z^2 + d \sqrt{a^2 + d^2 + z^2}, -az \right] + \text{ArcTan} \left[ d^2 + z^2 + d \sqrt{b^2 + d^2 + z^2}, bz \right] \right) + \frac{1}{70} (15a^5 c \sqrt{a^2 + c^2 + z^2} - 10a^3 c^3 \sqrt{a^2 + c^2 + z^2} + 15a^5 c \sqrt{a^2 + c^2 + z^2} + 6a^3 c^2 \sqrt{a^2 + c^2 + z^2} + 6ac^4 \sqrt{a^2 + c^2 + z^2} - 15b^5 c \sqrt{b^2 + c^2 + z^2} + 10b^3 c^3 \sqrt{b^2 + c^2 + z^2} - 15bc^5 \sqrt{b^2 + c^2 + z^2} - 6b^3 c^2 \sqrt{b^2 + c^2 + z^2} - 16bc^4 \sqrt{b^2 + c^2 + z^2} + 6ad^5 \sqrt{a^2 + d^2 + z^2} - 15a^5 d \sqrt{a^2 + d^2 + z^2} + 10a^3 d^3 \sqrt{a^2 + d^2 + z^2} - 15ad^5 \sqrt{a^2 + d^2 + z^2} - 6a^3 d^2 \sqrt{a^2 + d^2 + z^2} - 6ad^4 \sqrt{a^2 + d^2 + z^2} - 6ad^4 \sqrt{a^2 + d^2 + z^2} + 6ad^4 \sqrt{a^2 + d^2 + z^2} + 15b^5 \sqrt{b^2 + d^2 + z^2} + 10b^3 d^3 \sqrt{b^2 + d^2 + z^2} + 15bd^5 \sqrt{b^2 + d^2 + z^2} + 6b^3 d^2 \sqrt{b^2 + d^2 + z^2} + 6bd^4 \sqrt{b^2 + d^2 + z^2} - 6bd^4 \sqrt{b^2 + d^2 + z^2} - 3(5c^7 + 7c^5 z^2) \log \left[ a + \sqrt{a^2 + c^2 + z^2} \right] - 3(5a^7 + 7a^5 z^2) \log \left[ c + \sqrt{a^2 + c^2 + z^2} \right] + 15c^7 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + 21c^5 z^2 \log \left[ b + \sqrt{b^2 + c^2 + z^2} \right] + 15b^7 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] + 21b^5 z^2 \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] + 15d^7 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 21d^5 z^2 \log \left[ a + \sqrt{a^2 + d^2 + z^2} \right] + 15a^7 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] + 21a^5 z^2 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] - 15d^7 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - 21d^5 z^2 \log \left[ b + \sqrt{b^2 + d^2 + z^2} \right] - 15b^7 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] - 21b^5 z^2 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] ) \right) \]
\[
K_{z54,n} = \frac{1}{48} \left( \frac{9a^8 c^5}{\sqrt{a^2 + c^2 + z^2}} + \frac{3a^6 c^3}{\sqrt{a^2 + c^2 + z^2}} + \frac{2a^4 c^5}{\sqrt{a^2 + c^2 + z^2}} + \frac{8a^2 c^7}{\sqrt{a^2 + c^2 + z^2}} + \frac{16 c^9}{\sqrt{a^2 + c^2 + z^2}} + \frac{12 a^6 c z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^4 c^3 z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{12 a^2 c^5 z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{40 c^7 z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^2 c^3 z^4}{\sqrt{a^2 + c^2 + z^2}} + \frac{26 c^5 z^4}{\sqrt{a^2 + c^2 + z^2}} + \frac{c^3 z^6}{\sqrt{a^2 + c^2 + z^2}} + \frac{3 c z^8}{\sqrt{a^2 + c^2 + z^2}} + \frac{9 b^8 c}{\sqrt{a^2 + c^2 + z^2}} + \frac{3 b^6 c^3}{\sqrt{a^2 + c^2 + z^2}} + \frac{2 b^4 c^5}{\sqrt{b^2 + c^2 + z^2}} + \frac{8 b^2 c^7}{\sqrt{b^2 + c^2 + z^2}} + \frac{16 c^9}{\sqrt{b^2 + c^2 + z^2}} + \frac{12 b^6 c z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{b^4 c^3 z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{12 b^2 c^5 z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{40 c^7 z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{b^2 c^3 z^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{26 c^5 z^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{c^3 z^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{3 c z^8}{\sqrt{b^2 + c^2 + z^2}} + \frac{9 a^8 d}{\sqrt{b^2 + c^2 + z^2}} + \frac{3 a^6 d^3}{\sqrt{a^2 + d^2 + z^2}} + \frac{2 a^4 d^5}{\sqrt{a^2 + d^2 + z^2}} + \frac{8 a^2 d^7}{\sqrt{a^2 + d^2 + z^2}} + \frac{16 d^9}{\sqrt{a^2 + d^2 + z^2}} + \frac{12 a^6 d z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{a^4 d^3 z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{12 a^2 d^5 z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{40 d^7 z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{a^2 d^3 z^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{26 d^5 z^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{d^3 z^6}{\sqrt{a^2 + d^2 + z^2}} + \frac{3 d z^8}{\sqrt{a^2 + d^2 + z^2}} + \frac{9 b^8 d}{\sqrt{a^2 + d^2 + z^2}} + \frac{3 b^6 d^3}{\sqrt{b^2 + d^2 + z^2}} + \frac{2 b^4 d^5}{\sqrt{b^2 + d^2 + z^2}} + \frac{8 b^2 d^7}{\sqrt{b^2 + d^2 + z^2}} + \frac{16 d^9}{\sqrt{b^2 + d^2 + z^2}} + \frac{12 b^6 d z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{b^4 d^3 z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{12 b^2 d^5 z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{40 d^7 z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{b^2 d^3 z^4}{\sqrt{b^2 + d^2 + z^2}} + \frac{26 d^5 z^4}{\sqrt{b^2 + d^2 + z^2}} + \frac{d^3 z^6}{\sqrt{b^2 + d^2 + z^2}} + \frac{3 d z^8}{\sqrt{b^2 + d^2 + z^2}} + 3 (3 a^8 + 4 a^6 z^2 + z^8) \log \left[ c + \sqrt{a^2 + c^2 + z^2} \right] + 3 (3 b^8 + 4 b^6 z^2 + z^8) \log \left[ c + \sqrt{b^2 + c^2 + z^2} \right] + 9 a^8 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] + 12 a^6 z^2 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] + 3 z^8 \log \left[ d + \sqrt{a^2 + d^2 + z^2} \right] - 9 b^8 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] - 12 b^6 z^2 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] - 3 z^8 \log \left[ d + \sqrt{b^2 + d^2 + z^2} \right] \right) \right)
\]
\[ K'_{z_{05, n}} = \frac{1}{3} z \left( -2a^3 \sqrt{a^2 + c^2 + z^2} + ac^2 \sqrt{a^2 + c^2 + z^2} - 5az^2 \sqrt{a^2 + c^2 + z^2} + 
2b^3 \sqrt{b^2 + c^2 + z^2} - bc^2 \sqrt{b^2 + c^2 + z^2} + 5bz^2 \sqrt{b^2 + c^2 + z^2} + 
2a^3 \sqrt{a^2 + d^2 + z^2} - ad^2 \sqrt{a^2 + d^2 + z^2} + 5az^2 \sqrt{a^2 + d^2 + z^2} - 
2b^3 \sqrt{b^2 + d^2 + z^2} + bd^2 \sqrt{b^2 + d^2 + z^2} - 5bz^2 \sqrt{b^2 + d^2 + z^2} - 
3z^4 \log \left( a + \sqrt{a^2 + c^2 + z^2} \right) + 3z^4 \log \left( b + \sqrt{b^2 + c^2 + z^2} \right) + 
3z^4 \log \left( a + \sqrt{a^2 + d^2 + z^2} \right) - 3z^4 \log \left( b + \sqrt{b^2 + d^2 + z^2} \right) \right) \]
\[ K'_{215,n} = \frac{1}{15} z \left( -\frac{8 z^6}{\sqrt{a^2 + c^2 + z^2}} + \frac{8 b^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{24 b^4 z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{24 b^2 z^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{8 z^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{3 d^6}{\sqrt{a^2 + d^2 + z^2}} - \frac{d^4 z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{4 d^2 z^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{8 z^6}{\sqrt{a^2 + d^2 + z^2}} - \frac{8 b^6}{\sqrt{b^2 + d^2 + z^2}} - \frac{8 b^4 z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{24 b^2 d^4}{\sqrt{b^2 + d^2 + z^2}} - \frac{3 d^6}{\sqrt{b^2 + d^2 + z^2}} - \frac{24 b^4 z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{4 b^2 d^4}{\sqrt{b^2 + d^2 + z^2}} - \frac{8 z^6}{\sqrt{b^2 + d^2 + z^2}} + \frac{3}{\sqrt{a^2 + c^2 + z^2}} + \frac{3}{\sqrt{b^2 + c^2 + z^2}} \right) + \frac{8 a^6}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{a^2 + d^2 + z^2}} \right) + \frac{c^4}{\sqrt{a^2 + c^2 + z^2}} - \frac{b^2}{\sqrt{b^2 + c^2 + z^2}} + z^2 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} - \frac{1}{\sqrt{b^2 + c^2 + z^2}} \right) + \frac{4 c^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{2 a^2 z^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{b^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{2 b^2 z^2}{\sqrt{b^2 + c^2 + z^2}} + z^4 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{b^2 + c^2 + z^2}} \right) + \frac{a^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{d^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{8 d^2 z^2}{\sqrt{a^2 + d^2 + z^2}} + 24 z^4 \left( \frac{1}{\sqrt{a^2 + c^2 + z^2}} + \frac{1}{\sqrt{a^2 + d^2 + z^2}} \right) + \frac{4 a^4}{\sqrt{a^2 + d^2 + z^2}} + z^2 \left( \frac{6}{\sqrt{a^2 + c^2 + z^2}} + \frac{6}{\sqrt{a^2 + d^2 + z^2}} \right) \)
\[
K'_{z25,n} = \frac{1}{18} \left( \frac{8a^7}{\sqrt{a^2 + c^2 + z^2}} - \frac{4a^5c^2}{\sqrt{a^2 + c^2 + z^2}} + \frac{a^3c^4}{\sqrt{a^2 + c^2 + z^2}} - \frac{3ac^6}{\sqrt{a^2 + c^2 + z^2}} \right) - \frac{22a^5z^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{7a^3c^2z^2}{\sqrt{a^2 + c^2 + z^2}} - \frac{17a^3z^4}{\sqrt{a^2 + c^2 + z^2}} + \frac{3az^6}{\sqrt{a^2 + c^2 + z^2}} + \frac{8b^7}{\sqrt{b^2 + c^2 + z^2}} + \frac{4b^5c^2}{\sqrt{b^2 + c^2 + z^2}} - \frac{b^3c^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{3bc^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{22b^5z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{7b^3c^2z^2}{\sqrt{b^2 + c^2 + z^2}} + \frac{17b^3z^4}{\sqrt{b^2 + c^2 + z^2}} + \frac{3b^2z^6}{\sqrt{b^2 + c^2 + z^2}} + \frac{8a^7}{\sqrt{a^2 + d^2 + z^2}} + \frac{4a^5d^2}{\sqrt{a^2 + d^2 + z^2}} - \frac{a^3d^4}{\sqrt{a^2 + d^2 + z^2}} + \frac{3ad^6}{\sqrt{a^2 + d^2 + z^2}} + \frac{22a^5z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{7a^3d^2z^2}{\sqrt{a^2 + d^2 + z^2}} + \frac{17a^3z^4}{\sqrt{a^2 + d^2 + z^2}} - \frac{3az^6}{\sqrt{a^2 + d^2 + z^2}} + \frac{8b^7}{\sqrt{b^2 + d^2 + z^2}} + \frac{4b^5d^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{b^3d^4}{\sqrt{b^2 + d^2 + z^2}} - \frac{3bd^6}{\sqrt{b^2 + d^2 + z^2}} + \frac{22b^5z^2}{\sqrt{b^2 + d^2 + z^2}} + \frac{7b^3d^2z^2}{\sqrt{b^2 + d^2 + z^2}} - \frac{17b^3z^4}{\sqrt{b^2 + d^2 + z^2}} - \frac{3bz^6}{\sqrt{b^2 + d^2 + z^2}} + 3(c^6 + z^6)\log[a + \sqrt{a^2 + c^2 + z^2}] - 3c^6\log[b + \sqrt{b^2 + c^2 + z^2}] - \\
3z^6\log[b + \sqrt{b^2 + c^2 + z^2}] - 3(d^6 + z^6)\log[a + \sqrt{a^2 + d^2 + z^2}] + \\
3d^6\log[b + \sqrt{b^2 + d^2 + z^2}] + 3z^6\log[b + \sqrt{b^2 + d^2 + z^2}] \right)
\]
\[ K_{35, n}^' = \frac{1}{105} z \left( 16 z^6 \sqrt{a^2 + c^2 + z^2} + 40 b^6 \sqrt{b^2 + c^2 + z^2} + 64 b^4 z^2 \sqrt{b^2 + c^2 + z^2} + 8 b^2 z^4 \sqrt{a^2 + d^2 + z^2} - 16 z^6 \sqrt{a^2 + b^2 + z^2} - 30 d^6 \sqrt{a^2 + b^2 + z^2} - 16 z^6 \sqrt{a^2 + c^2 + z^2} - 30 d^6 \sqrt{a^2 + d^2 + z^2} - 16 d^4 z^2 \sqrt{a^2 + d^2 + z^2} + 6 b^2 d^2 \sqrt{a^2 + b^2 + d^2 + z^2} - 16 z^6 \sqrt{a^2 + d^2 + z^2} + 30 d^6 \sqrt{a^2 + d^2 + z^2} - 8 b^2 z^4 \sqrt{b^2 + d^2 + z^2} - 8 d^2 z^4 \sqrt{b^2 + d^2 + z^2} + 16 d^6 \sqrt{b^2 + d^2 + z^2} + 40 a^6 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + d^2 + z^2} \right) - 5 b^2 \sqrt{b^2 + c^2 + z^2} - 2 z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{b^2 + c^2 + z^2} \right) + 4 c^2 \left( 5 a^2 \sqrt{a^2 + c^2 + z^2} - 3 b^2 \sqrt{b^2 + c^2 + z^2} - 5 b^4 \sqrt{b^2 + c^2 + z^2} - 16 z^2 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + b^2 + z^2} \right) + 12 d^2 z^2 \sqrt{a^2 + d^2 + z^2} - 8 z^4 \left( \sqrt{a^2 + c^2 + z^2} - \sqrt{a^2 + b^2 + z^2} \right) \right) \right) \]
\[ K_{z, a} = \frac{1}{48 z} \left( \begin{array}{c}
-16a^9 + 8a^7c^2 + 2a^5c^4 + 3a^3c^6 + 9ac^8 \\
\sqrt{a^2 + c^2 + z^2} \sqrt{a^2 + c^2 + z^2} \sqrt{a^2 + c^2 + z^2} \sqrt{a^2 + c^2 + z^2} \\
40a^7z^2 - 12a^5c^2z^2 + 4a^3c^4z^2 - 12a^5c^6z^2 - 26a^5z^4 \\
\sqrt{a^2 + c^2 + z^2} \sqrt{a^2 + c^2 + z^2} \sqrt{a^2 + c^2 + z^2} \sqrt{a^2 + c^2 + z^2} \\
a^3c^2z^4 - 3az^6 + 3az^6 + 16b^9 + 8b^7c^2 \\
\sqrt{a^2 + c^2 + z^2} \sqrt{a^2 + c^2 + z^2} \sqrt{a^2 + c^2 + z^2} \sqrt{a^2 + c^2 + z^2} \\
2b^5c^4 - 3b^3c^6 + 9b^5 - 40b^7z^2 + 12b^5c^2z^2 \\
\sqrt{b^2 + c^2 + z^2} \sqrt{b^2 + c^2 + z^2} \sqrt{b^2 + c^2 + z^2} \sqrt{b^2 + c^2 + z^2} \\
b^3c^4z^2 - 12b^3c^2z^2 - 26b^5z^4 + b^3c^2z^4 + b^3z^6 \\
\sqrt{b^2 + c^2 + z^2} \sqrt{b^2 + c^2 + z^2} \sqrt{b^2 + c^2 + z^2} \sqrt{b^2 + c^2 + z^2} \\
3b^3z^8 + 16a^9 - 8a^7d^2 + 2a^5d^4 - 3a^3d^6 \\
\sqrt{b^2 + c^2 + z^2} \sqrt{b^2 + c^2 + z^2} \sqrt{b^2 + c^2 + z^2} \sqrt{b^2 + c^2 + z^2} \\
9ad^8 + 40a^7z^2 + 12a^5d^2z^2 - 3a^3d^4z^4 - 12a^5d^6z^2 \\
\sqrt{a^2 + d^2 + z^2} \sqrt{a^2 + d^2 + z^2} \sqrt{a^2 + d^2 + z^2} \sqrt{a^2 + d^2 + z^2} \\
26a^5z^4 + 3a^3d^2z^4 - 3az^8 + 16b^9 \\
\sqrt{a^2 + d^2 + z^2} \sqrt{a^2 + d^2 + z^2} \sqrt{a^2 + d^2 + z^2} \sqrt{a^2 + d^2 + z^2} \\
8b^7d^2 + 2b^5d^4 + 3b^3d^6 + 9bd^8 \\
\sqrt{b^2 + d^2 + z^2} \sqrt{b^2 + d^2 + z^2} \sqrt{b^2 + d^2 + z^2} \sqrt{b^2 + d^2 + z^2} \\
40b^7z^2 - 12b^5d^2z^2 + b^3d^4z^2 - 12b^5d^6z^2 \\
\sqrt{b^2 + d^2 + z^2} \sqrt{b^2 + d^2 + z^2} \sqrt{b^2 + d^2 + z^2} \sqrt{b^2 + d^2 + z^2} \\
26b^5z^4 + b^3d^2z^4 + b^3z^6 + 3bz^8 \\
\sqrt{b^2 + d^2 + z^2} \sqrt{b^2 + d^2 + z^2} \sqrt{b^2 + d^2 + z^2} \sqrt{b^2 + d^2 + z^2} \\
3(3c^8 + 4a^6z^2 + z^8) \log(a + \sqrt{a^2 + c^2 + z^2}) + 9c^8 \log(b + \sqrt{b^2 + c^2 + z^2}) + \\
12c^6z^2 \log(b + \sqrt{b^2 + c^2 + z^2}) + 9c^8 \log(b + \sqrt{b^2 + c^2 + z^2}) + \\
3(3d^8 + 4c^6z^2 + z^8) \log(a + \sqrt{a^2 + d^2 + z^2}) - 9d^8 \log(b + \sqrt{b^2 + d^2 + z^2}) - \\
12d^6z^2 \log(b + \sqrt{b^2 + d^2 + z^2}) - 3z^8 \log(b + \sqrt{b^2 + d^2 + z^2}) \right) \right)
\[ K'_{55,a} = \left( \frac{1}{1890} \left\{ \begin{array}{c}
\frac{600a^{10}}{\sqrt{a^2+z^2}} + \frac{280a^8z^2}{\sqrt{a^2+z^2}} + \frac{210a^6c^4}{\sqrt{a^2+z^2}} - \frac{280a^4d^6}{\sqrt{a^2+z^2}} - \frac{560a^2c^8}{\sqrt{a^2+z^2}} - \frac{1360a^8z^2}{\sqrt{a^2+z^2}} - \frac{400a^6c^2z^2}{\sqrt{a^2+z^2}} - \frac{90a^4c^4z^2}{\sqrt{a^2+z^2}} - \frac{520a^2c^6z^2}{\sqrt{a^2+z^2}} - \frac{560d^8z^2}{\sqrt{a^2+z^2}} - \frac{848d^6c^4z^4}{\sqrt{a^2+z^2}} + \frac{24a^2c^2z^4}{\sqrt{a^2+z^2}} + \frac{72a^4c^4z^4}{\sqrt{a^2+z^2}} + \frac{800d^6z^4}{\sqrt{a^2+z^2}} - \frac{16a^4z^6}{\sqrt{a^2+z^2}} + \frac{32a^2c^2z^6}{\sqrt{a^2+z^2}} - \frac{48a^4z^6}{\sqrt{a^2+z^2}} + \frac{64a^2c^8z^8}{\sqrt{a^2+z^2}} + \frac{64c^2z^8}{\sqrt{a^2+z^2}} - \frac{128z^{10}}{\sqrt{a^2+z^2}} \end{array} \right\} \right) + \right.

\left( \frac{1}{1890} \left\{ \begin{array}{c}
\frac{600b^{10}}{\sqrt{b^2+z^2}} - \frac{280b^8c^2}{\sqrt{b^2+z^2}} + \frac{210b^6c^4}{\sqrt{b^2+z^2}} - \frac{280b^4d^6}{\sqrt{b^2+z^2}} + \frac{560b^8c^8}{\sqrt{b^2+z^2}} + \frac{1360b^6c^2z^2}{\sqrt{b^2+z^2}} + \frac{800d^6z^4}{\sqrt{b^2+z^2}} + \frac{520b^2d^6z^2}{\sqrt{b^2+z^2}} - \frac{90b^4c^4z^2}{\sqrt{b^2+z^2}} + \frac{400b^2c^6z^2}{\sqrt{b^2+z^2}} + \frac{24b^4c^2z^4}{\sqrt{b^2+z^2}} + \frac{848b^6z^4}{\sqrt{b^2+z^2}} - \frac{72b^2c^4z^4}{\sqrt{b^2+z^2}} + \frac{800d^6z^4}{\sqrt{b^2+z^2}} + \frac{16b^4z^6}{\sqrt{b^2+z^2}} + \frac{32b^2c^2z^6}{\sqrt{b^2+z^2}} + \frac{48b^4z^6}{\sqrt{b^2+z^2}} + \frac{64b^2c^8z^8}{\sqrt{b^2+z^2}} + \frac{64c^2z^8}{\sqrt{b^2+z^2}} + \frac{128z^{10}}{\sqrt{b^2+z^2}} \end{array} \right\} \right) + \right.

\left( \frac{1}{1890} \left\{ \begin{array}{c}
\frac{a^2+d^2+z^2}{\sqrt{a^2+d^2+z^2}} \left( \frac{560a^{10}}{a^2+d^2+z^2} - \frac{280a^8d^2}{a^2+d^2+z^2} + \frac{70a^6d^4}{a^2+d^2+z^2} - \frac{70a^4d^6}{a^2+d^2+z^2} + \frac{280a^8d^8}{a^2+d^2+z^2} + \frac{560d^{10}}{a^2+d^2+z^2} + \frac{1360a^8d^2z^2}{a^2+d^2+z^2} - \frac{400a^6d^4z^2}{a^2+d^2+z^2} - \frac{30a^4d^6z^2}{a^2+d^2+z^2} - \frac{400a^2d^8z^2}{a^2+d^2+z^2} + \frac{280a^8d^8}{a^2+d^2+z^2} + \frac{560d^{10}}{a^2+d^2+z^2} + \frac{1360a^8d^2z^2}{a^2+d^2+z^2} - \frac{400a^6d^4z^2}{a^2+d^2+z^2} - \frac{30a^4d^6z^2}{a^2+d^2+z^2} - \frac{400a^2d^8z^2}{a^2+d^2+z^2} + \frac{32a^2d^6z^6}{a^2+d^2+z^2} + \frac{64a^4z^6}{a^2+d^2+z^2} + \frac{64d^2z^6}{a^2+d^2+z^2} + \frac{128z^{10}}{a^2+d^2+z^2} \end{array} \right\} \right) + \right.

\left( \frac{1}{1890} \left\{ \begin{array}{c}
\frac{b^2+d^2+z^2}{b^2+d^2+z^2} \left( \frac{560b^{10}}{b^2+d^2+z^2} - \frac{280b^8d^2}{b^2+d^2+z^2} + \frac{70b^6d^4}{b^2+d^2+z^2} - \frac{70b^4d^6}{b^2+d^2+z^2} + \frac{280b^8d^8}{b^2+d^2+z^2} + \frac{560d^{10}}{b^2+d^2+z^2} + \frac{1360b^8d^2z^2}{b^2+d^2+z^2} - \frac{400b^6d^4z^2}{b^2+d^2+z^2} - \frac{30b^4d^6z^2}{b^2+d^2+z^2} - \frac{400b^2d^8z^2}{b^2+d^2+z^2} + \frac{280b^8d^8}{b^2+d^2+z^2} + \frac{560d^{10}}{b^2+d^2+z^2} + \frac{1360b^8d^2z^2}{b^2+d^2+z^2} - \frac{400b^6d^4z^2}{b^2+d^2+z^2} - \frac{30b^4d^6z^2}{b^2+d^2+z^2} - \frac{400b^2d^8z^2}{b^2+d^2+z^2} + \frac{32b^2d^6z^6}{b^2+d^2+z^2} + \frac{64b^4z^6}{b^2+d^2+z^2} + \frac{64d^2z^6}{b^2+d^2+z^2} + \frac{128z^{10}}{b^2+d^2+z^2} \end{array} \right\} \right) + \right.\]