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High Accuracy Sensor Aided Inertial Navigation Systems

A Dissertation submitted in partial satisfaction of the requirements for the degree of

Doctor of Philosophy

in

Electrical Engineering

by

Arvind Ramanandan

June 2011

Dissertation Committee:

Dr. Jay A. Farrell, Chairperson
Dr. Matthew J. Barth
Dr. Ertem Tuncel
The Dissertation of Arvind Ramanandan is approved:

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Committee Chairperson

University of California, Riverside
I wish to thank my parents (Leela R. and Ramanandan N.) for always giving me their everything. I am immensely grateful to them. I wish to thank God for his role in the grand scheme of things.

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To Appa, Leela, Amma, Anu and Chinnu
ABSTRACT OF THE DISSERTATION

High Accuracy Sensor Aided Inertial Navigation Systems

by

Arvind Ramanandan

Doctor of Philosophy, Graduate Program in Electrical Engineering
University of California, Riverside, June 2011
Dr. Jay A. Farrell, Chairperson

Reliable and high accuracy (decimeter level) localization of a rover relative to a defined frame is an enabling technology for numerous Intelligent Transportation Systems (ITS) applications (e.g., automotive guidance, routing, lane departure warning). The goal of localization is to compute the navigation state of the rover in some defined frame of reference such that the expected errors in the estimate is within a given performance specification.

Inertial navigation is a popular navigation technique since it provides full six Degree-Of-Freedom (DOF) navigation information. Further inertial sensors have been studied for decades and have well understood error models. This dissertation discusses the theoretical and implementation aspects of certain sensor aided Inertial Navigation Systems (INS). Though the presentation can be easily generalized to all forms of INS, the primary focus of this dissertation will be on automotive INS.

This dissertation formulates the localization problem in a mathematically rigorous fashion and poses it as a nonlinear Bayesian estimation problem. The INS kinematic equations and linearized error state equations required by the Bayesian estimation solution are derived. Aiding techniques like GPS, Vision and stationary aiding are described and mathematically
formulated. Observability and performance analysis are presented for each of these aiding scenarios.

The later part of the dissertation defines and formulates the Near Real Time (NRT) estimation problem. The speed of modern computers allows significant processing beyond that required for a typical INS application. The excess processing power can be used to enhance reliability and accuracy through methods we refer to as NRT processing. Typically, even for an inexpensive IMU, an INS is designed such that it can maintain its specified accuracy for some time period (several seconds) even without aiding. This allows for aiding measurements to be processed in a more contemplative manner, within this time period, while the INS kinematic integration continues to supply users with state information. The contemplative method builds on optimal fixed interval smoothing to confirm or deny the validity of aiding measurements based on both the past and the future measurements. When aiding measurements are validated, the INS state is corrected from the time of validity of that aiding measurement up to the present time. In nonlinear systems, the NRT approach is built upon a Bayesian iterative framework.
4.1.5 Double Differential GPS ........................................... 30
4.2 GPS measurement residuals ........................................... 31
4.3 Observability analysis .................................................. 32
4.4 Dilution-Of-Precision (DOP) ........................................... 35
4.5 Localization results ..................................................... 36

5 Observability analysis of errors in a CDGPS-vision-INS 50
5.1 Introduction ............................................................. 50
5.2 Additional notation ..................................................... 51
5.3 Augmented state and error state vector ............................... 51
  5.3.1 Augmented state dynamic equation ............................... 52
5.4 Vision measurement model ............................................. 52
  5.4.1 Linearized measurement residual model ......................... 53
5.5 Observability analysis .................................................. 54
  5.5.1 Problem description ............................................... 54
  5.5.2 Observability of INS error with a fully calibrated camera ..... 55
  5.5.3 Observability of INS and errors in extrinsic parameters using a camera 55
  5.5.4 Observability of INS and errors in extrinsic parameters using camera and a GPS like sensor ........................................... 58

6 Stationary updates aided INS 60
6.1 Introduction ............................................................. 60
6.2 Literature review ....................................................... 61
6.3 Stationary update measurement model ................................ 63
  6.3.1 Zero velocity update .............................................. 63
  6.3.2 Zero angular rate update ......................................... 64
6.4 Detection of Stationarity using Inertial data ........................ 65
  6.4.1 Problem description ............................................... 65
  6.4.2 Theoretical Analysis .............................................. 66
  6.4.3 Selection of thresholds ........................................... 70
    6.4.3.1 Stationary (\(S(\omega) = 0\)) ................................. 72
    6.4.3.2 Moving (\(S(\omega) \neq 0\)) ................................... 73
  6.4.4 On vehicle Results ................................................. 76
    6.4.4.1 Selection of harmonics ..................................... 77
    6.4.4.2 Detection Results ........................................... 78
  6.4.5 Comparative analysis ............................................. 82
6.5 Observability analysis of INS error states ............................ 87
  6.5.1 Observability using Stationary updates ......................... 88
  6.5.2 Numerical demonstration of observability ...................... 92
6.6 Localization Performance Analysis .................................. 100
6.7 Conclusions ........................................................... 104
List of Figures

3.1 Unbounded growth of INS position error without aiding. ............... 18
3.2 Unbounded growth of INS velocity error without aiding. ............... 19
4.1 GPS aided INS test trajectory on I215 North. .......................... 37
4.2 Differential phase residuals for PRN 3. ................................. 38
4.3 Differential phase residuals for PRN 6. ................................. 39
4.4 Differential phase residuals for PRN 14. ............................... 40
4.5 Estimates of IMU accelerometer bias. ................................. 41
4.6 Standard deviation of errors in estimation of IMU accelerometer bias. 42
4.7 Raw gyroscope measurements during rotation. .......................... 43
4.8 Estimates of IMU gyroscope bias. ................................. 44
4.9 Standard deviation of errors in estimation of IMU gyroscope bias. 45
4.10 Standard deviation of errors in estimates of velocity. .................. 46
4.11 Standard deviation of errors in estimates of small angle rotation. .... 47
4.12 Standard deviation of errors in estimation of position. .................. 48
4.13 Standard deviation of errors in estimation of position. .................. 49
6.1 Analysis of a sequence of 128 samples of forward acceleration data. 69
6.2 A bound on the probability of false detection. .......................... 74
6.3 Trace of $k v_m$ for the $m = 1$ component of the roll-rate gyroscope. 79
6.4 Ground truth GPS estimate of speed. ................................. 81
6.5 Time series of the chosen harmonics for all six sensors. ............... 83
6.6 Stationary decisions vs. GPS estimate of speed. ....................... 84
6.7 Comparative analysis of detection performance for a stationary rover. 85
6.8 Comparative analysis of detection performance for a rover moving at 3.4mph. 86
6.9 Comparative analysis of detection performance for a rover moving at 10mph. 87
6.10 Uncertainty in $n \delta \hat{v}_p(0) = [\delta v_n \delta v_e \delta v_d]$ using all measurements in $S$. 95
6.11 Uncertainty in $n \hat{p}(0) = [\rho_n \rho_e \rho_d]$ using all measurements in $S$. 96
6.12 Uncertainty in yaw direction with GPS estimate of velocity. ........... 97
6.13 Uncertainty in $\hat{b}_g(0) = [b_g \ b'_g \ b''_g]$ using all measurements in $S$. 98
6.14 Uncertainty in $\hat{b}_a(0) = [b_a^x \ b_a^y \ b_a^z]$ using all measurements in $S$. 99
6.15 Position error uncertainty versus time with and without stationary updates. 101
## List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Components of GPS pseudorange measurement</td>
<td>26</td>
</tr>
<tr>
<td>4.2</td>
<td>Components of GPS carrier phase measurement</td>
<td>27</td>
</tr>
<tr>
<td>4.3</td>
<td>Expected PDOP under various GPS technologies</td>
<td>36</td>
</tr>
<tr>
<td>6.1</td>
<td>Sensitivities of harmonics to vehicle motion</td>
<td>78</td>
</tr>
<tr>
<td>6.2</td>
<td>Time of aiding and legend for various maneuvers</td>
<td>93</td>
</tr>
<tr>
<td>A.1</td>
<td>Symbols and values for constants</td>
<td>146</td>
</tr>
<tr>
<td>A.2</td>
<td>Abbreviations and acronyms</td>
<td>147</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

Localization with respect to a defined frame of reference is an age old problem. Examples of navigation with the celestial bodies as reference have been recorded as far as 15 BC [86]. Since then there has been an enormous progress in field navigation technology (e.g., Invention of the Compass, Mariner’s Astrolabe, Marine Chronometer, RADAR and GPS in chronological order). Concepts of Inertial Navigation Systems (INS) were introduced in the pioneering work of Dr. Robert Goddard as early as 1913 [88]. INS, developed mainly for rockets were put to extensive use during World War II for developing missile guidance systems [87]. Electromechanical accelerometers and gyroscopes developed in the 1960-70s were primarily used for applications like cruise missile guidance, autonomous submarine navigation, AHRS for torpedoes, flight controls and smart munitions [7]. The advent of low cost and smaller size Micro-Electro-Mechanical Systems (MEMS) based inertial sensors in the 1990s has led to their increased presence in low cost consumer devices. The limitations of localization using inexpensive inertial sensors are well known (e.g., potentially unbounded error growth), motivating the need for independent, high accuracy aiding sensors. Local-
ization using sensor aided INS is an open research problem with many active areas such as

- Aiding techniques (e.g., GPS [26], [89], Vision [74], [79], Stationary updates [69]).
- Algorithms (e.g., Particle filters [34], Unscented Kalman Filters [40], Multi-State Constraint Kalman Filter [59]).
- Applications (e.g., Autonomous land vehicle navigation [78], Pedestrian navigation [6], Indoor navigation [9], Automated mapping [77], [73]).

This dissertation discusses the theoretical and implementation aspects of certain sensor aided INS. Though the presentation can be easily generalized to all forms of INS, the primary focus of this dissertation will be on automotive INS.

Frames, notation and definitions used in this dissertation are described in Appendix A. For a list of often used abbreviations and acronyms see Appendix A.5.

1.1 Problem statement

Reliable and high accuracy (decimeter level) localization of a rover relative to a defined frame is an enabling technology for numerous Intelligent Transportation Systems (ITS) applications [1], [2] (e.g., automotive guidance, routing, lane departure warning). The goal of localization is to compute the navigation state of the rover in some defined frame of reference (say $n$) such that the expected errors in the estimate is within a given performance specification. It is a well known fact that a rover in a 3D world has 6 Degrees-Of-Freedom (DOF) (i.e., 3D position vector $^n_P_{nb} \in \mathbb{R}^3$ and attitude $^n q \in \mathbb{S}^3$ in Quaternion notation).
To fully describe the navigation state of a rover in the 3D world it is required to compute all 6 DOF. The following characteristics are desired for applications like automated vehicle guidance, high accuracy lane-level mapping etc.,

1. A performance specification of $\pm 1\sigma$ positioning and attitude accuracy in the order of $0.01 - 0.1m$ and $1$ deg respectively.

2. Robustness to errors in sensor measurements (e.g., Reject carrier phase GPS measurements with wrong integer ambiguity estimates).

3. Quantification of uncertainty in estimates of navigation state (e.g., error state covariance).

4. Graceful degradation in the absence of high quality measurements.

Achieving the outlined specifications is a challenge and an open area of research. This dissertation discusses the following sub-problems:

1. *Implementation of a real-time tightly integrated Carrier phase Differential GPS-INS (CDGPS-INS)*: GPS aided INS is the *de-facto* standard in outdoor localization. This is because it achieves performance specification outlined in Item 1 under ideal conditions (e.g., clear sky, differential correction availability) and has well understood conditions for state observability. Chapter 4 describes the theory and implementation of this approach in detail.

2. *Observability analysis of INS and Camera extrinsic parameters with an integrated CDGPS-vision-INS*: INS with feature-based sensor (e.g., vision, LIDAR, RADAR) aiding is a popular localization technique. One of the fundamental difficulties in
integrating any feature-based sensor with INS is the necessity to calibrate, with commensurate accuracy, the lever arm vector and rotation matrix (i.e., the sensor extrinsic parameters) from the body frame to the feature sensor frame. This calibration can occur during the system’s operation when the corresponding error state is observable. The objective of this work is to analyze observability of INS and the vision extrinsic parameters error states in an integrated CDGPS-vision-INS approach. This analysis is presented in detail in Chapter 5.

3. Detection of Stationarity and Observability analysis of an INS aided by Stationary updates: When the rover is known to be stationary, artificial “stationary” measurements (i.e., zero velocity and/or zero angular rate) decrease the rate of drift of the position and attitude by correcting velocity, attitude, and IMU sensor biases. Since these measurements are free and improve performance, they are attractive and have been suggested by many researchers. Implementation requires reliable, automated tests, using sensors already on the vehicle, to detect periods when the vehicle is stationary. Stationary measurements are not without risk. False application of stationary updates, while the vehicle is actually in motion, can severely deteriorate localization performance. Therefore, detection of time instances appropriate for such “stationary measurements” is a challenge worthy of careful consideration. In addition, it is preferable to understand the theoretical bounds to the utility of stationary updates through observability analysis. These problems are dealt with in Chapter 6.

4. Near Real Time estimation for INS: The speed of modern computers allows significant processing beyond that required for a typical INS application. Some sensor processing
errors might be unresolvable at any given time instant at which occurs, but when a
time series of such measurements are presented, it might be possible to resolve such
erors. This idea is called Near Real Time (NRT) estimation. NRT estimation can
give superior performance under certain conditions when algorithms like Extended
Kalman Filters fail. NRT estimation is the central theme of discussion in Chapter 7.

1.2 Dissertation outline

This dissertation can be described by the following outline:

Localization with an aided INS is fundamentally a nonlinear estimation problem. Chapter 2 describes a generic nonlinear estimation problem with stochastic inputs. Section 2.1 reviews some practical nonlinear estimation approaches. Section 2.2 describes the generalized Bayes’ solution and the linearized error state estimation approach. Chapter 2 concludes by proving that the generalized Bayesian solution and the Maximum-A-Posteriori (MAP) error state estimate are equivalent as long as linearization errors are small.

Chapter 3 describes the framework of an aided INS that will be used in this dissertation. Section 3.2 describes the IMU measurement model. Section 3.3 describes the INS kinematic equations, linearized error state equations and derives the state transition matrix for the linearized error state system.

Chapter 4 describes a tightly integrated CDGPS-INS. Section 4.1 describes the GPS pseudorange, carrier phase and Doppler measurement models and their corresponding error budget. Section 4.1 also briefly describes differential and double differential GPS procedures. Section 4.2 derives the linearized GPS measurement residuals. Section 4.3 reviews
the integrated GPS-INS observability results for some maneuvering scenarios described in Appendix A.4. Section 4.4 defines the GPS Dilution-Of-Precision (DOP) and describes the DOP achievable using various GPS aiding measurements. Section 4.5 describes some real world GPS-INS localization results.

Chapter 5 describes a tightly integrated CDGPS-vision-INS. The main focus is on observability of error states of the integrated approach as compared to CDGPS-INS and vision-INS. Section 5.3 describes the augmented error state dynamic equations. Section 5.4 describes the ideal perspective projection model and the vision measurement residual equations. Section 5.5 describes the observability analysis for some maneuvering scenarios described in Appendix A.4.

Stationary updates are useful in containing errors in velocity, gyroscope biases and some linear combination of accelerometer biases and attitude. Detection of the stationary condition is a challenge. In Chapter 6, we review existing stationary detection methods and propose a new frequency domain approach, using only IMU data, with specifications and analysis for land vehicles. Section 6.3 describes the stationary measurement model for the zero velocity and zero angular rate updates. Section 6.4.1 defines the stationary detection problem. Section 6.4.2 describes the theoretical analysis of the frequency domain approach. Section 6.4.3 describes probabilistic methods of selecting thresholds appearing in Section 6.4.2. Section 6.4.4 describes on-vehicle test results. Section 6.5 presents analytic and numeric evaluations of observability of INS error states with stationary updates. Section 6.5.1 proves various propositions for an INS aided by stationary updates for some maneuvering scenarios described in Appendix A.4. The null space of the continuous time Observability Gramian is evaluated for various motion scenarios typically occurring in a land vehicle. Sec-
tion 6.5.2 presents results on observability from a state estimation point of view. Section 6.6 demonstrates improvements in localization performance in an INS with stationary detection and aiding thus demonstrating the utility of stationary updates in inertial systems.

Chapter 7 introduces and motivates the Near Real Time (NRT) estimation strategy. Initialization of states of a nonlinear system is important, especially when utilizing any linearization based estimation algorithm like the EKF. Chapter 7 proposes a method to identify parts of the state of a nonlinear system, that when in error, causes significant deviation from the linearized model. As an example, a NRT based attitude initialization strategy for a 2D GPS-INS system is presented.

Chapter 8 concludes the dissertation, describes future work and provides a list of articles published during the course of this research.

1.3 Main contributions

The main contributions of this dissertation to the body of literature are enumerated below:

1. As outlined in Section 4.3, the observability properties of a GPS-INS are well known. It is interesting to know how the unobservable subspaces change when a vision sensor is integrated into GPS-INS. This question is addressed in Chapter 5.

2. Detection of stationary condition of the rover is a challenge worthy of careful consideration. Several detection algorithms have been proposed in literature with varying degrees of success. In Chapter 6, we formulate a frequency domain based stationary detection approach. Advantages of the proposed method over other methods are
discussed in Section 6.7.

3. Rigorous observability analysis is required to understand the theoretical limits of an aiding technique. This is unavailable for a stationary updates aided INS. Section 6.5 proves several propositions on this subject.

4. NRT estimation strategy is defined and motivated in Sections 7.1 and 7.2. This strategy improves robustness of INS to errors in aiding sensor estimates. Sections 7.3 contains an exposition of the application of NRT processing for the well known initialization problem in INS.
Chapter 2

A nonlinear estimation problem

Consider the evolution of state $x$ in the state space $\mathbb{R}^N$ according to some nonlinear vector of real analytic mappings $f : \mathbb{R}^{N+M} \rightarrow \mathbb{R}^N$ defined by

$$\dot{x} = f(x, u)$$  \hspace{1cm} (2.1)

where $u \in \mathbb{R}^M$ denotes the inputs to the system. If system dynamics $f$ and initial conditions are perfectly known, the state $x$ can be uniquely determined by integrating (2.1). But in real world scenarios, the initial condition $x(0)$ are unknown and we are only equipped with non-ideal sensors that make noisy measurements of $u$ denoted by $\tilde{u}$. An estimate of the state, $\hat{x}$, is computed by integrating $\dot{\hat{x}} = f(\hat{x}, \hat{u})$ with initial conditions $\hat{x}(0)$. But note that $\hat{x}$ is a stochastic process as the estimated input $\hat{u}$ and initial conditions $\hat{x}(0)$ are random in nature.

Assume that we are equipped with sensors that measure some function of the state $x$ modeled as

$$\tilde{y} = h(x) + n$$  \hspace{1cm} (2.2)
where $h : \mathbb{R}^N \rightarrow \mathbb{R}^P$ is a vector of real analytic functions and $n \in \mathbb{R}^P$ is additive noise. Many real world problems can be described by the generic equations in (2.1) and (2.2). The following section reviews some practical estimation approaches to estimate $x(t)$ given $(\tilde{u}(\tau), \tilde{y}(\tau))$ for all $\tau \in [0, t]$.

### 2.1 Literature review

Section 2.1.1 briefly reviews the Extended Kalman Filter (EKF), which estimates the error in $\hat{x}$, denoted by $\delta x$, which is optimal in the Meas Square Error sense under some assumptions. Section 2.1.2 reviews some popular nonlinear estimation methods.

#### 2.1.1 Linearization based methods

Linearization based estimation methods like the EKF seek to estimate the error accrued in integration of $\dot{x} = f(\hat{x}, \hat{u})$. The error in the estimate $\hat{x}(t)$ is defined as $\delta x(t) = x(t) - \hat{x}(t)$, and it evolves in time according to

$$\delta \dot{x}(t) = f(x(t), u(t)) - f(\hat{x}(t), \hat{u}(t)). \quad (2.3)$$

Linearization based algorithms approximate (2.3) by truncating the Taylor series expansion of $f$ around $(\hat{x}, \hat{u})$ to first order to derive an approximate error state evolution model as

$$\delta \dot{x}(t) = A(t)\delta x(t) + G(t)\delta u(t) \quad (2.4)$$

where $A(t), G(t)$ are time-varying matrices obtained by computing the Jacobian of $f$ with respect to $x$ and $u$ respectively. Similarly the error in the expected measurement is modeled to first order as

$$\delta y(t) = \tilde{y}(t) - \hat{y}(t) = H(t)\delta x(t) + n \quad (2.5)$$
where $\hat{y}(t) = h(\hat{x})$ and $H$ is the Jacobian of $h$ with respect to $x$.

Given (2.4) and (2.5), the state and measurement update equations for an EKF is well known [43]. The EKF estimator is optimal if linearization errors are small and sensor measurement noise in $\hat{u}$ and $\hat{y}$ are white and Gaussian (e.g. see Ch. 7 in [52]). Though the EKF algorithm is fast and a computationally inexpensive approach (esp. when using scalar sequential updates), if linearization errors in (2.4) are not negligible then the behavior of an EKF is unpredictable and linear propagation of error state covariance does not necessarily reflect the actual error in the state.

2.1.2 Nonlinear Sampling based methods

Alternative estimation paradigms like the Particle filter[4] do not require the system to be linear. Particle filters use Monte Carlo methods to approximate the a-posteriori Bayesian density by sampling it at random locations. Each sample of the a-posteriori density is assigned a weight based on how accurately it represents the state of the system. The system when all but a few particles have negligible weights (e.g. due to insufficiency of number of particles) is known to be degenerate. Large initial uncertainties can cause severe degeneracy of particles in a short time. For example, if the measurement noise is assumed to be Gaussian then the particles are penalized exponentially according to the magnitude of their measurement residuals, resulting in only a few particles with significant weight. A brute force method to mitigate degeneracy is to use a large number of particles but the disadvantage of this approach is that it causes a huge computational burden. Estimation methods like the Unscented Kalman Filter [40] seek to mitigate this issue of computational burden by deterministically choosing candidates or sigma points and propagating them
through the nonlinear function $f$. By carefully choosing sigma points, this algorithm is in effect generalizing the EKF by approximating the nonlinear function $f$ up to higher orders of estimation error [83]. Here again, large uncertainties in state initializations would require an increased number of sigma points to be able to model the higher order moments of the a-posteriori probability density of state.

### 2.2 Bayesian solutions

Under the Bayesian estimation philosophy (Ch. 10 in [45]), a MAP estimator of $x$ is defined as

$$
\hat{x} = \arg \max_{x \in \mathbb{R}^N} (p\{x | \tilde{y}, \tilde{u}\}) = \arg \max_{x \in \mathbb{R}^N} (p\{\tilde{y} | x, \tilde{u}\} p\{x, \tilde{u}\})
$$

(2.6)

where $p\{x | \tilde{y}, \tilde{u}\}$ denotes the probability density of state $x$ conditioned on the measurements $\tilde{y}$ and inputs $\tilde{u}$. Computation of the joint probability density $p\{\tilde{y} | x, \tilde{u}\}$ is tractable as the random noise appears additively in the model depicted in (2.2), however computing the stochastic properties of $x$, characterized by the joint density $p\{x, \tilde{u}\}$, is non-trivial due to the nonlinear nature of $f$.

Let the measurement of system inputs $u$ be modeled as

$$
\tilde{u} = u + b + \omega
$$

(2.7)

where $b$ and $\omega$ denote additive bias and random noise respectively. Sensor biases are nuisance parameters that should be estimated for improved system performance. Augmenting system state with the bias states $b$, we define the augmented state as $x^\top = \begin{bmatrix} x^\top & b^\top \end{bmatrix} \in \mathbb{R}^{n+b}$.
\[ \mathbb{R}^{N+M}. \text{We can rewrite (2.6) in terms of the augmented state vector } \bar{x} \text{ as} \]

\[ \hat{x} = \arg \max_{\bar{x} \in \mathbb{R}^{N+M}} (p(\bar{x}|\tilde{y}, \tilde{u})) = \arg \max_{\bar{x} \in \mathbb{R}^{N+M}} (p(\tilde{y}|\bar{x}, \tilde{u})p(\bar{x}, \tilde{u})). \quad (2.8) \]

### 2.2.1 Generalized Bayesian solution

In order to solve (2.8) we need to compute \( p(\tilde{y}|\bar{x}, \tilde{u}) \) and \( p(\bar{x}, \tilde{u}) \). Since \( n \) appears additively in (2.2), it is straightforward to compute

\[ p(\tilde{y}|\bar{x}, \tilde{u}) = p(\tilde{y}|\bar{x}) = p_n \{ \tilde{y} - h(x) \} \quad (2.9) \]

where \( p_n \) denotes the probability density of additive measurement noise. The joint density function of the state \( p(\bar{x}, \tilde{u}) \) can be derived as

\[ p(\bar{x}, \tilde{u}) = p(\bar{x}|\tilde{u}, \bar{x}(0))p(\tilde{u}|\bar{x}(0))p(\bar{x}(0)) \]

where \( p(\bar{x}(0)) \) is the \( a\text{-priori} \) density of the state. Assume that the biases \( b \) evolve as a random walk process according to

\[ \dot{b} = \omega_b \quad (2.10) \]

where \( \omega_b \) is random noise. In concept, the density \( p(\bar{x}|\tilde{u}, \bar{x}(0)) \) could be derived as

\[ p(\bar{x}|\tilde{u}, \bar{x}(0)) = \frac{\partial^2}{\partial \kappa_1 \partial \kappa_2} P \{ \int_0^t f(x, \hat{u})d\tau \leq \kappa_1 - x(0), b(\tau) \leq \kappa_2 \} \quad (2.11) \]

where \( \hat{u} = \bar{u} - b \) and

\[ b(t) = b(0) + \int_0^t \omega_b. \quad (2.12) \]

Unless \( f \) is a simple function, deriving (2.11) is not trivial. Further \( p(\bar{x}|\tilde{u}(0)) \) is non-stationary as it depends on system inputs \( u \). The following subsection discusses the linearization based approximations to simplify the computation of (2.8).
2.2.2 Standard linearized MAP estimation

Using \( \hat{x}(0) \) and \( \hat{u} \), an estimate of \( x \) is computed as

\[
\hat{x} = \hat{x}(0) + \int_{0}^{t} f(\hat{x}, \hat{u}) \, d\tau.
\]

(2.13)

The error in the estimate computed in (2.13) is

\[
\delta x = \delta x(0) + \int_{0}^{t} f(x, u) - f(\hat{x}, \hat{u}) \, d\tau.
\]

(2.14)

Since \( f \) is a vector of real analytic functions, we can express it as a Taylor series in the neighborhood of \((\hat{x}, \hat{u})\) as

\[
f(x, u) = f(\hat{x}, \hat{u}) + A\delta x + G(\delta b + \omega) + T_x
\]

(2.15)

where \( G = -\frac{\partial f}{\partial u} \), \( A = \frac{\partial f}{\partial x} \) evaluated at \((\hat{x}, \hat{u})\) and \( T_x \in \mathbb{R}^N \) denotes the higher order terms of the expansion. Substituting (2.15) into (2.14) yields

\[
\delta x = \delta x(0) + \int_{0}^{t} A\delta x + G(\delta b + \omega) + T_x \, d\tau.
\]

(2.16)

Using (2.10), the optimal estimate of the bias is

\[
\hat{b}(t) = \hat{b}(0).
\]

(2.17)

Using (2.17) and (2.12), the error in the optimal estimate is derived as

\[
\delta b(t) = \delta b(0) + \int_{0}^{t} \omega_b.
\]

(2.18)

Note that (2.16) and (2.18) are the unique solutions to the system that evolves according to

\[
\delta \dot{x} = A\delta x + G\omega + T_x.
\]

(2.19)
where \( \bar{A}^\top = \begin{bmatrix} A^\top & G^\top & 0 & 0 \end{bmatrix}, \bar{G}^\top = \begin{bmatrix} G^\top & 0 & I \end{bmatrix}, \bar{\omega}^\top = \begin{bmatrix} \omega^\top & \omega_b^\top \end{bmatrix} \) and \( \bar{T}_x^\top = \begin{bmatrix} T_x^\top & 0 \end{bmatrix} \in \mathbb{R}^{N+M} \). Similarly the error in expected measurement can be computed as

\[
\delta y = \tilde{y} - \hat{y} = \bar{H} \delta \bar{x} + n + T_y
\]  

(2.20)

where \( \bar{H} = \left[ \frac{\partial h}{\partial x} \ 0 \right] \) and \( T_y \in \mathbb{R}^P \) denotes linearization errors.

The main assumption in this section is that the error in \( (\hat{x}, \hat{u}) \) is sufficiently small so that \( T_x \) and \( T_y \) can be ignored. Under this assumption, (2.16) reduces to

\[
\delta x = \delta x_0 + \int_0^t A \delta x + G (\delta b + \omega) \ d\tau. \tag{2.21}
\]

Since \( \hat{x} \) is known we conclude

\[
p\{x|\tilde{y}, \tilde{u}\} = p_n\{\delta y|\delta x, \omega\}p(\delta x, \omega) = p_n\{\delta y|\delta \hat{x}\}p(\delta \hat{x})p(\omega). \tag{2.22}
\]

From (2.22) we conclude that, if linearization errors are small,

\[
\max_{x \in \mathbb{R}^{N+M}} p\{x|\tilde{y}, \tilde{u}\} = \max_{\delta x \in \mathbb{R}^{N+M}} p_n\{\delta y|\delta x\}p(\delta x). \tag{2.23}
\]

Denoting \( \delta \hat{x} \) as

\[
\delta \hat{x} = \arg \max_{\delta x \in \mathbb{R}^{N+M}} p_n(\delta y|\delta x)p(\delta x) \tag{2.24}
\]

the result in (2.23) can be formally stated as the following proposition.

**Proposition 1** When linearization errors are small, the estimate \( \delta \hat{x} \) satisfies (2.24) if and only if \( \hat{x} \) satisfies (2.8).

**Proof.** (\( \Rightarrow \)) Given that \( \hat{x} = x + \delta \hat{x} \) and \( \hat{x} \) satisfies (2.8), then \( p(\hat{x}|\tilde{y}, \tilde{u}) \geq p(\hat{x}^*|\tilde{y}, \tilde{u}) \) for all other estimators \( \hat{x}^* \), then from (2.22) we conclude \( p_n(\delta y|\delta \hat{x})p(\delta \hat{x}) \geq p_n(\delta y|\delta \hat{x}^*)p(\delta \hat{x}^*) \). Hence \( \delta \hat{x} \) satisfies (2.24). The converse can be proved likewise. \( \blacksquare \)
Chapter 3

Inertial Navigation Systems (INS)

3.1 Introduction

Dead reckoning using inertial sensors is a popular navigation technique. The IMU consists of 6 sensors comprising of 3 accelerometers and 3 gyroscopes. The sensitive axis of the suite of accelerometers (and gyroscopes) are mutually orthogonal. The sensitive axis of the accelerometers and gyroscopes are assumed to be coincident and referred to as the IMU sensor frame. In this dissertation, we choose the body frame $b$ of the rover to be the sensor frame of the IMU. Given the initial position and attitude, the INS is able to compute full 6DOF navigation information of the rover by integrating the kinematic equations.

Factors contributing to the prevalent use of INS for localization are:

1. The IMU is immune to external jamming.

2. There is a theoretical possibility that the INS requires no external reference to provide continuous navigation information after initialization.
3. The advent of MEMS sensors in the 1990s. MEMS based sensors have several favorable properties. The specifications of Analog Devices ADIS16334 MEMS IMU[3] are outlined below for the purpose of discussion,

(a) Low cost ($336), small footprint (22mm ×33mm ×11mm).

(b) Well understood and quantifiable error models [23], [27], [39].

(c) High frequency updates (> 300Hz).

(d) High dynamic range (±300deg/sec, ±50 m/s/s).

(e) High Bandwidth (330Hz).

4. Consumer driven demand for applications such as

(a) Navigation: routing, vehicle guidance & control [22], [89] etc.

(b) High accuracy mobile mapping [77], [73].

(c) Life-critical systems: Vehicle collision avoidances, automotive air bags etc.

(d) Hand-held devices: Cellphones, Cameras, Electronics readers etc.

In spite of these advantages, the error accrued by an INS performing dead reckoning with IMU information grows proportionally to at least the second power of integration time.

This is predominantly due to presence of sensor biases in IMU measurements. Figs. 3.1 and 3.2 shows 100 realizations of position and velocity error random processes obtained by integrating a 1D accelerometer during stationarity. It can be seen that the growth of error in velocity (position) is linear (quadratic). The thick black line indicates the sample ±1σ sample standard deviation. Section 3.2 discusses this issue in detail motivating the need for independent aiding sensors.
Figure 3.1: Unbounded growth of INS position error without aiding.
Figure 3.2: Unbounded growth of INS velocity error without aiding.
3.2 IMU Sensor Modeling

The IMU measurements $\mathbf{b}\mathbf{u} \in \mathbb{R}^6$ in the body frame can be modeled in continuous time as \[3.1\]

\[
\mathbf{b}\mathbf{u}(t) = \mathbf{b}\mathbf{u}(t) + \mathbf{b}\mathbf{b}(t) + \omega(t)
\]

where \[3.1\] \[3.1\] denotes ideal IMU measurement comprising of the specific force $\mathbf{b}\mathbf{f} \in \mathbb{R}^3$ and the angular rotation rate $\mathbf{b}\omega_{ib} \in \mathbb{R}^3$. The symbols $\mathbf{b}\mathbf{b}^\top = \begin{bmatrix} \mathbf{b}\mathbf{b}^\top_g & \mathbf{b}\mathbf{b}^\top_a \end{bmatrix}$ and $\omega^\top = \begin{bmatrix} \omega_g^\top & \omega_a^\top \end{bmatrix}$ denote the sensor bias and additive random noise respectively. The specific force vector $\mathbf{b}\mathbf{f}$ is derived as

\[
\mathbf{b}\mathbf{f} = \mathbf{b}\mathbf{a}_{ib} + \mathbf{b}\mathbf{R}^n\mathbf{g}
\]

where $\mathbf{n}\mathbf{g}$ is the gravity vector in the navigation frame. Throughout this dissertation we assume that $\mathbf{n}\mathbf{g}^\top = \begin{bmatrix} 0 & 0 & -g \end{bmatrix}$ (See Section A.5 for definition of $\mathbf{g}$). The additive sensor noise is assumed to be white and Gaussian with $\omega \sim \mathcal{N}(0, \mathbf{Q}_\omega)$.  

3.2.1 Sensor bias model

The IMU sensor biases are slowly changing with accelerometer (gyroscope) bias stability on the order of $1000 \times 10^{-6}$ m/s/s/Hr (0.0072 deg/Hr) \[3\]. In this dissertation, we assume a random walk model for the IMU sensor biases,

\[
\mathbf{b}\mathbf{b} = \omega_b
\]

where $\omega_b^\top = \begin{bmatrix} \omega_b^\top_g & \omega_b^\top_a \end{bmatrix}$ is white Gaussian noise with $\omega_b \sim \mathcal{N}(0, \mathbf{Q}_{\omega_b})$.  

20
3.3 INS kinematic equations

Let the navigation state be defined as $x^\top = \begin{bmatrix} \nabla p_{nb}^\top & \nabla v_{nb}^\top & \nabla q^\top & \nabla b_a^\top & \nabla b_g^\top \end{bmatrix} \in \mathbb{R}^6 \times \mathbb{S}^3$. The kinematic equations that describe the evolution of the navigation state with system inputs $u$ are (See Section 11.2.3 in [27])

$$
\begin{align*}
\nabla \dot p_{nb} &= \nabla v_{nb} \\
\nabla \dot v_{nb} &= \nabla R b f - \nabla g - 2 \left[ \nabla \omega_{ie} \times \right] \nabla v_{nb} \\
\nabla \dot R &= \nabla R \left( [b \omega_{ib} \times] - [b \omega_{ie} \times] \right). 
\end{align*}
$$

The eqns. (3.4 – 3.6) has the form of a generic nonlinear system described in (2.1). We augment the biases to the state to define the augmented state $\bar{x}$ as

$$
\bar{x}^\top = \begin{bmatrix} \nabla p_{nb}^\top & \nabla v_{nb}^\top & \nabla q^\top & \nabla b_a^\top & \nabla b_g^\top \end{bmatrix} \in \mathbb{R}^6 \times \mathbb{S}^3 \times \mathbb{R}^6.
$$

3.3.1 Linearized error state model

Using (3.7) we derive the error state vector $\delta \bar{x} \in \mathbb{R}^{15}$ as

$$
\delta \bar{x}^\top = \begin{bmatrix} \nabla \delta p_{nb}^\top & \nabla \delta v_{nb}^\top & \nabla \delta q^\top & \nabla \delta b_a^\top & \nabla \delta b_g^\top \end{bmatrix}.
$$

Using (3.3 – 3.6) we derive the linearized error state dynamic equations for $\delta \bar{x}$ are derived as

$$
\begin{align*}
\nabla \delta \dot p_{nb} &= \nabla \delta \dot v_{nb} \\
\nabla \delta \dot v_{nb} &= - [\nabla f \times] \nabla \rho - \nabla g \delta b_a - \nabla R \omega_a \\
\nabla \dot \rho &= \nabla R b \delta b_g + \nabla \omega_g \\
\nabla \delta \dot b_a &= \omega_{ba} \\
\nabla \delta \dot b_g &= \omega_{bg}
\end{align*}
$$
See Section C.1 for a detailed derivation of (3.9 − 3.13) from (3.3 − 3.6). Eqns. (3.9 − 3.13) can be written in matrix form as

\[ \delta \dot{X} = A(t)\delta \dot{x} + G(t)w \]  

where \( w^\top = [\omega^\top \quad \omega_b^\top] \in \mathbb{R}^{12} \) and \( A \in \mathbb{R}^{15 \times 15}, G \in \mathbb{R}^{15 \times 12} \) are suitably defined. Note that (3.14) has the same form as (2.19) when the linearization errors \( \bar{T}_x \) are ignored.

3.3.2 State transition matrix

Ignoring the IMU sensor noise, the linearized error state dynamic equation in (3.14) is reduced to

\[ \delta \dot{x}(t) = A(t)\delta \dot{x}(t). \]  

The set of differential equations given by (3.15) can be solved in closed form.

Let \( t_0 \geq 0 \) be given and \( \Phi(t, t_0) \) be the state transition matrix corresponding to the dynamic model in (3.15) such that for all \( t > t_0 \), \( \delta \bar{x}(t) = \Phi(t, t_0)\delta \bar{x}(t_0) \). For all \( t \geq t_0 \), the state transition matrix satisfies

\[ \dot{\Phi}(t, t_0) = A(t)\Phi(t, t_0). \]  

Since \( A \) is an upper triangular matrix, it is possible to solve (3.16) in closed form as

\[
\Phi(t, t_0) = \begin{bmatrix}
I & (t - t_0)I & P_t & T_t & -Q_t \\
0 & I & -S_t & M_t & -R_t \\
0 & 0 & I & R_t & 0 \\
0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I
\end{bmatrix}
\]  

(3.17)
where

\[ \mathcal{R}_t = \int_{t_0}^{t} R d\tau \quad \quad \mathcal{S}_t = \int_{t_0}^{t} [n f(\tau) \times] d\tau \]
\[ \mathcal{P}_t = -\int_{t_0}^{t} \mathcal{S}_s ds \quad \mathcal{Q}_t = \int_{t_0}^{t} \mathcal{R}_s ds \]
\[ \mathcal{M}_t = -\int_{t_0}^{t} [n f(s) \times] \mathcal{R}_s ds \quad \mathcal{T}_t = \int_{t_0}^{t} \mathcal{M}_r dr \]

See Section C.2 for a detailed derivation of (3.17) from (3.15).
Chapter 4

GPS aided INS

In this chapter we describe a tightly integrated Carrier phase Differential GPS - INS. Section 4.1 describes the GPS pseudorange, carrier phase and Doppler observables. Section 4.1 also describes differential and double differential GPS techniques with their corresponding error budgets. Section 4.2 describes the derivation of the GPS measurement residual model. Section 4.3 reviews observability analysis of INS error states using GPS measurements. Section 4.4 defines and describes GPS Dilution-Of-Precision achievable when aiding with various GPS observables. Section 4.5 provides some real world GPS localization results.

4.1 GPS measurement model

In this section, we will discuss the GPS measurement model for various observables and their corresponding error budgets. GPS satellites transmit information using carrier signals at two frequencies, they are referred to as the L1 signal (1575.42Mhz) and the L2 signal (1227.6Mhz). Each satellite uses a unique C/A PRN signal that is almost orthogonal
to signals from other GPS satellites. Hence all GPS satellites are able to transmit at the same carrier frequencies.

There are three kinds of measurement observables possible at each of these frequencies, they are:

1. **Pseudorange observable**: The pseudorange observable measures the time of travel of the signal from the satellite to the receiver by correlating the C/A code of the satellite with a locally generated one.

2. **Carrier phase observable**: Once the receiver acquires phase lock on the satellite signal, it is able to track incremental phase of the incoming signal. This incremental phase is called the carrier phase observable. Though the incremental phase is recorded, the number of integral wavelengths between the receiver and the satellite is unknown. This is unknown number is known as the carrier phase integer ambiguity.

3. **Doppler observable**: Once the receiver has acquired lock on the satellite signal, it is able to compute the change of the pseudorange observable over a period of time (≤ 1.0s). This change in pseudorange is defined as the Doppler observable.

The measurement model for each of these observables are described in Sections 4.1.1, 4.1.2 and 4.1.3. For more detailed information on GPS signals see Ch.8 in [27], [44], [58].

### 4.1.1 Pseudorange observable

The code pseudorange measurement of the $i^{th}$ satellite at the $j^{th}$ frequency ($j \in \{L_1, L_2\}$) by receiver $r$ located at $\mathbf{p}_{eb}$ is modeled as

\[
\tilde{\rho}_{r,j} = \rho \langle \mathbf{p}_r, \mathbf{p}_{es_i} \rangle + C \delta t^i + E^i + \frac{f_2}{f_1} I_r^i + T_r^i + M_{\rho,1}^i + \nu_{\rho,1}^i \tag{4.1}
\]
where $\mathbf{p}_r^\top = \begin{bmatrix} e_{\mathbf{p}_{eb}} & \Delta t_r \end{bmatrix}$ and the symbol $\rho(p_r, \hat{e}_{\mathbf{p}_{es_i}})$ is defined as the pseudorange between the receiver $r$ and satellite $i$. It is computed as

$$\rho(p_r, \hat{e}_{\mathbf{p}_{es_i}}) = ||e_{\mathbf{p}_{eb}} - \hat{e}_{\mathbf{p}_{es_i}}|| + C\Delta t_r.$$ 

Other terms in (4.1) are defined in Table 4.1.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Units</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{\mathbf{p}_{eb}} \in \mathbb{R}^3$</td>
<td>m</td>
<td>Receiver position in ECEF.</td>
</tr>
<tr>
<td>$\hat{e}<em>{\mathbf{p}</em>{es_i}} \in \mathbb{R}^3$</td>
<td>m</td>
<td>Estimate of the position of the $i^{th}$ satellite from the Ephemeris data.</td>
</tr>
<tr>
<td>$\Delta t_r$</td>
<td>s</td>
<td>Receiver clock bias.</td>
</tr>
<tr>
<td>$\delta t^i$</td>
<td>s</td>
<td>Residual satellite clock bias error after modeling drift.</td>
</tr>
<tr>
<td>$E^i$</td>
<td>m</td>
<td>Error pseudorange resulting from errors in estimated satellite position from Ephemeris data.</td>
</tr>
<tr>
<td>$I^i_r$</td>
<td>m</td>
<td>Dispersive atmospheric effects (Ionospheric delay).</td>
</tr>
<tr>
<td>$T^i_r$</td>
<td>m</td>
<td>Non-dispersive atmospheric effects (Tropospheric delay).</td>
</tr>
<tr>
<td>$M^i_{\rho,j}$</td>
<td>m</td>
<td>Multipath error between the receiver and $i^{th}$ satellite.</td>
</tr>
<tr>
<td>$\nu^i_{\rho,j}$</td>
<td>m</td>
<td>Random receiver noise in code measurement.</td>
</tr>
</tbody>
</table>

Table 4.1: Components of GPS pseudorange measurement

### 4.1.1.1 Pseudorange error budgeting

The typical satellite clock error residual has a standard deviation of about 2m. The typical error in estimation of satellite position using the ephemeris data has a standard deviation of about 3m. At the zenith the dispersive atmospheric delay error standard deviation can be about 1–3m at night at mid-latitudes while it can be about 5–15m
at mid afternoon [58]. The non-dispersive atmospheric delay can be decomposed into the dry tropospheric delay and the wet tropospheric delay. The dry tropospheric delay has a standard deviation of about $2.3 - 2.6\text{m}$ at the sea level. The standard deviation of the wet tropospheric delay is about $0 - 80\text{cm}$. The multipath error has a standard deviation of $0.1 - 5.0\text{m}$ and the random noise has a standard deviation of $0.1 - 0.7\text{m}$. The net User Range Error (URE) by using just the L1 signal is about $11\text{m}$. Detailed information can be found in Ch. 8 in [27].

4.1.2 Carrier phase observable

The carrier phase measurement of the $i^{th}$ satellite at the $j^{th}$ frequency ($j \in \{L_1, L_2\}$) by receiver $r$ located at $e_{\text{ref}}$ is modeled as

$$\lambda_j \hat{\phi}_{i,r,j} = \rho(p_r, e_{\text{ref}}) + E^i + C \delta t^i - \frac{f_2}{f_1} I^i + T^i + M^i_{\phi,1} + \nu^i_{\phi,1} + N^i_j \lambda_j. \quad (4.2)$$

Symbols that appear both in (4.2) and (4.1) have the same definition. Additional symbols appearing in (4.2) are defined in Table 4.2

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Units</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M^i_{\phi,j}$</td>
<td>m</td>
<td>Carrier phase signal multipath error.</td>
</tr>
<tr>
<td>$\nu^i_{\phi,j}$</td>
<td>m</td>
<td>Random measurement noise in carrier phase measurement.</td>
</tr>
<tr>
<td>$N^i_j$</td>
<td>Cycles</td>
<td>Integer ambiguity.</td>
</tr>
</tbody>
</table>

Table 4.2: Components of GPS carrier phase measurement
4.1.2.1 Carrier phase error budgeting

The random receiver noise occurring in the phase measurement equation has a standard deviation of about $0.005 - 0.01$ cycles. The typical error due to multipath in carrier phase measurements have standard deviation of $1 - 5$ cm. The statistics of tropospheric and ionospheric terms are same as that described in Section 4.1.1.1 \(^1\). Detailed information can be found in Ch. 8 in [27].

4.1.3 Doppler observable

The Doppler measurement model of the $i^{th}$ satellite at the $j^{th}$ frequency ($j \in \{L_1, L_2\}$) is derived by differentiating (4.2) as

$$\lambda_j \ddot{D}_i = \lambda_j \ddot{\phi}_j \approx g h_j (\epsilon v_{eb} - \epsilon v_{es}) + \epsilon \tag{4.3}$$

where $g h_j = \frac{h_{eb} T - e \dot{p}_{eb} T}{\| p_{eb} - e \dot{p}_{es} \|}$ denotes the unit vector from the receiver to the satellite, $\epsilon v_{eb}$ and $\epsilon v_{es}$ denote the rover and satellite velocity in the ECEF frame. It is assumed that (4.3) is corrected for satellite clock error drift using satellite clock model. Since atmospheric errors are slowly changing, their derivatives are assumed to be negligible. Errors due to multipath and receiver noise are grouped under $\epsilon$ and it is assumed that $\epsilon \sim N(0, \sigma^2_d)$ with $\sigma_d = 2$ cm.

4.1.4 Differential GPS

Errors in the code and carrier phase observables can be grouped into the following types:

---

\(^1\)The dispersive ionospheric medium affects the code and carrier phase measurements in the same magnitude but in the opposite sense because it advances the phase of the signal and delays the modulation. Thus it accounts for the difference in sign in front of the ionospheric error term.
1. Errors that are common to all receivers over a short baseline are known as common mode errors. Errors due to troposphere, ionosphere, satellite clock bias and satellite ephemeris fall into this category. The effect of common mode errors can be substantially reduced by a technique known as differential GPS.

2. Errors that are specific to the receiver/receiver location are called non-common mode errors. Errors due to receiver clock bias, multipath, receiver noise fall into this category.

Assume that a GPS receiver $B$ is setup such that the ECEF position of the phase center of it's antenna is known (say $\vec{e}_{p,B}$). This GPS receiver is henceforth referred to as the “base station”. For each satellite $i$ and frequency $j$, the base station $B$ computes the common mode errors by subtracting the computed pseudorange from the pseudorange measurements as

$$\delta \rho^i_{B,j} = \hat{\rho}^i_{B,j} - \tilde{\rho}^i_{B,j} = \|\vec{e}_{p,B} - \vec{e}_{es_j}\|$$

$$= C \Delta t_B + C \delta t^i + E^i + \frac{f_2}{f_1} I^i_r + T^i_r + M^i_{\phi,1} + \nu^i_{\rho,1}.$$

Similarly, the carrier phase differential correction is computed as

$$\delta \phi^i_{B,j} = C \Delta t_B + C \delta t^i + E^i - \frac{f_2}{f_1} I^i_r + T^i_r + M^i_{\phi,1} + \nu^i_{\phi,1} + N^i_j \lambda_j.$$

The base station broadcasts the differential correction $(\delta \rho^i_{B,j}, \delta \phi^i_{B,j})$ for each $j \in \{L_1, L_2\}$, $i = 1, \ldots, N_s$ to all rovers. The rovers in turn subtract the corrects from its measurements resulting in a more accurate pseudorange/carrier phase measurements resulting in the single differenced pseudorange/carrier phase measurements$^2$. The improved single differenced

$^2$Though carrier phase differential corrections removes atmospheric and ephemeris errors, it will not resolve integer ambiguities
pseudorange and carrier phase measurements, denoted by \((\Delta \rho_{r,b}^i, \Delta \phi_{r,b}^i)\) at rover \(r\) is derived as

\[
\Delta \rho_{r,b}^i = \tilde{\rho}_{r,j}^i - \delta \rho_{B,j}^i = \rho(p_r, e_{\text{ep}} i) + C \Delta t_B + \eta^i_{\rho}
\]

\[
\Delta \phi_{r,b}^i = \tilde{\phi}_{r,j}^i - \delta \phi_{B,j}^i = \rho(p_r, e_{\text{ep}} i) + C \Delta t_B + \lambda_j N^i_{r,B} + \eta^i_{\phi}
\]

where \(\eta^i_{\rho}\) and \(\eta^i_{\phi}\) denote residual atmospheric and ephemeris errors after differential correction. It is assumed that these errors are small over short baseline (~10 miles) but increase over longer baselines as the atmospheric errors become spatially de-correlated. Note that the single differenced pseudorange and carrier measurements are affected by the clock bias errors of both the rover and base clock. This can be eliminated by a process known as Double differencing.

### 4.1.5 Double Differential GPS

Given single differenced measurements at the rover \(r\) for satellites \(i_1\) and \(i_2\), the double differenced measurements are derived as

\[
\nabla \Delta \rho_{r,B}^{i_1,i_2} = \Delta \rho_{r,b}^{i_1} - \Delta \rho_{r,b}^{i_2} = \| e_{\text{ep}} p_{e_{s_j},i_1} - e_{\text{ep}} p_{e_{s_j},i_2} \| + \eta^i_{\rho}^{i_1,i_2}
\]

\[
\nabla \Delta \phi_{r,B}^{i_1,i_2} = \Delta \phi_{r,b}^{i_1} - \Delta \phi_{r,b}^{i_2} = \| e_{\text{ep}} p_{e_{s_j},i_1} - e_{\text{ep}} p_{e_{s_j},i_2} \| + \lambda_j N^i_{r,B} + \eta^i_{\phi}^{i_1,i_2}
\]

where \(\eta^i_{\rho}^{i_1,i_2} = \eta^i_{\rho}^{i_1} - \eta^i_{\rho}^{i_2}\) and \(\eta^i_{\phi}^{i_1,i_2} = \eta^i_{\phi}^{i_1} - \eta^i_{\phi}^{i_2}\). Note that the effects of receiver clock biases in the differential pseudorange and carrier phase measurements are eliminated from (4.4) and (4.5).
4.2 GPS measurement residuals

Linearizing (4.6) and (4.7) around \( e \hat{\mathbf{p}}_{eb} \) for each \( i = 1, \ldots, N_s \) and \( j \in \{L_1, L_2\} \) we derive

\[
\nabla \Delta \rho_{r,B}^{i_1,i_2} = \nabla \Delta \hat{\rho}_{r,B}^{i_1,i_2} + \left( g h_j^{i_1} - g h_j^{i_2} \right) e \delta \mathbf{p}_{eb} + \eta_\rho^{i_1,i_2} \tag{4.8}
\]

\[
\nabla \Delta \phi_{r,B}^{i_1,i_2} = \nabla \Delta \hat{\phi}_{r,B}^{i_1,i_2} + \left( g h_j^{i_1} - g h_j^{i_2} \right) e \delta \mathbf{p}_{eb} + \lambda_j N_{r,B}^{i_1,i_2} + \eta_\phi^{i_1,i_2} \tag{4.9}
\]

Since the navigation frame is known relative to the ECEF frame, the following relation can be derived

\[
e^{n} \mathbf{p}_{eb} = e^{n} R \left( n \mathbf{p}_{nb} - n \mathbf{p}_{ne} \right). \tag{4.10}
\]

Using (4.10) we can derive the error in \( e^{n} \hat{\mathbf{p}}_{eb} \) in terms of the error in \( n \hat{\mathbf{p}}_{nb} \) as

\[
e^{n} \delta \mathbf{p}_{eb} = e^{n} R \delta \mathbf{p}_{nb}. \tag{4.11}
\]

Substituting (4.11) into (4.8) and (4.9) we derive

\[
\nabla \Delta \rho_{r,B}^{i_1,i_2} = \nabla \Delta \hat{\rho}_{r,B}^{i_1,i_2} + \left( g h_j^{i_1} - g h_j^{i_2} \right) e^{n} R n \delta \mathbf{p}_{nb} + \eta_\rho^{i_1,i_2} \tag{4.12}
\]

\[
\nabla \Delta \phi_{r,B}^{i_1,i_2} = \nabla \Delta \hat{\phi}_{r,B}^{i_1,i_2} + \left( g h_j^{i_1} - g h_j^{i_2} \right) e^{n} R n \delta \mathbf{p}_{nb} + \lambda_j N_{r,B}^{i_1,i_2} + \eta_\phi^{i_1,i_2} \tag{4.13}
\]

Similarly, the Doppler measurement can be written in terms of \( n \mathbf{v}_{nb} \) from (4.3) as

\[
\lambda_j \tilde{D}^i = g h_j^{i,n} R n \mathbf{v}_{nb} + \epsilon. \tag{4.14}
\]

The pseudorange, carrier phase and Doppler, double differenced measurement residuals can be derived in terms of the INS error state vector \( \delta \mathbf{x} \) using (4.12), (4.13) and (4.14) as

\[
\delta \nabla \Delta \rho_{r,B}^{i_1,i_2} = g H_j^{i_1,i_2} \delta \mathbf{x} + \eta_\rho^{i_1,i_2} \tag{4.15}
\]

\[
\delta \nabla \Delta \phi_{r,B}^{i_1,i_2} = g H_j^{i_1,i_2} \delta \mathbf{x} + \lambda_j N_{r,B}^{i_1,i_2} + \eta_\phi^{i_1,i_2} \tag{4.16}
\]

\[
\lambda_j \delta \tilde{D}^{i_1} = g H_j^{i_1} \delta \mathbf{x} + \epsilon \tag{4.17}
\]

31
where
\[
\begin{align*}
\gamma H_{j_{i_1},i_2}^i &= \left[ \left( \gamma h_{j_{i_1}}^i - \gamma h_{j_{i_2}}^i \right) e_{\bar{n}} R \ 0 \ 0 \ 0 \right] \\
\nu H_{j_{i_1}}^i &= \left[ 0 \ \gamma h_{j_{i_1}}^i e_{\bar{n}} R \ 0 \ 0 \ 0 \right]
\end{align*}
\]
and the additive noise terms are assumed to be white and Gaussian such that \( \bar{\eta}_{\rho}^{i_1,i_2} \sim \mathcal{N}\left(0, \sigma_\rho^2\right), \bar{\eta}_{\phi}^{i_1,i_2} \sim \mathcal{N}\left(0, \sigma_\phi^2\right) \) and \( \epsilon \sim \mathcal{N}\left(0, \sigma_\epsilon^2\right) \) with \( \sigma_\rho = 5\text{m}, \sigma_\phi = 10^{-2}\text{m} \) and \( \sigma_D = 2 \times 10^{-2}\text{m/s} \)\(^3\). Further if integer ambiguities are not resolved then double differenced carrier phase residuals cannot be used due to the unknown \( \lambda_j n_{r,B}^{i_1,i_2} \) term occurring in (4.16). Estimation of integer ambiguities in an open area of research (See. [11], [13], [14]) and for the purpose of this dissertation we will assume that they have been resolved.

### 4.3 Observability analysis

Observability of INS error states using GPS measurements has been studied for decades and the maneuvering requirements for calibration of INS parameters (e.g., biases) using GPS measurements are well understood (e.g., See [17], [27], [30], [37], [38], [72]). This section derives observability conditions for certain common maneuvers occurring in land vehicle based INS and is included here for the sake of completeness.

Assume that there are \( N_s \) satellites in view with differential corrections available for all of them. Further assume that all integer ambiguities have been resolved, in which case, we have \( N_s - 1 \) equations of the form (4.16). Let us fix \( i_1 = 1 \), then for each \( i = 2, \ldots, N_s \)

\(^3\)When using a standard EKF implementation for estimating \( \delta \bar{x} \) using (4.15–4.17), a phase update using (4.16) should not be performed after a Doppler update given by (4.17), as the carrier phase measurement is the integral of the Doppler measurement over a second and hence the whiteness assumption of \( \bar{\eta}_{\phi}^{i_1,i_2} \) in (4.16) is invalidated.
we have:
\[ \delta \nabla \Delta \phi_{r,B}^{1,i} = g H_j^{1,i} \delta \bar{x} + \bar{\eta}_{\phi}^{1,i}, \]

(4.18)

Define the residual vector \( \delta y_g \in \mathbb{R}^{N_s-1} \) as
\[ \delta y_g^T = \begin{bmatrix} \delta \nabla \Delta \phi_{r,B}^{1,2} & \delta \nabla \Delta \phi_{r,B}^{1,3} & \ldots & \delta \nabla \Delta \phi_{r,B}^{1,N_s} \end{bmatrix}, \]

we write (4.18) in matrix form as
\[ \delta y_g = H_g \delta \bar{x} + n_g \]

(4.19)

where \( H_g^T = \begin{bmatrix} g H_j^{1,2} & \ldots & g H_j^{1,N_s} \end{bmatrix} \) and \( n_g^T = \begin{bmatrix} (\bar{\eta}_{\phi}^{1,2})^T & \ldots & (\bar{\eta}_{\phi}^{1,N_s})^T \end{bmatrix} \). It can be shown that \( \text{rank}(H_g) = 3 \), if \( N_s \geq 4 \) and if the vectors from the rover to the satellites are non-coplanar. Hence if this condition on the satellite geometry is satisfied and if linearization errors are small, then errors in \( \hat{b} \hat{p}_{nb} \) are observable. For the purpose of observability analysis, we can equivalently write (4.19) as
\[ \delta y_{gs} = H_{gs} \delta \bar{x} \]

(4.20)

where \( \delta y_{gs} = n \delta p_{nb}, H_{gs} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \end{bmatrix} \) and the measurement noise \( n_g \) has been ignored as it is irrelevant at this time.

Assume that \( N_s \geq 4 \) and the satellite geometry is such that the measurement residual equation can be reduced to (4.20) for all \( \tau \in [0, T] \). Further assume that GPS measurements of the form (4.20) are available at regular discrete time instants \( 0 = t_0 < t_1 < \cdots < t_M = T \) such that \( t_i - t_{i-1} = \Delta T \in \mathbb{R}^+ \), for \( 1 \leq i \leq M \). Then the discrete time observability gramian
defined in (B.8) can be reduced to

\[
O_g \simeq \begin{bmatrix}
H_{g^*} & 0 & 0 & 0 & 0 \\
0 & \Delta_T H_{g^*} & 0 & 0 & 0 \\
0 & 0 & -[n f(t_0) \times] \Delta^2_T & 0 & n_b R(t_0) \Delta^2_T \\
0 & 0 & \zeta^1 & \zeta^2 & \zeta^3 \\
0 & 0 & \zeta^1_{M-2} & \zeta^2_{M-2} & \zeta^3_{M-2}
\end{bmatrix}
\]  \quad (4.21)

where for each 1 \leq k \leq M - 2 we derive

\[
\begin{align*}
\zeta^1_k &= -[n f(t_k) \times] \Delta^2_T \\
\zeta^2_k &= -\sum_{i=1}^{k} [n f(t_k) \times] n_b R(t_i-1) \Delta^3_T \\
\zeta^3_k &= n_b R(t_k) \Delta^2_T.
\end{align*}
\]

Error in position estimates are observable at time \( t_0 \) and error in velocity estimates are observable time \( t_1 \) irrespective of rover maneuvers. Hence for \( M > 1 \) we can assume \( n \delta p_{nb} = n \delta v_{nb} = 0 \). Hence we analyze the reduced observability matrix \( O^* \) given by

\[
O^*_g = \begin{bmatrix}
-[n f(t_0) \times] \Delta^2_T & 0 & n_b R(t_0) \Delta^2_T \\
\zeta^1 & \zeta^2 & \zeta^3 \\
\zeta^1_{M-2} & \zeta^2_{M-2} & \zeta^3_{M-2}
\end{bmatrix}
\]  \quad (4.22)

The following propositions can be proved for some of the maneuvers described in Appendix A.4. For proof please refer to Appendix D.

**Proposition 2** If the rover is undertaking Maneuver 1 (See A.4.1) then \( \text{rank}(O^*_g) = 5 \)
and the unobservable subspace is spanned by
\[
\begin{bmatrix}
0 & u \\
\frac{b}{n} R(t_0)^n f(t_0) & 0 \\
0 & \frac{b}{n} R(t_0) [n f(t_0) \times] u
\end{bmatrix}
\]
where \( u = \{e_1, e_2, e_3\} \) with \( e_i \) being the \( i \)th standard basis for \( \mathbb{R}^3 \).

**Proposition 3** If the rover is undertaking Maneuver 2.a. (See A.4.2.1) then \( \text{rank}(O_g^*) = 7 \) and the unobservable subspace is spanned by
\[
\begin{bmatrix}
0 & n_d \\
\frac{b}{n} R(t_0)^n f(t_1) & 0 \\
0 & \frac{b}{n} R(t_0) [n f(t_0) \times] n_d
\end{bmatrix}
\]

**Proposition 4** If the rover is undertaking Maneuver 3 (See A.4.3) then \( \text{rank}(O_g^*) = 9 \).

Hence we have full state observability.

### 4.4 Dilution-Of-Precision (DOP)

The Maximum Likelihood Estimator (MLE) of error in position using (4.19) is given by
\[
n\delta \hat{p}_{nb} = \left( \bar{H}_g^\top \bar{H}_g \right)^{-1} \bar{H}_g^\top \delta y_g \quad (4.23)
\]
where \( \bar{H}_g^\top = \left[ \frac{n}{e} R\left( g h_j^1 - g h_j^2 \right)^\top \ldots \frac{n}{e} R\left( g h_j^1 - g h_j^{N_s} \right)^\top \right] \). Assuming \( n_g \sim N\left( 0, \sigma_\phi^2 I \right) \), we derive the covariance of the MLE estimate in (4.23) as
\[
P_p = E\{n\delta \hat{p}_{nb}, n\delta \hat{p}_{nb}^\top\} = \left( \bar{H}_g^\top \bar{H}_g \right)^{-1} \sigma_\phi^2. \quad (4.24)
\]
Setting \( G = \left( \bar{H}_g^\top \bar{H}_g \right)^{-1} \) we define various DOPs as
Vertical Dilution of Precision (VDOP) = $\sqrt{G_{33}}$

Horizontal Dilution of Precision (HDOP) = $\sqrt{G_{22} + G_{33}}$

Position Dilution of Precision (PDOP) = $\sqrt{G_{11} + G_{22} + G_{33}}$

Geometric Dilution of Precision (GDOP) = $\sqrt{G_{11} + G_{22} + G_{33} + G_{44}}$

where $G_{ij}$ denotes the element on the $i^{th}$ row and $j^{th}$ column in $G$. A metric for position and clock bias estimation accuracy can be defined as $RMS_{(pos+clk)} = \sigma_0 GDOP$. The value of GDOP entirely depends on the $\bar{H}_g$. The expected positional accuracy achievable by various GPS techniques described in Table 4.3.

<table>
<thead>
<tr>
<th>GPS technology</th>
<th>PDOP(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L1 pseudorange</td>
<td>10 – 25m</td>
</tr>
<tr>
<td>Differential L1 pseudorange</td>
<td>&lt; 1m</td>
</tr>
<tr>
<td>Double differential L1 pseudorange</td>
<td>&lt; 1m</td>
</tr>
<tr>
<td>Double differential carrier phase</td>
<td>1 – 10cm</td>
</tr>
</tbody>
</table>

Table 4.3: Expected PDOP under various GPS technologies

4.5 Localization results

The INS described in Chapter 3 was implemented on a digital computer aided by GPS observables described in Section 4.2. An EKF algorithm was used as an estimator. Localization results are displayed for a trajectory on US I215N freeway. Fig. 4.1 shows the trajectory overlaid on Google maps. The “1” marker denotes the starting position and “2” marker denotes the ending position. The total time taken by the rover to execute the trajectory was 450s. The rover was initially stationary for $t \in [0, 30]s$. 

36
Figure 4.1: GPS aided INS test trajectory on I215 North.
A total of 11 satellites were visible during the test. The double differential carrier phase residuals for satellite 3 is shown in Fig. 4.2. The histogram of the residuals is shown in the right subplot of Fig. 4.2. It can be seen that carrier phase residuals are not used for times $t \in [0, 54) \cup (110, 140)$s. This is because of satellite signal occlusion (resulting in loss of carrier signal phase lock) and the time required to resolve integer ambiguities. The carrier phase differential residuals for satellites 6 and 14 are shown in Figs. 4.3 and 4.4 respectively.
Figure 4.3: Differential phase residuals for PRN 6.
Figure 4.4: Differential phase residuals for PRN 14.
The IMU sensor bias estimates are shown in Fig. 4.5. The corresponding standard deviation of errors in accelerometer estimates are shown in Fig. 4.6.

It is interesting to see that at time $t = 65$, the standard deviation of accelerometer bias estimates decreases. This is the time when the rover undergoes rotation. It is possible to show that under such a maneuver, the accelerometer biases become observable. The raw gyroscope measurements for all $t \in [60, 70] \text{s}$ are shown in Fig. 4.7.

The IMU gyroscope bias estimates and the corresponding standard deviations are
Figure 4.6: Standard deviation of errors in estimation of IMU accelerometer bias.
Figure 4.7: Raw gyroscope measurements during rotation.
shown in Figs. 4.8 and 4.9.

The standard deviations of errors velocity are shown in Fig. 4.10. It can be seen that the uncertainty in rover velocity is no more than 15cm/s at any time.

The standard deviations of errors in estimates of small angle rotation is shown in Fig. 4.11. When the rover is stationary, errors in yaw are unobservable from GPS measurements. Hence we can see that the uncertainty in attitude corresponding to yaw is constant near 10deg, for all $t \in [0, 30]s$. At $t = 30s$, yaw is initialized using rover velocity, and hence the
Figure 4.9: Standard deviation of errors in estimation of IMU gyroscope bias.
Figure 4.10: Standard deviation of errors in estimates of velocity.
Figure 4.11: Standard deviation of errors in estimates of small angle rotation.

uncertainty in $n\hat{\rho}$ decreases to about 2deg. It can be seen that the uncertainty in rover attitude is no more than 0.5deg/s at any time after $t = 30s$. Further it is known that the rover was stopped at a traffic light for all $t \in [240, 450]s$, hence uncertainty in yaw increases as yaw is not observable under stationary conditions.

The standard deviations of errors in estimates of position are shown in Fig. 4.12. It is seen that errors in position are at the meter level for $t \in [0, 54]$. This is due to the fact that we were using noisy double differential pseudorange residuals instead of carrier phase
measurements. This could also be seen by the absence of carrier phase residuals in time $t \in [0, 54]s$ in Figs. 4.2, 4.3 and 4.4. Fig. 4.13 shows the in standard deviation for all $t > 54s$. It can be seen that errors in position estimates are at the decimeter level. It can be further seen that errors in down position estimates start growing in $t \in (110, 140)$. This is also attributed to usage of noisy double differential pseudorange residuals instead of carrier phase residuals.
Figure 4.13: Standard deviation of errors in estimation of position.
Chapter 5

Observability analysis of errors in a CDGPS-vision-INS

5.1 Introduction

One of the drawbacks of CDGPS-INS is the lack of observability of INS error states during certain maneuvers (e.g., rest, uniform motion). Maneuvering requirements for calibration of INS parameters (e.g., biases) using GPS measurements are well understood (e.g., See [17], [27], [30], [37], [38], [72]). Another popular land-vehicle based navigation method is feature-based navigation. There exists a spectrum of feature-based methods ranging from very little prior information (e.g., simultaneous localization and mapping, [19], [47], [59]) to using prior maps of surveyed features (e.g., [21], [35], [79]). One of the fundamental difficulties in integrating any feature-based sensor with INS is the necessity to calibrate, with commensurate accuracy, the lever arm vector \( \mathbf{p}_{bc} \in \mathbb{R}^3 \) and rotation quaternion \( \mathbf{q} \in \mathbb{S}^3 \), referred here as frame extrinsic parameters of the feature based sensor. The effects of errors
in the extrinsic parameters on accuracy of mapping roadway features are documented in [31]. In this dissertation, the feature based sensor is the camera.

5.2 Additional notation

Let $f \in \mathbb{R}^+$ be a given constant (focal length) and $U_f = \{(x, y, f)|x, y \in \mathbb{R}\}$ be identified with the image plane in the $c$ frame. Let $q_f$ be the perspective projection of $p_c$ onto the plane $U_f$. If $A_1, \ldots, A_N$ are arbitrary matrices, $\text{bd}(A_1, \ldots, A_N)$ is defined as a block diagonal matrix with $A_1, \ldots, A_N$ along the diagonal.

5.3 Augmented state and error state vector

In addition to the navigation states and biases we are interested in estimating the extrinsic parameters $(p_{bc}, q)$, hence we define the augmented state as

$$\bar{x}^\top = \begin{bmatrix} \bar{x}^\top & b_{bc}^\top & \bar{q}^\top \end{bmatrix} \in \mathbb{R}^6 \times \mathbb{S}^3 \times \mathbb{R}^3 \times \mathbb{S}^3.$$

The corresponding error state $\delta x \in \mathbb{R}^{21}$ is derived as

$$\delta x^\top = \begin{bmatrix} \delta \bar{x}^\top & b \delta p_{bc}^\top & b \rho^\top \end{bmatrix} \quad (5.1)$$

where $\rho$ characterizes the error in $R$ as a small angle rotation, such that

$$\hat{\epsilon} = \hat{\epsilon} \left( I - \left[ \rho \times \right] \right).$$
5.3.1 Augmented state dynamic equation

Since the extrinsic parameters are essentially constant once the camera is rigidly mounted onto the body, we assume the following model

\[ b \dot{p}_{bc} = 0 \]  \hspace{1cm} (5.2)
\[ \ddot{\theta} = 0. \]  \hspace{1cm} (5.3)

Using (3.14), (5.2) and (5.3) we derive the following linearized error state model for \( \delta x \):

\[ \delta \dot{x} = \bar{A} \delta x \]  \hspace{1cm} (5.4)

where sensor noises have been ignored as they are irrelevant to observability analysis and

\[ \bar{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}. \]

5.4 Vision measurement model

We use the ideal perspective projection model (See [81]) for the camera and assume that the intrinsic parameters (i.e., focal length, pixel dimensions and image centers) are estimated previously (See [8]). Let the coordinates of the \( j^{th} \) feature be denoted in the camera frame at time \( t \) as \( ^c p_{cj}(t)^T = \begin{bmatrix} x_j(t) & y_j(t) & z_j(t) \end{bmatrix} \). The image plane measurement at time \( t \) is given by the projection

\[ ^e \tilde{q}_{fj}(t) = \frac{1}{z_j(t)} \begin{bmatrix} x_j(t) \\ y_j(t) \end{bmatrix} \]  \hspace{1cm} (5.5)

where it was assumed that the intrinsic parameters of the camera are known and measurement noise ignored.

52
5.4.1 Linearized measurement residual model

Using (5.5), the measurement residual is modeled to first order as

\[ c^p_\delta q_{f_j}(t) = j^r \gamma(t) c^p_\delta p_{c_{f_i}}(t) \]  

(5.6)

where

\[ j^r \gamma(t) = \begin{bmatrix} \frac{1}{z_j(t)} & 0 & -\frac{x_j(t)}{z_j^2(t)} \\ 0 & \frac{1}{z_j(t)} & -\frac{y_j(t)}{z_j^2(t)} \end{bmatrix}. \]

If there are \( N_0 + 1 \) world features detected at each measurement time, then for each \( 0 \leq j \leq N_0 \) we can derive the following geometric relationships:

\[ b^p_{c_{f_j}} = b^r_{i} R \left( n^p_{n_{f_j}} - n^p_{n_{b}} \right) \]  

(5.7)

\[ c^p_{c_{f_j}} = c^r_{b} R \left( b^p_{b_{f_j}} - b^p_{b_{c}} \right). \]  

(5.8)

Using (5.7) and (5.8) we derive the linearized model relating \( c^p_\delta p_{c_{f_i}} \) to the augmented error state vector as

\[ c^p_\delta p_{c_{f_i}} = -c^r_{n} R n^p_{n_{b}} - c^r_{b} R b^p_{b_{c}} + \left[ c^p_{c_{f_i}} \times \right] c^r_{b} R b^p_{b_{f_i}} + \left[ c^p_{b_{f_i}} \times \right] c^r_{n} R n^p_{n_{b}} + c^r_{n} R n^p_{n_{f_i}}. \]  

(5.9)

Throughout this chapter we assume that features are surveyed in the navigation frame, hence \( n^p_{n_{f_i}} = 0 \), for each \( 0 \leq i \leq N_0 \).

Define the measurement vector of \( N_0 + 1 \) features at time \( t \) as

\[ c^p y(t) = \begin{bmatrix} c^p q_{f_0}(t)^\top & \cdots & c^p q_{f_{N_0}}(t)^\top \end{bmatrix}^\top \in \mathbb{R}^{2N_0}. \]  

(5.10)

Using (5.6) and (5.9) the relation between the residual measurement vector and the error state can be derived as

\[ c^p \delta y(t) = \Pi(t)c^p H(t) \delta x(t) \]  

(5.11)
where

$$\Pi(t) = bd\left(0, \ldots, N_0 \hat{\Upsilon}(t)\right)$$

$$^cH(t)^\top = \begin{bmatrix} n_0(t)^\top & \ldots & n_j(t)^\top & \ldots & n_{N_0}(t)^\top \\ t_0(t)^\top & \ldots & t_j(t)^\top & \ldots & t_{N_0}(t)^\top \end{bmatrix}$$

with

$$n_j(t) = \begin{bmatrix} -cR(t) & 0 & c_{bf_j}(t) \times cR(t) & 0 & 0 \end{bmatrix}$$

$$t_j(t) = \begin{bmatrix} -bR(t) & c_{cf_j}(t) \times bR(t) \end{bmatrix}.$$
5.5.2 Observability of INS error with a fully calibrated camera

This section deals with Problem 1 in Section 5.5.1. Assume that the extrinsic parameters are perfectly known, hence \( b\delta p_{bc} = b\rho = 0 \). The following proposition can be proved (See Appendix E.2 for proof).

**Proposition 5** Assuming the vision sensor is fully calibrated, then the INS error state is fully observable with, \( N_0 > 3 \) measurements at 3 time instants.

When the camera is fully calibrated, estimation of INS error is equivalent to estimation of camera motion from a series of images. This is a well studied problem in the literature (e.g., [5], [81]) \(^1\).

5.5.3 Observability of INS and errors in extrinsic parameters using a camera

This section deals with Problem 2 in Section 5.5.1 for certain roving maneuvers defined in Appendix A.4. The augmented error state vector and its dynamic equations are given by (5.1) and (5.4) respectively. Denote the state transition matrix of \( \delta\mathbf{x} \) by \( \Psi \) such that

\[
\delta\mathbf{x}(t_{k+1}) = \Psi(t_{k+1}, t_k)\delta\mathbf{x}(t_k)
\]  

(5.12)

where

\[
\Psi(t_{k+1}, t_k) = \begin{bmatrix}
\Phi(t_{k+1}, t_k) & 0 \\
0 & I
\end{bmatrix}
\]  

(5.13)

\(^1\)Note that observability of INS error states through vision measurements depends only on the feature locations and is completely independent of rover maneuvers.
and $\Phi$ is the state transition matrix of the INS error state derived in (3.17). At any measurement time instant $0 \leq t_k \leq t_M$, we derive the measurement vector as

$$
\delta y(t_k) = \begin{cases} 
\Pi(t_0)cH(t_0)\delta x(t_0) & k = 0 \\
P(t_k)cH(t_k) \prod_{i=k-1}^{0} \Psi(t_{i+1}, t_i)\delta x(t_0) & 1 \leq k \leq M 
\end{cases}
$$

(5.14)

Note that for each $0 \leq t_k \leq t_M$, the symbol $\delta y(t_k) \in \mathbb{R}^{2(N_0+1)}$ denotes a vector of measurement residuals corresponding to $N_0 + 1$ features. For each $0 \leq j \leq N_0$, we derive

$$
^c\delta q_{f_j}(t_k) = j\chi(t_k) \left\{ -^c_n R(t_k)^n \delta p_{nb}(t_0) - k \Delta_T^c n R(t_k)^n \delta v_{nb}(t_0) - ^b R^b \delta p_{bc}(t_0) \\
+ (-^c_n R(t_k)\eta_{k-1} + [^c p_{bf_j}(t_k)\times] \int_0^{t_k} R(t)\zeta_{k-1} \delta b_g(t_0)} \\
+ (-^c_n R\kappa_{k-1} + [^c p_{bf_j}(t_k)\times] \int_0^{t_k} R(t)\rho(t_0)} \\
-^c_n R(t_k)\mu_{k-1} x_5 + [^c p_{bf_j}\times] \int_0^{t_k} R^b \rho(t_0) \right\}
$$

(5.15)

where

$$
kappa_{k-1} = - \sum_{i=0}^{k-1} (k - i - 1) [^n f(t_i)\times] \Delta_T^2 \\
\zeta_{k-1} = \sum_{i=0}^{k-1} ^b R(t) \Delta_T \\
\eta_{k-1} = - \sum_{i=0}^{k-1} \left( \sum_{j=i}^{k-1} (k - j - 1) [^n f(t_j)\times] \right) \int_0^{t_i} R(t_{i-1}) \Delta_T^3 \\
\mu_{k-1} = \sum_{i=0}^{k-1} (k - i) ^b R(t_{i-1}) \Delta_T^2.
$$

For each measurement time $0 \leq t_k \leq t_M$ and feature $0 \leq j \leq N_0$ we derive

$$
^c b R = ^c n R(t_k)^n R(t_k) \\
^c p_{bf_j}(t_k) = ^c p_{cb} + ^c p_{bf_j}(t_k).
$$

(5.16)

(5.17)
Substituting (5.16) and (5.17) into (5.15) we derive

\[ c_δq_j(t_k) = \sum_{n} R(t_k) \left( n_δp_{nb}(t_k) + k_ν_δv_{nb}(t_k) + \kappa_{k-1} n_δp(\nu(\nu) + \eta_{k-1} b_δb_ν(t_k) \right. \]
\[ + \sum_{n} R(t_k) b_δp_{bc}(t_k) + \sum_{n} R(t_k) \left[ b_δp_{bc}(t_k) \times \right] b_δp(\nu(\nu) \]
\[ + \sum_{n} R(t_k) \left[ c_δp_{bf}(t_k) \times \right] c_δp(t_k) \left( n_δp(\nu(\nu) + \zeta_{k-1} b_δb_ν(t_k) + \eta_{k-1} b_δb_ν(t_k) \right) \right]. \] (5.18)

Writing (5.18) in matrix form we derive

\[ \delta y(t_k) = \mathcal{O}_c(t_k) \iota(t_k) \] (5.19)

where \( \mathcal{O}_c(t_k) = \Pi(t_k) \hat{\mathbf{H}}(t_k) \) with

\[ \hat{\mathbf{H}}(t_k) = \begin{bmatrix}
-\sum_{n} R(t_k) & \left[ c_δp_{bf}(t_k) \times \right] \sum_{n} R(t_k) \\
\vdots & \vdots \\
-\sum_{n} R(t_k) & \left[ c_δp_{bf}(t_k) \times \right] \sum_{n} R(t_k)
\end{bmatrix} \] (5.20)

and \( \iota(t_k)^\top = \begin{bmatrix} u(t_k)^\top & v(t_k)^\top \end{bmatrix} \) where \( u(t_k), v(t_k) \in \mathbb{R}^3 \) are defined by

\[ u(t_k) = \begin{bmatrix} I & k_ν_δI & \kappa_{k-1} & \eta_{k-1} & \mu_{k-1} & n_δR(t_k) & n_δR(t_k) \left[ b_δp_{bc}(t_k) \times \right] \end{bmatrix} \iota \]
\[ v(t_k) = \begin{bmatrix} 0 & 0 & I & \zeta_{k-1} & 0 & 0 & n_δR(t_k) \end{bmatrix} \iota \]

with \( \kappa_{-1} = \eta_{-1} = \mu_{-1} = \zeta_{-1} = 0 \). Using the structure of (5.20) and Proposition 20 from Appendix E we conclude that at every measurement time, \( \iota \) is fully observable. Hence assume \( \{ u(t_k), v(t_k) \}, k = 0, 1, \ldots, M \) are our new constructed measurements. Following propositions can be proved:

**Proposition 6** If the rover is undertaking **Maneuver 1** (See A.4.1) then there exists 6D
unobservable subspace spanned by

\[
\begin{bmatrix}
^n_b R(t_0) \left[ b_p c \times \right] ^n_b R(t_0)^\top u \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \left[ n_b R(t_0)^\top \left[ n f(t_0) \times \right] u \\
- \left[ b_p c \times \right] ^n_b R(t_0)^\top u \\
0 \\
- ^n_b R(t_0)^\top u
\end{bmatrix}
\]

where \( u = \{e_1, e_2, e_3\} \) with \( e_i \) being the \( i \)th standard basis of \( \mathbb{R}^3 \).

**Proposition 7** If the rover is undertaking **Maneuver 3** (See A.4.3) then there exists a 3D unobservable subspace spanned by

\[
\begin{bmatrix}
u^\top \\
0^\top \\
0^\top \\
0^\top \\
u^\top ^n_b R(t_0)^\top \\
- ^n_b R(t_0)^\top
\end{bmatrix}^\top
\]

where \( u = \{e_1, e_2, e_3\} \) with \( e_i \) being the \( i \)th standard basis of \( \mathbb{R}^3 \). Hence errors in INS position can not be resolved from errors in the lever arm.

### 5.5.4 Observability of INS and errors in extrinsic parameters using camera and a GPS like sensor

Assume that we have carrier phase measurements from \( N_s \) satellites and vision measurements of \( N_0 + 1 \) features. The measurement vector at time \( t_k \), \( 0 \leq k \leq M \) is given by

\[
\begin{bmatrix}
\delta y_g(t_k) \\
\delta y_c(t_k)
\end{bmatrix} = \mathcal{O}_{gy}(t_k) \delta x(t_0)
\]

(5.22)
where

\[
\mathcal{O}_{gc}(t_k) = \begin{bmatrix}
\mathcal{O}_g(t_k) & 0 & 0 \\
0 & 0 & \mathcal{O}_c
\end{bmatrix}.
\]

The following propositions can be proved:

**Proposition 8** If the rover is undertaking **Maneuver 1** (See A.4.1) then \(\text{rank}(\mathcal{O}_{gc}(t_k)) = 18\) and the unobservable subspace is spanned by

\[
\left[0^T \quad 0^T \quad u^T \quad 0^T \quad u^T[n f(t_0) \times]^T R(t_0) \quad u^T \widehat{b}_{\delta} R(t_0) \quad [n p_{\infty} \times]^T \quad -u^T \widehat{b}_{\delta} R(t_0)\right]^T
\]

where \(u = \{e_1, e_2, e_3\}\) with \(e_i\) being the \(i\)th standard basis of \(\mathbb{R}^3\).

**Proposition 9** If the rover is undertaking **Maneuver 3** (See A.4.3) then we have full state observability.

**Proof.** This proposition can be easily proved using Propositions 4 and 7. ■
Chapter 6

Stationary updates aided INS

6.1 Introduction

Sub-meter localization accuracies are required for control and guidance applications such as collision avoidance, lane departure warning, etc., [1]. Tightly integrated GPS-INS is able to achieve decimeter level positioning accuracy under favorable conditions (e.g., carrier phase processing with good satellite geometry and differential correction availability) [26], [27], [84]. However, in certain situations, such as urban canyons, where the line-of-sight to satellites is intermittent, GPS aiding may not be able to provide an absolute reference to INS during significant periods of time. In such situations, INS errors may diverge quickly. This motivates the need for alternative aiding signals.

Several authors have suggested stationary updates as an alternative low-cost aiding technique [18], [28], [32], [53], [61], [62], [90] etc. A reasonable model for a gyroscope is

\[ \ddot{y}(t) = y(t) + b_y(t) + n(t) \]  

(6.1)
where \( \tilde{y} \) is the sensor’s measurement of the signal \( y \). It is corrupted by a bias \( b_y \) and measurement noise \( n \). It can be seen that when \( y = 0 \), \( b_y \) is observable through measurement \( \tilde{y} \) and hence can be estimated directly. This is the principle of stationary or Zero Updates (ZUPT). Stationary updates are pseudo-measurements that are able to contain drift in the state estimates. Note that there is no actual stationary sensor; instead, software uses the data already available on the vehicle to decide that the vehicle is stationary. Then a measurement is imposed theoretically (e.g., velocity or angular rate equals zero). In this approach, the definition of which data to use and the formulation of the decision are critical.

6.2 Literature review

Stationary updates have a long history in the calibration and alignment of high performance systems (e.g., [24], [41], [49], [60], [91]), where controlled conditions are enforced to ensure that the system is immobile and vibrations are limited. Such controlled conditions, limited vibration, and known periods of stationarity are not applicable to land vehicle applications for driver assistance and intelligent transportation system (ITS) applications. The challenge in vehicle applications is to define reliable methods to detect the stationary condition in a general setting. Many methods to detect the stationary condition exist in the literature. These methods can be sorted into three main categories: manual [32], IMU-based [18], [28], [61], [90], and GPS based [53], [62]. In [32], the system detects the growth in positioning error covariance during GPS gaps and alerts the user to perform stationary updates manually. In [61], IMU sensors are used for pedestrian tracking with stationary updates as the only correcting measurement. Stationarity is established by detecting the lo-
cal minima of the norm of angular rate measurements. This approach is specific to walking pedestrians. It does not extend to land vehicles. In addition, it can miss significant periods of stationarity when the sensor is stationary without local minima (i.e., flat). The method in [28] thresholds the average accelerometer and gyroscope measurements over an interval to decide on stationarity. This approach is dependent on the IMU being nearly level and may fail if the biases or attitude are larger than expected. In [90], the author describes various pattern recognition approaches for detecting stationarity using accelerometer signals. In [62], the author uses Doppler measurements from a high sensitivity GPS receiver to apply zero velocity updates in an indoor scenario. In [18], the author thresholds the variance of the norm of the accelerometer measurements over an interval to decide on stationarity. The method in [53] declares the vehicle to be stationary when the GPS estimate of velocity is below a prescribed threshold for more than 20s. Estimation of GPS velocity requires at least 4 satellites, with appropriate satellite geometry, for at least two consecutive time instants. When these GPS conditions are satisfied, the artificial ‘stationary measurements’ are not needed. Therefore, in this article we focus on the detection of stationarity using only the IMU signals. Constant velocity scenarios will be directly discussed.

The outline of this chapter is as follows. Section 6.3 describes the stationary update measurement model. Section 6.4 presents our frequency domain approach for detection of stationarity. Section 6.4.1 provides a mathematical statement of the detection problem. Sections 6.4.2, 6.4.3, 6.4.4.1 and present a theoretical analysis of the proposed solution. We are primarily concerned with land vehicle navigation and hence we will use that information to simplify analysis. However, the approach can be easily generalized to other applications such as pedestrian navigation. Section 6.4.4.2 analyzes detection performance
of the proposed solution using field data. Section 6.5 of this chapter focuses on the INS observability analysis between stationary updates. Three main scenarios for vehicle motion will be of interest: (i) Stationary rover, (ii) Rover decelerating to a stop (ii) Rover starting and ending in stationarity. These motion scenarios are defined in Appendix A.4 and the observability gramian is analyzed in each case. Section 6.5.1 derives a basis for each unobservable subspace. The analysis discusses the observability results from a state estimation viewpoint and provides simulation results to support the theoretical conclusions. Section 6.6 demonstrates improvements in localization accuracies achievable by using stationary updates in terms of containing errors in position, velocity and attitude estimates. Section 6.7 concludes the chapter.

### 6.3 Stationary update measurement model

When the vehicle is stationary, the $b$ and $n$ frames have zero relative velocity and angular rate. Therefore, at those times when the vehicle is determined to be stationary, two artificial sensors will be used. One sensor measures the velocity (i.e., $^n v_{nb} = 0$). The other measures the angular rate (i.e., $^b \omega_{nb} = 0$). The state estimator will utilize the residual measurements and linearized measurement matrices that are defined next.

#### 6.3.1 Zero velocity update

In this case the signal is $^n v_{nb}$ and the imposed value of the measurement is $^n \tilde{v}_{nb} = 0$, hence the measurement residual is computed as

$$^n \delta v_{nb} = ^n \tilde{v}_{nb} - ^n \hat{v}_{nb} = -^n \hat{v}_{nb}. $$
The linearized model of the stationary velocity residual is

\[ n\delta v_{nb} = sH_v\delta \bar{x} \]

where \( sH_v = \begin{bmatrix} 0 & I & 0 & 0 \end{bmatrix} \).

### 6.3.2 Zero angular rate update

In this case, the measured signal is \( b\omega_{nb} \) and the measurement is \( b\bar{\omega}_{nb} = 0 \), hence the measurement residual is computed as

\[ b\delta \omega_{nb} = -b\bar{\omega}_{nb} + b\hat{b}_g. \]

The residual is modeled as a linear function of the error state as

\[ b\delta \omega_{nb} = b\omega_{nb} - b\bar{\omega}_{nb} \]

\[ = b\omega_{ib} - b\omega_{in} - b\bar{\omega}_{ib} + b\bar{\omega}_{in} \]

\[ = b\delta \omega_{ib} - b\delta \omega_{in} \] (6.2)

where

\[ b\delta \omega_{ib} = -b\delta b_g \] (6.3)

\[ b\delta \omega_{in} = [b\hat{R} e\omega_{ie} \times]n\hat{R} n\rho. \]

The term \( b\delta \omega_{in} \) is ignored because \( e\omega_{ie}^\top = \begin{bmatrix} 0 & 0 & 7.29 \times 10^{-5} \end{bmatrix} \) rad/s is smaller than the gyroscope measurement noise power spectral density for MEMS based instruments.

Substituting (6.3) into (6.2) we derive

\[ b\delta \omega_{nb} = sH_\omega\delta \bar{x} \]

where \( sH_\omega = \begin{bmatrix} 0 & 0 & 0 & -I & 0 \end{bmatrix} \).
6.4 Detection of Stationarity using Inertial data

The premise of our approach is that, for a vehicle in motion, even at constant speed, the IMU data will be non-constant and this change is useful for detecting non-stationarity. This section motivates and analyzes the idea both in theory and practice.

6.4.1 Problem description

The problem of stationarity detection from inertial data can be stated as follows: Given a finite sequence of \( N \in \mathbb{N} \) inertial measurements \( b\tilde{y}_i \), design a test

\[
F : \{ b\tilde{y}_i | 0 \leq i \leq N - 1 \} \rightarrow \{ 1, 0.1 \}
\]  

(6.4)

such that \( F = 1 \) implies that the rover is stationary during the interval \( t \in [0, N - 1] T_s \) and \( F = 0.1 \) implies otherwise.\(^1\)

Assume that the IMU is mounted on a land vehicle (e.g., a car). It is convenient to consider the signal of interest as having components with different frequency spectra,

\[
b\tilde{y} = s + e + \nu.
\]  

(6.5)

The portion of the motion represented as \( s \) is the large scale motion of the vehicle. The symbol \( e \) models vibrations caused by the engine (e.g., \( e = \alpha \cos(\omega_e t) \) where \( \omega_e \) is the frequency of rotations in rad/s). All other extraneous, unpredictable vibrations are grouped under \( \nu \). The characteristics of \( s, e \) and \( \nu \) are dependent on the motion of the vehicle. The specific characteristics of \( b\tilde{b} \) and \( \tilde{n} \) are not. The large scale motion \( s \) is directly caused by the driver via the steering wheel, brakes and accelerator. The motion is caused by the forces

\(^1\)Here we use the value 0.1 instead of the conventional 0.0 solely for the convenience of plotting using a logarithmic scale.
between the wheel and the non-smooth road. The chassis of the vehicle is a spring-mass-damper system that performs a low pass filtering operation between the driver, engine and road induced forces and the IMU. The resonant frequency of the chassis is typically $0 - 2$ Hz. The signal components $s, e,$ and $\nu$ are at the output of the chassis low pass filter. The frequency characteristics of $s$ are also band-limited due to the inertia of the vehicle.

Substituting (6.5) into (3.1) we model the inertial measurements as a function of time as

$$b\tilde{y} = s + e + b + n + \nu.$$  \hspace{1cm} (6.6)

The IMU provides discrete-time samples $b\tilde{y}_i = b\tilde{y}(iT_s)$ every $T_s$ seconds such that

$$b\tilde{y}_i = s(iT_s) + e(iT_s) + b(iT_s) + n(iT_s) + \nu(iT_s)$$  \hspace{1cm} (6.7)

where $T_s$ is the sampling time period of the IMU and $0 \leq i \leq N - 1$ is the sample index.

The physical intuition behind our approach is that the frequency domain structure of $s, e$ and $\nu$ is distinct for a moving vehicle relative (even at constant speed), to a stationary vehicle and that this difference is detectable. This is illustrated in Fig. 6.1. This idea and its theoretical analysis are considered next.

6.4.2 Theoretical Analysis

Let $b\tilde{y}^p$ be the periodic repetition of $b\tilde{y}$ as defined on page 450 in [64]. Denote the Discrete Fourier Transform (DFT) of $b\tilde{y}$ as $b\tilde{Y}$, which is interpreted as the Discrete Fourier Series (DFS) of the discrete-time periodic sequence $b\tilde{y}^p$. For $0 \leq m \leq N - 1$, the DFT of $b\tilde{y}$ is computed as

$$b\tilde{Y}(\omega_m) = \frac{1}{N} \sum_{i=0}^{N-1} b\tilde{y}_i \exp\{-j\omega_m i\}$$  \hspace{1cm} (6.8)
where $\omega_m = \frac{2\pi m}{N}$ and $j = \sqrt{-1}$. By the linearity property of the DFS,

$$b\hat{Y} = S + E + B + V$$

(6.9)

where $S, E, B, V$ represent the DFS of $s^p, e^p, b^p$ and $v = (\nu^p + n^p)$. Our approach uses the following assumptions:

1. The samples are drawn from an interval of duration $NT_s$ seconds. Since we desire to detect brief intervals of stationarity, $NT_s$ is small. The assumption is that the biases are constant over this interval. Therefore,

$$B(\omega_m) = \begin{cases} b^b & m = 0 \\ 0 & 1 \leq m \leq N - 1 \end{cases}$$

(6.10)

Note that for accelerometers the value of $b^b$ is affected by platform tilt.

2. The rate of engine revolution is constant at $K$ revolutions per minute over the window of length $T = NT_s$. For typical vehicles, $K \in [500, 5000]$. The resulting oscillatory motion has frequency $f_e \in [8, 85]$ Hz. During stationary periods, $f_e$ is in the lower portion of this range. Therefore, engine vibrations yield frequency components

$$E = \begin{cases} \alpha & \omega = \pm 2\pi f_e \\ 0 & \text{elsewhere} \end{cases}$$

(6.11)

3. The frequency content of $s$ is determined by the inertial and actuation characteristics of the vehicle and is band-limited with bandwidth less than $\bar{f}$. Typically, $\bar{f}$ is on the order of 10 Hz.

4. The measurement noise $n$ and other effects $\nu$ are assumed to be white with Gaussian statistics. Hence, $n \sim \mathcal{N}(0, \sigma_n^2 I), \nu \sim \mathcal{N}(0, \sigma_\nu^2 I)$ and $E\{n_{i_1} \nu_{i_2}^\top\} = 0$ for all $0 \leq i_1, i_2 \leq N - 1$ [64].
The pink, green, and violet background colors in Fig. 6.1 depict the approximate frequency ranges of \( B \), \( S \), and \( E \), respectively. The figure also shows the DFT of sampled acceleration sequences collected with the IMU rigidly mounted on a vehicle. The asterisks indicate the DFT samples for a period when the rover is stationary. The circles, squares and triangles indicate the DFT samples for a period when the rover is accelerating, decelerating and moving with a constant speed respectively. Constant speed was ensured by setting the land vehicle on cruise control mode on a flat road. The clear separation, in the \( S \) frequency band, between the DFT components for the stationary case (asterisks) and the non-stationary cases (circles, squares, and triangles) confirms the reasonableness of the frequency domain approach.

The DFT in Fig. 6.1 used \( N = 128 \) samples with an IMU sampling frequency of \( F_s = 136 \) Hz. The duration of each DFT interval was \( T = NT_s = 0.941s \). Therefore, the discrete frequencies are

\[
f_m = 1.0625m \text{ Hz, where } m = 0, \ldots, \frac{N}{2} - 1.
\]  

These parameters apply to all subsequent data used for illustrative purposes throughout this chapter.

When the rover is stationary the accelerometer measurement described in (6.6) reduces to

\[
{b^*} = b^b + b_n R^n g + \alpha \cos(\omega_e t + \phi_e) + n.
\]  

The DFT has a few interesting characteristics as depicted in Figure 6.1:

1. While stationary, there is a zero frequency component due to \( b^b + b_n R^n g \). Since the attitude and bias are unknown, the zero frequency component is not useful for
Figure 6.1: Analysis of a sequence of 128 samples of forward acceleration data.
detection of stationarity.

2. The engine causes vibrations of unknown magnitude with frequency in the range [8, 85] Hz. Frequency components in the range [8, 85] Hz are problematic for stationarity detection.

3. The remaining low frequency harmonics that are within the vehicle bandwidth are useful for discrimination of time periods when the vehicle is stationary. The low frequency components, \( f_m \in (0, 8) \) Hz or \( m = 1, \ldots, 7 \), provide a measure of the frequency content in \( s \).

The use of \( b\hat{Y}(2\pi f_m), m = 1, \ldots, 7 \) to detect stationarity enables the designer to disregard the effects of engine vibrations, biases and components of gravity. Selection of specific frequency components is considered in Section 6.4.4.1. Note that in applications, only the desired Fourier components need to be calculated. Those components are implementable by Eqn. (6.8) as an FIR filter.

### 6.4.3 Selection of thresholds

Let \( b\hat{y}^k = \{b\hat{y}_i^k, i = 0 \ldots N - 1\} \) be a sampled sequence of the \( k^{th} \) IMU sensor modeled as in (6.7). The DFS sample at \( \omega_m \) can be written as an FIR filtered version of \( b\hat{y}^k \):

\[
b\hat{Y}_k(\omega_m) = \frac{1}{N} \kappa_m^\top b\hat{y}^k = \left[ \frac{1}{N} \begin{array}{c} \omega \end{array} \right]_{N \times 1} \varphi_m \tag{6.14}
\]
where

\[ \kappa_m^\top = \begin{bmatrix} 1 & \exp\{-j\omega_m\} & \ldots & \exp\{-j\omega_m(N - 1)\} \end{bmatrix} \]

\[ k_\varphi_m^\top = \begin{bmatrix} a_k^m \quad b_k^m \end{bmatrix} \quad (6.15) \]

\[ a_k^m = \text{Re}\{\kappa_m^\top \bar{y}^k\} \quad (6.16) \]

\[ b_k^m = \text{Im}\{\kappa_m^\top \bar{y}^k\} \quad (6.17) \]

The DFS sample at frequency \( \omega_m \) is modeled as

\[ b_k \tilde{Y}_k(\omega_m) = S_k(\omega_m) + E_k(\omega_m) + B_k(\omega_m) + V_k(\omega_m). \quad (6.18) \]

If \( \omega_m \) is chosen in region \( S \) then according to our hypothesis, \( B_k(\omega_m) = 0 \) and \( E_k(\omega_m) = 0 \). Eqn. (6.18) reduces to

\[ b_k \tilde{Y}_k(\omega_m) = S_k(\omega_m) + V_k(\omega_m). \quad (6.19) \]

Assuming that \( s \) is deterministic but unknown, using (6.6), we derive that \( E\{b \bar{y}^k\} = s \) and \( \text{Cov}\{b \bar{y}^k\} = \sigma^2 I \) where \( \sigma^2 = (\sigma_n^2 + \sigma_v^2) \). Using (6.15) we derive \( E\{k_\varphi_m^\top\} = \begin{bmatrix} \text{Re}\{\kappa_m^\top\} & \text{Im}\{\kappa_m^\top\} \end{bmatrix} s \) and \( \text{Cov}\{k_\varphi_m\} = P \)

\[ k P_m = \sigma^2 \begin{bmatrix} \text{Re}(\kappa_m)\text{Re}(\kappa_m^\top) & \text{Re}(\kappa_m)\text{Im}(\kappa_m^\top) \\ \text{Im}(\kappa_m)\text{Re}(\kappa_m^\top) & \text{Im}(\kappa_m)\text{Im}(\kappa_m^\top) \end{bmatrix}. \]

To simplify the analysis that follows, decompose the symmetric matrix \( k P_m \) as \( k P_m = UD^2U^\top \), where \( D \) is diagonal and \( U \) is unitary. As a first step towards defining a test in the form of eqn. (6.4), define \( f: \mathbb{R}^2 \to [0, +\infty) \) as

\[ f(k_\varphi_m) = k_\varphi_m^\top (k P_m^{-1}) k_\varphi_m. \quad (6.20) \]
Using only the $k$-th sensor, the vehicle would be declared stationary if $f(k\varphi_m) < \lambda_k^2$. For any selected threshold value $\lambda_k > 0$, the probability of stationarity decision is

$$p = P\{f(k\varphi_m) < \lambda_k^2\} = P\{k\nu_m \nu_m^\top < \lambda_k^2\} \quad (6.21)$$

where

$$k\nu_m = D^{-1}U^\top k\varphi_m. \quad (6.22)$$

We are interested in deriving the probability of detecting stationarity, given by (6.21), under two scenarios: Stationary and Moving. The first case depicts the probability of successful detection while the second case depicts the probability of false detection.

6.4.3.1 Stationary ($S(\omega) = 0$)

Using $s = 0$ in (6.16) and (6.17) we derive

$$a^k_m = \sum_{k=0}^{N-1} (n_k + \nu_k) \cos(\omega k) \quad (6.23)$$

$$b^k_m = \sum_{k=0}^{N-1} (n_k + \nu_k) \sin(\omega k) \quad (6.24)$$

where $n \sim N(0, \sigma_n^2)$, $\nu \sim N(0, \sigma_\nu^2)$ with $E\{n_i, \nu_i\} = 0$ for all $0 \leq i_1, i_2 \leq N - 1$, as stated in Section 6.4.2. Therefore, because eqns. (6.23 – 6.24) are linear in $n$ and $\nu$, the random variable $k\varphi_m$ is a zero mean, jointly Gaussian random vector. Note that $k\nu_m$ defined in (6.22) is a zero mean Gaussian random vector with $E\{k\nu_m k\nu_m^\top\} = I$; hence $f(k\varphi_m)$ is a Rayleigh random variable with parameter $\chi = 1$, see page 147 in [27]. Hence for any desired value of $p > 0$, an appropriate value of the threshold $\lambda_k$ can be selected such that (6.21) is satisfied.
If we select harmonic $\omega^k_m \in S$ and $\lambda_k \in \mathbb{R}$ for each $k = 1, \ldots, 6$, where the $k$-th sensor indicates the vehicle is stationary if

$$k \varphi^\top_m \left( k P_m^{-1} \right)^k \varphi_m < \lambda^2$$

then joint probability of all six sensors indicating stationary is

$$p_{d/s} = P\{k \varphi^\top_m \left( k P_m^{-1} \right)^k \varphi_m < \lambda^2_k \mid k = 1, \ldots, 6\} = \prod_{k=1}^6 P\{k \varphi^\top_m \left( k P_m^{-1} \right)^k \varphi_m < \lambda^2_k \} = p^6.$$  \hfill (6.26)

It should be noted that when the designer selects exactly one harmonic in $S$ for each sensor, and the vehicle is stationary, then the random vectors $k_1 \varphi_m$ and $k_2 \varphi_m$ for $k_1 \neq k_2$ are statistically independent. Hence the joint probability can be reduced to the product shown in (6.26).

As an example, if the desired value of $p$ is 0.95, then selecting $\lambda_k = 2.4477$ for all $0 \leq k \leq 6$, yields $p_{d/s} = p^6 = 0.7351$ as the probability of correctly detecting stationary. The probability of missing a successful stationary detection is $\overline{p}_{d/s} = 1 - p_{d/s} = 0.2649$.

These probabilities can be manipulated by the designer by selection of $\lambda_k$. In applications, because stationary detections are not required at high rates and the goal is to minimize the probability of false stationary detections, $p$ could be selected significantly smaller (e.g., $p^6 = 0.01$ yields $\lambda = 1.1171$).

### 6.4.3.2 Moving ($S(\omega) \neq 0$)

Since $k v_m$ is a linear function of $k \varphi_m$, we have $k v_m \sim \mathcal{N}(k \mu_m, I)$, where

$$k \mu_m(t) = E\{k v_m(t)\} = D^{-1} U^\top E\{k \varphi_m(t)\}.$$
Figure 6.2 gives an intuitive illustration of the detection of motion using $k \varphi_m$. The blue dots represent 1000 samples of $k v_m$ while the vehicle is in motion. The mean $k \mu_m$ is at the point of the red arrow. The magenta circle around the origin depicts the $x^T x = \lambda^2$ region for $x \in \mathbb{R}^2$, $\lambda = 2.4477$. Stationarity is detected by (6.25) when all $k v_m$, $k = 1, \ldots, 6$ are inside the magenta circle. The blue square is centered at the origin with each side of length $2\lambda$. It is used to bound the probability of the blue dots lying inside the magenta circle. This is likely when $k \mu_m = 0$ and becomes increasingly unlikely as $k \mu_m$ (i.e., $s$) increases.
The following proposition shows that the probability of false detection can be made arbitrarily small as \( k\mu_m \) increases or \( \lambda \) decreases.

**Proposition 10** Let \( \lambda \in (0, \infty) \) be given. For any \( \epsilon > 0 \), there exists \( M_0 > 0 \) such that
\[
P\{k v_m^k v_m \leq \lambda^2\} < \epsilon
\]
for all \( k\mu_m = E\{k v_m\} \) such that \( ||k\mu_m|| > M_0 + \lambda \).

**Proof.** For the given \( \epsilon \) and \( \lambda \), choose \( M_0 \) such that
\[
M_0^2 > 2 \log \left(\frac{\lambda^2}{2\epsilon}\right).
\]
If \( ||k\mu_m|| > M_0 + \lambda \), then for \( k v_m^k v_m \leq \lambda^2 \)
\[
||k\mu_m|| > M_0 + ||k v_m|| > 0.
\] (6.27)
Therefore, defining \( l = (k v_m - k\mu_m)^\top (k v_m - k\mu_m) \),
\[
(k v_m - k\mu_m)^\top (k v_m - k\mu_m) = ||k v_m||^2 + ||k\mu_m||^2 - 2||k\mu_m|| ||k v_m||
\]
\[
= (||k\mu_m|| - ||k v_m||)^2
\]
\[
l > M_0^2.
\]
Thus for \( ||k\mu_m|| > M_0 + \lambda \) and \( k v_m^k v_m \leq \lambda^2 \),
\[
\exp\{-\frac{1}{2}(k v_m - k\mu_m)^\top (k v_m - k\mu_m)\} < \exp\left\{-\frac{1}{2}M_0^2\right\}.
\]
The probability \( P\{k v_m^k v_m \leq \lambda^2\} \) is
\[
P\{k v_m^k v_m \leq \lambda^2\} = \frac{1}{2\pi} \int_{D_\lambda} \exp\{-\frac{1}{2}l\} d^k v_m
\]
\[
< \frac{1}{2\pi} \int_{D_\lambda} \exp\{-\frac{1}{2}M_0^2\} d^k v_m
\]
\[
= \frac{1}{2} \exp\{-\frac{1}{2}M_0^2\} \lambda^2
\]
(6.28)
\[
< \epsilon
\]
where $D_\lambda$ denotes the disc $(k_\lambda)(k_{\lambda}) \leq \lambda^2$.

In fact, as illustrated in Figure 6.2, we can bound the probability of false detection $P\{k_\lambda(k_{\lambda}) \leq \lambda^2\}$ as

$$P\{k_\lambda(k_{\lambda}) \leq \lambda^2\} < P\{ \max_{j \in \{1, 2\}} k_{\lambda}(j) \leq \lambda\} = q$$

(6.29)

where $k_{\lambda}(j)$ denotes the component of $k_{\lambda}$ along $e_j$, the standard basis of $\mathbb{R}^2$. Note that the right side of (6.29) is the probability of the blue dots lying inside the blue square. This probability is

$$q = \frac{1}{4} \left( \text{erf} \left( \frac{\lambda - M \cos \beta}{\sqrt{2}} \right) + \text{erf} \left( \frac{\lambda + M \cos \beta}{\sqrt{2}} \right) \right)$$

$$\left( \text{erf} \left( \frac{\lambda - M \sin \beta}{\sqrt{2}} \right) + \text{erf} \left( \frac{\lambda + M \sin \beta}{\sqrt{2}} \right) \right)$$

(6.30)

where $k_{\mu_m} = M \begin{bmatrix} \cos \beta & \sin \beta \end{bmatrix}^\top$ and $M > 0$. Note that $||k_{\mu_m}|| = M$. From eqn. (6.30) we have $\lim_{M \to +\infty} q = 0$. From (6.28), (6.29) and (6.30) we see that the probability of false detection decreases as $\lambda$ decreases or the motion (i.e., $s$), as quantified through $M$, increases.

Given the analysis above which is for a single sensor, we are now in a position to define the function $F$ defined in (6.4). For each sensor $k$, let the selected harmonic and threshold be $\omega_m^k$ and $\lambda_k$ respectively, then the overall decision function is

$$F_{\lambda}(k_y) = \begin{cases} 1 & \max_{0 \leq k \leq 6} k_\lambda^{T}k_{\lambda} < \lambda^2 \\ 0.1 & \text{otherwise} \end{cases}$$

(6.31)

where $k_{\lambda}$ was defined in (6.22).

### 6.4.4 On vehicle Results

This section selects specific frequency components for each of the six sensors and analyzes actual on vehicle performance.
6.4.4.1 Selection of harmonics

The test of eqn. (6.31) combines the decisions on the six sensor measurements made independently for each sensor based on the variable \( k^m v_m \) defined in eqn. (6.22) using \( \varphi^k_m \). For each sensor, the designer may choose a parameter \( \lambda_k \) and a frequency \( f_{mk} \). For simplicity, we consider \( \lambda_k = \lambda \) for \( k = 1, \ldots, 6 \) where \( \lambda \) would be selected to satisfy eqn. (6.21) with a specified value of \( p \).

The symbol \( Y^k_m = b\hat{Y}_k(\omega_{mk}) \) denotes the component of the \( N \)-point DFT using samples from the \( k^{th} \) IMU sensor at frequency \( f_{mk} = \frac{m_k F_s}{N} \). Note that each sensor may use a different frequency component \( f_{mk} \). For the reasons outlined in Section 6.4.2, we will focus on the intermediate region \( S \) with frequencies defined as in eqn. (6.12).

False stationary measurements would corrupt the bias and attitude estimates; therefore, preventing false stationary detections is critical. Stationary vehicles tend to be stationary for several seconds, which allows many opportunities for a sliding window DFT to detect stationarity; therefore, given that stationary aiding measurements are only needed once every few seconds, detecting all instances of stationarity is not critical. This motivates selection of a small value for \( p \) and selection of frequency components that are sensitive to motion.

To demonstrate that this is possible and quite straightforward, we use data from an 1120s test run. GPS signals were available throughout the experiment; therefore, it was possible to estimate the vehicle velocity to use as ground truth for comparisons. Using the GPS estimated speed, 15 intervals of stationarity were identified. Using this data with known periods of motion and stationarity we collected statistics related to the sequence in
which harmonics in $S$ fail the stationarity test of (6.25) as the vehicle begins to move. Table 6.1 summarizes the statistics for this. For $m = 1, \ldots, 7$ the frequencies $f_m$ defined in eqn. (6.12) are listed in column two. The percentage of times when each harmonic is the first to fail the stationarity test is recorded in columns three through eight. For each sensor, the harmonic that most frequently fails the stationarity test first is deemed to be more sensitive to motion than others. By this methodology, we select the $m = 1$ component for sensors $a_x$, $a_y$, $g_x$, $g_y$, $g_z$ and the $k = 2$ component for $a_z$.

<table>
<thead>
<tr>
<th>$m$</th>
<th>Freq. Hz</th>
<th>$a_x$</th>
<th>$a_y$</th>
<th>$a_z$</th>
<th>$g_x$</th>
<th>$g_y$</th>
<th>$g_z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.0625</td>
<td>84.6</td>
<td>68.75</td>
<td>20</td>
<td>66.67</td>
<td>90</td>
<td>92.30</td>
</tr>
<tr>
<td>2</td>
<td>2.1250</td>
<td>0</td>
<td>12.5</td>
<td>80</td>
<td>8.33</td>
<td>10</td>
<td>7.69</td>
</tr>
<tr>
<td>3</td>
<td>3.1875</td>
<td>15.4</td>
<td>18.75</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>4.2500</td>
<td>0</td>
<td>0</td>
<td>16.67</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>5.3125</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>8.33</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>6.3750</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>7.4375</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 6.1: Sensitivities of harmonics to vehicle motion.

### 6.4.4.2 Detection Results

In this section, we consider the performance of test of eqn. (6.31) using the frequency components selected in the previous section. The test is implemented as six FIR filters at 136Hz. A tightly integrated GPS-INS is used for data comparison where the GPS aiding occurs at a 1Hz rate. In this section, the stationary measurements are not used to aid the INS.

The blue dots in Fig. 6.3 shows a trace of $k \mathbf{v}_m$ for the $m = 1$ component of the roll-rate gyroscope $g_x$(rad/s). The yellow circle depicts $k \mathbf{v}_m^T k \mathbf{v}_m < 1.1774^2$ and the magenta circle
Figure 6.3: Trace of $k \nu_m$ for the $m = 1$ component of the roll-rate gyroscope.
represents $k v_m^T k v_m < 2.4477^2$. Under stationarity condition, theoretically the blue dots ($k v_m$) will stay inside the yellow circle with probability 50%. The magenta circle depicts the $k v_m^T k v_m < 2.4477^2$ region. Under stationary condition, the magenta circle should enclose the blue dots 95% of time. The GPS-INS estimated speed during this time interval is shown in Fig. 6.4. The vehicle is initially stationary, accelerating forward during the last second of the graph. Figs. 6.3 and 6.4 contain larger red dots that indicate the values of $k v_m$ at 1.0 second intervals. The red dots are included to show the time dependence of the trace of $k v_m$. The green square shows the point where $k v_m^T k v_m$ is exiting the magenta circle. The times of the red and green markers in Fig. 6.3 are in one-one correspondence with those in Fig. 6.4.

The GPS-INS estimated speed of Fig. 6.4 is included to allow comparison with and verification of stationary decisions. The blue dots show the “ground truth” tangential speed computed using GPS observables. The red circles and the green square are in one-one correspondence with those in Fig. 6.3. In Fig. 6.4, the GPS correction instants and subsequent growth of the INS speed estimate due to noise and and attitude and bias errors are clearly evident. During the entire stationary interval, the signal $k v_m$ remains small. Once the vehicle accelerates, $k v_m$ clearly exits the magenta circle. Note that the roll gyro is not the strongest indicator of motion, forward and vertical acceleration and pitch rate are significantly more sensitive. The GPS-INS estimate of speed is 0.8 cm/s when the stationarity condition fails (i.e., motion is detected).

Fig. 6.5 shows the time series of the chosen harmonics for all six sensors for $t \in [300, 500]$. The $m = 1$ component were used for sensors $g_x$, $g_y$, $g_z$, $a_x$ and $a_y$. The $m = 2$ component was used for $a_z$. The time series of $k v_m^T k v_m$ is shown in green, magenta, cyan,
Figure 6.4: Ground truth GPS estimate of speed.
red, blue and black for $a_x$, $a_y$, $g_x$, $g_y$ and $g_z$ respectively. The threshold $\lambda^2$ is depicted by the dashed black line, where $\lambda = 2.4477$ each in its own color.

Fig. 6.6 shows the GPS-INS estimate of speed (blue) and the combined detection decision (black) of eqn. (6.31) for times $t \in [300, 500]$. These two figures should be considered together. From the estimated vehicle speed in Fig. 6.6 it is visually clear when the vehicle is moving or stationary. There are three intervals where the vehicle is stationary and three intervals where the vehicle is in motion. Fig. 6.6 shows that stationarity is detected frequently while the vehicle is stationary, but never detected while the vehicle is in motion. The GPS speed is at the cm/s level when the stationarity test fails with threshold $\lambda = 2.7744$. Figure 6.5 appears very busy. It shows that during periods of motion, specific sensors may give erroneous indications of stationarity, but the set of six sensors taken together never jointly make an error. The times at which motion is detected are $\sim 334$, $420$ and $472$ s. The GPS estimate of tangential speeds when the decision $F$ detects motion are $2$ cm/s, $1$ cm/s and $0.7$ cm/s respectively.

6.4.5 Comparative analysis

Figs. 6.7 - 6.9 compare the detection performance of the proposed algorithm to the algorithm in [18]. The red (black) dots denote the times at which the algorithm in [18] (our algorithm) report stationary. In all Figs. the blue curve denotes the INS speed computed by a GPS aided INS. In [18], the authors propose thresholding the variance of norm of accelerometer measurements, i.e., stationarity is reported if $\sigma^2 < p\sigma_s^2$, where $\sigma^2$ is the variance of $N$ samples of accelerometer measurements, $\sigma_s^2$ is defined as the variance of norms of the accelerometer measurements when the rover is known to be stationary and $p$
is a tuning parameter that can be changed according to the designer’s choice. The authors recommend $N$ to be such that it corresponds to a sequence of accelerometer measurements about $0.6 - 1.0$ s long. For our comparison we chose $N = 136$, such that it corresponds to a $1$s duration. Using periods of known stationarity we computed $\sigma_s^2 = 9.34 \times 10^{-4}(\text{m/s/s})^2$. Since [18] does not provide a method to select $p$, we choose $p = 1$. For reasons stated in Section 6.4.3, we pick the threshold $\lambda = 1.1171$ such that the probability of successful detection is $p^\lambda = 0.01$. Experimental data collected for Fig.6.7 on stationary rover, shows that the proposed algorithm detects stationary $0.87\%$ of the time, confirming the prediction.
Figure 6.6: Stationary decisions vs. GPS estimate of speed.
Fig. 6.7: Comparative analysis of detection performance for a stationary rover.

Fig. 6.8 shows the case when the rover is initially at rest and starts moving at a slow speed (3.4mph). It is known that the rover began to move at time $t = 30s$. It can be seen that [18] makes a false detection at time $t = 32s$, while the proposed algorithm makes no false detection.

Fig. 6.9 shows the case when the rover is initially at rest, accelerates to a cruising speed of 10mph. Here again, [18] detects motion later than the proposed algorithm resulting in some false detections at time $t = 48s$. 
Figure 6.8: Comparative analysis of detection performance for a rover moving at 3.4mph.
Figure 6.9: Comparative analysis of detection performance for a rover moving at 10mph.

At all higher speeds/accelerations, the detection performance of both algorithms are identical.

6.5 Observability analysis of INS error states

This section derives uses the observability gramian to define the unobservable subspaces of the state space using stationary measurements under various maneuvering scenarios. The continuous time observability gramian defined in (B.3) will be analyzed in the subsequent sections. In Section 6.5.1, we derive right null space of \( H \Phi \) under various motion and aiding
scenarios. Once the null space of $H\Phi$ is determined, we can use Proposition 18 to conclude the same for $O$.

### 6.5.1 Observability using Stationary updates

In this subsection we derive the observability conditions for the INS error states using only stationary updates. Errors in positions are not observable using stationary updates and hence the unobservable subspace is at least 3 dimensional. For this reason, we remove position error states from the original error state vector $\delta \bar{x}$. In all subsequent analysis in this subsection, reduced state vector, state transformation and measurement projection matrix are given by

$$ \delta \bar{x}_m^\top = \begin{bmatrix} \nu_b^\top & \rho^\top & b_y^\top & b_a^\top \end{bmatrix} $$
$$ \Phi_m = \begin{bmatrix} I & -S_\tau & M_\tau & R_\tau \\ 0 & I & R_\tau & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} $$
$$ H_m = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & -I & 0 \end{bmatrix} $$

where $\tau \in U$. Observability of error states is discussed for a subset of the motion scenarios outlined in Appendix A.4, using zero velocity and zero angular rate updates.

Following propositions can be proved (For proof see Appendix F):

**Proposition 11** If the rover is undergoing Maneuver 1 (See Appendix A.4.1) and stationary updates are available for all $\tau \in S = U$, then there exists a 3 dimensional unobservable
subspace spanned by
\[
\begin{bmatrix}
0^\top & e_i^\top & 0^\top & -e_i^\top [^n g \times]_b^n R
\end{bmatrix}^\top
\]  \hspace{1cm} (6.34)

where \(e_i\) for \(1 \leq i \leq 3\) is a basis for \(\mathbb{R}^3\).

Proposition 11 is a standard case often discussed in literature. It is known, see e.g., [30], that when the rover is stationary with GPS position updates for all \(\tau \in S\), we have a 4 dimensional unobservable subspace spanned by
\[
\begin{bmatrix}
0^\top & 0^\top & b^g^\top & 0^\top \\
0^\top & e_i^\top & 0^\top & -e_i^\top [^n g \times]_b^n R
\end{bmatrix}^\top.
\]  \hspace{1cm} (6.35)

Using Proposition 11, we see that with the addition of stationary updates, errors in the 4 dimensional subspace spanned by (6.35) are driven to a 3 dimensional unobservable subspace spanned by (6.34). For a level vehicle, the three dimensional null space is the yaw error and linear combinations of the tilt errors and the horizontal accelerometer biases.

**Proposition 12** Assume that the rover is undertaking Maneuver 2.b. (See Appendix A.4.2.2). If the rover is equipped with stationary updates for all \(\tau \in S\), there exists a 3 dimensional unobservable subspace spanned by
\[
\begin{bmatrix}
e_i^\top m^\top(\tau_1) & e_i^\top & 0^\top & -e_i^\top [^n g \times]_{b(\tau_1)}^n R
\end{bmatrix}^\top
\]  \hspace{1cm} (6.36)

where
\[
m(\tau_1) = \int_0^{\tau_1} \{ \kappa [^n R^b d^x] + [^n g \times] \} ds - \left\{ \int_0^{\tau_1} b^b_n R [^n g \times] ds \right\}_{b(\tau_1)}^n R. \]  \hspace{1cm} (6.37)

Note that Proposition 11 is a special case of Proposition 12. This is observed by setting \(\kappa = 0\) (i.e., Stationary for all \(\tau \in U\)) in (A.3). The basis derived in (6.34) can also be derived by setting \(\kappa = 0\) in (6.37).
Assume that the rover is undertaking Maneuver 4 (See Appendix A.4.4). Let \( v_3^\top = \begin{bmatrix} u_{31}^\top & u_{32}^\top & u_{33}^\top & u_{34}^\top \end{bmatrix} \) span the right null space of \( H_m \Phi_m \) for all \( \tau \in S_1 \cup S_2 \). By Proposition 11, if we are equipped with stationary updates in \( S_1 \), then the unobservable subspace is given by (6.34), hence

\[
\begin{align*}
    u_{31} &= 0 \\
    u_{33} &= 0 \\
    u_{34} &= b(t_1) R^{[n g \times]} u_{32}.
\end{align*}
\] (6.38)

For all \( \tau \in S_2 \) equating \( H_m \Phi_m v_3 = 0 \) we derive

\[
\int_0^\tau [n f \times] u_{32} - \frac{n}{b} R u_{34} ds = 0.
\] (6.41)

Differentiating (6.41) we derive

\[
u_{34} = b(t_2) R^{[n g \times]} u_{32}
\] (6.42)

for all \( \tau \in (t_2, t) \). Substituting (6.40) and (6.42) into (6.41) we derive

\[
\int_{t_1}^{t_2} [n f \times] u_{32} - \frac{n}{b} R u_{34} ds = 0.
\] (6.43)

The following result is needed before we can proceed analyzing Maneuver 4.

**Lemma 13** As \( ^n v_b = 0 \) for all \( \tau \in S_1 \cup S_2 \), then the following relation holds

\[
\int_{t_1}^{t_2} \kappa(\tau) \frac{n}{b} R \, d \tau = 0.
\]

Re-writing (6.43) using (A.4) we derive

\[
\int_{t_1}^{t_2} -[u_{32} \times] \left( \kappa(\tau) \frac{n}{b} R \, d + \frac{n}{b} R u_{34} \right) ds = 0.
\] (6.44)
Using Lemma 13 and (6.44) we derive

$$\int_{t_1}^{t_2} \left[ n^g \times u_{32} - \frac{n}{b} R u_{34} \right] ds = 0.$$  

(6.45)

The following proposition derives a basis for unobservable subspace for Maneuver 4.a. (See Appendix A.4.4.1).

**Proposition 14** If the rover is undertaking Maneuver 4.a. (See Appendix A.4.4.1), then there exists a 3 dimensional unobservable subspace spanned by

$$\gamma_i^\top = \begin{bmatrix} 0^\top & e_i^\top & 0^\top & -e_i^\top n^g \times n^b R \end{bmatrix}$$  

(6.46)

where $e_i$ for $1 \leq i \leq 3$ is a basis for $\mathbb{R}^3$.

The following proposition derives a basis for unobservable subspace for Maneuver 4.b. (See Appendix A.4.4.2).

**Proposition 15** If the rover is undertaking Maneuver 4.b., then the unobservable subspace is 1 dimensional spanned by

$$\begin{bmatrix} 0^\top & n^g^\top & 0^\top & 0^\top \end{bmatrix}^\top.$$  

(6.47)

As a corollary to Proposition 15, if we have velocity measurements (e.g., from GPS) in an arbitrarily small open subset of $M$ then the unobservable subspace is trivial as long as the pitch ($\theta$) of the rover is not $\pm \frac{\pi}{2}$ for any $\tau \in M$. Let the GPS velocity measurement residual be denoted as $n^\delta v_b$. The linearized measurement error model was derived in (4.17).

The following corollary can be proved:

**Corollary 16** In addition to the conditions in Proposition 15, if $\theta \neq \pm \frac{\pi}{2}$ for all $\tau \in M$ and velocity measurements are available in an arbitrary open neighborhood $U_g \subset M$, then the unobservable subspace is trivial.
6.5.2 Numerical demonstration of observability

The eqns. (3.10 – 3.13) are

\[
\begin{align*}
\dot{\delta v}_{nb} & = -[\hat{n}f \times] n \rho + \hat{n} \bar{R} b \delta b_a + \hat{n} \bar{R} \omega_a \\
\dot{\rho} & = \hat{b} \bar{R} b \delta b_g + \hat{b} \bar{R} \text{omega}_g \\
b \delta b_a & = \omega_{ba} \\
b \delta b_g & = \omega_{bg}
\end{align*}
\]

where for reasons stated earlier \(n \delta \hat{p}_{nb}\) has been ignored. Let \(w^\top = \begin{bmatrix} \omega^\top_a & \omega^\top_g & \omega^\top_{ba} & \omega^\top_{bg} \end{bmatrix}\) denote the process noise vector where \(\omega_a, \omega_g\) denote the accelerometer and gyroscope measurement noise and \(\omega_{ba}, \omega_{bg}\) denote the bias random walk parameters. Let \(Q = \text{Cov}(N)\) given by

\[
Q = \begin{bmatrix}
\sigma_a^2 I & 0 & 0 & 0 \\
0 & \sigma_g^2 I & 0 & 0 \\
0 & 0 & \sigma_{ba}^2 I & 0 \\
0 & 0 & 0 & \sigma_{bg}^2 I
\end{bmatrix}
\]

(6.48)

Similarly the measurement residual equations are modeled as

\[
\delta y = sH \delta x + n
\]

where \(\delta y^\top = \begin{bmatrix} n \delta v^\top_{nb} & n \delta \omega^\top_{nb} \end{bmatrix}, sH^\top = \begin{bmatrix} sH_v^\top & sH_\omega^\top \end{bmatrix}\) and \(n^\top = \begin{bmatrix} n_v^\top & n_\omega^\top \end{bmatrix}\) denotes additive white Gaussian measurement noise. It is assumed that \(n \sim \mathcal{N}(0, R)\) with

\[
R = \begin{bmatrix}
\sigma_v^2 I & 0 \\
0 & \sigma_\omega^2 I
\end{bmatrix}
\]

92
If the vehicle was in fact stationary then it would be correct to assume \( R = 0 \). However, corrections with \( R = 0 \) result in a \( P \) being positive semi-definite with zero eigenvalue corresponding to the direction of \( sH \). In reality, the vehicle is never stationary due to vibrations and passenger induced motion. The value of \( R \) should represent the sensitivity of the stationary decision. The variance of the zero velocity update is selected as \( \sigma_v^2 = 10^{-4}(\text{m/s})^2 \). The variance of the zero angular rate update is selected as \( \sigma_\omega^2 = 1 \times 10^{-6}(\text{rad/s})^2 \). In the numerical analysis that follows, we use the decrease in parts of the covariance matrix \( P \) to indicate which portions of the error state become observable.

For the purpose of simulation we set \( U = [0, 30] \) s. For motion scenarios described in Appendix A.4, the set \( S \) is described in Table 6.2.

<table>
<thead>
<tr>
<th>Maneuver</th>
<th>( S ), seconds</th>
<th>Legend</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( S = U )</td>
<td>Blue</td>
</tr>
<tr>
<td>2.b.</td>
<td>( S = [0, 15] )</td>
<td>Red</td>
</tr>
<tr>
<td>4.a. ( ^b\omega_{nb} = 0 )</td>
<td>( S_1 = [0, 10], S_2 = [20, 30] )</td>
<td>Black</td>
</tr>
<tr>
<td>4.b. ( ^b\omega_{nb} \neq 0 )</td>
<td>( S_1 = [0, 10], S_2 = [20, 30] )</td>
<td>Magenta</td>
</tr>
</tbody>
</table>

Table 6.2: Time of aiding and legend for various maneuvers.

The inertial measurements were integrated in discrete time at 100Hz. We assume we have stationary updates \(^n\dot{\omega}_{nb} = 0, ^b\dot{\omega}_{nb} = 0 \) at 1Hz for all discrete times \( \tau_k \in S \). We implemented a discrete time fixed-point smoother to compute \( E\{x(0)|^{n}\dot{\omega}_{nb}(\tau_k), ^b\dot{\omega}_{nb}(\tau_k), \forall \tau_k \in S\} \) [29], [52]. The error standard deviation of the smoothed estimates is plotted as a time series in Figures 6.10-6.14. For all motion scenarios, the standard deviation of the smoothed estimates of the initial conditions is plotted only at times when measurement updates are available, for example in Fig. 6.10, the magenta plot is absent during times (10, 20) because there are no stationary updates in Maneuver 4.b. at that time. It was assumed that the
initial orientation of the rover is such that $^b_o\dot{R}(0) = I$. The initial uncertainty was selected as

$$P(0) = \begin{bmatrix}
\sigma_p^2 I & 0 & 0 & 0 & 0 \\
0 & \sigma_v^2 I & 0 & 0 & 0 \\
0 & 0 & \sigma_\rho^2 I & 0 & 0 \\
0 & 0 & 0 & \sigma_g^2 I & 0 \\
0 & 0 & 0 & 0 & \sigma_f^2 I
\end{bmatrix}$$

where $\sigma_p = 1$m, $\sigma_v = 0.1$m/s, $\sigma_\rho = 10$deg, $\sigma_g = 10^{-5}$deg/s, and $\sigma_f = 10^{-3}$m/s/s.

Fig. 6.10 shows the uncertainty in the smoothed estimate of error in initial velocity. Since the observability conditions for errors in velocity in Maneuvers 1, 2.b., 4.a., 4.b. are the same, the blue, black and magenta plots overlap wherever defined. The red plot (Maneuver 2.b.) does not begin to decrease until $t = 15$s, when stationary measurements become available. Even for $t > 15$s, the standard deviation does not decrease to the same level as the other scenarios. This is because, even for $t > 15$s, the initial velocity is not observable; instead, the error state vector has converged to an new unobservable subspace. The lack of observability in Maneuver 2.b. is discussed in Proposition 12 which shows that the unobservable space includes a linear combination of velocity and yaw. The physical interpretation of this result, in this example, is that the knowledge that the vehicle is stationary for $t > 15$s and the inertial measurements for $t \in [0,15]$s does not uniquely identify the initial velocity, because the yaw angle is unknown. Therefore, the direction of the initial velocity cannot be determined from the available data.

Fig. 6.11 shows the uncertainty in the smoothed estimate of $^n\rho(0)$ using stationary measurements up to time $t$. In the interval $t \in [0,10]$s, there is no difference between
Figure 6.10: Uncertainty in $n\delta \hat{v}_b(0) = [\delta v_n \delta v_e \delta v_d]$ using all measurements in $S$. 

95
Figure 6.11: Uncertainty in $n \hat{\rho}^\top(0) = [\rho_n \rho_e \rho_d]$ using all measurements in $S$. 
Figure 6.12: Uncertainty in yaw direction with GPS estimate of velocity.

**Maneuvers 1, 2.b., 4.a., 4.b.** Hence the blue, black and magenta plots overlap in this time interval. Proposition 15 predicts that rotation along a single direction (yaw in this case) during $M$ causes the first two components of the attitude error to become observable. This is confirmed by the decrease in the first two components of the magenta plot compared to the black plot (**Maneuver 3.a.**), after the rover enters stationary at time 20s. Propositions 11, 12, 14 and 15 predict that initial errors in yaw direction are not observable. This is confirmed by the third subplot of Fig. 6.11, as the uncertainty in all scenarios stays constant.
Figure 6.13: Uncertainty in $b^g_\theta(0) = [b^g_\theta \ b^g_\phi \ b^g_\psi]$ using all measurements in $S$.

On the other hand, Corollary 16 proves that the initial uncertainty in the yaw estimate is observable if velocity measurements are available during some $U_g \in M$. This is supported by Fig. 6.12. The magenta plot shows the uncertainty in Maneuver 4.b., which is the same as the magenta plot in subplot 3 of Fig. 6.11. The green plot shows the uncertainty in the initial yaw estimate when velocity aiding is available at 3 time instants in M. The measurement standard deviation for the GPS velocity estimates is 2cm/s.

Fig. 6.13 shows the uncertainty in the smoothed estimate in initial gyroscope bias. All
Figure 6.14: Uncertainty in $\hat{b}_a(0) = [b_{ax} b_{ay} b_{az}]$ using all measurements in $S$.

Motion scenarios have the same gyro bias observability characteristics and hence all plots converge to the same value.

Fig. 6.14 shows the uncertainty in the smoothed estimate of initial accelerometer bias. It is seen that the black and blue plots overlap since the observability characteristics in Maneuver 1 and Maneuver 4.a. are similar. Since the roll ($\phi$), pitch ($\theta$) and yaw ($\psi$) were all initialized to zero, i.e., $\phi = \theta = \psi = 0$, the error in the third component of accelerometer bias is always observable as the error is along $^n g$ (ref. eqn. (6.34), (6.36),
(6.46), (6.47)). It is interesting to note that, in Maneuver 4.b. (magenta plot), the uncertainty in accelerometer bias in the first two directions decrease at time 20s, when the rover enters stationary after undertaking a rotation. Hence the magenta plot confirms the results in Proposition 15 that errors in accelerometer biases are observable if we have rotation in one direction.

6.6 Localization Performance Analysis

The utility of stationary updates is enhanced when GPS aiding is unavailable or sparse. This section demonstrates how stationary measurements aid in containing errors in position and attitude when GPS aiding is absent. When the rover comes to halt (e.g., at a traffic light) in an urban canyon, the absence of GPS aiding signals allows the position and attitude to drift. This drift is slowed or contained if stationary updates are used. In the following experiment, the rover was stationary and CDGPS measurements were used to aid the inertial system for \( t \in [0,8) \text{s} \). At \( t = 8 \text{s} \), GPS aiding was switched off. Fig. 6.15 graphs the position error standard deviation versus time, where the blue and red plots indicate the cases with and without stationary updates, respectively. Integer ambiguities in CDGPS measurements were successfully resolved during both cases by time \( t = 5 \text{s} \), hence centimeter level positioning accuracy was achieved in either case at that point. This is demonstrated by both red and blue plots during \( t \in (5,8) \text{s} \). At \( t = 8 \text{s} \) GPS aiding is switched off and we can see that growth of the red plot (without stationary aiding) is at least parabolic in time whereas there is no growth in the blue plot. This is because zero velocity updates correct the errors in the velocity estimates. Because the velocity errors are correlated with
Figure 6.15: Position error uncertainty versus time with and without stationary updates.
and in fact cause the position errors, the Kalman filter also corrects the position errors. If stationary updates were unavailable, the errors in position estimates grow to the meter level in 6s.

Fig. 6.16 shows the corresponding growth of uncertainty of velocity error estimates. Fig. 6.17 shows the growth in uncertainty of attitude errors. Since this experiment falls under the stationary rover category and the rover was initially assumed to be level, errors in yaw are not observable as demonstrated by the third subplot of Fig. 6.17. Without
stationary aiding, growth in uncertainty of roll and pitch errors (Fig. 6.17, subplot 1 & 2) is linear due to integration of errors in gyroscope biases. This is depicted by the red plot in subplots 1 & 2 of Fig. 6.17. Since errors in gyroscope biases can be estimated using stationary updates, we see that the standard deviation in roll and pitch errors attain steady state. This is depicted by the blue plot in subplots 1 & 2 of Fig. 6.17. This steady state value corresponds to the unobservable errors in the initial attitude that cannot be resolved from accelerometer biases.
6.7 Conclusions

Stationary or Zero updates offer the possibility to enhance the performance of an inertial navigation systems, in the absence of other aiding measurements (e.g., GPS), at no addition sensor cost. The challenge in the use of such zero updates is the automatic and reliable detection of instances of stationarity. In the design of such detection methods, elimination of false stationary detections is critical. This chapter reviewed existing methods for stationary detection and proposed a frequency domain solution. The new frequency domain algorithm allows the designer to remove the effects of bias and attitude error on the decision and to select frequency components that are sensitive to the motion of the vehicle. The chapter contains discussion of these issues that are specific to land vehicles and which be directly extended to other vehicle types or to pedestrian applications. The chapter analyzes and derives the probability of successful detection and shows that the probability of false detection decreases exponentially with an increase in the vehicle motion or the decrease in the decision threshold $\lambda$. Real world detection results are included and were compared to “ground truth” tangent plane speed estimates derived from GPS observables. This chapter also analyzed the observability characteristics of stationary updates for typical maneuvers on a land vehicle (e.g., stationary, stopping at a traffic light, accelerating from rest and then stopping). The chapter derived a basis for the unobservable subspace of the INS error state in each scenario. The theoretical conclusions with regard to observability are verified with simulation results. Finally we showed experimental results demonstrating decreased growth in position, velocity and attitude errors in a real-time INS when stationary updates are used.
Chapter 7

Near Real Time Estimation

7.1 Introduction

The speed of modern computers allows significant processing beyond that required for a typical INS application. The excess processing power can be used to enhance reliability and accuracy through methods we refer to as NRT processing. Typically, even for an inexpensive IMU, an INS is designed such that it can maintain its specified accuracy for some time period (several seconds) even without aiding. This allows for aiding measurements to be processed in a more contemplative manner, within this time period, while the INS kinematic integration continues to supply users with state information. The contemplative method builds on optimal fixed interval smoothing to confirm or deny the validity of aiding measurements based on both the past and the future measurements. When aiding measurements are validated, the INS state is corrected from the time of validity of that aiding measurement up to the present time. In nonlinear systems, the NRT approach is built upon a Bayesian iterative framework. This approach out-performs the EKF when portions of the
state vector have uncertainty that is large relative to the curvature of the nonlinearities. The computational requirements are significantly smaller than those necessary for Particle filter implementations.

7.2 Motivation

For the purpose of a motivating example consider a 1D dynamic system \( x \in \mathbb{R} \). Fig. 7.1 depicts a time line of evolution of this 1D system. Let this system be aided by measurements of the form \( \tilde{y} = h(x) \). Let the pink curve indicate the standard deviation of error in \( \hat{x} \). Assume that steady state standard deviation was achieved prior to time \( t_0 \). Assume that we have some validation algorithm that checks for processing errors in the sensor measurement \( \tilde{y} \). In Fig. 7.1, all measurements that pass the validation test are displayed in black and those that don’t are displayed in red. At time \( t_1 \), assume that some errors in measurement sensor processing occurred (e.g., invalid integer ambiguities, wrong feature data association). These errors in aiding sensor processing are detected and hence such measurements are usually discarded. Each time an aiding measurement is discarded the Kalman filter measurement update is skipped. As a result of this, errors in the state increases. It might be easier to resolve these sensor processing errors using all the measurements over the interval \( [t_1, t_2] \) than trying to resolve them at any single time instant \( \tau \in [t_1, t_2] \). Assume that it could be determined that these errors are observable from a time series of measurements \( y(\tau), t_1 \leq \tau \leq t_2 \). Then after correcting the sensor processing errors at time \( t_2 \), the Kalman filter can resume measurement updates at time \( t_2 \), thus decreasing the uncertainty in the state. Further if the sensor processing errors are such
that they are unchanging over \([t_1, t_2]\), then corrections could be applied to measurements in \([t_1, t_2]\) and an improved estimate (whose standard deviation is depicted by the broken blue curve) of the state could be derived at those times, by reintegrating the state from \(t_1\). This contemplative method of sensor processing is known as Near Real Time estimation. An accurate estimate of the state between times \([t_1, t_2]\) is obtained not in real time, but in near real time.

\[\begin{array}{|c|c|}
\hline
\text{Time (s)} & \text{Uncertainty} \\
\hline
0 & \text{Initial} \\
\hline
\end{array}\]

Figure 7.1: Invalid measurements due to incorrect sensor processing.

Several examples of the situation depicted in Fig. 7.1 in real world scenarios are enumerated below:

1. Assume that a GPS-INS is stationary in time \([t_0, t_1]\) and begins accelerating at time \(t_1\).
If the initial attitude estimate is wrong (say by 180 deg), then any linearization based estimation algorithm (e.g. EKF) is invalid as errors in linearization are not negligible. This results in large measurement error residuals for $t > t_1$. But errors in yaw are observable under motion with GPS aiding. Hence using GPS measurements in $[t_1, t_2]$, it is possible to initialize the yaw of the rover at time $t_1$. Section 7.3 describes this problem in detail and describes an NRT attitude initialization solution.

2. In the CDGPS-INS, it is essential to resolve integer ambiguities to be able to use high accuracy carrier phase measurement for INS aiding. After receiver has acquired lock with the satellite signals, computation of integer ambiguities take some time (say a few seconds). But once resolved, the integer ambiguities remain constant as long as the receiver maintains phase lock with the satellite signal. Assume that sometime at $t_1$, the receiver momentarily lost lock of the satellite signal due to occlusion. Further assume that after re-acquisition of lock, the integers were computed at $t_2$. If it was known that the signal phase lock was acquired at $t_1 + \epsilon$, $\epsilon > 0$ after occlusion then the integer ambiguities computed at $t_2$ could be used for all CDGPS measurements in $(t_1, t_2]$.

These are few examples where NRT sensor processing could be utilized to improve estimates by removing sensor processing errors. Section 7.3 describes INS initialization as an application of NRT processing in a 2D GPS-INS system in detail.
7.3 Initialization of Nonlinear systems

7.3.1 Introduction

The performance of any linearization based estimation algorithm like the EKF relies heavily on the accuracy of the nominal trajectory about which the system is linearized. When the linearization assumption does not hold, such an algorithm behaves in an unpredictable fashion and metrics of estimation error (i.e. state covariance) are invalid.

This section presents methods to identify in real-time those parts of the state vector whose uncertainties cause significant deviations from the linearized model and proposes a NRT approach to address the issue. One important class of applications is initialization of navigation systems; therefore, as an example we apply the results of the theory to a simplified 7 state, 2D GPS-INS. The near-real time approach is demonstrated in simulation.

7.3.2 Analysis

Consider a generic nonlinear state evolution and measurement equations described in (2.1) and (2.2). Since $f$ and $h$ vectors of real analytic functions, we can express it as a Taylor series in the neighborhood of $(\hat{x}, \hat{u})$ as

$$
\delta \dot{x} = \dot{A} \delta x + G \dot{\omega} + T_x
$$

(7.1)

$$
\delta y = \tilde{H} \delta x + n + T_y
$$

(7.2)

For the analysis presented here, we will consider only the contribution of the second order term in $T_x$ and $T_y$. Let $f_j : \mathbb{R}^N \rightarrow \mathbb{R}$ and $h_i : \mathbb{R}^N \rightarrow \mathbb{R}$, denote the $i^{th}$ and $j^{th}$ component of $f$ and $h$ respectively for $0 \leq i \leq N$, $0 \leq j \leq P$. Denote the $i^{th}$ ($j^{th}$) element of $T_x$ ($T_y$)
as $T_i^x \left( T_j^y \right)$ respectively. We derive

$$T_i^x = \frac{1}{2} \delta \bar{x}^\top J_i^x \delta \bar{x} \quad T_j^y = \frac{1}{2} \delta \bar{x}^\top J_j^y \delta \bar{x}$$

where $J_i^x \left( J_j^y \right)$ are the Hessian matrices of $f_i \left( h_j \right)$ computed with respect to $\bar{x}$.

The linearized model will be considered accurate if $T_i^x \left( T_j^y \right)$ are in the order of (in a statistical sense) $\mathbf{G} \tilde{\omega}(n)$. This idea is developed below. For each for each $0 \leq i \leq N$,

$$E\{\delta \bar{x}^\top J_i^x \delta \bar{x}\} = E \left\{ \text{trace} \left( \delta \bar{x}^\top J_i^x \delta \bar{x} \right) \right\} = E \left\{ \text{trace} \left( J_i^x \delta \bar{x} \delta \bar{x}^\top \right) \right\} = \text{trace} \left( J_i^x \mathbf{P} \right)$$

(7.3)

where $\mathbf{P}$ denotes the covariance matrix of error in $\tilde{x}$. Similarly $E\{\delta \bar{x}^\top J_j^y \delta \bar{x}\} = \text{trace} \left( J_j^y \mathbf{P} \right)$ for each $0 \leq j \leq P$. If $Q_i$ and $R_j$ denote the second order moments of $i^{th}$ and $j^{th}$ component of $\mathbf{G} \tilde{\omega}$ and $n$, we conclude that the linearized model is valid if

$$\text{trace} \left( J_i^x \mathbf{P} \right) < \gamma_i Q_i \quad \text{trace} \left( J_j^y \mathbf{P} \right) < \mu_j R_j$$

(7.4)

for each $0 \leq i \leq N$ and $0 \leq j \leq P$, where $\gamma_i$ and $\mu_j$ are designer specific parameters. The criteria in (7.4) allows us to identify those states where the estimation error as characterized by $\mathbf{P}$ is large enough relative to the curvature (i.e. $J_i^x$ or $J_j^y$) such that its effect is larger than noise. This is demonstrated in Section 7.3.7.1 for the specific case of a 2D GPS-INS.

One method to detect a discrepancy in the linearized model is to compare the probabilities of magnitudes of the observed measurement residuals computed from the linearized model to a designer specified threshold ($\lambda^2$). This threshold could be chosen using probabilistic principles such that it satisfies a specified performance criteria. If we observe measurement residuals whose probability of occurrence is very small according to our linearized model
then that might indicate an error in the linearized model. Section 7.3.3 builds more on this idea.

### 7.3.3 Validation of measurements

The linearized model for the measurement residual \( \delta \tilde{y} \in \mathbb{R}^P \) is derived by ignoring \( T_y \) in (2.20) as

\[
\delta y = \bar{H}\delta \bar{x} + n. \tag{7.5}
\]

If the linearized model is valid (i.e. \( \exists \Phi : \mathbb{R}^{N+M} \to \mathbb{R}^{N+M} \) such that \( \delta \bar{x}(t) = \Phi(t, 0)\delta \bar{x}(0) + \int \Phi G \omega d\tau \)) and uncertainty in initial state estimate \( \hat{\bar{x}}(0) \) is normally distributed as \( \hat{\bar{x}}(0) \sim \mathcal{N}(\bar{x}(0), P(0)) \) then \( \delta \bar{x}(t) \sim \mathcal{N}(0, P) \), where

\[
P = \Phi P(0)\Phi^\top + \int \Phi G Q \Phi^\top d\tau.
\]

Further, if the additive measurement noise \( n \) is white and Gaussian with \( n \sim \mathcal{N}(0, R) \), the residual in (7.5) is Gaussian whose covariance is derived as

\[
S = \bar{H}PH^\top + R. \tag{7.6}
\]

Using the procedure outlined in Section 4.9.1 in [27], we define a new variable \( v = \Sigma^{-1}U^\top \delta y \), where \( \Sigma, U \) are derived by resolving the positive symmetric matrix \( S \) using \( LDL^\top \) decomposition as \( S = U\Sigma^2U^\top \).

For a given threshold \( \lambda \in \mathbb{R}^+ \), we propose the following indicator function \( I \):

\[
I = \begin{cases} 
1 & v^\top v \leq \lambda^2 \\
0 & v^\top v > \lambda^2.
\end{cases} \tag{7.7}
\]

Using the fact \( \delta y \sim \mathcal{N}(0, S) \), we can derive the stochastic properties of \( v^\top v \). If linearization errors are indeed small then, \( v^\top v \) is Chi-square distribution with \( P \) degrees of freedom. The
probability density function of $v^\top v$ is given by

$$p_{v^\top v}(s) = \frac{1}{2^{P/2} \Gamma(P/2)} s^{P/2 - 1} \exp\left\{-\frac{s}{2}\right\}$$

for $s \geq 0$. Given $q \in (0, 1)$, the corresponding threshold $\lambda$ can be computed from the $P^{th}$ order Chi-square cumulative distribution function such that $P\{v^\top v \leq \lambda^2\} = q$.

### 7.3.4 NRT Estimation

Without loss of generality, let the state vector be organized as $\bar{x}^\top = [\zeta^\top_l \, \zeta^\top_{nl}]$.

The symbol $\zeta_{nl} \in \mathbb{R}^L$ represents the states with error large enough that, when they are observable, cause $I = 0$ in (7.7) to trigger. While $\zeta_l \in \mathbb{R}^{M+N-L}$ represent the remaining states, for which (7.7) indicates that linearization is valid.

This chapter describes a method by which, over intervals of time where portions of $\zeta_{nl}$ become observable, errors in $\zeta_{nl}$ within the observable subspace can be decreased to the level where the standard EKF method becomes practical. During such time intervals, due to the magnitude of $\zeta_{nl}$ linearization based techniques are not reliable.

![Figure 7.2: Timeline of NRT based initialization.](image)

Consider the situation depicted in Figure 7.2. In time $[t_0, t_1]$, the system is undertaking
a trajectory called **Maneuver 5**. Assume that during this time a significant portion of the error in $\zeta_{nl}$ lies in the unobservable subspace and hence cannot be detected by measurements $\tilde{y}$. During **Maneuver 5** the observable states are sufficiently accurate to allow the EKF approach to succeed. An example of this is a typical stationary GPS aided INS initialization where $\zeta_{nl}$ spans portions of the attitude and IMU biases. At time $t = t_1$, the system enters **Maneuver 6** when previously unobservable states become observable with errors too large to allow accurate linearization. Since $\zeta_{nl}(t_1)$ is observable using measurements in $\tilde{y}(t)$, $t_1 < t < t_2$, it is possible to initialize $\zeta_{nl}(t_1)$ such that $v^\top v \leq \lambda^2$ for all $t_1 < \tau < t_1$. This idea is outlined as Algorithm 1.

**Algorithm 1** Near-real time initialization

1: // $t_0 \leq \tau \leq t_1$

2: while $v^\top v < \lambda^2$ do

3: EKF: $\dot{x}(\tau) = \int f(\dot{x}, \dot{u}), \dot{b}(\tau) = \dot{b}(0), \delta \hat{x}(\tau) = E\{\delta \hat{x}(\tau)|\delta y(\tau)\}$

4: $v(\tau) \leftarrow \Sigma^{-1}U^\top \delta y(\tau)$

5: end while

6: // $t_1 \leq \tau \leq t_2$

7: if $O(\tilde{u}(\tau), \tilde{y}(\tau))$ then

8: Initialize: $\zeta_{nl}(t_1) \leftarrow I(\tilde{u}(\tau), \tilde{y}(\tau))$ using a MAP estimator.

9: end if

10: Re-integrate via EKF: $\dot{x}(\tau) = \int f(\dot{x}, \dot{u}), \dot{b}(\tau) = \dot{b}(t_0), \delta \dot{x}(\tau) = E\{\delta \dot{x}|\delta y\}$

In Algorithm 1, the indicator function $O : \mathbb{R}^{M+P} \times [t_1, t_2] \to \{0, 1\}$ returns 1 if $\dot{x}(t_0)$ is observable from measurements $(\tilde{u}(\tau), \tilde{y}(\tau)), t_1 < \tau < t_2$ and returns 0 otherwise. This condition is critical, for otherwise trying to initialize an unobservable state from noisy
measurements can lead to erroneous results. The function $I : \mathbb{R}^{M+P} \times [t_1, t_2] \rightarrow \mathbb{R}^L$ initializes $\zeta_{nl}(t_1)$ based on observables $(\tilde{u}(\tau), \tilde{y}(\tau))$. A method to achieve this is through nonlinear least squares minimization, provided a unique global minima exists for the given cost function (existence is guaranteed if the cost function is continuous and the domain is compact, uniqueness requires proof) and achievable in a finite number of iterations. There are methods that achieve off-line over long time intervals (e.g. Bundle adjustment [80], Square root SAM [20]), but we are interested in doing the nonlinear optimization on-line over short intervals, to drive observable states to the point where second order errors are small. A possible initialization function is described for the specific case of a 2D GPS-INS discussed in Section 7.3.5.

**7.3.5 A 2D GPS-INS example**

Consider a rover in a 2D world whose navigation state can be fully described by

$$s \mathbf{x}^\top = \begin{bmatrix} n^p_{nb} & n^v_{nb} & \psi \end{bmatrix}$$

where $n^p_{nb} \in \mathbb{R}^2$, $n^v_{nb} \in \mathbb{R}^2$, $\psi \in [-\pi, \pi]$ denote the position, velocity and attitude of the rover in the 2D world. We assume that the rover moves in the forward direction (defined relative to the body frame) without slipping in other directions. The rover is equipped with a 1D accelerometer and a yaw rate gyroscope that measures forward acceleration and yaw rate respectively. The inertial measurements are modeled as

$$b\ddot{a}_f = b a_f + b_a + n_a$$

$$b\ddot{\omega}_y = b \omega_y + b_{\omega} + n_{\omega}$$
where \( b_a \) denotes the forward acceleration and \( b_{\omega y} \) denote the yaw rate about the body frame, \( b_a, b_\omega \) and \( n_a, n_\omega \) represent the biases and additive noise in those sensors respectively.

### 7.3.5.1 2D INS Kinematics & GPS Measurement model

The kinematic equations are

\[
\begin{align*}
\dot{n}\vec{p}_{nb} & = \dot{n}\vec{v}_{nb} \\
\dot{n}\vec{v}_{nb} & = b_{\omega y} \Lambda_1^{T} b\vec{v}_{nb} + n\vec{R}_{b} b\vec{a}_{nb}
\end{align*}
\]

\[\dot{\psi} = b_{\omega y}\]

where \( b\vec{a}_{nb} = \begin{bmatrix} b_a f & 0 \end{bmatrix} \) and

\[ n\vec{R} = \begin{bmatrix} \cos \psi & -\sin \psi \\
\sin \psi & \cos \psi \end{bmatrix}, \quad \Lambda_1 = \begin{bmatrix} -\sin \psi & \cos \psi \\
-\cos \psi & -\sin \psi \end{bmatrix}. \]

Assume a random walk model for the sensor biases

\[
\begin{align*}
\dot{b}_a & = n_{b_a} \\
\dot{b}_\omega & = n_{b_\omega}
\end{align*}
\]

where \( n_{b_a} \) and \( n_{b_\omega} \) are random processes. The augmented navigation state \( \vec{x} \) is derived as

\[
\vec{x}^{T} = \begin{bmatrix} \vec{x}^{T} & b_\omega & b_a \end{bmatrix}.
\]

Equations \((7.8 - 7.12)\) correspond to \( f \) in the generic nonlinear model described in (2.1).

As stated earlier, it is difficult to compute the stochastic properties of \( \vec{x} \) due to the nonlinear nature of \((7.8 - 7.10)\). The EKF propagates the error covariance using a linearized approximation of the error state evolution. Let the augmented error state vector be defined
\[ \delta \dot{x}^\top = \begin{bmatrix} n \delta p_{nb}^\top & n \delta v_{nb}^\top & \delta \psi & \delta b_\omega & \delta b_a \end{bmatrix}. \]

Using (7.8 – 7.10) and (7.11 – 7.12) the error state dynamic equations equations are derived to first order as

\[ \begin{align*}
  n \delta p_{nb} &= n \delta v_{nb} \quad \quad (7.13) \\
  n \delta \dot{v}_{nb} &= b_\omega \Lambda_1^\top n \dot{R} \delta v_{nb} - \Lambda_1^\top n \dot{b}_\omega \delta v_{nb} + \Lambda_2 (\delta b_a + n_a) + \Lambda_3 \delta \psi \quad \quad (7.14) \\
  \delta \dot{\psi} &= -\delta b_\omega - n_\omega \quad \quad (7.15) \\
  \delta \dot{b}_\omega &= n_b_\omega \quad \quad (7.16) \\
  \delta \dot{b}_a &= n_b_a \quad \quad (7.17)
\end{align*} \]

where \( \Lambda_2 = - \begin{bmatrix} \cos \hat{\psi} & \sin \hat{\psi} \end{bmatrix} \) and

\[ \Lambda_3 = b_\omega \Lambda_1^\top \Lambda_1 n \dot{b}_\omega \hat{b}_\omega - n_b \dot{R} \delta \dot{b}_\omega b_\omega + \Lambda_1^\top \hat{a}_{nb} \]

which can be written in matrix form as

\[ \delta \dot{x}(t) = \dot{A}(t) \delta x(t) + G(t) \omega(t) \quad (7.18) \]

where \( \Lambda_2^\top = \begin{bmatrix} n_\omega & n_a & n_{ba} & n_{ba} \end{bmatrix} \) such that \( N \sim \mathcal{N}(0, Q) \) and \( Q \) is the noise power spectral density.

### 7.3.6 GPS Measurement updates

Assume that there are 2 satellites in this 2D world at known locations \( n p_j, j \in \{1, 2\} \).

The range measurement at time \( t \) corresponding to the \( j^{th} \) satellite is modeled as

\[ \tilde{y}_j(t) = ||n p_{nb}(t) - n p_j|| + \omega(t) \]
where $\omega \sim \mathcal{N}(0, R)$ is additive Gaussian white noise. The $j^{th}$ measurement residual, defined as $\delta y = y - \tilde{y}$ is modeled as a function of $\delta \bar{x}$

$$\delta y_j(t) = H_j \delta \bar{x}(t) + \omega(t) \quad (7.19)$$

where $H = \begin{bmatrix} \frac{n_{p_n}(t) - n_{p_j}}{\|n_{p_n}(t) - n_{p_j}\|} & 0 & 0 & 0 \end{bmatrix}$.

### 7.3.6.1 Observability conditions

Define the following rover maneuvers during the time $[t_0, t_2]$:

1. **Maneuver 5**: Rover is at rest for all $\tau \in [t_0, t_1]$.

2. **Maneuver 6**: Rover undergoes non-zero constant acceleration along a straight line for all $\tau \in [t_1, t_2]$.

The following proposition is stated but not proved:

**Proposition 17** The rover state is not fully observable from range measurements during **Maneuver 5**. The rover state is fully observable when **Maneuver 5** is followed by **Maneuver 6** using range measurements.

Further using linearized observability analysis, we can show that for during **Maneuver 5** with at least ranging measurements from two satellites with non-collinear line-of-sight vectors, $n_{p_{nb}}$ and $n_{v_{nb}}$ are observable. The attitude and gyroscope bias states are unobservable until the vehicle accelerates.

### 7.3.6.2 Literature review

Initial uncertainty in position and velocity affect the error state evolution linearly (see (7.13)) and hence it is accounted for by linear covariance propagation models. If the rover
is initially known to be at rest then a reliable estimate of gyroscope biases can be obtained by averaging gyroscope measurements. Hence, yaw is the only state with a potential to have large uncertainty, because it is unobservable through standard range measurements. In this idealized 2D system, we are able to estimate the forward accelerometer biases also, but in real-world systems, this is not possible as errors in alignment are coupled with the accelerometer biases. Significant research has been done on problem of attitude initialization.

Methods in literature can be broadly classified into the following categories

1. **Using additional sensors:** It is known that in any linearization based aided INS, attitude can be a significant source of nonlinearity, hence a significant amount of research has been conducted into attitude initialization. In [60], the author uses 2 tilt sensors to measure initial roll and pitch and estimates accelerometer biases while the vehicle is stationary. Yaw and heading is initialized after the rover begins to move. The approaches in [55] and [85], use the accelerometer measurements to initialize roll and pitch and a magnetic compass to initialize yaw. In a related work [75], the author uses a sun sensor to initialize yaw. Approaches in [48], [50], [51] etc., use double differenced carrier phase measurements (after resolving integer ambiguities) from three or more GPS antennas mounted at at known relative locations to estimate 3D attitude.

2. **Techniques:** In [36], the author uses a Particle Filter to solve the in flight misalignment problem where initial uncertainty is large. In [63], [76], large heading errors are accommodated by alternative modeling (i.e. estimating the sine and cosine of yaw separately by augmenting them in the state vector). The approach in [33] uses
standard fixed point smoother to initialize the system in near-real time. Particle filters have also been considered as a solution to the aided INS problem. A 3 state 2D GPS aided odometer based navigation system requires about 2000 particles to achieve lane-level accuracy [54]. A 7 state 2D GPS aided INS simulation outlined in Section 7.3.7 required 5000 particles to provide similar navigation performance to the near-real time initialization approach. The number of particles required to approximate the \textit{a-posteriori} density grows exponentially with the increase in state dimension. In general if \( m \) particles are used sample a single dimensional probability density and the \textit{a-posteriori} density is \( d \) dimensional then the number of required particles is \( m^d \).

\subsection*{7.3.7 Simulation Results}

The purpose of the this subsection to demonstrate the use of the methodology in Section 7.3 of this chapter to identify the states whose uncertainty causes significant deviation from the ideal linearized model. In the simulation the power spectral density of the accelerometer and gyroscope measurements noise were assumed to be \( 10^{-3}(\text{m/s/s})^2\text{ Hz}^{-1} \) and \( 10^{-7}(\text{rad/s})^2\text{ Hz}^{-1} \). The bias random walk parameters in (7.11 - 7.12) were assumed to be \( n_{b_w} \sim \mathcal{N}(0, \sigma_{b_w}), n_{b_a} \sim \mathcal{N}(0, \sigma_{b_a}) \) where \( \sigma_{b_a}^2 = 10^{-7}(\text{m/s/s})^2\text{ Hz}^{-1} \), \( \sigma_{b_w}^2 = 10^{-11}(\text{rad/s})^2\text{ Hz}^{-1} \) for the accelerometer and gyroscope respectively. The raw IMU measurements are shown in Figure 7.3. It can be shown that the full linearized error state is observable at time \( t = 19 \text{ s} \). The initial uncertainty was assumed to be Gaussian with \( \text{diag}(P(0)) = \begin{bmatrix} 0.5 & 0.5 & 0.1 & 0.1 & 0.67\pi & 10^{-3} & 5 \times 10^{-3} \end{bmatrix} \) with all correlation set to zero. The error in initial yaw was 120 deg.
Figure 7.3: The IMU measurements as the rover executed the desired trajectory.
7.3.7.1 Nonlinear effects of states

Given the kinematic system described by (7.8 - 7.10), the exact error state dynamic is derived as (2.19). From (7.13 - 7.17) we see that only (7.14) is an approximation (i.e. truncated at the first order of Taylor series) and the rest are exact, hence $\bar{T}_x$ has the form

$$\bar{T}_x^\top = \begin{bmatrix} 0 & T_x^3 & T_x^4 & 0 & 0 \end{bmatrix}.$$ Considering the contribution of only the second order terms, we derive

$$T_x^3 = \frac{1}{2}\delta\bar{x}J_x^3\delta\bar{x}^\top, \quad T_x^4 = \frac{1}{2}\delta\bar{x}J_x^4\delta\bar{x}^\top$$ (7.20)

where

$$J_x^{3\top} = \begin{bmatrix} 0_{7 \times 4} & C_3 & 0_{7 \times 2} \end{bmatrix},$$

$$J_x^{4\top} = \begin{bmatrix} 0_{7 \times 4} & C_4 & 0_{7 \times 2} \end{bmatrix}$$

and

$$C_3^{\top} = \begin{bmatrix} 0 & \hat{\omega}_y & 0 & \alpha^b\hat{v}_{nb} & v_1 & 0 \end{bmatrix},$$

$$C_4^{\top} = \begin{bmatrix} 0 & 0 & \hat{\omega}_y & \beta^b\hat{v}_{nb} & v_2 & 0 \end{bmatrix}$$

with

$$\alpha = \frac{1}{2} \begin{bmatrix} -\sin \psi^h \hat{\omega}_y + \cos \psi & -\cos \psi^h \hat{\omega}_y - \sin \psi \end{bmatrix},$$

$$bm\beta = \frac{1}{2} \begin{bmatrix} \cos \psi^h \hat{\omega}_y + \sin \psi & -\sin \psi^h \hat{\omega}_y + \cos \psi \end{bmatrix}$$

$$n\hat{v}_{nb}^\top = \begin{bmatrix} v_1 & v_2 \end{bmatrix}.$$  

Similarly nonlinear effects of states on the $j^{th}$ measurement is denoted by $T_y^j = \frac{1}{2}\delta^n p_{nb}^\top C_y^n\delta p_{nb}$, where $C_y = \frac{I}{\rho} - \frac{(n p_{nb} - n p_j)(n p_{nb} - n p_j)^\top}{\rho^2}$ with $\rho = \|n p_{nb} - n p_j\|^2$. Note that in real world scenarios $\rho \approx 20 \times 10^6$ m, hence effects of $T_y^j$ are at the $10^{-2}$ m level.
even if error in position $n\delta p_{nb} \approx 100m$. Hence (7.4) is satisfied with $\mu = 1$ even for large errors in $n\delta p_{nb}$.

For a motion scenario defined by the raw IMU measurements in Fig. 7.3, the absolute value of $\text{trace}(J_3^3 P)$ and $\text{trace}(J_4^4 P)$ (black dots) along with $Q_3$ and $Q_4$ (red dots) are depicted in Fig. 7.4, where $Q_3$ ($Q_4$) is the third(fourth) diagonal term in

$$Q_d = \int_0^t \Phi G Q G^\top \Phi^\top d\tau.$$  

The EKF propagation of the error covariance matrix $P$ takes at least 30 s to decrease to the point where (7.4) is satisfied. This is attributed to the fact that as the rover begins to move, errors in attitude, biases become observable and are corrected by the EKF resulting in smaller values of $\text{diag}(P)$. Note that this does not imply that errors in those states are small, because the EKF uses an invalid linearized error model. Convergence of the EKF will depend on the magnitude of the initial attitude error. Fig. 7.5 shows the corresponding measurement residuals (blue dots) along with the residual covariance (red dots). It can be seen that attitude errors cause large measurement residuals, easily detected by (7.7), when the rover begins to move at $t = 10$ s.

### 7.3.7.2 Initialization of $\zeta_{nl}$

In the previous section it was determined that for a 2D GPS-INS, $\zeta_{nl} = \psi$. This section presents results of initializing $\zeta_{nl}$ by finding the MAP estimate via nonlinear optimization. First we generated $N_p$ candidates of the state $\tilde{x}^i$, $0 \leq i \leq N_p$ such that they differ only along the $\psi$ direction. Assuming that $\psi \sim U[-\pi, \pi]$, the attitudes of the candidates were deterministically chosen at equal intervals in that range. The candidates were integrated through $[t_1, t_2]$ and the GPS measurements residuals generated by the $i^{th}$ candidate was
computed as $\delta Y_i^\top = \begin{bmatrix} \delta y_i^\top (t_1) & \ldots & \delta y_i^\top (t_1 + j\Delta T) & \ldots & \delta y_i^\top (t_2) \end{bmatrix}$ where $1 \leq j \leq n$ is the index on discrete time GPS measurements. We assign weights inversely proportional to the square-norm of the measurement residuals, i.e. the candidates are weighted according to $w_i = (\delta Y_i^\top \delta Y_i)^{-1}$, and the top two candidates with maximum weights are selected (say $\psi_m, \psi_n$). In the next iteration, $N_p$ candidates are chosen with attitudes deterministically chosen between $\psi_m$ and $\psi_n$. This iterative procedure continues till $|\psi_m - \psi_n| < \epsilon$, where $\epsilon$ is a chosen threshold (in this case $\epsilon = 3$ deg) selected to guarantee accuracy of the linearized
Fig. 7.5: GPS residuals before the NRT initialization procedure.

error model.

Fig. 7.6 shows the normalized weights plotted against the attitudes of candidates during various iterations. The black dots depict weights of candidates whose attitudes are deterministically selected. The red line indicates the true initial attitude (45 deg). In this case, the weighted average $\psi_c$, after coarse initialization is computed as 42.1 deg. In our simulations, the attitude converged in just 3 iterations when $N_p = 5$.

Fig. 7.7 depicts the measurement residuals (blue dots) along with its $\pm 1\sigma$ standard
deviation (red dots) derived by re-integrating the state after NRT initialization. The proposed approach work even for yaw errors as high as 180 deg, whereas an EKF may quickly become unstable with such large attitude errors.
Figure 7.7: GPS residuals after the NRT initialization procedure.
Chapter 8

Conclusion

This dissertation discussed some theoretical and implementation aspects of INS aided with GPS, vision and stationary updates. As stated in the introduction, main contributions of this dissertation to the body of literature are:

1. As outlined in Section 4.3, the observability properties of a GPS-INS are well known. It is interesting to know how the unobservable subspaces change when a vision sensor is integrated into GPS-INS. This question was addressed in Chapter 5.

2. Detection of stationary condition of the rover is a challenge worthy of careful consideration. Several detection algorithms have been proposed in literature with varying degrees of success. In Chapter 6, we formulated a frequency domain based stationary detection approach. Advantages of the proposed method over other methods were discussed in Section 6.7.

3. Rigorous observability analysis is required to understand the theoretical limits of an aiding technique. This is unavailable for a stationary updates aided INS. Section 6.5
proved several propositions on this subject.

4. NRT estimation strategy was defined and motivated in Sections 7.1 and 7.2. This strategy improves robustness of INS to errors in aiding sensor estimates. Sections 7.3 contained an exposition of the application of NRT processing for the well known initialization problem in INS.

8.1 Publications

The following articles were published as a result of the research described in this dissertation:

8.1.1 Journal publications


8.1.2 Conference publications


8.2 Future Work

Following problems are interesting in the area of NRT processing:

1. AutonomouS underwater navigation using Long Base Line (LBL) transceivers:
Assume that there are \( N_T \) transceivers located at known positions under water. The location of the \( k^{th} \) transceiver is denoted as \( n_{p_{nt_k}}, \ 1 \leq k \leq N_T \) and is depicted in Fig. 8.1 as a black pentagram. The symbol \( x(t) \) denotes the state of the rover as a function of time \( t \). The solid blue curve in Fig. 8.1 indicates the rover trajectory.

The rover state generating the pulse is indicated by the black square and the corresponding paths to the transponders are marked by straight solid magenta lines. The rover states receiving the pulses are marked by red squares and the return paths are marked by straight dashed magenta lines.

The rover transmits a query ping at some time \( t_0 \geq 0 \)s. The transceiver detects the query ping, waits for a known period of time \( t_p \) and sends a reply ping. The sensor at the rover detects the reply ping from the \( k^{th} \) transceiver at time \( t_k > t_0 \)s and scales it with the speed of sound in water. The measurement is modeled as

\[
\tilde{r}(t_k) = r(t_k) + n_r(t_k)
\]  

(8.1)

where \( r(t_k) = ||n_{p_{nt_k}} - n_{p_b(t_0)}|| + ||n_{p_{nt_k}} - n_{p_b(t_k)}|| \). The symbol \( n_r \) denotes the additive measurement noise. The measurement process is depicted in Fig. 8.1. Note that the LBL measurement at time \( t \) is a function of the rover state at time \( t \) and \( t_0 \). Since the conventional EKF algorithm propagates the probability density of \( x(t) \), the LBL measurement at \( t \) can only be used by ignoring the correlation between the states at times \( t_0 \) and \( t \). This approach is theoretically incorrect and might lead to severe degradation in localization performance. Instead, if the EKF can be modified such that it propagates the joint probability density of \( x(t_0) \) and \( x(t) \) then residual
measurements at time $t$ could be modeled as

$$\delta r(t_k) = H_l(t_0)\delta x(t_0) + H_l(t_k)\delta x(t_k) + n_r(t_k)$$

where $H_l(t_0) = \frac{\partial r}{\partial x(t_0)}$, $H_l(t_k) = \frac{\partial r}{\partial x(t_k)}$. This is an example of NRT estimation.

2. Improving Robustness of Aided Inertial Navigation Systems with NRT Processing: The expected performance (e.g., availability, accuracy, observability of errors) of an INS integrated with GPS or feature based sensors have been extensively researched over the last couple of decades. One of the primary challenges of such integrated approaches is validation of aiding measurements. Consider the following
examples of errors in aiding sensor measurement processing: (i) Errors in phase measurement due to incorrect integer ambiguity estimates. (ii) Errors in feature position measurement on an image due to incorrect data association. Integer ambiguity resolution and feature data association are usually assumed to have been performed before the state estimation process and hence such errors are not accounted for in the stochastic error models. Metrics like Mahalanobis distance can be used to detect such errors in aiding measurements. Once such measurement residuals are identified (say at time $t_0$) they are usually discarded resulting in a waste of potentially useful information. But on the other hand, using erroneous residuals that are large (in a stochastic sense) from an otherwise highly accurate measurement sensor (i.e., low measurement noise covariance) can severely deteriorate performance as they can potentially introduce errors in sensor calibration estimates.

Corruption of measurement by errors described above have a special characteristic that could possibly be exploited. In the case of integer ambiguities in phase measurements, as long as the receiver maintains phase lock with the satellite signal, the integer ambiguity is a fixed constant. Similarly, in the case of feature based aiding, if the feature were viewed from different camera poses, it might be possible to perform better data association. In both of these cases, there is a possibility in a future time (say $t_1$, $t_1 > t_0$), that these unmodeled errors are resolved. Hence due their special characteristic, we have the capability to derive useful information of a past state. But even if such ambiguities are resolved at time $t_1$, under the conventional EKF estimation framework, one cannot use the information from time $t_0$, to correct the state at time $t_1$ (assuming that some other measurement updates have been applied.
between \( t_0 \) and \( t_1 \) as it violates the white noise requirement of the EKF.

Instead, for all \( t > t_0 \), if we propagate the joint probability density of the states at times \( t \) and \( t_0 \), when ambiguities in measurements are resolved at time \( t = t_1 \), we can update the states with measurement at time \( t_0 \) by maximizing the joint *a-posteriori* probability density, resulting in the MAP estimate of the states at \( t_0 \) and \( t_1 \). Since the states at time \( t_0 \) and \( t_1 \) are correlated, we automatically obtain an improved estimate of the state at time \( t_1 \). In effect, we have used a measurement from the past to improve the state estimate without violating any of the EKF assumptions.
Bibliography


Appendix A

Frames, Notations, Definitions & Constants

A.1 Frames

Let the symbols $b$, $c$, $n$, $e$ and $i$ in either the superscript or subscript denote the body, camera, navigation, Earth Centered Earth Fixed (ECEF) and inertial frames respectively. The ECEF and inertial frames are described in detail in Section 2.2 in [27]. The navigation frame $n$, is a constant tangent plane with known position and orientation relative to the ECEF frame. The body frame $b$ is defined to be coincident with the Inertial Measurement Unit (IMU) sensor frame. Hence all inertial measurements are represented in the body frame.
A.2 Notations

All matrices and vectors are represented in boldface notation. All scalars are represented in non-boldface notation. The symbol $x_{yz}$ is used to express the vector from points $y$ to $z$ represented in the $x$ frame. Let $a$ and $b$ frames be such that their origins coincide and their relative orientation be known. If $^a v$ denotes a vector $v$ in the $a$ frame, then the rotation matrix $^b a R$ is used to represent it in the $b$ frame as $^b v = ^b a R^a v$. This rotation can be equivalently represented in terms of the Quaternion $^b a q \in S^3$, where $S^3$ represents the unit circle in 4D Euclidean space (See Appendix D in [27]). There exists a well defined invertible function that maps a quaternion to it’s direction cosine matrix (See eqns. (D.13) and (D.15) on p.504 in [27]). In context of this equivalence, the rotation matrix $^b a R$ and the corresponding quaternion $^b a q$ are used interchangeably in this dissertation. The symbol $^c \dot{x}$ denotes the time derivative of $^c x$ vector in the $c$ frame. The symbol $^c \omega_{ab}$ denotes the angular rate of rotation of frame $b$ relative to frame $a$ as represented in the $c$ frame. The symbol $^c a_{ab}$ denotes the acceleration of frame $b$ relative to frame $a$ represented in the $c$ frame. For any vector $v$, the symbol $[v \times]$ denotes the matrix cross product form for $v$ such that $v \times u = [v \times]u$. The symbols $\hat{x}$, $\tilde{x}$ denote an estimate and measurement of $x$ respectively. The estimation error in $\hat{x}$ is computed as $\delta x = x - \hat{x}$. The vector $^a \rho$ parameterizes the small angle rotation from the computed to the actual $a$ frame such that

$$^b a R = ^b a \hat{R} (I - [^a \rho \times])$$

(A.1)

for any arbitrary known frame $b$. Given matrices $A$ and $B$ of same dimensions, the equivalence relation $A \simeq B$ holds if and only if there exists an invertible matrix $C$ such that $B = CA$. 

142
A.3 Definitions

Given random variables $a, b \in \mathbb{R}$, the symbol $P(a \leq b)$ denotes the probability of the event $a \leq b$ and the symbol $p_a(a)$ denotes the marginal probability density of $a$. The subscript in the probability density function is ignored if the context is clear. For a given estimator, define a loss function $L$ as

$$L(\hat{x}) = ||x - \hat{x}||^2.$$ 

The estimator $\hat{x}$ is defined to be optimal if $\hat{x} = \arg \min_{x} L(x)$ (i.e. if it minimizes the Mean Square Error).

A.4 Definition of roving maneuvers

Since we are concerned with a land vehicle based INS in this dissertation, the linear error state dynamic system of interest is of the form (3.15). Determining observability conditions for an aided INS undertaking an arbitrary trajectory is non-trivial. Hence, we analyze observability only for certain special maneuvers often encountered in real world, land vehicle based INS applications. Assume that the maneuver is executed in the time interval $U = [0, T]$ over which the inertial measurements $^b f$ and $^b \omega_{ib}$ are defined. Let $M \subseteq U$ be the times when the rover is in motion and $S = U \setminus M$ be the time when the rover is stationary. The vector $^b d^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ denotes the forward axis of a land vehicle (e.g., a car) in the body frame. Since the rover is a non-holonomic system it can accelerate only along the vehicle forward axis ($^b d$) as long as there is no wheel slip. We are interested in analyzing the observability of the state $\delta x(0)$ with measurements $\delta y(\tau), 0 \leq \tau \leq T$. The definitions of maneuvers of interest are as follows:
A.4.1 Maneuver 1

This is defined as the scenario where the rover is stationary for all $\tau \in U$; hence, $S = U$ and $M = \emptyset$. The specific force in the navigation frame is given by $^{n}f = ^{n}g$ and the rotation $^{n}bR$ is time invariant for all $\tau \in U$.

A.4.2 Maneuver 2

This is defined as the scenario where the rover accelerates for some part of $U$ and is at rest state for some other part of $U$. This can be further divided into two sub-scenarios:

A.4.2.1 Maneuver 2.a.

Let $t_1$ and $t_2$ be such that $0 < t_1 < t_2 = T$. Let $S = [0, t_1)$ and $M = [t_1, t_2]$. Assume that the rover is at rest for all $\tau \in S$, undergoes constant acceleration without rotation along $^{b}d$ for all $\tau \in M$. The specific force in the navigation frame is given by

$$^{n}f = \begin{cases} \kappa(\tau) ^{n}bR ^{b}d + ^{n}g & \tau \in M \\ ^{n}g & \tau \in S \end{cases}$$

(A.2)

such that the vectors $^{n}f(0)$ and $^{n}f(\tau)$ are not collinear for each $\tau \in M$ and $\kappa$ is a smooth real valued function whose support lies in $M$.

A.4.2.2 Maneuver 2.b.

Let the rover be in motion in $M = [0, t_1)$, where $t_1 > 0$ is known. Hence $S = U \setminus M = [t_1, T]$. The specific force in the navigation frame is given by

$$^{n}f = \begin{cases} \kappa(\tau) ^{n}bR ^{b}d + ^{n}g & \tau \in M \\ ^{n}g & \tau \in S \end{cases}$$

(A.3)
where $\kappa$ is a smooth real valued function whose support lies in $M$. The rotation $^b_R$ is time invariant in $S$ and possibly time varying in $M$.

### A.4.3 Maneuver 3

This is defined as the scenario where the rover accelerates from a rest state and then decelerates to a stop along a straight line without rotation. Assume that this motion occurs during three intervals $S = [0, t_1)$, $M_1 = [t_1, t_2]$ and $M_2 = (t_2, T]$ such that $0 < t_1 < t_2 < T$. Assume that the rover is at rest for all $\tau \in S$, undergoes constant acceleration $\kappa_1(\tau)^n d$ for all $\tau \in M_1$ and finally undergoes constant acceleration $\kappa_2(\tau)^n d$ for all $\tau \in M_2$ where $\kappa_1$ and $\kappa_2$ are smooth functions whose support lie in $M_1$ and $M_2$ respectively. Further there exist a $\tau_1 \in M_1$ and $\tau_2 \in M_2$ such that $\kappa(\tau_1) \neq 0$, $\kappa(\tau_2) \neq 0$ and $\kappa_1(\tau_1) \neq \kappa_2(\tau_2)$.

### A.4.4 Maneuver 4

This is defined as the scenario where the rover starts and ends in stationarity. Let $t_1, t_2$ be such that $0 < t_1 < t_2 < T$. Assume that the rover was stationary during $S_1 = [0, t_1]$, $S_2 = [t_2, T]$ and in motion during $M = U \setminus (S_1 \cup S_2)$. The specific force in the navigation frame is given by

$$
^n f = \begin{cases} 
  \kappa(\tau)^n b_R^b d + ^n g & \tau \in M \\
  ^n g & \tau \in S_1 \cup S_2
\end{cases}.
$$

Depending on the rotation of the rover during $M$, this scenario can be divided in two sub-scenarios as described:

#### A.4.4.1 Maneuver 4.a.

If there is no rotation for all $\tau \in M$ then $^b_R$ is time invariant for all $\tau \in U$. 

145
A.4.4.2 Maneuver 4.b.

If the rover undergoes rotation during $M$ then $\text{^n}_b \mathbf{R}$ is time varying and its behavior is governed by

$$\text{^n}_b \dot{\mathbf{R}} = \text{^n}_b \mathbf{R} \left[ \text{^b}_n \omega_{nb} \times \right].$$

Usually the vehicle can turn only along one axis (i.e. yaw rate), hence $\text{^b}_n \omega_{nb}$ is along a single direction.

A.5 Constants

The following values are used for constants in this dissertation:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Units</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g$</td>
<td>9.78</td>
<td>m/s/s</td>
<td>Magnitude of gravity at the equator</td>
</tr>
<tr>
<td>$c$</td>
<td>$2.99792458 \times 10^8$</td>
<td>m/s</td>
<td>Speed of light</td>
</tr>
<tr>
<td>$f_1$</td>
<td>$1575.42 \times 10^6$</td>
<td>Hz</td>
<td>GPS L1 frequency</td>
</tr>
<tr>
<td>$f_2$</td>
<td>$1227.60 \times 10^6$</td>
<td>Hz</td>
<td>GPS L2 frequency</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>$\frac{c}{f_1} \approx 19.0 \times 10^{-2}$</td>
<td>m</td>
<td>GPS L1 wavelength</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>$\frac{c}{f_2} \approx 24.4 \times 10^{-2}$</td>
<td>m</td>
<td>GPS L2 wavelength</td>
</tr>
<tr>
<td>$\omega_{ie}$</td>
<td>$7.2921151467 \times 10^{-5}$</td>
<td>rad/s</td>
<td>Magnitude of Earth’s rotation rate</td>
</tr>
</tbody>
</table>

Table A.1: Symbols and values for constants

A.6 List of abbreviations & acronyms

The following list abbreviations and acronyms are often used in this dissertation:
<table>
<thead>
<tr>
<th>Acronym (or) Abbreviation</th>
<th>Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>CDGPS</td>
<td>Carrier phase Differential GPS</td>
</tr>
<tr>
<td>DOF</td>
<td>Degree(s) Of Freedom</td>
</tr>
<tr>
<td>EKF</td>
<td>Extended Kalman Filter</td>
</tr>
<tr>
<td>GPS</td>
<td>Global Positioning System</td>
</tr>
<tr>
<td>IMU</td>
<td>Inertial Measurement Unit</td>
</tr>
<tr>
<td>INS</td>
<td>Inertial Navigation Systems</td>
</tr>
<tr>
<td>MAP</td>
<td>Maximum-A-Posteriori</td>
</tr>
<tr>
<td>NRT</td>
<td>Near Real Time</td>
</tr>
</tbody>
</table>

Table A.2: Abbreviations and acronyms
Appendix B

Observability analysis of Linear systems

B.1 Continuous time observability gramian

Consider the following linear state propagation and measurement model

\[ \delta \dot{x}(t) = A(t) \delta x(t) \]  \hspace{1cm} (B.1)
\[ \delta y(t) = H(t) \delta x(t) \]  \hspace{1cm} (B.2)

where \( \delta x \in \mathbb{R}^{N \times 1} \), \( A \in \mathbb{R}^{N \times N} \), \( \delta y \in \mathbb{R}^{M \times 1} \) and \( H \in \mathbb{R}^{M \times N} \). Let \( \Phi(t, \tau) \in \mathbb{R}^{N \times N} \) denote the state transition matrix corresponding (B.1), such that \( \delta x(t) = \Phi(t, \tau) \delta x(\tau) \). The state at time \( t_0 \), denoted by \( \delta x(t_0) \) is observable from measurements \( \delta y(t) = H(t) \delta x(t) \), \( t \geq t_0 \), if and only if the observability gramian, given by

\[ O = \int_{t_0}^{t} \Phi(t, \tau)^T H^T H \Phi(t, \tau) d\tau \]  \hspace{1cm} (B.3)
is non-singular [42]. Further (B.3) is non-singular if and only if

\[ H\Phi(t,\tau)x(\tau) = 0 \Rightarrow x(\tau) = 0 \]  \hspace{1cm} (B.4)

for all \( \tau \in [t_0, t] \)

**Proposition 18** A vector \( x \in \text{null}(H\Phi(t,\tau)) \) for all \( \tau \in (t_0, t) \) if and only if \( x \in \text{null}(O) \).

**Proof.** \((\Rightarrow)\) Assume that there exists a vector \( x \) such that for all \( \tau \in (t_0, t) \), \( x \in \text{null}(H\Phi(t,\tau)) \). Then

\[ Ox = \int_{t_0}^{t} \Phi(t,\tau)^\top H^\top H\Phi(t,\tau)x d\tau = 0. \]

Hence \( x \in \text{null}(O) \).

\((\Leftarrow)\) On the other hand, if \( x \in \text{null}(O) \), then \( x^\top Ox = 0 \). We can rewrite \( x^\top Ox \) as

\[
x^\top Ox = \int_{t_0}^{t} x^\top \Phi(t,\tau)^\top H^\top H\Phi(t,\tau)x d\tau
= \int_{t_0}^{t} ||H\Phi(t,\tau)x||^2 d\tau
= 0.
\]

From (B.5) we conclude \( H\Phi(t,\tau)x = 0 \) for all \( \tau \in D \), where \( D \) is dense subset of \( (t_0, t) \).

But since \( H\Phi x \) is continuous in \( t \) its behavior on \( (t_0, t) \) is fully described by its behavior on \( D \). Therefore \( H\Phi(t,\tau)x = 0 \) for all \( \tau \in (t_0, t) \). Thus \( x \in \text{null}(H\Phi(t,\tau)) \) for all \( \tau \in (t_0, t) \).

**B.2 Discrete time observability gramian**

Equivalently, the continuous time models can be discretized by partitioning the time \([0, T]\) as \( t_0 = 0, t_1 = \Delta_T, \ldots, t_i = i\Delta_T, \ldots t_M = M\Delta_T = T \) at regular intervals \( \Delta_T \) indexed.
as $0 \leq i \leq M$. For each $0 \leq i \leq M$, the measurement model has the form

$$\delta y_i = H_i \delta x_i.$$  

(B.6)

Assume that over the $i^{th}$ time interval, the evolution of $\delta x$ as described by the state transformation matrix $\Phi(t_{i+1}, t_i)$ such that

$$\delta x_{i+1} = \Phi(t_{i+1}, t_i) \delta x_i.$$  

(B.7)

The state $\delta x_0$ is observable if and only if there exists a finite $M$ such that $\delta x_0$ can be uniquely determined using measurements $\delta y_i$ for $i = 0, \ldots, M$. Let $\delta Y^\top = \begin{bmatrix} \delta y_0^\top & \ldots & \delta y_M^\top \end{bmatrix}$, then

$$\delta Y = \mathcal{O} \delta x_0$$  

(B.8)

where $\mathcal{O}^\top = \begin{bmatrix} \mathcal{O}_0^\top & \ldots & \mathcal{O}_M^\top \end{bmatrix}$ denotes the observability matrix. The observability matrix is defined as

$$\mathcal{O}_k = \begin{cases} H_0 & k = 0 \\ H_k \prod_{i=0}^{k-1} \Phi(t_{i+1}, t_i) & 0 < k \leq M \end{cases}$$  

(B.9)

where $\prod_{i=0}^{k-1} \Phi(t_{i+1}, t_i) = \Phi(t_k, t_{k-1}) \Phi(t_{k-1}, t_{k-2}) \ldots \Phi(t_1, t_0)$. The vector $\delta x_0$ is an unique, non-zero solution to (B.8) if and only if $\mathcal{O}$ has full column rank [15].
Appendix C

INS error state dynamics

C.1 Linearized error state dynamic equations

This section derives the linearized error state model (3.9 – 3.13) from (3.3 – 3.6). Using (3.4), it can be easily seen that

\[ n\delta \dot{p}_{nb} = n\dot{p}_{nb} - n\dot{\hat{p}}_{nb} = n\dot{v}_{nb} - n\dot{\hat{v}}_{nb} = n\delta v_{nb} \]

which corresponds to (3.9). Using (A.1) we derive

\[ b_n R = b_n \dot{R} (I - [^n \rho \times ]) \]  \hspace{1cm} (C.1)

Differentiating (C.1) we derive,

\[ b_n \dot{R} = b_n \dot{\dot{R}} (I - [^n \rho \times ]) - b_n \dot{R} [^n \dot{\rho} \times ] \] \hspace{1cm} (C.2)

Substituting \( b_n \dot{R} = b_n R [^n \omega_{bn} \times ] \) into (C.2), after simplification we derive

\[ n\dot{\rho} = -b_n R^b \delta \omega_{nb} \] \hspace{1cm} (C.3)
Noting that $b\hat{\delta}\omega_{nb} = b\hat{\delta}\omega_{ni} + b\hat{\delta}\omega_{ib}$, $b\hat{\delta}\omega_{ni} = 0$ and
\[b\hat{\delta}\omega_{ib} = -b\hat{\delta}b - \omega_g\] (C.4)
we can derive (3.11) by substituting (C.4) into (C.3). Further from (3.10) we derive
\[n\hat{\delta}v_{nb} = n\hat{\delta}p_{nb}(t_0) + \int_{t_0}^{t} n\hat{\delta}v_{nb} d\tau\] (C.7)
\[n\hat{\delta}v_{nb}(t) = n\hat{\delta}v_{nb}(t_0) + \int_{t_0}^{t} \left[ n\hat{\delta}b_{a} + [n\hat{\omega}_a \times n\hat{\delta}v_{nb}] \right] \] (C.8)
\[n\hat{\rho}(t) = n\hat{\rho}(t_0) + \int_{t_0}^{t} n\hat{\rho} \delta b_{g} d\tau\] (C.9)
\[b\hat{\delta}b_{a}(t) = b\hat{\delta}b_{a}(t_0)\] (C.10)
\[b\hat{\delta}b_{g}(t) = b\hat{\delta}b_{g}(t_0)\] (C.11)
Substituting (C.11) into (C.9) we derive
\[n\hat{\rho}(t) = n\hat{\rho}(t_0) + \sum_{i} b\hat{\delta}b_{g}(t_0)\] (C.12)
\[\text{Noting that } n\omega_{ie} = e\hat{\omega}_e \hat{\omega}_e \text{ and } e\hat{\omega}_e = 7.2921151467 \times 10^{-5} \text{ rad/s, the last term is ignored.} \]
Hence (3.10) is derived from (C.6). Eqns. (3.12 – 3.13) can be immediately derived from the sensor bias models in (3.3).

C.2 Integration of ideal linearized error state equations

Ignoring the noise terms in (3.9 – 3.13) we derive
\[n\hat{\delta}p_{nb} = n\hat{\delta}p_{nb}(t_0) + \int_{t_0}^{t} n\hat{\delta}v_{nb} d\tau\] (C.7)
\[n\hat{\delta}v_{nb}(t) = n\hat{\delta}v_{nb}(t_0) + \int_{t_0}^{t} \left[ n\hat{\delta}b_{a} + [n\hat{\omega}_a \times n\hat{\delta}v_{nb}] \right] \] (C.8)
\[n\hat{\rho}(t) = n\hat{\rho}(t_0) + \int_{t_0}^{t} n\hat{\rho} \delta b_{g} d\tau\] (C.9)
\[b\hat{\delta}b_{a}(t) = b\hat{\delta}b_{a}(t_0)\] (C.10)
\[b\hat{\delta}b_{g}(t) = b\hat{\delta}b_{g}(t_0)\] (C.11)
Substituting (C.11) into (C.9) we derive
\[n\hat{\rho}(t) = n\hat{\rho}(t_0) + \sum_{i} b\hat{\delta}b_{g}(t_0)\] (C.12)
where $\mathcal{R}_t = \int_{t_0}^{t} nR d\tau$. Substituting (C.10) and (C.12) into (C.8) we derive

$$n \delta v_{n,b}(t) = n \delta v_{n,b}(t_0) - \mathcal{S}_t n \rho(t_0) + \mathcal{M}_t \delta b_g(t_0) - \mathcal{R}_t \delta b_a(t_0)$$  \hspace{1cm} (C.13)

where $\mathcal{S}_t = \int_{t_0}^{t} n f(\tau) \times d\tau, \mathcal{M}_t = - \int_{t_0}^{t} n f(s) \times \mathcal{R}_s ds$. Substituting (C.13) into (C.7) to derive

$$n \delta p_{n,b}(t) = n \delta p_{n,b}(t_0) + (t - t_0) n \delta v_{n,b}(t_0) + \mathcal{P}_t n \rho(t_0) + \mathcal{T}_t \delta b_g(t_0) - \mathcal{Q}_t \delta b_a(t_0).$$  \hspace{1cm} (C.14)

Writing (C.10 − C.14) in matrix form we derive (3.17).
Appendix D

Proofs in chapter 4

D.1 Proof of Proposition 2

**Proof.** If the rover is undertaking Maneuver 1 (See A.4.1) then $b^nR(t_k) = b^nR(t_0)$, $n^f(t_k) = n^f(t_0)$ for all $0 \leq k \leq M$. Substituting this into (4.22), it is easy to show that the unobservable subspace is spanned by

\[
\begin{bmatrix}
0 & u \\
b^nR(t_0)^n f(t_0) & 0 \\
0 & b^nR(t_0)^n [n^f(t_0) \times] u
\end{bmatrix}
\]

where $u = \{e_1, e_2, e_3\}$ with $e_i$ being the $i^{th}$ standard basis for $\mathbb{R}^3$. ■
D.2 Proof of Proposition 3

**Proof.** The observability matrix in (4.22) can be reduced to the following

\[
O^*_g \simeq \begin{bmatrix}
- [^n f(t_0) \times] \Delta^2_T & 0 & \frac{b}{b} R(t_0) \Delta^2_T \\
- [^n d \times] \Delta^2_T & 0 & 0 \\
0 & - [^n f(t_1) \times] \frac{n}{b} R(t_0) \Delta^3_T & 0
\end{bmatrix}
\]

by linear row operations. It can be seen that \( \text{rank}(O^*_g) = 7 \). It can be easily shown that the vectors

\[
\begin{bmatrix}
0 \\
[^n d] \\
\frac{b}{b} R(t_0) ^n f(t_1) \\
0
\end{bmatrix} = \begin{bmatrix}
[^n d] \\
\frac{b}{b} R(t_0) [^n f(t_0) \times] ^n d
\end{bmatrix}
\]

span the 2D unobservable subspace. ■

D.3 Proof of Proposition 4

**Proof.** If the rover is undertaking Maneuver 3 (See A.4.3), the observability matrix in (4.22) can be reduced to

\[
O^*_g \simeq \begin{bmatrix}
- [^n f(t_0) \times] \Delta^2_T & 0 & \frac{n}{b} R(t_0) \Delta^2_T \\
- [^n d \times] \Delta^2_T & 0 & 0 \\
0 & - [^n f(t_1) \times] \frac{n}{b} R(t_0) \Delta^3_T & 0 \\
0 & - [^n d \times] \Delta T^2 & 0 \\
0 & - [^n f(t_2) \times] \frac{n}{b} R(t_0) \Delta^3_T & 0
\end{bmatrix}
\]

It can be shown that \( \text{rank}(O^*_g) = 9 \). Thus we have full state observability. ■
Appendix E

Analysis and proofs in chapter 5

E.1 Camera measurement analysis

To analyze observability of states from equations of the form (5.5) consider the measurement vector given by (5.10). It can be rewritten as

\[ \begin{bmatrix}
    c^q f_0 \\
    \vdots \\
    c^q f_{N_0}
\end{bmatrix} = f
\begin{bmatrix}
    \pi \circ c^p cf_0 \\
    \vdots \\
    \pi \circ c^p cf_{N_0}
\end{bmatrix} \] (E.1)

where \( \pi : B \rightarrow \mathbb{R}^2 \) is the projection defined for each \( x = (x_1, x_2, x_3) \in B \) as

\[ \pi(x) = \frac{1}{x_3} \begin{bmatrix}
    x_1 \\
    x_2
\end{bmatrix} \] (E.2)

with the domain \( B = \{ x \in \mathbb{R}^3 | x_3 > 0 \} \). For each feature \( j \), the vector \( c^p cf_j \) can be viewed as a vector of real analytic function \( c^p cf_j : \mathbb{R}^3 \times \mathbb{S}^3 \rightarrow B \) in \( (n^p nc, c^q) \) such that

\[ c^p cf_j (n^p nc, c^q) = c_n R (c^q) \left( n^p_n f_j - n^p nc \right) \] (E.3)
where \( \hat{c}_n \mathbf{R}(\hat{c}_n q) \) denotes the direction cosine matrix representation of the quaternion \( \hat{c}_n q \).

Expanding (E.3) in Taylor series to first order, we derive

\[
\hat{c}_p c_{cfj}(n_p n_f, c_{nc}) \approx \hat{c}_p c_{cfj}(\hat{n}_p n_f, \hat{c}_n q) + \hat{c}_h j \delta \alpha \quad (E.4)
\]

where

\[
\hat{c}_h j = \left[ -\frac{\hat{c}_n \mathbf{R}(\hat{c}_n q)}{n} \right] \begin{bmatrix} \hat{n}_p n_f - \hat{n}_p n_c \end{bmatrix} \quad (E.5)
\]

and \( \delta \alpha = \left[ \delta \mathbf{p}^\top n_{nc} \mathbf{p}_c \right] \). The symbol \( n_{nc} \mathbf{p}_c \) represents the error in \( \hat{c}_n q \) in terms of a small angle rotation such that

\[
\hat{c}_n \mathbf{R}(\hat{c}_n q) = \frac{c}{n} \mathbf{R}(\hat{c}_n q) (\mathbf{I} - [n_{nc} \times]) \quad (E.6)
\]

From (E.1) and (E.4) we derive

\[
\hat{c}_y \approx \hat{c}_y + \Pi \hat{c}_H \delta \alpha 
\]

where \( \Pi = \mathbf{bd} \left( \mathbf{0}_Y, \mathbf{1}_Y, \ldots, \mathbf{N}_0 \mathbf{Y} \right) \) and \( \hat{c}_H = \left[ \hat{c}_h^\top_0 \ldots \hat{c}_h^\top_{N_0} \right] \). The following is a known result in literature.

**Proposition 19** [65] Given image plane feature locations \( \hat{c}_y \) and feature locations in navigation frame \( n_p n_f \) for \( j = 0, 1, \ldots, N_0 \), with \( N_0 \geq 3 \) such that \( \{n_p n_{f_0}, n_p n_{f_1}, \ldots, n_p n_{f_{N_0}}, n_p n_{nc}\} \) are not coplanar, then there exists a unique \( (n_p n_{nc}, \hat{c}_n q) \neq 0 \) such that (E.1) is satisfied.

The following proposition will be used in subsequent analysis.

**Proposition 20** The matrix \( \Pi \hat{c}_H \) has full column rank if \( N_0 \geq 3 \) and the set of points \( \{n_p n_{f_0}, n_p n_{f_1}, \ldots, n_p n_{f_{N_0}}, n_p n_{nc}\} \) are not coplanar.

**Proof.** If the conditions in the statement are satisfied then by Proposition 19 there exists an unique \( (n_p n_{nc}, \hat{c}_n q) \) such that (E.1) is satisfied. If there exists a \( v^\top = \left[ v_1^\top \ v_2^\top \right] \in \mathbb{R}^6 \)
such that \( v \neq 0 \) and \( \Pi^c Hv = 0 \) then \( {}^n p_{nc} = {}^n p_{nc} + v_1 \) and \( c_q(n_c R(q_c)(I - [v_2 \times])) \) is another solution, contradicting the uniqueness guaranteed by Proposition 19. ■

### E.2 Proof of Proposition 5

**Proof.** Assuming the vision sensor is fully calibrated \( b\delta p_{bc} = b\rho = 0 \). The measurement residuals at the first three time instants are given by

\[
\begin{bmatrix}
\delta y(t_0) \\
\delta y(t_1) \\
\delta y(t_2)
\end{bmatrix} =
\begin{bmatrix}
\Pi(t_0)^c H(t_0) \\
\Pi(t_1)^c H(t_1) \Phi(t_1, t_0) \\
\Pi(t_2)^c H(t_0) \Phi(t_2, t_1) \Phi(t_1, t_0)
\end{bmatrix}
\delta \bar{x}(t_0).
\]  

(E.8)

At \( t = t_0 \), the measurement vector is given by

\[
\delta y(t_0) = \Pi(t_0)^c H(t_0) \delta \bar{x}(t_0)
\]  

(E.9)

where \( \Pi(t_0) = bd(0, \ldots, N_0 \bar{\Upsilon}(t_0)) \) and

\[
^c H(t_0) =
\begin{bmatrix}
-\gamma_q R(t_0) & 0 & \left[ ^n p_{bf_0}(t_0) \times \right] \gamma_n R(t_0) & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
-\gamma_n R(t_0) & 0 & \left[ ^n p_{bfx_0}(t_0) \times \right] \gamma_n R(t_0) & 0 & 0
\end{bmatrix}.
\]

Using Proposition 20, it can be concluded that \( rank(\Pi(t_0)^c H(t_0)) = 6 \) and hence errors in position and attitude are fully observable at time \( t = t_0 \). Let the errors in position and attitude computed at time \( t = t_0 \) be denoted as \( (^n \delta p_{ab}, ^n \rho) \). Construct a new measurement vector as

\[
\delta \bar{y}(t_1) = \delta y(t_1) - \Pi(t_1)^c H(t_1) \Phi(t_1, t_0) \nu_0
\]  

(E.10)
where $\nu_1^\top = \begin{bmatrix} n^\top \delta p_{nb}^c & 0^\top & n^\top \rho^c & 0^\top & 0^\top \end{bmatrix}$. At time $t = t_1$ the measurement vector $\delta \hat{y}$ can be expressed as

$$\delta \hat{y}(t_1) = \Pi(t_1)\mathcal{H} \delta \nu_1$$

where

$$\mathcal{H} = \begin{bmatrix} -c_n^c R(t_1) & [^c p_{bf_0}(t_1) \times]^c_n R(t_1) & 0 \\ \vdots & \vdots & \vdots \\ -c_n^c R(t_1) & [^c p_{bf_{N_0}}(t_1) \times]^c_n R(t_1) & 0 \end{bmatrix} \delta \nu_1$$

and $\delta \nu_1^\top = \begin{bmatrix} \Delta T_n^\top \delta v_{nb}^n & b^\top \delta b_g^b & n^\top R(t_1) & b^\top \delta b_a^b \end{bmatrix}$. The last column of $\mathcal{H}$ are all zeros, hence errors in accelerometer biases are unobservable at time $t = t_1$. Using Proposition 20 it can be concluded that errors in velocity and gyro biases are observable at time $t = t_1$. Assume position, velocity, attitude and gyro error states are estimated using measurements at times $t = t_0, t_1$. Using a construction similar to (E.10) we derive a new measurement $\delta \hat{y}(t_2)$ such that $\delta \hat{y}(t_2) = \Pi(t_2)\mathcal{I}^\top R_{n}^c R(t_2)_{b}^b R(t_0)^b \delta b_a^b$, where $\mathcal{I}^\top = \begin{bmatrix} I & \ldots & I \end{bmatrix}$. It can be easily shown that errors in accelerometer biases are fully observable at $t = t_2$. Hence assuming the vision sensor is fully calibrated, then the INS error state is fully observable with, $N_0 > 3$ measurements at 3 time instants.
E.3 Proof of Proposition 6

Proof. If the rover is undertaking Maneuver 1 (See A.4.1) then the observability matrix can be reduced to

\[
\mathbf{O}_c(t_k) \approx \begin{bmatrix}
I & 0 & 0 & 0 & 0 & \nabla_{b} R(t_0) & \nabla_{b} R(t_0) \left[ b_{pbc} \times \right] \\
0 & \Delta_T I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \left[ \nabla f(t_0) \times \right] \Delta_T^2 & 0 & \nabla_{b} R(t_0) \Delta_T^2 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & \nabla_{b} R(t_0) \\
0 & 0 & 0 & \nabla_{b} R(t_0) & 0 & 0 & 0
\end{bmatrix}
\] (E.11)

From rows 2 and 5 we see that errors in velocity and gyro biases are observable. Let \( D \) be a sub-matrix of (E.11) composed of rows 1, 3 and 4 such that

\[
D = \begin{bmatrix}
I & 0 & 0 & 0 & 0 & \nabla_{b} R(t_0) & \nabla_{b} R(t_0) \left[ b_{pbc} \times \right] \\
0 & \Delta_T I & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \left[ \nabla f(t_0) \times \right] \Delta_T^2 & 0 & \nabla_{b} R(t_0) \Delta_T^2 & 0 & 0 \\
0 & 0 & I & 0 & 0 & 0 & \nabla_{b} R(t_0) \\
0 & 0 & 0 & \nabla_{b} R(t_0) & 0 & 0 & 0
\end{bmatrix}
\] (E.12)

The matrix \( D \) can be reduced to reduced row echelon form as follows

\[
D \approx \begin{bmatrix}
I & 0 & 0 & 0 & \nabla_{b} R(t_0) & \nabla_{b} R(t_0) \left[ b_{pbc} \times \right] \\
0 & 0 & I & 0 & 0 & 0 & \nabla_{b} R(t_0) \\
0 & 0 & 0 & 0 & I & 0 & \Box
\end{bmatrix}
\] (E.13)

where \( \Box \) is the placeholder for the terms resulting from reduced row echelon reduction. From (E.13), we conclude that \( \text{rank}(D) = 9 \), hence \( \text{rank}(O_c) = 15 \) and the 6D unobservable
subspace is spanned by

\[
\begin{bmatrix}
  n_b R(t_0) \begin{bmatrix} p_{bc} \times \end{bmatrix} n_b R(t_0)^\top \mathbf{u} & 0 \\
  0 & 0 \\
  0 & \mathbf{u} \\
  0 & 0 \\
  0 & n_b R(t_0)^\top \begin{bmatrix} f(t_0) \times \end{bmatrix} \mathbf{u} \\
  - \begin{bmatrix} p_{bc} \times \end{bmatrix} n_b R(t_0)^\top \mathbf{u} & - \begin{bmatrix} p_{bc} \times \end{bmatrix} n_b R(t_0)^\top \mathbf{u} \\
  0 & -n_b R(t_0)^\top \mathbf{u}
\end{bmatrix}
\]

where \( \mathbf{u} = \{e_1, e_2, e_3\} \) with \( e_i \) being the \( i \)th standard basis of \( \mathbb{R}^3 \).

\( \blacksquare \)

### E.4 Proof of Proposition 7

**Proof.** If the rover is undertaking Maneuver 3 (See A.4.3), then the observability matrix can be reduced to

\[
\mathcal{O}_c \simeq \begin{bmatrix}
  I & 0 & 0 & 0 & 0 & 0 & n_b R(t_0) & n_b R(t_0) \begin{bmatrix} p_{bc} \times \end{bmatrix} \\
  0 & \Delta_T I & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & - \begin{bmatrix} f(t_0) \times \end{bmatrix} \Delta_T^2 & 0 & n_b R(t_0) \Delta_T^2 & 0 & 0 & 0 \\
  0 & 0 & - \begin{bmatrix} f(t_1) \times \end{bmatrix} \Delta_T^2 & 0 & n_b R(t_0) \Delta_T^2 & 0 & 0 & 0 \\
  0 & 0 & - \begin{bmatrix} f(t_2) \times \end{bmatrix} \Delta_T^2 & 0 & n_b R(t_0) \Delta_T^2 & 0 & 0 & 0 \\
  0 & 0 & I & 0 & 0 & 0 & n_b R(t_0) & 0 \\
  0 & 0 & 0 & n_b R(t_0) & 0 & 0 & 0 & 0
\end{bmatrix}
\]

(E.14)
It can be shown that \( \text{rank}(O_2) = 18 \) in (E.14) and the 3D unobservable subspace is spanned by

\[
\begin{bmatrix}
    u^\top & 0^\top & 0^\top & 0^\top & -u^\top R(t_0) & 0^\top
\end{bmatrix}^\top
\]

(E.15)

where \( u = \{e_1, e_2, e_3\} \) with \( e_i \) being the \( i^{th} \) standard basis of \( \mathbb{R}^3 \). Hence errors in INS position can not be resolved from errors in the lever arm.

### E.5 Proof of Proposition 8

**Proof.** If the rover is undertaking Maneuver 1 (See A.4.1), then the observability matrix can be reduced to

\[
O_{gc}(t_k) \approx \begin{bmatrix}
    I & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & I & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & n_b R(t_0) & n_b R(t_0) \times [p_{bc} \times] \\
    0 & 0 & -[n f(t_0) \times] \Delta_T^2 & 0 & n_b R(t_0) \Delta_T^2 & 0 & 0 \\
    0 & 0 & I & 0 & 0 & 0 & n_b R(t_0) \\
    0 & 0 & 0 & n_b R(t_0) & 0 & 0 & 0
\end{bmatrix}
\]

(E.16)

It can be easily shown that that \( \text{rank}(O_{gc}(t_k)) = 18 \) and the 3D unobservable subspace is spanned by

\[
\begin{bmatrix}
    0^\top & 0^\top & u^\top & u^\top [n f(t_0) \times] \Delta_T^2 & n_b R(t_0) & u^\top [p_{bc} \times] \Delta_T^2 & -u^\top n_b R(t_0)
\end{bmatrix}^\top
\]

where \( u = \{e_1, e_2, e_3\} \) with \( e_i \) being the \( i^{th} \) standard basis of \( \mathbb{R}^3 \). ■
Appendix F

Proofs in chapter 6

F.1 Proof of Proposition 11

Proof. Under the conditions of the proposition the state transition $\Phi_m$ in (6.32) can be simplified as

$$
\Phi_m = \begin{bmatrix}
I & -\tau [^n g \times] & -\frac{\tau^2}{2} [^n g \times]^n_b R & \tau^n_b R \\
0 & I & \frac{n_b R\tau}{2} & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}.
$$

(F.1)

Using (6.33) and (F.1) we derive

$$
H_m \Phi_m = \begin{bmatrix}
I & -[^n g \times]\tau & -\frac{1}{2}[^n g \times]^n_b R\tau^2 & \frac{n_b R\tau}{2} \\
0 & 0 & -I & 0 \\
0 & 0 & 0 & I
\end{bmatrix}.
$$

(F.2)
Let \( v_1^\top = \begin{bmatrix} u_{11}^\top & u_{12}^\top & u_{13}^\top & u_{14}^\top \end{bmatrix} \) span the right null space of \((F.2)\) for all \( \tau \in S \). Using \( H_m \Phi_m v_1 = 0 \) we derive

\[
\begin{align*}
    u_{13} &= 0 \quad \text{(F.3)} \\
    u_{11} - \tau \int_0^\tau [n \times u_{12}] + \tau \int_0^n Ru_{14} &= 0. \quad \text{(F.4)}
\end{align*}
\]

Substituting \( \tau = 0 \) into \((F.4)\) we derive

\[ u_{11} = 0. \quad \text{(F.5)} \]

Differentiating \((F.4)\), we derive

\[ u_{14} = \int_0^n R [n \times u_{12}] \]

\[ \quad \text{(F.6)} \]

for all \( \tau \in (0, t) \). Using \( (F.3), (F.5) \) and \( (F.6) \) we derive a basis of unobservable subspace as \((6.34)\). ■

**F.2 Proof of Proposition 12**

**Proof.** Let \( v_2^\top = \begin{bmatrix} u_{21}^\top & u_{22}^\top & u_{23}^\top & u_{24}^\top \end{bmatrix} \) span the right null space of \( H_m \Phi_m \) in \( S \). Equating \( H_m \Phi_m v_2 = 0 \) we derive

\[
\begin{align*}
    u_{23} &= 0 \quad \text{(F.7)} \\
    u_{21} - \int_0^\tau [n \times u_{22}] ds + \int_0^n Ru_{24} ds &= 0 \quad \text{(F.8)}
\end{align*}
\]

for all \( \tau \in S \). Differentiating \((F.8)\) with respect to \( \tau \) we derive

\[
-\int_0^n f(\tau) \times u_{22} + \int_0^n b(\tau) Ru_{24} = 0 \quad \text{(F.9)}
\]

for all \( \tau \in (t_1, t) \). Therefore, we can rewrite \((F.8)\) as

\[
\begin{align*}
    u_{21} - \int_0^{t_1} [n \times u_{22}] ds + \int_0^{t_1} n Ru_{24} ds &= 0. \quad \text{(F.10)}
\end{align*}
\]

164
Since \( n f(\tau) = n g \) and \( n b(\tau) R = n b(t_1) R \) for all \( \tau \in S \) we rewrite (F.9) as
\[
\mathbf{u}_{24} = n b(t_1) R [n g x] \mathbf{u}_{22}.
\] (F.11)

Substituting (F.11) and (A.3) into (F.10) we derive
\[
\mathbf{u}_{21} = m(t_1) \mathbf{u}_{22}
\] (F.12)

where \( m(t_1) \) is given by (6.37). From (F.7), (F.11) and (F.12) a basis for the unobservable subspace is computed as (6.36). ■

F.3 Proof of Lemma 13

**Proof.** By assumption we know that
\[
n^0 v_b(t_2) = 0.
\] (F.13)

Further, for all \( \tau \in U \) we derive
\[
n^0 \dot{v}_b = \kappa(\tau) b R^b d.
\] (F.14)

Using (F.14) we derive
\[
n^0 v_b(t_2) = \int_0^{t_2} n^0 \dot{v}_b d\tau = \int_{t_1}^{t_2} \kappa(\tau) b R^b d d\tau.
\] (F.15)

The change in limits is because \( n^0 \dot{v}_b = 0 \) for all \( \tau \in S_1 \). Using (F.13) and (F.15), the desired result is derived. ■

F.4 Proof of Proposition 14

**Proof.** Using (6.45) we derive
\[
(t_2 - t_1)[n g x] \mathbf{u}_{32} - (t_2 - t_1) b R \mathbf{u}_{34} = 0.
\] (F.16)
Since \( t_2 - t_1 > 0 \), eqn. (F.16), (6.38) and (6.39) gives us the required basis for the unobservable subspace. ■

F.5 Proof of Proposition 15

**Proof.** From (6.40) and (6.42) we derive

\[
\mathbf{u}_{34} = \mathbf{b}_{(t_2)}^{b(t_1)} \mathbf{R} \mathbf{u}_{34}. \quad (F.17)
\]

Note that \( \mathbf{u}_{34} \) is an eigenvector of \( \mathbf{b}_{(t_2)}^{b(t_1)} \mathbf{R} \) with eigenvalue 1. Since we have rotation only along \( \mathbf{b}_\omega \) by Euler’s theorem we conclude that \( \mathbf{u}_{43} \) can only be a scalar multiple \( p \in \mathbb{R} \) of the angular rate vector:

\[
\mathbf{u}_{34} = p \mathbf{b}_\omega \quad (F.18)
\]

where \( p \) is a constant. Note that, for all \( \tau \in M \)

\[
\mathbf{n}_{b(\tau)} \mathbf{R} = \int_{t_1}^{\tau} \mathbf{R} d\tau = \int_{t_1}^{\tau} \mathbf{R}_b [\mathbf{b}_\omega \times] d\tau. \quad (F.19)
\]

Substituting (F.18) and (F.19) into (6.45) we derive

\[
(t_2 - t_1)[\mathbf{n}_g \times] \mathbf{u}_{32} = p \int_{t_1}^{t_2} \int_{t_1}^{s} \mathbf{R}_b [\mathbf{b}_\omega \times] \mathbf{b}_\omega dr ds = 0. \quad (F.20)
\]

Since \( t_2 - t_1 > 0 \), using (F.20) we conclude

\[
\mathbf{u}_{32} = q \mathbf{n}_g \quad (F.21)
\]

where \( q \) is a constant. Substituting (F.21) into (6.40) we derive

\[
\mathbf{u}_{34} = 0. \quad (F.22)
\]

From (6.38), (6.39), (F.21) and (F.22) we derive the unobservable subspace to be (6.47). ■
F.6 Proof of Corollary 16

**Proof.** For all $\tau \in U_g$, equating $H_g \Phi_m v_3 = 0$ we derive

$$
\int_0^\tau \left[ n f \times \right] u_{32} - n b R u_{34} ds = 0. \quad (F.23)
$$

Substituting (F.22) and (F.21) into (F.23) we derive

$$
q \int_0^\tau \kappa \left[ n d \times \right] g ds = 0 \quad (F.24)
$$

where the limits are changed as $n f = n g$ for all $\tau \in S_1$. Since $\kappa \neq 0$ in $M$ and $\theta \neq \pm \frac{\pi}{2}$ (i.e. $n d$ and $n g$ cannot be collinear), we conclude $q = 0$. Substituting $q = 0$ into (F.21), we see that the unobservable subspace is trivial. ■