The effective mass of an accelerating dislocation

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The Effective Mass of
an Accelerating Dislocation

A dissertation submitted in partial satisfaction of the
requirements for the degree Doctor of Philosophy

in

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by

Luqun Ni

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2005
The dissertation of Luqun Ni is approved, and it is acceptable in quality and form for publication on microfilm:

[Signatures]

University of California, San Diego

2005
To my parents and my wife
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VITA

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ABSTRACT OF THE DISSERTATION

The Effective Mass of an Accelerating Dislocation

by

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Macroscopic dynamic plastic deformation is a consequence of the motion of dislocations at the microscopic level. To determine the effective mass of an accelerating dislocation is a fundamental problem in continuum mechanics and plasticity. An equally important and closely related problem is to find the force needed to accelerate a dislocation. Both open problems have attracted a great deal of interest, however solutions are still not satisfactory. It is well-known that in Eshelby’s theory, the configurational force on a static defect is defined as the negative gradient of the total energy with respect to the position of the defect. It has not been clarified whether that definition will still be adequate for the dynamic case. We propose a definition of the dynamic configurational force and self-force by using the change of the total Lagrangian of the mechanical system. And by carefully treating the discontinuities and singularities, a rigorous consistent discussion on the definition and expression of the dynamic configurational force on moving elastic defect is presented. The effective mass of an accelerating dislocation is then derived from the inertial part of the self-force divided by the acceleration. For moving screw and edge dislocations, we prove new theorems on the near field behaviors, so that the full determination of the near field expansions up to the $O(1)$ terms are achieved. The solution and evaluation of the self-force and effective mass for moving screw and edge dislocations is obtained.
Chapter I

Introduction

Macroscopic dynamic plastic deformation is a consequence of the motion of dislocations at the microscopic level. To determine the effective mass of an accelerating dislocation is a fundamental problem in continuum mechanics and plasticity. An equally important and closely related problem is to find the force needed to accelerate a dislocation. Both open problems have attracted a great deal of interest, however solutions are still not satisfactory.

Several versions of the effective mass of a moving dislocation exist in the literature, which are all approximations based on an assumption that the motion is uniform, (Hirth et al., 1998), as well as a recent paper Fedelich (2004). Eshelby’s theory of the configurational force on an elastic defect was established in the fifties of the last century (Eshelby, 1951, 1970, 1975, etc.). More recently, Eshelby’s theory has been developed into an active branch of engineering science as the configurational mechanics of materials, see Maugin (1993), Kienzler and Herrmann (2000), Gurtin (2000), and Kienzler and Maugin (2001).

According to Eshelby, the configurational force on a static defect is defined as the negative gradient of the total energy with respect to the position of the defect,

$$ F_i \delta \xi_i = -\delta E. \quad (I.1) $$

In the case that the stress field is solely created by the defect itself, the con-
figurational force on the defect is called the self-force. Eshelby showed that the configurational force on a static defect is equal to a “path independent” integral of the energy-momentum tensor (see Chapter III). A satisfactory generalization of the configurational force to the dynamic case had not been reached by Eshelby. In Eshelby (1970), he commented that: “*It is difficult to fit moving defects rigorously into the P_{ij}, g_l formalism*”, where $P_{ij}$ is the spacial components of the energy-momentum tensor and $g_l$ is the pseudo-momentum. It has not been clarified that whether Eshelby’s definition of the configurational force will still be adequate for the configurational force on a moving defect or inhomogeneity.

The dislocation is an elastic defect in the material. When a dislocation is moving respect to the material, the ensuing force on the dislocation is a configurational force. The effective mass of an accelerating dislocation can then be determined from the inertial part of the self-force divided by the acceleration, i.e.,

$$m_e = \frac{F^{in}}{\dot{v}(t)},$$

where $m_e$ is the effective mass of an accelerating dislocation, $v(t)$ is the instantaneous velocity of the moving dislocation, and $F^{in}$ is the inertial part of the self-force on the dislocation, which linearly depends on the acceleration $\dot{v}(t)$. We will clarify the definition of the dynamic configurational force, and give a mathematically rigorous treatment of discontinuities and singularities. Then we will solve and evaluate the self-forces on moving screw and edge dislocations, further to determine the effective masses of the accelerating screw and edge dislocations. The discussion is within the framework of linear elasticity and continuum mechanics.

Stroh (1962) has questioned the validity of Eshelby’s definition (I.1) for the dynamics case, and stated that “*The nature of the force on a dislocation at rest has been fully discussed by Eshelby (1951), who has particularly emphasized the need for thorough treatment. His conclusions are that (a) the force should be defined as the derivative of the energy with respective to dislocation displacement; from it follows that (b) the force in the slip plane is just $\sigma b$ per unit length of the dislocation, where $\sigma$ is the resolved shear stress. It seems to have been accepted quite uncritically that both statements (a) and (b) apply also in the dynamic case, without
realizing that here that they are in fact inconsistent." Instead of using the total energy of the system, Stroh used the Lagrangian of the interacting dislocations to derive the interacting force between two dislocations, so that (a) was "modified by replacing the energy with the Lagrangian", and it was found that "(b) remains true".

Inspired by Stroh (1962) (see also Lothe, 1992; Hirth, et al., 1998), we propose (in Chapter IV) a definition of the dynamic configurational force on a moving inhomogeneity by using the change of the total Lagrangian of the mechanical system, i.e.,

$$\delta \xi \int_{t_1}^{t_2} L_{\text{total}} dt = \int_{t_1}^{t_2} F_l \delta \xi_l dt,$$

where $\delta \xi_l$ is the infinitesimal translation of the elastic inhomogeneity, and $\delta \xi$ represents the change due to the infinitesimal translation $\delta \xi_l$, assuming that the elastic medium is the infinite whole space, and there is no body force. Here an inhomogeneity means a region where the elasticity properties are different from the otherwise homogeneous elastic body, which can be a piecewise, or point-wise difference (see Maugin, 1993). When the elastic system is static, then $L = -W$ and all quantities are independent of time $t$, (I.3) is clearly reduced to Eshelby’s definition (I.1). From the definition (I.3), the following expression of the configurational force on a moving elastic inhomogeneity is derived

$$F_l = -\int_{R^3} \left( \frac{\partial L}{\partial x_l} \right)_{\exp} dx^3,$$

which was first reported by Rogula (1977). From that expression, the following "contour-independent" integral expression of the force on a moving inhomogeneity is then obtained,

$$F_l = \int_V \frac{\partial}{\partial t} \left[ \rho \dot{u}_i u_{i,l} \right] dV + \int_S \left[ (W - T) \delta_{ij} - u_{i,l} \sigma_{ij} \right] dS_j,$$

which a new expression.

In the derivation of (I.4), certain regularity of the elastic field is assumed, such as the elastic field should be sufficiently smooth. However, in general, the
elastic field may not be continuous over the boundary of the inhomogeneity. The discontinuous field will be considered as a limiting case of the smooth field. By using a convolution with Friedrichs’ regularization function (Friedrichs, 1953; see also Yosida, 1980), the discontinuous field can be smoothed to an infinitely differentiable field. A rigorous treatment of such limiting process is then given. We show that for the non-singular discontinuous case, the “contour-independent” integral expression (I.5) is still valid. As an illustration of that expression, an example of the dynamic effective normal force on a moving interface is obtained, which generalizes Eshelby’s result of the effective normal force on a static interface (Eshelby, 1970).

An elastic defect may be a point singularity, a line, or surface singularity. At the point or line singularity, the stress or displacement may become infinite; on the surface singularity, the stress or displacement becomes infinite, or, discontinuous (see Eshelby, 1951). We carefully treat the involved singularities (in Chapter V). To define the configurational force on a moving defect, as in the conventional way, we exclude a small neighborhood around the singularity. Such neighborhood is then considered as an inhomogeneity with jump discontinuities on its boundary. The dynamic configurational force on such inhomogeneity as we have already discussed is well-defined and given by a “contour-independent” integral expression. The configurational force on the moving defect is then defined as the limit of the force on that non-singular inhomogeneity as the small neighborhood shrinks upon to the singularity.

We prove a necessary and sufficient condition for the existence of the limit in the definition of the dynamic configurational force on a moving defect so that the force is well-defined. If the limit exists when the small neighborhood which shrinks upon the singularity can be chosen arbitrarily, then the volume integral in the “contour-independent” integral expression converges as an improper integral. If the limit exists only when the small neighborhood is chosen to be symmetric as required in the definition of the integral of the Cauchy type, then the volume integral converges as an integral of the Cauchy type. When the condition fails to be satisfied, then the defined force diverges and becomes infinite. We propose
two ways to treat such divergence: (i) Regularize the involved divergent integrals by use of methods based on the theory of distributions; (ii) Modify the physical model, e.g., smear the singularity, such that for the modified model, the dynamic configurational force for that model is well-defined.

Examples of the force on moving defects are presented. For the driving force on an advancing crack, we show that the necessary and sufficient condition for the existence of the volume integral as an integral of the Cauchy type is satisfied. As a result, the driving force is well-defined, and Bui’s dynamic J-integral of an advancing crack follows from the “contour-independent” integral expression for the dynamic configurational force. From Eshelby’s result, it is seen that the driving force of an advancing crack does not include an inertial part, so that an advancing crack has no effective mass.

By the presented treatment, it becomes clear that only when the definition of the dynamic configurational force is clarified, and discontinuities and singularities are rigorously treated, then we are able to derive and evaluate the self-force on the moving dislocation. Since the effective mass is determined by the self-force as in (I.2), we may concentrate on evaluating the self-force. To derive and evaluate the self-force, according to the definition, the “contour-independent” integral expression, and the necessary and sufficient condition, an essential issue is to obtain the near field expansions for the elastic field variables.

For the moving screw dislocations (see Chapter VI), we may write the near field expansions

\[ u_{3,j} = u_{3,j}^0 + f_{3,j}(\theta, t) \ln \epsilon + g_{3,j}(\theta, t) + h.o.t., \]  

(I.6)

for \( j = 1, 2, t \) and \( \epsilon > 0 \), where \( \epsilon = \sqrt{(x - l(t))^2 + y^2} \) and \( \theta = \tan^{-1}(y/(x - l(t))) \) with \( l(t) \) as the current position of the dislocation and \((x, y)\) as the field point coordinates. \( u_{3,j}^0 \) are the leading terms which coincide with the corresponding solutions of the uniform motion, and for convenience, functions \( f_{3,j}(\theta, t) \) and \( g_{3,j}(\theta, t) \) are called the near field coefficients. To find those six near field coefficients, a routine approach is first to derive the exact solutions of \( u_{3,j} \), then use singular perturbation methods to obtain the near field expansions. However, that routine
method requires a large number of operations, which is almost prohibitive. We use an entirely different approach, with surprising simplicity, to prove that the full determination of such six functions of the near field coefficients can be reduced to an evaluation of two near field constants, which are stated in two important new theorems.

In Theorem 1, we show that the partial differentiations of the near field coefficients with respect to $\theta$, $f'_{3j}$, satisfy a homogeneous system of linear equations with a non-vanishing determinant of the matrix of coefficients, so that $f'_{3j}$ are all zero. Further we prove that $f_{31} = f_{3t} = 0$, and $f_{32}(\theta, t) = f_{32}(t)$ is independent of $\theta$. With Theorem 1, we are able to solve the self-force on a moving screw dislocation. In Theorem 2, we prove that $g'_{3j}$ satisfy an inhomogeneous system of linear equations with a non-zero determinant, so that $g'_{3j}(\theta, t)$, and further by integration, $g_{3j}(\theta, t)$, can be solved explicitly. Six functions of $f_{3j}(\theta, t)$ and $g_{3j}(\theta, t)$ are solved in terms of two near field constants $f_{32}(t)$ and $g_{32}(0, t)$, regarding $t$ as a parameter. $f_{32}$ has been evaluated by Callias and Markenscoff (1988). By using a corollary of Callias and Markenscoff (1988), after a lengthy calculation, $g_{32}(0, t)$ is evaluated explicitly (detail see Appendix A). Hence, the complete evaluations of self-force and effective mass for moving screw dislocation are achieved.

It is seen that the self-force derived is divergent logarithmically. We then treat the divergence by regularizing the divergent volume integral in the expression of the self-force based on the theory of distributions (see Chapter VII), and treat the divergence based on a smearing method which smooths the singularity at the core of the dislocation (see Chapter VIII). For both cases, complete solutions and explicit evaluations are obtained. These results for the moving screw dislocation are similarly extended to the moving edge dislocation. It is proved that for a moving edge dislocation, the full determination of twelve functions of near field coefficients are reduced to the evaluation of five near field constants, which are illustrated in Theorem 3 and Theorem 4. Therefore, the solution of the self-force and effective mass problem for a moving screw or edge dislocation is completely obtained.
Chapter II

Preliminaries

II.A Elastic Field of Screw Dislocation

II.A.1 Static Screw Dislocation

Consider an infinitely long Volterra screw dislocation line situated at the z-axis in a three dimensional Euclidean space, which is occupied by an isotropic elastic medium. The Burgers vector \((0, 0, b)\) parallels to the z-direction. The displacement field \(u\) has only one nonzero component \(u_3\), which depends on the spatial variables \(x\) and \(y\), and is independent of \(z\). We define \(u(x, y) \equiv u_3(x, y)\). \(u\) satisfies the equilibrium equation, for \(y \neq 0\),

\[
\frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0, \tag{II.1}
\]

with the discontinuity condition at \(y = 0\),

\[
u(x, 0^+) - u(x, 0^-) = -\frac{b}{2}[H(x) - H(-x)], \tag{II.2}
\]

where \(H(\cdot)\) is the Heaviside step function.

It is well-known that the displacement \(u\) is solved by

\[
u = -\frac{b}{2\pi} \tan^{-1} \left( \frac{x}{y} \right), \tag{II.3}
\]
which is uniquely determined apart from a constant term. The non-zero components of the stress fields are

\[ \sigma_{13} = \mu u_1 = -\frac{b}{2\pi} \frac{y}{x^2 + y^2}, \quad (\text{II.4}) \]

\[ \sigma_{23} = \mu u_2 = \frac{b}{2\pi} \frac{x}{x^2 + y^2}, \quad (\text{II.5}) \]

where \( \mu \) is the Lame constant.

From the equation and the discontinuity condition, it is seen that \( u(x, y) \) is odd in \( y \). Hence it is equivalent to consider a half-space problem, for \( y > 0 \)

\[ \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0, \quad (\text{II.6}) \]

with the boundary condition

\[ u(x, 0) = -b/2 \text{H}(x). \quad (\text{II.7}) \]

II.A.2 Screw Dislocation in Uniform Motion

When a screw dislocation as described in the last subsection is uniformly moving along the \( x \)-axis with a constant velocity \( v \), then the displacement \( u(x, y, t) \equiv u_3(x, y, t) \) satisfies the following equation of motion, for \( y \neq 0 \),

\[ \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} = \frac{1}{c_2^2} \frac{\partial^2 u(x, y, t)}{\partial t^2}, \quad (\text{II.8}) \]

and the discontinuity condition at \( y = 0 \),

\[ u(x, 0^+, t) - u(x, 0^-, t) = -\frac{b}{2}[\text{H}(x - vt) - \text{H}(-x + vt)], \quad (\text{II.9}) \]

where \( c_2 = \sqrt{\mu/\rho} \) is the shear wave speed, and \( \rho \) the mass density of the material.

According to Frank (1942), the displacement field is identical to that of a screw dislocation at rest, apart from a "Lorentz contraction", i.e., the displacement is given by

\[ u(x, y, t) = -\frac{b}{2\pi} \tan^{-1}\left(\frac{x - vt}{\gamma y}\right) \quad (\text{II.10}) \]

where \( \gamma \equiv \sqrt{1 - v^2/c_2^2} \).
Hence the stresses are given by

$$\sigma_{13} = -\frac{b\mu}{2\pi} \frac{\gamma y}{[\gamma^2 y^2 + (x - vt)^2]^2}, \quad \text{(II.11)}$$

$$\sigma_{23} = \frac{b\mu}{2\pi} \frac{\gamma(x - vt)}{[\gamma^2 y^2 + (x - vt)^2]^2}, \quad \text{(II.12)}$$

and

$$\frac{\partial u}{\partial t} = \frac{b\mu}{2\pi} \frac{v\gamma y}{[\gamma^2 y^2 + (x - vt)^2]^2}. \quad \text{(II.13)}$$

### II.A.3 Screw Dislocation in Non-Uniform Motion

The non-uniformly moving screw dislocation was discussed in Eshelby (1951, 1953), Nabarro (1951), and Kiusalaas and Mura (1964). A closed-form solution of the stress $\sigma_{32}$ for a non-uniformly moving screw dislocation starting from rest was given in Markenscoff (1980).

Assume that a screw dislocation line is situated on the $z$-axis as described in the previous subsections. It is at rest until time $t = 0$ when it begins to move along the $x$-axis according to $x = l(t)$. The equation of motion is, for $y \neq 0$,

$$\frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u(x, y, t)}{\partial t^2}, \quad \text{(II.14)}$$

with the discontinuity condition at $y = 0$,

$$u(x, o^+, t) - u(x, o^-, t) = -\frac{b}{2} [H(x - l(t)) - H(l(t) - x)]. \quad \text{(II.15)}$$

Similar to the static problem, the above problem can also be reduced to a half-space problem. As in Markenscoff (1980), by using Laplace transform in time and two-side Laplace transform in space, the problem was solved in the transformed space. In the inversion, the Cagniard-de Hoop technique was used to change the integral contour. So that the inverse Laplace transform can be obtained simply by inspection. The exact solution for the stress $\sigma_{32}$ for non-uniformly moving screw dislocation starting from rest is then obtained as

$$\sigma_{13} = \frac{b\mu}{2\pi} \int_0^\infty \frac{(t - \eta(\xi))(x - \xi)H(t - \eta(\xi) - r/c_2)}{r^4[(t - \eta(\xi))^2 - r^2/c_2^2]^{1/2}} d\xi$$

$$- \frac{b\mu}{2\pi} y^2 \frac{\partial}{\partial t} \int_0^\infty \frac{(t - \eta(t))^2H(t - \eta(t) - r/c_2)}{r^4[(t - \eta(t))^2 - r^2/c_2^2]^{1/2}} d\xi + \frac{b\mu}{2\pi} \frac{x}{x^2 + y^2}, \quad \text{(II.16)}$$
where \( r^2 = (x - \xi)^2 + y^2 \), and \( \eta(\xi) = t \) is the inverse function of \( \xi = l(t) \).

The stress given in the last equation shows that the stress at a field point \((x, y)\) and time \( t \) is the superposition of all wavelets from the dislocation which had time to reach the point. In the integrals, the root of Heaviside function \( H(t - \eta(\xi) - r/c_2) \) determines the upper limit of the integration, which represents the last position from which the wavelet is able to reach the field point in time \( t \).

II.B Leading Terms of Near-Field Expansion Solutions

The near-field expansion solution is the asymptotic expansion in \( \epsilon \) of the solution at the field point which is in a \( \epsilon \)-neighborhood of the dislocation. Assume that the field point \((x, y)\) is in an \( \epsilon \)-neighborhood of the dislocation at \((x_0, y_0)\), such that

\[
(x - x_0)^2 + (y - y_0)^2 = \epsilon^2,
\]

or,

\[
x = x_0 + \epsilon \cos \theta, \quad y = y_0 + \epsilon \sin \theta,
\]

for \( 0 \leq \theta < 2\pi \) and \( \epsilon > 0 \).

For a non-uniformly moving screw dislocation, the most singular terms of the expansions of the stress field \( \sigma_{13}, \sigma_{23}, \) and \( \partial u_3/\partial t \) are found to be the corresponding solutions of the steady-state motion with the instantaneous velocity as the uniform velocity (Clifton and Markenscoff, 1981; Markenscoff and Ni, 1993). That can be proved without using the exact field solutions. The following is a detailed proof.

We shall prove that the leading terms for \( \sigma_{31}, \sigma_{32}, \) and \( \dot{u}_3 \) are

\[
\sigma_{31} \sim \mu \frac{\partial u_0}{\partial x} = -\frac{b\mu \gamma}{2\pi} \left( \frac{y}{(x - l(t))^2 + \gamma^2 y^2} \right) = -\frac{b\mu \gamma}{2\pi \cos^2 \theta + \gamma^2 \sin^2 \theta} \frac{1}{\epsilon}.
\]

(II.19)
\[ \sigma_{32} \sim \mu \frac{\partial u_0}{\partial y} = \frac{b \mu}{2\pi} \frac{\gamma (x - l(t))}{(x - l(t))^2 + \gamma^2 y^2} = \frac{b \mu}{2\pi \cos^2 \theta + \gamma^2 \sin^2 \theta} \epsilon. \] (II.20)

\[ \frac{\partial u}{\partial t} \sim - v(t) \frac{\partial u_0}{\partial \eta_1} = \frac{b}{2\pi} \frac{v(t) \gamma y}{(x - l(t))^2 + \gamma^2 y^2} \frac{v(t) \gamma \sin \theta}{2\pi \cos^2 \theta + \gamma^2 \sin^2 \theta} \epsilon. \] (II.21)

where and in the sequel \( v(t) \equiv \dot{l}(t) \).

To show (1)-(3), we make the following change of variables,
\[ \epsilon \eta_1 = x - l(t), \quad \epsilon \eta_2 = y, \quad \epsilon \eta_3 = z, \quad t' = t, \] (II.22)
here \( \epsilon > 0 \) is an arbitrary infinitesimal parameter. The derivatives are transformed as
\[ \frac{\partial}{\partial x} = \frac{1}{\epsilon} \frac{\partial}{\partial \eta_1}, \] (II.23)
\[ \frac{\partial}{\partial t} = - \frac{v(t)}{\epsilon} \frac{\partial}{\partial \eta_1} + \frac{\partial}{\partial t'}, \] (II.24)
and
\[ \epsilon^2 \frac{\partial^2}{\partial t'^2} = v(t) \frac{\partial^2}{\partial \eta_1^2} - 2v(t) \epsilon \frac{\partial^2}{\partial \eta_1 \partial t'} - v(t) \epsilon \frac{\partial}{\partial \eta_1} + \epsilon^2 \frac{\partial^2}{\partial t'^2}. \] (II.25)

The equation of motion (II.14) is changed to
\[ \frac{\partial^2 u}{\partial \eta_1^2} + \frac{\partial^2 u}{\partial \eta_2^2} = \frac{1}{c^2} \left[ v(t) \frac{\partial^2 u}{\partial \eta_1^2} - 2v(t) \epsilon \frac{\partial^2 u}{\partial \eta_1 \partial t'} - \dot{v}(t) \frac{\partial u}{\partial \eta_1} + \epsilon^2 \frac{\partial^2 u}{\partial t'^2} \right], \] (II.26)
with the discontinuity condition
\[ u(\epsilon \eta_1, 0^+, t') - u(\epsilon \eta_1, 0^-, t') = -\frac{b}{2} [H(\eta_1) - H(-\eta_1)], \] (II.27)
since $H(\epsilon \eta_1) = H(\eta_1)$ for $\epsilon > 0$. 

Denote by $u_0(\eta_1, \eta_2, t')$ the $O(1)$ term in the expansion of $u(\epsilon \eta_1, \epsilon \eta_2, t')$ as $\epsilon \to 0$. In (II.26), set $\epsilon \to 0$, it follows the equation for $u_0$

$$\frac{\partial^2 u_0}{\partial \eta_1^2} + \frac{\partial^2 u_0}{\partial \eta_2^2} = -\frac{i^2 \partial^2 u_0}{c^2 \partial \eta_1^2},$$

(II.28)

which is rewritten as

$$\frac{\partial^2 u_0}{\partial \eta_1^2} + \frac{\partial^2 u_0}{\partial \eta_2'^2} = 0,$$

(II.29)

for $\eta'_2 = \gamma \eta_2$ and $\gamma = \sqrt{1 - v^2(t)/c^2}$. The discontinuity condition for $u_0$ is

$$u_0(\eta_1, 0^+, t') - u_0(\eta_1, 0^-, t') = -\frac{b}{2} [H(\eta_1) - H(-\eta_1)].$$

(II.30)

(II.29) and (II.30) are exactly the equation and discontinuity condition for a static screw dislocation. Hence $u_0$ is given by

$$u_0 = -\frac{b}{2\pi \tan^{-1}(\eta_1/\gamma \eta_2)}.$$

(II.31)

The most singular term of $\partial u/\partial x$ is

$$\frac{\partial u_0}{\partial x} = \frac{1}{\epsilon} \frac{\partial u_0}{\partial \eta_1} - \frac{b}{2\pi \epsilon \eta_1^2 + \gamma^2 \eta_2^2}.$$

(II.32)

As defined in (II.22), $x - l(t) = \epsilon \eta_1$ and $y = \epsilon \eta_2$, where $\eta_1$ and $\eta_2$ are independent variables. Using the polar coordinates, $\eta_1 = \delta \cos \theta$ and $\eta_2 = \delta \sin \theta$. In the final step of (II.32), we may set $\delta = 1$, then

$$(x - l(t)) = \epsilon \cos \theta, \quad y = \epsilon \sin \theta.$$ 

Hence, for the stress $\sigma_{31}$, we have the leading term

$$\sigma_{31} \sim \mu \frac{\partial u_0}{\partial x} = -\frac{b \mu}{2\pi} \frac{\gamma y}{(x - l(t))^2 + \gamma^2 y^2} = -\frac{b \mu}{2\pi} \frac{\gamma \sin \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta}$$

$$= \frac{1}{\epsilon},$$

(II.33)

which is exactly the corresponding solution for steady-state moving screw dislocation. (1) is then proved.
In the same manner, we can prove (2), the $1/\epsilon$ term of the stress $\sigma_{23}$ is given by

$$\sigma_{32} \sim \mu \frac{\partial u_0}{\partial y} = \frac{b\mu}{2\pi} \frac{\gamma(x - l(t))}{(x - l(t))^2 + \gamma^2 y^2} = \frac{b\mu}{2\pi} \frac{\gamma \cos \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta} \frac{1}{\epsilon}. \tag{II.34}$$

To prove (3), note that the $1/\epsilon$ term of $\partial u/\partial t$ is given by

$$\frac{\partial u_0}{\partial t} = \frac{\partial u_0}{\partial \eta_1} \frac{\partial \eta_1}{\partial t} + \frac{\partial u_0}{\partial \eta'} \frac{\partial \eta'}{\partial t} = -\frac{v(t) \partial u_0}{\epsilon} \frac{\partial u_0}{\partial \eta_1} + \frac{\partial u_0}{\partial t}. \tag{II.35}$$

There, only the first term is of order of $1/\epsilon$, the second term on the right hand side is in order of $O(1)$. So that, the leading term of $\partial u/\partial t$ is given by

$$\frac{\partial u}{\partial t} \sim -\frac{v(t) \partial u_0}{\epsilon} \frac{\partial u_0}{\partial \eta_1} = \frac{b}{2\pi} \frac{v(t) \gamma y}{(x - l(t))^2 + \gamma^2 y^2} \frac{b}{2\pi} \frac{v(t) \gamma \sin \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta} \frac{1}{\epsilon}. \tag{II.36}$$

II.C Historical Perspective

II.C.1 Effective Mass of a Moving Dislocation

In the literature, the existing results about the effective mass of a moving dislocation are based on the energy calculation of uniformly moving dislocations, assuming that the steady-state motion is possible. As pointed out in Kocks, Argon, and Ashby (1975), the effective (inertial) mass of a moving dislocation per unit length of the dislocation “could, in principle, be calculated by deriving the stress necessary to provide uniform acceleration of a straight dislocation in the absence of any damping and any element glide resistance in a given lattice at a given temperature. ‘This has not been done, partly because of the difficulty of describing the force on a moving dislocation.’” (Nabarro, 1967) Instead one may calculate the kinetic energy of a straight dislocation moving at uniform velocity.”

Frank (1949) calculated the total energy of a uniformly moving screw dislocation, when the dislocation line is parallel to the $z$-axis, and moving in the
x-direction,

\[ E = E_0/(1 - \beta^2)^{\frac{1}{2}} \]  \hspace{1cm} (II.37)

where \( E_0 \) is the energy of a screw dislocation at rest, and \( \beta = v/c_2 \) with \( v \) as the uniformly moving speed of the dislocation. For low velocity \( v \), i.e., \( v/c_2 << 1 \),

\[ E = E_0(1 + \frac{1}{2} \frac{v^2}{c_2^2} + ...) \simeq E_0(1 + \frac{1}{2} \frac{v^2}{c_2^2}). \]  \hspace{1cm} (II.38)

Then,

\[ \frac{1}{2} \frac{v^2}{c_2^2} E_0 \simeq E - E_0 \]  \hspace{1cm} (II.39)

is the kinetic energy. The effective mass \( m_{\text{screw}} \) is obtained as

\[ m_{\text{screw}} = \frac{E_0}{c_2^2}. \]  \hspace{1cm} (II.40)

That is analogous to the Einstein’s relation

\[ m_0 = \frac{E_0}{c^2}, \]  \hspace{1cm} (II.41)

where \( m_0 \) is the rest mass, \( E_0 \) the rest energy, and \( c \) the light speed.

For the edge dislocation, the energy can not be expressed by a relativistic relation as (II.40) for the screw dislocation. Weertman (1961) obtained a relation between the effective masses of screw and edge dislocations at low speed motion

\[ m_{\text{edge}} = [1 + \left( \frac{c_2}{c_1} \right)^4] m_{\text{screw}}, \]  \hspace{1cm} (II.42)

where \( c_1 = \sqrt{(\lambda + 2\mu)/\rho} \) is the longitudinal wave speed.

There are several definitions for the effective mass of a dislocation moving at a uniform high speed. They are not necessarily consistent with each other (see, e.g., Hirth, Zbib, and Lothe, 1998).

Noting an analogy of the energy expression for the screw dislocation and relativistic Einstein kinematics, Weertman (1961) proposed a definition for the effective mass for screw dislocations at uniform motion with low or high speed,

\[ m_{\text{screw}} = \frac{W}{c_2^2}, \]  \hspace{1cm} (II.43)
where $W$ is the total energy.

Sakamoto (1991) defined the effective mass of an edge dislocation moving at a uniform velocity $v$ by calculating the momentum, where $v$ may be a low or high speed. Starting from the momentum

$$p = \int f \, dt,$$  

(II.44)

where $f$ is defined by the change of the total energy

$$\delta E = f \delta x,$$  

(II.45)

Sakamoto (1991) derived that

$$\frac{dp}{dt} = \frac{dE}{dv} \frac{dv}{dt} = \frac{dE}{dv} \frac{dt}{dx}.$$  

(II.46)

So that

$$\frac{dp}{dt} = \frac{1}{v} \frac{dE}{dv}.$$  

(II.47)

From that and the relation

$$p = mv,$$  

(II.48)

it follows

$$m = \frac{p}{v} = \frac{1}{v} \int_0^v \frac{1}{v} \frac{dE}{dv} \, dv.$$  

(II.49)

According to Weertman (1961), the expression of the total energy for the uniformly moving edge dislocation is given by

$$E_{\text{edge}} = \frac{E_0}{2v^2} \left(16 \gamma_1 + 8 \gamma_1^{-1} - 14 \gamma - 12 \gamma^{-1} + 2 \gamma^{-3}\right),$$  

(II.50)

where $E_0$ is the energy at rest, $\gamma_1 = (1 - v^2/c_1^2)^{1/2}$ with the longitudinal wave speed $c_1 = \sqrt{(\lambda + 2\mu)/\rho}$, and again $\gamma = (1 - v^2/c_2^2)^{1/2}$ with the shear wave speed $c_2 = \sqrt{\mu/\rho}$. By use of the total energy (II.50), (II.49) was calculated in Sakamoto (1991) to give the effective mass of a uniformly moving edge dislocation as

$$m_{\text{edge}} = \frac{(E_0)_{\text{edge}}}{c^2} \left[4c_2^2/c_1^4 \gamma_1 + 4c_2^2 \gamma_1/c_1^2 V^2 + 8\gamma_1/V^4 - 2/\gamma 
+ V^2/\gamma^3 - 68\gamma/15V^2 - 64\gamma/15V^2 + 16\gamma^2/15V^2 
- 16\gamma/5V^6 + 8\gamma^3/5V^6 + 8\gamma^5/5V^6\right].$$  

(II.51)
where \( V = v/c_2 \).

Hirth, Zbib, and Lothe (1998) gave another definition for the effective mass of the dislocation moving at high constant speed. They again start with the relation

\[
\delta E = f \delta x, \quad (II.52)
\]
or equivalently,

\[
dW = F dx = F v dt, \quad (II.53)
\]

where \( E \) is the total energy, and \( f \) is the force. On the other hand, it is easy to see that

\[
dW = \frac{\partial E}{\partial v} \frac{\partial v}{\partial t} dt. \quad (II.54)
\]

Compare (II.53) with (II.54) to result that

\[
F = \frac{1}{v} \frac{\partial E}{\partial v}. \quad (II.55)
\]

Hence the effective mass is given as

\[
m = \frac{1}{v} \frac{\partial E}{\partial v}. \quad (II.56)
\]

By using the total energy (II.37) for the uniformly moving screw dislocation as given in Frank (1949), and the total energy (II.50) of the moving edge dislocation as given in Weertman (1981), from (II.56), the effective masses for the uniformly moving screw and edge dislocations are given respectively as

\[
m_{\text{screw}} = \frac{(E_0)_{\text{screw}} c^2}{v^2} \gamma^{-1}, \quad (II.57)
\]

and

\[
m_{\text{edge}} = \frac{(E_0)_{\text{edge}} c^2}{v^4} (-8 \gamma_1 - 20 \gamma_1^{-1} + 4 \gamma_1^{-3} + 7 \gamma + 25 \gamma^{-1} - 11 \gamma^{-3} + 3 \gamma^{-5}). (II.58)
\]

For more definitions of the effective mass, see Hirth, Zbib, and Lothe (1998).
II.C.2 Force on a Moving Dislocation: Eshelby’s Results

Based on a perfect analogy between the two-dimensional electromagnetic field and the elastic field of the screw dislocation, under certain limitation of the acceleration of the motion, Eshelby (1953) gave an approximate estimation of the force on a moving screw dislocation, which accelerated from rest and moves at a constant acceleration in an infinite elastic solid.

By an analogy with two-dimensional electromagnetic field, near the center of the screw dislocation with the Burgers vector \( b \) in the direction parallel to \( z \)-axis and moving along the direction of the \( x \)-axis, the stress field \( \sigma_{zy} \) was approximately given by

\[
\sigma_{zy} = \frac{\mu b}{2\pi} \left\{ \frac{x'}{r'^2} - \frac{1}{2x_0} \ln \frac{2x_0(t-s_0)}{s_0^2} - \frac{t-s_0}{x_0s_0} - \frac{s_0}{2x_0(t+s_0)} \right\}, \tag{II.59}
\]

where \( x_0 \) is the initial \( x \)-coordinate of the dislocation, \( \xi(t) \) is the current position of the center of the dislocation, the shear speed \( c_2 \) is taken to be 1, \( s_0 = \sqrt{t^2 - (x - \xi(t))^2} \), and

\[
x' = x - \xi(t), \quad r'^2 = \frac{x'^2}{\gamma^2(t)} + y^2, \tag{II.60}
\]

with \( \gamma^2 = \sqrt{1 - \dot{\xi}^2} \).

Eshelby assumed that Peierls law relating stresses and displacement at the slip plane should be satisfied. For a screw dislocation with Burgers vector \( b \) at the slip plane \( y = a/2 \), the Peierls law is

\[
\sigma_{zy}(y = a/2) = -\frac{\mu b}{2\pi a} \sin \left[ \frac{4\pi}{b} u(y = a/2) \right], \tag{II.61}
\]

where \( u \) is the \( z \)-component of the displacement field.

Note that in the stress \( \sigma_{zy} \) (II.59), the first term is the stress produced by a stead-state moving dislocation which at time \( t \) coincides in position and velocity with the accelerated dislocation. The Peierls law (II.61) is satisfied by the elastic solution for a stead-state moving screw dislocation with velocity \( v \),

\[
u = \frac{b}{2\pi} \tan^{-1} \left[ \frac{y\sqrt{1 - v^2/t^2}}{x - vt} \right]. \tag{II.62}
\]
Then Eshelby pointed that the Peierls law would be satisfied if we could impress on every point of the material a displacement equal to the difference between the displacement of the uniformly moving and accelerated dislocation. The required displacement can be produced by a uniform applied stress which is equal and opposite to the last three terms of the right hand side of (II.59). On the other hand, according to the analogy between the electromagnetic field and the elastic field of the screw dislocation, the force on the moving screw dislocation is

\[ F_x = b \sigma_{xy}^\text{applied}. \]  

(II.63)

So that, noting that \( \partial^2 \xi / \partial t^2 = \gamma^2 c_2^2 / x_0 \) and here \( c_2 \) is taken to be 1, from (II.59) and (II.63), it follows that

\[ F_x = \gamma^{-3} b \rho b^2 \ln f(t) \frac{\partial^2 \xi}{\partial t^2} + g(t), \]  

(II.64)

where

\[ f(t) = \frac{32 x_0^2 (t - s_0)}{a s_0 s_0}, \]  

(II.65)

\[ g(t) = \frac{s_0}{2(t + s_0)} + \frac{t - s_0}{s_0}. \]  

(II.66)

And from (II.64), the effective mass for a screw dislocation moving at a constant acceleration is

\[ m_{\text{screw}} = \gamma^{-3} b \rho b^2 \ln f(t). \]  

(II.67)

### II.C.3 Force on a Dislocation in Transient Uniform Motion

Clifton and Markenscoff (1981) obtained the drag force on moving screw and edge dislocations which jump from rest to a velocity \( v_d \) and then move at that constant velocity. Their analysis was based on an integral expression for the energy flux for an advancing crack by Atkinson and Eshelby (1965) and the exact solutions for the transient motion of non-uniformly moving screw and edge dislocations by
Markenscoff (1980), and Markenscoff and Clifton (1981), respectively. The force on the moving dislocation is defined as

$$F = \frac{\dot{E}}{v_d},$$

where $\dot{E}$ is the energy flux. According to Atkinson and Eshelby (1965), the energy flux through a closed surface $\Sigma = \partial V$, which enclosing a moving crack is given by energy release rate,

$$\dot{E} = \int_{\Sigma} \left[ \dot{u}_i \sigma_{ij} n_j + \frac{1}{2} (\sigma_{ij} u_{i,j} + \rho \dot{u}_i \dot{u}_i) \nu_n \right] ds,$$

where $\sigma$ and $u_i$ are the stress and displacement field, $\rho$ is the density of the solid material, $n$ is the outer normal of the surface $\Sigma$, and $\nu_n = (\nu, n)$ with $\nu$ as the velocity of the moving crack.

Clifton and Markenscoff pointed out that for a moving dislocation, which jumps from rest and moves along the $x$–axis in a constant velocity $v_d$, the integral of the energy flux is independent of the choices of the surface which surrounds the center of the moving dislocation as the surface shrinks to the center of the dislocation. Suppose that $S_{d,1}$ and $S_{d,2}$ are such surfaces, and $R^*$ is the volume between them, then the difference of the integrals of the energy flux over $S_{d,1}$ and $S_{d,2}$ is equal to the volume integral (Freund, 1972)

$$\int_{R^*} \left[ \rho \dot{u}_i (\ddot{u}_i + v_d \dot{u}_i,1) + \sigma_{ij} (\ddot{u}_i + v_d u_{i,1},1) \right] dv.$$  

Note that the integral (II.70) is zero for a steady-state moving dislocation, since the field solutions for a steady-state moving dislocation are all in the form as $f(x - vt)$, the integrand vanishes. For the moving dislocation which jumps from rest to a constant velocity, the only singular terms in the expansions of the near field solutions about the current position of the dislocation are the same as the terms for steady motion in the velocity $v_d$. Hence, they concluded that the integration (II.70) vanishes when the radii of $S_{d,i}$ for $i = 1, 2$ go to zero, and one can define uniquely the energy flux

$$\dot{E}_0 = \lim_{S_d \to 0} \dot{E}_{S_d}.$$
Using the near field expansions from the exact solutions of the transient motion of screw and edge dislocations, Clifton and Markenscoff (1981) calculated the energy flux of (II.71) and (II.69), based on a circular contour and rectangular contour, where $R^*$ is considered as a two-dimensional area between two closed curves $S_{d1}$ and $S_{d2}$. From the definition (II.68), they obtained the drag forces on moving screw and edge dislocations with Burgers vectors $b_z$ and $b_x$ respectively, which jump from rest and move in a constant velocity $v_d$.

$$F_{\text{screw}} = \mu b_z^2 \left[ \frac{1 - (1 - v_d^2/c_2^2)^{\frac{1}{2}}}{(1 - v_d^2/c_1^2)^{\frac{1}{2}}} \right];$$ (II.72)

$$F_{\text{edge}} = \frac{\mu b_x^2}{2\pi v_d} \left[ \frac{12 - 8\alpha^2}{\beta^2(1 - \alpha^2)} - \frac{(2 - \beta^2)(6 - 7\beta^2)}{\beta^2(1 - \beta^2)^{\frac{1}{2}} - 2(1 - c_2^2/c_1^2)} \right],$$ (II.73)

where

$$\alpha = \frac{v_d}{c_1}, \quad \beta = \frac{v_d}{c_2},$$ (II.74)

assuming that $v_d < c_2 < c_1$.

### II.D  Remark on the Energy Flux Method

The energy flux method for calculating the force on a moving dislocation used in Clifton and Markenscoff (1981) can not apply to the case when the dislocation is moving non-uniformly.

As in Clifton and Markenscoff (1981), using the expression (II.69) as the energy flux is in fact assuming that the energy flux expression for an advancing crack can be used for a moving screw dislocation. As pointed by Freund (1972), “in any case, path independence of” the integral of the energy flux “should be checked before it is used for any particular problem”

Here we show that for the case of a screw dislocation moving non-uniformly, the “path independent” does not hold, even as the integral contour is very close and shrinking to the core of the dislocation as in Clifton and Markenscoff (1981). As explained in last subsection, we only need to prove that the area integral (II.70) does not go to zero as $S_{d,1}$ and $S_{d,2}$ approach to zero.
Let a dislocation move along the $x$–axis according to $x = l(t)$ with a non-constant velocity $v(t) = \dot{l}(t)$. By using the most singular terms of the near field expansions of a non-uniformly moving screw dislocation, we calculate the leading term of the integral (II.70),

$$I \equiv \int_{R^*} [\rho \dot{u}_3 + v_d \dot{u}_{3,1} + \sigma_{3j}(\dot{u}_3 + v_d u_{3,1})_j] dv,$$

where repeated $j$ is summed for $j = 1, 2$, and $R^*$ is the area between two circles $S_{d,1}$ and $S_{d,2}$ with radii $r_1$ and $r_2$, respectively, such that $r_1 < r_2$. Our calculation will show that $I$ does not converge to zero when $r_1, r_2 \to 0$.

(II.70) is equivalently rewritten as

$$I = \int_{R^*} [\rho \dot{u}_3 (\partial_t + v(t) \partial_x) \dot{u}_3) + \mu u_{3,j} (\partial_t + v(t) \partial_x) u_{3,j}] dv, \quad (\text{II.75})$$

where again the repeated $j$ is summed for $j = 1, 2$.

We note that for a function $f(\epsilon, \theta, t)$ with $\epsilon^2 = (x - l(t))^2 + y^2$ and $\theta = \tan^{-1}(y/(x - l(t)))$, from

$$(\partial_t + v(t) \partial_x) \epsilon = (\partial_t + v(t) \partial_x) \theta = 0, \quad (\text{II.76})$$

it follows that

$$(\partial_t + v(t) \partial_x) f(\epsilon, \theta, t) = (\partial f/\partial t)|_{\text{exp}}, \quad (\text{II.77})$$

where the explicit partial differentiation $(\partial f/\partial t)|_{\text{exp}}$ means the derivative with respect to $t$ is taken when $\epsilon$ and $\theta$ are fixed.

Hence we have

$$\dot{u}_3 (\partial_t + v(t) \partial_x) \dot{u}_3 = \dot{u}_3 (\partial u_3 / \partial t)|_{\text{exp}}, \quad (\text{II.78})$$

and

$$u_{3,j} (\partial_t + v(t) \partial_x) u_{3,j} = u_{3,j} (\partial u_{3,j} / \partial t)|_{\text{exp}}, \quad (\text{II.79})$$

for $j = 1, 2$.

Using the leading term (II.36) of the near field expansion of $\dot{u}_3$,

$$\dot{u}_3 \sim \frac{b}{2\pi} \frac{v(t) \gamma \sin \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta} \frac{1}{\epsilon}, \quad (\text{II.80})$$
where $\gamma = \sqrt{1 - v^2(t)/c^2}$ is a function of $t$, we have that
\[
\left( \frac{\partial u_3}{\partial t} \right)_{\text{exp}} \sim \frac{\dot{v}(t) b \sin \theta}{2\pi \epsilon} \left[ \frac{1 - 2\beta^2 + \beta^2 \sin^2 \theta}{\gamma \left( \cos^2 \theta + \gamma^2 \sin^2 \theta \right)^2} \right],
\]
(II.81)
and
\[
\ddot{u}_3 (\partial_t + v(t) \partial_x) \dot{u}_3 \sim \frac{v(t) \dot{v}(t) b \sin^2 \theta}{4\pi^2 \epsilon^2} \left[ \frac{1 - 2\beta^2 + \beta^2 \sin^2 \theta}{\left( \cos^2 \theta + \gamma^2 \sin^2 \theta \right)^3} \right],
\]
(II.82)
where $\beta = v(t)/c^2$.

Similarly, using the leading terms of the near field expansions of $u_{3,1}$ and $u_{3,2}$,
\[
u_{3,1} = -\frac{b}{2\pi \epsilon} \frac{\gamma \sin \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta},
\]
(II.83)
and
\[
u_{3,2} = -\frac{b}{2\pi \epsilon} \frac{\gamma \cos \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta},
\]
(II.84)
we obtain
\[
(\partial_t + v(t) \partial_x)u_{3,1} \sim -\frac{b \gamma \cos \theta}{2\pi \epsilon} \left[ \frac{\cos^2 \theta - \gamma^2 \sin^2 \theta}{\left( \cos^2 \theta + \gamma^2 \sin^2 \theta \right)^2} \right],
\]
(II.85)
\[
(\partial_t + v(t) \partial_x)u_{3,2} \sim \frac{b \gamma \sin \theta}{2\pi \epsilon} \left[ \frac{\cos^2 \theta - \gamma^2 \sin^2 \theta}{\left( \cos^2 \theta + \gamma^2 \sin^2 \theta \right)^2} \right]
\]
(II.86)
and
\[
u_{3,1}(\partial_t + v(t) \partial_x)u_{3,1} + u_{3,2}(\partial_t + v(t) \partial_x)u_{3,2}
\sim \frac{b^2 \gamma^2}{4\pi^2 \epsilon^2} \left[ \frac{\cos^2 \theta - \gamma^2 \sin^2 \theta}{\left( \cos^2 \theta + \gamma^2 \sin^2 \theta \right)^3} \right].
\]
(II.87)

Using those results in (II.75), the area integral $I$ is rewritten as
\[
I \sim I_1 + I_2,
\]
(II.88)
where $I_1$ and $I_2$ are defined by
\[
I_1 \equiv \int_{r_1}^{r_2} \int_0^{2\pi} \rho b^2 v(t) \dot{v}(t) \sin^2 \theta \frac{1 - 2\beta^2 + \beta^2 \sin^2 \theta}{4\pi^2 \epsilon^2} \left[ \frac{1 - 2\beta^2 + \beta^2 \sin^2 \theta}{\left( \cos^2 \theta + \gamma^2 \sin^2 \theta \right)^3} \right] \epsilon d\epsilon d\theta,
\]
(II.89)
and

\[
I_2 = \int_{r_1}^{r_2} \int_0^{2\pi} \mu b^2 \gamma \int_0^{2\pi} \frac{\cos^2 \theta - \gamma^2 \sin^2 \theta}{4\pi^2 \varepsilon (\cos^2 \theta + \gamma^2 \sin^2 \theta)^3} \varepsilon d\varepsilon d\theta,
\]

respectively.

The integrals \(I_1\) and \(I_2\) are calculated and given as

\[
I_1 = \frac{\rho b^2 v(t) \dot{v}(t)}{8\pi \gamma^3} (2 - \beta^2) \ln(r_2/r_1),
\]

\[
I_1 = \frac{\rho b^2 v(t) \dot{v}(t)}{8\pi \gamma^3} \beta^2 \ln(r_2/r_1).
\]

In the calculation, the following integrals have been evaluated

\[
\int_0^{2\pi} \frac{\sin^2 \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^3} d\theta = \frac{\pi (4 - \beta^2)}{4\gamma^5},
\]

\[
\int_0^{2\pi} \frac{\cos^2 \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^3} d\theta = \frac{\pi (4 - 3\beta^2)}{4\gamma^3},
\]

and

\[
\int_0^{2\pi} \frac{\sin^4 \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^3} d\theta = \frac{3\pi}{4\gamma^5}.
\]

And we have calculated the following two integral

\[
\int_0^{2\pi} \frac{\sin^2 \theta (1 - 2\beta^2 + \beta^2 \sin^2 \theta)}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^3} d\theta = \frac{\pi (2 - \beta^2)}{2\gamma^3},
\]

and

\[
\int_0^{2\pi} \frac{\cos^2 \theta - \gamma^2 \sin^2 \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^3} d\theta = -\frac{\pi \beta^2}{2\gamma^3},
\]

also used the relation

\[
\mu \gamma \dot{\gamma} = -\frac{\mu v(t) \dot{v}(t)}{c^2} = -\rho v(t) \dot{v}(t).
\]

Consequently, we obtain

\[
I \sim I_1 + I_2 = \frac{\rho b^2 v(t) \dot{v}(t)}{4\pi \gamma^3} \ln(r_2/r_1),
\]
which implies that the leading term of $I$ does not go to zero as $r_1$ and $r_2$ approach to zero, since the ratio $r_2/r_1$ may take any value.

Hence, the energy flux integral (II.69) is not “contour-independent” in the case of a non-uniformly moving dislocation. Therefore, the energy flux method used in Clifton and Markenscoff (1981) is not appropriate for defining and calculating the force on a non-uniformly moving dislocation.

We note that dislocation is a defect in material, and the motion of a dislocation in material changes the configuration. Eshelby’s theory of the configurational forces on elastic defects may be suitable for this purpose.
Chapter III

Configurational Force on a Static Elastic Defect

III.A Introduction

Eshelby (1951) introduced the concept of the force on an elastic imperfection. An imperfection is an elastic defect, singularity, or inhomogeneity. In a series of papers (1970, 1975, and 1977), Eshelby further expanded his theory. The total energy in material is determined by externally applied forces and the presence of imperfections in the material. When the applied forces are absent or kept fixed, then the total energy is a function of the parameters describing the configuration of the imperfection. According to Eshelby (1951, 1970), the configurational force on the imperfection is defined as the negative gradient of the total energy with respect to the position of the imperfection. Namely, the configurational force $F_l$ on an elastic imperfection satisfies

$$ F_l \delta \xi_l = -\delta E, \quad (III.1) $$

where $l$ is not summed, $\delta \xi_l$ is a virtual translation of the imperfection relative to the material, and $\delta E$ is the corresponding change of the total energy of the system.

Eshelby (1951) derived the conservation laws in elastostatics,

$$ \frac{\partial P_{ij}}{\partial x_j} = 0, \quad (III.2) $$
where $j = 1, 2, 3$, are summed, $l = 1, 2, 3$ not summed, and

$$P_{lj} = W \delta_{lj} - \sigma_{jk} u_{k,l} \tag{III.3}$$

is the energy-momentum tensor, with $\sigma_{jk}$ and $u_k$ as the stress and displacement components, and $W = 1/2 \sigma_{jk} u_{j,k}$ the elastic potential energy density.

Eshelby proved that the configurational force on a defect can be expressed as an integral of the energy-momentum tensor over the surface enclosing the defect,

$$F_l = \int_S P_{lj} dS_j, \tag{III.4}$$

where the repeated indices are summed for $j = 1, 2, 3$, $S$ is the surface enclosing the defect, and $dS_j = n_j dS$ with $(n_1, n_2, n_3)$ as the outer normal of the surface. The force given in the last equation is well-defined, since as shown in Eshelby (1975), the integral is “contour-independent”, i.e., it is independent of the choices of the surface $S$ if it encloses the defect, and is assumed to be sufficiently smooth.

### III.B Energy Momentum Tensor and Conservation Laws

The concept of the energy momentum tensor is from field theory. The elastic energy momentum tensor, and the following derivation are due to Eshelby.

Assume that there exists the Lagrangian density $L$ in the elastic system, which is a function of the displacement $u_i$, its first derivative $u_{i,j}$, and the independent spatial variable $x_i$,

$$L = L(u_i, u_{i,j}, x_j) \tag{III.5}$$

where $i, j = 1, 2, 3$. $L = T - W$, with $T$ as the kinetic energy density, and $W$ the potential energy density. In elastostatics, $T = 0$ and $L = -W$.

The Euler-Lagrange equations

$$\frac{\partial}{\partial x_j} \left[ \frac{\partial L}{\partial u_{i,j}} \right] - \frac{\partial L}{\partial u_i} = 0, \tag{III.6}$$
are the equation of equilibrium

\[ \sigma_{ij,j} = 0, \]  \hspace{1cm} (III.7)

where the repeated indices \( j \) are summed.

Consider now the partial derivative of the Lagrangian density with respect to \( x_i \), and noting that in elastostatics \( L = -W \), we have

\[ \frac{\partial W}{\partial x_i} = \frac{\partial W}{\partial u_i} u_{i,l} + \frac{\partial W}{\partial u_{i,j}} \frac{\partial u_{i,j}}{\partial x_l} + (\frac{\partial W}{\partial x_l})_{\text{exp}} \]  \hspace{1cm} (III.8)

where \((\partial W/\partial x_l)_{\text{exp}}\) denotes the explicit partial derivative with respect to \( x_l \) when \( u_i, u_{i,j}, \) and \( x_k, k \neq l \) are kept fixed.

Taking into account of the following relations

\[ \frac{\partial u_{i,j}}{\partial x_l} = \frac{\partial^2 u_i}{\partial x_j \partial x_l}, \]  \hspace{1cm} (III.9)

(III.8) is rewritten as

\[ \frac{\partial W}{\partial x_l} = \left( \frac{\partial W}{\partial u_i} - \frac{\partial}{\partial x_j} \left( \frac{\partial W}{\partial u_{i,j}} \right) u_{i,l} + \frac{\partial}{\partial x_j} \left( \frac{\partial W}{\partial u_{i,j}} u_{i,l} \right) + (\frac{\partial W}{\partial x_l})_{\text{exp}} \right). \]  \hspace{1cm} (III.10)

In the last equation, using the Euler-Lagrange Equation (III.6) and regrouping terms, we have that

\[ \frac{\partial P_{lj}}{\partial x_j} = (\frac{\partial W}{\partial x_l})_{\text{exp}}, \]  \hspace{1cm} (III.11)

where

\[ P_{lj} = W \delta_{lj} - \sigma_{jk} u_{k,l} \]  \hspace{1cm} (III.12)

is the elastic energy-momentum tensor defined in (III.3).

The explicit partial derivative of the Lagrangian density with respect to the spatial variable \( x_l \) vanishes in the region where the Lagrangian density is invariant under translation in the direction of \( x_l \), such as, the material is homogeneous in the region. Then, from (III.11), we obtain the conservation laws in elastostatics, for \( l = 1, 2, 3 \),

\[ \frac{\partial P_{lj}}{\partial x_j} = 0. \]  \hspace{1cm} (III.13)
In the above discussion, \( u_i \) is required to be continuously differentiable. For the Volterra dislocation, although \( u_i \) has a jump discontinuity, except at the core of the dislocation, \( u_{i,j} \) is continuous. Hence, in view of that the elastic Lagrangian is not explicitly dependent of \( u_i \), the conservation laws in elastostatics discussed above apply to the Volterra dislocation in the region that does not contain the dislocation core.

## III.C Static Configurational Force on an Elastic Defect

If there are two disjoint smooth surfaces \( S_1 \) and \( S_2 \) enclosing the same defect, with \( S_2 \subset S_1 \). In the volume \( V \) between \( S_1 \) and \( S_2 \), material is homogeneous and contains no singularity, hence,

\[
\frac{\partial P_{ij}}{\partial x_j} = 0. \tag{III.14}
\]

By Gauss divergence theorem

\[
\int_V \frac{\partial P_{ij}}{\partial x_j} dv = \int_{S_1} P_{ij}dS_j - \int_{S_2} P_{ij}dS_j = 0, \tag{III.15}
\]

\[
\int_{S_1} P_{ij}dS_j = \int_{S_2} P_{ij}dS_j. \tag{III.16}
\]

Therefore, each of those two integrals defines a “contour-independent” invariant, which has a physical scale of force. In his series of papers (1951, 1970, and 1975), Eshelby used different ways to prove that the surface integral of the energy momentum tensor is exactly the configurational force on the defect defined in (III.1). Hence, the configurational force on a static elastic defect is given by

\[
F_l = \int_{S_1} P_{ij}dS_j \tag{III.17}
\]

A more generalized result for the dynamic force will be derived in the forthcoming discussion, where (III.17) for the static configurational force will be derived as a special case.
Eshelby (1970) also gave an expression for the force on an inhomogeneity over a region \( V \),

\[
F_l = \int_V \left( \frac{\partial W}{\partial x_l} \right) \exp dV. \tag{III.18}
\]

In plane fracture mechanics, Rice (1968), without knowing Eshelby’s work, introduced an important “path-independent integral”, named as J-integral, to represent the energy release rate on the crack tip.

Gunther (1962), and Knowles and Sternberg (1972) independently showed that the conservation laws (III.13) follow from an application of Neother’s theorem on invariant variational principles (Neother, 1981), as a consequence of the invariants under a coordinate translation of the elastic potential energy. In addition, two new types of conservation laws both for linear and finite elastostatics were also derived, which correspond to invariants under a rotation of coordinates, and under a family of coordinate scale changes. Budiansky and Rice (1973) pointed out that those two new types of conservation laws are associated with cavity rotation and cavity expansion. The corresponding integrals are named as L-integral and M-integral, respectively.

### III.D Examples of Static Configuration Forces

#### III.D.1 Force on a Screw Dislocation in an Infinite Homogeneous Isotropic Elastic Medium

Consider an infinitely long screw dislocation line in a homogeneous whole space of an isotropic elastic solid. The configurational force on the dislocation is zero, since there is no characteristic length. When a small translation is given to the dislocation line in any direction, the configuration of the system remains unchanged, so does the total energy. The force on the dislocation is then zero.
III.D.2 Force on a Screw Dislocation in a Homogeneous Isotropic Half-Space

In the case that an infinitely long screw dislocation lies in a homogeneous isotropic half-space, the configurational force on the dislocation is not zero. There now exists a characteristic length: the distance to the boundary of the half-space. A translation of the dislocation line may change its position relative to the boundary.

Let the screw dislocation line situate on the $x, z$-plane in the half-space $x \leq 0$ with a free boundary $x = 0$. A screw dislocation line has a Burgers vector of $(0, 0, b)$, parallels to the $z$-axis, and intersects to $x$-axis at $x = -l$. The configurational force on the dislocation has been calculated as an image force, e.g., in Lardner (1974). Where, the force is calculated as the minus the rate of change of the energy with the dislocation position. In order to have a finite energy, a hollow-core model for the Volterra dislocation was used. The force per unit length on the screw dislocation in a half-space was calculated to be

$$ F_1 = \frac{\mu b^2}{4\pi l}, \quad (\text{III.19}) $$

(Lardner 1974).

Eshelby’s surface integral of the configurational force involves no singularities. The defect is enclosed by the surface, but not on the integral surface. So the evaluation of the convergent surface integral is straightforward. We will calculate the force for this example, and show the effectiveness of Eshelby’s formula.

The non-zero components of the stress field are given as (Lardner, 1974)

$$ \sigma_{31} = -\frac{\mu b}{2\pi} \left( \frac{y}{(x + l)^2 + y^2} - \frac{y}{(x - l)^2 + y^2} \right), \quad (\text{III.20}) $$

$$ \sigma_{32} = -\frac{\mu b}{2\pi} \left( \frac{x + l}{(x + l)^2 + y^2} - \frac{x - l}{(x - l)^2 + y^2} \right). \quad (\text{III.21}) $$

Using Eshelby’s integral, the configurational force is

$$ F_1 = \int_S P_{1j} dS_j. \quad (\text{III.22}) $$
where $S$ is taken to be the surface of the cylinder of radius $r$ with the dislocation line as the axis of the cylinder, and a unit height in the $z$-direction. Hence,

$$dS_j = n_j dS = r n_j d\theta,$$

(III.23)

where $n_1 = \cos \theta$ and $n_2 = \sin \theta$. The configurational force is given by

$$F_1 = \int_0^{2\pi} r [P_{11} \cos \theta + P_{12} \sin \theta] d\theta.$$

(III.24)

The components $P_{11}$ and $P_{12}$ of the energy-momentum tensor are

$$P_{11} = W - \sigma_{31} u_{3,1} = \frac{1}{2\mu} (\sigma_{32}^2 - \sigma_{31}^2),$$

(III.25)

$$P_{12} = -\sigma_{32} u_{3,1} = -\frac{1}{\mu} \sigma_{32} \sigma_{31},$$

(III.26)

where

$$W = \frac{1}{2} \sigma_{3j} u_{3,j} = \frac{1}{2\mu} (\sigma_{31}^2 + \sigma_{32}^2).$$

(III.27)

So that (III.24) is written as

$$F_1 = \int_0^{2\pi} \frac{r}{2\mu} [(\sigma_{32}^2 - \sigma_{31}^2) \cos \theta - 2\sigma_{32} \sigma_{31} \sin \theta] d\theta.$$

(III.28)

Using the polar coordinates

$$x + l = r \cos \theta, \quad y = r \sin \theta,$$

(III.29)

the stresses (III.20) and (III.21) is reduced to

$$\sigma_{31} = -\frac{\mu b}{\pi r} \frac{2l \cos \theta (l - r \cos \theta) + lr}{4l^2 + r^2 - 4rl \cos \theta},$$

(III.30)

$$\sigma_{32} = \frac{\mu b \sin \theta}{\pi r} \frac{l(l - r \cos \theta)}{4l^2 + r^2 - 4rl \cos \theta}.$$  

(III.31)

Substituting the last two expressions of stresses into (III.28), we have

$$F_1 = \frac{2\mu b^2}{\pi^2} \frac{1}{r} \int_0^{2\pi} G(r, \theta) d\theta,$$

(III.32)
where
\[ G = \frac{2l^2(l - r \cos \theta)(l \cos \theta + r \sin^2 \theta) + 1/2lr^2 \cos \theta}{(4l^2 + r^2 - 4lr \cos \theta)^2}. \] (III.33)

In view of that the integral expression of \( F_1 \) is independent of the choice of the surface, especially, independent of \( r \), then
\[ F_1 = \frac{\mu b^2}{\pi^2} \lim_{r \to 0} \frac{1}{r} \int_0^{2\pi} G(r, \theta) \, d\theta. \] (III.34)

In view of
\[ \int_0^{2\pi} G(0, \theta) \, d\theta = 1/8 \int_0^{2\pi} \cos \theta \, d\theta = 0, \] (III.35)
from (III.34), we have
\[ F_1 = \frac{\mu b^2}{\pi^2} \int_0^{2\pi} \frac{dG(0, \theta)}{dr} \, d\theta \] (III.36)
A straightforward calculation shows that the configurational force on the screw dislocation in a half-space is evaluated as
\[ F_1 = \frac{\mu b^2}{\pi^2} \int_0^{2\pi} \frac{1}{8l} [2 \cos^2 \theta - \cos 2\theta] \, d\theta = \frac{\mu b^2}{4\pi l}, \] (III.37)
which is identical to Lardner’s result (III.19).

### III.D.3 J-Integral in Fracture Mechanics

Rice’s J-integral (Rice, 1968) in fracture mechanics can be derived from Eshelby’s integral of the configurational force. Assume that a crack is on the \( x \)-axis in \( x, y \)-plane from \(-\infty \) to 0. Regarded as a very flat hole, a crack is actually a planar defect. However, at least in two-dimensional case, the tip of a crack can be considered as a defect or a singularity in its own right (Eshelby, 1970). Therefore, \( F_1 \) given by the integral of the components of energy-momentum tensor \( P_{ij} \) around the loop \( C \) enclosing the crack tip gives the force on the crack tip
\[ F_1 = \int_C P_{ij} \, dS_j = \int_C [W \delta_{ij} - u_{i,1} \sigma_{ij}] \, dS_j, \] (III.38)
which is the energy release rate of Rice’s J-integral.
Chapter IV

Dynamic Configurational Force on a Moving Inhomogeneity

IV.A Introduction

As discussed in the previous chapter, according to Eshelby, the configurational force on a static defect is defined as the negative gradient of the total energy with respect to the position of the defect,

\[ F_l \delta \xi_l = -\delta E. \]  

(IV.1)

It has not been clarified that whether this definition will still be adequate for the configurational force on a moving defect or on a moving inhomogeneity.

Stroh (1962) has questioned the validity of Eshelby’s definition (IV.1) for the dynamics case, and stated that

“The nature of the force on a dislocation at rest has been fully discussed by Eshelby (1951), who has particularly emphasized the need for thorough treatment. His conclusions are that (a) the force should be defined as the derivative of the energy with respective to dislocation displacement; from it follows that (b) the force in the slip plane is just \( \sigma b \) per unit length of the dislocation, where \( \sigma \) is the resolved shear stress. It seems to have been accepted quite uncritically that both statements (a) and (b) apply also in the dynamic case, without realizing that here
that they are in fact inconsistent.”

Instead of using the total energy of the system, Stroh used the interacting Lagrangian of the dislocations to derive the interacting force between two dislocations, and showed that (a) was “modified by replacing the energy with the Lagrangian”, it was found that “(b) remains true”.

Inspired by Stroh (1962) (see also Lothe, 1992; Hirth, et al., 1998), we propose a definition of the dynamic configurational force on a moving inhomogeneity by using the change of the total Lagrangian of the mechanical system, i.e.,

$$\delta \varepsilon \int_{t_1}^{t_2} L_{\text{total}} dt = \int_{t_1}^{t_2} F_l \delta \xi_l dt,$$ (IV.2)

where $\delta \xi_l$ is the infinitesimal translation of the elastic inhomogeneity, and $\delta \varepsilon$ represents the change due to the infinitesimal translation $\delta \xi_l$, assuming that the elastic medium is the infinite whole space, and there is no body force.

When the elastic system is static, then $L = -W$ and all quantities are independent of time $t$, (IV.2) is clearly reduced to Eshelby’s definition (IV.1).

From the definition (IV.2), the following expression of the configurational force on a moving elastic inhomogeneity will be derive

$$F_l = - \int (\frac{\partial L}{\partial x_l})_{\text{exp}} dx^3,$$ (IV.3)

from which, further, a “contour-independent” integral expression of the force on a moving inhomogeneity is obtained.

In the derivation of (IV.3), certain regularity of the elastic field is assumed, such as the elastic field should be twice continuously differentiable. However, in general, the elastic field may not be continuous over the boundary of the inhomogeneity. The discontinuous field will be considered as limiting case. A rigorous treatment of such limiting process will be given. We show that for non-singular discontinuous case the “contour-independent” integral expression will still hold.

We start this chapter with a brief review of the conservation laws in elastodynamics, which are important for the forthcoming discussion.
IV.B Eshelby’s Conservation Laws in Elastodynamics

In this section, we review Eshelby’s $4 \times 4$ dynamic energy momentum tensor and the conservation laws in elastodynamics (Eshelby, 1970). The derivation parallels to that for elastostatics.

Assume that the Lagrangian density $L$ is a function of the displacement $u_i$, its first derivative $u_{i,\alpha}$, and the independent variable $x_\alpha$, i.e.,

$$L = L(u_i, u_{i,\alpha}, x_\alpha), \quad (IV.4)$$

where $i, j = 1, 2, 3$, $\alpha = 0, 1, 2, 3$, $x_i$ for $i = 1, 2, 3$, are spatial independent variables, $x_0 = t$ is the time variable, and $u_{i,0} = \dot{u}_i$.

For linear elastodynamics,

$$L = \frac{1}{2}[\rho \dot{u}_j \dot{u}_j - C_{jkr} \dot{u}_r \dot{u}_j]. \quad (IV.5)$$

The Euler-Lagrange equation

$$\frac{\partial}{\partial x_\alpha} \left[ \frac{\partial L}{\partial u_{i,\alpha}} \right] - \frac{\partial L}{\partial u_i} = 0, \quad (IV.6)$$

is the field equation of motion

$$\sigma_{ij,j} = \frac{\partial}{\partial t} [\rho \dot{u}_i]. \quad (IV.7)$$

To derive the conservation laws in elastodynamics, as in Eshelby (1970), we take the partial derivative of the Lagrangian density with respect to $x_\alpha$. It gives

$$\frac{\partial L}{\partial x_\alpha} = \frac{\partial L}{\partial u_i} u_{i,\alpha} + \frac{\partial L}{\partial u_{i,\beta}} \frac{\partial u_{i,\beta}}{\partial x_\alpha} + (\frac{\partial L}{\partial x_\alpha})_{\exp}, \quad (IV.8)$$

where the repeated indices $i$ are summed from 1 to 3, and the repeated indices $\beta$ are summed from 0 to 3, and the last term in the equation, as before, the explicit partial derivative denotes the partial derivative with respect to $x_\alpha$ when $\dot{u}_i$, $u_{i,j}$, and $x_\beta$, $\beta \neq \alpha$ are assumed to be fixed.
Manipulating the second term on the right hand side, (IV.8) is rewritten as

\[ \frac{\partial L}{\partial x_\alpha} = \frac{\partial L}{\partial u_i} \frac{\partial}{\partial x_\beta} \left( \frac{\partial L}{\partial u_{i,\beta}} \right) u_{i,\alpha} + \frac{\partial}{\partial x_\beta} \left( \frac{\partial L}{\partial u_{i,\beta}} u_{i,\alpha} \right) + \frac{\partial L}{\partial x_\alpha} \exp. \]  

(IV.9)

In the last equation, using the Euler-Lagrange Equation (IV.6) and re-grouping terms, we obtain

\[ \frac{\partial P_{\alpha\beta}}{\partial x_\beta} = -(\frac{\partial L}{\partial x_\alpha})_\exp \]  

(IV.10)

where \( P_{\alpha\beta} \) is the \( 4 \times 4 \) energy-momentum tensor for \( \alpha, \beta = 0, 1, 2, 3 \), defined as

\[ P_{\alpha\beta} = u_{i,\alpha} \frac{\partial L}{\partial u_{i,\beta}} - L \delta_{\alpha\beta}. \]  

(IV.11)

In elastodynamics, \( L = T - W \). The sixteen components of the energy-momentum tensor are listed below.

\[ P_{00} = \dot{u}_i \frac{\partial L}{\partial \dot{u}_i} - L = T + W = H, \]  

(IV.12)

which is the Hamiltonian energy density.

\[ P_{0l} = \dot{u}_i \frac{\partial L}{\partial u_{i,l}} = -\dot{u}_i \sigma_{il} \]  

(IV.13)

is the energy flow vector for \( l = 1, 2, 3 \).

\[ P_{l0} = \frac{\partial L}{\partial u_i} u_{i,l} = \rho \ddot{u}_i u_{i,l} \]  

(IV.14)

is the pseudo-momentum vector for \( l = 1, 2, 3 \).

\[ P_{ij} = (W - T) \delta_{ij} - u_{i,l} \sigma_{ij} \]  

(IV.15)

is the spatial components of the energy momentum tensor for \( l, j = 1, 2, 3 \). (IV.10) is then written as

\[ -(\frac{\partial L}{\partial x_\alpha})_\exp = \frac{\partial P_{\alpha\beta}}{\partial x_\beta} = \frac{\partial P_{\alpha0}}{\partial \dot{t}} - \frac{\partial P_{\alpha j}}{\partial x_j}. \]  

(IV.16)

In the equation, using (IV.12)-(IV.15), we obtain that for \( \alpha = 0, \)

\[ -(\frac{\partial L}{\partial \dot{t}})_\exp = \frac{\partial P_{0\beta}}{\partial x_\beta} = \frac{\partial H}{\partial \dot{t}} + \frac{\partial}{\partial x_j}[\dot{u}_i \sigma_{ij}]; \]  

(IV.17)
and for \( \alpha = l = 1, 2, 3 \),

\[
-(\frac{\partial L}{\partial x_l})_{\text{exp}} = \frac{\partial P_{l\beta}}{\partial x_\beta} = \frac{\partial}{\partial t}[\rho \ddot{u}_i u_i,i] + \frac{\partial}{\partial x_j}[(W - T)\delta_{ij} - u_{i,l}\sigma_{ij}] .
\] (IV.18)

When the Lagrangian density does not explicitly depend on the independent variable \( x_\alpha \) for \( \alpha = 0, 1, 2, 3 \), for instance, the elastic constants are not explicitly dependent of the position or time, then, from (IV.10), we have the following conservation laws

\[
\frac{\partial P_{\alpha\beta}}{\partial x_\beta} = 0,
\] (IV.19)

where \( \alpha, \beta = 0, 1, 2, 3 \).

When \( \alpha = 0 \), the conservation laws give the energy conservation law

\[
\frac{\partial H}{\partial t} - \frac{\partial}{\partial x_j}[(\dot{u}_i\sigma_{ij})] = 0,
\] (IV.20)

where \( i, j = 1, 2, 3 \) are summed.

When \( \alpha = l = 1, 2, 3 \), we have the main conservation laws

\[
\frac{\partial}{\partial t}[(\rho \ddot{u}_i u_i,i] + \frac{\partial}{\partial x_j}[(W - T)\delta_{ij} - u_{i,l}\sigma_{ij}] = 0,
\] (IV.21)

where again \( i, j = 1, 2, 3 \) are summed.

**IV.C Noether’s Theorem: Fletcher’s Conservation Laws in Elastodynamics**

Based on Noether’s theorem of the invariant variational principle in field theory (1918), Fletcher (1976) obtained a set of conservation laws in elastodynamics, which especially included the above discussed conservation laws.

Noether’s theorem is stated as follows: If a given set of differential equations can be identified as the Euler-Lagrange equations corresponding to a variational problem for an action integral and the action integral remains invariant under an n-parameter group of infinitesimal transformations, then there is an associated set of n conservation laws, which are satisfied by all solutions of the original differential equations. (Fletcher, 1976).
Especially, for the translational invariance, i.e., if the action integral
\[ \mathcal{L} = \int L(u_i, u_{i,\alpha}, x_\alpha) d^4x \]  
(IV.22)
is invariant under the coordinates translation
\[ x_\alpha \rightarrow x_\alpha + \epsilon a_\alpha, \]  
(IV.23)
for \( \alpha = 0, 1, 2, 3 \) and provided \( u_i \) are fixed, then it follows the above discussed conservation laws (IV.20) and (IV.21).

In Fletcher (1976), the integral form of the conservation laws corresponding to (IV.20) and (IV.21) are given as well
\[ \frac{d}{dt} \int_D H dV - \int_{\partial D} \dot{u}_i \sigma_{ij} dS_j = 0, \]  
(IV.24)
\[ \frac{d}{dt} \int_D \rho \dot{u}_i u_{i,l} dV + \int_{\partial D} [(W - T) \delta_{ij} - u_{i,l} \sigma_{ij}] dS_j = 0. \]  
(IV.25)

**IV.D Definition of the Configurational Force on a Moving Inhomogeneity**

Inspired by Stroh (1962) (see also Lothe, 1992; Hirth, et al., 1998), we define the configurational force on a moving elastic inhomogeneity according to the change of the total Lagrangian of the whole material space and over a time period when the inhomogeneity experiences a virtue infinitesimal translation. Here the inhomogeneity means a region where the elasticity properties are different from the otherwise homogeneous elastic body, such differences of the elastic properties can be piecewise, or point-wise.

Consider an otherwise homogeneous elastic solid which contains a region of inhomogeneity and such that the mechanical system having a Lagrangian density \( L \). Suppose that the region of inhomogeneity is given a virtual infinitesimal translation. To describe such translation, for each time \( t \), we choose a “typical representative” point \( \xi \) in the region of inhomogeneity (see Stroh, 1962). The translation of the inhomogeneity is then characterized by the translation of the
point $\xi$. Here we assume that under such virtual infinitesimal translation of the inhomogeneity, the shape of the region of the inhomogeneity remains unchanged.

The Lagrangian density $L$ then depends on $\xi$ and is denoted as

$$L = L(x_\alpha, u_i, u_{i,\alpha}; \xi), \tag{IV.26}$$

where again $i = 1, 2, 3$, $\alpha = 0, 1, 2, 3$, $x_i$ for $i = 1, 2, 3$, are spatial independent variables, $x_0 = t$ is the time variable, and $\xi$ characterizes the position of the inhomogeneity. The virtual infinitesimal translation of the inhomogeneity will translate $\xi$ to $\xi + \delta\xi$. The Lagrangian density is then changed to

$$L(x_\alpha, u_i, u_{i,\alpha}; \xi + \delta\xi). \tag{IV.27}$$

Without loss of generality, here we assume that the infinitesimal translation is along one coordinate, i.e., $\delta\xi_l = \delta\xi_l$ for $l = 1, 2, 3$ fixed. A general infinitesimal translation is a linear combination of $\delta\xi_l$ for $l = 1, 2, 3$, which corresponds to the linear combination of the configurational force $F_l$ for $l = 1, 2, 3$.

We define the configurational force on a moving inhomogeneity by using the change of the total Lagrangian under a virtual infinitesimal translation. The force $F_l$ is then defined by

$$\delta\xi \int_{t_1}^{t_2} L_{\text{total}} dt = \int_{t_1}^{t_2} F_l \delta\xi_l dt, \tag{IV.28}$$

and

$$L_{\text{total}} = \int_{R^3} L \, dx^3, \tag{IV.29}$$

where $\delta\xi_l$ is the virtual infinitesimal translation of the elastic inhomogeneity, and $\delta\xi$ represents the change due to the infinitesimal translation $\delta\xi_l$, assuming that the elastic medium is the infinite whole space, and there is no body force. We note that the infinitesimal change $\delta\xi_l$ is a function of time $t$. As in Stroh (1962), we assume that it vanishes at the initial and final time, $\delta\xi_l(t_i) = 0$ for $i = 1, 2$. 
IV.E Dynamic Configurational Force for Sufficiently Smooth Elastic Field

We assume that the Lagrangian density

\[ L = L(u_j, \dot{u}_j, u_{j,i}, x_i, t; \xi) \]  \hspace{1cm} (IV.30)

is sufficiently smooth in all its components. Especially the assumption implies that the Lagrangian density is continuous over the boundary of the inhomogeneity. The case where the Lagrangian is not continuous over the boundary of the inhomogeneity is considered as a limiting case, and will be treated later on.

From (IV.28) and (IV.29), we have

\[ \delta \xi \int_{t_1}^{t_2} \int_{\mathcal{R}^3} L dx^4 = \int_{t_1}^{t_2} \int_{\mathcal{R}^3} f_l \delta \xi_t dx^4, \]  \hspace{1cm} (IV.31)

where \( f_l \) is the force density. From (IV.27), the change of the total Lagrangian is given as

\[ \delta \xi \int_{t_1}^{t_2} \int_{\mathcal{R}^3} L dx^4 = \int_{t_1}^{t_2} \int_{\mathcal{R}^3} [L(x_\alpha, u_i, u_{i,\alpha}; \xi + \delta \xi_l) - L(x_\alpha, u_i, u_{i,\alpha}; \xi)] dx^4. \]  \hspace{1cm} (IV.32)

In order to compare the Lagrangians under the integration, we try to bring them to the same configuration in terms of the position of the inhomogeneity. To do so, for the integral of \( L(x_\alpha, u_i, u_{i,\alpha}; \xi + \delta \xi_l) \), we make a change of the integral variable in the \( x_\ell \)-direction:

\[ x_\ell \rightarrow \tilde{x}_\ell = x_\ell + \delta \xi_\ell(t), \]  \hspace{1cm} (IV.33)

and the other independent variables remain unchanged. The Jacobian of the change of the variables is clearly unit. Under the change of integral variables,

\[ \int_{\mathcal{R}^3} L(x_\alpha, u_i, u_{i,\alpha}; \xi + \delta \xi_l) dx^3 = \int_{\mathcal{R}^3} L(x_\alpha - \delta \xi_l, u_i(x_\alpha - \delta \xi_l), u_{i,\beta}(x_\alpha - \delta \xi_l); \xi) dx^3, \]  \hspace{1cm} (IV.34)
where, for simplicity, in the integral on the right hand side, the tildes over the new integral variables \( \tilde{x}_j \) are suppressed and still written as \( x_j \), and the difference \( x_\alpha - \delta \xi_l \) denotes \( x_1 - \delta \xi_l \) when \( \alpha = l \); and denotes \( x_\alpha \) otherwise. Combining (IV.32) and (IV.34), we write

\[
\delta \xi \int_{t_1}^{t_2} \int_{\mathbb{R}^3} L \, dx^4 = \int_{t_1}^{t_2} \int_{\mathbb{R}^3} \Delta_\xi L \, dx^4
\]  

(IV.35)

where \( \Delta_\xi L \) is defined by

\[
\Delta_\xi L \equiv L(x_\alpha - \delta \xi_l, u_i(x_\alpha - \delta \xi_l), u_{i,\beta}(x_\alpha - \delta \xi_l); \xi) - L(x_\alpha, u_i(x_\alpha), u_{i,\beta}(x_\alpha); \xi).
\]  

(IV.36)

Assuming that the elastic field is sufficiently smooth, in Taylor’s expansion of \( \Delta_\xi L \), dropping higher order terms of \( \delta \xi_l \), we have

\[
\Delta_\xi L = (-\delta \xi_l) \left( \frac{\partial L}{\partial x_l} \right)_{\text{exp}} + \frac{\partial L}{\partial u_i} \delta u_i + \frac{\partial L}{\partial u_{i,\beta}} \delta u_{i,\beta},
\]  

(IV.37)

where repeated indices \( i \) and \( \beta \) are summed from 1 to 3, and 0 to 3, respectively.

We note that

\[
\delta u_i = (-\delta \xi_l) u_{i,l},
\]  

(IV.38)

\[
\delta u_{i,j} = u_{i,j}(x_k - \delta \xi_l) - u_{i,j}(x_k) = (-\delta \xi_l) u_{i,jl}
\]  

(IV.39)

\[
\delta \dot{u}_i = \frac{\partial}{\partial t} u_i(x_k - \delta \xi_l, t) - \frac{\partial}{\partial t} u_i(x_k, t)
\]  

\[= \frac{\partial}{\partial t} [u_{i,j}(x_k - \delta \xi_l(t), t) - u_{i,j}(x_k, t)] = \frac{\partial}{\partial t} ((-\delta \xi_l(t)) u_{i,l}].
\]  

(IV.40)

By using those relations, (IV.37) is reduced to

\[
\Delta_\xi L = (-\delta \xi_l) \left( \frac{\partial L}{\partial x_l} \right)_{\text{exp}} + (-\delta \xi_l) \frac{\partial L}{\partial u_i} u_{i,l}
\]

\[+ (-\delta \xi_l) \frac{\partial L}{\partial u_{i,j}} u_{i,jl} + \frac{\partial L}{\partial \dot{u}_i} \frac{\partial}{\partial t} [(-\delta \xi_l(t)) u_{i,l}]
\]  

(IV.41)

Observe that

\[
\frac{\partial L}{\partial u_{i,j}} u_{i,jl} = \frac{\partial}{\partial x_j} \left[ \frac{\partial L}{\partial u_{i,j}} u_{i,l} \right] - \frac{\partial}{\partial x_j} \left[ \frac{\partial L}{\partial u_{i,j}} u_{i,l} \right]
\]  

(IV.42)
and
\[
\frac{\partial L}{\partial \dot{u}_i} \frac{\partial}{\partial t} [(-\delta \xi_l(t))u_{i,l}] = \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{u}_i} (-\delta \xi_l(t))u_{i,l} - \frac{\partial}{\partial t} \frac{\partial L}{\partial u_i} (-\delta \xi_l(t))u_{i,l}. \tag{IV.43}
\]

(IV.41) is then reduced to
\[
\Delta \xi L = (-\delta \xi_l)(\frac{\partial L}{\partial x_l})_{\text{exp}} + (-\delta \xi_l) \frac{\partial}{\partial u_i} \frac{\partial L}{\partial \dot{u}_i} + \frac{\partial}{\partial \dot{u}_i} \frac{\partial L}{\partial u_i} [(-\delta \xi_l(t))u_{i,l}] + \frac{\partial}{\partial x_j} \frac{\partial L}{\partial u_{i,j}} u_{i,l} + \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{u}_i} (-\delta \xi_l(t))u_{i,l}. \tag{IV.44}
\]

In the equation, using the Euler-Lagrange equation
\[
\frac{\partial L}{\partial u_i} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{u}_i} - \frac{\partial}{\partial x_j} \frac{\partial L}{\partial u_{i,j}} = 0, \tag{IV.45}
\]
we obtain
\[
\Delta \xi L = (-\delta \xi_l)(\frac{\partial L}{\partial x_l})_{\text{exp}} + (-\delta \xi_l) \frac{\partial}{\partial x_j} \frac{\partial L}{\partial u_{i,j}} u_{i,l} + \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{u}_i} (-\delta \xi_l(t))u_{i,l}. \tag{IV.46}
\]

In view of that
\[
\int_{R^3} \frac{\partial}{\partial x_j} \frac{\partial L}{\partial u_{i,j}} u_{i,l} dx^3 = 0, \tag{IV.47}
\]
since the strains vanish at the infinity, and
\[
\int_{t_1}^{t_2} \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{u}_i} (-\delta \xi_l(t))u_{i,l} dt = 0, \tag{IV.48}
\]
since \(\delta \xi_l(t_i) = 0\) for \(i = 1, 2\), we have
\[
\int_{t_1}^{t_2} \Delta \xi L dx^3 dt = \int_{t_1}^{t_2} (-\delta \xi_l)(\frac{\partial L}{\partial x_l})_{\text{exp}} dx^3 dt. \tag{IV.49}
\]
So that from (IV.29), (IV.32), (IV.36), and (IV.49), it follows
\[
\int_{t_1}^{t_2} \left\{ \int_{R^3} \left[ (\frac{\partial L}{\partial x_l})_{\text{exp}} + f_l \right] dx^3 \right\} \delta \xi_l dt = 0. \tag{IV.50}
\]

By the Lagrange Fundamental Lemma in the calculus of variations, from (IV.50) and the fact that \(\delta \xi_l(t) = 0\) at \(t = t_1, t_2\), the configurational force on a moving inhomogeneity is obtained to be
\[
F_l = \int_{R^3} f_l dx^3 = - \int_{R^3} (\frac{\partial L}{\partial x_l})_{\text{exp}} dx^3. \tag{IV.51}
\]
IV.F “Contour-Independent” Integral Expression

In view of the fact that outside the region of inhomogeneity,

\[ \left( \frac{\partial L}{\partial x_l} \right)_{\text{exp}} = 0 \]  \hspace{1cm} (IV.52)

hence the integral domain in (IV.51) may be replaced by any volume \( V \) which encloses the region of inhomogeneity. (IV.51) then reduces to

\[ F_l = -\int_V \left( \frac{\partial L}{\partial x_l} \right)_{\text{exp}} dV. \]  \hspace{1cm} (IV.53)

Here and below, unless stated otherwise, all volumes are assumed to be bounded, convex, and regular region, or, can be decomposed into a finite number of such regions, such that each of them has a smooth boundary, and satisfies the requirement for the domain in order to apply Gauss divergence theorem.

As it has been shown in (IV.18),

\[ -\left( \frac{\partial L}{\partial x_l} \right)_{\text{exp}} = \frac{\partial P_{l\beta}}{\partial x_{\beta}} = \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] + \frac{\partial}{\partial x_j} [(W - T) \delta_{ij} - u_{i,j} \sigma_{ij}]. \]  \hspace{1cm} (IV.54)

From that and (IV.53), it follows

\[ F_l = \int_V \left\{ \frac{\partial P_{l\beta}}{\partial x_{\beta}} \right\} dV = \int_V \left\{ \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] + \frac{\partial}{\partial x_j} [(W - T) \delta_{ij} - u_{i,j} \sigma_{ij}] \right\} dV. \]  \hspace{1cm} (IV.55)

Apply Gauss divergence theorem in the last equation, we have the following integral expression of the configurational force on a moving inhomogeneity

\[ F_l = \int_V \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] dV + \int_S [(W - T) \delta_{ij} - u_{i,j} \sigma_{ij}] dS_j, \]  \hspace{1cm} (IV.56)

where \( V \) is a volume enclosing the inhomogeneity, \( S = \partial V \), and \( dS_j = n_j dS \) with \( n_j \) the outer normal of \( S \).

The integral expression in (IV.56) is “contour-independent”, which means that for any two volumes \( V_1 \) and \( V_2 \) such that \( V_2 \supset V_1 \), both enclose the same elastic
inhomogeneity,

\[
\int_{V_2} \frac{\partial}{\partial t} \left[ \rho \dot{u}_i u_{i,l} \right] dV + \int_{\partial V_2} \left[ (W - T) \delta_{ij} - u_{i,l} \sigma_{ij} \right] dS_j
\]

\[
= \int_{V_1} \frac{\partial}{\partial t} \left[ \rho \dot{u}_i u_{i,l} \right] dV + \int_{\partial V_1} \left[ (W - T) \delta_{ij} - u_{i,l} \sigma_{ij} \right] dS_j.
\]

(IV.57)

When the elastic field is sufficiently smooth, the proof of that “contour-independence” is trivial, since both sides of (IV.57) equal to the integral over the inhomogeneity as seen from (IV.53). A general proof is given below which will be also applicable when the elastic field is discontinuous, or for the configurational force on an elastic defect, which will be discussed in the sequel.

To prove (IV.57), it suffices to prove

\[
\int_{V_2 \setminus V_1} \frac{\partial}{\partial t} \left[ \rho \dot{u}_i u_{i,l} \right] dV + \int_{\partial(V_2 \setminus V_1)} \left[ (W - T) \delta_{ij} - u_{i,l} \sigma_{ij} \right] dS_j
\]

\[
= \int_{S_1} \left[ (W - T) \delta_{ij} - u_{i,l} \sigma_{ij} \right] dS_j.
\]

(IV.58)

Noting that the conservation laws (IV.21) holds in the homogeneous region \(V_2 \setminus V_1\). Integrating the equation (IV.21) over the region \(V_2 \setminus V_1\), and using Gauss divergence theorem, we have

\[
\int_{V_2 \setminus V_1} \frac{\partial}{\partial t} \left[ \rho \dot{u}_i u_{i,l} \right] dV + \int_{\partial(V_2 \setminus V_1)} \left[ (W - T) \delta_{ij} - u_{i,l} \sigma_{ij} \right] dS_j = 0.
\]

(IV.59)

In view of that the boundary of \(V_2 \setminus V_1\) is

\[
\partial(V_2 \setminus V_1) = S_2 \cup (-S_1),
\]

(IV.60)

where \(S_i = \partial V_i\) for \(i = 1, 2\), (IV.58) follows from (IV.59).

**IV.G Dynamic Configurational Force for a Discontinuous Elastic Field**

In the above discussion, we assume that the field variables and the Lagrangian are sufficiently smooth, and thus continuous over the boundary of the
inhomogeneity. However, in general, the elastic field may not be continuous over the boundary of the inhomogeneity. In this section, we will show that when the field variables are discontinuous over the boundary but not singular, then the problem can be treated as a limiting case. A rigorous limiting process will be described. By use of a convolution with Friedrichs’ regularization function (Friedrichs, 1953; see also Yosida, 1980), the discontinuous field can be smoothed to an infinitely differentiable field. For that regularized field, the configurational force is well-defined as discussed in previous sections. The change of the total Lagrangian of the discontinuous field will be proved to be the limit of the change of the corresponding Lagrangian of the infinitely differentiable field, as the parameter of the regularization goes to zero. We then show that for that discontinuous case, the “contour-independent” integral expression for the dynamic configurational force (IV.56) on the moving inhomogeneity will hold as well.

IV.G.1 Regularization of a Discontinuous Elastic Field

Consider an inhomogeneity over a closed subset $V_I$ in $R^3$. $V_I$ may be bounded or unbounded, and is contained in a volume $V \subset R^3$. Recall that the Lagrangian density for elastodynamics is expressed in the form

$$L = \frac{1}{2} \left[ \rho \dddot{u}_j \dot{u}_j - C_{jkr,s} u_{r,s} u_{j,k} \right]. \quad \text{(IV.61)}$$

Let the mass density $\rho(x,t)$, the elasticity tensor $C_{ijkl}(x,t)$, and displacement $u_i(x,t)$ be sufficiently smooth with respect to $x_i$ for $i = 1, 2, 3$ and to the time $t$ in $V_I^0$, i.e., the interior of $V_I$, and $R^3 \setminus V_I$, respectively, however there are non-singular jumps on the boundary $\partial V_I$; where $x$ denotes three-dimensional vector. We remark that the singular case can be treated in a similar way as elastic defect which will be discussed in next chapter.

We assume now that $V_I$ and $V$ are bounded regions. The case of unbounded regions will be treated later on. Let $\Omega \supset V$ be a bounded open set in $R^3$. Without loss of generality, we may define $u_i$, $\rho$, and $C_{ijkl}$ to be zero outside $\Omega$ and on $\partial \Omega$, and remain to be sufficiently smooth in $R^3 \setminus V_I$. We construct the
corresponding infinitely differentiable functions as follows.

\[
(u_i)_a = (u_i) \ast \theta_a = \int \limits_\Omega u_i(x - y)\theta_a(y)dy^3 = \int \limits_\Omega u_i(y)\theta_a(x - y)dy^3,
\]

(IV.62)

where \(a > 0\), \(\ast\) is the symbol of convolution, for simplicity, \(x, y\) denote three-dimensional vectors, and the Friedrichs' regularization function \(\theta_a\) is defined by (Friedrichs, 1953, see, also Yosida, 1980)

\[
\theta_a(x) = h_a^{-1} \exp((|x|^2/a^2 - 1)^{-1}) \quad \text{for} \quad |x| = \sqrt{\sum x_i^2} < a \\
\theta_a(x) = 0 \quad \text{for} \quad |x| = \sqrt{\sum x_i^2} \geq a,
\]

(IV.63)

where \(h_a\) is a normalization constant such that

\[
\int \limits_{R^3} \theta_a(x)dx^3 = 1.
\]

(IV.64)

By the way, note that \(\theta_a\) is a three-dimensional \(\delta\)-sequence (see, e.g., Kanwal, 1983).

Similarly, we define

\[
\rho_a = \rho \ast \theta,
\]

(IV.65)

and

\[
(C_{ijkl})_a = C_{ijkl} \ast \theta.
\]

(IV.66)

From the definitions it is clearly seen that \((u_i)_a\), \(\rho_a\), and \((C_{ijkl})_a\) are infinitely differentiable functions on \(R^3\). For that regularized field, the configurational force on the inhomogeneity is well-defined, which, as will be shown below, will converge to the configurational force for the discontinuous field as \(a \to 0\).

\section{Limiting Process: Inhomogeneity over a Bounded Region}

Define \(V_O \equiv \Omega \setminus V_I\). We shall prove following statements for the interior of \(V_I\) and \(V_O\), respectively,
(1) If \( f(x, t) \) is continuously differentiable in \( V^0_I \) and \( V^0_O \), and for \( t \geq 0 \), then when \( a \) is sufficiently small, we have

\[
\frac{\partial}{\partial x_j} (f)_a = \left( \frac{\partial}{\partial x_j} f \right)_a, \tag{IV.67}
\]

and

\[
\frac{\partial}{\partial t} (f)_a = (\dot{f})_a. \tag{IV.68}
\]

Especially, for \( i, j = 1, 2, 3 \),

\[
\frac{\partial}{\partial x_j} (u_i)_a = (u_{i,j})_a, \quad \frac{\partial}{\partial t} (u_i)_a = (\dot{u}_i)_a, \tag{IV.69}
\]

and

\[
\frac{\partial}{\partial t} (u_{i,j})_a = (\dot{u}_{i,j})_a, \quad \frac{\partial}{\partial t} (\rho)_a = (\dot{\rho})_a. \tag{IV.70}
\]

(2) If \( f \) is continuous in \( V^0_I \) and \( V^0_O \) and for \( t \geq 0 \), then as \( a \to 0 \),

\[
(f)_a \to f, \tag{IV.71}
\]

point-wise.

In particular,

\[
(u_i)_a \to u_i, \tag{IV.72}
\]

\[
(u_{i,l})_a \to u_{i,l}, \tag{IV.73}
\]

\[
(\rho)_a \to \rho, \tag{IV.74}
\]

\[
(C_{ijkl})_a \to C_{ijkl}, \tag{IV.75}
\]

point-wise as \( a \to 0 \).
(3) Based on Statement 1 and 2, we shall show that as \( a \to 0 \),

\[
\delta_\xi \int_{t_1}^{t_2} (\mathcal{L})_a dt \to \delta_\xi \int_{t_1}^{t_2} \mathcal{L} dt. \tag{IV.76}
\]

(4)

\[
F_l = \lim_{a \to 0} \int_V \frac{\partial}{\partial t} [(\rho)_a (\dot{u}_i)_a (u_{i,l})_a] dV + \int_S [(W_a - T_a) \delta_{lj} - (u_{i,l})_a (\sigma_{ij})_a] dS_j, \tag{IV.77}
\]

where \( (\sigma_{ij})_a \) is defined by \( (C_{ijrs})_a (u_{r,s})_a \).

(5) We then have the “contour-independent” integral expression for the force on a moving inhomogeneity for the discontinuous field, which is in the same form of that for the smooth field,

\[
F_l = \int_V \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] dV + \int_S [(W - T) \delta_{lj} - u_{i,l} \sigma_{ij}] dS_j. \tag{IV.78}
\]

Now we prove the above statements.

**Proof**

(1)

From (IV.62), we have

\[
\frac{\partial f_a(x)}{\partial x_j} = \int_{B_a(x)} f(y) \frac{\partial \theta_a(x - y)}{\partial x_j} dy. \tag{IV.79}
\]

For \( x \in V_I^0 \), and a sufficiently small \( a \), then \( B_a(x) \subset V_I^0 \), where we define \( B_a(x) = \{ y \mid |x - y| \leq a \} \) is a ball with respect to the norm \(| \cdot |\). By assumption, for \( y \in B_a(x) \), \( f(y) \) is continuously differentiable. Then using the integration by parts, and noting that \( \theta_a(x - y) = 0 \) when \( y \in \partial B_a(x) \), we have

\[
\frac{\partial f_a(x)}{\partial x_j} = \int_{B_a(x)} \frac{\partial f(y)}{\partial y_j} \theta_a(x - y) dy = (\frac{\partial f(x)}{\partial x_j})_a. \tag{IV.80}
\]
(IV.68) is trivial, since the differentiation with respect to \( t \) under the integral is permissive.

(2)

It suffices to prove the statement for \( x \in V_I^o \). If \( x \in V_I^o \), then for a small \( a \), \( B_a(x) \subset V_I^o \). In view of that \( f \) is continuous at \( x \), we may find a sufficiently small number \( a \), such that for any \( 0 < \epsilon << 1 \) and every \( y \in B_a(x) \),

\[
|f(y) - f(x)| < \epsilon. \tag{IV.81}
\]

Then, in view of that \( \theta_a(x - y) \) is zero for \( y \) outside \( B_a(x) \), we have

\[
|f_a(x) - f(x)| \leq \int_{B_a(x)} |f(y) - f(x)|\theta_a(x - y)dy
\leq \epsilon \int_{B_a(x)} \theta_a(x - y)dy = \epsilon, \tag{IV.82}
\]

We have then proved that \( f_a \to f \) point-wise in \( V_I^o \), as \( a \to 0 \).

(3)

It suffices to prove that

\[
\lim_{a \to 0} \int_{\Omega} L_a dx^3 = \int_{\Omega} L dx^3. \tag{IV.83}
\]

By definition, the Lagrangian of the infinitely differentiable field is written as

\[
L_a = \frac{1}{2} \left[ \dot{\rho}_a \frac{\partial (u_i)_a}{\partial t} \frac{\partial (u_i)_a}{\partial t} - (C_{ijkl})a \frac{\partial (u_j)_a}{\partial x_j} \frac{\partial (u_k)_a}{\partial x_l} \right]. \tag{IV.84}
\]

From Statement (1), it follows that for sufficiently small \( a \),

\[
L_a = \frac{1}{2} \left[ \rho_a \dot{u}_a \dot{u}_a - (C_{ijkl})a (u_{ij})a (u_{kl})a \right]. \tag{IV.85}
\]

which, according to Statement (2), will point-wise converge to \( L \), in the interior of \( V_I \) and \( V_O \), i.e.,

\[
\lim_{a \to 0} L_a = \frac{1}{2} \left[ \rho \dot{u} \dot{u} - (C_{ijkl}) (u_{ij}) (u_{kl}) \right] = L. \tag{IV.86}
\]

So that \( L_a \to L \), almost everywhere in \( \Omega \).
We note that if \( f \leq M \) is bounded, then
\[
|f_a(x)| = |\int f \theta_a dx^3| \leq M |\int \theta_a dx^3| = M, \tag{IV.87}
\]
i.e., \( f_a \) is bounded as well. \( L_a \) is thus bounded, since \( L \) is bounded. Therefore, from Lebesque convergence theorem ¹ (see, e.g., EDM, 221c, p.841, 1993)
\[
\lim_{a \to 0} \int_\Omega L_a dx^3 = \int_\Omega \lim_{a \to 0} L_a dx^3 = \int L dx^3. \tag{IV.88}
\]
Statement (3) is then proved.

(4)

As discussed in previous section, for the infinitely differentiable field, we have
\[
\int_{t_1}^{t_2} \delta \xi (\mathcal{L})_a dt = \int_{t_1}^{t_2} \delta \xi_t (F_i)_a dt = \int_{t_1}^{t_2} \delta \xi_t \left\{ \int V \frac{\partial}{\partial t} \left[ \rho_{a}(\dot{u}_i)_a (u_i)_a \right] dV + \int_S \left[ (W_a - T_a) \delta_{ij} - (u_{i,l})_a (\sigma_{ij})_a \right] dS_j \right\} dt. \tag{IV.89}
\]
On the other hand, from the definition of the configurational force,
\[
\int_{t_1}^{t_2} \delta \xi \mathcal{L} dt = \int_{t_1}^{t_2} \delta \xi_t F_i dt. \tag{IV.90}
\]
Compare (IV.89) with (IV.90), and note that from Statement (3)
\[
\lim_{a \to 0} \int_{t_1}^{t_2} \delta \xi (\mathcal{L})_a dt = \int_{t_1}^{t_2} \delta \xi \mathcal{L} dt, \tag{IV.91}
\]
¹If \( \lim_{n \to \infty} f_n(x) \) exists almost everywhere on \( E \), and there exists a \( \phi(x) \) such that \( |f(x)| \leq \phi(x) \) and \( \int_E \phi < \infty \) (for example, if \( \mu(E) < \infty \) and \( |f_n(x)| < M \), then
\[
\lim_{n \to \infty} \int_E f_n = \int_E \left( \lim_{n \to \infty} f_n \right).
\]
we have

\[
\int_{t_1}^{t_2} \delta \xi_1 F_l dt = \int_{t_1}^{t_2} \delta \xi_1 \lim_{a \to 0} \left\{ \int_V \frac{\partial}{\partial t} [\rho_a(\dot{u}_{i_a})_a(u_{i,l})_a] dV + \int_S [(W_a - T_a) \delta_{ij} - (u_{i_l})_a(\sigma_{ij})_a] dS_j \right\} dt.
\]

(IV.92)

Consequently,

\[
F_l = \lim_{a \to 0} \left\{ \int_V \frac{\partial}{\partial t} [\rho_a(\dot{u}_{i_a})_a(u_{i,l})_a] dV + \int_S [(W_a - T_a) \delta_{ij} - (u_{i,l})_a(\sigma_{ij})_a] dS_j \right\},
\]

(IV.93)

which completes the proof of Statement (4).

(5)

To prove Statement (5), it suffices to prove

\[
\lim_{a \to 0} \int_V \frac{\partial}{\partial t} [\rho_a(\dot{u}_{i_a})_a(u_{i,l})_a] dV = \int_V \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] dV,
\]

(IV.94)

and

\[
\lim_{a \to 0} \int_S [(W_a - T_a) \delta_{ij} - (u_{i,l})_a(\sigma_{ij})_a] dS_j = \int_S [(W - T) \delta_{ij} - u_{i,l} \sigma_{ij}] dS_j.
\]

(IV.95)

Both equalities follow from Lebesgue convergence theorem, in view of that the integrands are bounded, and converge almost everywhere on bounded sets \( V \) and \( S \), respectively.

Hence for the discontinuous case, we obtain the same integral expression of the dynamic configurational force on a moving inhomogeneity

\[
F_l = \int_V \frac{\partial}{\partial t} [\rho(\dot{u}_i u_{i,l})] dV + \int_S [(W - T) \delta_{ij} - u_{i,l} \sigma_{ij}] dS_j.
\]

(IV.96)

where \( V \) is the volume enclosing the inhomogeneity which is over a bounded region, and has a smooth boundary \( S = \partial V \).
IV.G.3 Limiting Process: Inhomogeneity over an Unbounded Region

Under the same assumptions as the discussion in the previous subsection, we assume that now $V_I$ and $V$ are unbounded, and all the involved integrals are convergent. In other words, it is assumed that $\rho$, $C_{ijkl}$ are bounded in $R^3$, and $u_{i,j}$, $\dot{u}_i$, and their certain derivatives are infinitesimal near the infinity. Then, for all the involved integrals and for any given $0 < \epsilon << 1$, and sufficiently large $M > 0$, we write

$$\int_D G dv = \int_{D_M} G dv + O(\epsilon), \quad \text{(IV.97)}$$

where $G$ represents the integrand of any of the involved integrals, $D$ is the domain of the integral, which may be a volume or a surface, and for sufficiently large positive number $M$, $D_M \equiv D \cap B_M$ with $B_M$ as the $M$-ball defined as

$$B_M = \{x| |x| \leq M\}. \quad \text{(IV.98)}$$

For the bounded open set $\Omega_M$, $V_M$ is the volume which enclosing the inhomogeneity over the region $(V_I)_M$, and $S_M = \partial V_M$, from the conclusion obtained in previous subsection, we have

$$\int_{\Omega_M} f_I dV dt = \int_{V_M} \frac{\partial}{\partial t}[\rho \ddot{u}_i u_{i,l}]dV + \int_{S_M} [(W - T)\delta_{ij} - u_{i,l}\sigma_{ij}] dS_j \quad \text{(IV.99)}$$

Combing this equation with (IV.97), we have

$$\int_{\Omega} f_I dV dt = \int_{V} \frac{\partial}{\partial t}[\rho \ddot{u}_i u_{i,l}]dV + \int_{S} [(W - T)\delta_{ij} - u_{i,l}\sigma_{ij}] dS_j \quad \text{dt} + O(\epsilon). \quad \text{(IV.100)}$$

The integrals in the equation are not dependent of $\epsilon$. Thus it concludes that

$$F_I = \left\{ \int_V \frac{\partial}{\partial t}[\rho \ddot{u}_i u_{i,l}]dV + \int_{S} [(W - T)\delta_{ij} - u_{i,l}\sigma_{ij}] dS_j \right\} dt. \quad \text{(IV.101)}$$
Hence we have complete the proof that for the discontinuous field, which has a jump non-singular discontinuity, the same “contour-independent” integral expression (IV.101) of the dynamic configurational force on a moving inhomogeneity as for the sufficiently smooth field is valid, where $V$ is the volume enclosing the inhomogeneity which may be bounded or unbounded.

### IV.H Example: Dynamic Effective Normal Force on a Moving Interface

Eshelby (1970) discussed the static effective normal force on an interface. When a region $V_I$ inside the elastic material matrix undergoes some kind of transformation, then the elastic properties inside $V_I$ becomes different from those of the matrix. For example, that can be the coherent twined region developed, so there is a change of form and elastic properties; or the material is the same, but the crystal axes are differently oriented in the two parts of the material, then in the linear case, the elasticity constants are different when referred to the same axes (Eshelby, 1970). By analyzing the change of the total energy due to the virtual shift of the interface, Eshelby derived the effective normal force density on the interface is given by

$$ f = [W] - (t_n)_i \left[ \frac{\partial u_i}{\partial n} \right], \quad \text{(IV.102)} $$

where $(t_n)_i$ is the surface traction at the interface, and $[\cdot]$ means the jump across the interface.

We now consider the corresponding dynamic problem. By using the integral expression of the configurational force on a moving inhomogeneity derived in this chapter, we generalize Eshelby’s result to the dynamic case. Now $V_I$ is considered as the region of the inhomogeneity in the matrix, and assume that within the region $V_I$ the elastic properties are homogeneous, but different from those of the matrix. From the integral expression (IV.56), we have the configurational force on
the inhomogeneity

\[
F_l = \int_V \frac{\partial}{\partial t} P_{l0} dV + \int_S P_{lj} dS_j, \tag{IV.103}
\]

where \( V \supset V_I \) is the volume enclosing the inhomogeneity \( V_I \), and \( S_V = \partial V \). Define that \( V_O = V \setminus V_I \). We have

\[
F_l = (\int_{V_I} + \int_{V_O}) \frac{\partial}{\partial t} P_{l0} dV + \int_{S_V} P_{lj} dS_j. \tag{IV.104}
\]

In view of the fact that in the region \( V_I \) and \( V_O \), the material is homogeneous, respectively, the main conservation laws (IV.21) apply. Then, in the region \( V_O \), using the main conservation laws we have that

\[
\int_{V_O} \frac{\partial}{\partial t} P_{l0} dV + \int_{S_V} P_{lj} dS_j = \int_{S_+} P_{lj}^A dS_j, \tag{IV.105}
\]

where \( S_+ \) denotes the positively oriented interface \( \partial V_I \) with the out-normal pointing to the region \( V_O \). Likewise, applying the conservation laws to the region \( V_I \), we have

\[
\int_{V_I} \frac{\partial}{\partial t} P_{l0} dV = \int_{S_-} P_{lj}^B dS_j, \tag{IV.106}
\]

where \( S_- \) denotes the interface with a negative orientation, i.e., the out-normal of it pointing to the region \( V_I \).

Substituting (IV.105) and (IV.106) into (IV.104), we have

\[
F_l = \int_{S_+} (P_{lj}^O - P_{lj}^I) dS_j = \int_{S_+} [P_{lj}] dS_j, \tag{IV.107}
\]

where \([·]\) means the jump across the boundary \( S_+ = \partial V_I \),

\[
[P_{lj}] \equiv P_{lj}^O - P_{lj}^I, \tag{IV.108}
\]

\( P_{lj}^O \) and \( P_{lj}^I \) denote the energy momentum tensor evaluated from \( V_O \) and \( V_I \), respectively.

The effective normal force \( F \) on the interface is the equal to

\[
F = F_l n_I = \int_S n_I [P_{lj}] n_j dS, \tag{IV.109}
\]
where repeated indices \( l \) and \( j \) are summed. In view of that

\[
P_{lj} = (W - T)\delta_{lj} - u_{i,l}\sigma_{ij}, \tag{IV.110}
\]

from (IV.109), it follows that

\[
F = \int_S \left\{ [W - T] - [(n_l u_{i,l}) (\sigma_{ij} n_j)] \right\} dS
= \int_S \left\{ [W - T] - \left[ \frac{\partial u_i}{\partial n} (t_n)_i \right] \right\} dS \tag{IV.111}
\]

where \((t_n)_i\) is the traction on the interface. The density of the effective normal force on the interface is

\[
f = [W - T] - \left[ \frac{\partial u_i}{\partial n} (t_n)_i \right]. \tag{IV.112}
\]

here, in elastodynamics, \((t_n)_i\) is not necessarily continuous across the interface.

When the problem is static, \(T\) vanishes and \([ (t_n)_i ] = 0\), i.e., the traction is continuous on the interface, then (IV.112) is reduced to

\[
f = [W] - \left[ \frac{\partial u_i}{\partial n} (t_n)_i \right], \tag{IV.113}
\]

which is the Eshelby’s result (IV.102) for the static effective normal force on the interface.
Chapter V

Dynamic Configurational Force on a Moving Defect

V.A Introduction

An elastic defect may be a point singularity, a line, or surface singularity. At the point or line singularity, the stress or displacement may become infinite; on the surface singularity, the stress or displacement becomes infinite, or, discontinuous (see Eshelby, 1951). To define the configurational force on a moving defect, as a conventional way, we may exclude a small neighborhood around the singularity. Such neighborhood is then considered as an inhomogeneity with jump discontinuities on its boundary. As discussed in last chapter, the dynamic configurational force on such inhomogeneity is well-defined and given by a “contour-independent” integral expression. The configurational force on the moving defect is then defined as the limit of the force on that non-singular inhomogeneity as the small neighborhood shrinks upon to the singularity.

We give a necessary and sufficient condition for the existence of the limit in the definition of the dynamic configurational force on a moving defect and so that the force is well-defined. When the limit exists when the small neighborhood which shrinks upon to the singularity can be chosen arbitrary, then the volume integral in the “contour-independent” integral expression converges as an improper
integral. If the limit exists only when the small neighborhood is chosen to be symmetric as required in the definition of the integral of the Cauchy type, then the volume converges as an integral of the Cauchy type. When the condition fails to be satisfied, then the defined force becomes infinite. There are two possible ways to treat such divergence: (i) Regularize the involved divergent integrals by use of methods based on the theory of distributions; (ii) Modify the physical model, e.g., smear the singularity, such that for the modified model, the dynamic configurational force for that model is well-defined.

V.B Definition of the Dynamic Configurational Force on a Moving Defect

As a conventional way, we may exclude a small neighborhood $N_\epsilon$ of the singularity, which may be symmetric or asymmetric, such as a small circle, a narrow cylinder, or other infinitesimal body. The region of the small neighborhood $N_\epsilon$ of the singularity is then considered as a non-singular inhomogeneity, there exists only a jump discontinuity on the boundary of $N_\epsilon$. As discussed in previous chapter, the dynamic configurational force on such inhomogeneity is well-defined, and given by

$$F_\epsilon^i = \int_V \frac{\partial}{\partial t} [\rho \dot{u}_i] dV + \int_S [(W - T) \delta_{ij} - u_{i,j} \sigma_{ij}] dS_j,$$  \hspace{1cm} (V.1)

where $V$ contains $N_\epsilon$, and $S = \partial V$. In view of that the region of $N_\epsilon$ is void, (V.1) is then rewritten as

$$F_\epsilon^i = \int_{V \setminus N_\epsilon} \frac{\partial}{\partial t} [\rho \dot{u}_i] dV + \int_S [(W - T) \delta_{ij} - u_{i,j} \sigma_{ij}] dS_j,$$  \hspace{1cm} (V.2)

where $S = \partial V$.

The configurational force on the singularity is then defined as the limit of the force $F_\epsilon^i$ when $N_\epsilon$ shrinks upon the singularity, namely,

$$F_i = \lim_{\epsilon \to 0} \{ \int_{V \setminus N_\epsilon} \frac{\partial}{\partial t} [\rho \dot{u}_i] dV + \int_S [(W - T) \delta_{ij} - u_{i,j} \sigma_{ij}] dS_j \}.$$  \hspace{1cm} (V.3)
Here we assume that the defect is fully enclosed by \( V \), i.e., \( N_\epsilon \) is in the interior of \( V \), so that the surface integral on \( S \) in (V.3) is in fact independent of \( \epsilon \). Mathematically, the limit

\[
\lim_{\epsilon \to 0} \int_{V \setminus N_\epsilon} \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] dV, \tag{V.4}
\]

is exactly the definition of the singular volume integral

\[
\int_V \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] dV. \tag{V.5}
\]

The force on the defect is also given by the “contour-independent” integral expression

\[
F_l = \int_V \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] dV + \int_S [(W - T)\delta_{ij} - u_{i,l}\sigma_{ij}] dS_j. \tag{V.6}
\]

Here, the volume integral is in general a singular integral.

Recall that if a function \( f(x) \) on \( \mathbb{R}^n \) has a singularity at \( x_0 \in V \subset \mathbb{R}^n \). Here, for simplicity, \( x \) and \( x_0 \) denote \( n \)-dimension variables. The improper integral is defined as

\[
\int_V f(x) dx = \lim_{\epsilon \to 0} \int_{V \setminus N_\epsilon(x_0)} f(x) dx, \tag{V.7}
\]

if the limit exists for any arbitrary \( \epsilon \)-neighborhood \( N_\epsilon(x_0) \) at \( x_0 \).

When the limit in (V.7) does not exist for arbitrary \( \epsilon \)-neighborhood \( N_\epsilon(x_0) \), but exists when the \( \epsilon \)-neighborhood \( N_\epsilon \) of the singular point is taken to be symmetric, specifically, in two dimensional case, a circular disc with a radius of \( \epsilon \), and an \( \epsilon \)-ball in three dimensional case. The value of the limit is called the Cauchy principal value of a divergent integral, the integral is called an integral of the Cauchy type, which is denoted as

\[
P.V. \int_V f(x) dx. \tag{V.8}
\]

The concept of Cauchy principal value may also be easily extended to the case when the integral has a curve or surface singularity.

Hence, The singular integral (V.5) may exist in following senses
1. As a usual improper integral

If the limit (V.4) exists for arbitrary small neighborhood $N_\epsilon$ of the singularity $x_0$ as $\epsilon$ approaches to zero, where $\epsilon$ is the maximum distance from $x$ to $x_0$ for all $x$ in the neighborhood, then the limit approaches to an improper integral in the usual sense. In that case, the configurational force on the defect is given by

$$F_l = \int_V \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] dV + \int_S [(W - T)\delta_{lj} - u_{i,l} \sigma_{ij}] dS_j, \quad (V.9)$$

where the volume integral is an improper integral in the usual sense, $V$ is a volume enclosing the defect, and $S = \partial V$ is a smooth surface.

2. As an integral of the Cauchy type

If the limit (V.4) does not exist for arbitrary neighborhood $N_\epsilon$, and however exists when $N_\epsilon$ is a circular disc with radius $\epsilon$ in two dimensional space, or an $\epsilon$-ball in higher dimensional space, i.e.,

$$N_\epsilon = B_\epsilon \equiv \{ x \mid |x - x_0| \leq \epsilon \} \quad (V.10)$$

where the norm $| \cdot |$ in $R^n$ is defined by $|x| = \sqrt{\sum_{i=1}^{n} x_i^2}$, for $n \geq 2$, then the limit approaches to an integral of the Cauchy type,

$$\lim_{\epsilon \to 0} \int_{V \setminus B_\epsilon} \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] dV = \mathcal{P} \mathcal{V} \int_V \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] dV. \quad (V.11)$$

The force on a moving defect is given as

$$F_l = \mathcal{P} \mathcal{V} \int_V \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] dV + \int_S [(W - T)\delta_{lj} - u_{i,l} \sigma_{ij}] dS_j. \quad (V.12)$$
V.C A Necessary and Sufficient Condition

It is clear that the limit in (V.3) may not always exist, or the singular volume integral (V.5) may not converge. We now establish a necessary and sufficient condition for the existence of the limit (V.3), equivalently, the existence of the volume integral, so that a finite dynamic configurational force on a moving defect is defined.

Note that the volume $V \setminus N_\epsilon$ is a homogeneous region. Applying the conservation laws (IV.21) in the region $V \setminus N_\epsilon$, we obtain

$$
\int_{V \setminus N_\epsilon} \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] dV + \int_{S} \left[ (W - T) \delta_{lj} - u_{i,l} \sigma_{ij} \right] dS_j
= \int_{S_\epsilon} \left[ (W - T) \delta_{lj} - u_{i,l} \sigma_{ij} \right] dS_j, \tag{V.13}
$$

where $S_\epsilon = \partial N_\epsilon$. The limit of the left hand side exists if and only if the limit on the right hand side exists. Hence, we have the necessary and sufficient condition:

(1) The dynamic configurational force on a moving defect is well-defined and expressed by

$$
F_l = \int_{V} \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] dV + \int_{S} \left[ (W - T) \delta_{lj} - u_{i,l} \sigma_{ij} \right] dS_j, \tag{V.14}
$$

if and only if the limit

$$
\lim_{\epsilon \to 0} \int_{S_\epsilon} \left[ (W - T) \delta_{lj} - u_{i,l} \sigma_{ij} \right] dS_j \tag{V.15}
$$

exists, for $S_\epsilon = \partial N_\epsilon$ and $N_\epsilon$ is an arbitrary $\epsilon$-neighborhood.

(2) The dynamic configurational force on a moving defect is well-defined and expressed by

$$
F_l = \mathcal{P} \mathcal{V} \int_{V} \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,l}] dV + \int_{S} \left[ (W - T) \delta_{lj} - u_{i,l} \sigma_{ij} \right] dS_j, \tag{V.16}
$$

if and only if

$$
\lim_{\epsilon \to 0} \int_{S_\epsilon} \left[ (W - T) \delta_{lj} - u_{i,l} \sigma_{ij} \right] dS_j \tag{V.17}
$$
exists, for \( S_\epsilon = \partial B_\epsilon \) and \( B_\epsilon \equiv \{ x | |x - x_0| \leq \epsilon \} \) is an \( \epsilon \)-ball, as required in the definition of the integral of the Cauchy type.

When the condition does not hold, then the configurational force given by the limit

\[
F_l = \lim_{\epsilon \to 0} \{ [ \int_{V \setminus N_\epsilon} \frac{\partial}{\partial t} (\rho \dot{u}_i u_{ij}) dV + \int_S ((W - T)\delta_{ij} - u_{i,j}\sigma_{ij}) dS_j ] \} \tag{V.18}
\]

will be infinite.

In view of (V.13), for any case, the configurational force is equivalently given as

\[
F_l = \lim_{\epsilon \to 0} \int_{S_\epsilon} ((W - T)\delta_{ij} - u_{i,j}\sigma_{ij}) dS_j. \tag{V.19}
\]

There are two possible ways to deal with such divergence: (1) Regularize the divergent integral based on the theory of distributions, (2) Smearing method. Next we will discuss those methods in detail.

\section*{V.D Regularizing the Divergent Integral Based on the Theory of Distributions}

When the volume integral in (V.18) does not exist either in sense of usual improper integral, or in the sense of the Cauchy principal value, then the divergent integral may possibly be regularized based on the theory of distributions.

Distributions are linear continuous functionals on the space of the fundamental functions (or, test functions). One example of the space of fundamental functions is \( \mathcal{K} = C_0^\infty(R^n) \), i.e., the space of infinitely differentiable functions with compact supports in \( R^n \). A locally integral function \( f \) on \( R^n \) corresponds to a linear continuous functional on \( \mathcal{K} \), which is defined as

\[
(f, \phi) = \int_{R^n} f(x)\phi(x) dx, \tag{V.20}
\]

where \( \phi \in \mathcal{K} \), and \( x \) denotes an \( n \)-dimensional vector.
If \( f \) is not locally integrable, then the integral (V.20) does not exist for all \( \phi \in \mathcal{K} \). Suppose that \( f \) only has isolated singularities. Then the regularization \( \mathcal{R}eg.f \) of \( f \) satisfies the following conditions:

1. \( \mathcal{R}eg.f \) is a linear continuous functional on \( \mathcal{K} \).

2. For all \( \phi \in \mathcal{K} \) which vanishes in a neighborhood of a singularity \( x_0 \) of \( f \),

\[
(\mathcal{R}eg.f, \phi) = \int_{\mathbb{R}^n} f(x)\phi(x)dx. \tag{V.21}
\]

For example (Gel’fand and Shilov, Vol. I, 1964), the function \( f_0(x) = 1/x \) on \( \mathbb{R} \) is not locally integrable. A regularization of \( f_0 \) is defined by

\[
(\mathcal{R}eg.f_0, \phi) = \int_{-\infty}^{-a} \frac{\phi(x)}{x} dx + \int_{-a}^{b} \frac{\phi(x) - \phi(0)}{x} dx + \int_{b}^{\infty} \frac{\phi(x)}{x} dx \tag{V.22}
\]

where \( a, b \) are arbitrary positive numbers. It is easy to verify that the conditions (1) and (2) for a regularization are satisfied, namely, \( \mathcal{R}eg.f_0 \) is a linear continuous functional defined on all \( \phi \in \mathcal{K} \), and for any \( \phi \in \mathcal{K} \) vanishing in a neighborhood of \( x_0 = 0 \), it coincides with the integral of \( f_0 \) and \( \phi \),

\[
\int_{-\infty}^{\infty} \frac{\phi(x)}{x} dx. \tag{V.23}
\]

There are several ways to construct a regularization:

1. **The Cauchy principal value.**

\[
(\mathcal{R}eg.f, \phi) = \mathcal{P}.\mathcal{V}. \int f(x)\phi(x)dx, \tag{V.24}
\]

if it exists.

2. **The Hadamard Finite part.**

\[
(\mathcal{R}eg.f, \phi) = \mathcal{F}.\mathcal{P}. \int f(x)\phi(x)dx, \tag{V.25}
\]
if the right hand side exists, which denotes the Hadamard finite part of a divergent integral. The Hadamard finite part of a divergent integral is defined by removing the divergent part of the integral and keep the finite part. (see, Hadamard, 1952; Kanwal, 1983; or, Estrada and Kanwal, 1994)

As an example (Estrada and Kanwal, 1994), for \( a > 0 \), consider that

\[
\int_{0}^{a} f(x)dx
\]  

(V.26)

is a one-dimensional divergent integral of \( f \), which has a singularity at \( x = 0 \), otherwise \( f \) is integrable. Define

\[
F(\epsilon) = \int_{\epsilon}^{a} f(x)dx.
\]  

(V.27)

Then, \( F(\epsilon) \) can be decomposed into two parts, i.e.,

\[
F(\epsilon) = F_1(\epsilon) + F_2(\epsilon).
\]  

(V.28)

\( F_1(\epsilon) \) is the divergent part as \( \epsilon \to 0 \), which may be a linear combination of some set of functions, e.g., \( \epsilon^{-\lambda} \) for \( \lambda > 0 \), or \( \ln \epsilon \). While \( F_2(\epsilon) \) is the finite part, such that

\[
\lim_{\epsilon \to 0} F_2(\epsilon) = A.
\]  

(V.29)

Then, the Hadamard finite part of the divergent integral (V.26) is defined as

\[
\mathcal{F.P.} \int_{0}^{a} f(x)dx = \lim_{\epsilon \to 0} F_2(\epsilon) = A.
\]  

(V.30)

The finite part integral arises naturally in problems of bridged cracks in fracture mechanics, see e.g., Nemat-Nasser and Hori (1987), Willis and Nemat-Nasser (1990), Hori and Nemat-Nasser (1990), Nemat-Nasser and Hori (1999), and Ni and Nemat-Nasser (2000).
If \( f(x) \) has an algebraic singularity at \( x_0 \), i.e., \( f(x)|x-x_0|^m \) is locally integrable for an integer \( m > 0 \), then

\[
(\text{Reg}. f, \phi) = \int_{U \setminus B_a} f(x)\phi(x)dx + \int_{B_a} f(r, \theta_j)[\phi(r, \theta_j) - \sum_{|k| < m-n+1} \frac{1}{k!} \partial_r^k \phi(x_0, \theta_j)r^k]r^{n-1}drd\Omega, \tag{V.31}
\]

where \( B_a \) is a closed ball of radius \( a \) at \( x_0 \), \( a \) is an arbitrary positive number, \( r \) and \( \theta_j \), for \( 1 \leq j \leq n-1 \) are the spherical coordinate variables at \( x_0 \), and \( d\Omega \) is the area element of the unit ball at \( x_0 \). This method of regularization will be used in Chapter VIII.

For the regularization of a singular distribution defined by those three ways, it is easy to verify that the conditions (1) and (2) of a regularization are satisfied.

For a divergent integral, we may decompose the integrand into two factors with one in the function space \( \mathcal{K} \) and for the other its regularization is available. Then we view the divergent integral as a linear continuous functional, corresponding to a singular distribution, on an element in the infinitely differentiable test function space, which gives the regularization of the divergent integral.

Regarding the definition of the dynamic configurational force on a moving defect, we may denote the integrand of the divergent volume integral by

\[
g = \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}], \tag{V.32}
\]

and define by \( g_\epsilon \) the restriction of \( g \) on \( V \setminus N_\epsilon \). Then \( g_\epsilon \) as a locally integrable function is a distribution on \( \mathcal{K} \). When \( \epsilon \to 0 \), as a functional associated with the distribution \( g_\epsilon \), the integral of \( g_\epsilon \) will converges to the regularization of the divergent integral of \( g \), if \( g \) can be regularized.

By regularizing the divergent integral of \( g \) based on the theory of distributions, the configurational force on a moving elastic defect is given by

\[
F_l = \text{Reg}. \int_V \frac{\partial}{\partial t}[\rho \dot{u}_i u_{i,l}]dV + \int_S [(W - T)\delta_{lj} - u_{i,l} \sigma_{ij}]dS_j, \tag{V.33}
\]
where now the volume integral, if it does not converge in the usual sense, is un-
derstand as the regularization of the divergent integral in the sense of the regular-
ization of a functional associated with a singular distribution, if the regularization
exists.

Here we would like to point out two important facts. First, it is certain
that not every non-locally integrable function can have its regularization. How-
ever, most singularities in solid mechanics are algebraic singularities which can
have their regularization, e.g., as seen in (V.31). Secondly, in the above discus-
sion, for instance, in (V.22) and (V.31), there are arbitrary constants \( a, b \) and \( a \),
respectively. Although the conditions (1) and (2) of the definition of the regu-
larization can be verified with no trouble from those arbitrariness, it implies that
regularization of a singular distribution is not unique. Gel’fand and Shilov (Vol.
I, 1964) proved that: the regularization of a singular distribution, if it exists, is
uniquely determined apart from a linear combination of the delta function and its
derivatives concentrated at the singularities. In other words, if \( g_1 \) and \( g_2 \) are two
regularizations of \( f \) which is singular at \( x_0 \), then

\[
g_1 - g_2 = \sum_{|j|<k} c_j D^j \delta(x - x_0). \tag{V.34}
\]

**V.E The Smearing Method**

Eshelby (1951) pointed out that “Singularities with infinite self-energy
can be regarded as limiting case of singularities with finite self-energy, and when
we make the passage to the limit the expression for the force is still valid”.

In Al’shitz, Indenbom, and Shtol’berg (1971) a smearing method was
used. “The divergence of the elastic field on the dislocation axis is eliminated by
smearing the nucleus of the dislocation over a region of radius \( r_0 \approx a \)”, where \( a \) is
the period of the function of energy per unit length of the dislocation.

Eshelby (1977) discussed the smearing technique in the calculation of
the configurational force for a moving crack. In the calculation, the Dirac delta
function is replaced by a delta series.
J. Weertman and T. R. Weertman (1980) stated: “A discrete dislocation cannot exist in a real crystal because a real crystal cannot contain an infinite amount of energy nor can it support infinite stresses”. “If a discrete dislocation is ‘smeared-out’ over a localized region on its glide or climb plane, the infinite stresses and self-energy can be eliminated”.

If the singularity may be smeared in some way, such that the force on the smeared singularity is well-defined. We call such approach a smearing method.

\[ V.35 \]

**V.F Example: Self-Force on a Steadily Moving Screw Dislocation**

Consider a screw dislocation parallel to the \( z \)-direction is moving in a steady motion along the \( x \)-direction relative to an isotropic elastic material. The material is supposed to occupy the whole infinite space. According to Frank (1949), the non-zero component of the displacement filed is

\[
u_3 = -\frac{b}{2\pi} \tan^{-1}\left(\frac{x - vt}{\gamma y}\right),
\]

where \( \gamma = \sqrt{1 - v^2/c_2^2} \) with \( c_2 \) as the speed of the shear stress wave. We thus have the following field solutions

\[
u_{3,1} = -\frac{b}{2\pi} \frac{\gamma y}{(x - vt)^2 + \gamma^2 y^2},
\]

\[
u_{3,2} = \frac{b}{2\pi} \frac{\gamma(x - vt)}{(x - vt)^2 + \gamma^2 y^2},
\]

and

\[
u_3 = \frac{b}{2\pi} \frac{\gamma y v}{(x - vt)^2 + \gamma^2 y^2}.
\]

Choose the integral volume \( V \) to be a cylinder of a radius \( r \) around the dislocation line, and with a unit height in the \( z \)-direction. At time \( t \), the dislocation line is located at the position \( x = vt, y = z = 0 \). We shall show that the necessary
and sufficient condition (V.17) is satisfied. Hence (V.12) can be used to define and calculate the self force on the steadily moving screw dislocation.

It suffices to calculate the force $F_1$, since the calculation of $F_2$ is analogous. To verify the necessary and sufficient condition (V.17), we define by $V\epsilon$ a cylinder similar to $V$ with an infinitesimal radius $\epsilon$. And consider the surface integral over $S_\epsilon \equiv \partial V\epsilon$,

$$
\int_{S_\epsilon} [(W - T)\delta_{ij} - u_{3,1}\sigma_{3j}]dS_j = \int_{S_\epsilon} [(W - T) - u_{3,1}\sigma_{31}]dS_1 - \int_{S_\epsilon} u_{3,1}\sigma_{32}dS_2, \quad \text{(V.39)}
$$

where $dS_1 = \epsilon \cos \theta d\theta$, $dS_2 = \epsilon \sin \theta d\theta$, $\theta = \tan^{-1}(y/(x - vt))$ and $0 \leq \theta < 2\pi$.

From (V.36)- (V.38), we have

$$
W = \frac{\mu}{2}[u_{3,1}^2 + u_{3,2}^2] = \frac{\mu b^2}{8\pi^2}[\frac{\gamma^2}{((x - vt)^2 + \gamma^2y^2)^2}]
$$

$$
= \frac{\mu b^2}{8\pi^2} \frac{\gamma^2}{\epsilon^2 \cos^2 \theta + \gamma^2 \sin^2 \theta}, \quad \text{(V.40)}
$$

$$
T = \frac{\rho}{2}u_{3,2}^2 = \frac{\rho b^2}{8\pi^2}[\frac{\gamma^2y^2v^2}{((x - vt)^2 + \gamma^2y^2)^2}]
$$

$$
= \frac{\rho b^2}{8\pi^2} \frac{\gamma^2y^2v^2 \sin^2 \theta}{[\cos^2 \theta + \gamma^2 \sin^2 \theta]^2}, \quad \text{(V.41)}
$$

$$
u_{3,1}\sigma_{31} = \frac{\mu u_{3,1}}{2} = \frac{\mu b^2}{4\pi^2}[\frac{\gamma^2y^2}{((x - vt)^2 + \gamma^2y^2)^2}]
$$

$$
= \frac{\mu b^2}{4\pi^2} \frac{\gamma^2y^2 \sin \theta}{[\cos^2 \theta + \gamma^2 \sin^2 \theta]^2}, \quad \text{(V.42)}
$$

and

$$
u_{3,1}\sigma_{32} = \frac{\mu v}{4\pi^2}[\frac{\gamma y(x - vt)}{((x - vt)^2 + \gamma^2y^2)^2}]
$$

$$
= \frac{\mu b^2}{4\pi^2} \frac{\gamma y(x - vt) \cos \theta}{[\cos^2 \theta + \gamma^2 \sin^2 \theta]^2}. \quad \text{(V.43)}
$$

Substituting those calculation into (V.39) and using the symmetry in $\theta$, we obtain that

$$
\int_{S_\epsilon} [(W - T)\delta_{ij} - u_{3,1}\sigma_{3j}]dS_j = 0. \quad \text{(V.44)}
$$

Therefore the necessary and sufficient condition is satisfied and the self force of a steadily moving screw dislocation is zero.
V.G  Example: Driving Force on an Advancing Crack

In an infinite isotropic elastic medium, consider a straight-through crack with the crack surface on the $x, z$-plane with the crack front at $x = x_0$ parallel to the $z$-direction, moving in the $x$-direction. To find the driving force on the crack front per unit length, all volumes are taken with a unit length in the $z$-direction. It then suffices to examine a plane problem on the $x, y$-plane. Consider an area $V$ containing the crack tip at $x = x_0$, such that the boundary $S \equiv \partial V$ consists of the upper and lower lips of the crack, $\Gamma^+$ and $\Gamma^-$, and a curve $\Gamma$ enclosing the crack tip and connecting the upper lip and the lower lip of the crack respectively.

V.G.1  The Configurational Force

As discussed before, by excluding an infinitesimal neighborhood $V_\epsilon$ at the crack tip, we define the configurational force on the moving crack as the limit of the force on the inhomogeneity over $V_\epsilon$ as $\epsilon \to 0$.

The dynamic configurational force in the $x$-direction on the inhomogeneity over $V_\epsilon$ is well-defined as discussed in the previous chapter, and expressed as

$$F^\epsilon_1 = \int_{V \setminus V_\epsilon} \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,1}] dV + \int_{S \setminus S_\epsilon} [(W - T)\delta_{1j} - u_{i,1}\sigma_{ij}] dS_j.$$  \hspace{1cm} (V.45)

In the expression, the domain of the area integral $V \setminus V_\epsilon$ is the area between $\Gamma$ and $S_\epsilon \equiv \partial V_\epsilon$, the boundary of the infinitesimal neighborhood. The domain of the curve integral $S \setminus S_\epsilon$ consists of $\Gamma$, $\Gamma^\pm_\epsilon \equiv \Gamma^\pm \setminus V_\epsilon$. The dynamic configurational force on the crack tip of the moving crack is defined by the limit

$$F_1 = \lim_{\epsilon \to 0} F^\epsilon_1$$

$$= \lim_{\epsilon \to 0} \left\{ \int_{V \setminus V_\epsilon} \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,1}] dV + \int_{\Gamma \cup \Gamma^\pm_\epsilon} [(W - T)\delta_{1j} - u_{i,1}\sigma_{ij}] dS_j \right\} \hspace{1cm} (V.46)$$

In view of that $(n_1, n_2, n_3) = (0, 1, 0)$ on the crack lips $\Gamma^\pm$, $dS_1 = n_1 dS = 0$
and

\[
\int_{\Gamma^\pm} (W - T)\delta_{ij} dS_j = 0. \tag{V.47}
\]

From the boundary conditions \(\sigma_{ij} = 0\) on the crack lips \(\Gamma^\pm\), it follows

\[
\int_{\Gamma^\pm} u_{i,1}\sigma_{ij} dS_j = 0. \tag{V.48}
\]

Then (V.46) is reduced to

\[
F_1 = \lim_{\epsilon \to 0} F_{1}^\epsilon
= \lim_{\epsilon \to 0} \{ \int_{V\setminus V^\epsilon} \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,1}] dV \} + \int_{\Gamma} [(W - T)\delta_{ij} - u_{i,1}\sigma_{ij}] dS_j \tag{V.49}
\]

V.G.2 The Necessary and Sufficient Condition

The necessary and sufficient condition for a finite well-defined dynamic configurational force on a moving defect is

\[
\lim_{\epsilon \to 0} \int_{S^\epsilon} [(W - T)\delta_{ij} - u_{i,1}\sigma_{ij}] dS_j < \infty. \tag{V.50}
\]

We now show that for the advancing crack, the necessary and sufficient condition (V.50) is satisfied, if \(S_\epsilon\) is taken as an infinitesimal circle.

The asymptotic behavior of the near field solutions around the crack tip shows that near the crack tip,

\[
\dot{u}_i \sim r^{-1/2}, \quad u_{i,1} \sim r^{-1/2}, \quad \sigma_{ij} \sim r^{-1/2},
\]

where \(r\) is the distance between the field point and the crack tip. Hence the integrand in (V.50) is asymptotically

\[
[(W - T)\delta_{ij} - u_{i,1}\sigma_{ij}] \sim r^{-1}, \tag{V.51}
\]

as \(r \to 0\)

Assume that \(S_\epsilon = \{(r(\theta)\cos \theta, r(\theta)\sin \theta); 0 \leq \theta < 2\pi\}\) is an arbitrary smooth curve, with \(\text{Max}\{r(\theta)\} = \epsilon\). The length element \(dS\) is expressed as

\[
dS = \sqrt{r^2 + (dr/d\theta)^2} d\theta. \tag{V.52}
\]
Then for an arbitrary infinitesimal neighborhood \( V_\epsilon \), the limit (V.50) may not exist, since \( r^{-1}dr/d\theta \) may have no limit as \( r \to 0 \). However if \( V_\epsilon \) is taken as an infinitesimal circle with radius \( \epsilon \) for which the length element \( dS = rd\theta \), then the limit (V.50) exists.

Therefore the necessary and sufficient condition for a finite well-defined configurational force is then satisfied when \( V_\epsilon \) is taken to be an infinitesimal circle. Hence, the driving force on the advancing crack is well-defined and expressed by

\[
F_1 = \mathcal{P} \cdot \mathcal{V} \int_V \frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,1}] dV + \int_\Gamma [(W - T)\delta_{1j} - u_{i,1}\sigma_{ij}] dS_j, \tag{V.53}
\]

or equivalently,

\[
F_1 = \lim_{\epsilon \to 0} \int_{S_\epsilon} [(W - T)\delta_{1j} - u_{i,1}\sigma_{ij}] dS_j, \tag{V.54}
\]

by (V.50).

**V.G.3 A Direct Proof of the Existence of the Cauchy Type Integral**

The existence of the limit in (V.49) can also be proved directly. Where the integrand is expressed as

\[
\frac{\partial}{\partial t} [\rho \dot{u}_i u_{i,1}] = \rho \ddot{u}_i u_{i,1} + \rho \dot{u}_i \dot{u}_{i,1} = \rho \ddot{u}_i u_{i,1} + \frac{1}{2} \frac{\partial}{\partial x_1} [\rho \dddot{u}_i]. \tag{V.55}
\]

The integration of the second term, by Gauss divergence theorem, is converted to surface integrals, disregard the constant factors \( \rho/2 \),

\[
\int_{V \setminus V_\epsilon} \frac{\partial}{\partial x_1} [\dot{u}_i \dot{u}_i] dV = \int_\Gamma [\dot{u}_i \dot{u}_i] n_1 dS + \int_{S_\epsilon} [\dot{u}_i \dot{u}_i] n_1 dS + \int_{\Gamma_\epsilon^\pm} [\dot{u}_i \dot{u}_i] n_1 dS. \tag{V.56}
\]

The limit of those integrals on the right hand side exist as \( \epsilon \to 0 \), since the first integral is over the far field then independent of \( \epsilon \), the second integral is bounded, as on \( S_\epsilon \), \( \dot{u}_i \dot{u}_i \) is in order of \( \epsilon^{-1} \) and \( dS = \epsilon d\theta \), and the third integral vanishes since \( n_1 = 0 \) on \( \Gamma_\epsilon^\pm \).
As for the integral of the first term on the right hand side of (V.55), the limit
\[
\lim_{\epsilon \to 0} \int_{V\setminus V_{\epsilon}} \rho \dddot{u}_{i_{1}}u_{i} dV
\] (V.57)
exists if
\[
\lim_{\epsilon \to 0} \int_{V_{\epsilon}} \rho \dddot{u}_{i_{1}}u_{i} \, dV = 0.
\] (V.58)
From Nishioka and Atluri (1983), when \( V_{\epsilon} \) is an infinitesimal circle, by symmetry, the integral in (V.58) is zero, i.e., as \( \epsilon \to 0 \),
\[
\int_{-\pi}^{\pi} \int_{0}^{\epsilon} \rho \dddot{u}_{i_{1}}u_{i} \, r \, dr \, d\theta \to 0.
\] (V.59)
So that the limit in in (V.49) exists and
\[
\lim_{\epsilon \to 0} \left\{ \int_{V\setminus V_{\epsilon}} \frac{\partial}{\partial t} [\rho \dddot{u}_{i_{1}}] u_{i} \, dV \right\} = \mathcal{P} \cdot V. \int_{V} \frac{\partial}{\partial t} [\rho \dddot{u}_{i_{1}}] u_{i} \, dV.
\] (V.60)

V.G.4 Bui’s Dynamic J-Integral of an Advancing Crack

In the homogeneous region \( V_{I} \equiv V \setminus V_{\epsilon} \) the field variables are smooth, the time derivative of the integral of \( A_{1} \equiv \rho \dddot{u}_{i_{1}}u_{i} \) over \( V_{I} \) is written as
\[
\frac{d}{dt} \int_{V_{I}} A_{1} \, dV = \int_{V_{I}} \left[ \frac{\partial}{\partial t} A_{1} + \nabla \cdot (\mathbf{U} A_{1}) \right] dV,
\] (V.61)
(see e.g., Fung, 1965), where \( \mathbf{U} = (\mathbf{U}, 0, 0) \) is the velocity of the advancing crack, and assume that \( V \) is moving with the crack.

In (V.49), using (V.61) and Gauss divergence theorem over the region \( V_{I} = V \setminus V_{\epsilon} \), we obtain the driving force on the moving crack tip
\[
F_{1} = \lim_{\epsilon \to 0} \left\{ \frac{d}{dt} \int_{V_{I}} [\rho \dddot{u}_{i_{1}}] u_{i} \, dV - \int_{\partial V_{I}} [\rho \dddot{u}_{i_{1}}] UdS_{1} \right\}
+ \int_{\Gamma} [(W - T)\delta_{1j} - u_{i_{1}}\sigma_{ij}] dS_{j},
\] (V.62)
or, equivalently,

\[ F_1 = \frac{d}{dt} \int_{V_I} [\rho \dot{u}_i u_{i,1}] dV - \int_{\partial V_I} [\rho \dot{u}_i u_{i,1}] UdS_1 + \int_{\Gamma} [(W - T)\delta_{1j} - u_{i,1}\sigma_{ij}] dS_j, \]  

(V.63)

where again \( V_I = V \setminus V_\epsilon \) and \( V_\epsilon \) is vanishing small circle. (V.63) is the generalized dynamic J-integral of a moving crack in transient loading in fracture mechanics given in Bui (1977) and Maugin (1993). In the above derivation, it has been assumed that the volume \( V \) is moving with the crack in a velocity \( U = (U, 0, 0) \), and the infinitesimal circle \( V_\epsilon \) is rigid during the motion.

In (V.62), \( \partial V_I \) consists of \( \Gamma, S_\epsilon \), and \( \Gamma_\epsilon^\pm \). Note that again \( dS_1 = 0 \) on \( \Gamma_\epsilon^\pm \),

\[ \int_{\Gamma_\epsilon^\pm} [\rho \dot{u}_i u_{i,1}] UdS_1 = 0. \]  

(V.64)

Further, the integral over the small circle \( S_\epsilon \) vanishes as well, i.e.,

\[ \lim_{\epsilon \to 0} \int_{S_\epsilon} [\rho \dot{u}_i u_{i,1}] UdS_1 = 0, \]  

(V.65)

that important result was reported by Bui (1977) and Maugin (1993): as stated in Maugin (1993, p.169), “According to Bui (1977, p.167) and an analysis of Afanasev and Cherepanov to which he refers, a term such as \( \rho \dot{u} \cdot u_{i,1} u_1 \) integrated over a vanishing small circle yields zero.”

The integral over \( \partial V_I \) is then converted to the integral over \( \Gamma \),

\[ \lim_{\epsilon \to 0} \int_{\partial V_I} [\rho \dot{u}_i u_{i,1}] UdS_1 = \int_{\Gamma} [\rho \dot{u}_i u_{i,1}] UdS_1. \]  

(V.66)

V.G.5 The Energy Release Rate

For an advancing crack, the energy release rate at the tip of the crack has been discussed by many authors, e.g., Atkinson and Eshelby (1968), Eshelby (1970), and Maugin (1993). Atkinson and Eshelby (1968) derived the expression of the energy release rate of an advancing crack. Eshelby (1970) re-derived the expression and, in the linear elasticity framework, proved the driving force on
the tip of crack is equal to the energy release rate. Maugin (1993) proved the equivalence for the non-linear case.

In this subsection, we shall give a simple proof of the equivalence of the driving force and the energy release rate of the advancing crack.

It is well known that the energy release rate of an advancing crack is determined by

\[ vG = \lim_{\epsilon \to 0} \int_{S_\epsilon} [\sigma_{ij} \dot{u}_i + v(W + T)\delta_{1j}] dS_j, \] (V.67)

where \( G \) is the energy release rate, \( S_\epsilon \) is the closed curve which circles and approaches the tip of the crack as \( \epsilon \to 0 \).

As in Atkinson and Eshelby (1968), and Eshelby (1970), for the advancing crack, it is possible to write the displacement in the form as

\[ u_i(x, y, t) = u^0_i(x - vt, y) + u'_i(x, y, t), \] (V.68)

such that near the crack tip, \( u'_i \) is much smaller than \( u^0_i \). Then we have the relation

\[ \frac{\partial u_i}{\partial t} \approx -v \frac{\partial u_i}{\partial x}, \] (V.69)

which is substituted into (V.67), and implies that

\[ G = \lim_{\epsilon \to 0} \int_{S_\epsilon} [(W + T)\delta_{1j} - \sigma_{ij}u_{i,1}] dS_j. \] (V.70)

Let \( S_\epsilon \) be an infinitesimal circle around the crack tip. Compare the expression (V.70) for the energy release rate with the expression (V.54) of the driving force, we have

\[ G - F_1 = 2 \lim_{\epsilon \to 0} \int_{S_\epsilon} T dS_1 = \lim_{\epsilon \to 0} \int_{S_\epsilon} [\rho \dot{u}_i \dot{u}_i] dS_1. \] (V.71)

By using (V.69), the last limit is converted to

\[ \lim_{\epsilon \to 0} \int_{S_\epsilon} [\rho \dot{u}_i u_{i,1}] vdS_1, \] (V.72)

which is zero according to (V.65). So that \( G = F_1 \), the proof of the equivalence of the energy release rate and the driving force of the advancing crack is completed.
V.G.6 Advancing Crack Has No Effective Mass

Eshelby (1970) considered the simple case of a crack tip moving arbitrarily under anti-plane strain. It is found that the acceleration of the tip only enters the higher order term of the solution of the displacement, i.e., the term of $O(R^{3/2})$, where $R$ is the distance from the current position to the crack tip. Hence after taking the limit of $R \to 0$, the acceleration does not affect the energy release rate and the driving force. The crack tip then has no inertia, and a moving crack has no effective mass.
Chapter VI

Effective Mass of an Accelerating Screw Dislocation

VI.A Introduction

The effective mass of a moving dislocation is determined by the inertial part of the self-force on the dislocation. In this chapter, we shall use the definition of the configurational force on a moving elastic defect given in Chapter V to calculate the self-force on a moving screw dislocation in an infinite elastic medium. From the equation of motion itself, instead of the solution of the equation, we prove two important new theorems about the near field behavior of the stresses and the field velocity, which play key role in the calculation of the self-force and effective mass. From those theorems, the near field expansions of $u_{3,j}$, for $j = 1, 2, t$, are completely determined up to the $O(1)$ terms. Complete explicit solutions of the self-force and effective mass are derived. It will be seen that the self-force on an accelerating screw dislocation is infinite, and diverges logarithmically.

VI.B The Main Problem

Let a Volterra screw dislocation situate on the $z$-axis at rest for $t \leq 0$ in an infinite, homogeneous, isotropic elastic solid. For $t > 0$, it moves according
to \( x = l(t) \) in the (positive) \( x \)-direction, where \( l(t) \) is an arbitrarily given smooth function such that

\[
0 < \frac{dl(t)}{dt} < c_2 = \sqrt{\mu/\rho}
\]

(VI.1)

where \( c_2 \) is the shear wave speed. The condition (IX.1) means that the dislocation moves in a subsonic motion along the positive \( x \)-direction.

The main problem is to find the effective mass per unit length of the moving dislocation. We shall calculate the self-force on per unit length of the dislocation, then the effective mass is derived from the inertial part of the self-force. Self-force is the configurational force solely due to the elastic field created by the moving dislocation itself. Hence in the calculation, we assume that there are no other external or internal forces, neither the drag, dissipative, or damping effects in the problem.

VI.C The Field Equation and Solution

VI.C.1 Equation of the Displacement Field

As discussed in Chapter 2, the motion of a non-uniformly moving screw dislocation starting from rest satisfies the Navier’s equation of elastodynamics for \( y \neq 0 \),

\[
\frac{\partial^2 u_3(x, y, t)}{\partial x^2} + \frac{\partial^2 u_3(x, y, t)}{\partial y^2} = \frac{1}{c_2^2} \frac{\partial^2 u_3(x, y, t)}{\partial t^2},
\]

and the discontinuity condition at \( y = 0 \),

\[
u_3(x, 0^+, t) - u_3(x, 0^-, t) = -\frac{b}{2} [H(x - l(t)) - H(l(t) - x)],
\]

(VI.3)

where \( H(\cdot) \) is the Heaviside step function.

In view of the oddness of the displacement \( u_3(x, y, t) \) in \( y \) and the symmetric property of the Navier’s equation, the problem is reduced to a half-space mixed initial-boundary-value problem with the same Navier’s equation (VI.2) for \( y > 0 \), the initial conditions

\[
u_3(x, y, 0) = u_3^*(x, y) = -\frac{b}{2\pi} \tan^{-1}(x/y),
\]

(VI.4)
and
\[ \frac{\partial}{\partial t}[u_3(x, y, 0)] = 0, \]  
(VI.5)

where \( u_3(x, y) \) is the solution of a static screw dislocation at the \( z \)-axis in the infinite whole space, and the boundary condition on \( y = 0 \)
\[ u_3(x, 0, t) = -\frac{b}{2}H(x - l(t)). \]  
(VI.6)

**VI.C.2 Solution of the Field Equation**

To reduce the problem to a mixed initial-boundary-value problem with homogeneous initial conditions, we decompose the problem into two problems:

**Problem I** The static screw dislocation with the boundary condition
\[ u(x, 0, t) = -\frac{b}{2}H(x). \]  
(VI.7)

**Problem II** The moving screw dislocation with the boundary condition:
\[ u(x, 0, t) = -\frac{b}{2}[H(x - l(t)) - H(x)]. \]  
(VI.8)

The solution of the original problem is the sum of the solutions of Problem I and Problem II. The solution of Problem I is the static solution
\[ u = -\frac{b}{2\pi}\tan^{-1}\left(\frac{x}{y}\right). \]  
(VI.9)

Problem II is a mixed initial-boundary-value problem with the homogeneous initial conditions
\[ u_3(x, y, 0) = \frac{\partial}{\partial t}[u_3(x, y, 0)] = 0, \]  
(VI.10)

and the boundary conditions (VI.8). Follow the approach of solving the stress \( \sigma_{32} \) used in Markenscoff (1980), we now solve Problem II for the displacement \( u_3 \) by using Laplace transform with respect to \( t \) and two-side Laplace transform with respect to \( x \), and a Cagniard-de Hoop technique to inverse the Laplace transform.
The solution of the displacement field will be useful in the discussion of the self-force and effective mass based on a smear method.

We sketch the procedure as follows. By using the Laplace transform with respect to $t$,

$$\tilde{u}(x, y, s) = \int_0^\infty u(x, y, t)e^{-st}dt,$$  \hspace{1cm} (VI.11)

with $s > 0$, and the initial conditions (VI.10), we transform the equation (??) to

$$\frac{\partial^2 \tilde{u}}{\partial x^2} + \frac{\partial^2 \tilde{u}}{\partial y^2} = s^2 \frac{c^2}{c^2} \tilde{u}.$$  \hspace{1cm} (VI.12)

The boundary condition (VI.8) is transformed to

$$\tilde{u}(x, 0, s) = \frac{b}{2s}e^{-s\eta(x)}H(x),$$  \hspace{1cm} (VI.13)

where $\eta(x)$ is the inverse function of $x = l(t)$.

Further using two-side Laplace transform with respect to $x$

$$U(\lambda, y, s) = \int_{-\infty}^\infty \tilde{u}(x, y, s)e^{-s\lambda x}dx,$$  \hspace{1cm} (VI.14)

where $\lambda$ is a complex variable. The equation (VI.12) is transformed into

$$s^2\lambda^2U + \frac{\partial^2 U}{\partial y^2} = \frac{s^2}{c^2}U,$$  \hspace{1cm} (VI.15)

which is then reduced to

$$\frac{\partial^2 U}{\partial y^2} = s^2(d^2 - \lambda^2)U,$$  \hspace{1cm} (VI.16)

where $d \equiv 1/c_2$.

(VI.16) is solved as

$$U(\lambda, y, s) = C(\lambda, s)e^{-s\hat{\beta}(\lambda)y},$$  \hspace{1cm} (VI.17)

where $\hat{\beta}(\lambda) \equiv (d^2 - \lambda^2)^{1/2}$ such that $\text{Re}\hat{\beta} > 0$, and the function $C(\lambda, s)$ is determined by

$$C(\lambda, s) = U(\lambda, 0, s) = \int_{-\infty}^\infty \tilde{u}(x, 0, s)e^{-s\lambda x}dx$$

$$= \frac{b}{2s} \int_0^\infty e^{-s\eta x - s\lambda x}dx.$$  \hspace{1cm} (VI.18)
Apply the inverse Laplace transform to $U(\lambda, y, s)$, we have

$$
\tilde{u}(x, y, s) = \frac{s}{2\pi i} \int_{Br} C(\lambda, s)e^{-s\beta(\lambda)y}e^{s\lambda x}d\lambda,
$$

where $Br$ means the Bromwich contour of the inverse Laplace transform.

Substitute (VI.18) into the last equation, we have

$$
\tilde{u}(x, y, s) = \frac{b}{2\pi} \int_{Br} \left[ \int_0^\infty e^{-s\eta \xi - s\lambda \xi}d\xi \right] e^{-s\beta(\lambda)y}e^{s\lambda x}d\lambda.
$$

Now apply the Cagniard-de Hoop technique to the integral of $\lambda$, i.e., change the Bromwich integral contour to a contour on the complex $\lambda$ plane which is determined by

$$
\text{Re}[-\lambda(x-\xi) + \hat{\beta}(\lambda)y] = \tau > 0,
$$

$$
\text{Im}[-\lambda(x-\xi) + \hat{\beta}(\lambda)y] = 0.
$$

$\lambda$ and $\hat{\beta}$ are solved out and expressed as

$$
\lambda_{\pm} = \frac{1}{r^2} \left[-\tau(x-\xi) \pm iy\sqrt{\tau^2 - r^2d^2}\right],
$$

$$
\hat{\beta}(\lambda_{\pm}) = \frac{1}{r^2} \left[\tau y \pm i(x-\xi)\sqrt{\tau^2 - r^2d^2}\right],
$$

where $r^2 = (x-\xi)^2 + y^2$.

The new contour is parameterized by $\tau$, and $\tilde{u}$ is written as

$$
\tilde{u}(x, y, s) = \frac{b}{2\pi} \int_0^\infty e^{-s\eta \xi} \left[ \int_{\tau d} e^{-s\tau}d\tau \right] d\xi.
$$

It is easy to find that

$$
\text{Im}\left(\frac{\partial \lambda_{\pm}}{\partial \tau}\right) = \frac{y\tau}{r^2\sqrt{\tau^2 - r^2d^2}}.
$$

Then (VI.25) is rewritten as

$$
\tilde{u}(x, y, s) = \frac{b}{2\pi} \int_0^\infty \int_0^\infty \frac{y\tau}{r^2\sqrt{\tau^2 - r^2d^2}} e^{-s\eta \xi}d\xi d\tau.
$$
From the last equation, the inverse Laplace transform can be obtained by simple inspection and gives the solution of Problem II

\[ u(x, y, t) = \frac{b}{2\pi} \int_0^\infty \frac{y(t - \eta(\xi))}{r^2\sqrt{\tau^2 - r^2d^2}} H(t - \eta(\xi) - rd)d\xi. \] (VI.28)

The final solution of \( u(x, y, t) \) is the sum of solutions of problem I and Problem II, and given by

\[ u(x, y, t) = \frac{b}{2\pi} \int_0^\infty \frac{y(t - \eta(\xi))}{r^2\sqrt{\tau^2 - r^2d^2}} H(t - \eta(\xi) - rd)d\xi - \frac{b}{2} \tan^{-1}(x/y). \] (VI.29)

The uniqueness of the solution for the mixed initial-boundary-value problem with this discontinuous boundary condition in our problem can be proved by the method of the energy integral in an analogous way as for the continuous boundary condition, see e.g., Achenbach (1984).

**VI.D Singularities**

We summarize the singularities involved in the configuration.

1. The near field singularities

The leading terms of the near field solutions are described in Chapter 2.

\[ \sigma_{31} = \mu u_{3,1} \sim -\frac{b\mu}{2\pi} \frac{\gamma y}{(x - l(t))^2 + \gamma^2 y^2} \sim \frac{1}{r}, \] (VI.30)

where \( r = \sqrt{(x - l(t))^2 + y^2} \), \( \gamma = \sqrt{1 - v^2(t)/c_2^2} \), and \( v(t) = \dot{l}(t) \).

\[ \sigma_{32} = \mu u_{3,2} \sim \frac{b\mu}{2\pi} \frac{\gamma(x - l(t))}{(x - l(t))^2 + \gamma^2 y^2} \sim \frac{1}{r}, \] (VI.31)

\[ \dot{u}_3 \sim \frac{b}{2\pi} \frac{v(t)\gamma y}{(x - l(t))^2 + \gamma^2 y^2} \sim \frac{1}{r}. \] (VI.32)

2. The discontinuity across the slip plane

(a) \( u_3(x, 0^+, t) - u_3(x, 0^-, t) = -\frac{b}{2}[H(x - l(t)) - H(l(t) - x)]. \) (VI.33)
(b) \[ \sigma_{31}(x, 0^+, t) - \sigma_{31}(x, 0^-, t) = -\mu b \delta(x - l(t)), \] (VI.34)

where \( \delta(\cdot) \) is the Dirac delta function.

(c) In view of the fact that \( u_3 \) is odd in \( y \), \( \sigma_{32} = \mu u_{3,2} \) is even in \( y \), \( \sigma_{32} \) is continuous on the slip plane \( y = 0 \) except at the core of the dislocation \( (x, y) = (l(t), 0) \).

(d) \[ \dot{u}_3(x, 0^+, t) - \dot{u}_3(x, 0^-, t) = b v(t) \delta(x - l(t)). \] (VI.35)

3. The far field behavior

(a) At the wave front

There is no singularity at the wave front unless the motion is jump started at the time \( t = 0 \), i.e., \( v(0) = \dot{l}(0) \neq 0 \). (see Markenscoff, 1980, and 1982) For an accelerating moving dislocation for which \( v(t) = at \) with time-dependent non-uniform acceleration \( a = a(t) \), it is seen that \( v(0) = 0 \) and thus there is no singularity at the wave front.

(b) At the infinity

The field variables \( \sigma_{31}, \sigma_{32}, \) and \( \dot{u}_3 \) are all zero when \( x^2 + y^2 > c_2^2 t^2 \).

VI.E Definition of the Self-Force on a Moving Screw Dislocation

Dislocations are defects in material. A dislocation moving in material changes the configuration. As discussed in Chapter V, the configurational force on an elastic defect may be defined as the limit of the force on the inhomogeneity over the infinitesimal neighborhood around the defect when it is shrinking upon the defect.

Choose \( V \) as a cylindrical volume around the screw dislocation line at \( (x, y) = (l(t), 0) \) at time \( t \), and with a unit length in the \( z \)-direction. The infinitesimal volume \( V_\epsilon \) around the dislocation line is chosen to be a cylinder with a radius
$\epsilon$ for $0 < \epsilon << 1$. The force on the moving dislocation is defined as the limit of the force on the inhomogeneity $V_\epsilon$ when $V_\epsilon$ is shrinking upon the dislocation,

$$F_l = \lim_{\epsilon \to 0} \left\{ \int_{V \setminus V_\epsilon} \frac{\partial}{\partial t} [\rho \dot{u}_i u_i] dV + \int_S [(W - T) \delta_{ij} - u_{i,j} \sigma_{ij}] dS_j \right\}. \quad \text{(VI.36)}$$

When the limit exists, then the force is rewritten as a “contour-independent” integral

$$F_l = \mathcal{P} \mathcal{V} \int_V \frac{\partial}{\partial t} [\rho \dot{u}_3 u_{3,i}] dv + \int_S [(W - T) \delta_{jl} - \sigma_{3j} u_{3,l}] dS_j, \quad \text{(VI.37)}$$

where $S = \partial V$, and $\mathcal{P} \mathcal{V}$. denotes the Cauchy principal value of the integral.

We shall show that for a non-uniformly moving screw dislocation, the necessary and sufficient condition given in previous chapter for the existence of the volume integral of the Cauchy type in (VI.37) does not hold, and the self-force diverges in an order of $\ln \epsilon$ as $\epsilon \to 0$.

As shown in Chapter V, the necessary and sufficient condition for the existence of the volume integral of the Cauchy type in the definition of the self-force is, for $l = 1$,

$$\lim_{\epsilon \to 0} I_\epsilon \equiv \lim_{\epsilon \to 0} \int_{S_\epsilon} [(W - T) \delta_{1j} - \sigma_{3j} u_{3,1}] dS_j < \infty, \quad \text{(VI.38)}$$

where $S_\epsilon = \partial V_\epsilon$. And the self-force is equivalently given by the limit of $I_\epsilon$ as $\epsilon \to 0$,

$$F_1 = \lim_{\epsilon \to 0} I_\epsilon. \quad \text{(VI.39)}$$

For moving screw dislocation, the surface integral $I_\epsilon$ is explicitly given by

$$I_\epsilon = \int_{S_\epsilon} [(W - T) \delta_{1j} - \sigma_{3j} u_{3,1}] dS_j$$

$$= \int_0^{2\pi} \frac{1}{2} [\mu u_{3,1}^2 + \mu u_{3,2}^2 - \rho u_{3,3}^2] \cos \theta \epsilon d\theta - \int_0^{2\pi} \mu u_{3,1} (u_{3,1} \cos \theta + u_{3,2} \sin \theta) \epsilon d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} [\mu u_{3,1}^2 - \mu u_{3,2}^2 - \rho u_{3,3}^2] \cos \theta \epsilon d\theta - \int_0^{2\pi} \mu u_{3,1} u_{3,2} \sin \theta \epsilon d\theta. \quad \text{(VI.40)}$$

To find the limit of $I_\epsilon$ as $\epsilon \to 0$, we must know the near field properties of field variables $u_{3,1}$, $u_{3,2}$, and $\dot{u}_3$. 
In next two section, we shall prove a proposition about global properties of $u_{3,1}$ and $\dot{u}_3$, and two important theorems about near field behaviors of $u_{3,1}$, $u_{3,2}$, and $\dot{u}_3$, respectively, which will play important role in the calculation of the self-force and effective mass.

**VI.F Global Properties of $u_{3,1}$ and $\dot{u}_3$**

We are going to prove that $u_{3,1}$ and $\dot{u}_3$ consist of two parts, respectively, i.e., the corresponding steady-state solution with the instantaneous velocity $v(t) = \dot{l}(t)$, and the remaining part. The discontinuities $u_{3,1}$ and $\dot{u}_3$ are due to the steady-state solution , and the remaining parts of the solutions are continuous in $y$ for every $x$, and converge to zero as $y \to 0$.

Consider the strain field $u_{3,1}$. We shall prove that, for all $x$, $u_{3,1}(x, y, t)$ can be expressed as

$$u_{3,1}(x, y, t) = -\frac{b}{2\pi} \frac{\gamma y}{(x - l(t))^2 + \gamma y^2} + G(x, y, t),$$

where $G(x, y, t)$ is a continuous odd function in $y$, and satisfies

$$\lim_{y \to 0} G(x, y, t) = 0,$$

for every $x$.

From (VI.3), we have the jump condition for $u_{3,1}$,

$$u_{3,1}(x, 0^+, t) - u_{3,1}(x, 0^-, t) = -b \delta(x - l(t)).$$

where $\delta(\cdot)$ is the Dirac delta function. As we have known from Chapter II that the leading term of the near field expansion of $u_{3,1}$ is exactly the solution of a uniformly moving screw dislocation and written as

$$-\frac{b}{2\pi} \frac{\gamma \sin \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta} \epsilon = -\frac{b}{2\pi} \frac{\gamma y}{(x - l(t))^2 + \gamma^2 y^2}.$$

We note that the right hand side is in fact a delta series, i.e., as $y$ goes to zero,

$$-\frac{b}{2\pi} \frac{\gamma y}{(x - l(t))^2 + \gamma^2 y^2} \to -\frac{b}{2} \delta(x - l(t)).$$

(VI.44)
If we define
\[ 2G(x, y, t) = u_{3,1}(x, y, t) - u_{3,1}(x, -y, t) + \frac{b}{\pi} \frac{\gamma y}{(x - l(t))^2 + \gamma^2 y^2}, \]  \hspace{1cm} (VI.45)
then in view of (VI.43) and (VI.44), it follows that for each \( x \in (-\infty, \infty), \)
\[ \lim_{y \to 0} G(x, y, t) = 0. \]  \hspace{1cm} (VI.46)

On the other hand, we know that \( G(x, y, t) \) is an odd function in \( y \). Hence, (VI.46) implies that \( G(x, y, t) \) is continuous in \( y \).

A similar conclusion holds for \( \dot{u}_3 \), i.e., \( \dot{u}_3 \) can be expressed as
\[ \dot{u}_3(x, y, t) = \frac{\dot{l}(t) \gamma y}{(x - l(t))^2 + \gamma^2 y^2} + G_1(x, y, t), \]  \hspace{1cm} (VI.47)
where \( G_1(x, y, t) \) is odd and continuous in \( y \), and such that
\[ \lim_{y \to 0} G_1(x, y, t) = 0, \]  \hspace{1cm} (VI.48)
for every \( x \).

\section*{VI.G Two Theorems on Near Field Behaviors of the Field Variables}

For the near field expansion of \( u_{3,2} \), Callias and Markenscoff (1988) showed that
\[ u_{3,2} \approx -\frac{b}{2\pi \cos^2 \theta + \gamma^2 \sin^2 \theta} \frac{1}{\epsilon} + f_{32} \ln \epsilon + g_{32} + \text{h.o.t.}, \]  \hspace{1cm} (VI.49)
the logarithmic term in \( \epsilon \) is given explicitly
\[ f_{32} = -\frac{b}{4\pi c_2 \gamma^2} \dot{\psi}(t). \]  \hspace{1cm} (VI.50)

Except the leading terms, the near field expansions for \( u_{3,1} \) and \( \dot{u}_3 \) are not available in the literature.
As shown in Chapter II, the most singular terms in the near field expansions of the field variables \( u_{3,1} \), \( u_{3,2} \), and \( \dot{u}_3 \) are in the order of \( 1/\epsilon \). Hence, in general, we have the following near field expansions,

\[
\begin{align*}
  u_{3,1} &= u^0_{3,1} + f_{31}(\theta, t) \ln \epsilon + g_{31}(\theta, t) + h.o.t., \\
  u_{3,2} &= u^0_{3,2} + f_{32}(\theta, t) \ln \epsilon + g_{32}(\theta, t) + h.o.t., \\
  \dot{u}_3 &= \dot{u}^0_{3,1} + f_{33}(\theta, t) \ln \epsilon + g_{33}(\theta, t) + h.o.t.,
\end{align*}
\]

where \( \epsilon = \sqrt{(x-l(t))^2 + y^2} > 0 \), \( \theta = \tan^{-1}(y/(x-l(t))) \). In the above near field expansions, \( u^0_{3,1} \), \( u^0_{3,2} \), and \( \dot{u}^0_{3,1} \) are the leading terms in the expansions, which, as we know, are equal to the corresponding steady-state solutions given in Chapter II:

\[
\begin{align*}
  u^0_{3,1} &= -\frac{b}{2\pi} \frac{\gamma y}{(x-l(t))^2 + \gamma^2 y^2} \\
  &= -\frac{b}{2\pi} \frac{\gamma \sin \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta} \frac{1}{\epsilon}, \\
  u^0_{3,2} &= \frac{b}{2\pi} \frac{\gamma (x-l(t))}{(x-l(t))^2 + \gamma^2 y^2} \\
  &= \frac{b}{2\pi} \frac{\gamma \cos \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta} \frac{1}{\epsilon}, \\
  \dot{u}^0_3 &= \frac{b}{2\pi} \frac{v(t) \gamma y}{(x-l(t))^2 + \gamma^2 y^2} \\
  &= \frac{b}{2\pi} \frac{v(t) \gamma \sin \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta} \frac{1}{\epsilon},
\end{align*}
\]

where \( v(t) \equiv \dot{l}(t) \). We call \( f_{3j} \) and \( g_{3j}, j = 1, 2, \), in the above near field expansions to be the near field coefficients.

We are going to prove the following important theorems.

**Theorem 1**
Let the near field coefficients $f_{3j}(\theta, t)$, $j = 1, 2, t$, be defined in the near field expansions (VI.51) - (VI.53). Then the partial differentiations of $f_{3j}$ with respect to $\theta$, $f'_{3j}(\theta, t)$, satisfy the homogeneous system of linear equations,

$$
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
v \sin \theta & 0 & \sin \theta \\
-\mu \sin \theta & \mu \cos \theta & -\rho v \sin \theta
\end{bmatrix}
\begin{bmatrix}
f'_{31} \\
f'_{32} \\
f'_{3t}
\end{bmatrix} = 0. \quad (VI.57)
$$

So that

$$f'_{31} = f'_{32} = f'_{3t} = 0. \quad (VI.58)$$

Furthermore, it follows that

$$f_{31}(\theta, t) = f_{32}(\theta, t) = 0, \quad (VI.59)$$

and

$$f_{32}(\theta, t) = f_{32}(t) \quad (VI.60)$$

is independent of $\theta$.

**Theorem 2**

Let the near field coefficients $g_{3j}(\theta, t)$, $j = 1, 2, t$, be defined in the near field expansions (VI.51) - (VI.53). Then the partial differentiations of $g_{3j}$ with respect to $\theta$, $g'_{3j}(\theta, t)$, satisfy the inhomogeneous system of linear equations,

$$
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
v \sin \theta & 0 & \sin \theta \\
-\mu \sin \theta & \mu \cos \theta & -\rho v \sin \theta
\end{bmatrix}
\begin{bmatrix}
g'_{31} \\
g'_{32} \\
g'_{3t}
\end{bmatrix} = \begin{bmatrix}
f_{32} \cos \theta \\
-U_{31} \\
\rho U_{3t} - \mu f_{32} \sin \theta
\end{bmatrix}. \quad (VI.61)
$$

where

$$U_{3j} = \epsilon \frac{\partial}{\partial t} [u^0_{3,j}]|_{\text{exp}}, \quad (VI.62)$$

for $j = 1, t$, and the explicit partial differentiation with respect to $t$ means the partial differentiation with respect to $t$ when $\epsilon$ and $\theta$ are assumed to be fixed.

Furthermore, when $f_{32}$ is given by

$$f_{32} = -\frac{b \dot{v}(t)}{4\pi c_s^2 \gamma^3}, \quad (VI.63)$$
where \( \gamma = \sqrt{1 - v^2/c^2} \), then
\[
g'_{32} = \frac{bv \cos \theta \sin \theta}{2\pi c^2 \gamma} \left[ \frac{(2 - 3\gamma^2) \cos^2 \theta + \gamma^2(\gamma^2 - 2) \sin^2 \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^3} \right] \\
- \frac{bv^2 \cos \theta \sin \theta}{4\pi c^2 \gamma^3 (\cos^2 \theta + \gamma^2 \sin^2 \theta)}; \tag{VI.64}
\]
\[
g'_{3t} = \frac{b v^2}{4\pi c^2 \gamma^3} \left[ \frac{\cos^2 \theta - \gamma^2 \sin^2 \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^2} + \frac{2\gamma^2(3\gamma^2 \sin^2 \theta - \cos^2 \theta)}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^3} \right], \tag{VI.65}
\]
\[
g'_{31} = f_{32} - g'_{32} \tan \theta. \tag{VI.66}
\]

From Theorem 1, for the near field coefficients \( f_{3j} \) with respect to the \( \ln \epsilon \) terms in the near field expansions of \( u_{3,j} \), the only \( f_{32}(\theta, t) = f_{32}(t) \) is non-zero, and \( f_{31} = f_{3t} = 0 \). From Theorem 2, \( g'_{3j}, j = 1, 2, t, \) are explicitly expressed in \( f_{32} \). Then by integration of \( g'_{3j} \), the near field coefficients \( g_{3j} \) can be determined,
\[
g_{3j}(\theta, t) = \int_0^\theta g'_{3j}(\omega, t) d\omega + g_{3j}(0, t). \tag{VI.67}
\]

By an analogous reasoning as in the proof of Theorem 1, \( g_{31}(0, t) = g_{3t}(0, t) = 0 \).

Therefore, the near field behavior of the moving screw dislocation, up to the order of \( O(1) \), i.e., the six functions of the near field coefficients \( f_{3j}(\theta, t) \) and \( g_{3j}(\theta, t) \), for \( j = 1, 2, t \), are completely determined if we know two near field constants \( f_{32}(t) \) and \( g_{32}(0, t) \), here \( t \) may be considered as a parameter.

Both theorems will play key role in solving and evaluating the self-force and effective mass.

**Proof of Theorem 1**

It suffices to consider \( y > 0 \), since the discussion for \( y < 0 \) is analogous.

For \( y > 0 \), \( u_{3,i} \) for \( i = 1, 2, t \), are continuously differentiable. From the relation
\[
\frac{\partial u_{3,1}}{\partial y} = \frac{\partial u_{3,2}}{\partial x}, \tag{VI.68}
\]
and the expansions (VI.51) and (VI.52), it follows that
\[
\frac{\partial u_{3,1}^0}{\partial y} + f'_{31} \frac{\cos \theta \ln \epsilon}{\epsilon} + f_{31} \frac{\sin \theta}{\epsilon} + g'_{31} \frac{\cos \theta}{\epsilon} \\
= \frac{\partial u_{3,2}^0}{\partial x} - f'_{32} \frac{\sin \theta \ln \epsilon}{\epsilon} + f_{32} \frac{\cos \theta}{\epsilon} - g'_{32} \frac{\sin \theta}{\epsilon} + h.o.t., \tag{VI.69}
\]
where the following relations are used
\[
\frac{\partial \epsilon}{\partial x} = \frac{x}{\sqrt{(x - l(t))^2 + y^2}} = \cos \theta,
\]
\[
\frac{\partial \epsilon}{\partial y} = \frac{y}{\sqrt{(x - l(t))^2 + y^2}} = \sin \theta,
\]
\[
\frac{\partial \theta}{\partial x} = \frac{-y}{(x - l(t))^2 + y^2} = \frac{-\sin \theta}{\epsilon},
\]
\[
\frac{\partial \theta}{\partial y} = \frac{(x - l(t))}{(x - l(t))^2 + y^2} = \frac{\cos \theta}{\epsilon}.
\]

It is clear that for the steady-state solutions
\[
\frac{\partial u_{3,1}}{\partial y} = \frac{\partial u_{3,2}}{\partial x}, \quad (VI.70)
\]

Then in (VI.69), compare the like terms of \( \ln \epsilon/\epsilon \), we have
\[
f'_{31} \cos \theta + f'_{32} \sin \theta = 0. \quad (VI.71)
\]

Similarly, from
\[
\frac{\partial u_{3,1}}{\partial t} = \frac{\partial u_{3}}{\partial x}, \quad (VI.72)
\]

and the expansions (VI.51) and (VI.53), it follows that
\[
\frac{\partial u_{3,1}}{\partial t} + f'_{31} \frac{v(t) \sin \theta \ln \epsilon}{\epsilon} - f_{31} \frac{v(t) \cos \theta}{\epsilon} + g'_{31} \frac{v(t) \sin \theta}{\epsilon} = \frac{\partial u_{3,1}}{\partial x} - f'_{31} \frac{\sin \theta \ln \epsilon}{\epsilon} + f_{31} \frac{\cos \theta}{\epsilon} - g'_{31} \frac{\sin \theta}{\epsilon} + h.o.t., \quad (VI.73)
\]

where the following relations are used
\[
\frac{\partial \theta}{\partial t} = \frac{v(t)y}{(x - l(t))^2 + y^2} = \frac{v(t) \sin \theta}{\epsilon},
\]
\[
\frac{\partial \epsilon}{\partial t} = -\frac{v(t)(x - l(t))}{(x - l(t))^2 + y^2} = \frac{-v(t) \cos \theta}{\epsilon}.
\]

Noting that for the steady-state solutions
\[
\frac{\partial u_{3,1}}{\partial t} - \frac{\partial u_{3,1}}{\partial x} = \left( \frac{\partial u_{3,1}}{\partial t} \right)_{\text{exp}}. \quad (VI.74)
\]
And it is easy to check that
\[
\left( \frac{\partial u^0_{3,1}}{\partial t} \right)_{\text{exp}} \sim O(1/\epsilon).
\] (VI.75)

Then compare the like terms of \( \ln \epsilon/\epsilon \) in (VI.73), we have
\[
v(t)f'_{31} \sin \theta + f'_{3t} \sin \theta = 0.
\] (VI.76)

To establish the third equation for \( f'_{31}, f'_{32}, f'_{3t} \), we use the equation of motion
\[
\mu \frac{\partial u^0_{3,1}}{\partial x} + \mu \frac{\partial u^0_{3,2}}{\partial y} = \rho \frac{\partial u^0_{3,t}}{\partial t},
\] (VI.77)

or together with the expansions (VI.51)-(VI.53). Then, we have that
\[
\mu \frac{\partial u^0_{3,1}}{\partial x} + \mu \frac{\partial u^0_{3,2}}{\partial y} = \rho \frac{\partial u^0_{3,t}}{\partial t} + \rho \left( f'_{3t} v(t) \sin \theta \ln \frac{\epsilon}{\epsilon} \right.
\]
\[
- f'_{31} \frac{\sin \theta}{\epsilon} + f'_{32} \frac{\cos \theta}{\epsilon} + f'_{32} \frac{\sin \theta}{\epsilon} \epsilon + g'_{32} \frac{\cos \theta}{\epsilon} \epsilon + h.o.t.
\] (VI.78)

For the steady-state solutions,
\[
\rho \frac{\partial u^0_{3,t}}{\partial t} - \mu \left( \frac{\partial u^0_{3,1}}{\partial x} + \frac{\partial u^0_{3,2}}{\partial y} \right) = \rho \left( \frac{\partial u^0_{3,t}}{\partial t} \right)_{\text{exp}}.
\] (VI.79)

It is easy to show that the term on the right hand side is in the order of \( O(1/\epsilon) \).
Hence compare the like terms of \( \ln \epsilon/\epsilon \) in (VI.78), we have the third equation
\[
- \mu f'_{31} \sin \theta + \mu f'_{32} \cos \theta - \rho v f'_{3t} \sin \theta = 0.
\] (VI.80)

Hence, (VI.71), (VI.76), and (VI.80) form a homogeneous system for the unknowns \( f'_{3j}, \quad j = 1, 2, t, \)

\[
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
v \sin \theta & 0 & \sin \theta \\
- \mu \sin \theta & \mu \cos \theta & - \rho v \sin \theta
\end{bmatrix}
\begin{bmatrix}
f'_{31} \\
f'_{32} \\
f'_{3t}
\end{bmatrix} = \mathbf{0}.
\] (VI.81)
The determinant of the matrix of the coefficients in (VI.81) is calculated to be

\[
\begin{vmatrix}
\cos \theta & \sin \theta & 0 \\
v \sin \theta & 0 & \sin \theta \\
-\mu \sin \theta & \mu \cos \theta & -\rho \sin \theta
\end{vmatrix} = (\rho v^2 \sin^2 \theta - \mu) \sin \theta \neq 0,
\]

(VI.82)

for \(\theta \neq \pi, 0\), since \(v < c_2 = \sqrt{\mu/\rho}\). Therefore, we conclude that

\[f'_{31} = f'_{32} = f'_{3t} = 0,
\]

(VI.83)

for \(\theta \neq \pi, 0\). \(f_{31}, f_{32}\) and \(f_{3t}\) are then independent of \(\theta\) for \(y > 0\) and \(y < 0\), respectively.

Combining those conclusion with the global properties of \(u_{3,1}\) and \(u_3\) discussed in previous subsection, i.e., \(u_{3,1}\) and \(u_3\), thus \(f_{3,1}\) and \(f_{3,t}\) as well, are continuous across \(y = 0\), and odd in \(y\), which can be seen by using (VI.42) and (VI.48) in the near field. Then we have

\[f_{31} = f_{3t} = 0.
\]

(VI.84)

Similarly, in view of that \(u_{3,2}\), thus \(f_{32}\) as well, is an even function in \(y\), (VI.83) implies that

\[f_{32}(\theta, t) = f_{32}(t)
\]

(VI.85)

is independent of \(\theta\) and only a function of \(t\). Theorem 1 is therefore proved.

**Proof of Theorem 2**

In (VI.69), (VI.73), and (VI.78), noting that \(f'_{3j} = 0\) and \(f_{31} = f_{3t} = 0\), equate the like terms of \(1/\epsilon\), we obtain the inhomogeneous system of linear equations

\[
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
v \sin \theta & 0 & \sin \theta \\
-\mu \sin \theta & \mu \cos \theta & -\rho \sin \theta
\end{bmatrix}
\begin{bmatrix}
g'_{31} \\
g'_{32} \\
g'_{3t}
\end{bmatrix} =
\begin{bmatrix}
f_{32} \cos \theta \\
-U_{31} \\
\rho U_{3t} - \mu f_{32} \sin \theta
\end{bmatrix},
\]

(VI.86)

where

\[U_{3j} = \epsilon \frac{\partial}{\partial t} [u^0_{3,j}]_{\text{exp}},
\]

(VI.87)
for \( j = 1, t. \)

The matrix of coefficients of the system of linear equations in (VI.86) is exactly same as that of the system (VI.81), so that the determinant of the matrix is known to be

\[
det = (\rho v^2 \sin^2 \theta - \mu) \sin \theta = -(\cos^2 \theta + \gamma^2 \sin^2 \theta) \sin \theta. \tag{VI.88}
\]

\( g'_{32} \) and \( g'_{3t} \) are solved to be

\[
g'_{32} = \frac{\cos \theta}{c^2_2} \left[ \frac{v U_{31} - U_{3t} + v^2 f_{32} \sin \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta} \right], \tag{VI.89}
\]

\[
g'_{3t} = \frac{\rho v U_{3t} \sin^2 \theta - \mu U_{31} - \mu v f_{32} \sin \theta}{\mu \sin \theta (\cos^2 \theta + \gamma^2 \sin^2 \theta)}. \tag{VI.90}
\]

\( U_{31} \) and \( U_{3t} \) are written as

\[
U_{31} = \frac{b \dot{v}}{2 \pi c^2_2 \gamma} \left[ \frac{\cos^2 \theta - \gamma^2 \sin^2 \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^2} \right], \tag{VI.91}
\]

\[
U_{3t} = \frac{b \dot{v} \sin \theta}{2 \pi \gamma} \left[ \frac{(2 \gamma^2 - 1) \cos^2 \theta + \gamma^2 \sin^2 \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^2} \right]. \tag{VI.92}
\]

Substitute those expressions into (VI.64) and (VI.90), when \( f_{32} \) is given by (VI.63), we obtain that

\[
g'_{32} = \frac{b \dot{v} \cos \theta \sin \theta}{2 \pi c^2_2 \gamma} \left[ \frac{(2 - 3 \gamma^2) \cos^2 \theta + \gamma^2 (\gamma^2 - 2) \sin^2 \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^3} \right] - \frac{b \dot{v} v^2 \cos \theta \sin \theta}{4 \pi c^2_2 \gamma^3 (\cos^2 \theta + \gamma^2 \sin^2 \theta)}, \tag{VI.93}
\]

\[
g'_{3t} = \frac{b \dot{v} \cos \theta - \gamma^2 \sin^2 \theta}{4 \pi c^2_2 \gamma^3 (\cos^2 \theta + \gamma^2 \sin^2 \theta)^2} + \frac{2 \gamma^2 (3 \gamma^2 \sin^2 \theta - \cos^2 \theta)}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^3}, \tag{VI.94}
\]

From the first equation in the system (VI.86), it follows that

\[
g'_{31} = f_{32} - g'_{32} \tan \theta. \tag{VI.95}
\]

Theorem 2 is proved.
VI.H Evaluation of $g_{3j}(\theta, t)$

In this section, we shall evaluate the near field coefficients $g_{3j}$ by integrating $g'_{3j}$ which are explicitly given in Theorem 2. Namely, the near field coefficients $g_{3j}(\theta, t)$ will follow from

$$g_{3j}(\theta, t) = \int_0^\theta g'_{3j}(\theta, t) d\theta + g_{3j}(0, t).$$  \hfill (VI.96)

VI.H.1 Evaluation of $g_{32}(\theta, t)$

By substitute the explicit expression (VI.89) for $g'_{32}(\theta, t)$, into the equation (VI.96) for $j = 2$, we shall obtain the explicit solution of $g_{32}(\theta, t)$.

We calculate the integration

$$\int_0^\theta g'_{32}(\theta, t) d\theta = \frac{b\dot{v}}{2\pi c^2 \gamma} \int_0^\theta \frac{\cos \theta \sin \theta [(2 - 3\gamma^2) \cos^2 \theta + \gamma^2 (\gamma^2 - 2) \sin^2 \theta]}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^3} d\theta$$

$$- \frac{b\dot{v}v^2}{4\pi c^2 \gamma^3} \int_0^\theta \cos \theta \sin \theta \frac{1}{\cos^2 \theta + \gamma^2 \sin^2 \theta} d\theta$$

$$= \frac{b\dot{v}}{2\pi c^2 \gamma} [(2 - 3\gamma^2) \cos^2 \theta - \gamma^4 \sin^2 \theta]$$

$$+ \frac{b\dot{v}}{8\pi c^2 \gamma^3} \ln(\cos^2 \theta + \gamma^2 \sin^2 \theta).$$  \hfill (VI.97)

Hence, it follows that

$$g_{32}(\theta, t) = \frac{b\dot{v}}{2\pi c^2 \gamma} [(2 - 3\gamma^2) \cos^2 \theta - \gamma^4 \sin^2 \theta]$$

$$+ \frac{b\dot{v}}{8\pi c^2 \gamma^3} \ln(\cos^2 \theta + \gamma^2 \sin^2 \theta) + g_{32}(0, t),$$  \hfill (VI.98)

where the term $g_{32}(0, t)$ will be evaluated later on.

The near field coefficient $g_{32}$ obtained here will be used in the calculation of the self-force and effective mass.

VI.H.2 Evaluation of $g_{3t}(\theta, t)$

Noting that $g_{3t}(\theta, t)$ is odd in $\theta$, by the same analysis as used in the proof of Theorem 2 based on the global properties of $u_{3t}$, we conclude that $g_{3t}(0, t) = 0$. 
Hence, it follows that

$$g_{3t}(\theta, t) = \int_0^\theta g'_{3t}(\theta, t)d\theta. \quad (VI.99)$$

Using the expression (VI.90) in the integration, we obtain that

$$g_{3t}(\theta, t) = \int_0^\theta g'_{3t}(\theta, t)d\theta = \frac{bv\dot{v}}{4\pi c^2 \gamma^3} \left[ \frac{2}{\gamma} \tan^{-1}(\gamma \tan(\theta)) - \frac{(1 - 2\gamma^2) \cos^2 \theta + 3\gamma^2 \sin^2 \theta \sin \theta \cos \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^2} \right]. \quad (VI.100)$$

The near field coefficient $g_{3t}$ obtained here will be used in the calculation of the self-force and effective mass.

**VI.H.3 Evaluation of $g_{31}(\theta, t)$**

Similarly, in view of that $g_{31}(0, t) = 0$, we have that

$$g_{31}(\theta, t) = \int_0^\theta g'_{31}(\theta, t)d\theta. \quad (VI.101)$$

By using (VI.95) and (VI.89), we have that

$$g_{31}(\theta, t) = \int_0^\theta g'_{31}(\theta, t)d\theta = \int_0^\theta [f_{32} - g_{32} \tan \theta]d\theta$$

$$= f_{32}(t)\theta + \frac{b\dot{v}}{4\pi c^2 \gamma^3} \int_0^\theta \sin^2 \theta \left[ \frac{7\gamma^2 - 5}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^3} \right]d\theta$$

$$= -\frac{b\dot{v}}{4\pi c^2 \gamma^3} \left[ \theta - \frac{1}{\gamma} \tan^{-1}(\gamma \tan(\theta)) \right]$$

$$- \frac{b\dot{v}}{8\pi c^2 \gamma^3} \left[ \frac{2 \cos^2 \theta + \gamma^2 (3 - 2\gamma^2) \sin^2 \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^2} \right]. \quad (VI.102)$$

The near field coefficient $g_{31}(\theta, t)$ will not be used in the calculation below.
VI.1 Solution of the Self-Force

With the near field properties shown in previous subsection, we are ready to find the self-force which is given as the limit of the surface integral $I_\epsilon$ given by (IX.107) as $\epsilon \to 0$,

$$F_1 = \lim_{\epsilon \to 0} I_\epsilon,$$

and $I_\epsilon$ is explicitly written as

$$I_\epsilon = \int_{S} [(W - T)\delta_{1j} - \sigma_{3j}u_{3,1}]dS_j$$

$$= \int_{0}^{2\pi} \frac{1}{2} [\mu u_{3,2}^2 - \mu u_{3,1}^2 - \rho \dot{u}_{3}^2] \cos \theta \epsilon d\theta - \int_{0}^{2\pi} \mu u_{3,1} u_{3,2} \sin \theta \epsilon d\theta. \quad (VI.104)$$

From Theorem 1, the near field expansions of $u_{3,i}$ for $i = 1, 2, t$ are now rewritten as

$$u_{3,1} = \frac{b}{2\pi} \frac{\gamma \sin \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta \epsilon} + g_{31}(\theta, t) + O(\epsilon), \quad (VI.105)$$

$$u_{3,2} = \frac{b}{2\pi} \frac{\gamma \cos \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta \epsilon} + f_{32}(t) \ln \epsilon + g_{32}(\theta, t) + O(\epsilon), \quad (VI.106)$$

$$\dot{u}_3 = \frac{b}{2\pi} \frac{v(t) \gamma \sin \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta \epsilon} + g_{3t}(\theta, t) + O(\epsilon), \quad (VI.107)$$

where again $\gamma = \sqrt{1 - v^2/c^2}$, $\epsilon = \sqrt{x - l(t))^2 + y^2}$, and $\theta = \tan^{-1}(y/(x - l(t)))$.

We now substitute those expressions into the integral $I_\epsilon$ to calculate the self-force.

From (VI.105) - (VI.107), we have, as $\epsilon \to 0$,

$$u_{3,1}^2 = \frac{b^2}{4\pi^2} \frac{\gamma^2 \sin^2 \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta \epsilon^2} \frac{1}{\epsilon^2}$$

$$- \frac{b}{\pi} \frac{\gamma \sin \theta g_{31}(\theta, t)}{\cos^2 \theta + \gamma^2 \sin^2 \theta \epsilon} + g_{31}^2(\theta, t) + O(\epsilon), \quad (VI.108)$$

$$\dot{u}_3^2 = \frac{b^2}{4\pi^2} \frac{v^2(t) \gamma^2 \sin^2 \theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta \epsilon^2} \frac{1}{\epsilon^2}$$

$$+ \frac{b}{\pi} \frac{v(t) \gamma \sin \theta g_{3t}(\theta, t)}{\cos^2 \theta + \gamma^2 \sin^2 \theta \epsilon} + g_{3t}^2(\theta, t) + O(\epsilon) \quad (VI.109)$$
\[
\begin{align*}
\left(\frac{\gamma^2 \cos^2 \theta}{f_{32} \ln \epsilon + g_{32}}\right) & = \frac{1}{\epsilon^2} + \frac{b \gamma \cos \theta (f_{32} \ln \epsilon + g_{32})}{\cos^2 \theta + \gamma^2 \sin^2 \theta} + (f_{32} \ln \epsilon + g_{32})^2 + O(\epsilon), \\
\end{align*}
\]

and

\[
\begin{align*}
\left(\frac{\gamma^2 \cos \theta \sin \theta}{f_{32} \ln \epsilon + g_{32}}\right) & = \frac{1}{\epsilon^2} + \frac{b \gamma \sin \theta (f_{32} \ln \epsilon + g_{32})}{\cos^2 \theta + \gamma^2 \sin^2 \theta} + (f_{32} \ln \epsilon + g_{32}) + O(\epsilon).
\end{align*}
\]

Hence, the corresponding integrals are obtained to be

\[
\begin{align*}
\int_{0}^{2\pi} u_{3,1} \cos \theta \epsilon d\theta & = -\frac{b}{\pi} \int_{0}^{2\pi} \frac{\gamma \sin \theta \cos \theta f_{31}(\theta, t)}{\cos^2 \theta + \gamma^2 \sin^2 \theta} d\theta + O(\epsilon), \\
\int_{0}^{2\pi} u_{3,2} \cos \theta \epsilon d\theta & = \frac{b}{\pi} \int_{0}^{2\pi} \frac{v(t) \gamma \sin \theta \cos \theta g_{31}(\theta, t)}{\cos^2 \theta + \gamma^2 \sin^2 \theta} d\theta + O(\epsilon), \\
\int_{0}^{2\pi} u_{3,2} \cos \theta \epsilon d\theta & = \frac{b}{\pi} \int_{0}^{2\pi} \frac{\gamma \cos \theta (f_{32} \ln \epsilon + g_{32})}{\cos^2 \theta + \gamma^2 \sin^2 \theta} d\theta + O(\epsilon), \\
\int_{0}^{2\pi} u_{3,1} u_{3,2} \sin \theta \epsilon d\theta & = -\frac{b}{2\pi} \int_{0}^{2\pi} \frac{\gamma \sin \theta [\sin \theta (f_{32} \ln \epsilon + g_{32}) - \cos \theta g_{31}]}{\cos^2 \theta + \gamma^2 \sin^2 \theta} d\theta + O(\epsilon).
\end{align*}
\]

Using (VI.112), (VI.113), (VI.114) and (VI.115) in (IX.107), we have that

\[
\begin{align*}
I_\epsilon & = \frac{\mu b}{2\pi} \int_{0}^{2\pi} \frac{\gamma (f_{32} \ln \epsilon + g_{32}(\theta, t))}{\cos^2 \theta + \gamma^2 \sin^2 \theta} d\theta \\
& - \frac{\rho b}{2\pi} \int_{0}^{2\pi} \frac{\gamma v(t) \sin \theta \cos \theta g_{31}(\theta, t)}{\cos^2 \theta + \gamma^2 \sin^2 \theta} d\theta + O(\epsilon).
\end{align*}
\]
In view of the integral
\[
\int_0^{2\pi} \frac{\gamma d\theta}{\cos^2 \theta + \gamma^2 \sin^2 \theta} = 2\pi, \tag{VI.117}
\]
and \( \rho = \mu/c^2 \), (VI.116) is reduced to
\[
I_\epsilon = \mu bf_{32} \ln \epsilon + \frac{\mu b\gamma}{2\pi} \int_0^{2\pi} \frac{c^2 g_{32}(\theta, t) - v(t) \sin \theta \cos \theta g_{34}(\theta, t)}{c^2[\cos^2 \theta + \gamma^2 \sin^2 \theta]} d\theta + O(\epsilon), \tag{VI.118}
\]
where again,
\[
f_{32} = \frac{b}{4\pi c^2 \gamma^3} \dot{v}(t),
\]
and \( g_{32}(\theta, t) \) and \( g_{34}(\theta, t) \) are the near field coefficients.

Therefore, if the motion is non-uniform, i.e., \( \dot{v}(t) \neq 0 \),
\[
\lim_{\epsilon \to 0} I_\epsilon = -\frac{\mu b^2 \dot{v}(t)}{4\pi c^2 \gamma^3} \ln \epsilon + \frac{\mu b\gamma}{2\pi} \int_0^{2\pi} \frac{c^2 g_{32}(\theta, t) - v(t) \sin \theta \cos \theta g_{34}(\theta, t)}{c^2[\cos^2 \theta + \gamma^2 \sin^2 \theta]} d\theta \tag{VI.119}
\]
is divergent. Hence, the self-force on an accelerating Volterra screw dislocation is infinite, which diverges as \( \ln \epsilon \) as \( \epsilon \to 0 \) and in terms of the near field coefficients which is given by
\[
F_1 \sim -\frac{\mu b^2 \dot{v}(t)}{4\pi c^2 \gamma^3} \ln \epsilon + \frac{\mu b\gamma}{2\pi} \int_0^{2\pi} \frac{c^2 g_{32}(\theta, t) - v(t) \sin \theta \cos \theta g_{34}(\theta, t)}{c^2[\cos^2 \theta + \gamma^2 \sin^2 \theta]} d\theta. \tag{VI.120}
\]

### VI.J Evaluation of the Self-Force

We now substitute the explicit expressions of \( g_{32} \) and \( g_{34} \) into (VI.120) to evaluate the self-force.

Using the expression (VI.89), we calculate the first part of the integration,
\[
\int_0^{2\pi} \frac{g_{32}(\theta, t)}{\cos^2 \theta + \gamma^2 \sin^2 \theta} d\theta = \frac{b\dot{v}(1 - 3\gamma^2)}{8c^2 \gamma^4} + \frac{b\dot{v}}{2c^2 \gamma^4} \ln\left(\frac{2\gamma}{1 + \gamma}\right) + \frac{2\pi}{\gamma} g_{32}(0, t). \tag{VI.121}
\]
Using the expression (VI.90), we obtain the second part of the integration,
\[
\frac{2\pi}{c_2^2(\cos^2 \theta + \gamma^2 \sin^2 \theta)} \int_0^\infty v_{32}(\theta, t) \sin \theta \cos \theta d\theta
\]
\[
= \frac{b\dot{v}}{c_2^2 \gamma^4} \ln \left( \frac{1 + \gamma}{2} \right) + \frac{b\dot{v}(1 - \gamma)}{8c_2^2 \gamma^4 (1 + \gamma)^2} \left[ 3(1 - \gamma^2) + \gamma(5 - r^2) \right].
\]
(VI.122)

Substitute (VI.121) and (VI.122) into the expression (VI.120) of the self-force, we obtain
\[
F_1 \sim -\mu b^2 \dot{v} \ln \epsilon + \mu bg_{32}(0, t)
\]
\[
+ \frac{\mu b^2 \dot{v}}{4\pi c_2^2 \gamma^3} \left[ \ln(\gamma(1 + \gamma)/2) - \frac{4 + \beta^4 - \beta^2(7 + 2\gamma)}{(1 + \gamma)^2} \right].
\]
(VI.123)

In the expression, \(g_{32}(0, t)\) is still unknown. We shall evaluate it in next subsection.

**VI.J.1 Evaluation of \(g_{32}(0)\)**

In this section, we shall give the results of the calculation of \(g_{32}(0)\). The detailed calculation will be presented in Appendix A. The calculation is done based on the near field expansion of the stress \(u_{3,2}\) at the positions \((x, y) = (l(t) + \epsilon, 0)\), as \(\epsilon \to 0\). The calculation will use the exact solution of \(u_{32}\) for a non-uniformly moving screw dislocation starting from rest given by Markenscoff (1980),
\[
u_{32} = \frac{b}{2\pi} \int_0^\infty \frac{(t - \eta(\xi))(x - \xi)^2 H(t - \eta(\xi) - r/c)}{r^4[(t - \eta(\xi))^2 - r^2/c^2]^{1/2}} d\xi
\]
\[
- \frac{b}{2\pi} y^2 \frac{\partial}{\partial t} \int_0^\infty \frac{(t - \eta(\xi))^2 H(t - \eta(\xi) - r/c)}{r^4[(t - \eta(\xi))^2 - r^2/c^2]^{1/2}} d\xi + \frac{b}{2\pi} \frac{x}{x^2 + y^2},
\]
(VI.124)

where \(c \equiv c_2 = \sqrt{\mu/\rho}\), \(r^2 = (x - \xi)^2 + y^2\)and \(\eta(\xi) = \tau\) is the inverse function of \(\xi = l(\tau)\).

In (VI.124), the third term gives the following simple term contribution to \(g_{32}(0)\),
\[
\frac{b}{2\pi l(t)}.
\]
(VI.125)

We then focus on the integrations in the expression.
For \( x = l(t) + \epsilon \), the integrations in (VI.124) only have a square root singularities, and the limit of \( y \to 0 \) can be taken under the integrations. After taking the limit, the second term in (VI.124) vanishes, and the first integral in (VI.124) is reduced to

\[
II_\epsilon = \frac{b}{2\pi} \int_0^\xi \frac{t - \eta(\xi)}{(x - \xi)^2[(t - \eta(\xi))^2 - (x - \xi)^2/c^2]^{1/2}} d\xi, \quad (VI.126)
\]

where \( x = x_0 + \epsilon \).

To find the near field expansion of (VI.126), we express its integrand as a function of \( \epsilon \), the integral \( II_\epsilon \) is rewritten as

\[
II_\epsilon = \frac{b}{2\pi} \int_0^{\xi_0} h(s, \epsilon, x, t) ds, \quad (VI.127)
\]

where \( s \equiv \xi_0 - \xi \) with \( \xi_0 \) as the zero of the argument of the Heaviside step function \( H(t - \eta(\xi) - r/c) \) when \( y = 0 \), i.e., \( \xi_0 \) is the root of the equation

\[
c(t - \eta(\xi_0)) = (x - \xi_0), \quad (VI.128)
\]

\( h \) is defined by

\[
h(s, \epsilon, x, t) = \frac{(t - \tau)}{(x - \xi)^2[(t - \tau)^2 - (x - \xi)^2/c^2]^{1/2}}
\]

\[
= \frac{1}{(s + \epsilon A_0(t, \epsilon))^2[s^2(\psi^2 - 1/c^2) + 2\epsilon s A_0(t, \epsilon)(\psi - 1/c)]^{1/2}}; \quad (VI.129)
\]

\( A_0(t, \epsilon) \) is defined by

\[
A_0(t, \epsilon) \equiv (t - \eta(\xi_0))/\epsilon; \quad (VI.130)
\]

and \( \psi(s) = \psi(s, \xi_0) \) is defined by

\[
\psi(s, \xi_0) = \frac{\eta(\xi_0) - \eta(s)}{s} = \frac{\eta(\xi_0) - \eta(\xi_0 - s)}{s}. \quad (VI.131)
\]

We note that if using simple Taylor expansion for factor \( 1/(s + \epsilon A_0)^2 \), which will give

\[
\frac{1}{(s + \epsilon A_0)^2} = \frac{1}{s^2} + O(\epsilon), \quad (VI.132)
\]
the integral will be divergent. Hence a special technique is needed in the calculation.

**A Special Technique for Asymptotic Expansions**

Now we shall use a Corollary of the theorem given by Callias and Markenscoff (1988) to do asymptotic expansion for $II_\epsilon$, which states as follows.

**Corollary:**

Let $f(s, y)$ be such that

1. $f \in C^\infty((0, p] \times [0, \infty))$
2. $|\partial^k_s f(s, z)| \leq y^k h_k(y)$, for all $s, y$,
   $$k = 0, 1, 2, ...$$

   where $\int_0 h_k(1/s)ds < \infty$, for each $\epsilon > 0$.

Then we have as $s \to 0^+$,

$$\int_0^p f(s, \epsilon/s)ds \sim \int_0^p f(s, 0)ds + \sum_{m=1}^{\infty} e^m \left\{ L_m(f) + u_m(f; p) + \frac{1}{m!(m-1)!} \partial^{m-1}_s \partial^m_y f(0, 0) \sum_{j=1}^{m-1} \frac{1}{j} \right\}$$

$$+ \sum_{m=1}^{\infty} e^m \ln \epsilon \frac{-1}{m!(m-1)!} \partial^{m-1}_s \partial^m_y f(0, 0)$$

$$+ \sum_{m=1}^{\infty} e^m \left\{ \sum_{j=1}^{m-1} (-1)^{m+j} \frac{(j-1)!}{m!(m-1)!} \frac{1}{p^j} \partial^{m-1-j}_s \partial^m_y f(p, 0) \right\}, \quad (VI.133)$$

where

$$L_m(f) = -\frac{1}{(m-1)!} \int_0^\infty \ln \zeta \partial_\zeta [\zeta^m \partial^{m-1}_x R_{m+1}(0, 1/\zeta)]d\zeta, \quad (VI.134)$$

$R_{m+1}(x, y)$ is the remainder of $f(x, y)$ in the Taylor expansion about $y = 0$ after $m + 1$ terms, i.e.,

$$R_{m+1} = f(x, y) - \sum_{k=0}^{m} \frac{1}{k!} \partial^k_y f(x, 0) y^k, \quad (VI.135)$$
and

\[ u_m(f; p) = -\frac{1}{m!(m-1)!} \int_0^p \ln x \partial_x^m \partial_y^m f(x, 0) dx. \] (VI.136)

We have verified that for the integral \( II \) the conditions required in the Corollary are satisfied. After a lengthy calculation, we obtain the evaluation of \( g_{32}(0) \) as given below, where the term \( b/(2\pi l(t)) \) contributed by the third term in (VI.124) is also included.

**Evaluation of \( g_{32}(0) \)**

\[
\begin{align*}
g_{32}(0) &= \frac{b}{2\pi} \left\{ \frac{c\dot{v}(t)}{4(c^2 - v^2(t))^{3/2}} \ln \left( \frac{cv}{2(c^2 - v^2)} \right) \\
&\quad + \frac{\dot{v}\sqrt{c^2 - v^2}}{(c - v)^3} \frac{v(c - v)}{c(c + v)} - \frac{c^2(c + 2v)}{v^2(c + v)^3} \\
&\quad + \frac{\dot{v}}{(c - v)^3} \left( \frac{c(2c - 5v)}{2(c - v)} + \frac{3v^2}{2c} \right) \ln \left( \frac{c + \sqrt{c^2 - v^2}}{v} \right) \\
&\quad + \frac{\ddot{v}}{2c^2} \int_0^t [\ln(l(t) - l(\tau)) \frac{l(t) - l(\tau)}{v^2} \\
&\quad - \frac{1}{l(t)[1 - \frac{\tau^2(l(t) - l(\tau))^2}{c^2}]^{1/2}} + \frac{1}{l(t)} \\
&\quad + \frac{1}{2c^2} \int_0^t \frac{1}{v^2 \left[ 1 - \frac{(l(t) - l(\tau))^2}{c^2} \right]^{5/2}} \\
&\quad \left[ \frac{t - \tau}{(l(t) - l(\tau))^2} P_2 + \frac{1}{c^2(l(t) - l(\tau))} P_1 \right] d\tau \right\}, \tag{VI.137}
\end{align*}
\]

where \( P_1 \) and \( P_2 \) are defined by

\[ P_1 = v^2 \left[ 2\left( \frac{\omega(t, \tau)}{t - \tau} \right) + \dot{v} - \ddot{v}(\theta) [v^2 - \frac{(l(t) - l(\tau))^2}{t - \tau}] \right], \tag{VI.138} \]

\[ P_2 \equiv v^2 \omega^2(t, \tau) - \omega(t) \left( \frac{l(t) - l(\tau)}{t - \tau} \right) [2\omega(t, \tau) + \dot{v}(t - \tau) - \ddot{v}(\theta)(l(t) - l(\tau)) \omega(t, \tau)], \tag{VI.139} \]
and \( \tau \leq \theta \leq t \), and \( \omega(t, \tau) \) is defined by

\[
\omega(t, \tau) \equiv v(t) - \frac{l(t) - l(\tau)}{t - \tau}.
\]  

(VI.140)

In (VI.137), except the last three terms, all other terms has an explicit factor of \( \dot{v} \), which are related to the inertial part of the self-force, and will contribute to the effective mass of the moving screw dislocation. The integrations in the expression depend on the history of the motion.

VI.J.2 Evaluation of the Self-Force

Summarize the results obtained in above subsections, we write the evaluation of the self-force on the moving screw dislocation

\[
F_1 = F_1^{in} + F_1^{non},
\]  

(VI.141)

where \( F_1^{in} \) and \( F_1^{non} \) are the inertial part and the remain part of the self-force, are given, for \( t > 0 \), respectively as follows.

\[
F_1^{in} \sim \frac{\mu b^2 \dot{v}(t)}{2\pi} \left\{ \frac{-\ln \epsilon}{2c^2\gamma^3} + \frac{c}{4(c^2 - v^2(t))^3/2} \ln \left( \frac{cv}{2(c^2 - v^2)} \right) \right. \\
+ \frac{\sqrt{c^2 - v^2}}{(c - v)^3} \frac{v(c - v)}{c(c + v)} - \frac{c^2(c + 2v)}{v^2(c + v)} \left. \right\} \\
+ \frac{1}{c(v - c)^3} \left[ \frac{v^2}{2c} \ln \left( \frac{c + \sqrt{c^2 - v^2}}{v} \right) \right. \\
+ \frac{\dot{v}}{2\epsilon^2} \int_0^t \ln(l(t) - l(\tau)) \frac{l(t) - l(\tau)}{l^2} \] \\
\left[ \frac{v(t) - l(t) - l(\tau)}{(t - \tau)^2} \right] \frac{c^2 + v(t) \left( \frac{l(t) - l(\tau)}{t - \tau} \right) + \left( \frac{l(t) - l(\tau)}{t - \tau} \right)^2}{1 - \left( \frac{l(t) - l(\tau)}{t - \tau} \right)^2 / c^2^{5/2}} d\tau \right. \\
+ \frac{\mu b^2 \dot{v}}{4\pi c^2\gamma^3} \ln(\gamma(1 + \gamma)/2) - \frac{4 + \beta^4 - \beta^2(7 + 2\gamma)}{(1 + \gamma)^2},
\]  

(VI.142)

\[
F_1^{non} = \frac{\mu b^2}{2\pi} \left\{ \frac{1}{l(t)[1 - \frac{\tau(t)}{l(t)}]^{1/2}} + \frac{1}{l(t)} \right. \\
+ \frac{1}{2\epsilon^2} \int_0^t \frac{1}{\sqrt{v^2(1 - \left( \frac{\tau - \tau}{l(t) - l(\tau)} \right)^2 / c^2^{5/2}} \ln(l(t) - l(\tau))}{l(t)} \right. \\
\] 

(VI.142)
\[
\left\{ \frac{t - \tau}{(l(t) - l(\tau))^2} P_2 + \frac{1}{c^2(l(t) - l(\tau))} P_1 \right\} \d\tau, \tag{VI.143}
\]

where \( P_1 \) and \( P_2 \) are defined by

\[
P_1 \equiv v^2 [2(\frac{\omega(t, \tau)}{t - \tau}) + \dot{v}] - \ddot{v}(\theta)[v^2 - (\frac{l(t) - l(\tau)}{t - \tau})^2], \tag{VI.144}
\]

\[
P_2 \equiv v^2 \omega^2(t, \tau) - v(t)(\frac{l(t) - l(\tau)}{t - \tau})[2\omega(t, \tau) + \dot{v}(t - \tau)] - \ddot{v}(\theta)(l(t) - l(\tau))\omega(t, \tau), \tag{VI.145}
\]

and \( \tau \leq \theta \leq t \), and \( \omega(t, \tau) \) is defined by

\[
\omega(t, \tau) \equiv v(t) - \frac{l(t) - l(\tau)}{t - \tau}. \tag{VI.146}
\]

### VI.K Effective Mass of an Accelerating Screw Dislocation

From (VI.142), the effective mass of an accelerating screw dislocation is obtained as, for \( t > 0 \),

\[
m_e \approx \frac{\mu b^2}{2\pi} \left\{ \frac{-\ln \epsilon}{2c^2 \gamma^3} + \frac{c}{4(c^2 - v^2(t))^{3/2}} \ln \left( \frac{c v}{2(c^2 - v^2)} \right) \right. \]
\[
+ \frac{\sqrt{c^2 - v^2}}{(c - v)^3} \left[ \frac{v(c - v)}{c(c + v)} - \frac{c^2(c + 2v)}{v^2(c + v)} \right] \]
\[
+ \frac{1}{(c - v)^3} \frac{c(2c - 5v)}{2c} + \frac{3v^2}{2c} \ln \left( \frac{c + \sqrt{c^2 - v^2}}{v} \right) \]
\[
+ \frac{1}{2c^2} \int_0^t \ln(l(t) - l(\tau)) \frac{l(t) - l(\tau)}{v^2} d\tau \]
\[
\left. \cdot \left[ \frac{v(t) - l(t) - l(\tau)}{(t - \tau)^2} \right] \left[ c^2 + v(t)(\frac{l(t) - l(\tau)}{t - \tau}) + (\frac{l(t) - l(\tau)}{t - \tau})^2 \right] \left[ 1 - (\frac{l(t) - l(\tau)}{t - \tau})^2/c^2 \right]^{5/2} d\tau \right\} \]
\[
+ \frac{\mu b^2}{4\pi c^2 \gamma^3} \ln(\gamma(1 + \gamma)/2) - \frac{4 + \beta^2 - \beta^2(7 + 2\gamma)}{(1 + \gamma)^2}. \tag{VI.147}
\]
VI.L Conclusion

In this chapter, by using the definition given in last chapter, we calculated the self-force on the moving screw dislocation. The effective mass of an accelerating screw dislocation is determined by the inertial part of the self-force. It is shown that for Volterra screw dislocation, the self-force and the effective mass per unit length of the moving screw dislocation are divergent logarithmically. A complete explicit solutions for the self-force and the effective mass are obtained.

The calculation is based on the definition and the expression of the self-force given in last chapter, and based on two new theorems of the near field behaviors of the field solutions. An asymptotic corollary by Callias and markenscoff (1988) is used to calculate a near field value. Two important new theorems on the properties of the near field coefficients play essential role in the calculation. From the theorems, the near field expansions of the field solutions $u_{3,j}$, $j = 1, 2, t$ are explicitly determined, and so that a complete evaluations of the self-force and effective mass are achieved.

To deal with the divergence of the self-force and effective mass for the non-uniformly moving dislocation, in the following chapters, we shall discuss the self-force and effective mass based on (i) the theory of distributions; (ii) a smearing method.
Chapter VII

Effective Mass of an Accelerating Screw Dislocation Based on Theory of Distributions

VII.A Introduction

In last chapter, it is shown that the self-force and effective mass for an accelerating moving screw dislocation are divergent as \( \ln \epsilon \) as \( \epsilon \) goes to zero. In this chapter, we shall use the method based on the theory of distributions as described in Chapter V, to regularize the divergent volume integral in the “contour-independent” integral expression for the self-force. In view of that the singularities involved in moving dislocations are algebraic, i.e., the singularity is at most as singular as \(|x - x_0|^{-m}\), for an integer \( m \), where \( x_0 = l(t) \) is the position of the core of dislocation, so that the regularization is possible. We shall then obtain the evaluations of the self-force and effective mass, which are now finite and depend on the parameter of the regularization.
VII.B Regularization of the Divergent Volume Integral

Specifically, here we seek to use (V.31) in Chapter V to regularize the divergent volume integral in (V.18),
\[ \int_V g dv \equiv \int_V \frac{\partial}{\partial t} [\rho \dot{u}_3 u_{3,i}] dv. \] (VII.1)

As discussed in last chapter, for a moving dislocation, we choose V to be a cylindrical volume around the dislocation line. The problem may be considered to be two-dimensional, all volumes and surfaces have an unite length in the z-direction. Hence, the regularization is performed in the two-dimensional framework. Namely, we may choose a sufficiently small positive number \( a \), and define the regularization of the integral through the regularization of a singular distribution, specifically by using

\[ (\text{Reg} \, g, \phi) = \int_{V \setminus B_a} g(x) \phi(x) dV + \int_{B_q} g(r, \theta) [\phi(r, \theta) - \sum_{|k|<m} \frac{\partial^k}{\partial r^k} \phi(x_0, \theta) r^k/k!] r dr d\theta, \] (VII.2)

where \( B_q \) is a closed circle of radius \( q \) at the core of dislocation \( x_0 \), \( r \) and \( \theta \) are the circular coordinate variables at \( x_0 \). The infinitely differentiable function \( \phi(x) \in \mathcal{K} = C_0^\infty \) is chosen to be identical to 1 on \( V \) with a support in a small neighborhood of \( V \). Then, the regularization of (VII.1) is written as

\[ \text{Reg.} \, \int_V g(x) dv = \int_{V \setminus B_q} g(x) dv. \] (VII.3)

VII.C The Self-Force

The self-force on a moving screw dislocation in the \( x \)-direction is then expressed as

\[ F_1 = \int_{V \setminus B_q} g(x) dV + \int_S [(W - T)\delta_{ij} - \sigma_{3j} u_{3,j}] dS_j, \] (VII.4)
where \( S = \partial V \). By using the main conservation laws (IV.21) over the homogeneous region \( V \setminus B \), (VII.4) is further reduced to

\[
F_1 = \int_{S_q} [(W - T)\delta_{ij} - \sigma_{3j}u_{3j}]dS_j,
\]

(VII.5)

where \( S_q \equiv \partial B_q \). That integration is exactly \( I_\epsilon \) in (VI.40) in last chapter when \( \epsilon = q \). In other words, in fact we have

\[
F_1 = I_q,
\]

(VII.6)

where \( I_q = I_\epsilon \) for \( \epsilon = q \) which is defined in last chapter as in (VI.40), and has already been calculated in last chapter and expressed as, in terms of the near field coefficients, for \( t > 0 \),

\[
F_1 = -\frac{\mu b^2}{4\pi c^2 \gamma^3} \frac{\dot{v}(t)}{c \sqrt{c^2 - v^2}} \ln a + \frac{\mu b r_\gamma}{2\pi} \int_0^{2\pi} \frac{c^2 g_{32}(\theta, t) - v(t) \sin \theta \cos \theta g_{33}(\theta, t)}{c^2 \gamma^2 \sin^2 \theta} d\theta + O(q).
\]

(VII.7)

Now \( 0 < q << 1 \) is a fixed number, and the self-force \( F_1 \) is not divergent.

The explicit expression of \( F_1 \) based on the theory of distributions is as follows, where \( q \) is the regularization parameter.

\[
F_1 = F_1^{in} + F_1^{non} + h.o.t,
\]

(VII.8)

where \( F_1^{in} \) and \( F_1^{non} \) are the inertial part and the remain part of the self-force, are given, for \( t > 0 \), respectively as follows.

\[
F_1^{in} \sim \frac{\mu b^2 \dot{v}(t)}{2\pi} \left\{ -\frac{\ln q}{2 c^2 \gamma^3} + \frac{c}{4(c^2 - v^2(t))^{3/2}} \ln \left( \frac{cv}{2(c^2 - v^2)} \right) \right. \\
+ \frac{\sqrt{c^2 - v^2}}{(c - v)^3} \left[ \frac{v(c - v)}{c(c + v)} - \frac{c^2(c + 2v)}{v^2(c + v)} \right] \\
+ \frac{1}{(c - v)^3} \left[ \frac{c(2c - 5v)}{2(c - v)} + \frac{3v^2}{2c} \right] \ln \left( \frac{c + \sqrt{c^2 - v^2}}{v} \right) \\
+ \frac{\dot{v}}{2c^2} \ln(l(t) - l(\tau)) \frac{l(t) - l(\tau)}{v^2} \\
\left. + \frac{v}{(t - \tau)^2} \left[ \frac{v(t) - \frac{l(t) - l(\tau)}{t - \tau}}{(t - \tau)^2} \right] \left[ \frac{c^2 + v(t)\left( \frac{l(t) - l(\tau)}{t - \tau} \right) + \left( \frac{l(t) - l(\tau)}{t - \tau} \right)^2}{1 - \left( \frac{l(t) - l(\tau)}{t - \tau} \right)^2/c^2} \right]^{3/2} d\tau \right\} \\
+ \frac{\mu b^2 \dot{v}}{4\pi c^2 \gamma^3} \left[ \ln(\gamma(1 + \gamma)/2) - \frac{4 + \beta^4 - \beta^2(7 + 2\gamma)}{(1 + \gamma)^2} \right],
\]

(VII.9)
\[ F_{1}^{\text{non}} = \frac{\mu b^2}{2\pi} \left\{ -\frac{1}{l(t)[1 - \frac{c_v(t)}{c^2}]^{1/2}} + \frac{1}{l(t)} \right. \\
+ \frac{1}{2c^2} \int_0^t \frac{1}{v^2} \frac{\ln(l(t) - l(\tau))}{1 - (\frac{t - \tau}{l(t) - l(\tau)})^2/c^2}^{3/2} \\
\left. \left[ \frac{t - \tau}{l(t) - l(\tau)} P_2 + \frac{1}{c^2(l(t) - l(\tau))} P_1 \right] d\tau \right\}, \quad (\text{VII.10}) \]

where \( P_1 \) and \( P_2 \) are defined by

\[ P_1 = v^2[2(\frac{\omega(t, \tau)}{t - \tau}) + \dot{\nu} - \ddot{\nu} \theta[v^2 - \frac{l(t) - l(\tau)}{t - \tau}]], \quad (\text{VII.11}) \]

\[ P_2 \equiv v^2 \omega^2(t, \tau) - v(t)(\frac{l(t) - l(\tau)}{t - \tau})[2\omega(t, \tau) + \dot{\nu}(t(t - \tau)) - \ddot{\nu}(\theta)(l(t) - l(\tau))\omega(t, \tau)], \quad (\text{VII.12}) \]

and \( t \geq \theta \leq t \), and \( \omega(t, \tau) \) is defined by

\[ \omega(t, \tau) \equiv v(t) - \frac{l(t) - l(\tau)}{t - \tau}. \quad (\text{VII.13}) \]

### VII.D  The Effective Mass

From (VII.9), the effective mass of a moving screw dislocation is obtained as, for \( t > 0 \),

\[ m_e = \frac{\mu b^2}{2\pi} \left\{ -\ln q \frac{c}{2c^2 \gamma^3} + \frac{c}{4(c^2 - v^2(t))^{3/2}} \ln(\frac{c v}{2(c^2 - v^2)}) \\
+ \frac{\sqrt{c^2 - v^2} v(c - v)}{(c - v)^3} \left[ \frac{c(c + 2v)}{v^2(c + v)} \right] \\
+ \frac{1}{(c - v)^3} \left[ \frac{2c - 5v}{2(c - v)} \right] + \frac{3v^2}{2c} \ln(\frac{c + \sqrt{c^2 - v^2}}{v}) \\
+ \frac{\dot{\nu}}{2c^2} \int_0^t \ln(l(t) - l(\tau)) \frac{l(t) - l(\tau)}{v^2} \right\} \]

\[ \left[ \frac{v(t) - \frac{l(t) - l(\tau)}{t - \tau}}{(t - \tau)^2} \right] \left[ c^2 + v(t)(\frac{l(t) - l(\tau)}{t - \tau}) + \frac{(l(t) - l(\tau))^2}{c^2} \right] d\tau \]

\[ + \frac{\mu b^2}{4\pi c^2 \gamma^3} \ln(\gamma(1 + \gamma)/2) - \frac{4 + 2\beta(7 + 2\gamma)}{(1 + \gamma)^2} + \text{h.o.t.} \quad (\text{VII.14}) \]
VII.E Conclusion

Using the method of the regularization for a divergent integral based on
the theory of distributions discussed in Chapter V, we eliminate the divergence in
the expression of the self-force and effective mass on an accelerating moving screw
dislocation.

Take advantage of the calculation results established in last chapter, we
readily obtain the convergent explicit evaluation of the effective mass and self-force.

It is seen that this mathematical treatment can be in comparison with
the “cut-off” physical model of the Volterra screw dislocation in the theory of
dislocation.
Chapter VIII

Effective Mass of an Accelerating Screw Dislocation Based on a Smearing Method

VIII.A Introduction

In this chapter, we propose a smearing method to smooth the core of screw dislocation so that a convergent self-force and effective mass are well-defined. The smeared elastic field is examined. The displacement field is solved explicitly. All field solutions are infinitely differentiable except having a finite jump discontinuity on the slip plane of the moving screw dislocation. The self-force is explicitly evaluated. The effective mass of an accelerating screw dislocation based on the smearing method is obtained explicitly.

VIII.B A Smearing Method

Recall that the Navier’s equation of elastodynamics for a moving screw dislocation, for \( y \neq 0 \),

\[
\frac{\partial^2 u_3(x, y, t)}{\partial x^2} + \frac{\partial^2 u_3(x, y, t)}{\partial y^2} = \frac{1}{c_s^2} \frac{\partial^2 u_3(x, y, t)}{\partial t^2},
\]

(VIII.1)
and the discontinuity condition is
\[ u_3(x, 0^+, t) - u_3(x, 0^-, t) = -\frac{b}{2}[H(x - l(t)) - H(l(t) - x)]. \tag{VIII.2} \]

Note that the discontinuity condition (VIII.2) is equivalently rewritten as
\[ u_3(x, 0^+, t) - u_3(x, 0^-, t) = -\frac{b}{2}[H(x - l(t)) - H(l(t) - x)] \ast \delta(x), \tag{VIII.3} \]
where \( \ast \) is the symbol of the convolution, which is defined by
\[ [f \ast h](x) \equiv \int_{-\infty}^{\infty} f(x - \xi)h(\xi)d\xi. \]

Follow Eshelby's smearing technique (1977), we replace the delta function in the discontinuity condition (VIII.3) by a delta sequence
\[ g_a(x) \equiv \frac{1}{\pi} \frac{a}{x^2 + a^2}, \tag{VIII.4} \]
where \( a > 0 \) is the smearing parameter, and define the smeared field variable by
\[ \hat{u}_3(x, y, t) = u_3 \ast g_a. \tag{VIII.5} \]

Then the discontinuity condition for the smeared displacement \( \hat{u}_3(x, y, t) \) is,
\[ \hat{u}_3(x, 0^+, t) - \hat{u}_3(x, 0^-, t) = -\frac{b}{2}[H(x - l(t)) - H(l(t) - x)] \ast g_a(x) \]
\[ = -\frac{b}{\pi} \tan^{-1}\left(\frac{x - l(t)}{a}\right). \tag{VIII.6} \]

That can be seen from
\[ H(l(t) - x) = 1 - H(x - l(t)), \tag{VIII.7} \]
\[ H(x - l(t)) - H(l(t) - x) = 2H(x - l(t)) - 1, \tag{VIII.8} \]
and
\[ H(x - l(t)) \ast g_a(x) = \int_{-\infty}^{\infty} H(x - l(t) - \xi) \frac{1}{\pi} \frac{a}{\xi^2 + a^2}d\xi \]
\[ = \int_{-\infty}^{(x-l(t))} \frac{1}{\pi} \frac{a}{\xi^2 + a^2}d\xi = \frac{1}{\pi} \tan^{-1}\left(\frac{x - l(t)}{a}\right) + \frac{1}{2}. \tag{VIII.9} \]
We now show that \( \hat{u}_3(x, y, t) \) is infinitely differentiable with respect to \( x, y, \) and \( t, \) for \( y \neq 0 \) and all \( x, t, \) and satisfies the Navier’s equation of elastodynamics.

It suffices to consider, for instance, the first order differentiation with respect to \( x \)
\[
\frac{\partial}{\partial x} [\hat{u}_3] = \frac{\partial}{\partial x} [u_3 \ast g_a] = \frac{\partial}{\partial x} \int_{-\infty}^{\infty} u_3(\xi, y, t) \frac{a}{\pi (x - \xi)^2 + a^2} d\xi.
\]
(VIII.10)

It is clear that the integrand of the improper integral in the last equation is well-behaved at infinity. Hence for \( y \neq 0, \) in a neighborhood of \( y, \) the uniform convergence for the improper integral is easy to verify. So that it is permissible to interchange the order of the differentiation and integration in the last integral of (VIII.10) and gives that
\[
\frac{\partial \hat{u}_3}{\partial x} = \int_{-\infty}^{\infty} u_3(\xi, y, t) \frac{1}{\pi} \frac{\partial}{\partial x} \left[ \frac{a}{(x - \xi)^2 + a^2} \right] d\xi.
\]
(VIII.11)

After integration by parts, we obtain
\[
\frac{\partial \hat{u}_3}{\partial x} = (u_{3,1}) = u_{3,1} \ast g_a.
\]
(VIII.12)

Similarly, we have
\[
\frac{\partial \hat{u}_3}{\partial y} = \hat{u}_{3,2} = (u_{3,2}) = u_{3,2} \ast g_a,
\]
(VIII.13)
\[
\frac{\partial^2 \hat{u}_3}{\partial x^2} = (u_{3, jj}) = u_{3, jj} \ast g_a,
\]
(VIII.14)

for \( j = 1, 2; \) and
\[
\frac{\partial^2 \hat{u}_3}{\partial t^2} = (u_{3,tt}) = u_{3,tt} \ast g_a.
\]
(VIII.15)

Based on those relations and the fact that \( g_a \) is independent of \( t, \) we see that \( \hat{u}_3(x, y, t) \) satisfies the the Navier’s equation of elastodynamics for \( y \neq 0 \)
\[
\frac{\partial^2 \hat{u}_3(x, y, t)}{\partial x^2} + \frac{\partial^2 \hat{u}_3(x, y, t)}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 \hat{u}_3(x, y, t)}{\partial t^2}.
\]
(VIII.16)
VIII.C Equation of the Smeared Field

By the symmetry of the smeared displacement field \( \hat{u}_3(x, y, t) \), i.e., for \( y \neq 0 \)
\begin{equation}
\hat{u}_3(x, -y, t) = u_3(x, -y, t) \ast g_a = -u_3(x, y, t) \ast g_a = -\hat{u}_3(x, y, t),
\end{equation}
the smeared field equation can be reduced to a mixed initial-boundary-value problem on the half-space \( y \geq 0 \). The mixed problem has the equation, for \( y > 0 \)
\begin{equation}
\frac{\partial^2 \hat{u}_3(x, y, t)}{\partial x^2} + \frac{\partial^2 \hat{u}_3(x, y, t)}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 \hat{u}_3(x, y, t)}{\partial t^2},
\end{equation}
with the initial conditions
\begin{equation}
\hat{u}_3(x, y, 0) = u_3(x, y, 0) \ast g_a = -\int \frac{b}{2\pi} \tan^{-1}(\xi/y) \frac{1}{\pi(x - \xi)^2 + y^2} d\xi
\end{equation}
\begin{equation}
= \frac{b}{2\pi} [\tan^{-1}(y + a) - \frac{\pi}{2}],
\end{equation}
\begin{equation}
\frac{\partial \hat{u}_3(x, y, 0)}{\partial t} = 0,
\end{equation}
and the boundary condition at \( y = 0 \), from (VIII.6),
\begin{equation}
\hat{u}_3(x, 0, t) = -\frac{b}{2\pi} \tan^{-1}\left(\frac{x - l(t)}{a}\right).
\end{equation}

VIII.D Solution of the Smeared Field

We shall give the explicit solution of the smeared displacement field \( \hat{u}_3(x, y, t) \) by calculating the convolution
\begin{equation}
\hat{u}_3(x, y, t) = u(x, y, t) \ast \frac{a}{\pi(x^2 + a^2)},
\end{equation}
where, by (VI.29), the displacement field of the Volterra screw dislocation \( u(x, y, t) \) is given as the sum of an integral expression and the corresponding solution of a static screw dislocation.

As given in (VIII.19), the convolution regarding the static solution is written as
\begin{equation}
I_s = -\frac{b}{2\pi} \tan^{-1}(x/y) \ast \frac{a}{\pi(x^2 + a^2)} = \frac{b}{2\pi} [\tan^{-1}(\frac{y + a}{x}) - \frac{\pi}{2}].
\end{equation}
It is seen that this part of the convolution is infinitely differentiable when \( y > 0 \) and \( a > 0 \), continuous for \( y \geq 0 \).

We are going to calculate the convolution about the integral expression in the solution of \( u_3(x,y,t) \).

\[
I_d = \frac{by}{2\pi} \int_0^\infty \frac{(t - \eta(\xi))H(t - \eta(\xi) - rd)}{r^2 \sqrt{(t - \eta(\xi))^2 - r^2 d^2}} \frac{d\xi}{\pi(x^2 + a^2)}, \tag{VIII.24}
\]

where again \( r = \sqrt{(x - \xi)^2 + y^2} \) and \( d = 1/c_2 \).

The convolution \( I_d \) is written as

\[
I_d = \frac{by}{2\pi} \int_0^\infty (t - \eta(\xi)) \int_{-\infty}^{\infty} \frac{aH(t - \eta(\xi) - \hat{r}d)}{\hat{r}^2 \sqrt{(t - \eta(\xi))^2 - \hat{r}^2 d^2}[(x - \xi)^2 + a^2]} d\hat{r} d\xi, \tag{VIII.25}
\]

where \( \hat{r}\equiv\sqrt{((\xi - \xi)^2 + y^2} \). Change variable \( p = \zeta - \xi \), we rewrite \( I_d \) as

\[
I_d = \frac{by}{2\pi} \int_0^\infty (t - \eta(\xi))I_\xi d\xi \tag{VIII.26}
\]

where

\[
I_\xi \equiv \int_{-\infty}^{\infty} \frac{aH(t - \eta(\xi) - \hat{r}d)}{\hat{r}^2 \sqrt{(t - \eta(\xi))^2 - \hat{r}^2 d^2}[(x - \xi)^2 + a^2]} dp, \tag{VIII.27}
\]

with now \( r = \sqrt{p^2 + y^2} \).

Note that

\[
H(t - \eta(\xi) - rd) = H(\alpha_0^2 - p^2)H(t - \eta(\xi) - yd), \tag{VIII.28}
\]

where \( \alpha_0 \equiv \sqrt{(t - \eta(\xi))^2 - y^2 d^2}/d \). Thus \( I_\xi \) is reduce to

\[
I_\xi = \int_{-\alpha_0}^{\alpha_0} \frac{aH(t - \eta(\xi) - yd)}{(p^2 + y^2) \sqrt{(\alpha_0 - p)(\alpha_0 + p)[(x - \xi)^2 + a^2]}} dp. \tag{VIII.29}
\]

The integral in last equation can be evaluated by calculating the residues of the integrand at \( p = \pm yi \) and \( p = (x - \xi) \pm ai \). It is then found that

\[
I_\xi = \frac{2\pi H(t - \eta(\xi) - yd)}{(x - \xi)^2 + (y - a)^2)((x - \xi)^2 + (y + a)^2)} \left\{ \frac{[s_1 s_2 + a^2 + \alpha_0^2 - (x - \xi)^2]((x - \xi)^2 + y^2 - a^2) + 4a^2(x - \xi)^2}{\sqrt{2s_1 s_2(s_1 s_2 + a^2 + \alpha_0^2 - (x - \xi)^2)}} \right. \\
\left. + \frac{ab((x - \xi)^2 - y^2 + a^2)}{y(t - \eta(\xi))} \right\}, \tag{VIII.30}
\]
where \( s_1 \equiv \sqrt{a^2 + (x - \xi + \alpha_0)^2} \) and \( s_2 \equiv \sqrt{a^2 + (x - \xi - \alpha_0)^2} \).

Therefore,

\[
\dot{u}_3(x, y, t) = I_s + I_d
= \frac{b}{2\pi} \left[ \tan^{-1}\left( \frac{y + a}{x} \right) - \frac{\pi}{2} \right] - \frac{by}{2\pi} \int_0^\infty (t - \eta(\xi)) I_\xi d\xi,
\]

(VIII.31)

with \( I_\xi \) given by (VIII.30).

From the step function \( H(t - \eta(\xi) - yd) \) in \( I_\epsilon \), it is seen that

\[
t - \eta(\xi) \geq yd.
\]

(VIII.32)

Then when \( y > 0 \), \( t - \eta(\xi) > 0 \) and equivalently \( x - \xi > 0 \). Therefore, for \( y \geq 0 \), in the denominator of \( I_\epsilon \), the factors

\[
((x - \xi)^2 + (y - a)^2)((x - \xi)^2 + (y + a)^2) > 0,
\]

(VIII.33)

so never vanish for \( a > 0 \).

On the other hand, in view of that

\[
(s_1 s_2)^2 = (a^2 + (x - \xi + \alpha_0)^2)(a^2 + (x - \xi - \alpha_0)^2)
= a^4 + 2a^2((x - \xi)^2 + \alpha_0^2) + ((x - \xi)^2 - \alpha_0^2)^2,
\]

(VIII.34)

and

\[
\alpha_0^2 = \alpha_0^2(t, y, \xi) = \left( \frac{t - \eta(\xi)}{b} \right)^2 - y^2 \geq 0,
\]

(VIII.35)

we know that \( s_1 s_2 \neq 0 \) and

\[
s_1 s_2 + a^2 + \alpha_0^2 - (x - \xi)^2 > 0,
\]

(VIII.36)

and never vanish. So that \( I_\epsilon \) is not singular and always continuous in \( x, y, t \) for \( y \geq 0 \).

Furthermore, from the expression (VIII.30) and (VIII.34), it is seen that \( I_\xi \) is a function of \( \xi, x, y, t, a \) and \( \alpha_0^2 \). Hence noting the expression (VIII.35) of \( \alpha_0^2 \) and

\[
\frac{\partial \alpha_0^2}{\partial t} = 2\left( \frac{t - \eta(\xi)}{b} \right),
\]

(VIII.37)
\[
\frac{\partial \alpha_0^2}{\partial y} = -2y,
\]  
(VIII.38)

we conclude that \( \hat{u}_3(x, y, t) \) is infinitely differentiable with respect to \( x, y, t \), and its any partial differentiations are continuous and always bounded \( y \geq 0 \), which are the properties we need in the forthcoming discussion.

**VIII.E A Surface Singularity**

As we know from the discussion in previous subsections, the smeared field \( \hat{u}_3 \) is infinitely differentiable for \( y > 0 \) and \( y < 0 \), and continuous and bounded when \( y \rightarrow 0^+ \) or \( y \rightarrow 0^- \). However, as shown in the discontinuity condition

\[
\hat{u}_3(x, 0^+, t) - \hat{u}_3(x, 0^-, t) = -\frac{b}{\pi} \tan^{-1}\left(\frac{x - l(t)}{a}\right),
\]  
(VIII.39)

\( \hat{u}_3 \) is not continuous across the slip plane \( y = 0 \) and has a finite jump.

The stress \( \hat{u}_{3,1} \) and the field speed \( \hat{u}_{3,t} \) are also have finite jumps across the slip plane \( y = 0 \),

\[
\hat{u}_{3,1}(x, 0^+, t) = -\hat{u}_{3,1}(x, 0^-, t) = \frac{1}{2} g_a(x - l(t)) = \frac{1}{2\pi} \frac{a}{(x - l(t))^2 + a^2}, \quad (VIII.40)
\]

and

\[
\hat{u}_{3,t}(x, 0^+, t) = -\hat{u}_{3,t}(x, 0^-, t) = -\frac{1}{2} v(t) g_a(x - l(t)),
\]

\[
= -\frac{1}{2\pi} \frac{av(t)}{(x - l(t))^2 + a^2}, \quad \text{(VIII.41)}
\]

While \( \hat{u}_{3,2} \) is continuous across \( y = 0 \), since it is even in \( y \).

Therefore, there is a surface singularity on the slip plane \( y = 0 \), on which, the displacement field \( \hat{u}_3 \), the stress \( \mu \hat{u}_{3,1} \), and the field velocity \( \hat{u}_{3,t} \) have finite jumps, respectively.

**VIII.F Definition of the Self-Force**

According to the previous discussion, in the configuration of the moving smeared screw dislocation there exists a surface singularity on the slip plane: the
On the slip plane, \( \hat{u}_3, \hat{u}_{3,1} \) and \( \hat{u}_{3,t} \) have finite jump discontinuities. The discontinuity is concentrated around the dislocation line as the smearing parameter \( a \) approaches to zero. In order to keep the smeared dislocation moving non-uniformly on the slip plane, a force opposite and equal to the self-force on the slip plane must be applied on the dislocation. The effective mass of the moving dislocation will then be determined accordingly by the inertial part of the self-force.

To define the force on the slip plane, using a similar approach as we treat the point singularity in Chapter V, we exclude an infinitesimal neighborhood of the surface singularity, which is the infinite strip \( V_{\delta} \) with height of \( \delta \) both in the positive and negative \( y \)-directions for \( \delta \ll 1 \), and the surfaces \( \partial V_{\delta} = S_{\delta} \cup S_{-\delta} \) with \( S_{\delta} \) and \( S_{-\delta} \) parallel and symmetric with distances of \( \pm \delta \) to the slip plane. Here and below, all the volumes and surfaces are assumed to be of unit length in the \( z \)-direction. Then we consider the infinite strip \( V_{\delta} \) as an inhomogeneity. In an analogous way as discussed in Chapter V, we define the force \( F_1 \) in the \( x \)-direction on the surface singularity by the integral expression

\[
F_1 = \lim_{\epsilon \to 0} \int \frac{\partial}{\partial t} [\rho \dot{\hat{u}}_{3,1}] dV + \int_{S_A \cup S_{-A}} [(\dot{W} - \dot{T})\delta_{j1} - \dot{\sigma}_{3j}\hat{u}_{3,1}] dS_j. \quad (\text{VIII.42})
\]

where \( V_A \) is an infinite strip with a height of \( 2A \), for \( A > \delta \), and with the boundaries \( \partial V_A = S_A \cup S_{-A} \), parallel and symmetric to the slip plane with distances of \( \pm A \).

In view of that the smeared field are sufficiently smooth for \( y \geq 0 \) and \( y \leq 0 \) respectively, and field variables \( \hat{u}_{3,j} \) for \( j = 1, 2, t \) are well-behaved near infinity, the volume integral in (VIII.42)

\[
\int_{V_A} \frac{\partial}{\partial t} [\rho \dot{\hat{u}}_{3,1}] dV \quad \text{(VIII.43)}
\]

exists as an improper integral. So that the self-force is rewritten as

\[
F_1 = \int \frac{\partial}{\partial t} [\rho \dot{\hat{u}}_{3,1}] dV + \int_{S_A \cup S_{-A}} [(\dot{W} - \dot{T})\delta_{j1} - \dot{\sigma}_{3j}\hat{u}_{3,1}] dS_j. \quad (\text{VIII.44})
\]

Due to the same consideration about the smeared field, Gauss divergence theorem on the infinite strips described above is still valid. Hence the integral expression (VIII.44) is “contour-independent”.

\( (x, z) \)-plane.
Because of the “contour-independence” of the integral expression, we may set $A \to 0$ in (VIII.44). Further, note that on the surfaces $S_A$ and $S_{-A}$, $(dS_1, dS_2, dS_3) = (0, dx, 0)$. Thus we have

$$ F_1 = \lim_{A \to 0} \left\{ \int_{V_A} \frac{\partial}{\partial t} [\rho \hat{\mu} \hat{u}_3 \hat{u}_3]dV + \int_{-\infty}^{\infty} [\hat{\sigma}_{32}(x, -A, t) \hat{u}_{3,1}(x, -A, t) - \hat{\sigma}_{32}(x, A, t) \hat{u}_{3,1}(x, A, t)] dx \right\}. \quad (VIII.45) $$

Taking into account that the volume integral over $V_A$ is convergent, when $A \to 0$, we have

$$ \lim_{A \to 0} \int_{V_A} \frac{\partial}{\partial t} [\rho \hat{\mu} \hat{u}_3 \hat{u}_3]dV = 0. \quad (VIII.46) $$

Further considering that $\hat{\sigma}_{32} = \mu \hat{u}_{3,2}$ is an even function of $y$, and $\hat{u}_{3,1}$ is odd in $y$, we reduce (VIII.45) to

$$ F_1 = \lim_{A \to 0^+} I_{SA}, \quad (VIII.47) $$

where $I_{SA}$ is defined by

$$ I_{SA} \equiv -2\mu \int_{-\infty}^{\infty} \hat{u}_{32}(x, A, t) \hat{u}_{3,1}(x, A, t) dx. \quad (VIII.48) $$

**VIII.G Evaluation of the Self-Force**

Using (VIII.47) and (VIII.48), we seek to find the limit

$$ F_1 = \lim_{A \to 0^+} I_{SA} $$

$$ -2\mu \lim_{A \to 0^+} \int_{-\infty}^{\infty} \hat{u}_{32}(x, A, t) \hat{u}_{3,1}(x, A, t) dx. \quad (VIII.49) $$

We shall evaluate the self-force in four steps.

(1) We prove that $\hat{u}_{3,1}$ is expressed as

$$ \hat{u}_{3,1}(x, y, t) = -\frac{b}{2\pi} \frac{(\gamma y + a)}{(x - l(t))^2 + (\gamma y + a)^2} + G(x, y, t), \quad (VIII.50) $$

where again $\gamma = \sqrt{1 - v^2/c^2}$, $a$ is the smearing parameter, and $G(x, y, t) \to 0$ as $y \to 0$ uniformly for $|x| < M$, for any positive $M$. 
We show that
\[
I_{SA} = -2\mu \int_{-\infty}^{\infty} \hat{u}_{3,2} \hat{u}_{3,1} dx \\
\sim -\mu b [u_{3,2} \ast g(2a+\gamma A)](l(t)),
\]  
(VIII.51)

where

\[
g(2a+\gamma A) = \frac{2a + \gamma A}{\pi(x^2 + (2a + \gamma A)^2)}.
\]

We calculate the convolution \(u_{3,2} \ast g(2a+\gamma A)\), and show that as \(A \to 0\),
\[
I_{SA} \sim \mu b f_{32} \ln(2a) + \mu b g_{32}(0) + O(a\ln a),
\]  
(VIII.52)

where \(f_{32}\) and \(g_{32}\) are the near field coefficients defined in Chapter VI.

We obtain that
\[
F_1 = \frac{\mu b \hat{v}}{4\pi e^2 \gamma} \ln(2a) + \mu b g_{32}(0) + O(a\ln a),
\]  
(VIII.53)

where \(g_{32}(0)\) has been evaluated in Chapter VI.

Now we proceed to show Steps (1) - (4).

**Step (1)**

From (VI.41) and (VI.42), \(u_{3,1}\) is expressed by
\[
u_{3,1} = -\frac{b}{2\pi} \frac{\gamma y}{(x - l(t))^2 + \gamma^2 y^2} + g(x, y, t),
\]  
(VIII.54)

with \(g(x, y, t)\) an odd continuous function in \(y\) and satisfying
\[
\lim_{y \to 0} g(x, y, t) = 0,
\]  
(VIII.55)

for every \(x\).

\(\hat{u}_{3,1}\) is then written as
\[
\hat{u}_{3,1} = u_{3,1} \ast g_a \\
= [\frac{-b}{2\pi} \frac{\gamma y}{(x - l(t))^2 + \gamma^2 y^2} + g(x, y, t)] \ast \frac{a}{\pi(x^2 + a^2)}.
\]  
(VIII.56)
Calculate the first convolution, for $y > 0$,

$$\frac{\gamma y}{(x - l(t))^2 + \gamma^2 y^2} \ast \frac{a}{x^2 + a^2} = \int_{-\infty}^{\infty} \frac{\gamma y}{(\xi - l(t))^2 + \gamma^2 y^2} \frac{a}{(x - \xi)^2 + a^2} d\xi$$

$$= \int_{-\infty}^{\infty} \frac{\gamma y}{\eta^2 + \gamma^2 y^2} \frac{a}{(x - l(t) - \eta)^2 + a^2} d\eta$$

$$= \frac{\pi (\gamma y + a)}{(x - l(t))^2 + (\gamma y + a)^2} = \pi^2 g(\gamma y + a)(x - l(t)), \quad (VIII.57)$$

where the following integral result is used, for $p > 0$,

$$\int_{-\infty}^{\infty} \frac{p}{x^2 + p^2} \left[ \frac{a}{(z - x)^2 + a^2} \right] dx = \frac{\pi(p + a)}{z^2 + (p + a)^2}. \quad (VIII.58)$$

Then we obtain

$$\hat{u}_{3,1}(x, y, t) = -\frac{b}{2\pi} \frac{(\gamma y + a)}{(x - l(t))^2 + (\gamma y + a)^2} + G(x, y, t), \quad (VIII.59)$$

where $G(x, y, t)$ is defined by the integral

$$G(x, y, t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(\xi, y, t)}{(x - \xi)^2 + a^2} d\xi. \quad (VIII.60)$$

As we know that

$$\lim_{y \to 0} \hat{u}_{3,1}(x, y, t) = -\frac{b}{2\pi} \frac{a}{(x - l(t))^2 + a^2}, \quad (VIII.61)$$

and for $|x| \leq M$ such convergence is uniform, where $M$ may be any positive number. Hence, from (VIII.59) and (VIII.61),

$$\lim_{y \to 0} G(x, y, t) = \lim_{y \to 0} \left[ \hat{u}_{3,1}(x, y, t) + \frac{b}{2\pi} \frac{(\gamma y + a)}{(x - l(t))^2 + (\gamma y + a)^2} \right] = 0. \quad (VIII.62)$$

The convergence is again uniform for $|x| \leq M$.

**Step (2)**

Substitute (VIII.59) into the integral expression of $I_{SA}$, we have

$$I_{SA} = -2 \int_{-\infty}^{\infty} \mu \hat{u}_{3,2} \hat{u}_{3,1} dx$$
\[ \begin{align*}
&= \frac{\mu b}{\pi} \int_{-\infty}^{\infty} \hat{u}_{3,2}(x, A, t) \frac{(\gamma A + a)}{(x - l(t))^2 + (\gamma A + a)^2} dx \\
&- 2\mu \int_{-\infty}^{\infty} \mu \hat{u}_{3,2}(x, A, t) G(x, A, t) dx \\
&\equiv I_{S1} + I_{S2},
\end{align*} \]

(VIII.63)

where \( I_{S1} \) and \( I_{S2} \) are defined correspondingly.

For the integral \( I_{S2} \), in view of the well behavior of the integrand at infinity, the improper integral is uniformly convergent with respect to \( A \). Inasmuch as \( \hat{u}_{3,2} \) is bounded and as shown in (VIII.62), \( G(x, A, t) \to 0 \), as \( A \to 0 \), uniformly for \( |x| \leq M \) where \( M \) may be any positive number. Therefore, the limit of \( A \to 0 \) may be taken under the integral, and

\[ \lim_{A \to 0} I_{S2} = 0. \] (VIII.64)

As for the integral \( I_{S1} \),

\[ I_{S1} = \frac{\mu b}{\pi} \int_{-\infty}^{\infty} \hat{u}_{3,2}(x, A, t) \frac{(\gamma A + a)}{(x - l(t))^2 + (\gamma A + a)^2} dx \\
= \mu b[\hat{u}_{3,2} \ast g_{(a+\gamma A)}](l(t)). \] (VIII.65)

Noting that \( \hat{u}_{3,2} = u_{3,2} \ast g_{a} \) and applying the associativity property of the convolution, we have

\[ \begin{align*}
I_{S1} &= \mu b[\hat{u}_{3,2} \ast g_{(a+\gamma A)}](l(t)) = \mu b[(u_{3,2} \ast g_{a}) \ast g_{(a+\gamma A)}](l(t)) \\
&= \mu b[u_{3,2} \ast (g_{a} \ast g_{(a+\gamma A)})](l(t)) \\
&= \mu b[u_{3,2} \ast g_{(2a+\gamma A)}](l(t)),
\end{align*} \]

(VIII.66)

where from (VIII.58),

\[ g_{a} \ast g_{(a+\gamma A)} = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{a}{\eta^2 + a^2} \frac{a + \gamma A}{(x - \eta)^2 + (a + \gamma A)^2} d\eta \\
= \frac{1}{\pi} \frac{(2a + \gamma A)}{(x)^2 + (2a + \gamma A)^2} = g_{(2a+\gamma A)}. \] (VIII.67)
Hence, as $A \to 0$,

$$I_{SA} \sim I_{S1} = \mu b [u_{3,2} \ast g_e](l(t))$$

$$= \frac{\mu b}{\pi} \frac{1}{\pi} \int_{-\infty}^{\infty} u_{3,2}(\xi, A, t) \frac{e}{(\xi - l(t))^2 + e^2 d\xi}$$

$$= \frac{\mu b}{\pi} \frac{1}{\pi} \int_{-\infty}^{\infty} u_{3,2}(\eta + l(t), A, t) \frac{e}{\eta^2 + e^2 d\eta},$$  \hspace{1cm} (VIII.68)

where $e \equiv 2a + \gamma A$.

**Step (3)**

From the meaning of the asymptotic expansion, there is a sufficiently small number $\zeta > 0$, such that when $e^2 = \eta^2 + A^2 \leq \zeta^2$, for $A \geq 0$, $u_{3,2}(\eta + l(t), A, t) = u_{3,2}(\epsilon, \theta, t)$ has the asymptotic expansion

$$u_{3,2} = \frac{b}{2\pi} \frac{\gamma \eta}{\eta^2 + \gamma^2 A^2} + f_{32} \ln \epsilon + g_{32}(\theta) + p_4(\epsilon, \theta),$$  \hspace{1cm} (VIII.69)

where $0 \leq \theta = \tan^{-1}(A/\eta) \leq \pi$, $f_{32}$ and $g_{32}$ are the near field coefficients, and $|p_4| \leq M(\theta)\epsilon$,  \hspace{1cm} (VIII.70)

and $M(\theta)$ is bounded function.

We decompose the integral domain in $I_{S1}$ into two parts: (i) the near field: $\eta^2 + A^2 \leq \zeta^2$; (ii) the far field: $\eta^2 + A^2 > \zeta^2$ and write the integral $I_{S1}$ as

$$I_{S1} \equiv I_N + I_F$$

$$= \frac{\mu b}{\pi} \int_{-q}^{q} \left[ \frac{b}{2\pi} \frac{\gamma \eta}{\eta^2 + \gamma^2 A^2} + f_{32} \ln \epsilon + g_{32}(\theta) + p_4(\epsilon, \theta) \right] \frac{e}{\eta^2 + e^2} d\eta$$

$$+ \frac{\mu b}{\pi} \int_{\eta>q} u_{3,2}(\eta + l(t), A, t) \frac{e}{\eta^2 + e^2} d\eta,$$  \hspace{1cm} (VIII.71)

where $q = \sqrt{\zeta^2 - A^2}$, $I_N$ and $I_F$ are the integral over the near field and far field, and defined accordingly.

We shall show that for $a << 1$,

(i) $\lim_{A \to 0} I_F = O(a).$  \hspace{1cm} (VIII.72)
\[
\lim_{A \to 0} I_N = \mu b f_{32} \ln(2a) + \mu b g_{32}(0) + O(a \ln a) \quad \text{(VIII.73)}
\]

To show (VIII.72), we note that for \( \eta > q = \sqrt{\zeta^2 - A^2} \), or equivalently,
\( \eta^2 + A^2 > \zeta^2 \), \( u_{3,2}(\eta + l(t), A, t) \) is bounded, i.e., for \( \eta > q \),
\[
\frac{\mu b}{\pi} |u_{3,2}(\eta + l(t), A, t)| \leq M_1, \quad \text{(VIII.74)}
\]
for some bound \( M_1 > 0 \). From the definition of \( I_F \) and (VIII.74), it follows that
\[
|I_F| \leq M_1 \int_{\eta > q} \frac{e}{\eta^2 + \epsilon^2} d\eta
\]
\[
= 2M_1 \int_{q}^{\infty} \frac{e}{\eta^2 + \epsilon^2} d\eta = 2M_1 \int_{q/e}^{\infty} \frac{d\zeta}{\zeta^2 + 1}
\]
\[
= 2M_1 e/q + \text{h.o.t. of } e, \quad \text{(VIII.75)}
\]
where again \( e = 2a + \gamma A \) and \( q = \sqrt{\zeta^2 - A^2} \). Hence, as \( A \to 0 \), \( I_F \) is of the order of \( O(a) \) for \( a << 0 \).

For \( I_N \), we write
\[
I_N = \mu b \frac{\pi}{2} \int_{-q}^{q} \frac{b}{2\pi \eta^2 + \gamma^2 A^2} + f_{32} \ln \epsilon + g_{32}(\theta) + p_{4}(\epsilon, \theta) \frac{e}{\eta^2 + \epsilon^2} d\eta. \quad \text{(VIII.76)}
\]

By symmetry, the integral of the first term is zero, i.e.,
\[
\int_{-q}^{q} \left[ \frac{b}{2\pi \eta^2 + \gamma^2 A^2} \frac{e}{\eta^2 + \epsilon^2} \right] d\eta = 0. \quad \text{(VIII.77)}
\]

The integral of the second term is
\[
\frac{\mu b}{\pi} \int_{-q}^{q} f_{32} \ln(\eta^2 + A^2) \frac{e}{\eta^2 + \epsilon^2} d\eta = \frac{\mu b}{\pi} \int_{-\infty}^{\infty} f_{32} \ln(\eta^2 + A^2) \frac{e}{\eta^2 + \epsilon^2} d\eta - \frac{\mu b}{\pi} \int_{q}^{\infty} f_{32} \ln(\eta^2 + A^2) \frac{e}{\eta^2 + \epsilon^2} d\eta, \quad \text{(VIII.78)}
\]
where again \( f_{32} \) is independent of \( \eta \).

For the first integral in (VIII.78),
\[
\frac{\mu b}{\pi} \int_{-\infty}^{\infty} f_{32} \ln(\eta^2 + A^2) \frac{e}{\eta^2 + \epsilon^2} d\eta = \mu b f_{32} \ln(A + \epsilon) = \mu b f_{32} \ln(2a + 2A). \quad \text{(VIII.79)}
\]
In the second integral in (VIII.78), by change of integral variables, we obtain
\[
| \int_{q}^{\infty} \ln(A^2 + \eta^2) \frac{e}{\eta^2 + e^2} d\eta | = | \int_{0}^{e/q} \ln(A^2 + e^2/\xi^2) \frac{1}{1 + \xi^2} d\xi |
\]
\[
\leq \int_{0}^{e/q} \left| \frac{2 \ln(\xi)}{1 + \xi^2} \right| d\xi + \int_{0}^{e/q} \left| \frac{\ln(A^2 \xi^2 + e^2)}{1 + \xi^2} \right| d\xi,
\]
(VIII.80)
the last two integrals are of the order of \( O(a \ln a) \) as \( A \to 0 \), since
\[
\int_{0}^{e/q} \left| 2 \ln(\xi) \right| d\xi \leq \int_{0}^{e/q} 2 \ln(\xi) d\xi = |e/q(1 - \ln(e/q)| = O(e/q \ln(e/q)); (VIII.81)
\]
and
\[
\int_{0}^{e/q} \left| \ln(A^2 \xi^2 + e^2) \right| d\xi \leq \int_{0}^{e/q} \ln(A^2(e/q)^2 + e^2) d\xi = O(e/q \ln(e)).
\]
(VIII.82)
The integral of the third term in (VIII.77) is rewritten as
\[
\frac{\mu b}{\pi} \int_{-q}^{q} g_{32}(\theta) \frac{e}{\eta^2 + e^2} d\eta = \frac{\mu b}{\pi} \int_{-\delta}^{\delta} g_{32}(\theta) \frac{e}{\eta^2 + e^2} d\eta + \frac{\mu b}{\pi} \left[ \int_{-q}^{-\delta} + \int_{-\delta}^{-q} \right] g_{32}(\theta) \frac{e}{\eta^2 + e^2} d\eta,
\]
(VIII.83)
where \( \delta = o(a) \) is a small number. Then
\[
| \int_{-\delta}^{\delta} g_{32}(\theta) \frac{e}{\eta^2 + e^2} d\eta | \leq 2M_0 \int_{0}^{\delta} \frac{e}{\eta^2 + e^2} d\eta = 2M_0 \tan^{-1}(\delta/e) = O(a), (VIII.84)
\]
since \( g_{32} \) is bounded in the near field.
For the second integral on the right hand side of (VIII.83), the limit of \( A \to 0 \) may be taken under the integral, so we have
\[
\lim_{A \to 0} \frac{\mu b}{\pi} \int_{-q}^{q} g_{32}(\theta) \frac{e}{\eta^2 + e^2} d\eta = \frac{\mu b}{\pi} \int_{-\delta}^{\delta} g_{32}(\theta) \frac{e}{\eta^2 + e^2} d\eta + o(a)
\]
\[
= \frac{\mu b}{2} [g_{32}(0) + g_{32}(\pi)] \tan^{-1}(\zeta/a) + O(a)
\]
\[
= [g_{32}(0) + g_{32}(\pi)](\frac{\pi}{2} - e/q) + O(a)
\]
\[
= \frac{\pi}{2} [g_{32}(0) + g_{32}(\pi)] + O(a).
\]
(VIII.85)
Furthermore, as $A \to 0$,

$$| \int_{-q}^{q} p_1 \frac{e^{\eta^2 + e^2}}{\eta^2 + e^2} d\eta | \leq M_2 \int_{-q}^{q} \frac{e^{\eta^2 + e^2}}{\eta^2 + e^2} d\eta = M_2 e \ln(q^2 + e^2) = O(a). \quad (\text{VIII.86})$$

Hence, combing the above results, we obtain, for $a \ll 1$,

$$\lim_{A \to 0} II_1 = \frac{\mu b f_{32}}{2} \ln(2a) + \frac{\mu b}{2} [g_{32}(0) + g_{32}(\pi)] + O(a \ln a). \quad (\text{VIII.87})$$

The evaluation (VIII.72) and (VIII.73) are then proved.

Consequently, the self-force is thus given by

$$F_1 = \lim_{A \to 0} I_{SA} = \lim_{A \to 0} [I_{S1} + I_{S2}] = \lim_{A \to 0} [I_F + I_N]$$

$$= \frac{\mu b f_{32}}{2} \ln(2a) + \frac{\mu b}{2} [g_{32}(0) + g_{32}(\pi)] + O(a \ln a). \quad (\text{VIII.88})$$

Note that

$$g_{32}(\pi, t) = \int_{0}^{\pi} g'_{32}(\theta, t) d\theta + g_{32}(0), \quad (\text{VIII.89})$$

where $g'_{32}(\theta, t)$ is given in (VI.89) in Chapter VI as

$$g'_{32} = -\frac{b \dot{v} \cos \theta \sin \theta}{4\pi c^2 \gamma} \frac{(7\gamma^2 - 5) \cos^2 \theta - \gamma^2 (\gamma^2 - 3) \sin^2 \theta}{(\cos^2 \theta + \gamma^2 \sin^2 \theta)^3}. \quad (\text{VIII.90})$$

It is then easy to see that

$$\int_{0}^{\pi} g'_{32}(\theta, t) d\theta = 0, \quad (\text{VIII.91})$$

it follows that $g_{32}(\pi) = g_{32}(0)$. Therefore,

$$F_1 = \frac{\mu b f_{32}}{2} \ln(2a) + \mu b g_{32}(0) + O(a \ln a), \quad (\text{VIII.92})$$

which completes Step (4).
VIII.H The Self-Force

Summarize the results obtained in above subsections, we write the evaluation of the self-force on the moving screw dislocation

\[ F_1 = F_1^{\text{in}} + F_1^{\text{non}}, \] (VIII.93)

where \( F_1^{\text{in}} \) and \( F_1^{\text{non}} \) are the inertial part and the remain part of the self-force, are given, for \( t > 0 \), respectively as follows.

\[ F_1^{\text{in}} \sim \frac{\mu b^2 \dot{v}(t)}{2\pi} \left\{ - \ln(2a) + \frac{c}{2(c^2 - v^2(t))^{3/2}} \ln \left( \frac{cv}{2(c^2 - v^2(t))} \right) \right. \]
\[ + \frac{\sqrt{c^2 - v^2}}{(c - v)^3} \left[ \frac{c(c + 2v)}{v^2(c + v)} - \frac{c^2(c + 2v)}{v^2(c + v)} \right] \]
\[ + \frac{1}{(c - v)^3} \left[ \frac{c(2c - 5v)}{2(c - v)} + \frac{3v^2}{2c} \ln \left( \frac{c + \sqrt{c^2 - v^2}}{v} \right) \right] \]
\[ + \frac{\dot{v}}{2c^2} \int_0^t \ln(l(t) - l(\tau)) \frac{[l(t) - l(\tau)]}{v^2} \left( \frac{\omega(t, \tau)}{t - \tau} \right)^2 \left[ 1 - \frac{(l(t) - l(\tau))^2}{c^2} \right]^{5/2} d\tau \}, \] (VIII.94)

\[ F_1^{\text{non}} = \frac{\mu b^2}{2\pi} \left\{ - \frac{1}{l(t)[1 - \frac{l(t)}{c^2}]^{1/2}} + \frac{1}{l(t)} \right. \]
\[ + \frac{1}{2c^2} \int_0^t \frac{1}{v^2} \frac{\ln(l(t) - l(\tau))}{[1 - \frac{(l(t) - l(\tau))^2}{c^2}]^{5/2}} \right. \]
\[ \left. \left[ \frac{t - \tau}{(l(t) - l(\tau))^2} P_2 + \frac{1}{c^2(l(t) - l(\tau))^2} P_1 \right] d\tau \right\}, \] (VIII.95)

where \( P_1 \) and \( P_2 \) are defined by

\[ P_1 = v^2 \frac{2(\omega(t, \tau) + \dot{v})}{t - \tau} - \frac{\dot{v}}{2} \left[ \frac{(l(t) - l(\tau))^2}{l(t) - l(\tau)} \right], \] (VIII.96)

\[ P_2 \equiv v^2 \omega^2(t, \tau) - v(t) \frac{(l(t) - l(\tau))(l(t) - l(\tau))}{t - \tau} \left[ 2\omega(t, \tau) + \dot{v}(t) (t - \tau) \right] \]
\[ - \dot{v}(\theta)(l(t) - l(\tau))\omega(t, \tau), \] (VIII.97)
and \( \tau \leq \theta \leq t \), and \( \omega(t, \tau) \) is defined by

\[
\omega(t, \tau) \equiv v(t) - \frac{l(t) - l(\tau)}{t - \tau}.
\]  

(VIII.98)

In those expressions, the integrals represent the terms dependent on the history of motion.

### VIII.I The Effective Mass

The effective mass of an accelerating screw dislocation is defined as, for \( t > 0 \),

\[
m_e = \frac{F_{1}}{\dot{v}},
\]  

(VIII.99)

where \( F_{1}^{in} \) is the inertial part of the self-force.

\[
m_e \sim \frac{\mu b^2}{2\pi} \left\{ \frac{-\ln(2a)}{2c^2} + \frac{c}{4(c^2 - v^2)^{3/2}} \ln\left(\frac{cv}{2(c^2 - v^2)}\right) \right. \\
+ \frac{\sqrt{c^2 - v^2} [v(c - v)]}{(c - v)^3} \left[ \frac{c^2(c + 2v)}{c(c + v)} - \frac{v^2(c + v)}{2v} \right] \\
+ \frac{1}{(c - v)^3} \left[ \frac{c(2c - 5v)}{2(c - v)} + \frac{3v^2}{2c} \ln\left(\frac{c + \sqrt{c^2 - v^2}}{v}\right) \right] \\
+ \frac{1}{2c^2} \int_{0}^{t} \left[ \ln(l(t)) - l(\tau) \right] \frac{1}{\frac{v^2}{l(t) - l(\tau)}} \\
\left[ \frac{v(t) - l(t) - l(\tau)}{l(t) - l(\tau)} \right] \left[ c^2 + v(t) \left( \frac{l(t) - l(\tau)}{l(t) - l(\tau)} \right) + \left( \frac{l(t) - l(\tau)}{l(t) - l(\tau)} \right)^2 \right] d\tau \right\}. 
\]  

(VIII.100)

The integral term depends on the history of motion.

### VIII.J Conclusion

In this chapter, we propose a smearing method which smoothes the singular core of the screw dislocation. Then eliminate the divergence in the self-force.
Explicit results of self-force and effective mass for a moving screw dislocation are obtained.

It is seen that the results based on the smearing method and that based on the theory of distributions are consistent for the leading logarithmic terms, if matching the parameters, and have one term difference for the next order terms.
Chapter IX

Effective Mass of an Accelerating Edge Dislocation

IX.A Introduction

In this chapter, we shall use the definition of the configurational force on a moving elastic defect discussed in Chapter V to calculate the self-force on a moving edge dislocation in an infinite elastic medium, and then the effective mass of an accelerating edge dislocation is determined by the inertial part of the self-force. Two theorems on the near field coefficients are established. Then in order to completely determine all the twelve functions of the near field coefficients, up to the order of $O(1)$, we only need to evaluate five near field constants. Based on those theorems, the self-force of a moving edge dislocation is derived. In an analogous way, the near field coefficients can be solved explicitly, so that self-force and effective mass can be evaluated.

IX.B The Main Problem

A Volterra gliding edge dislocation with the Burgers vector $(b, 0, 0)$ parallel to the $x$-direction is situated on the $z$-axis at rest for $t \leq 0$ in an infinite homogeneous isotropic elastic solid. For $t > 0$, it moves non-uniformly according
to \( x = l(t) \) in the (positive) \( x \)-direction, where \( l(t) \) is an arbitrarily given smooth function such that

\[
0 < \frac{dl(t)}{dt} < c_2 < c_1
\]  

(IX.1)

where \( c_2 = \sqrt{\mu/\rho} \) is the speed of shear wave, and \( c_1 = \sqrt{(\lambda + 2\mu)/\rho} \) is the speed of longitudinal wave.

The main problem is to determine the effective mass of an accelerating edge dislocation per unit length of the dislocation by calculating the inertial part of the self-force. Again, we assume that there are no other external or internal forces, neither the drag, dissipative, or damping effects in the problem.

**IX.C  Elastic Field of Edge Dislocation**

**IX.C.1  Static Edge Dislocation**

Consider an infinitely long Volterra gliding edge dislocation situated at the \( z \)-axis in a three dimensional Euclidean space occupied by an isotropic elastic medium. The Burgers vector \((b, 0, 0)\) parallels to the \( x \)-direction. The displacement field has only nonzero components \( u_1 \) and \( u_2 \), which depend on the spatial variables \( x \) and \( y \), is independent of \( z \).

The equation of equilibrium is written as

\[
\frac{\partial \sigma_{ij}}{\partial x_j} = 0, 
\]  

(IX.2)

and the stress field \( \sigma_{ij} \) is determined by

\[
\sigma_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \lambda \delta_{ij} \frac{\partial u_k}{\partial x_k}, 
\]  

(IX.3)

where \( \delta_{ij} \) is Kronecker delta, and the repeated indices \( k \) are summed from 1 to 3.

Substitute (IX.3) into (IX.2), the equation of equilibrium is rewritten as a system of equations, for \( y \neq 0 \),

\[
(\lambda + 2\mu) \frac{\partial^2 u_1}{\partial x^2} + \mu \frac{\partial^2 u_1}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 u_2}{\partial x \partial y} = 0, 
\]  

(IX.4)
\[ \mu \frac{\partial^2 u_2}{\partial x^2} + (\lambda + 2\mu) \frac{\partial^2 u_2}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 u_1}{\partial x \partial y} = 0. \] (IX.5)

The displacement field satisfies the discontinuity condition at \( y = 0 \),

\[ u_1(x, 0^+) - u_1(x, 0^-) = -\frac{b}{2} [H(x) - H(-x)], \] (IX.6)

where \( H(\cdot) \) is the Heaviside step function. And the stress field satisfies, at \( y = 0 \),

\[ \sigma_{22} = 0. \] (IX.7)

The displacement field solutions of static edge dislocation are well-known, and given as, see e.g., Hirth and Lothe (1982),

\[ u_1(x, y) = \frac{b}{2\pi} \left[ \tan^{-1}\left( \frac{y}{x} \right) + \frac{xy}{2(1-\nu)(x^2 + y^2)} \right], \] (IX.8)

\[ u_2(x, y) = -\frac{b}{2\pi} \left[ \frac{1 - 2\nu}{4(1-\nu)} \ln(x^2 + y^2) + \frac{x^2 - y^2}{4(1-\nu)(x^2 + y^2)} \right], \] (IX.9)

where \( \nu = \lambda/2(\lambda + \mu) \) is the Poisson’s ratio.

**IX.C.2 Edge Dislocation in Uniform Motion**

Consider now the edge dislocation described above moving in the \( x \)-direction at a uniform speed \( v \). The displacements \( u_1 \) and \( u_2 \) depend on the spatial variables \( x, y \), and the time \( t \). The equation of motion is written as

\[ \frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}. \] (IX.10)

Substitute (IX.3) into (IX.10), the equation of motion is rewritten as a system of equations, for \( y \neq 0 \),

\[ (\lambda + 2\mu) \frac{\partial^2 u_1}{\partial x^2} + \mu \frac{\partial^2 u_1}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 u_2}{\partial x \partial y} = \rho \frac{\partial^2 u_1}{\partial t^2}, \] (IX.11)

\[ \mu \frac{\partial^2 u_2}{\partial x^2} + (\lambda + 2\mu) \frac{\partial^2 u_2}{\partial y^2} + (\lambda + \mu) \frac{\partial^2 u_1}{\partial x \partial y} = \rho \frac{\partial^2 u_2}{\partial t^2}. \] (IX.12)

The discontinuity condition at \( y = 0 \) is

\[ u_1(x, 0^+, t) - u_1(x, 0^-, t) = -\frac{b}{2} [H(x - vt) - H(-x + vt)], \] (IX.13)
where $H(\cdot)$ is the Heaviside step function. And the condition at $y = 0$ for $\sigma_{22}$ is

$$\sigma_{22}(x, 0, t) = 0.$$  (IX.14)

From Eshelby (1949), see also Weertman and Weertman (1983), the displacement field solutions are given by

$$u_1 = \frac{bc_2^2}{\pi v^2} \left[ \tan^{-1}\left( \frac{\gamma_1 y}{x - vt} \right) - \frac{\alpha^2}{\gamma_2} \tan^{-1}\left( \frac{\gamma_2 y}{x - vt} \right) \right],$$  (IX.15)

$$u_2 = \frac{bc_2^2}{2\pi v^2} \left[ \gamma_1 \ln\left( (x - vt)^2 + \gamma_1^2 y^2 \right) - \frac{\alpha^2}{\gamma_2} \ln\left( (x - vt)^2 + \gamma_2^2 y^2 \right) \right],$$  (IX.16)

where $\gamma_1 = \sqrt{1 - v^2/c_1^2}$, $\gamma_2 = \sqrt{1 - v^2/c_2^2}$, and $\alpha^2 = 1 - v^2/2c_2^2$. It is easy to see that when $v = 0$, (IX.15) and (IX.16) reduce to the displacement solutions of static edge dislocation (IX.8) and (IX.9).

The stress field solutions are given by, see e.g., Weertman and Weertman (1983),

$$\sigma_{12} = \frac{2\mu b c_2^2}{\pi v^2 \gamma_2} \left[ \frac{\gamma_1 \gamma_2}{x^2 + \gamma_1^2 y^2} - \frac{\alpha^4}{x^2 + \gamma_2^2 y^2} \right],$$  (IX.17)

$$\sigma_{11} = \frac{2\mu b c_2^2 y}{\pi v^2} \left[ -\frac{\gamma_1 (\gamma_1^2 + 1 - \alpha^2)}{x^2 + \gamma_1^2 y^2} + \frac{\alpha^2 \gamma_2}{x^2 + \gamma_2^2 y^2} \right],$$  (IX.18)

$$\sigma_{22} = \frac{2\mu b c_2^2 y}{\pi v^2} \left[ \frac{\gamma_1}{x^2 + \gamma_1^2 y^2} - \frac{\gamma_2}{x^2 + \gamma_2^2 y^2} \right],$$  (IX.19)

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}).$$  (IX.20)

**IX.C.3 Edge Dislocation in Non-Uniform motion**

The non-uniformly moving edge dislocation is governed by the same system of equation of motion (IX.11) and (IX.12) for $y \neq 0$, with the discontinuity condition at $y = 0$,

$$u_1(x, 0^+, t) - u_1(x, 0^-, t) = -\frac{b}{2} \left[ H(x - l(t)) - H(-x + l(t)) \right],$$  (IX.21)

and the condition for the stress

$$\sigma_{22}(x, 0, t) = 0.$$  (IX.22)
From the discontinuity condition and the system of equations of motion, it follows that \( u_1 \) is odd in \( y \), and \( u_2 \) is even in \( y \). The problem can be reduced to a mixed initial-boundary-value problem in the half-space \( y \geq 0 \), which satisfies the equations (IX.11) and (IX.12) for \( y > 0 \). The initial conditions are

\[
\begin{align*}
    u_i(x, y, 0) &= u_i^s(x, y), \\
    \frac{\partial}{\partial t} u_i(x, y, 0) &= 0,
\end{align*}
\]

where \( i = 1, 2 \) and \( u_i^s(x, y) \) are the displacement solutions for static edge dislocation. The boundary conditions are

\[
\begin{align*}
    u_1(x, 0, t) &= -\frac{b}{2} H(x - l(t)), \\
    \sigma_{22}(x, 0, t) &= 0.
\end{align*}
\]

**IX.D Leading Terms of Near-Field Expansion Solutions**

As discussed in Chapter II, the near-field expansion solution is the asymptotic expansion in \( \epsilon \) of the solution at the field point which is in a \( \epsilon \)-neighborhood of the dislocation.

In an analogous way as discussed for the moving screw dislocation in Chapter II, we can prove that for a non-uniformly moving edge dislocation, the most singular terms of the expansions of the field \( u_{i,j} \) and \( u_t \) for \( i = 1, 2 \) are as well the \( 1/\epsilon \) terms, and equal to the corresponding solutions of the steady-state motion with the instantaneous velocity \( v(t) = \dot{l}(t) \) as the uniform velocity. We present the results as follows, and do not repeat the proof.

The leading terms of the near field expansions of \( u_{i,j} \) for \( i = 1, 2, j = 1, 2, t \), are expressed as below, here the index \( j = t \) means the partial differentiation with respect to \( t \),

\[
\begin{align*}
    u_{1,1}^0 &= \frac{bc_2}{\pi v^2} \frac{-\gamma_1}{(x - l(t))^2 + \gamma_1 y^2} + \frac{\alpha^2 \gamma_2}{(x - l(t))^2 + \gamma_2 y^2},
\end{align*}
\]
\[ u_{2,2}^0 = \frac{bc_2^2 y}{\pi v^2} \left( \frac{\gamma_1^3}{(x - l(t))^2 + \gamma_1^2 y^2} - \frac{\alpha^2 \gamma_2}{(x - l(t))^2 + \gamma_2^2 y^2} \right), \quad \text{(IX.28)} \]
\[ u_{1,2}^0 = \frac{bc_2^2(x - l(t))}{\pi v^2} \left( \frac{\gamma_1}{(x - l(t))^2 + \gamma_1^2 y^2} - \frac{\alpha^2 \gamma_2}{(x - l(t))^2 + \gamma_2^2 y^2} \right), \quad \text{(IX.29)} \]
\[ u_{2,1}^0 = \frac{bc_2^2(x - l(t))}{\pi v^2} \left( \frac{\gamma_1}{(x - l(t))^2 + \gamma_1^2 y^2} - \frac{\alpha^2 \gamma_2}{(x - l(t))^2 + \gamma_2^2 y^2} \right), \quad \text{(IX.30)} \]
and
\[ \dot{u}_1^0 = -vu_{1,1}^0, \quad \text{(IX.31)} \]
\[ \dot{u}_2^0 = -vu_{2,1}^0. \quad \text{(IX.32)} \]

In the polar coordinates at the core of dislocation,
\[ x - l(t) = \epsilon \cos \theta, \quad y = \epsilon \sin \theta, \]
the above expressions are rewritten as
\[ u_{1,1}^0 = \frac{bc_2^2}{\pi v^2} \left[ \frac{-\gamma_1}{\cos^2 \theta + \gamma_1^2 \sin^2 \theta} + \frac{\alpha^2 \gamma_2}{\cos^2 \theta + \gamma_2^2 \sin^2 \theta} \right] \frac{\sin \theta}{\epsilon}, \quad \text{(IX.33)} \]
\[ u_{2,2}^0 = \frac{bc_2^2}{\pi v^2} \left[ \frac{\gamma_1^3}{\cos^2 \theta + \gamma_1^2 \sin^2 \theta} - \frac{\alpha^2 \gamma_2}{\cos^2 \theta + \gamma_2^2 \sin^2 \theta} \right] \frac{\sin \theta}{\epsilon}, \quad \text{(IX.34)} \]
\[ u_{1,2}^0 = \frac{bc_2^2}{\pi v^2} \left[ \frac{\gamma_1}{\cos^2 \theta + \gamma_1^2 \sin^2 \theta} - \frac{\alpha^2 \gamma_2}{\cos^2 \theta + \gamma_2^2 \sin^2 \theta} \right] \frac{\cos \theta}{\epsilon}, \quad \text{(IX.35)} \]
\[ u_{2,1}^0 = \frac{bc_2^2}{\pi v^2} \left[ \frac{\gamma_1}{\cos^2 \theta + \gamma_1^2 \sin^2 \theta} - \frac{\alpha^2 \gamma_2}{\cos^2 \theta + \gamma_2^2 \sin^2 \theta} \right] \frac{\cos \theta}{\epsilon}. \quad \text{(IX.36)} \]

The leading terms in the near field expansions of the stress field \( \sigma_{ij} \) for \( i, j = 1, 2 \) are then given by
\[ \sigma_{12}^0 = \mu(u_{1,2}^0 + u_{2,1}^0), \quad \text{(IX.37)} \]
\[ \sigma_{11}^0 = (\lambda + 2\mu)u_{1,1}^0 + \lambda u_{2,2}^0, \quad \text{(IX.38)} \]
\[ \sigma_{22}^0 = \lambda u_{1,1}^0 + (\lambda + 2\mu)u_{2,2}^0. \quad \text{(IX.39)} \]

It is trivial that those expressions are equivalent to (IX.17) - (IX.19).
IX.E Global properties of $u_{1,1}, u_{2,2}$ and $\dot{u}_1$

From the discontinuity condition (IX.21), it follows that

$$u_{1,1}(x, 0, t) = -\frac{b}{2} \delta(x - l(t)), \quad (IX.40)$$

and

$$\dot{u}_1(x, 0, t) = \frac{b}{2} v(t) \delta(x - l(t)). \quad (IX.41)$$

We define

$$u_{1,1}(x, y, t) = u_{1,1}^0(x, y, t) + G_{11}(x, y, t), \quad (IX.42)$$

and

$$\dot{u}_1(x, y, t) = \dot{u}_{1,1}^0(x, y, t) + G_{11}(x, y, t). \quad (IX.43)$$

From (IX.27) and (IX.31), it is seen that

$$\lim_{y \to 0} u_{1,1}^0(x, y, t) = -\frac{b}{2} \delta(x - l(t)), \quad (IX.44)$$

and

$$\lim_{y \to 0} \dot{u}_{1,1}^0(x, y, t) = \frac{b}{2} v(t) \delta(x - l(t)). \quad (IX.45)$$

Hence we conclude that

$$\lim_{y \to 0} G_{11} = \lim_{y \to 0} G_{1t} = 0, \quad (IX.46)$$

for every $x$ and $t$, and $G_{11}$ and $G_{1t}$ are odd continuous function in $y$ for $-\infty < y < \infty$.

About $u_{2,2}$, we note that from the boundary condition (IX.26),

$$\sigma_{22}(x, 0, t) = 0, \quad (IX.47)$$

and the relation

$$\sigma_{22} = \lambda u_{1,1} + (\lambda + 2\mu) u_{2,2}, \quad (IX.48)$$
it follows
\[ \lim_{y \to 0} u_{2,2}(x, y, t) = -\frac{\lambda}{\lambda + 2\mu} \lim_{y \to 0} u_{1,1}(x, y, t). \] (IX.49)
Combining that relation with (IX.40), we derive that
\[ u_{2,2}(x, 0^+, t) = \frac{\lambda b}{2(\lambda + 2\mu)} \delta(x - l(t)). \] (IX.50)
In (IX.28), let \( y \to 0 \), we have
\[ \lim_{y \to 0} u_{2,2}^0(x, y, t) = \frac{\lambda b}{2(\lambda + 2\mu)} \delta(x - l(t)). \] (IX.51)
Hence if we define
\[ u_{2,2}(x, y, t) = u_{2,2}^0(x, y, t) + G_{22}(x, y, t), \] (IX.52)
then \( G_{22} \) is odd and continuous in \( y \), and satisfies
\[ \lim_{y \to 0} G_{2,2} = 0, \] (IX.53)
for every \( x \) and \( t \).

IX.F Two Theorems on the Near Field Coefficients

In an analogous way as the discussion for the moving screw dislocation, we shall prove two important theorems on the near field coefficients. The definitions of the near field coefficients are the same as in the case of the moving screw dislocations, and will be given below.

In view of that the most singular terms of the near field are in the order of \( 1/\epsilon \), in general, the near field expansions are written as
\[ u_{i,j} = u_{i,j}^0 + f_{ij}(\theta, t) \ln \epsilon + g_{ij}(\theta, t) + h.o.t., \] (IX.54)
where \( i = 1, 2, j = 1, 2, t \), the index \( j = t \) means the partial differentiation with respect to \( t \), \( u_{i,j}^0 \) are the leading terms of the near field expansions which are the
corresponding solutions of the steady-state moving edge dislocation as given in Section C, and \( f_{ij} \) and \( g_{ij} \) are the near field coefficients.

We are going to prove the following important theorems.

**Theorem 3**

Let the near field coefficients \( f_{ij}(\theta, t) \), \( i = 1, 2, j = 1, 2, t \), be defined in the near field expansions (IX.54). Then the partial differentiations of \( f_{ij} \) with respect to \( \theta \), \( f'_{ij}(\theta, t) \), satisfy the homogeneous system of linear equations,

\[
Q \phi = 0, \quad (IX.55)
\]

where \( Q \) is the matrix defined by

\[
Q = \begin{bmatrix}
\cos \theta & \sin \theta & 0 & 0 & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta & 0 & 0 \\
0 & 0 & v \sin \theta & 0 & 0 & \sin \theta \\
v \sin \theta & 0 & 0 & 0 & \sin \theta & 0 \\
-(\lambda + 2\mu) \sin \theta & \mu \cos \theta & (\lambda + \mu) \cos \theta & 0 & -\rho v \sin \theta & 0 \\
0 & (\lambda + \mu) \sin \theta & \mu \sin \theta & -(\lambda + 2\mu) \cos \theta & 0 & \rho v \sin \theta \\
\end{bmatrix},
\]

and \( \phi \) is defined by

\[
\phi = (f'_{11}, f'_{12}, f'_{21}, f'_{22}, f'_{1t}, f'_{2t})^T. \quad (IX.56)
\]

So that

\[
f'_{ij} = 0, \quad (IX.57)
\]

for \( i = 1, 2, j = 1, 2, t \).

Furthermore, it follows that

\[
f_{11}(\theta, t) = f_{22}(\theta, t) = f_{1t}(\theta, t) = 0, \quad (IX.58)
\]

and

\[
f_{12}(\theta, t) = f_{12}(t), \quad (IX.59)
\]

\[
f_{21}(\theta, t) = f_{21}(t), \quad (IX.60)
\]
\[ f_{2t}(\theta, t) = f_{2t}(t) \]  

(IX.61)

are independent of \( \theta \).

**Theorem 4**

Let the near field coefficients \( g_{ij}(\theta, t), i = 1, 2, j = 1, 2, t, \) be defined in the near field expansions (IX.54). Then the partial differentiations of \( g_{ij} \) with respect to \( \theta \), \( g'_{ij}(\theta, t) \), satisfy the inhomogeneous system of linear equations,

\[ Q\psi = \zeta, \]  

(IX.62)

where \( Q \) is the matrix defined in Theorem 3, \( \psi \) is defined by

\[ \psi = (g'_{11}, g'_{12}, g'_{21}, g'_{22}, g'_{1t}, g'_{2t})^T, \]  

(IX.63)

and \( \zeta \) is defined by

\[ \zeta = (a_1, a_2, a_3, a_4, a_5, a_6)^T, \]  

(IX.64)

and

\[ a_1 = f_{12}(t) \cos \theta, \]

\[ a_2 = -f_{21}(t) \sin \theta, \]

\[ a_3 = -U_{11}(\theta, t), \]

\[ a_4 = -U_{21}(\theta, t) + v f_{21} \cos \theta + f_{2t} \cos \theta, \]

\[ a_5 = \rho U_{1t}(\theta, t) - \mu f_{12} \sin \theta, \]

\[ a_6 = -\rho U_{2t}(\theta, t) + (\lambda + \mu) f_{12} \cos \theta, \]

where

\[ U_{ij} \equiv \epsilon \frac{\partial}{\partial t} [u_{i,j}^0]_{\text{exp}}, \]  

(IX.65)
the explicit partial differentiation with respect to \( t \) means the partial differentiation with respect to \( t \) when \( \epsilon \) and \( \theta \) are assumed to be fixed. Therefore \( g'_{ij} \) can be solved explicitly in terms of \( f_{12}, f_{21}, \) and \( f_{2t} \).

**Proof of Theorem 3**

It suffices to consider \( y > 0 \). \( u_{i,j} \) are continuously differentiable when \( y > 0 \). In the relation

\[
u_{1,12} = u_{1,21},
\]

(IX.66)

using the expansions for \( u_{1,1} \) and \( u_{1,2} \), we have

\[
u_{1,12}^0 + \frac{\partial}{\partial y} [f_{11} \ln \epsilon + g_{11}]
= u_{1,21}^0 + \frac{\partial}{\partial x} [f_{12} \ln \epsilon + g_{12}] + h.o.t.
\]

(IX.67)

using

\[
u_{1,12}^0 = u_{1,21}^0,
\]

(IX.68)

and the partial differentiations

\[
\begin{align*}
\frac{\partial \epsilon}{\partial x} & = \cos \theta, & \frac{\partial \epsilon}{\partial y} & = \sin \theta, \\
\frac{\partial \theta}{\partial x} & = \frac{\sin \theta}{\epsilon}, & \frac{\partial \theta}{\partial y} & = \frac{\cos \theta}{\epsilon},
\end{align*}
\]

from (IX.67), we derive

\[
f'_{11} \frac{\cos \theta \ln \epsilon}{\epsilon} + f_{11} \frac{\sin \theta}{\epsilon} + g'_{11} \frac{\cos \theta}{\epsilon}
= -f'_{12} \frac{\sin \theta \ln \epsilon}{\epsilon} + f_{12} \frac{\cos \theta}{\epsilon} - g'_{12} \frac{\sin \theta}{\epsilon} + h.o.t.
\]

(IX.69)

Compare the like terms of \( \ln \epsilon / \epsilon \) in last equation, we obtain

\[
f'_{11} \cos \theta + f'_{12} \sin \theta = 0.
\]

(IX.70)

Similarly starting from

\[
u_{2,12} = u_{2,21},
\]

(IX.71)
and

\[ u_{2,12}^0 + \frac{\partial}{\partial y} [f_{21} \ln \epsilon + g_{21}] \]

\[ = u_{2,21}^0 + \frac{\partial}{\partial x} [f_{22} \ln \epsilon + g_{22}] + h.o.t., \]  

(IX.72)

using the near field expansions, we have

\[ f'_{21} \frac{\cos \theta \ln \epsilon}{\epsilon} + f_{21} \frac{\sin \theta}{\epsilon} + g'_{21} \frac{\cos \theta}{\epsilon} \]

\[ = -f'_{22} \frac{\sin \theta \ln \epsilon}{\epsilon} + f_{22} \frac{\cos \theta}{\epsilon} - g_{22} \frac{\sin \theta}{\epsilon} + h.o.t. \]  

(IX.73)

It follows that

\[ f'_{21} \cos \theta + f'_{22} \sin \theta = 0. \]  

(IX.74)

From the relation

\[ u_{1,1}^0 = u_{1,1}, \]  

(i.e.,

\[ \frac{\partial u_{1,1}}{\partial t} = \frac{\partial \dot{u}_1}{\partial x}, \]  

(IX.76)

and the expansions for \( u_{1,1} \) and \( u_{1,t} \), it follows that

\[ \frac{\partial}{\partial t} [u_{1,1}^0] + \frac{\partial}{\partial x} [f_{11} \ln \epsilon + g_{11}] \]

\[ = \frac{\partial}{\partial x} [u_{1,1}^0] + \frac{\partial}{\partial x} [f_{11} \ln \epsilon + g_{11}] + h.o.t. \]  

(IX.77)

using the differentiations

\[ \frac{\partial \theta}{\partial t} = \frac{v \sin \theta}{\epsilon}, \quad \frac{\partial \epsilon}{\partial t} = -\frac{v \cos \theta}{\epsilon}, \]  

(IX.78)

and the relations

\[ \dot{u}_1^0 = -v(t)u_{1,1}^0, \]  

(IX.79)

and

\[ \frac{\partial}{\partial t} [u_{1,1}^0] = -v(t) \frac{\partial}{\partial x} [u_{1,1}^0] + \frac{\partial}{\partial t} [u_{1,1}^0]_{\exp} \]

\[ = \frac{\partial}{\partial x} [u_{1,1}^0] + \frac{\partial}{\partial t} [u_{1,1}^0]_{\exp}, \]  

(IX.80)
from (IX.77), we have

\[
\frac{\partial}{\partial t}[u_{1,1}^0]|_{\text{exp}} + f_1' \frac{v \sin \theta \ln \epsilon}{\epsilon} - f_1 \frac{v \cos \theta}{\epsilon} + g_1' \frac{v \sin \theta}{\epsilon} = -f_1' \frac{\cos \theta \ln \epsilon}{\epsilon} + f_1 \frac{\cos \theta}{\epsilon} - g_1' \frac{\sin \theta}{\epsilon} + \text{h.o.t.} \quad \text{(IX.81)}
\]

It is easy to verify that \( (\partial u_{1,1}^0/\partial t)|_{\text{exp}} \) is of the order of \( 1/\epsilon \). Hence in (IX.81), compare the like terms of \( \ln \epsilon / \epsilon \), we derive that

\[
v f_1' \sin \theta + f_1' \sin \theta = 0. \quad \text{(IX.82)}
\]

In the same way, using the relation

\[
u_{2,1t} = u_{2,1t}, \quad \text{(IX.83)}
\]

and

\[
\frac{\partial}{\partial t}[u_{0,2,1}]|_{\exp} + f_2' \frac{v \sin \theta \ln \epsilon}{\epsilon} - f_2 \frac{v \cos \theta}{\epsilon} + g_2' \frac{v \sin \theta}{\epsilon} = -f_2' \frac{\cos \theta \ln \epsilon}{\epsilon} + f_2 \frac{\cos \theta}{\epsilon} - g_2' \frac{\sin \theta}{\epsilon} + \text{h.o.t.}, \quad \text{(IX.84)}
\]

we derive the equation

\[
v f_2' \sin \theta + f_2' \sin \theta = 0. \quad \text{(IX.85)}
\]

To find additional equations for \( f_{ij} \), \( i = 1, 2 \), and \( j = 1, 2, t \), consider the equations of motion (IX.11) and (IX.12).

(IX.11) is rewritten as

\[
(\lambda + 2\mu) \frac{\partial}{\partial x}[u_{1,1}] + (\lambda + \mu) \frac{\partial}{\partial y}[u_{2,1}] + \mu \frac{\partial}{\partial y}[u_{1,2}] = \rho \frac{\partial}{\partial t}[u_{1,t}]. \quad \text{(IX.86)}
\]

Substituting the expansions (IX.54) into the last equation, and noting that

\[
(\lambda + 2\mu) \frac{\partial}{\partial x}[u_{1,1}^0] + (\lambda + \mu) \frac{\partial}{\partial y}[u_{2,1}^0] + \mu \frac{\partial}{\partial y}[u_{1,2}^0] = \rho \frac{\partial}{\partial t}[u_{1,t}]|_{\nu=\text{constant}}, \quad \text{(IX.87)}
\]

and

\[
\frac{\partial}{\partial t}[u_{1,t}] = \frac{\partial}{\partial t}[u_{1,t}]|_{\nu=\text{constant}} + \frac{\partial}{\partial t}[u_{1,t}]|_{\exp}, \quad \text{(IX.88)}
\]
we derive that

\[-(\lambda + 2\mu)[f'_{11}\frac{\sin \theta \ln \epsilon}{\epsilon} - f_{11}\frac{\cos \theta}{\epsilon} + g'_{11}\frac{\sin \theta}{\epsilon}] - (\lambda + \mu)[f'_{21}\frac{\cos \theta \ln \epsilon}{\epsilon} + f_{21}\frac{\sin \theta}{\epsilon} + g'_{21}\frac{\cos \theta}{\epsilon}]
+ \mu[f'_{12}\frac{\cos \theta \ln \epsilon}{\epsilon} + f_{12}\frac{\sin \theta}{\epsilon} + g'_{12}\frac{\cos \theta}{\epsilon}]
= \rho(f'_{1t}\frac{v \sin \theta \ln \epsilon}{\epsilon} - f_{1t}\frac{v \cos \theta}{\epsilon} + g'_{1t}\frac{v \sin \theta}{\epsilon} + \frac{\partial}{\partial t}\langle u_{1,t}\rangle_{\exp}) + h.o.t.\]  

(IX.89)

It is easy to verify that

\[\frac{\partial}{\partial t}\langle u_{1,t}\rangle_{\exp} \sim O\left(\frac{1}{\epsilon}\right).\]  

(IX.90)

Compare the like terms of \(\ln \epsilon/\epsilon\) in (IX.89), we obtain the fifth equation for \(f'_{ij}\)

\[-(\lambda + 2\mu)\sin \theta f'_{11} + (\lambda + \mu)\cos \theta f'_{21}\mu \cos \theta f'_{12} - \rho v \sin \theta f'_{1t} = 0.\]  

(IX.91)

Similarly, start from (IX.12), we derive the sixth equation for \(f'_{ij}\)

\[\mu \sin \theta f'_{21} + (\lambda + \mu)\sin \theta f'_{12} - (\lambda + 2\mu)\cos \theta f'_{22} + \rho v \sin \theta f'_{2t} = 0.\]  

(IX.92)

(IX.70), (IX.74), (IX.82), (IX.85), (IX.91), and (IX.92) form a homogeneous system of linear equations for the unknowns \((f'_{11}, f'_{12}, f'_{21}, f'_{22}, f'_{1t}, f'_{2t})\)

\[Q\psi = 0,\]  

(IX.93)

where the matrix \(Q\) is defined by

\[
\begin{bmatrix}
\cos \theta & \sin \theta & 0 & 0 & 0 & 0 \\
0 & 0 & \cos \theta & \sin \theta & 0 & 0 \\
0 & 0 & v \sin \theta & 0 & 0 & \sin \theta \\
v \sin \theta & 0 & 0 & 0 & \sin \theta & 0 \\
-(\lambda + 2\mu) \sin \theta & \mu \cos \theta & (\lambda + \mu) \cos \theta & 0 & -\rho v \sin \theta & 0 \\
0 & (\lambda + \mu) \sin \theta & \mu \sin \theta & -(\lambda + 2\mu) \cos \theta & 0 & \rho v \sin \theta \\
\end{bmatrix}
\]

and \(\psi = (f'_{11}, f'_{12}, f'_{21}, f'_{22}, f'_{1t}, f'_{2t})^T\).

The determinant of the coefficient matrix \(Q\) is calculated to be

\[\det Q = -\sin^2 \theta[(\lambda + 2\mu) - \rho v^2 \sin^2 \theta][\mu \cos^2 \theta + (\mu - \rho v^2) \sin^2 \theta] < 0,\]  

(IX.94)
for $\theta \neq 0, \pi$, since $(\lambda + 2\mu) > \mu > \rho v^2$ for $v < c_2 < c_1$. Therefore, we conclude that

$$f'_{ij}(\theta, t) = 0,$$  \hspace{1cm} (IX.95)

for $i = 1, 2$, $j = 1, 2, t$, $y > 0$ and $y < 0$, respectively.

By a similar reasoning as in the discussion for moving screw dislocation, from the global properties of $u_{1,1}$, $u_{2,2}$, and $u_{1,t}$, i.e., $G_{11}, G_{22}$, and $G_{1t}$ are continuous odd function in $y$. Especially, in the near field, $f_{11}, f_{22}$, and $f_{1t}$ are odd continuous function in $y$, which implies that

$$f_{11} = f_{22} = f_{1t} = 0.$$  \hspace{1cm} (IX.96)

On the other hand, from that $u_{1,2}, u_{2,1}, u_{2,t}$ and $f_{12}, f_{21}, f_{2t}$ are even in $y$. Note that for a fixed $t$, $f_{12}, f_{21}, f_{2t}$ are constants for $y > 0$ and $y < 0$, respectively. That fact implies that each of them is a constant for all $y$ and an arbitrarily fixed $t$. Theorem 3 is then proved.

**Proof of Theorem 4**

From Theorem 3, in (IX.69), (IX.73), (IX.81), (IX.84), (IX.89), and a corresponding relation derived from the second equation of motion (IX.12), the leading terms are of the order of $1/\epsilon$. Compare the like terms of $1/\epsilon$, we obtain an inhomogeneous system of linear equations for $g'_{ij}$. The matrix of the coefficients of the system is exactly $Q$ defined in Theorem 3. It is easy to verify that the system of equations is written as

$$Q \psi = \zeta,$$  \hspace{1cm} (IX.97)

where $\psi$ is defined by

$$\psi = (g'_{11}, g'_{12}, g'_{21}, g'_{22}, g'_{1t}, g'_{2t})^T,$$  \hspace{1cm} (IX.98)

and $\zeta$ is defined by

$$\zeta = (a_1, a_2, a_3, a_4, a_5, a_6)^T,$$  \hspace{1cm} (IX.99)

and

$$a_1 = f_{12}(t) \cos \theta,$$
\[ a_2 = -f_{21}(t) \sin \theta, \]
\[ a_3 = -U_{11}(\theta, t), \]
\[ a_4 = -U_{21}(\theta, t) + v f_{21} \cos \theta + f_{2t} \cos \theta, \]
\[ a_5 = \rho \dot{U}_{1t}(\theta, t) - \mu f_{12} \sin \theta, \]
\[ a_6 = -\rho \dot{U}_{2t}(\theta, t) + (\lambda + \mu) f_{12} \cos \theta, \]

and
\[ U_{ij} \equiv \epsilon \frac{\partial}{\partial t} [u_{ij,0}]|_{\text{exp}}. \tag{IX.100} \]

\( g'_{ij} \) can be solved explicitly in terms of \( f_{12}, f_{21}, \) and \( f_{2t} \). \( g_{ij}(\theta, t) \) can be determined by integrating \( g'_{ij} \), for \( i = 1, 2 \) \( j = 1, 2, t \), respectively. Theorem 4 is proved.

Theorem 3 means that there are no \( \ln \epsilon \) terms in the near field expansions of \( u_{1,1}, u_{2,2}, \) and \( \dot{u}_1; \) and in the near field expansions of \( u_{1,2}, u_{2,1}, \) and \( \dot{u}_2 \), the coefficient functions of the \( \ln \epsilon \) terms are all independent of \( \theta \), and only functions of \( t \).

Based on Theorem 4, \( g'_{ij} \) are solved out explicitly, and by integrating \( g'_{ij} \), \( g_{ij} \) is obtained explicitly as well.

\[ g_{ij}(\theta, t) = \int_0^\theta g'_{ij}(\omega, t) d\omega + g_{ij}(0, t). \tag{IX.101} \]

By an analogous reasoning as used in the proof of Theorem 3, in view of that \( g_{11}, g_{22}, g_{1t} \) are odd in \( y \), we conclude that
\[ g_{11}(0, t) = g_{22}(0, t) = g_{1t}(0, t) = 0. \tag{IX.102} \]

Furthermore, examine the fourth equation in the system (IX.97), i.e.,
\[ v g'_{11} \sin \theta + g'_{1t} \sin \theta = a_4 = -U_{21}(\theta, t) + v f_{21} \cos \theta + f_{2t} \cos \theta. \tag{IX.103} \]

In the last equation, set \( \theta = 0 \), we have that
\[ v f_{21} + f_{2t} = U_{21}(0, t). \tag{IX.104} \]
So that \( f_{21} \) and \( f_{2t} \) are not independent.

Therefore, in order to completely determine the twelve functions of the near field coefficients \( f_{ij} \) and \( g_{ij} \), we only need to evaluate the five near field constants, which are \( f_{21}(t), f_{12}(t), g_{12}(0, t), g_{21}(0, t), \) and \( g_{2t}(0, t) \), where \( t \) may be considered as a parameter.

### IX.G Solution of the Self-Force

As similar to the derivation of the self-force on the moving screw dislocation, choose \( V \) to be a cylindrical volume around the edge dislocation line at \((x, y) = (l(t), 0)\) at time \( t \), and with a unit length in the \( z \)-direction. The infinitesimal volume \( V_\epsilon \) around the dislocation line is chosen to be a cylinder with a radius \( \epsilon \) for \( 0 < \epsilon << 1 \). Then the self-force on the moving edge dislocation is defined as the limit of the force on the inhomogeneity \( V_\epsilon \) when \( V_\epsilon \) is shrinking upon the dislocation,

\[
F_l = \lim_{\epsilon \to 0} \left\{ \int_{V_\epsilon \setminus V} \frac{\partial}{\partial t} \left[ \rho \dot{u}_{i,l} \right] dV + \int_S \left[ (W - T) \delta_{ij} - u_{i,l} \sigma_{ij} \right] dS \right\}. \tag{IX.105}
\]

where \( S = \partial V \). Equivalently, the self-force \( F_1 \) for \( l = 1 \), is expressed as

\[
F_1 = \lim_{\epsilon \to 0} I_\epsilon = \lim_{\epsilon \to 0} \int_{S_\epsilon} \left[ (W - T) \delta_{1j} - \sigma_{3j} u_{3,1} \right] dS_j, \tag{IX.106}
\]

where \( S_\epsilon = \partial V_\epsilon \).

\( I_\epsilon \) is explicitly given by

\[
I_\epsilon = \int_{S_\epsilon} \left[ (W - T) \delta_{1j} - \sigma_{ij} u_{i,1} \right] dS_j
= \int_0^{2\pi} \left[ (W - T) - \sigma_{11} u_{i,1} \right] \cos \theta \epsilon d\theta - \int_0^{2\pi} \sigma_{12} u_{i,1} \sin \theta \epsilon d\theta
= \frac{1}{2} \int_0^{2\pi} \left[ \sigma_{12} (u_{1,2} - u_{2,1}) + \sigma_{22} u_{2,2} - \sigma_{11} u_{1,1} - \rho (\dot{u}_1^2 + \dot{u}_2^2) \right] \cos \theta \epsilon d\theta
- \int_0^{2\pi} (\sigma_{12} u_{1,1} + \sigma_{22} u_{2,2}) \sin \theta \epsilon d\theta. \tag{IX.107}
\]
Using the near field expansions of \( u_{i,j} \) (IX.54) for \( i = 1, 2 \) and \( j = 1, 2, t \), and (IX.38) - (IX.37), we have the near field expansions of the stresses

\[
\sigma_{11} = (\lambda + 2\mu)u_{1,1} + \lambda u_{2,2} \\
= (\lambda + 2\mu)u_{1,1}^0 + \lambda u_{2,2}^0 + (\lambda + 2\mu)g_{11} + \lambda g_{22} + O(\epsilon)
\]

\[
\sigma_{11} = \sigma_{11}^0 + (\lambda + 2\mu)g_{11} + \lambda g_{22} + O(\epsilon), \quad (IX.108)
\]

\[
\sigma_{22} = \lambda u_{1,1} + (\lambda + 2\mu)u_{2,2} \\
= \sigma_{11}^0 + \lambda g_{11} + (\lambda + 2\mu)g_{22} + O(\epsilon), \quad (IX.109)
\]

\[
\sigma_{12} = \mu(u_{1,2} + u_{2,1}) \\
= \sigma_{12}^0 + \mu(f_{12} + f_{21}) \ln \epsilon + \mu(g_{12} + g_{22}) + O(\epsilon). \quad (IX.110)
\]

Observe now the integral of the term of \( \epsilon \sigma_{12} u_{1,2} \cos \theta \) in (IX.107). From (IX.110) and (IX.54), it follows that

\[
\sigma_{12} u_{1,2} = (\sigma_{12}^0 + \mu(f_{12} + f_{21})) \ln \epsilon + \mu(g_{12} + g_{21}) + O(\epsilon)
\]

\[
= \sigma_{12}^0 u_{1,2} + \mu u_{1,2}^0 (f_{12} + f_{21}) \ln \epsilon + \sigma_{12}^0 f_{12} + O(1). \quad (IX.111)
\]

Examine the integral of the first term on the most right hand side of last equation. By using the expressions (IX.35) and (IX.36), we have that

\[
\int_0^{2\pi} \sigma_{12}^0 u_{1,2}^0 \cos \theta d\theta = 0, \quad (IX.112)
\]

Therefore, to \( I_\epsilon \), the contributions from the integral of the term \( \sigma_{12} u_{1,2} \) are integrals of the second and third term on the right hand side of (IX.111), i.e.,

\[
\int_0^{2\pi} [\mu u_{1,2}^0 (f_{12} + f_{21}) + \sigma_{12}^0 f_{12}] \ln \epsilon \cos \theta \epsilon d\theta, \quad (IX.113)
\]

which will contribute a \( \ln \epsilon \) term to \( I_\epsilon \), and

\[
\int_0^{2\pi} [\mu u_{1,2}^0 (g_{12} + g_{21}) + \sigma_{12}^0 g_{12}] \cos \theta \epsilon d\theta, \quad (IX.114)
\]
which will contribute a $O(1)$ term to $I_\epsilon$.

In a similar manner, we find the corresponding terms which contribute to $I_\epsilon$ for the remaining part in (IX.107). The results are listed below

<table>
<thead>
<tr>
<th>Term</th>
<th>$\ln \epsilon$</th>
<th>$O(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{12}u_{1,2}$</td>
<td>$\mu u_{1,2}^0 (f_{12} + f_{21}) + \sigma_{12}^0 f_{12}$</td>
<td>$\mu u_{1,2}^0 (g_{12} + g_{21}) + \sigma_{12}^0 g_{12}$</td>
</tr>
<tr>
<td>$\sigma_{12}u_{2,1}$</td>
<td>$\mu u_{2,1}^0 (f_{12} + f_{21}) + \sigma_{12}^0 f_{12}$</td>
<td>$\mu u_{2,1}^0 (g_{12} + g_{21}) + \sigma_{12}^0 g_{12}$</td>
</tr>
<tr>
<td>$\sigma_{22}u_{2,2}$</td>
<td>0</td>
<td>$u_{2,2}^0 (\lambda g_{11} + (\lambda + 2 \mu) g_{22}) + \sigma_{22}^0 g_{22}$</td>
</tr>
<tr>
<td>$\sigma_{11}u_{1,1}$</td>
<td>0</td>
<td>$u_{1,1}^0 ((\lambda + 2 \mu) g_{11} + \lambda g_{22}) + \sigma_{11}^0 g_{11}$</td>
</tr>
<tr>
<td>$\sigma_{12}u_{1,1}$</td>
<td>$\mu u_{1,1}^0 (f_{12} + f_{21})$</td>
<td>$\mu u_{1,1}^0 (g_{12} + g_{21}) + \sigma_{12}^0 g_{12}$</td>
</tr>
<tr>
<td>$\sigma_{22}u_{2,1}$</td>
<td>$\sigma_{22}^0 f_{21}$</td>
<td>$\mu u_{2,1}^0 (\lambda g_{11} + (\lambda + 2 \mu) g_{22}) + \sigma_{22}^0 g_{22}$</td>
</tr>
<tr>
<td>$\dot{u}_1^2$</td>
<td>0</td>
<td>$2g_{11}u_1^0$</td>
</tr>
<tr>
<td>$\dot{u}_2^2$</td>
<td>$2f_{21}\dot{u}_2^0$</td>
<td>$2g_{22}\dot{u}_2^0$</td>
</tr>
</tbody>
</table>

Substitute those results into the integration (IX.107) and use the integrals

$$
\frac{\pi}{2} \int_0^\frac{\pi}{2} \frac{d\theta}{\cos^2 \theta + \beta^2 \sin^2 \theta} = \frac{\pi}{2\beta} \tag{IX.115}
$$

$$
\frac{\pi}{2} \int_0^\frac{\pi}{2} \frac{\cos^2 \theta d\theta}{\cos^2 \theta + \beta^2 \sin^2 \theta} = \frac{\pi}{2(1 + \beta)} \tag{IX.116}
$$

$$
\frac{\pi}{2} \int_0^\frac{\pi}{2} \frac{\sin^2 \theta d\theta}{\cos^2 \theta + \beta^2 \sin^2 \theta} = \frac{\pi}{2\beta(1 + \beta)} \tag{IX.117}
$$

we obtain that

$$
I_\epsilon = F_{\ln \epsilon}(t) \ln \epsilon + \int_0^{2\pi} F_{O(1)}(\theta, t) d\theta + O(\epsilon), \tag{IX.118}
$$

where

$$
F_{\ln \epsilon}(t) = b\mu f_{12}(t) + 2b\mu f_{21}(t) \left[ \frac{1}{1 + \gamma_1} + \frac{\epsilon^2}{v(t)} \left( \frac{\alpha^2}{\gamma_2} - 1 \right) \right] + \frac{2b\mu f_{21}(t)}{v(t)} \left[ \frac{\gamma_1}{1 + \gamma_1} - \frac{\alpha^2}{\gamma_2(1 + \gamma_2)} \right], \tag{IX.119}
$$
and
\[ F_{O(1)} = \frac{bc_2^2\mu}{\pi v^2(t)} \left( g_{12} - g_{21} \right) \left[ \frac{\gamma_1}{\cos^2 \theta + \gamma_1^2 \sin^2 \theta} - \frac{\alpha^2\gamma_2}{\cos^2 \theta + \gamma_2^2 \sin^2 \theta} \right] + \frac{b\mu\gamma_1}{\pi} \left[ \frac{g_{21} \sin^2 \theta - g_{22} \sin \theta \cos \theta}{\cos^2 \theta + \gamma_1^2 \sin^2 \theta} \right] + \frac{b\alpha^2 \mu g_{21} \cos^2 \theta + ((\lambda + \mu)g_{11} + (\lambda + 2\mu)g_{22}) \sin \theta \cos \theta}{\cos^2 \theta + \gamma_2^2 \sin^2 \theta} \right] + \frac{b\mu g_{11}(\theta, t)}{\pi v(t)} \left[ \frac{\gamma_1}{\cos^2 \theta + \gamma_1^2 \sin^2 \theta} - \frac{\alpha^2\gamma_2}{\cos^2 \theta + \gamma_2^2 \sin^2 \theta} \right] + \frac{b\mu g_{21}(\theta, t)}{\pi v(t)} \left[ \frac{\gamma_1}{\cos^2 \theta + \gamma_1^2 \sin^2 \theta} - \frac{\alpha^2}{\gamma_2(\cos^2 \theta + \gamma_2^2 \sin^2 \theta)} \right], \tag{IX.120} \]

where again \( \gamma_i = \sqrt{1 - v^2/c_i^2} \) for \( i = 1, 2 \), and \( \alpha = \sqrt{1 - v^2/2c_0^2} \).

Consequently, the self-force on a moving gliding edge dislocation is expressed as,
\[ F_1 \sim F_{\ln \epsilon} \ln \epsilon + \int_0^{2\pi} F_{O(1)}(\theta, t) d\theta, \tag{IX.121} \]
as \( \epsilon \to 0 \), where \( F_{\ln \epsilon} \) and \( F_{O(1)}(\theta, t) \) are given by (IX.119) and (IX.120). It is seen that if \( F_{\ln \epsilon} \) does not vanish then the self-force for a moving edge dislocation will be divergent in an order of \( \ln \epsilon \) as \( \epsilon \to 0 \). In that case, to deal with the divergence, method based on the theory of distributions and based on smearing procedures can be used. The discussions are analogous to those for the case of an accelerating screw dislocation.

**IX.H Evaluation of the Self-Force**

As seen in previous sections, to evaluate the self-force on a moving gliding edge dislocation, it is necessary to evaluate the terms in the near field coefficients, \( f_{12}, f_{21}, g_{12}(0, t), g_{21}(0, t) \) and \( g_{21}(0, t) \). All those terms have not been evaluated in the literature.

We propose a general approach to calculate those values. We illustrated that approach by calculating \( f_{12} \). The approach is based on the Mura (1963)
formula for \( u_{n,m} \) using the Green’s function of a unit impulse load in a full space, and Callias, Markenscoff, and Ni (1990) analysis of the near field of a dislocation loop. The algebra is lengthy and tedious. The detailed analysis and calculation is recently published (Ni and Markenscoff, 2003). It is found that

\[
f_{12}(t) = \frac{3b\mu \dot{v}(t)}{\rho \pi v^4} \left[ \frac{(2\beta_2^2 - 1)(3\beta_2^2 - 4)}{\gamma_2} - \frac{(2\beta_1^2 - 1)(3\beta_1^2 - 4)}{\gamma_1} \right] \\
+ \frac{b\mu \dot{v}(t)}{\rho \pi v^4} \left[ \frac{\beta_1^4}{\gamma_1^3} - \frac{\beta_2^4(1 - 4\beta_2^2)}{2\gamma_2^3} \right].
\]

(IX.122)

**IX.I Conclusion**

The self-force on a moving edge dislocation is solved by virtue of the definition of the self-force discussed in Chapter VI, and the two new theorems on the near field coefficients. The effective mass hence can be derived from the inertial part of the self-force.
Chapter X

Conclusions

In summary we have achieved that

- Based on the change of the total Lagrangian of the elastic system, a clear definition of the dynamic configurational force on a moving elastic inhomogeneity and defect is given.

- In the definition, the discontinuities and singularities are rigorously treated.

- A new “contour-independent” integral expression for the dynamic configurational force is established, which is used to calculate the self-force on moving screw and edge dislocations.

- A complete evaluation of the near field expansions for the elastic field of accelerating screw and edge is obtained, which is the foundation of the evaluation of the self-force and effective mass for moving screw and edge dislocations. The method of evaluating the near field expansions is elegant and powerful, and may widely apply to solving other problems.

- Explicit and complete solutions for the self-force and effective mass for accelerating screw and edge dislocations are established.

- The self-force and effective mass evaluation for moving dislocations based on the theory of distributions and a smearing method (ramp-core) is explicitly achieved, which eliminates the involved divergences in Volterra dislocations.
We also like to remark the following:

- The “contour-independent” integral expression given in Chapter IV and V for the dynamic configurational force can be similarly applied to calculate the self-force and effective mass of an accelerating dislocation loop, which is a real three-dimensional problem. The integral expression of the self-force can be reduced to a surface integral over a closed thin tube with infinitesimal radius around the dislocation loop. For the near field behavior of a dislocation loop, see Markenscoff and Ni (1990). The method of full determination of the near field expansions of the elastic field of an accelerating straight dislocation presented in Chapter VI and IX can be extended to the case of an accelerating dislocation loop.

- Although our discussion is in the framework of linear elasticity, in principle, the same approach and ideas apply as well to the nonlinear case, since the main conservation law which has been frequently used in our discussion remains valid for non-linear elasticity, where \( x_k \) are material coordinates, \( W \) is the strain energy density per unit undeformed volume, \( \rho \) the mass per unit undeformed volume, \( \sigma_{ij} \) the components of the Piola-Kirckhoff stress tensor (see Fletcher, 1976).

- Our discussion is limited to the isotropic elasticity. In principle, the same approach and ideas can be extended to the anisotropic case. For the near field behavior of an accelerating dislocation in an anisotropic solid, see Markenscoff and Ni (1993).

- In the ramp-core model used in our discussion of the smearing method (Chapter VIII), we assume that the core parameter \( a \) is independent of time. A closed form solution of \( u_{3,2} \) for a time dependent ramp-core model was given in Markenscoff and Ni (2001), which can be explicitly evaluated, when the function of the core parameter \( a(t) \) is given, and by a similar analysis the self-force and effective mass can be also derived.
The analysis presented here can lead to the equation of motion for a moving dislocation where the self-force is equated to an externally applied loading (Eshelby, 1953).
Appendix A

Evaluation of $g_{32}(0, t)$

A.A Introduction

In this appendix, the $O(1)$ terms $g_{32}(0)$ will be calculated. The simple Taylor expansion will lead to divergent results, and a special method of asymptotic expansion given by Callias and Markenscoff (1988) is used in calculation.

The terms $g_{32}(0)$ will be calculated from the near field expansion of the stress $\sigma_{32}$ at the positions $(x, y) = (l(t) + \epsilon, 0)$ as $\epsilon \to 0$, respectively. The calculation will use the analytic solution of the stress $\sigma_{32}$ for a non-uniformly moving screw dislocation starting from rest given by Markenscoff (1980),

$$
\sigma_{32} = \frac{b\mu}{2\pi} \int_0^\infty \frac{(t - \eta(\xi))(x - \xi)^2 H(t - \eta(\xi) - r/c)}{r^4[(t - \eta(\xi))^2 - r^2/c^2]^{1/2}} d\xi
$$

$$
-\frac{b\mu}{2\pi} \frac{y^2}{\partial t} \int_0^\infty \frac{(t - \eta(\xi))^2 H(t - \eta(\xi) - r/c)}{r^4[(t - \eta(\xi))^2 - r^2/c^2]^{1/2}} d\xi + \frac{b\mu}{2\pi} \frac{x}{x^2 + y^2},
$$

(A.1)

where $c \equiv c_2 = \sqrt{\mu/\rho}$, $r^2 = (x - \xi)^2 + y^2$ and $\eta(\xi) = \tau$ is the inverse function of $\xi = l(\tau)$.

The third term in the above expression will give no contribution to the self-force or effective mass, since it represents the solution of a static screw dislocation.

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A.B Simplify the Integrations

We shall show that for \( x = l(t) + \epsilon \), the integrations in (A.1) only have a square root singularity, and the limit of \( y \to 0 \) can be taken under the integrations, so that the only non-zero integral, i.e., the first integral in (A.1) will be reduced to a simpler form.

In (A.1), the upper limit of the integrals is determined by \( H(t - \eta(\xi_0) - r/c) \), i.e., the root \( \xi_0 \) of the equation

\[
 c(t - \eta(\xi_0)) = \sqrt{(x - \xi_0)^2 + y^2}. \tag{A.2}
\]

The physical meaning of \( \xi_0 \) is the last position from which the wavelet of the moving screw dislocation can reach the field point \( (x, y) = (l(t) + \epsilon, y) \) at time \( t \).

It is seen that \( \xi < \xi_0 < x_0 \equiv l(t) < x = x_0 + \epsilon \), since \( x_0 \) is the position of the dislocation at time \( t \), its wavelet has no time to reach \( x = x_0 + \epsilon \). It follows that \( x_0 + \epsilon - \xi \) and \( r = \sqrt{(x_0 + \epsilon - \xi)^2 + y^2} \) are never zero. Then singularities of the integrals in (A.1) are only due to the factor

\[
 \sqrt{(t - \eta(\xi_0))^2 - r^2/c^2} \tag{A.4}
\]

in the denominator. The improper integrals in (A.1) converge uniformly for \( y \) in a small neighborhood of \( y = 0 \). It is therefore permissible to interchange the limit of \( y \to 0 \) and the integration. Taking limit of \( y \to 0 \) under the integrals of (A.1), we have

\[
 II_\epsilon \equiv \frac{b\mu}{2\pi} \int_0^\xi \frac{t - \eta(\xi)}{(x - \xi)^2((t - \eta(\xi))^2 - (x - \xi)^2/c^2)^2} d\xi, \tag{A.5}
\]

where \( x = x_0 + \epsilon \).

To find the near field expansion of (A.5), we express its integrand as a function of \( \epsilon \) as follows: (1) Express \( t - \eta(\xi) \) in \( \epsilon \), (2) Express \( x - \xi \) in \( \epsilon \), (3) Express \( (t - \eta(\xi))^2 - (x - \xi)^2/c^2 \) in \( \epsilon \), and then (4) Express the integrand in \( \epsilon \).
Define $\eta(\xi_0) = \tau_0$, or equivalently, $l(\tau_0) = \xi_0$; and $\tau = \eta(\xi)$. Here for $y = 0$, $\xi_0$ is the root of
\[ c(t - \eta(\xi_0)) = x_0 + \epsilon - \xi_0, \quad \text{(A.6)} \]
or, equivalently,
\[ c(t - \tau_0) = l(t) + \epsilon - l(\tau_0), \quad \text{(A.7)} \]

In view of the meaning of $\xi_0$, i.e., the last position from there the wavelet of dislocation can reach the point $x_0 + \epsilon$, so that when $\epsilon \to 0$, $\xi_0$ will coincide with $x_0$. That implies as $\epsilon \to 0$, $\xi_0 = l(\tau_0) \to x_0 = l(t)$, and $\tau_0 \to t$.

Let both sides of (A.6) be divided by $t - \tau_0$, and take limit of $\epsilon \to 0$, we have
\[ c = \lim_{\epsilon \to 0} \left[ \frac{l(t) - l(\tau_0)}{t - \tau_0} + \frac{\epsilon}{t - \tau_0} \right] = v(t) + \lim_{\epsilon \to 0} \left( \frac{\epsilon}{t - \tau_0} \right), \quad \text{(A.8)} \]
where $v(t) \equiv \dot{l}(t)$. Since the motion is subsonic, i.e., $v(t) < c$, so that from (A.8), we have
\[ \lim_{\epsilon \to 0} \left( \frac{t - \tau_0}{\epsilon} \right) = \frac{1}{c - v(t)}. \quad \text{(A.9)} \]
Then we have
\[ t - \tau_0 = \frac{\epsilon}{c - v} + o(\epsilon) \equiv \epsilon A_0(t, \epsilon), \quad \text{(A.10)} \]
and
\[ x_0 - \xi_0 = c\epsilon A_0(t, \epsilon) - \epsilon, \quad \text{(A.11)} \]
where $A_0(t, \epsilon)$ is written as
\[ A_0(t, \epsilon) = a_1 + a_2\epsilon + \ldots, \quad \text{(A.12)} \]
and
\[ a_1 = \frac{1}{c - v(t)}. \quad \text{(A.13)} \]
From (A.6), it follows that
\[ a_2 = -\frac{\ddot{\psi}(t)}{2(c - v(t))^3}. \]  
(A.14)

Examine \( t - \tau = t - \eta(\xi) \), we have that
\[ t - \tau = (t - \tau_0) + (\tau_0 - \tau) = \epsilon A_0(t, \epsilon) + (\eta(\xi_0) - \eta(\xi)). \]  
(A.15)

Define \( s \equiv \xi_0 - \xi \), then
\[ t - \tau = \epsilon A_0(t, \epsilon) + s\psi(s, \xi_0), \]  
(A.16)

where \( \psi \) is defined by
\[ \psi(s, \xi_0) = \frac{\eta(\xi_0) - \eta(\xi)}{s} = \frac{\eta(\xi_0) - \eta(\xi_0 - s)}{s}. \]  
(A.17)

For sufficiently small \( s \), \( \psi(s, \xi_0) \) has the Taylor expansion in \( s \),
\[ \psi(s, \xi_0) = \frac{1}{2} \eta''(\xi_0)s + \ldots. \]  
(A.18)

with
\[ \eta'(\xi_0) = \frac{1}{v(\tau_0)}, \quad \eta''(\xi_0) = -\frac{\ddot{\psi}(\tau_0)}{[v(\tau_0)]^3}. \]  
(A.19)

(2) Express \( x - \xi \) in \( \epsilon \)
\[ x - \xi = x_0 + \epsilon - \xi_0 + s. \]  
(A.20)

Using (A.11) in the equation, we have
\[ x - \xi = c\epsilon A_0(t, \epsilon) + s. \]  
(A.21)

(3) Express \((t - \tau)^2 - (x - \xi)^2/c^2\) in \( \epsilon \)

From (A.16) and (A.20), we have
\[ (t - \tau)^2 - (x - \xi)^2/c^2 = (\epsilon A_0(t, \epsilon) + \psi s)^2 - (c\epsilon A_0(t, \epsilon) + s)^2/c^2 \]
\[ = s^2(\psi^2 - 1/c^2) + 2\epsilon s A_0(t, \epsilon)(\psi - 1/c). \]  
(A.22)
(4) Express the integrand in $\epsilon$

Therefore, the integral $II_\epsilon$ is rewritten as

$$II_\epsilon = \frac{b\mu}{2\pi} \int_0^{\xi_0} h(s, \epsilon, x, t) ds, \quad (A.23)$$

where $h$ is defined by

$$h(s, \epsilon, x, t) = \frac{(t - \tau)}{(x - \xi)^2[(t - \tau)^2 - (x - \xi)^2/c^2]}$$

$$= \frac{s\psi(s) + \epsilon A_0(t, \epsilon)}{(s + \epsilon A_0(t, \epsilon))^2[s^2(\psi^2 - 1/c^2) + 2s\epsilon A_0(t, \epsilon)(\psi - 1/c)]^{1/2}}. \quad (A.24)$$

We note that if using simple Taylor expansion for factor $1/(s + \epsilon A_0)^2$, which will give

$$\frac{1}{(s + \epsilon A_0)^2} = \frac{1}{s^2} + O(\epsilon), \quad (A.25)$$

then the integral will be divergent. This implies that a special technique is needed for our calculation.

### A.C A Special Technique for Asymptotic Expansions

#### A.C.1 A Corollary

Now we shall use a Corollary of the theorem given by Callias and Markenscoff (1988), to do asymptotic expansion for $II_\epsilon$, which states as follows.

**Corollary:**

Let $f(s, y)$ be such that

(i) $f \in C^\infty((0, p] \times [0, \infty))$

(ii) $|\partial^k_s f(s, z)| \leq y^k h_k(y)$, for all $s, y$,

$k = 0, 1, 2, ...$

where $\int_0^\epsilon h_k(1/s) ds < \infty$, for each $\epsilon > 0$. 
Then we have as $s \to 0^+$,
\[
\int_0^p f(s, \epsilon/s) ds \sim \int_0^p f(s, 0) ds
\]
\[
+ \sum_{m=1}^\infty \epsilon^m \left\{ L_m(f) + u_m(f; p) + \frac{1}{m!(m-1)!} \partial_{y}^{m-1} \partial_y f(0, 0) \sum_{j=1}^{m-1} \frac{1}{j} \right\}
\]
\[
+ \sum_{m=1}^\infty \epsilon^m \ln \epsilon \frac{-1}{m!(m-1)!} \partial_{y}^{m-1} \partial_y f(0, 0)
\]
\[
+ \sum_{m=1}^\infty \epsilon^m \left\{ \sum_{j=1}^{m-1} (-1)^{m+j} \frac{(j-1)!}{m!(m-1)!} p^j \partial_{y}^{m-j} \partial_y f(p, 0) \right\},
\]
where
\[
L_m(f) = -\frac{1}{(m-1)!} \int_0^\infty \ln \zeta \partial_\zeta\left[ \zeta^m \partial_{x}^{m-1} R_{m+1}(0, 1/\zeta) \right] d\zeta,
\]
\[
R_{m+1}(x, y)
\]
is the remainder of $f(x, y)$ in the Taylor expansion about $y = 0$ after $m + 1$ terms, i.e.,
\[
R_{m+1} = f(x, y) - \sum_{k=0}^{m} \frac{1}{k!} \partial_y^k f(x, 0) y^k,
\]
and
\[
u_m(f; p) = -\frac{1}{m!(m-1)!} \int_0^p \ln x \partial_x^m \partial_y f(x, 0) dx.
\]

\section*{A.C.2 Apply the Corollary}

To use this corollary for our calculation, we bring the integrand (A.24) into the form $f(s, \epsilon/s)$ as the corollary required.
\[
h = \frac{s \psi(s) + \epsilon A_0(t, \epsilon)}{(s + \epsilon \epsilon A_0(t, \epsilon))^2[s^2(\psi^2 - 1/c^2) + 2 \epsilon \epsilon A_0(t, \epsilon)(\psi - 1/c)]^{1/2}}
\]
\[
= \frac{y^2}{e^2} \left\{ \frac{\psi(s) + y A_0(t, \epsilon)}{(1 + cy A_0(t, \epsilon))^2[(\psi^2 - 1/c^2) + 2y A_0(t, \epsilon)(\psi - 1/c)]^{1/2}} \right\}.
\]
If we denote that
\[
h = \frac{1}{e^2} f(s, y, \epsilon),
\]
then the integral \(II_\epsilon\) becomes
\[
II_\epsilon = \frac{b\mu}{2\pi} \int_0^{\xi_0} h d\xi = \frac{b}{2\pi \epsilon^2} \int_0^{\xi_0} f(s, y, \epsilon) ds.
\] (A.32)

\(f(s, y, \epsilon)\) has a Taylor expansion in \(\epsilon\),
\[
f(s, y, \epsilon) = \sum_{k=0}^{\infty} f_k(s, y) \epsilon^k.
\] (A.33)

We write
\[
h = \sum_{k=0}^{\infty} \frac{1}{\epsilon^2} f_k(s, y) \epsilon^k.
\] (A.34)

We observe that

(i) The Taylor expansion in (A.34) can be naturally extended over to the whole interval \([0, x_0]\). For instance, in the terms of (A.34),
\[
\psi_0(s) = \eta(x_0) - \eta(x_0 - s)
\] (A.35)
is well-defined on \([\xi_0, x_0]\), and the same for its derivatives.

(ii) Over the interval \([\xi_0, x_0]\), the right hand side of (A.34) is not singular on \([\xi_0, x_0]\). So that
\[
\int_{\xi_0}^{x_0} \sum_{k=0}^{\infty} \frac{1}{\epsilon^2} f_k(s, y) \epsilon^k ds = O(\epsilon),
\] (A.36)
as \(x_0 - \xi_0 = O(\epsilon)\).

Hence, to find the \(O(1)\) term of \(\epsilon\) in the near field expansion of (A.5), we may equivalently consider the integral
\[
III_\epsilon \equiv \frac{b\mu}{2\pi} \int_0^{x_0} \sum_{k=0}^{\infty} \frac{1}{\epsilon^2} f_k(s, y) \epsilon^k ds,
\] (A.37)
and we need to find

(A) The order \(\epsilon^2\) term of \(\int_0^{x_0} f_0(s, y) ds;\)
(B) The order $\epsilon$ term of $\int_0^{x_0} f_1(s, y) ds$;

(C) The order $O(1)$ term of $\int_0^{x_0} f_2(s, y) ds$.

We have checked the condition (i) and (ii) in the corollary, which is a prerequisite for applying the corollary, and proved that those conditions are satisfied by the integrands $f_k$, for $k = 0, 1, 2, \ldots$ Here, we omit the tedious proof.

Now we proceed to calculate (A), (B), (C), and further to evaluate $g_{32}(0)$.

**A.C.3 Evaluation of (A)**

We seek to find the order $\epsilon^2$ term of the asymptotic expansion of the integral, as $\epsilon \to 0$,

$$
\int_0^{x_0} f_0(s, y) ds, \tag{A.38}
$$

where $y = \epsilon/s$. $f_0$ is expressed as

$$
f_0 = y^2 \hat{f}_0(s, y), \tag{A.39}
$$

with $\hat{f}_0$ is defined by

$$
\hat{f}_0 \equiv \frac{\psi_0(s) + y a_1}{(1 + cy a_1)^2[(\psi_0(s)^2 - 1/c^2) + 2y a_1(\psi_0(s) - 1/c)]^{\frac{1}{2}}}, \tag{A.40}
$$

where again $a_1 = 1/(c - v)$, and

$$
\psi_0(s) = \psi(s)|_{\epsilon=0} = \frac{\eta(x_0) - \eta(\xi)}{x_0 - \xi} = \eta'(x_0) - \frac{1}{2} \eta''(x_0)s + \ldots \tag{A.41}
$$

According to the corollary, from (A.26), the order $\epsilon^2$ term of the asymptotic expansion of the integral (A.38) is given by

$$
L_2(f_0) + v_2(f_0, x_0) + \frac{1}{2} \partial_s \partial^2_y f_0(0, 0) - \frac{1}{2x_0} \partial^2_y f_0(x_0, 0), \tag{A.42}
$$

where

$$
L_2(f_0) = -\int_0^\infty \partial_\zeta [\zeta^2 \partial_\zeta R_3(0, 1/\zeta)] d\zeta, \tag{A.43}
$$
$R_3$ is the remainder of Taylor expansion of $f_0(s, y)$ in $y$,

$$R_3(s, y) = f_0(s, y) - [f_0(s, 0) + y\partial_y f_0(s, 0) + \frac{1}{2} \partial_y^2 f_0(s, 0)y^2], \quad (A.44)$$

and

$$u_2(f_0, x_0) = -\frac{1}{2} \int_0^{x_0} \ln{s} \partial_y \partial_y^2 f_0(s, 0) ds. \quad (A.45)$$

We shall evaluate the four terms in (A.42) as follows: (1) Evaluate $L_2^2(f_0)$, (2) Evaluate $u_2(f_0, x_0)$, (3) Contribution from $\frac{1}{2} \partial_s \partial_y^2 f_0(0, 0)$, (4) Contribution from the fourth term.

(1) Evaluate $L_2^2(f_0)$

From the expression (A.39) of $f_0$, it is easy to see that

$$f_0(s, 0) = y\partial_y f_0(s, 0) = 0, \quad (A.46)$$

and

$$R_3(s, y) = f_0(s, y) - y^2 \hat{f}_0(s, 0) = y^2[\hat{f}_0(s, y) - \hat{f}_0(s, 0)], \quad (A.47)$$

where $\hat{f}_0$ is defined in (A.40).

To compute $\partial_s R_3(s, y)$, we find that

$$\partial_s [\hat{f}_0(s, y)] = -\frac{\psi'_0(s)/c^2}{[\psi^2_0(s) - 1/c^2 + 2ya_1(\psi_0(s) - 1/c)]^{3/2}}. \quad (A.48)$$

Then

$$\partial_s R_3(0, y) = -\frac{y^2}{c^2} \left\{ \frac{\psi'_0(0)}{[\psi^2_0(0) - 1/c^2 + 2ya_1(\psi_0(0) - 1/c)]^{3/2}} \right. \right.$$

$$- \left. \frac{\psi'_0(0)}{[\psi^2_0(0) - 1/c^2]^{3/2}} \right\}. \quad (A.49)$$

To compute $\partial_\zeta[\zeta^2 \partial_s R_3(0, 1/\zeta)]$, only the first term in (A.49) gives contribution. So that

$$\partial_\zeta[\zeta^2 \partial_s R_3(0, 1/\zeta)] = -\partial_\zeta[\frac{\psi'_0(0)/c^2}{[\psi^2_0(0) - 1/c^2 + 2ya_1(\psi_0(0) - 1/c)]^{3/2}}] \right.$$

$$\left. = -\left(\frac{3\hat{v}}{2c^3v^2}\right) \frac{\zeta^{1/2}}{[\zeta(1/v^2 - 1/c^2) + 2a_1(1/v - 1/c)]^{5/2}}, \quad (A.50)$$
where
\[ \psi_0(0) = \frac{1}{v(t)}, \quad \psi_0'(0) = \frac{\dot{v}(t)}{2(v(t))^3} \] (A.51)
are used.

Thus
\[
L_2(f_0) = -\int_0^\infty \partial_\zeta [\zeta^2 \partial_s R_3(0, 1/\zeta)] d\zeta
\]
\[
= \left( \frac{3\dot{v}}{2c^3 v^4} \right) \int_0^\infty \frac{\zeta^{1/2} \ln \zeta}{[\zeta(1/v^2 - 1/c^2) + 2a_1(1/v - 1/c)]^{3/2}} d\zeta. \tag{A.52}
\]
The last integral can be worked out. Consequently, we obtain the evaluation of \( L_2 \)
\[
L_2(f_0) = \frac{c\dot{v}}{2(c^2 - v^2)^{3/2}} \left\{ 1 + \frac{1}{2} \ln \left[ \frac{c\dot{v}}{2(c^2 - v^2)} \right] \right\}. \tag{A.53}
\]

(2) Evaluate \( u_2(f_0, x_0) \)

It is found that
\[
\partial_s^2 f_0(s, 0) = \frac{\psi_0(s)}{[\psi_0^2(s) - 1/c^2]^{1/2}}, \tag{A.54}
\]
\[
\partial_s \partial_s^2 f_0(s, 0) = -\frac{2\psi_0'(s)/c^2}{[\psi_0^2(s) - 1/c^2]^{3/2}}, \tag{A.55}
\]
where \( \psi(s) \) is defined in (A.17), and rewritten as
\[
\psi_0(s) = \frac{\eta(x_0) - \eta(\xi)}{x_0 - \xi} = \frac{t - \tau}{l(t) - l(\tau)}, \tag{A.56}
\]
and \( \psi_0' \) is given by
\[
\psi_0' = -\frac{\partial_t \partial_\zeta \partial s}{\partial_\zeta \partial s} = -\frac{(t - \tau)v(\tau) - (l(t) - l(\tau))}{v(\tau)(l(t) - l(\tau))^2}. \tag{A.57}
\]
Further, we find that
\[
\partial_s^2 \partial_s^2 f_0(s, 0) = -\partial_s \left[ \frac{2\psi_0'(s)/c^2}{[\psi_0^2(s) - 1/c^2]^{3/2}} \right]
\]
\[
= \frac{\psi_0'}{c^2[\psi_0^2 - 1/c^2]^{5/2}}. \tag{A.58}
\]
In the expression, $\psi_0''$ is given by

$$\psi_0'' = \frac{2v^3(t - \tau) - 2v^2(l(t) - l(\tau)) + \dot{v}(l(t) - l(\tau))^2}{v^3(l(t) - l(\tau))^2}, \quad (A.59)$$

with $v = \dot{l}(\tau)$; and $3\psi_0^2 - \psi_0^2\psi''$ is expressed as

$$3\psi_0^2 - \psi_0^2\psi'' = \frac{1}{v^3(l(t) - l(\tau))} \left\{ v^3(t - \tau)^2 - 4v^2(t - \tau)(l(t) - l(\tau)) 
+ 3v(l(t) - l(\tau))^3 - \dot{v}(t - \tau)(l(t) - l(\tau))^3 \right\}. \quad (A.60)$$

Because of the motion is subsonic, we have $c > v$,

$$\int_{\tau}^{t} c dt' > \int_{\tau}^{t} v(t') dt', \quad (A.61)$$

and

$$c(t - \tau) > l(t) - l(\tau). \quad (A.62)$$

So that

$$\psi_0^2 - 1/c^2 = \left( \frac{t - \tau}{l(t) - l(\tau)} \right)^2 - 1/c^2 > 0. \quad (A.63)$$

By using the expressions given here, it is easy to check the integral

$$u_2(f_0, x_0) = -\frac{1}{2} \int_{0}^{x_0} \ln s \left[ \frac{\psi_0(s) - 1/c^2}{\psi_0(s) - 1/c^2} \right] ds$$

$$= -\frac{1}{2} \int_{0}^{x_0} \ln \left[ \frac{\psi_0(s) - 1/c^2}{\psi_0(s)} \right] ds$$

$$= \frac{1}{2} \int_{0}^{t} \ln(l(t) - l(\tau)) [\partial_{\tau}^2\partial_y f_0(s, 0)]v(\tau) d\tau \quad (A.64)$$

is convergent.

Further, (A.59) is rewritten as

$$\psi_0'' = \frac{1}{v^3(l(t) - l(\tau))^2} \left[ 2v^2(t - \tau)(v - \frac{l(t) - l(\tau)}{t - \tau}) + \dot{v}(l(t) - l(\tau))^2 \right]$$

$$= \frac{1}{v^3(l(t) - l(\tau))^2} \left\{ v^2(t - \tau)^2[2\frac{\omega(t, \tau)}{t - \tau} + \dot{v}] \right. \right.$$  

$$\left. \left. - \dot{v}(t - \tau)^2[v^2 - (\frac{l(t) - l(\tau)}{t - \tau})^2] \right\}, \quad (A.65)$$

$$\psi_0'' = \frac{1}{v^3(l(t) - l(\tau))^2} \left[ 2v^2(t - \tau)(v - \frac{l(t) - l(\tau)}{t - \tau}) + \dot{v}(l(t) - l(\tau))^2 \right]$$

$$= \frac{1}{v^3(l(t) - l(\tau))^2} \left\{ v^2(t - \tau)^2[2\frac{\omega(t, \tau)}{t - \tau} + \dot{v}] \right. \right.$$  

$$\left. \left. - \dot{v}(t - \tau)^2[v^2 - (\frac{l(t) - l(\tau)}{t - \tau})^2] \right\}, \quad (A.65)$$
where again \( v = \dot{l}(t), \ \dot{v} = \ddot{l}(t) \), and \( \omega(t, \tau) \) is defined by
\[
\omega(t, \tau) \equiv v - \frac{l(t) - l(\tau)}{t - \tau}. \tag{A.66}
\]

It is easy to verify that in (A.65) every term is integrable. Assume now \( \dot{v}(\tau) \) has the Taylor expansion
\[
\dot{v}(\tau) = \dot{v}(t) + \ddot{v}(\theta)(\tau - t), \tag{A.67}
\]
where the Lagrange remainder is used for \( \theta \in [\tau, t] \). \( \psi''_0 \) is written as
\[
\psi''_0 = \frac{P_1}{v^3(l(t) - l(\tau))^3} + \frac{\dot{v}(t)[v^2 - (\frac{l(t) - l(\tau)}{t - \tau})^2]}{v^3(l(t) - l(\tau))^3}, \tag{A.68}
\]
where \( P_1 \) is defined by
\[
P_1 \equiv \frac{v^2(t - \tau)^2[2(\omega(t, \tau) + \dot{v}) - \dot{v}(\theta)(t - \tau)]^{2}v^2 - (\frac{l(t) - l(\tau)}{t - \tau})^2}. \tag{A.69}
\]

We note that terms in (A.68) are integrable on \([0, t]\), the second term explicitly depends on the acceleration \( \dot{v} \).

Similarly, \( 3\psi''_0^2 - \psi\psi''_0 \) is rewritten as
\[
3\psi''_0^2 - \psi\psi''_0 = \frac{P_2}{v^3(l(t) - l(\tau))^4} + \frac{\dot{v}(t)\omega(t, \tau)(t - \tau)^2(l(t) - l(\tau))}{v^3(l(t) - l(\tau))^4}, \tag{A.70}
\]
where
\[
P_2 \equiv \frac{v^2(t - \tau)^2\omega^2(t, \tau) - v(t - \tau)(l(t) - l(\tau))[2\omega(t, \tau) + \dot{v}(t - \tau)]}{v^3(l(t) - l(\tau))}\omega(t, \tau). \tag{A.71}
\]

It is clear that each term in (A.70) is integrable over \([0, t]\).

Hence from
\[
u_2(f_0, x_0) = -\frac{1}{2} \int_0^{x_0} \ln s \partial_s^2 \partial_y^2 f_0(s, 0) ds
\]
\[
= \frac{1}{2} \int_0^t \ln(l(t) - l(\tau))\left[\psi_0[3\psi''_0^2 - \psi\psi''_0] + \psi''_0/c^2 \right] v(\tau) d\tau, \tag{A.72}
\]
and using the expressions obtained above, we may combine the terms which explicitly depend on the acceleration, and have that
\[
u_2(f_0, x_0) = \dot{v}(t)U_1(t) + U_2(t), \tag{A.73}
\]
where

\[ U_1 = \frac{1}{2c^2} \int_0^t \left[ \frac{(t-\tau)^3}{v^2(l(t)-l(\tau))^4} \right] \left\{ \frac{(v - \frac{l(t)-l(\tau)}{t-\tau})[v^2 + v(l(t)-l(\tau)) + (\frac{l(t)-l(\tau)}{t-\tau})^2]}{[(\frac{l(t)-l(\tau)}{t-\tau})^2 - 1/c^2]^{3/2}} \right\} d\tau, \] (A.74)

\[ U_2 = \frac{1}{2c^2} \int_0^t \frac{1}{v^2} \left[ \frac{\ln(l(t)-l(\tau))}{[(\frac{l(t)-l(\tau)}{t-\tau})^2 - 1/c^2]^{5/2}} \right] \frac{t-\tau}{(l(t)-l(\tau))^3} P_2 + \frac{1}{c^2(l(t))(l(\tau))^3} P_1 d\tau, \] (A.75)

where \( P_1 \) and \( P_2 \) are defined in (A.91) and (A.92), respectively. \( U_1 \) and \( U_2 \) depend on the history of the motion of the dislocation.

(3) Contribution from \( \frac{1}{2} \partial_s \partial_y f_0(0,0) \)

From (A.55),

\[ \partial_s \partial_y f_0(s,0) = -\frac{2\psi_0'(s)/c^2}{\psi_0^2(s) - 1/c^2]^{3/2}}. \] (A.76)

As in (A.56) and (A.57), \( \psi(s) \) and \( \psi_0' \) are given by

\[ \psi_0(s) = \frac{\eta(x_0) - \eta(\xi)}{x_0 - \xi} = \frac{t-\tau}{l(t) - l(\tau)}, \] (A.77)

and

\[ \psi_0' = -\frac{\partial \tau \partial \psi_0}{\partial \xi \partial \tau} = -\frac{(t-\tau)v(\tau) - (l(t) - l(\tau))}{v(\tau)(l(t) - l(\tau))^2}. \] (A.78)

In the above expressions, taking limit of \( s \to 0 \), or equivalently, \( \xi \to x_0 \), we obtain

\[ \psi_0(0) = 1/v(t), \quad \psi_0'(0) = -1/2\dot{v}/v^3; \] (A.79)

and

\[ \frac{1}{2} \partial_s \partial_y f_0(0,0) = -\frac{\dot{v}c}{2(c^2 - v^2)^{3/2}}. \] (A.80)

(4) Contribution from the forth term

From (A.39),

\[ f_0 = y^2 \hat{f}_0(s,y), \] (A.81)
and
\[ \tilde{f}_0 \equiv \frac{\psi_0(s) + y_{a_1}}{(1 + cy_{a_1})^2[(\psi_0(s)^2 - 1/c^2) + 2y_{a_1}(\psi_0(s) - 1/c)^2]^{\frac{1}{2}}} \]  
(A.82)

we have the forth term in (A.42),
\[ -\frac{1}{2x_0} \partial_y^2 f_0(x_0, 0) = -\frac{1}{x_0} \tilde{f}_0(x_0, 0) = -\frac{1}{x_0} \frac{\psi_0(x_0)}{[\psi_0(x_0)^2 - 1/c^2]^{1/2}}. \]  
(A.83)

Here \( \psi_0(x_0) = \psi_0(s)|_{s=0} \) is equal to
\[ \psi_0(x_0) = \frac{\eta(x_0) - \eta(0)}{x_0} = \frac{t}{l(t)}. \]  
(A.84)

Hence
\[ -\frac{1}{2x_0} \partial_y^2 f_0(x_0, 0) = -\frac{ct}{(c^2t^2 - l^2(t))^{1/2}}. \]  
(A.85)

(5) Summary

We therefore complete the calculation for (A), i.e., the \( \epsilon^2 \) terms of the asymptotic expansion of \( \int_0^x f_0(s, y) ds \) as \( \epsilon \to 0 \), which are given by
\[ L_2(f_0) + u_2(f_0, x_0) + \frac{1}{2} \partial_s \partial_y^2 f_0(0, 0) - \frac{1}{2x_0} \partial_y^2 f_0(x_0, 0). \]  
(A.86)

Those terms are evaluated as follows.
\[ L_2(f_0) = \frac{c\dot{v}(t)}{2(c^2 - v^2(t))^{3/2}} [1 + \frac{1}{2} \ln \left( \frac{cv}{2(c^2 - v^2(t))} \right)]. \]  
(A.87)
\[ u_2(f_0, x_0) = \dot{v}(t)U_1(t) + U_2(t), \]  
(A.88)

where
\[ U_1 = \frac{1}{2c^2} \int_0^t \left[ \frac{1}{v^2(l(t) - l(\tau))} \frac{(t - \tau)^3}{v^2(l(t) - l(\tau))^{1/2}} \right] \left[ \frac{(v - l(t) - l(\tau))}{l(t) - l(\tau)} \right] \]  
\[ \left. \frac{(v - l(t) - l(\tau))^2 + \left( \frac{l(t) - l(\tau)}{l(t) - l(\tau)} \right)^2}{[(l(t) - l(\tau))^2 - 1/c^2]^{1/2}} \right] d\tau, \]  
(A.89)
\[ U_2 = \frac{1}{2c^2} \int_0^t \frac{1}{v^2} \left[ \frac{\ln(l(t) - l(\tau))}{[(l(t) - l(\tau))^2 - 1/c^2]^{5/2}} \right] \right. \]  
\[ \left. \left[ \frac{l(t) - l(\tau)}{l(t) - l(\tau)} \right] P_2 + \frac{1}{c^2(l(t)l(\tau))^3} P_1 \right] d\tau, \]  
(A.90)
with
\[ P_1 = v^2(t - \tau)^2[2(\omega(t, \tau) + v) - \dot{v}(t - \tau)^2[v^2 - (\frac{1(t) - l(\tau)}{t - \tau})^2], \] (A.91)

and
\[ P_2 = v^2(t - \tau)^2(\omega^2(t, \tau) - v(t - \tau)(l(t) - l(\tau))[2\omega(t, \tau) + \dot{v}(t - \tau)] - \dot{v}(t - \tau)^2(l(t) - l(\tau))\omega(t, \tau). \] (A.92)

\[ \frac{1}{2} \partial_y \partial_y f_0(0, 0) = -\frac{\dot{v}c}{2(c^2 - v^2)^{3/2}.} \] (A.93)

\[ -\frac{1}{2x_0} \partial_y^2 f_0(x_0, 0) = -\frac{ct}{l(t)(c^2t^2 - l^2(t))^{1/2}.} \] (A.94)

### A.C.4 Evaluation of (B)

(1) Expression of \( f_1 \)

Now we shall evaluate the order \( \epsilon \) term of \( \int_0^{x_0} f_1(s, y)ds \), where \( f_1(s, y) \) is defined by the \( \epsilon \) term in the Taylor expansion (A.33) of \( f(s, y, \epsilon) \). \( f \) is defined by (A.31) and (A.30), given by

\[ f = y^2\left\{ \frac{\psi(s) + yA_0(t, \epsilon)}{(1 + cyA_0(t, \epsilon))^2[(\psi^2 - 1/c^2) + 2yA_0(t, \epsilon)(\psi - 1/c)]^{1/2}} \right\}. \] (A.95)

Then \( f_1(s, y) = [\partial_y f(s, y, \epsilon)]|_{\epsilon=0} \) can be computed to be

\[ f_1(s, y) = y^2 \left\{ \frac{\psi(s) + yA_0(t, \epsilon)}{(1 + cyA_0(t, \epsilon))^2[(\psi^2(s) - 1/c^2) + 2yA_0(t, \epsilon)(\psi - 1/c)]^{1/2}} \right\} - \frac{2cyA_1(\psi_0(s) + ya_1)}{(1 + cyA_2)^2[(\psi_0^2(s) - 1/c^2) + 2yA_0(s)(\psi - 1/c)]^{3/2}} \]

\[ -\frac{(\psi_0(s) + ya_1)(\psi_0(s) + ya_1)(\psi_0(s) - ya_2(\psi_0 - 1/c))}{(1 + cyA_2)^2[(\psi_0^2(s) - 1/c^2) + 2yA_0(s)(\psi - 1/c)]^{3/2}} \] (A.96)

where \( a_1 = [A_0]|_{\epsilon=0} \) and \( a_2 = [\partial_y A_0]|_{\epsilon=0} \) are defined in (A.13) and (A.14), respectively. It is easy to verify that \( f_1 \) is integrable over \([0, x_0]\).

According to the Corollary, the \( \epsilon \)-terms of the integral of \( f_1 \) is given by

\[ L_1(f_1) + u_1(f_1, x_0) + \partial_y f_1(0, 0) + \frac{1}{x_0} \partial_y f_1(x_0, 0), \] (A.97)
where
\[ L_1(f_1) = -\int_0^\infty \ln \xi \partial_\xi [\xi R_2(0, 1/\xi)] d\xi, \]  
(A.98)

and
\[ u_1(f_1, x_0) = -\int_0^{x_0} \ln s \partial_s \partial_y f_1(s, 0) ds. \]  
(A.99)

We shall see that except the first term, the rest three terms gives trivial contribution.

(2) **Trivial Terms**

From (A.96),
\[ \partial_y f_1(s, 0) = 0, \]  
(A.100)

hence in (A.97) the last two terms give zero contribution. Further
\[ u_1(f_1, x_0) = -\int_0^{x_0} \ln s \partial_s \partial_y f_1(s, 0) ds = 0. \]  
(A.101)

(3) **Non-Trivial Term: \(L_1\)**

Then, only \(L_1(f_1)\) gives non-zero contribution.
\[
L_1(f_1) = -\int_0^\infty \ln \xi \partial_\xi [\xi R_2(0, 1/\xi)] d\xi
= -\xi \ln \xi R_2(0, 1/\xi) + \int_0^\infty R_2(0, 1/\xi) d\xi.
\]  
(A.102)

In the equation,
\[ R_2(s, y) = f_1(s, y) - [f_1(s, 0) + \partial_y f_1(s, 0)y] = f_1(s, y). \]  
(A.103)

From the expression of \(f_1\), it follows that, as \(\xi \to 0\),
\[ R_2(s, 1/\xi) = f(s, 1/\xi) \sim 1/\sqrt{\xi}, \]  
(A.104)
and as $\xi \to \infty$,

$$R_2(s, 1/\xi) = f(s, 1/\xi) \sim 1/\xi^2.$$  \hspace{1cm} (A.105)

So that the improper integral $L_1(f_1)$ is convergent. We denote that

$$L_1(f_1) = \int_{0}^{\infty} f_1(0, 1/\xi) d\xi \equiv I_{(1)} + I_{(2)} + I_{(3)},$$  \hspace{1cm} (A.106)

where

$$I_{(1)} = \int_{0}^{\infty} \frac{\xi \psi_e + a_2}{\sqrt{\xi(\xi + ca_1)^2[(\psi_0^2 - 1/c^2)\xi + 2a_1(\psi_0 - 1/c)]^{1/2}}} d\xi,$$  \hspace{1cm} (A.107)

$$I_{(2)} = -2ca_2 \int_{0}^{\infty} \frac{\xi \psi_0 + a_1}{\sqrt{\xi(\xi + ca_1)^2[(\psi_0^2 - 1/c^2)\xi + 2a_1(\psi_0 - 1/c)]^{1/2}}} d\xi,$$  \hspace{1cm} (A.108)

and

$$I_{(3)} = -\int_{0}^{\infty} \frac{(\xi \psi_0 + a_1)(\psi_e(\xi \psi_0 + a_1) + a_2(\psi_0 - 1/c))}{\sqrt{\xi(\xi + ca_1)^2[(\psi_0^2 - 1/c^2)\xi + 2a_1(\psi_0 - 1/c)]^{3/2}}} d\xi.$$  \hspace{1cm} (A.109)

Those three integrals can be evaluated explicitly,

$$I_{(1)} = \frac{\dot{v}}{2(c - v)^3} \left[ (v - 2c) \ln\left(\frac{c + \sqrt{c^2 - v^2}}{v}\right) + \frac{\sqrt{c^2 - v^2}(c - v)^2}{cv} + \frac{c\sqrt{c^2 - v^2}}{v}\right],$$  \hspace{1cm} (A.110)

$$I_{(2)} = \frac{\dot{v}}{2} \left[ \frac{\sqrt{c^2 - v^2}}{(c - v)^3} - \frac{v(c + v)(2c + v)}{(c^2 - v^2)(c - v)^3} \ln\left(\frac{c + \sqrt{c^2 - v^2}}{v}\right),$$  \hspace{1cm} (A.111)

and

$$I_{(3)} = -\frac{2c^3 \dot{v}}{c(c - v)(c^2 - v^2)^{3/2}} \left[ \frac{(c_v)^2(2c + v)}{4c^2v} + \frac{3(c + v)^2(c - v)}{4c^3} + \frac{v^3(c + v)}{c^4} \right.$$

$$\left. - \frac{(c + v)(2c^2 - cv + v^2)\sqrt{c^2 - v^2}}{4c^3(c - v)} \ln\left(\frac{c + \sqrt{c^2 - v^2}}{v}\right) \right].$$  \hspace{1cm} (A.112)

The $\epsilon$-terms of the integral of $f_1$ is

$$I_{(1)} + I_{(2)} + I_{(3)} = \frac{\dot{v}\sqrt{c^2 - v^2}v(c - v)}{(c - v)^3 \left[ \frac{c(c + v)}{c(c + v)} - \frac{c^2(c + 2v)}{v^2(c + v)} \right]$$

$$+ \frac{\dot{v}}{(c - v)^3} \frac{c(2c - 5v)}{2(c - v)} + \frac{3v^2}{2c} \ln\left(\frac{c + \sqrt{c^2 - v^2}}{v}\right).$$  \hspace{1cm} (A.113)
A.C.5 Evaluation of (C)

To evaluate the $O(1)$ terms of the integral

$$\int_{0}^{x_0} f_2(s, y) ds,$$  \hspace{1cm} (A.114)

we use the Corollary again, and the value is given by

$$\int_{0}^{x_0} f_2(s, 0) ds.$$  \hspace{1cm} (A.115)

From the definition $f_2(s, y) = \frac{1}{2} \frac{\partial^2 f(s, y, \epsilon)}{\partial \epsilon^2}|_{\epsilon=0}$, and it is easy to see that $f_2$ has a factor of $y^2$. Thus $f_2(s, 0) = 0$, and

$$\int_{0}^{x_0} f_2(s, 0) ds = 0.$$  \hspace{1cm} (A.116)

A.D Conclusion

Combing results obtained in previous discussion, we obtain the evaluation of $g_{32}(0)$,

$$g_{32}(0) = \frac{b}{2\pi} \left\{ \begin{array}{l}
\frac{c\dot{v}(t)}{4(c^2-v^2(t))^{3/2}} \ln \left( \frac{cv}{2(c^2-v^2)} \right) \\
+ \frac{\dot{v} \sqrt{c^2-v^2} v(c-v)}{(c-v)^3} \left[ \frac{c(c+v)}{v^2(c+v)} - \frac{c^2(c+2v)}{v^2(c+v)} \right] \\
+ \frac{\dot{v}}{(c-v)^3} \left[ \frac{c(2c-5v)}{2(c-v)} + \frac{3v^2}{2c} \ln \left( \frac{c + \sqrt{c^2-v^2}}{v} \right) \right] \\
+ \frac{\dot{v}}{2c^2} \int_{0}^{t} \ln(l(t)-l(\tau)) \frac{l(t)-l(\tau)}{v^2} d\tau \\
- \frac{v(t) - \frac{l(t)-l(\tau)}{\tau}}{(t-\tau)^2} \int_{0}^{\tau} \left[ c^2 + v(t) \left( \frac{l(t)-l(\tau)}{l(t)-l(\tau)} \right)^2 + \left( \frac{l(t)-l(\tau)}{l(t)-l(\tau)} \right)^2 \right] d\tau \\
- \frac{1}{l(t)[1 - \frac{c^2[t]}{c^2(t)^2}]^{1/2}} + \frac{1}{l(t)} \end{array} \right\}$$
\[
\frac{1}{2c^2} \int_0^t \frac{1}{v^2} \left[ 1 - \frac{(t-\tau)^2/c^2}{(l(t) - l(\tau))^2} \right]^{5/2} \left[ \frac{t-\tau}{(l(t) - l(\tau))^2} P_2 + \frac{1}{c^2(l(t) - l(\tau))} P_1 \right] d\tau, \tag{A.117}
\]

where \( P_1 \) and \( P_2 \) are defined by

\[
P_1 = v^2 \left[ 2 \left( \frac{\omega(t, \tau)}{t - \tau} \right) + \dot{v} \right] - \ddot{v}(\theta) \left[ v^2 - \left( \frac{l(t) - l(\tau)}{t - \tau} \right)^2 \right], \tag{A.118}
\]

\[
P_2 \equiv v^2 \omega^2(t, \tau) - v(t) \left( \frac{l(t) - l(\tau)}{t - \tau} \right) \left[ 2 \omega(t, \tau) + \dot{v}(t)(t - \tau) \right] - \ddot{v}(\theta)(l(t) - l(\tau))\omega(t, \tau), \tag{A.119}
\]

and \( \tau \leq \theta \leq t \), and \( \omega(t, \tau) \) is defined by

\[
\omega(t, \tau) \equiv v(t) - \frac{l(t) - l(\tau)}{t - \tau}. \tag{A.120}
\]

In (A.117), except the last two terms, all other terms has an explicit factor of \( \dot{v} \), which will contribute to the effective mass of the moving screw dislocation. The integral terms in the expression depend on the history of motion.
Bibliography


