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The Two Moment Closure In Radiation Transport Resurrected †

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ABSTRACT

Previously, it was found that the set of first two angular moments of the Boltzmann equation coupled to a nonlinear closure is numerically unstable for a rather large class of closures. I present solutions of this set of equations using the maximum entropy closure which belongs to this 'forbidden' class. The solutions are accurate, stationary and stable. This proof of existence by construction would seem to cast doubt on the generality of the conclusions reached previously, which doomed the two moment approach with physically motivated closures, and indicates that additional constraints beyond the usual ones on the admissibility of particular closures are not called for.
1 Introduction

In practical numerical applications of radiation transport, solving the Boltzmann equation is often too costly. This is for example the case in numerical simulations of supernova explosions and the early neutrino driven neutron star formation and cooling phase, in which neutrino transport is coupled to a dynamically evolving stellar environment. Instead, Eddington moment approaches which consider a hierarchy of angular moments of the Boltzmann equation, at each order containing angular moments of the distribution function of at least one order higher, are used. At any order of truncation there is always one variable more than there are equations, and an additional relation between the Eddington factors is required to close the set.

Ideally, the first two moments of the Boltzmann equation are considered, taking account of the conservation of radiation energy and momentum, respectively. For a spherically symmetric system, a static matter background in Newtonian gravity and excluding inelastic scattering and pair processes, these are given by

\[
\begin{align*}
\partial_t e + \frac{1}{r^2} \partial_r \left( r^2 e f \right) &= \kappa_a \left( e^0 - e \right), \\
\partial_t F + \partial_r P + \frac{(3P - e)}{r} &= -\kappa_{tot} F.
\end{align*}
\] (1.1) (1.2)

The variables are the radiation energy density (distribution) \( e \), the zeroth angular moment of the distribution function, the radiation flux \( F = ef \), containing the first Eddington moment \( f \), and the radiation pressure, in this case the \( P_{rr} = P = p\ e \) component of the pressure tensor containing the second Eddington moment \( p \). The quantities \( \kappa_a \) and \( \kappa_{tot} \) denote the absorption opacity and the total opacity, including elastic scattering on heavy particles \( \kappa_a \). The equilibrium particle distribution at ambient matter temperature \( T \) and equilibrium chemical potential \( \mu_{\nu} \) is denoted by \( e^0 = (\exp[(\omega - \mu_{\nu})/T] - 1)^{-1} \), for fermions. The Eddington factors \( e, f, p \) as well as the opacities are functions of the particle energy \( \omega \), radial position \( r \) and time \( t \).

In the two-moment closure, or P-1, approach, the coupled set of equations (1.1,1.2) together with a closure \( p = p(e,f) \) is solved. With the use of a nonlinear variable Eddington factor \( p(e,f) \) the (numerical) properties of the ensuing set of nonlinear partial differential equations are neither a priori transparent, nor is numerical stability guaranteed. Previously, Körner and Janka (1992) found this approach to be numerically unstable for a large class of closures and analyzed its behaviour to uncover the cause of the insta-
bility. From this analysis they derived an additional constraint on closures to guarantee numerical stability.

In the following paragraph their analysis is reviewed and slightly generalized. A summary of its conclusions and predictions is given. In paragraph 3 the closure that was used in the calculation is introduced and shown to be of the 'unstable' class. In paragraph 4 the results of transport calculations on a simplified neutron star stellar background are presented, and it is demonstrated that the solutions are accurate, stable and stationary. The solutions are shown to be 'regular' according to the definition of Körner and Janka (1992), stable against perturbations, and calculable using standard numerical methods.

2 Stability Analysis

We are interested in the stationary solution, and therefore drop the time-derivatives in (1.1,1.2). With the closure, we may write out the spatial derivative in (1.2) as

$$\partial_r P = p(e, f) \partial_r e + e \left[ \frac{\partial p}{\partial f} \partial_r f + \frac{\partial p}{\partial e} \partial_r e \right].$$

Eliminating $\partial_r f$ from Eq. (1.2) with Eq. (1.1) we arrive at

$$J_1(e, f, p) \partial_r e = J_2(e, f, p)$$

with

$$J_1(e, f, p) \equiv p - f \frac{\partial p}{\partial f} + e \frac{\partial p}{\partial e},$$

$$J_2(e, f, p) \equiv \frac{\partial p}{\partial f} \left[ \frac{2e f}{r} - \kappa_a (e^0 - e) \right] - e \frac{3p - 1}{r} - \kappa_{tot} e f.$$

Körner and Janka (1992) argued that the set will develop a singularity when $J_1(r) = 0$ somewhere in the star, unless $J_2$ is zero simultaneously. The solution for which the latter occurs is called 'regular'. They predict the existence of the regular solution in principle, but argue that it is unstable against perturbations and in practice is unaccessible by common discrete mesh methods.

The exposition given here is not restricted to one-dimensional closures of the form $p(f)$, but allows the more general two-dimensional form $p(e, f)$. The stability criterion may be phrased slightly differently: for many closures $J_1(r)$ will change sign somewhere in the star, say at $r = r_0$. The solution is regular if $J_2(r)$ changes sign at the same point. In general, $\partial_r e$ will be negative because radiation occupation density is usually
a (monotonically) decreasing function of radial position in most stellar environments. In order for the sign of \( \partial_r e \) to be consistent, according to equation (2.2) \( J_1 \) and \( J_2 \) need to change sign simultaneously in the stationary case. Note however, that the spatial derivative of \( e \) is of course not determined through equation (2.2) in practical calculations.

Since the closure is in general nonlinear, deciding on the stability of a particular solution of the set of equations with analytical rigor is impossible, because no analytical stability analysis for a set of nonlinear partial differential equations exists. Therefore, no definitive conclusions about the instability of the set can be made, since such conclusions can be inferred from numerical experimentation at best. In contrast, positive proof of stability can be given numerically by construction of a stable regular solution, which meets the following requirements:

- For a given closure, \( J_1 \) must change sign, so that it is of the ‘unstable’ class.
- The solution must be stationary: all time-derivatives must be small compared to other terms in the equations.
- The solution must be regular and stable: the system must converge towards it and not away from it, and the Eddington factors should show no ‘wiggles’.
- The solution must be accurate: all equations must be satisfied to some predefined (high) accuracy.

### 3 Fermi-Dirac Maximum Entropy Closure

The maximum entropy Fermi-Dirac closure (MEC-FD) has been derived and discussed by Cernohorsky and Bludman (1994), and is given by

\[
p(e, f) = \frac{2(1 - e)(1 - 2e)}{3} \chi\left(\frac{f}{1 - e}\right) + \frac{1}{3}
\]  

(3.1)

where

\[
\chi(x) = 1 - 3 x / q(x)
\]  

(3.2)

with \( q(x) \) the inverse of the Langevin function \( x = \coth q - 1/q \).

The lowest order polynomial approximation to equation (3.2), which has the correct behaviour in the free-streaming and diffusive limits and contains no free parameters, is

\[
\chi(x) = x^2(3 - x + 3x^2)/5
\]  

(3.3)
It is accurate to 2% and because of its simplicity has been used in the numerical experiment instead of the exact expression (3.2).

Without going into the details and properties of MEC-FD, for which the reader is referred to Janka, Dgani and van den Horn (1992) and Cernohorsky and Bludman (1994), it suffices to note here that it takes into account the Fermi-Dirac quantum statistics of the neutrino radiation, in particular the exclusion principle which for a given occupation density $0 \leq e \leq 1$ limits the flux $f \leq 1 - e$ and the pressure $p \leq 1 - 2e + (4/3)e^2$.

![Figure 1: $J_1$ as a function of flux-saturation $x = f/(1-e)$ for, from bottom to top, $e = 0.001$ (full thin line), $e = 0.01 e = 0.99$ (dotted), $e = 0.1 e = 0.9$ (dashed), $e = 0.2 e = 0.8$ (long-dashed), $e = 0.3 e = 0.7$ (dash-dotted), $e = 0.4 e = 0.6$ (thick dotted), $e = 0.5$ (thick full line). $J_1$ is $e \to 1 - e$-invariant : $J_1(e, x) = J_1(1 - e, x)$.](image)

To show that $J_1$ changes sign in MEC-FD, $J_1(e, f, p)$ is plotted as a function of flux saturation $x = f/(1-e)$, in figure 1. As occupancy decreases, for $e \neq 0.5$ there is always a value of $x$ for which $J_1$ changes sign, whereas for $x < 2/3$ it is positive for any $e$. Moving outward to lower matter density in a neutron star, the neutrino mean free path increases, the occupancy $e$ decreases and $f$ increases, so that $J_1$ generally changes sign somewhere within the star.
4 Numerical Results

The transport calculation is spectral, so that all Eddington factors are functions of particle energy \( \omega \), as are the opacities. For the simplified problem at hand there is no bin-coupling, and the transport is monochromatic. For the proof it is sufficient to demonstrate the stability of a single energy bin, although the calculation is done for a spectrum of 14 bins between 0.7 and 215 MeV.

Figure 2: \( J_1(\rho) \) (thick full line) and \( 10^5 \times J_2(\rho) \) (thin full line) as functions of radial position for the \( \omega = 8.77 \) MeV bin, in the stationary state. The two functions change sign simultaneously at \( \rho_0 = 43 km \).

Although arguably details of the numerical method are decisive for stability, I will not describe the numerical details of the calculation in this letter, but delegate that to a more elaborate forthcoming paper in which the results of transport using different closures are compared, and their physical impact is discussed. The conservation equations (1.1,1.2), together with the closure (3.1) are solved as a set of three separate equations. The energy balance equation is cast into conservative form, the remaining equations are treated as in Cernohorsky and van Weert (1992). The solution package is fully implicit. The matrix inversion uses a sophisticated sparse-matrix package due to Blinnikov and Bartunov (1993), based on a method by Østerby and Zlatev (1983). Given a fixed stellar background and starting from an initial guess for the Eddington factor profiles, the equations are evolved.
in time until the stationary solution is reached. The results and discussion all refer to the stationary state and focus on the energy bin $\omega = 8.7$ MeV. This bin contributes most to the emergent luminosity, and lies near the peak of the emitted spectrum. The stellar background model is model M0 from Cernohorsky and van Weert (1992).

Figure 3: The Eddington factors $e$ (full), $f$ (short dashed) and $p$ (long dashed) as functions of radial position for the $\omega = 8.77$ MeV bin in the stationary state. The vertical dotted line at $r_0 = 43$ km denotes the point where $J_1$ and $J_2$ change sign, the position of the 'singularity'. There is no hint of unstable or singular behaviour.

The functions $J_1(r)$ and $J_2(r)$ are plotted as functions of positions in figure 2. The curves change sign simultaneously, between $42.8 < r_0 < 43.1$ km, up to spatial resolution of the grid. The solution is regular. In Fig. 3 the Eddington factors $e$, $f$ and $p$ are plotted as functions of radial position.

The equations are satisfied to better than one part in $10^7$, the Eddington factors are stationary to better than one part in $10^6$. Normalization is in both cases on the largest term in the equations, other than the time-derivative. The pressure $\partial_t p/p$ is stationary to 13 decimal places. The accuracies cited are for the whole 14-bin ensemble. For the 8.7 MeV bin the accuracies are about 2 orders of magnitude better. There are no signs of instability.
5 Conclusions

I have demonstrated that for MEC-FD, $J_1(r)$ changes sign. According to Körner and Janka (1992), the use of MEC-FD as a closure for the conservation equations should be numerically unstable, and a stationary solution unattainable with standard numerical methods.

Instead, the system naturally converges on the regular solution, the equations are satisfied and the Eddington factors are stationary to high accuracy and show no hint of instability.

The stability is not due to the two-dimensional nature of MEC-FD. To verify this I have implemented in addition to MEC-FD two one-dimensional phenomenological Monte-Carlo closures, $\Lambda_{MC1}$ and $\Lambda_{MC2}$ from Janka (1992), for which $J_1$ also changes sign. These are found to behave similarly to MEC-FD, their solutions are regular and stable.

The analysis of Körner and Janka (1992) correctly predicts the existence of the regular solution, which is accurately realised by the set of equations, thus avoiding the singularity.

I conclude that the two-moment P-1 approach with closures for which $J_1$ changes sign can be formulated in a numerically stable fashion. In addition to the usual moment conditions as discussed in Levermore (1984), the additional constraint on admissible closures, $J_1(e, f, p) > 0$ for all $e, f$ and $p$, which excludes many physically interesting ones, is unwarranted.
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