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Authors
Cao, X
Lu, M
Wan, D
et al.

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LINEARIZED WENGER GRAPHS

XIANG CAO, MEI LU, DAQING WANG, LI-PING WANG, QIANG WANG

Abstract. Motivated by recent extensive studies on Wenger graphs, we introduce a new infinite class of bipartite graphs of the similar type, called linearized Wenger graphs. The spectrum, diameter and girth of these linearized Wenger graphs are determined.

1. Introduction

Let \( \mathbb{F}_q \) be a finite field of order \( q \) such that \( p \) is prime and \( q = p^e \) a prime power. All graph theory notions can be found in Bollobás [2]. Recently, a class of bipartite graphs called Wenger graphs which are defined over \( \mathbb{F}_q \) has attracted a lot of attention because of their nice graphical properties [5, 11, 12, 16, 18, 19, 20, 21]. For example, the number of edges of these graphs meets the lower bound of Turán number of the cycle with length 4, 6, 10 \([21]\). The original definition was introduced by Wenger [21] for \( p \)-regular bipartite graphs and then was extended by Lazbnik and Ustimenko [11] for arbitrary prime power \( q \). An equivalent representation of these graphs appeared later in Lazebnik and Viglione [13] and then a more general class of graphs was defined in [19], on which we concentrate in this paper.

Let \( m \geq 1 \) be a positive integer and \( g_k(x, y) \in \mathbb{F}_q[x, y] \) for \( 2 \leq k \leq m + 1 \). Let \( \mathcal{P} = \mathbb{F}_q^{m+1} \) and \( \mathcal{L} = \mathbb{F}_q^{m+1} \) be two copies of the \((m+1)\)-dimensional vector space over \( \mathbb{F}_q \), which are called the point set and the line set respectively. Let \( \mathcal{G} = G_q(g_2, \ldots, g_{m+1}) = (V, E) \) be the graph with vertex set \( V = \mathcal{P} \cup \mathcal{L} \) and the edge set \( E \) is defined as follow: there is an edge from a point \( P = (p_1, p_2, \ldots, p_{m+1}) \in \mathcal{P} \) to a line \( L = [l_1, l_2, \ldots, l_{m+1}] \in \mathcal{L} \), denoted by \( P \sim L \) (we force \( \mathcal{G} \) to be a undirected graph by removing the arrows), if the following \( m \) equalities hold:

\[
\begin{align*}
    l_2 + p_2 &= g_2(p_1, l_1) \\
    l_3 + p_3 &= g_3(p_1, l_1) \\
    &\vdots \\
    l_{m+1} + p_{m+1} &= g_{m+1}(p_1, l_1).
\end{align*}
\]

If \( g_k(x, y), k = 2, \ldots, m + 1 \), are all monomials, the graph is called a monomial graph; see [6]. If \( g_k(x, y) = x^{k-1}y, k = 2, \ldots, m + 1 \), then the graph is just the original Wenger graph in [5], also denoted by \( W_m(q) \). It was shown in [11] that the automorphism group of \( W_m(q) \) acts transitively on each of \( \mathcal{P} \) and \( \mathcal{L} \), and on the set of edges of \( W_m(q) \). In other words, the graphs \( W_m(q) \) are point-, line-, and edge-transitive. It is also shown that, see [12], \( W_1(q) \) is

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vertex-transitive for all \( q \), and that \( W_2(q) \) is vertex-transitive for even \( q \). For all \( m \geq 3 \) and \( q \geq 3 \), and for \( m = 2 \) and all odd \( q \), the graphs \( W_m(q) \) are not vertex-transitive. Another result of [12] is that \( W_m(q) \) is connected when \( 1 \leq m \leq q - 1 \), and disconnected when \( m \geq q \), in which case it has \( q^{m-q+1} \) components, each isomorphic to \( W_{q-1}(q) \). In [20], Viglione proved that the diameter of \( W_m(q) \) is \( 2m + 2 \) when \( 1 \leq m \leq q - 1 \). In [5], Cioabă, Lazebnik and Li determined the spectrum of \( W_m(q) \).

In this paper we focus on the basic properties of some extensions of Wenger graphs defined as in Equation (1.1). In Section 2 we first study the spectrum of a general class of graphs such that polynomials \( g_k(x, y) \in \mathbb{F}_q[x, y] \) are defined by \( g_k(x, y) = f_k(x)y \), and the mapping \( \vartheta : \mathbb{F}_q \to \mathbb{F}_q^{m+1} : u \mapsto (1, f_2(u), \cdots, f_{m+1}(u)) \) is injective. The eigenvalues of such a graph are determined, however, their multiplicities are reduced to counting certain polynomials with a given number of roots over finite fields. The latter problem is an interesting number theoretical problem, which is expected to be difficult in general. A complete solution in interesting special cases is already significant. In particular, we introduce a new class of bipartite graphs called linearized Wenger graphs. These graphs are denoted by \( L_m(q) \), which are defined by Equation (1.1) together with \( g_k(x, y) = x^{p^{k-2}}y, k = 2, \cdots, m + 1 \). Using results on linearized polynomials over finite fields, we are able to explicitly determine the spectrum of such graphs when \( m \geq e \) in Section 3. Finally we obtain the diameter and girth of linearized Wenger graphs in Section 4 and Section 5, respectively. As a consequence, when \( m = e \), this provides a new class of infinitely many connected \( p^e \)-regular expander graphs of \( q^{2m+2} \) vertices with optimal diameter \( 2(m + 1) \) when either the prime \( p \) or the exponent \( e \) goes to infinity.

2. THE SPECTRUM OF GENERAL WENGER GRAPHS

In this section we study the basic properties of the class of graphs \( \mathcal{G} \) defined by \( g_k(x, y) = f_k(x)y \), where \( g_k(x, y) \) is a product of a polynomial in terms of \( x \) and the linear polynomial\( y \), for \( 2 \leq k \leq m + 1 \).

**Proposition 2.1.** The graph \( \mathcal{G} = G_q(f_2(x)y, \ldots, f_{m+1}(x)y) \) is \( q \)-regular.

**Proof.** Given a point \( P \) and a line \( L \) in \( V \), by definition, \( P = (p_1, p_2, \cdots, p_{m+1}) \) is adjacent to \( L = [l_1, l_2, \cdots, l_{m+1}] \) if and only if the following \( m \) equalities hold:

\[
\begin{align*}
\begin{cases}
    l_2 + p_2 &= f_2(p_1)l_1 \\
    l_3 + p_3 &= f_3(p_1)l_1 \\
    \vdots & \vdots \\
    l_{m+1} + p_{m+1} &= f_{m+1}(p_1)l_1.
\end{cases}
\end{align*}
\]

(2.1)

When the point \( P \) is prescribed, (2.1) implies that one can uniquely solve \( l_k \ (k \geq 2) \) from \( l_1 \), and thus (2.1) has \( q \) solutions. Similarly, when the point \( L \) is prescribed, (2.1) implies that one can uniquely solve \( p_k \ (k \geq 2) \) from \( p_1 \), and thus (2.1) has \( q \) solutions. \( \square \)
Since $\mathcal{G}$ is a bipartite graph, its adjacency matrix is of the form:

$$A = \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix}$$

with a matrix $N$ and

$$A^2 = \begin{pmatrix} NN^T & 0 \\ 0 & N^T N \end{pmatrix}. \quad (2.2)$$

In order to consider the properties of $\mathcal{G}$, we define a graph $H$ as follows: the vertex set is $\mathbb{F}_q^{m+1}$ containing all lines in $\mathcal{G}$, any two lines $L = [l_1, l_2, \ldots, l_{m+1}]$ and $L' = [l'_1, l'_2, \ldots, l'_{m+1}]$ are adjacent if and only if they share a common neighbor point $P = (p_1, p_2, \ldots, p_{m+1})$ in the graph $\mathcal{G}$ defined above.

Moreover, one can check that the graph $H$ is a Cayley graph with the generating set

$$S = \{(t, t_1, t_2, \ldots, t_{m+1}) | t \in \mathbb{F}_q^*, u \in \mathbb{F}_q\}.$$ 

Indeed, $L \sim L'$ if and only if $l_k - l'_k = f_k(p_1, l_1) - f_k(p_1, l'_1) = f_k(l_1 - l'_1)$ for $2 \leq k \leq m + 1$.

Furthermore, if $B$ is the adjacency matrix of $H$ then

$$NN^T = B + qI, \quad (2.3)$$

where $I$ is the identity matrix. Let us denote all eigenvalues of $H$ by $\lambda_1(B), \ldots, \lambda_{q^{m+1}}(B)$.

Since $N^T N$ and $NN^T$ have the same eigenvalues, one can check that the eigenvalues of $\mathcal{G}$ are $\pm \sqrt{\lambda_i(B) + q}, i = 1, 2, \ldots, q^{m+1}$.

Now let us assume the mapping $\vartheta : \mathbb{F}_q \rightarrow \mathbb{F}_q^{m+1}; u \mapsto (1, f_2(u), \ldots, f_{m+1}(u))$ is injective. Then we know that $|S| = q(q - 1)$. Our first result is the following

**Theorem 2.2.** Let $\mathcal{G}$ be defined in $(1.1)$ with the assumptions that $g_k(x,y) = f_k(x)y$ for $k = 2, \ldots, m+1$ and the mapping $\vartheta : \mathbb{F}_q \rightarrow \mathbb{F}_q^{m+1}$ defined by $u \mapsto (1, f_2(u), \ldots, f_{m+1}(u))$ is injective. For all prime power $q$ and positive integer $m$, the eigenvalues of $\mathcal{G}$, counted with multiplicities, are

$$\pm \sqrt{qN_{F_w}}, w = (w_1, w_2, \ldots, w_{m+1}) \in \mathbb{F}_q^{m+1},$$

where $F_w(u) = w_1 + w_2f_2(u) + \cdots + w_{m+1}f_{m+1}(u)$ and $N_{F_w} = |\{u \in \mathbb{F}_q : F_w(u) = 0\}|$. For $0 \leq i \leq q$, the multiplicity of $\pm \sqrt{q_i}$ is

$$n_i = |\{w \in \mathbb{F}_q^{m+1} : N_{F_w} = i\}|.$$

Moreover, the number of connected components of $\mathcal{G}$ is

$$q^{m+1} - \text{rank}_{\mathbb{F}_q}(1, f_2, \ldots, f_{m+1}).$$

Therefore $\mathcal{G}$ is connected if and only if $1, f_2, \ldots, f_{m+1}$ are $\mathbb{F}_q$-linearly independent.

**Proof.** Let $\zeta_p$ be a primitive $p$-th root of unity, and for every $w := (w_1, w_2, \ldots, w_{m+1}) \in \mathbb{F}_q^{m+1}$, we define a character $\psi_w : \mathbb{F}_q^{m+1} \rightarrow \mathbb{C}^*$ by

$$\psi_w : u = (u_1, u_2, \ldots, u_{m+1}) \mapsto \zeta_p^{\text{tr}(w_1u_1 + w_2u_2 + \cdots + w_{m+1}u_{m+1})}.$$
where \( \text{tr} \) is the absolute trace map. As described in [13], the eigenvalues of the Cayley graph \( H \) are

\[
\psi_w(S) := \sum_{t \in \mathbb{F}_q, u \in \mathbb{F}_q} \zeta_p^{\text{tr}(t(w_1 + w_2 f_2(u) + \ldots + w_{m+1} f_{m+1}(u)))}, w \in \mathbb{F}_q^{m+1}.
\] (2.4)

Denote by \( F_w(u) \) the function \( w_1 + w_2 f_2(u) + \ldots + w_{m+1} f_{m+1}(u) \) and \( N_{F_w} = |\{u \in \mathbb{F}_q : F_w(u) = 0\}| \). Then it follows that

\[
\psi_w(S) = \sum_{t \in \mathbb{F}_q, u \in \mathbb{F}_q} \zeta_p^{\text{tr}(tF_w(u))} = \sum_{t \in \mathbb{F}_q, F_w(u) = 0} \zeta_p^{\text{tr}(tF_w(u))} + \sum_{t \in \mathbb{F}_q, F_w(u) \neq 0} \zeta_p^{\text{tr}(tF_w(u))} = (q - 1) N_{F_w} + (-1)(q - N_{F_w}) = q (N_{F_w} - 1).
\]

Thus this derives that the eigenvalues of \( \mathcal{S} \) are

\[
\pm \sqrt{q N_{F_w}}, w \in \mathbb{F}_q^{m+1},
\] (2.5)

where \( N_{F_w} = |\{u \in \mathbb{F}_q : F_w(u) = 0\}| \). For example, when \( w = (0, \ldots, 0) \) we have \( N_{F_0} = q \) which implies that \( \mathcal{S} \) has \( \pm q \) as its eigenvalues. Moreover, for any \( w \neq 0 \), it is easy to see that \( N_{F_w} \leq \deg(F_w) \leq \max\{\deg(f_2), \ldots, \deg(f_{m+1})\} \).

The number of connected components of \( \mathcal{S} \) is

\[
|\{w : F_w(x) \equiv 0 \text{ for all } x \in \mathbb{F}_q\}| = q^{m+1 - \text{rank}_{\mathbb{F}_q}(1, f_2, \ldots, f_{m+1})}.
\] (2.6)

Therefore \( \mathcal{S} \) is connected if and only if \( 1, f_2, \ldots, f_{m+1} \) are \( \mathbb{F}_q \)-linearly independent.

\[ \square \]

**Remark 1.** The computation of the multiplicities \( n_k \)’s is obviously an interesting number theoretical problem. One cannot expect a simple closed formula for \( n_k \)’s in general. Among the most interesting case is when the \( f_k(x) \)’s are given by monomials in \( x \). When the \( f_k \)’s are consecutive monomials (the original Wenger graph), there is indeed a simple formula for \( n_k \)’s. When the \( f_k \)’s are not consecutive monomials, the problem is more difficult. The linearized Wenger graph considered in next section deals with the first non-trivial example of non-consecutive monomials.

3. **The spectrum of linearized Wenger graphs**

Let \( q = p^c \) and \( m \) be a positive integer as before. We focus on the linearized Wenger graph \( L_m(q) \) from now on where \( f_k(x) = x^{p^{k-2}}, k = 2, \ldots, m + 1 \). The goal of this section is to explicitly compute the spectrum of \( L_m(q) \) by determining the explicit formula of \( N_{F_w} \) and \( n_k \) in Theorem 2.2. The computation involved in linearized Wenger graphs is more complicated since the degrees of \( f_k(x) = x^{p^{k-2}}, k = 2, \ldots, m + 1 \) are high and not consecutive as in Wenger graphs.

We first give a basic lemma which will be used in the rest of the paper. It is an old result with the first derivation of the formula due to Landsberg [9, p.455]; see also Lemma 2.1 in [10].

**Lemma 3.1.** The number of \( l \times n \) matrices over \( \mathbb{F}_q \) with rank \( k \) is

\[
\frac{\prod_{i=0}^{k-1}(q^i - q^j)(q^n - q^l)}{\prod_{i=0}^{k-1}(q^i - q^j)}.
\]
Proof. For a fixed $k$-dimensional subspace $W \in \mathbb{F}_q^l$, the number of $l \times n$ matrices with $W$ as the column space is equal to the number of $k \times n$ matrices of rank $k$. Such a matrix is given by the $k$ linearly independent row vectors of length $n$. The number of those is $\prod_{i=0}^{k-1} (q^n - q^i)$. The number of $k$-dimensional subspaces of $\mathbb{F}_q^l$ is $\prod_{i=0}^{k-1} (q^l - q^i)$ and the product is the number of rank $k$ matrices.

When $m = e$, the functions $1, x, \ldots, x^{p^{m-1}}$ are $\mathbb{F}_q$-linearly independent and so $L_m(q)$ is connected. For every $w = (w_1, w_2, \ldots, w_{m+1}) \in \mathbb{F}_q^{m+1}$, define $F_w(x) = w_1 + w_2x + w_3x^p + \cdots + w_{m+1}x^{p^{m-1}}$. By Theorem 2.2, the eigenvalues of the linearized Wenger graph $L_m(q)$, counting multiplicities, are

$$\pm \sqrt{qN_{F_w}}, w \in \mathbb{F}_q^{m+1},$$

where $N_{F_w} = |\{u \in \mathbb{F}_q : F_w(u) = 0\}| = |\{u \in \mathbb{F}_q : \bar{F}_w(u) = 0\}|$, where $\bar{F}_w(x) = w_2x + \cdots + w_{m+1}x^{p^{m-1}}$ is an $\mathbb{F}_p$-linearized polynomial. If $-w_1 \notin \text{Im}(\bar{F}_w)$, then $N_{F_w} = 0$. Otherwise, this also implies that

$$N_{F_w} = p\dim_{\mathbb{F}_p}(\ker(\bar{F}_w)).$$

Choosing a fixed basis of $\mathbb{F}_q/\mathbb{F}_p$ as $\alpha_1, \ldots, \alpha_e$, we know that every $p$-linear polynomial $\bar{F}_w(x)$ can be written as

$$\bar{F}_w(x) = \text{tr}(\beta_1 x)\alpha_1 + \text{tr}(\beta_2 x)\alpha_2 + \cdots + \text{tr}(\beta_e x)\alpha_e,$$  

(3.1)

where $\beta_1, \ldots, \beta_e$ are elements in $\mathbb{F}_q$ uniquely determined by $w_2, \ldots, w_{m+1}$. By Theorem 2.2 in [10], we have $\dim_{\mathbb{F}_p}(\ker(\bar{F}_w)) = i$ if and only if $\text{rank}_{\mathbb{F}_p}(\beta_1, \ldots, \beta_e) = e - i$. For $0 \leq i \leq e$, there are exactly

$$\prod_{j=0}^{e-i-1} (p^e - p^j)^2 \prod_{j=0}^{e-i-1} (p^e - p^j)$$

different $w_2, \ldots, w_{m+1}$ such that $\dim_{\mathbb{F}_p}(\ker(\bar{F}_w)) = i$ by Lemma 3.1. There are $p^{e-i}$ choices for $-w_1$ in the image set of $\bar{F}_w$, therefore the multiplicity of the eigenvalue $\pm \sqrt{q^p}$ is

$$n_p = p^{e-i} \prod_{j=0}^{e-i-1} (p^e - p^j)^2 \prod_{j=0}^{e-i-1} (p^e - p^j).$$  

(3.2)

Now, counting each $-w_1$ not in the image set of $\bar{F}_w$ such that $\dim_{\mathbb{F}_p}(\ker(\bar{F}_w)) = i$ for $1 \leq i \leq e$, the multiplicity of the eigenvalue 0 is

$$n_0 = \sum_{i=1}^{e} (p^e - p^{e-i}) \prod_{j=0}^{e-i-1} (p^e - p^j)^2 \prod_{j=0}^{e-i-1} (p^e - p^j).$$  

(3.3)

When $m > e$, one checks that $\text{rank}_{\mathbb{F}_q}(1, x, x^p, \ldots, x^{p^{m-1}}) = e + 1$ and thus we obtain the following result:

**Theorem 3.2.** Let $m \geq e$. The linearized Wenger graph $L_m(q)$ has $q^{m-e}$ components. The distinct eigenvalues are

$$0, \pm \sqrt{q^p}, 0 \leq i \leq e.$$  

For $0 \leq i \leq e$, the multiplicity of the eigenvalue $\pm \sqrt{q^p}$ is $q^{m-e}n_p$, where $n_p$ is given by (3.2). The multiplicity of the eigenvalue 0 is $q^{m-e}n_0$ where $n_0$ is given by (3.3).
When \( m = e \), these linearized Wenger graphs are connect \( q \)-regular \((q, \epsilon)\)-expander graphs with edge expansion \( \epsilon > \frac{q\sqrt{q^{p-1}}}{2} = \frac{q^{1/p}^{(p-1)/2}(p^{1/2}-1)}{2} \). As to expander graphs, we refer to [28] for more details.

When \( m < e \), the linearized Wenger graph \( L_m(q) \) is connected, however, we do not know a closed formula for the multiplicities of the eigenvalues \( \pm \sqrt{q^p} \). We leave this as an open problem.

4. THE DIAMETER OF LINEARIZED WENGER GRAPHS

Recall that a sequence of vertices \( v_1, \ldots, v_s \) in a simple graph \( G = (V, E) \) defines a path of length \( s - 1 \) if \((v_i, v_{i+1}) \in E \) for every \( i, 1 \leq i \leq s - 1 \). The distance between \( v_i \) and \( v_j \) is the number of edges in a shortest path joining \( v_i \) and \( v_j \). The diameter of a graph \( G \) is the maximum distance between any two vertices of \( G \). In [20] it is shown that the diameter of the Wenger graph \( W_m(q) \) is \( 2m + 2 \) when \( 1 \leq m \leq q - 1 \). In this section, we assume that \( m \leq e \) so that the linearized Wenger graphs are connected. We now explicitly determine the diameter of the linearized Wenger graph \( L_m(q) \).

Theorem 4.1. If \( m \leq e \), the diameter of the linearized Wenger graph \( L_m(q) \) is \( 2(m + 1) \).

Before proceeding to the proof of the above theorem, we give the following lemma.

Lemma 4.2. If \( x_1, \ldots, x_m \) in \( \mathbb{F}_q \) are \( \mathbb{F}_p \)-linearly independent, then

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_m \\
x_1^p & x_2^p & \cdots & x_m^p \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{p^{m-2}} & x_2^{p^{m-2}} & \cdots & x_m^{p^{m-2}}
\end{vmatrix} \neq 0.
\]

Proof. First it is easy to see that

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_m \\
x_1^p & x_2^p & \cdots & x_m^p \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{p^{m-2}} & x_2^{p^{m-2}} & \cdots & x_m^{p^{m-2}}
\end{vmatrix} = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
0 & x_2 - x_1 & \cdots & x_m - x_1 \\
0 & (x_2 - x_1)^p & \cdots & (x_m - x_1)^p \\
\vdots & \vdots & \ddots & \vdots \\
0 & (x_2 - x_1)^{p^{m-2}} & \cdots & (x_m - x_1)^{p^{m-2}}
\end{vmatrix}.
\]

Since \( x_1, \ldots, x_m \) are \( \mathbb{F}_p \)-linearly independent, \( x_2 - x_1, \ldots, x_m - x_1 \) are \( \mathbb{F}_p \)-linearly independent. By induction,

\[
\begin{vmatrix}
x_2 - x_1 & \cdots & x_m - x_1 \\
(x_2 - x_1)^p & \cdots & (x_m - x_1)^p \\
\vdots & \vdots & \ddots & \vdots \\
(x_2 - x_1)^{p^{m-2}} & \cdots & (x_m - x_1)^{p^{m-2}}
\end{vmatrix} \neq 0,
\]

the proof is complete. \( \square \)

Proof of Theorem 4.1. First we consider the distance between any two vertices \( L \) and \( L' \) in \( \mathcal{L} \) of the linearized Wenger graph \( L_m(q) \). If \( L_1 P_1 \ldots P_n L_{s+1} \) is a path in \( L_m(q) \) between \( L = L_1 \) and \( L' = L_{s+1} \), where \( L_i = [l_i^{(i)}, \ldots, l_i^{(i)}]_{m+1} \) and \( P_i = (p_i^{(i)}, \ldots, p_i^{(i)}) \), we have

\[
l_k^{(i+1)} - l_k^{(i)} = (l_k^{(i+1)} - l_1^{(i)}) (p_k^{(i)})^{p^k - 1}, k = 2, \ldots, m + 1, i = 1, \ldots, s.
\]
Therefore there are elements \( t_i = l_1^{(i+1)} - l_1^{(i)} \), \( x_i = p_1^{(i)} \in \mathbb{F}_q \), \( 1 \leq i \leq s \), such that

\[
(L_{s+1} - L_1)^T = t_1 \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_1^{m-1} \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ x_2 \\ \vdots \\ x_2^{m-1} \end{pmatrix} + \cdots + t_s \begin{pmatrix} 1 \\ x_s \\ \vdots \\ x_s^{m-1} \end{pmatrix}.
\] (4.1)

Take \( s = m + 1 \) and choose \( x_1, \ldots, x_{m+1} \in \mathbb{F}_q \) such that \( x_2 - x_1, \ldots, x_{m+1} - x_1 \) are \( \mathbb{F}_p \)-linearly independent. Then by Lemma 4.2, the coefficient matrix of Eq. (4.1) is nonsingular, and thus Eq. (4.1) has a unique solution for \( t_1, t_2, \ldots, t_s \). Thus the distance of any two vertices in \( \mathcal{L} \) is at most \( 2(m+1) \).

Similarly, let us consider any two vertices \( P \) and \( P' \) in \( \mathcal{Q} \) of \( L_m(q) \). Let \( P_1 L_1 \cdots L_s P_{s+1} \) be a path in \( L_m(q) \) between \( P = P_1 \) and \( P' = P_{s+1} \), where \( L_i = [l_1^{(i)}, \ldots, l_{m+1}^{(i)}] \) and \( P_i = (p_1^{(i)}, \ldots, p_{m+1}^{(i)}) \). Then we have

\[
p_k^{(i+1)} - p_k^{(i)} = l_1^{(i)} (p_1^{(i+1)} - p_1^{(i)}) k^{-2}, k = 2, \ldots, m+1, i = 1, \ldots, s.
\]

Similarly, if we take \( s = m + 1 \) and choose \( p_i \in \mathbb{F}_q \) such that \( p_1^{(i+1)} - p_1^{(i)} \), \( 1 \leq i \leq m \) are \( \mathbb{F}_p \)-linearly independent, then we can find unique solution for \( l_1^{(1)}, \ldots, l_{m+1}^{(m)} \). Hence the distance of any two vertices in \( \mathcal{Q} \) is at most \( 2(m+1) \).

Finally, we consider the distance between a vertex \( P = (p_1, \ldots, p_{m+1}) \in \mathcal{Q} \) and a vertex \( L \in \mathcal{L} \). First we choose any line \( L_1 \in \mathcal{L} \) such that it is adjacent to \( P \). From the earlier discussion, there exists a path from \( L_1 \) to \( L \) with distance at most \( 2(m+1) \). We modify the earlier construction so that the path goes through the vertex \( P \). Namely, In Eq. (4.1), we let \( x_1 = p_1 \) and choose the rest of \( x_i's \) so that \( x_2 - x_1, \ldots, x_{m+1} - x_1 \in \mathbb{F}_q \) are \( \mathbb{F}_p \)-linearly independent. Then there is a unique solution \( \{t_1, \ldots, t_s\} \) and so there is a path between \( L_1 \) and \( L \) with length at most \( 2(m+1) \) passing through \( P \). Therefore the distance of \( P \) and \( L \) is less than or equal to \( 2(m+1) \). Hence the diameter of \( L_m(q) \) is always at most \( 2(m+1) \).

On the other hand, we now show that the distance \( 2(m+1) \) can be reached. Indeed, choose two vertices \( L_1 \) and \( L_{s+1} \) such that \( L_{s+1} - L_1 = [0, \ldots, 0, 1] \). We can show that the distance between them is at least \( 2(m+1) \). Otherwise, suppose there is a path from \( L_1 \) to \( L_{s+1} \) with distance \( 2s \leq 2m \). Then Eq. (4.1) has a solution with \( 1 \leq s \leq m \). We show that this is impossible.

If either \( x_1, \ldots, x_s \) are \( \mathbb{F}_p \)-linearly independent and \( s < m \), or \( x_1, \ldots, x_s \) are \( \mathbb{F}_p \)-linearly dependent, then the last \( m \) rows of (4.1) always can be reduced to

\[
\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = t_1' \begin{pmatrix} x_1' \\ (x_1')^p \\ \vdots \\ (x_1')^{p^{m-1}} \end{pmatrix} + t_2' \begin{pmatrix} x_2' \\ (x_2')^p \\ \vdots \\ (x_2')^{p^{m-1}} \end{pmatrix} + \cdots + t_k' \begin{pmatrix} x_k' \\ (x_k')^p \\ \vdots \\ (x_k')^{p^{m-1}} \end{pmatrix}.
\] (4.2)
where \( x_1', \ldots, x_k' \) are \( \mathbb{F}_p \)-linearly independent and \( k < m \). Because the determinant of the coefficient matrix of the system from the first \( k \) rows is not zero by Lemma 4.2, we must have \( t_i' = 0 \) for all \( i \)'s, which contradicts with \( t_1'(x_1')^{p^{m-1}} + \ldots + t_k'(x_k')^{p^{m-1}} = 1 \).

If \( x_1, \ldots, x_s \) are \( \mathbb{F}_p \)-linearly independent and \( s = m \), then the determinant of the coefficient matrix of the system from the first \( m \) rows in Eq. (4.1) are not zero by Lemma 4.2. Again we must have \( t_i = 0 \) for all \( i \)'s, which also contradicts with \( t_1x_1^{p^{m-1}} + \ldots + tsx_s^{p^{m-1}} = 1 \). The proof is now complete.

\[ \square \]

5. The girth of linearized Wenger graphs

In graph theory, the girth of a graph is the length of a shortest cycle contained in the graph. In [9], Shao et al proved the Wenger graphs have girth 8, and moreover, if \( m \geq 3 \), then for any integer \( l \) with \( l \neq 5, 4 \leq l \leq 2p \) (where \( p \) is the character of the finite field \( \mathbb{F}_q \)) and any vertex \( v \) in the Wenger graph \( W_m(q) \), there is a cycle of length \( 2l \) in \( W_m(q) \) passing through the vertex \( v \). The existence of the cycles of certain even length plays an important role in the study of the accurate order of the Turán number in extremal graph theory. See [3, 4, 15, 17]. In this section, we consider the girth of linearized Wenger graphs \( L_m(q) = (V, E) \).

Let \( P = (p_1, \ldots, p_{m+1}), P' = (p'_1, \ldots, p'_{m+1}) \) be two distinct points in \( V \). Suppose that \( P \) and \( P' \) share a common neighbor \( L = [l_1, \ldots, l_{m+1}] \), then

\[ P - P' = (p_1 - p'_1, l_1(p_1 - p'_1), l_1(p_1 - p'_1)^p, \ldots, l_1(p_1 - p'_1)^{p^{m-1}}). \tag{5.1} \]

In other words, \( P - P' \) has the form \( (u, lu, lu^p, \ldots, lu^{p^{m-1}}) \). Conversely, if \( P - P' \) has the form \( (u, lu, lu^p, \ldots, lu^{p^{m-1}}) \) with \( u \neq 0 \), we show that there exists a unique \( L \in V \) such that \( L \) is a common neighbor of \( P \) and \( P' \). Indeed, let \( l_1 = l \). Since \( l_1p_1^{k-2} - p_k = l_1(p'_1)^{p^{k-2}} - p'_k, k = 2, \ldots, m + 1 \), we can define \( l_k = l_1p_1^{p^{k-2}} - p_k, k = 2, \ldots, m + 1 \) and then the point \( L = [l_1^p, \ldots, l_{m+1}^p] \) is a common neighbor of \( P, P' \). Moreover, if both \( L = [l_1, \ldots, l_{m+1}] \) and \( L' = [l'_1, \ldots, l'_{m+1}] \) are common neighbors of \( P, P' \), then by definition, \( l_1 = l_1' = l \) and \( l_k = l_k' = l_1p_1^{p^{k-2}} - p_k = l_1p'_1^{p^{k-2}} - p'_k, k = 2, \ldots, m + 1 \). Thus \( L = L' \).

We summarize the above discussion as follows:

**Lemma 5.1.** In the linearized Wenger graph \( L_m(q) \), two distinct points \( P = (p_1, \ldots, p_{m+1}) \) and \( P' = (p'_1, \ldots, p'_{m+1}) \) have a common neighbor if and only if \( P - P' \) has the form \( (u, lu, lu^p, \ldots, lu^{p^{m-1}}) \) with \( u \in \mathbb{F}_q^*, l \in \mathbb{F}_q \). Moreover, if \( P - P' \) has the form \( (u, lu, lu^p, \ldots, lu^{p^{m-1}}) \) with \( u \in \mathbb{F}_q^*, l \in \mathbb{F}_q \), then \( P, P' \) have a unique common neighbor.

As a consequence, we have

**Corollary 5.2.** There is no cycle of length 4 in the linearized Wenger graph \( L_m(q) \).

**Proof.** If \( P_1L_1P_2L_2P_3 \) or \( L_1P_1L_2P_2L_1 \) is a cycle of length 4 in the linearized Wenger graph, then \( L_1, L_2 \) are common neighbors of \( P_1, P_2 \), which is contrary to Lemma 5.1. \[ \square \]

Since the girth of the linearized Wenger graphs is even, the girth of the linearized Wenger graphs is at least 6 by Corollary 5.2. Furthermore, if \( P_1L_1P_2L_2P_3 \ldots L_4P_1 \) is a cycle of length
2t in the linearized Wenger graph $L_m(q)$, then there are elements $u_1, u_2, \ldots, u_t \in \mathbb{F}_q^*$ and $c_1, c_2, \ldots, c_t \in \mathbb{F}_q$ such that

$$
P_t - P_1 = (u_1, c_1 u_1, c_1 u_1^p, \ldots, c_1 u_1^{p^{m-1}})
$$

and thus

$$
\begin{align*}
L_1 - L_1 &= (u_1 + u_2 + \ldots + u_t, c_1 u_1 + c_2 u_2 + \ldots + c_t u_t, c_1 u_1^{p-1} + c_2 u_2^{p-1} + \ldots + c_t u_t^{p^{m-1}}) = 0.
\end{align*}
$$

The converse of this result does not hold since $P_1 L_1 P_2 L_2 P_3 \ldots P_t P_1$ may not be a cycle. For example, in linearized Wenger graph $L_1(11)$, choose $P_1 = (0, 0)$, $P_2 = (-1, -1)$, $P_3 = (-2, 0)$, $P_4 = P_1 = (0, 0)$, $P_5 = (-1, -2)$, $P_6 = (-2, -8)$, $L_1 = (1, 0)$, $L_2 = (-1, 2)$, $L_3 = (0, 0)$, $L_4 = (2, 0)$, $L_5 = (6, -4)$, and $L_6 = (4, 0)$. Then there are $u_1 = u_2 = u_3 = u_5 = 1, u_3 = u_6 = -2$, $c_1 = 1$, $c_2 = -1$, $c_3 = 0$, $c_4 = 2$, $c_5 = 6$, $c_6 = 4$ such that Eq. (5.2) and (5.3) hold. However, $P_1 L_1 \ldots P_6 P_1$ is not a cycle in $W_1(11)$.

Therefore, in order to study cycles of length $2t$ in linearized Wenger graphs, we first try to solve Eq. (5.2) and (5.3). If there are no $u_i$'s and $c_i$'s satisfying Eq. (5.2) and (5.3), then there is no cycle with length $2t$ in $L_m(q)$. Otherwise, construct $P_1, \ldots, P_t$ and $L_1, \ldots, L_t$ as follows:

Let $P_i = (p_1(i), \ldots, p_m(i)), L_i = [l_1(i), \ldots, l_m(i)], i = 1, \ldots, t$, where

$$
p_1(i) - p_1(i+1) = u_i, i = 1, 2, \ldots, t-1, p_1(i) - p_1(1) = u_t,
$$

and

$$
l_1(i) = c_i, l_k(i) = l_1(i) p_1(i)^{p^{k-2}} - p_k(i), k = 2, \ldots, m+1.
$$

If both $P_1, \ldots, P_t$ are distinct and $L_1, \ldots, L_t$ are also distinct, then $P_1 L_1 P_2 L_2 P_3 \ldots P_t P_1$ is a cycle of length $2t$ in $W_m(q)$. Otherwise, we choose new solutions $u_i$'s and $c_i$'s, and test these new vertices. If there are always two $P_i$'s (or two $L_i$'s) which are the same in the above construction for all $u_i$'s and $c_i$'s satisfying Eq. (5.2) and (5.3), then there is no cycle with length $2t$ in $L_m(q)$.

Using the above technique, in the following we give the girth of linearized Wenger graphs.

**Theorem 5.3.** Let $q = p^e$ and $m \geq 1$, $e \geq 1$ and $p$ be an odd prime, or $m = 1$, $e \geq 2$ and $p = 2$. Then the girth of the linearized Wenger graph $L_m(q)$ is 6.

**Proof.** Case 1. $m \geq 1$, $e \geq 1$ and $p$ is an odd prime. By Corollary 5.2 it is enough to construct a cycle with length 6 in this case. Indeed, let $u_1 = u_2 = 1, u_3 = -2, c_1 = 1$, $c_2 = -1, c_3 = 0, P_1 = (0, 0, \ldots, 0), P_2 = (-1, -1, \ldots, -1), P_3 = (-2, 0, \ldots, 0), L_1 = [1, 0, \ldots, 0], L_2 = [-1, 2, 2, \ldots, 2], L_3 = [0, 0, \ldots, 0]$. Then $P_1 L_1 P_2 L_2 P_3 L_3 P_1$ is a cycle with length 6.

Case 2. $e \geq 2$, $m = 1$ and $p = 2$. For an element $\beta \in \mathbb{F}_q^*$ and $\text{tr}(\beta) = 0$, there exists some $\alpha \in \mathbb{F}_q^*$ such that $\alpha^2 + \alpha = \beta$. Put $u_1 = \alpha^2, u_2 = \alpha, u_3 = \beta, c_1 = 0, c_2 = -\alpha^{-1}\beta$ and $c_3 = 1$. One can construct a cycle $P_1 L_1 P_2 L_2 P_3 L_3 P_1$ of length 6, where $P_1 = (0, 0), P_2 = (\alpha^2, 0), P_3 = (\beta, \beta), L_1 = [0, 0], L_2 = [\alpha^{-1}, \alpha \beta]$ and $L_3 = [1, 0]$. 

\qed
Theorem 5.4. Let \( q = p^e, \ p = 2 \) and either \( e = m = 1 \) or \( e \geq 1, \ m \geq 2 \). Then the girth of the linearized Wenger graph \( \mathbb{L}_m(q) \) is 8.

Proof. First we need to show that there is no cycle of length 6 in \( \mathbb{L}_m(q) \) in these two cases. For the case of \( e = 1 \) and \( p = 2 \), there is no \( u_i \in \mathbb{F}_q^*, \ 1 \leq i \leq 3 \), such that Eq (5.3) holds. Hence there is no cycle with length 6 in this case. Assume that there is a cycle \( P_1P_2P_3P_4 \) of length 6 in \( \mathbb{L}_m(q) \) for the case of \( e \geq 2, \ m \geq 2 \) and \( p = 2 \). Then there are elements \( u_1, u_2, u_3 \in \mathbb{F}_q^* \), \( c_1, c_2, c_3 \in \mathbb{F}_q \) such that Eq (5.2) and (5.3) hold.

Eliminating \( c_1 \) among two successive equations of the last \( m - 1 \) equations in Eq. (5.3), we get

\[
\begin{align*}
\begin{cases}
  u_1 + u_2 + u_3 = 0 \\
  c_1 u_1 + c_2 u_2 + c_3 u_3 = 0 \\
  c_2(u_2^2 - u_2 u_1) + c_3(u_3^2 - u_3 u_1) = 0 \\
  \vdots \\
  c_2(u_2^{2m-1} - u_2^{2m-2} u_1^{2m-2}) + c_3(u_3^{2m-1} - u_3^{2m-2} u_1^{2m-2}) = 0.
\end{cases}
\end{align*}
\]

Further simplifying Eq. (5.4) by using \( u_1 + u_2 + u_3 = 0 \) and \( u_1, u_2, u_3 \in \mathbb{F}_q^* \), we get

\[
\begin{align*}
\begin{cases}
  u_1 + u_2 + u_3 = 0 \\
  c_1 u_1 + c_2 u_2 + c_3 u_3 = 0 \\
  c_2 + c_3 = 0 \\
  \vdots \\
  c_2 + c_3 = 0.
\end{cases}
\end{align*}
\]

Therefore, by symmetry, Eq. (5.3) has only the solution \( c_1 = c_2 = c_3 \). Then we have \( \mathbb{L}_1 = \mathbb{L}_3 \) since they share the common vertex \( P_1 \), which contradicts to the earlier assumption.

In the following we can construct a cycle \( P_1P_2P_3P_4 \) in both cases: Put \( u_1 = u_2 = u_3 = u_4 = 1 \) and \( c_1 = c_3 = 0, \ c_2 = c_4 = 1 \). Let \( P_1 = (0, 0, 0, \ldots, 0), \ P_2 = (1, 0, 0, \ldots, 0), \ P_3 = (0, 1, 1, \ldots, 1), \ P_4 = (1, 1, 1, \ldots, 1), \ L_1 = [0, 0, 0, \ldots, 0], \ L_2 = [1, 1, 1, \ldots, 1], \ L_3 = [0, 1, 1, \ldots, 1], \ L_4 = [1, 0, 0, \ldots, 0]. \) Then it is straightforward to check \( P_1P_2P_3P_4 \) is indeed a cycle of length 8. Hence we complete the proof. \( \square \)

6. Open Problems

There are several open problems about linearized Wenger graphs. First finding an explicit formula for the eigenvalue multiplicities \( n_{\lambda} \)'s of the linearized Wenger graphs when \( m < e \) is an open problem. Constructing even cycles with specific length in linearized Wenger graphs is also interesting. In addition, it would be desirable to find new classes of \( f_k(x) \) such that the explicit spectrum of these new types of Wenger graphs can be determined by Theorem 2.2.

References

Xiwang Cao is with the School of Mathematical Sciences, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China, email: xwcao@nuaa.edu.cn

Mei Lu is with Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China, email: mlu@math.tsinghua.edu.cn

Daqing Wan is with Department of Mathematics, University of California, Irvine, CA 92697-3875, USA, email: dwan@math.uci.edu

Li-Ping Wang is with Institute of Information Engineering, Chinese Academy of Sciences, Beijing 100093, Beijing, China, email: wangliping@iie.ac.cn

Qiang Wang is with School of Mathematics and Statistics, Carleton University, 1125 Colonel By Drive, Ottawa, Ontario K1S 5B6, Canada. email: wang@math.carleton.ca