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Using Twisted Alexander Polynomials to Detect Fiberability

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

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in

Mathematics

by

Azadeh Rafizadeh

June 2011

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To my husband, Samuel Chamberlin,

who believed in me more than I believed in myself.

To my father, Rahmatollah Rafizadeh,

even though he was deprived of higher education because of religious discrimination,

his respect for knowledge has been my inspiration.
The purpose of this dissertation is to discuss how certain algebraic invariants of 3-manifolds, the twisted Alexander polynomials, can be effectively used in the study of fiberability and the Thurston norm of links. The links to which we have applied this technique belong to the class of graph links. For graph links, D. Eisenbud and W. Neumann introduced splice diagrams and developed a method to use the combinatorial information included in splice diagrams to determine fiberability and the Thurston norm, [3]. We use twisted Alexander polynomials to prove that the exterior of a certain graph knot, whose splice diagram is given, is not fibered. Then we consider three 2-component graph links built out of this knot. For these links we use the same technique, involving twisted Alexander polynomials, to discuss their fiberability and Thurston norm. This allows us to demonstrate the effectiveness of twisted Alexander polynomials in this context (links in homology spheres different from $S^3$), where no calculations exist in the literature.
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Chapter 1

Background and Preliminaries

1.1 Introduction

In classical knot theory, a knot is a copy of $S^1$ embedded in the 3-dimensional sphere $S^3$. Different invariants have been developed in the last decades to try to differentiate one knot from another. One of the earliest algebraic invariants was the Alexander polynomial, which was defined by J. Alexander in 1923 [16]. Even though the Alexander polynomial is a strong invariant of knots, it is not a complete one; meaning that there are knots that are known to be different, but have identical Alexander polynomials. Alexander polynomials are also used to determine fiberability. However, the information they provide is not necessarily complete.

In 1990, X. S. Lin introduced a generalization of these polynomials, called twisted Alexander polynomials for knots [14]. His definition was later generalized to 3-manifolds by B. Jiang and S. Wang [11], M. Wada [18], P. Kirk and C. Livingston [13], and J. Cha [2].
Twisted Alexander polynomials can be used to investigate other properties of 3-manifolds, such as fiberability. In particular, S. Friedl and S. Vidussi have used them in relation with fiberability of 3-manifolds [8], as we will see in Section 2.3. These polynomials can also be helpful in investigating the Thurston norm. We will discuss the Thurston norm in Section 1.4.

The main purpose of this thesis has been to find explicit applications of the relationship between twisted Alexander polynomials and fiberability. In particular, we have studied a knot $K$ that is included in a homology sphere $\Sigma$ (different from the 3-dimensional sphere $S^3$), and three different 2-component links that have $K$ as one of their components. In such cases the Wirtinger presentation can not be used directly to find the fundamental group of the exterior of the knot $K$ or of the links $L_\alpha$, $L_\beta$, and $L_\gamma$ (see Chapter 2).

Our knot $K$ is the result of gluing the exteriors of two right-handed trefoil knots to a 3-component link, the necklace (see Figure 3.2) in a special way that is called splicing. Using the Wirtinger presentation for the three pieces together with the splicing relations, we have calculated the fundamental group of the exterior of the knot $K$. We have used a similar technique for the afore mentioned links to calculate the fundamental group of their exterior. Since we are studying graph links, we present them in terms of their splice diagrams, which are discussed in detail in Section 1.3. Splice diagrams, which were introduced by D. Eisenbud and W. Neumann, encode all of the information about graph links [3]. Using the combinatorial information included in the splice diagram, Eisenbud and Neumann show that $K$ is not fibered. However, the technique they use to show this fact only applies to graph links, whereas we recast this result using twisted Alexander polynomials. The
method of this thesis can theoretically be applied to any 3-manifold and a number of cases (mostly knots in $S^3$ with few crossings) have been discussed in the literature. We have used the computer program *Knottwister* created by S. Friedl [4]. Knottwister requires the fundamental group of the 3-manifold, $N$, along with a cohomology class $\phi$. It uses Fox differential calculus to compute the Alexander polynomial and twisted Alexander polynomials via representations of the fundamental group. For details on Fox calculus see [12]. As we will see in detail in Chapter 3, the technique used in this thesis to determine fiberability does not depend on the fact that $K$ is a graph knot.

1.2 Graph Links

In this section, we will introduce graph links and discuss the necessary definitions and concepts to clarify how splice diagrams represent graph links. The first concept to define is Seifert fibration, for which we follow the definition of [10].

**Definition 1** A Seifert fibration is a triple $(M, F, \pi)$, where $M$ is an oriented 3-manifold, $F$ is a surface, oriented or unoriented, and $\pi : M \to F$ such that $(M, F, \pi)$ is “almost” a locally trivial $S^1$-bundle, namely for every $x \in F$, there exists a $D^2$ neighborhood of $x$ such that $\pi^{-1}(D^2) \cong D^2 \times S^1$ and

$$
\pi : D^2 \times S^1 \to D^2 \text{ is defined by } (rt_1, t_2) \mapsto rt_1^\alpha t_2^\beta
$$

where $t_i \in S^1$, $r \in [0,1]$, $\alpha, \beta \in \mathbb{Z}$, and $\gcd(\alpha, \beta) = 1$. Here, the values of $\alpha$ and $\beta$ depend on $x$. We call a fiber singular or exceptional if the value of $\alpha$, called multiplicity, associated to this fiber is not equal to $\pm 1$. Otherwise, we call the fiber regular.
For us, an \( n \)-component link is an embedding of a disjoint collection of \( n \) copies of \( S^1 \) in a homology sphere \( \Sigma \). (A homology sphere is an \( n \)-manifold whose homology groups are the same as the homology groups of the \( n \)-sphere, \( S^n \).) The knot \( K \), and the links \( L_\alpha \), \( L_\beta \), and \( L_\gamma \), which we will introduce in Sections 2.1 and 2.2, are contained in homology spheres different from \( S^3 \).

**Definition 2** A Seifert link is an \( n \)-component link \( L = (\Sigma, K) \), where \( K = S_1 \cup \ldots \cup S_n \subset \Sigma \), \( S_i \)'s being copies of \( S^1 \), and \( \Sigma \) a homology sphere, whose exterior \( \Sigma_0 = \Sigma \setminus \text{int}(\nu(K)) \) admits a Seifert fibration, when \( \nu(K) \) denotes a neighborhood of \( K \) and \( \text{int}(\nu(K)) \) is its interior.

We know by Lemma 7.1 in [3] that \( \Sigma \) itself must be Seifert fibered and (with one family of exceptions corresponding to the necklaces), the link components are singular or regular fibers of the fibration. (For examples of what we call a necklace, see Figures 3.2 and 3.10.)

We can specialize the above description of Seifert fibered spaces to obtain homology spheres as follows. We choose \( F \) to be \( S^2 \) and for any choice of an \( n \)-tuple \( (\alpha_1, \ldots, \alpha_n) \) of singular fibers with multiplicity \( \alpha_i \) we get a Seifert fibered homology sphere by choosing coefficients \( \beta_i \) to be determined module \( \alpha_i \) by the following equation:

\[
\sum_{i=1}^{n} \beta_i \alpha_1 \ldots \alpha_i \ldots \alpha_n = 1.
\]

Each Seifert fibered homology sphere is homeomorphic to \( \varepsilon \Sigma(\alpha_1, \ldots, \alpha_n) \) for some \( n \) when \( \varepsilon = \pm 1 \). For the canonical orientation, \( \varepsilon = 1 \) and for the opposite orientation, \( \varepsilon = -1 \) [3]. For example, \( \Sigma(p, q, 1, \ldots, 1) \) is \( S^3 \) for all coprime integers \( p, q \) and \( \Sigma(2, 3, 5) \) is the Poincaré
homology sphere. The only case when an \( \alpha_i \) may be zero is the case \( \Sigma(0,1,...,1) \), which gives \( S^3 \) (see [3]).

We can denote a Seifert link as

\[
(\varepsilon \Sigma(\alpha_1, ..., \alpha_n), \pm S_1 \cup ... \cup \pm S_m), \ m \leq n
\]

where the \( S_i \) represent singular or regular fibers of the Seifert fibration with their canonical orientation. By allowing the \( \alpha_i \) to be negative, we can recast all Seifert links as

\[
(\pm \Sigma(\alpha_1, ..., \alpha_n), S_1 \cup ... \cup S_m)), \ m \leq n.
\]

Now, we describe the splicing as an operation.

**Definition 3** Given two links \( L = (\Sigma, K) \) and \( L' = (\Sigma', K') \), let \( S \subset K \) and \( S' \subset K' \). Let \( \mu \) and \( \lambda \) be the standard meridian and longitude of \( S \) and \( \mu' \) and \( \lambda' \) that of \( S' \). Consider \( \Sigma'' = (\Sigma \setminus \text{int}(\nu(S))) \cup (\Sigma' \setminus \text{int}(\nu(S'))) \), which is formed by identifying \( \lambda \) with \( \mu' \) and \( \lambda' \) with \( \mu \). This operation is well-defined [3]. The link \( (\Sigma'', (K \setminus S) \cup (K' \setminus S')) \) is called the splice of the links \( L \) and \( L' \) along \( S \) and \( S' \).

Note that if \( K \) and \( K' \) have \( n \) and \( m \) components respectively, \( K'' \) has \( (n + m) - 2 \) components. Using the Mayer-Vietoris sequence, it can be shown that \( \Sigma'' \) is also a homology sphere.

**Remark 4** Some other operations of links, such as disjoint sums, connected sums, and cabling can be viewed as special cases of splicing. For details, see Proposition 1.1 in [3].

Now we are ready to define what a graph link is.
**Definition 5** A graph link is a link that is the result of splicing two or more Seifert links, and a graph knot is a one-component graph link.

### 1.3 Splice Diagrams

Following Eisenbud and Neumann, we shall represent graph links by using certain diagrams, which are called “splice diagrams”. Splice diagrams encode all the information about graph links. If we consider the minimal version of a splice diagram, there is a one to one correspondence between splice diagrams and graph links. (The concept of minimality of diagrams is discussed in detail in Theorem 8.1 in [3]. Using this theorem, it is straightforward to determine when a diagram is minimal. In what follows, all our splice diagrams will be minimal.)

The building blocks of splice diagrams are Seifert links. The following diagram corresponds to the Seifert link \((\pm \Sigma(\alpha_1, ..., \alpha_n), S_1 \cup ... \cup S_m))\). It is a Seifert fibered homology sphere with the first \(m\) fibers removed; \(S_i\) is regular if \(\alpha_i = 1\), and singular otherwise.

![Figure 1.1: Seifert Link \((\pm \Sigma(\alpha_1, ..., \alpha_n), S_1 \cup ... \cup S_m))\)](image)

Figure 1.1: Seifert Link \((\pm \Sigma(\alpha_1, ..., \alpha_n), S_1 \cup ... \cup S_m))\)
It is worth mentioning that the unknot and the Hopf link have splice diagrams that follow suitable modifications of the same rules. Other than these two exceptions, the splice diagram of every Seifert link is made out of three different parts, which we will explain next.

1. Nodes:

\[ \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\varepsilon \\
\alpha_m \\
\end{array} \]

Figure 1.2: A Node

Here, \( m \geq 3 \), the \( \alpha_i \)'s are pairwise coprime integers, and \( \varepsilon \) is a sign. If we consider the canonical orientation \( \varepsilon \) is positive; otherwise it is negative. Each node represents a Seifert link. So if the minimal diagram has \( k \) nodes, it is the splice diagram of a graph link that is the result of splicing \( k \) Seifert links together.

2. Boundary vertices:

In a splice diagram, these represent singular fibers of a fibration that are not components of the link. Splicing cannot happen along these vertices.
3. Arrowhead vertices:

These correspond to actual link components. They are regular fibers (when $\alpha_i = 1$) or singular fibers (when $\alpha_i \neq 1$) of the Seifert fibration of the link being represented. Splicing can happen along these vertices.

As mentioned before, every graph link is the result of splicing two or more Seifert links together to obtain a diagram of the following form.

![Figure 1.3: Example of a Splice Diagram](image)

We will now describe how this is represented in terms of splice diagrams. Recall that we can represent Seifert links $L^{(1)} = (\Sigma^{(1)}, K^{(1)})$ and $L^{(2)} = (\Sigma^{(2)}, K^{(2)})$ by the following diagrams.

![Figure 1.4: Seifert Links Before Splicing](image)
where $S^{(1)}$ and $S^{(2)}$ are components of $K^{(1)}$ and $K^{(2)}$ respectively, along which we do splicing. The graph link that is the result of this splicing is the following.

\[ \Gamma^{(1)} \xrightarrow{S^{(1)}} \Gamma^{(2)} \]

Figure 1.5: The Resulting Graph Link

As we see here, the components along which splicing happens, disappear as arrowhead vertices, and appear as an edge in the diagram of the resulting graph link.

When we speak of “the vertices of the diagram” we will include nodes as well as boundary vertices and arrowhead vertices.

1.4 Thurston Norm and Fiberability

Given a knot, we define its Seifert surface to be a connected, embedded, oriented, compact surface $F$ such that the given knot is the boundary of $F$. The genus of a knot is the least genus of all its Seifert surfaces. As we will see in this chapter, genus is important in determining fiberability for knots. The generalization of genus to a general 3-manifold such as links contained in homology spheres, is the Thurston norm. In this section, we discuss the Thurston norm on homology. To begin, we need to introduce the idea of links with multiplicity.
**Definition 6** By a multilink on \( L = (\Sigma, K) \) we mean \( L \) together with an integer multiplicity \( m_i \) associated with each component \( S_i \). We write the multilink

\[
L(m_1, ..., m_n) = (\Sigma, m_1 S_1 \cup ... \cup m_n S_n).
\]

By associating to the collection \( \{m_i\} \) the class \( m = (m_1, ..., m_n) \in H^1(\Sigma \setminus K; \mathbb{Z}) \cong \mathbb{Z}^n \), we can think of a multilink as a link together with an element of \( H^1(\Sigma \setminus K; \mathbb{Z}) \).

For a connected CW complex \( X \) let \( \chi(X) \) be its Euler characteristic, and let \( \chi_-(X) = \max(-\chi(X), 0) \). More generally, let \( \chi(X) = \sum \chi_-(X_i) \) where \( X_i \)'s are connected components of \( X \). Thus we are ready to define the Thurston norm.

**Definition 7** If \( N \) is a 3-manifold, the Thurston norm for \( m \in H^1(N; \mathbb{Z}) \) is defined to be

\[
\|m\|_T := \min \{ \chi_-(F) | F \subset N, PD[F] = m \}.
\]

By \( PD[F] \) we mean the Poincaré dual to the class \([F]\), where \([F]\) \in \( H_2(N, \partial N; \mathbb{Z}) \). This norm can be extended uniquely to \( H^1(N; \mathbb{R}) \) (see [17] for details).

The unit ball of the Thurston norm is by definition the subset of \( H^1(N; \mathbb{R}) \) of elements \( \phi \in H^1(N; \mathbb{R}) \) such that \( \|\phi\|_T \leq 1 \). See Figures 2.5, 2.8, 2.11 for example of the Thurston unit ball.

Before we can define what it means for a multilink to be fibered, we need to talk about the Seifert surface for a multilink. We follow the definition given in [3]. Given a multilink \( L(m) = (\Sigma, m_1 S_1 \cup ... \cup m_n S_n) \), its Seifert surface is an embedded oriented surface \( F_0 \subset \Sigma \setminus K \) such that \( F_0 \subset F_0 \subset K \) and \( F_0 \) intersects \( \nu(S_i) \) for each \( i \) as follows.

1. If \( m_i \neq 0 \), \( F_0 \cap \nu(S_i) \) consists of \( |m_i| \) leaves meeting along \( S_i \). In this case, the orientation of \( F_0 \) is such that \( \partial F_0 \cap \nu(S_i) = m_i S_i \).
2. If $m_i = 0$, then $F_0 \cap \nu(S_i)$ consists of discs transverse to $S_i$. In this case, the orientation of $F_0$ is such that intersection number of $S_i$ with each of these discs is the same, either 1 or $-1$ for each disc.

Let $\Sigma_0 = \Sigma \setminus (\nu(K))$ be the link exterior. We also call $F = F_0 \cap \Sigma_0$ a Seifert surface, since it determines and is determined by $F_0$ up to isotopy. It has the following properties:

1. $F$ is an oriented surface, properly embedded in $\Sigma_0$, that is $F \cap \partial \Sigma_0 = \partial F$ transversally.

2. $F \cap \partial(\nu(S_i)) = d_i S_i(p_i, q_i)$, $d_i$ parallel copies of the $(p_i, q_i)$-cable on $S_i$, where $d_ip_i = m_i$, $\gcd(p_i, q_i) = 1$, and $d_iq_i = -l(\sum_{j \neq i} m_j S_j, S_j)$ (see [3] for more details), where $l$ is the linking number (see [16]).

**Definition 8** The multilink $L(m) = (\Sigma, m_1 S_1 \cup ... \cup m_n S_n)$ is said to be a fibered multilink if there exits a fibration $\Lambda : \Sigma_0 \rightarrow S^1$ of the link exterior $\Sigma_0 = \Sigma \setminus \text{int}(\nu(K))$, all of whose fibers are Seifert surfaces for $L(m)$. Equivalently, the requirement is that if we identify $H^1(\Sigma_0) = [\Sigma_0, S^1]$, the group of homotopy classes of maps of $\Sigma_0$ to $S^1$, then $m \in [\Sigma_0, S^1]$ contains a fibration $\Lambda$.

D. Eisenbud and W. Neumann explain a theory that enables one to determine the Thurston norm and fiberability of graph links using the combinatorial information included on their splice diagrams [3]. To discuss this theory, we will need some preliminary definitions.

Define $l_{ij} := (\text{product of all signs of nodes on } \sigma_{ij}) \times (\text{product of all edge weights adjacent to these nodes but not on } \sigma_{ij})$, when $\sigma_{ij}$ is the path between the vertices $i$ and $j$. 

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Example: Given the following splice diagram, we calculate $l_{ij}$.

![Splice Diagram](image)

Figure 1.6: An Example

In this example, $l_{ij} = (+1 \cdot -1 \cdot -1) \cdot (1 \cdot 1 \cdot 2 \cdot 3 \cdot 1) = +6$

The following theorems, which are due to Eisenbud-Neumann [3], state how to obtain information about linking number, Thurston norm, and fiberability of graph links, using their splice diagrams. From now on, we will assume that all links are irreducible (i.e. they cannot be written as the disjoint sum of two other links when neither is the empty link). Also, all splice diagrams will be minimal. Theorem 9 states how the linking number of any two components in a graph link can be computed.

**Theorem 9** (Eisenbud-Neumann, [3]) *If the vertices of a splice diagram $\Gamma$ are numbered as $v_1, \ldots, v_n, v_{n+1}, \ldots, v_k$ with $v_1, \ldots, v_n$ being the arrowheads and $v_{n+1}, \ldots, v_k$ being the remaining vertices, then $l_{ij} = l(S_i, S_j)$ for $1 \leq i < j \leq k$, and $m(S_j) = m_1 l_{1j} + \ldots + m_n l_{nj}$ for $n + 1 \leq j \leq k$.***
We can also find the Thurston norm of a multilink by considering its (minimal) splice diagram by Theorem 10 as follows.

**Theorem 10** (Eisenbud-Neumann, [3]) Assume \( L(\Gamma) \) is irreducible, and not the unknot. Then the Thurston norm of \( L(\Gamma(m)) = L(\Gamma(m_1, \ldots, m_n)) \) is:

\[
\|m\|_T = \sum_{j=n+1}^{k} (\delta_j - 2)|m(S_j)| = \sum_{j=n+1}^{k} (\delta_j - 2)|m_1l_{1j} + \ldots + m_nl_{nj}|
\]

where \( \delta_j \)'s denote the degree of the \( j \)th vertex in \( \Gamma \) (so \( \delta_j = 1 \) for a boundary vertex or an arrowhead, and \( \delta_j \geq 3 \) for a node).

The following theorem states how one can determine fiberability of a multilink, given its splice diagram.

**Theorem 11** (Eisenbud-Neumann, [3]) Assume \( L(\Gamma) \) is irreducible and \( \Gamma \) is a minimal splice diagram for \( L(\Gamma) \), and \( \Gamma \) is not the empty link or the Hopf link. Then \( L(\Gamma(m)) \) is a fibered multilink if and only if \( m(S_j) = m_1l_{1j} + \ldots + m_nl_{nj} \) is nonzero for \( j = n+1, \ldots, k \).

### 1.5 Twisted Alexander Polynomials

In this section, we will define twisted Alexander polynomial, which is a generalization of Alexander polynomial. As we will see in the following chapters, Alexander polynomials provide information about the Thurston norm and fiberability. However, sometimes the information they provide is not decisive. For such cases, twisted Alexander polynomials are employed to obtain more information. The following discussion is consistent with that appearing in [6].
Let $N$ be a compact, orientable 3-manifold, whose boundary (if any) is union of
tori, and $\pi_1(N)$ its fundamental group. Define $H$ to be the maximal free abelian quotient
of $\pi_1(N)$. Hence $H \simeq \mathbb{Z}^n$ when $n$ is the first Betti number of $N$. For the rest of this section
we will let $F = \mathbb{Z}$ (to define the single variable twisted Alexander polynomial), or $F = H$
to define the multi-variable twisted Alexander polynomial). Now, take $\phi \in \text{Hom}(H, F)$
to be a nontrivial homomorphism. Also, let $\alpha : \pi_1(N) \to \text{Gl}_k(\mathbb{Z})$ be a representation for
some positive integer $k$. Define $\mathbb{Z}^k[F] := \mathbb{Z}^k \otimes_\mathbb{Z} \mathbb{Z}[F]$. The elements of $\mathbb{Z}^k[F]$ are column
matrices with $k$ entries. Each entry is a (multivariable) polynomial with integer coefficients.
Combining $\alpha$ and $\phi$ will give us the following representation:

$$
\alpha \otimes \phi : \pi_1(N) \to \text{Gl}_k(\mathbb{Z}[F])
$$

$$
g \mapsto (v \mapsto t^{\phi(g)} \alpha(g)v)
$$

Here, $v \in \mathbb{Z}^k[F]$, and $t$ stands for $(t_1, \ldots, t_n)$ in the multivariable case. Hence $\mathbb{Z}^k[F]$ is a left
$\mathbb{Z}[\pi_1(N)]$-module. At the same time, it is a right $\mathbb{Z}[F]$-module as we can multiply by any
element of $\mathbb{Z}[F]$ on the right. The two module structures are compatible.

Now, consider $\tilde{N}$, the universal cover of $N$. $\pi_1(N)$ acts on $\tilde{N}$ by deck transforma-
tions. The chain groups $C_*(\tilde{N})$ are in a natural way right $\mathbb{Z}[\pi_1(N)]$-modules. We can form
by tensor product the chain complex

$$
C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} (\mathbb{Z}^k[F])
$$

Now define

$$
H_1(N; \mathbb{Z}^k[F]) := H_1(C_*(\tilde{N}) \otimes_{\mathbb{Z}[\pi_1(N)]} \mathbb{Z}^k[F])
$$

This inherits the structure of right $\mathbb{Z}[F]$-module, and is called the twisted Alexander module.
The $\mathbb{Z}[F]$-module $H_1(N, \mathbb{Z}^k[F])$ is a finitely presented module since $\mathbb{Z}[F]$ is Noetherian (see [12]). Hence $H_1(N, \mathbb{Z}^k[F])$ has a free resolution

$$\mathbb{Z}[F]^r \to \mathbb{Z}[F]^s \to H_1(N, \mathbb{Z}^k[F]) \to 0$$

Let $S$ be the $s \times r$ matrix presenting the homomorphism going from $\mathbb{Z}[F]^r$ to $\mathbb{Z}[F]^s$. We may assume that $r \geq s$.

**Definition 12** The twisted Alexander polynomial of $(N, \alpha, \phi)$ is defined to be the order of $H_1(N, \mathbb{Z}^k[F])$, which is the greatest common divisor of the $s \times s$ minors of the $s \times r$ matrix $S$. This polynomial is denoted as $\Delta_{N,\phi}^\alpha$. This polynomial is well-defined up to units of $\mathbb{Z}[F]$, which are monic monomials, meaning single term Laurent polynomials with coefficient $\pm 1$.

If $\alpha$ is trivial, the above definition gives ordinary Alexander polynomial. We will denote the ordinary Alexander polynomial as $\Delta_{N,\phi}$ for a general 3-manifold, and $\Delta$ for knots in the following chapters. Similarly, we can define polynomials out of the modules $H_0(N, \mathbb{Z}^k[F])$ and $H_2(N, \mathbb{Z}^k[F])$. 

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Chapter 2

Motivating Calculations

In what follows, we will introduce a knot $K$ and three 2-component links $L_\alpha$, $L_\beta$, and $L_\gamma$. Since all of these are graph links they have splice diagrams. Using their splice diagrams and Theorems 11, and 15, we will determine their fiberability and calculate their Alexander polynomials. In addition we will use Theorem 10 to calculate the Thurston norm for a general cohomology class $\phi$. As we have access to the splice diagrams for these links, they will provide a good testing ground for the application of twisted Alexander polynomials to study fiberability and the Thurston norm. We will use twisted Alexander polynomials to do this in Chapter 3.

The following theorem of C. McMullen shows the ability of the Alexander polynomial to provide information on the Thurston norm and fiberability for a general 3-manifold $N$. If $\phi = (m_1, ..., m_n)$, div($\phi$) is the greatest common divisor of $m_1, ..., m_n$. 

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**Theorem 13** (McMullen, [15]) Let $N$ be a compact, connected, orientable 3-manifold whose boundary (if any) is a union of tori. Then for any $\phi \in H^1(N; \mathbb{Z})$

\[ \deg(\Delta_{N,\phi}) \leq \|\phi\|_T + \begin{cases} 
0, & b_1(N) \geq 2 \\
\text{div}(\phi) \cdot (1 + b_3(N)), & b_1(N) = 1 
\end{cases} \]

(If $N$ has boundary, $b_3(N) = 0$ and if it has no boundary, $b_3(N) = 1$.)

If $\phi$ is fibered, $\Delta_{N,\phi}$ is monic, and equality holds.

Observe that the last part of Theorem 13 generalizes the following classical theorem of Neuwirth.

**Theorem 14** (Neuwirth, [16]) If $K$ is a fibered knot, then its Alexander polynomial, $\Delta_K$ is monic and $\deg(\Delta_K) = 2g(K)$ when $g(K)$ denotes the genus of the knot.

### 2.1 Introducing the Graph Knot $K$

The converse of Theorem 13 is not true as we will discuss in what follows. For this, we will introduce a graph knot $K$, whose properties can be analyzed using the results we have already mentioned. The knot $K$ has the following splice diagram.

![Splice Diagram of the Knot $K$](image)
As we will see in detail in Chapter 3, according to the splicing instructions determined by its splice diagram, this knot is the result of splicing two copies of the right handed trefoil and a three component necklace. The following theorem of Eisenbud-Neumann allows us to compute the ordinary Alexander polynomial for the knot $K$, since it is a graph knot.

**Theorem 15** (Eisenbud-Neumann, [3]) Let $\Gamma$ be a splice diagram with arrowhead vertices $v_1,...,v_n$ and remaining vertices $v_{n+1},...,v_k$ representing a link $L = L(\Gamma) = (\Sigma,S_1 \cup \cdots \cup S_n)$.

Let $n \geq 1$, that is, the link is non-empty. Then the multi-variable Alexander polynomial is:

$$
\Delta_L(t_1, ..., t_n) = \begin{cases} 
\prod_{j=n+1}^k (t_1^{l_{1j}}t_2^{l_{2j}}\cdots t_n^{l_{nj}} - 1)^{d_j-2}, & n \geq 2 \\
(t_1 - 1) \prod_{j=2}^k (t_1^{l_{1j}} - 1)^{d_j-2}, & n = 1.
\end{cases}
$$

The single variable Alexander polynomial for such a link, given $\phi = (m_1, ..., m_n)$ is:

$$
\Delta_{L,\phi} = \begin{cases} 
(t^d - 1) \cdot \Delta_L(t^{m_1}, ... t^{m_n}), & n \geq 2 \\
\Delta_L(t^{m_1}), & n = 1
\end{cases}
$$

where $d = \gcd(m_1, ..., m_n)$.

To do calculations using this theorem, the convention is to cancel the terms $(t^0 - 1)$ and $(t^0 - 1)^{-1}$ against each other. We will use the splice diagram of $K$ to calculate its Alexander polynomial, its genus, and to show it is not fibered.
Proposition 16 The genus of the knot $K$ is 1, it has Alexander polynomial equal to $t^2 - t + 1$, and the pair $(\Sigma \setminus \text{int } (\nu(K)), \phi)$ is not fibered for any $\phi$.

Proof. In order to prove this proposition, we will use the theorems stated in this section. As we can see in the diagram, there is one arrowhead vertex, so we will call this vertex $v_1$ as in Figure 2.2. As mentioned before, we include nodes in vertices. Hence this knot has 8 vertices. So $n = 1$, and $k = 8$. First, we will find $l_{ij}$ for $i = 1$, and $1 < j \leq 8$:

\[
\begin{align*}
l_{12} &= 0 \cdot 3 = 0 \\
l_{13} &= 0 \cdot 2 \cdot 3 = 0 \\
l_{14} &= 0 \cdot 2 = 0 \\
l_{15} &= 0 \cdot 1 = 0 \\
l_{16} &= 1 \cdot 2 \cdot 3 = 6 \\
l_{17} &= 1 \cdot 3 = 3 \\
l_{18} &= 1 \cdot 2 = 2
\end{align*}
\]

as can be read off the following diagram.

![Diagram](image.png)

Figure 2.2: Vertices of the Knot $K$
For boundary vertices and arrowhead vertices, $\delta_i = 1$. For this particular knot, each node has 3 arrowhead vertices and/or boundary vertices attached to it. So we have the following valued for $\delta_i$ where $1 < i \leq 8$:

$\delta_2 = \delta_4 = \delta_7 = \delta_8 = 1$ and $\delta_3 = \delta_5 = \delta_6 = 3$. Now we can use Theorem 15 to compute the Alexander polynomial:

$$\Delta = (t - 1)(t^0 - 1)^{-1}(t^0 - 1)(t^0 - 1)^{-1}(t^0 - 1)(t^6 - 1)(t^3 - 1)^{-1}(t^2 - 1)^{-1}.$$ 

Using the convention mentioned, we get

$$\Delta = \frac{(t - 1)(t^6 - 1)}{(t^3 - 1)(t^2 - 1)} = \frac{t^3 + 1}{t + 1} = t^2 - t + 1.$$ 

To find the genus of the knot, we calculate the Thurston norm of class $\phi = (1) \in H^1(\Sigma \setminus \nu(K), \mathbb{Z}) \simeq \mathbb{Z}$. By Theorem 10,

$$\|\phi\|_T = \|(1)\|_T = \sum_{j=2}^{8} (\delta_j - 2)|l_{1j}|$$

$$= -l_{12} + l_{13} - l_{14} + l_{15} + l_{16} - l_{17} - l_{18}$$

$$= -0 + 0 + 0 + 6 - 2 - 3 = 1.$$ 

So this knot has genus equal to 1 as claimed since $\|\phi\|_T = 2g - 1$. It remains to show it is not fibered. To show this, we use Theorem 11. As we can see, some of the terms in the summation are zero (it would be sufficient to have one be zero). Therefore by Theorem 11, the class $(1)$ is not fibered. So no class is fibered. ■

The ordinary Alexander polynomial of the knot $K$ is monic. Indeed, it is equal to the Alexander polynomial of the trefoil knot, which is fibered. Also the degree of its ordinary Alexander polynomial is $2g$. However, it is not fibered by Theorem 11. So in this example, the ordinary Alexander polynomial contains no information about fiberability.
2.2 2-Component Links Containing $K$ as a Component

As we saw in the last section, $K$ is a knot with some interesting properties. In this section, we will see examples of links including $K$ as one of their components. We will add a component to this knot in one of the three following ways, and observe what happens with the resulting 2-component links.

Case 1. First, we put the second arrowhead vertex on the last node. The following is the splice diagram of this 2-component link. From now on, we call this link $L_\alpha$. Since this link contains the knot $K$ as a component, we can denote it as $L_\alpha = K_\alpha \cup K$, when $K_\alpha$ is the new component of the link.

![Splice Diagram of the Link $L_\alpha$]

As we discussed earlier, for a general 3-manifold $N$, fiberability is a property of the pair $(N, \phi)$, when $\phi \in H^1(N; \mathbb{Z})$. In the following proposition we state and prove some properties of the link $L_\alpha$. 

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Proposition 17 The 2-component link \( L_\alpha \) in Figure 2.3 has the following properties:

1. Its multivariable Alexander polynomial is:

\[
\Delta_{L_\alpha}(t_1, t_2) = (t_1^{12} - t_1^6 + 1)(t_1^4 t_2^4 + t_1^2 t_2^2 + 1)(t_1^3 t_2^3 + 1).
\]

2. For a general \( \phi = (p, q) \), the Thurston norm is:

\[
\|\phi\|_T = 7|p + q| + 12|p|.
\]

3. If \( N \) is the exterior of the link, the pairs \( (N, (0,1)) \) and \( (N, (1,-1)) \) are not fibered.

Proof. To use the splice diagram to do the calculations, we need to enumerate the vertices. Consider the following figure.

As we can see from Figure 2.4, \( \delta_3 = \delta_5 = \delta_8 = \delta_9 = 1 \), since they represent arrowhead vertices or boundary vertices. \( \delta_4 = \delta_6 = 3 \), since they represent nodes with 3 vertices attached to them, and \( \delta_7 = 4 \) since it has 4 vertices attached to it. We also see that
\[ l_{ij} \text{ equal:} \]
\[ l_{13} = 18, \ l_{14} = 36, \ l_{15} = 12, \ l_{16} = 6 = l_{17}, \ l_{18} = 2 \]
\[ l_{19} = 3, \ l_{23} = l_{24} = l_{25} = l_{26} = 0, \ l_{27} = 6 \]
\[ l_{28} = 2, \ \text{and} \ l_{29} = 3. \]

Using this information and Theorem 15 we calculate the multivariable Alexander polynomial:
\[
\Delta_{L_\alpha}(t_1, t_2) = \prod_{j=3}^{9} (t_1^{l_{1j}} \cdot t_2^{l_{2j}} - 1)^{(\delta_j - 2)} =
\]
\[
(t_1^{18} - 1)^{-1}(t_1^{36} - 1)(t_1^{12} - 1)^{-1}(t_1^{6} - 1)
\]
\[
(t_1^{36} - 1)^2(t_2^{2} - 1)^{-1}(t_1^{3}t_2^{3} - 1)^{-1}
\]
\[
= (t_1^{12} - t_1^{6} + 1)(t_1^{4}t_2^{4} + t_1^{2}t_2^{2} + 1)(t_1^{3}t_2^{3} + 1).
\]

To find the Thurston norm for a general class \( \phi = (p, q) \), we use Theorem 10:
\[
\|\phi\|_T = \|(p, q)\|_T = \prod_{j=3}^{9} (\delta_j - 2) \cdot |pl_{1j} + ql_{2j}|
\]
\[
= -|18p| + |36p| - |12p| + |6p| + 2|6p + 6q| - |2p + 2q| - |3p + 3q|
\]
\[
= 7|p + q| + 12|p|.
\]

To show the last part of the proposition, we notice that for \( \phi = (p, q) \), if \( p = -q \) or \( p = 0 \), then \( \phi \) is not fibered by Theorem 11, since 0s get created. This only happens when \( p = \pm 1 \) and \( q = \mp 1 \) or if \( p = 0 \). Since the classes \((0, 1)\) and \((1, -1)\) fall in one of these categories, neither one of them is fibered. ■
The following figure shows the Thurston unit ball for $L_\alpha$.

![Thurston Unit Ball for $L_\alpha$.](image)

**Figure 2.5:** The Thurston Unit Ball for $L_\alpha$

**Remark 18** From the previous proposition, we can observe that for the class $\phi = (1, -1)$ the single variable Alexander polynomial is

$$\Delta_{L_\alpha, \phi} = 6(t - 1)(t^{12} - t^6 + 1).$$

Even though $\deg(\Delta_{L_\alpha, \phi}) = \|\phi\|_T + 1$, the polynomial is not monic. So Theorem 13 states that this class is not fibered. However, for $\phi = (0, 1)$, we get

$$\Delta_{L_\alpha, \phi} = (t - 1)(t^4 + t^2 + 1)(t^3 + 1) = (t^6 - 1)(t^2 - t + 1)$$

for the Alexander polynomial.
In this case, the Alexander polynomial is monic, and \( \deg(\Delta_{L_\alpha, \phi}) = 8 \). According to Theorem 13, this result is compatible with fiberability, but we showed in Proposition 17 that it is not fibered.

**Case 2.** A second link containing \( K \) is obtained by adding the second arrowhead vertex to the middle node. The following is the splice diagram of this link \( K_\beta \cup K \), call it \( L_\beta \).

![Splice Diagram of the Link \( L_\beta \)](image)

**Proposition 19** For the link \( L_\beta \) the following are true:

1. The multivariable Alexander polynomial vanishes.

2. The Thurston norm of the class \( \phi = (p, q) \) on \( L_\beta \) is \( |p + q| \).

3. No cohomology class \( \phi \) on \( L_\beta \) is fibered.
**Proof.** To do the necessary calculations, we number the vertices of this link as follows.

![Figure 2.7: Vertices of the Link $L_\beta$](image)

Considering this diagram, we see that $\delta_4 = \delta_7 = 3$, $\delta_6 = 4$, $\delta_3 = \delta_5 = \delta_8 = \delta_9 = 1$.

Also, we see that $l_{13} = l_{14} = l_{15} = l_{16} = l_{23} = l_{24} = l_{25} = l_{26} = 0$, $l_{17} = l_{27} = 6$, $l_{18} = l_{28} = 2$, and $l_{19} = l_{29} = 3$. Using this information and Theorem 15, we calculate the multivariable Alexander polynomial:

$$
\Delta_{L_\beta}(t_1, t_2) = \prod_{j=3}^{9} (t_1^{l_{1j}} \cdot t_2^{l_{2j}} - 1)^{\delta_j - 2} = \\
(t_1^{0} t_2^{0} - 1)^{-1}(t_1^{0} t_2^{0} - 1)^{1}(t_1^{0} t_2^{0} - 1)^{-1}(t_1^{0} t_2^{0} - 1)^{2} \\
(t_1^{0} t_2^{6} - 1)(t_1^{2} t_2^{0} - 1)^{-1}(t_1^{3} t_2^{0} - 1)^{-1} = 0.
$$
Now we use Theorem 10 to find the Thurston norm of the class \( \phi = (p, q) \):

\[
\| \phi \|_T = \| (p, q) \|_T = \prod_{j=3}^{9} (\delta_j - 2) \cdot |p l_{1j} + q l_{2j}| =
\]

\[
-|0| + |0| - |0| + 2 |0| + |6p + 6q| - |2p + 2q| - |3p + 3q| = |p + q|
\]

as claimed. Observe that the first four terms in the Thurston norm become 0 independent of choice of \((p, q)\). Hence by Theorem 11 no class on the link \( L_\beta \) is fibered. ■

For the link \( L_\beta \), the following is the Thurston unit ball.

![Diagram of the Thurston Unit Ball for \( L_\beta \)](image)

Figure 2.8: The Thurston Unit Ball for \( L_\beta \)
Case 3. A third link, $L_\gamma = K_\gamma \cup K$ is obtained by adding an arrowhead vertex on the first node. The following is the splice diagram of this link.

![Splice Diagram of the Link $L_\gamma$](image)

**Figure 2.9: Splice Diagram of the Link $L_\gamma$**

**Proposition 20** The link $L_\gamma$ has the following properties.

1. The Alexander polynomial vanishes.
2. The Thurston norm for a class $\phi = (p, q)$ on this link is $7|p| + |6p + q|$.
3. No class $\phi$ on this link is fibered.

**Proof.** Consider the diagram for this link, appearing on the next page. The necessary information for calculations is as follows:

\[\delta_3 = \delta_5 = \delta_8 = \delta_9 = 1, \delta_4 = 4, \text{ and } \delta_6 = \delta_7 = 3.\]

We also see that $l_{12} = l_{16} = l_{23} = l_{24} = l_{25} = l_{26} = 0, l_{13} = 3, l_{14} = 6, l_{15} = 2, l_{17} = 36, l_{18} = 12, l_{19} = 18, l_{27} = 6, l_{28} = 2, \text{ and } l_{29} = 3.$
Therefore, by Theorem 15 the multivariable polynomial is:

\[
\Delta_{L_\gamma}(t_1, t_2) = \prod_{j=3}^{9} \left( t_1^{l_{1j}} \cdot t_2^{l_{2j}} - 1 \right)^{\delta_j - 2} =
\]

\[
(t_1^3 - 1)^{-1}(t_1^6 - 1)^2(t_1^2 - 1)^{-1}(t_1^{l_0} - 1)
\]

\[
(t_1^{l_3} - 1)(t_1^{l_0}t_2 - 1)^{-1}(t_1^{l_3} - 1)^{-1} = 0.
\]

Next we will use Theorem 10 to find the Thurston norm of \(\phi = (p, q)\):

\[
\|\phi\|_T = \|(p, q)\|_T =
\]

\[
\prod_{j=3}^{9} (\delta_j - 2) \cdot |pl_{1j} + ql_{2j}| =
\]

\[
-3|p| + 2|6p| - 2|p| + |0| + 36|p + 6q| - 12|p + 2q| - 18|p + 3q| =
\]

\[
7|p| + |6p + q|
\]

as claimed. Since one of the terms becomes 0 independent of choice of \((p, q)\), no class is fibered on this link by Theorem 11. \(\blacksquare\)
The following is the Thurston unit ball for $L_\gamma$.

Figure 2.11: Thurston Unit Ball for $L_\gamma$

In cases 2 and 3, we notice that the Alexander polynomial vanishes. We also notice that the Thurston norm is non-trivial. Therefore, in these cases the Alexander polynomial does not provide a meaningful bound for the Thurston norm.

### 2.3 Twisted Alexander Polynomials and Fiberability

S. Friedl and T. Kim have generalized the result in Theorem 13 by considering the collection of twisted Alexander polynomials in the following theorem.
Theorem 21 (Friedl-Kim, [5]) Let $N$ be a 3-manifold different from $S^1 \times D^2$ and $S^1 \times S^2$. Let $\phi \in H^1(N;\mathbb{Z})$ be non-trivial such that $(N,\phi)$ fibers over $S^1$. Then for every representation $\alpha : \pi_1(N) \to \text{Gl}_k(\mathbb{Z})$, 

$$\Delta^0_{N,\phi} \text{ is monic and } \text{deg}(\Delta^0_{N,\phi}) = k\|\phi\|_T + \text{deg}(_{N,\phi,0}) + \text{deg}(_{N,\phi,2}).$$ (2.1)

Here, $k$ is the size of the representation $\alpha$ and $\Delta_{N,\phi,0}$ and $\Delta_{N,\phi,2}$ are determined by the Alexander modules $H_0(N;\mathbb{Z}[F])$ and $H_2(N;\mathbb{Z}[F])$.

Theorem 21 leads one to believe that the collection of twisted Alexander polynomials gives stronger obstructions to fiberability. This, in fact is confirmed by the following theorem of S. Friedl and S. Vidussi.

Theorem 22 (Friedl-Vidussi, [8]) Let $N$ be a compact, connected, orientable 3-manifold whose boundary (if any) is a union of tori. Let $\phi$ be a primitive class in $H^1(N;\mathbb{Z})$, i.e. $\text{div}(\phi) = 1$. Then if $\phi$ is not fibered, there is a representation $\alpha : \pi_1(N) \to \text{Gl}_k(\mathbb{Z})$ for which the conditions in (2.1) are not satisfied.

For knots of genus 1, this result can be enhanced to show that, there is some representation $\alpha$ for which the twisted Alexander polynomial vanishes [7].

The proof of Theorem 22 is not constructive, in that it does not give us a way to find such an $\alpha$. We have found explicit representations for the knot $K$ and the link $L_\alpha$ for which (2.1) is violated. This will be the focus of the next chapter.
Chapter 3

Main Results

For the calculations up to this point, we have used the fact that the knot \( K \) and the links \( L_\alpha, L_\beta, \) and \( L_\gamma \) are graph links. Following the techniques of D. Eisenbud and W. Neumann, we have used the splice diagram to find the Alexander polynomial. We have also used the splice diagram to show the fiberability of these links. The technique of Eisenbud and Neumann is not available for other classes of links and it is in general non-trivial to decide the fiberability of links. The results that follow show how twisted Alexander polynomials can be effective to study this problem.

3.1 Fundamental Group of the Exterior of \( K \)

To compute the twisted Alexander polynomials of a 3-manifold, we need its fundamental group. For a knot in \( S^3 \), one can use the Wirtinger presentation of any blackboard projection of the knot. (For details on Wirtinger presentation for knots and links in \( S^3 \) see [12].) Given the knot \( K \subset \Sigma \), this method is not directly available, as we do not have access
to any blackboard presentation. The route we will follow uses instead Seifert-Van Kampen theorem and the decomposition of the knot exterior into three components reflected by the splice diagram of $K$ given in Figure 2.1.

From now on, for the sake of simplicity, when we talk about the fundamental group of the exterior of a link or a knot $L$, we will call it the fundamental group of $L$. We will be following this convention in our notation as well. For example, we will denote the fundamental group of the exterior of the knot $K$ as $\pi_1(K)$ instead of $\pi_1(\Sigma \setminus (\nu(K)))$.

**Lemma 23** The exterior of the knot $K$ has the following fundamental group:

$$\pi_1(K) = \langle x, y, s, t, b | xyx = yxy, stbst = bstb, $$

$$xs = sx, xt = tx, s = x^{-1}yx^2yx^{-3}, x = (st)^{-1}b(st)^2b(st)^{-3} \rangle.$$

Note: This is not the most simplified version of the fundamental group. However, no further simplification is necessary for our purposes.

**Proof.** First, we will look at the three building blocks of the splice diagram. If we separate the middle node from the rest, we get the following splice diagram.

![Splice Diagram of the 3-Component Necklace](image-url)
The three-component necklace that this splice diagram represents is the one in Figure 3.2. The arrowhead vertex with weight 0 is the main loop, and the ones with weight 1 are the two hanging loops. We will call the main loop $N_0$, the loop hanging on the left $N_1$ and the one hanging on the right $N_2$. The following is its projection.

$$\bigwedge \bigwedge \bigwedge$$

Figure 3.2: 3-Component Necklace

When we speak of meridians of various components from now on, we mean the curve that has a fixed point at infinity and loops around each arc with linking number 1. To avoid making diagrams busy, we will put the names of the meridians on the arc and will not include the actual meridians in pictures. For this necklace, let $\mu(N_1) = s$, and $\mu(N_2) = t$ be the meridians of $N_1$ and $N_2$. Also since $N_0$ is made of two arcs $m$ and $n$, we can choose as meridian of this component either $m$ or $n$. Using the Wirtinger presentation for links, we see that the fundamental group of this link is:

$$\langle m, n, s, t | tnt^{-1}m^{-1} = 1, sms^{-1}n^{-1} = 1, nsn^{-1}s^{-1} = 1, mtm^{-1}t^{-1} = 1 \rangle.$$
Now, we simplify the group. The first relation gives $m = tnt^{-1}$, so we can eliminate $m$ as a generator. If we replace $m$ by $tnt^{-1}$ in the second relation and use the third relation, we get $tn = nt$. Using these two relations in the fourth relation will give us: $tn = nt$, which is equivalently redundant. Hence, simplifying the fundamental group of the three component necklace, we get

$$\pi_1(N) = \langle n, s, t | ns = sn, nt = tn \rangle.$$ 

This is consistent with the fact that the exterior of this necklace in $S^3$ is a 2-punctured disk cross the circle $S^1$. Hence its fundamental group is $\mathbb{Z} \oplus (\mathbb{Z} * \mathbb{Z})$. Since there will be splicing along the component $N_1$, we note that the longitude of this component is $\lambda(N_1) = n$, and its meridian is $\mu(N_1) = s$.

The node on the left represents the right-handed trefoil knot with the canonical orientation, since the latter is the $(2, 3)$ cable on the unknot as we can read from its splice diagram (Proposition 7.3 in [3]). We will call it $T_L$. The following diagram is the node on the left.

![Splice Diagram of the Trefoil on the Left](image)

Figure 3.3: Splice Diagram of the Trefoil on the Left

Considering the following projection of the right-handed trefoil, we can use Wirtinger presentation for knots to calculate the fundamental group.
Doing so will give us the following group:

\[ \pi_1(T_L) = \langle x, y, z | yx = xz, xz = yz, zy = yx \rangle. \]

We can simplify this presentation to get:

\[ \pi_1(T_L) = \langle x, y | yxy = yxy \rangle. \]

For this knot, we will choose the meridian to be \( \mu(T_L) = x \). Then by the details discussed in Remark 3.13 of [1], the longitude will be \( \lambda(T_L) = zyx = x^{-1}yx^{-3}yx^{-3}. \)

Splicing on the left, we identify the longitude of the trefoil knot with the meridian of \( N_1 \) and the meridian of the trefoil knot with the longitude of \( N_1 \). Doing so will yield relations \( s = x^{-1}yx^2yx^{-3} \) and \( x = n \) respectively.

The node on the right is another copy of the right-handed trefoil knot, we will call it \( T_R \). Its splice diagram is the following.
Figure 3.5: Splice Diagram of the Trefoil on the Right

We use the Wirtinger presentation to calculate the fundamental group for the right-handed trefoil knot in Figure 3.6. Doing so, we get the following group:

$$\pi_1(T_R) = \langle a, b \mid aba = bab \rangle.$$  

when \( c = a^{-1}ba \). If we choose the meridian to be \( a \), then the longitude is \( caba^{-3} = a^{-1}ba^2ba^{-3} \). The splicing on the right happens along the \( N_0 \) component of the necklace, with meridian \( \mu(N_0) = n \) and longitude \( \lambda(N_0) = st \). Hence after splicing on the right, we will have the relations \( st = a \) and \( n = a^{-1}ba^2ba^{-3} \).

Figure 3.6: Trefoil Knot on the Right
Given the fundamental groups of each of the building blocks, along with the relations due to the splicing, the Seifert-Van Kampen Theorem states that the fundamental group of the knot $K$ is:

$$\pi_1(K) = \langle x, y, n, t, a, b | xyx = yxy, aba = bab, ns = sn, nt = tn, x = n, s = x^{-1}yx^2yx^{-3}, st = a, n = a^{-1}ba^2ba^{-3} \rangle.$$ 

Replacing $n$ by $x$ and $a$ by $st$ gives:

$$\pi_1(K) = \langle x, y, s, t, b | xyx = yxy, bst = bstb, sx = xs, xt = tx, s = x^{-1}yx^2yx^{-3}, x = (st)^{-1}b(st)^2b(st)^{-3} \rangle$$

as claimed.

### 3.1.1 Finding an Explicit Representation $\alpha$ that Shows $K$ is not Fibered

In this section, we will discuss an explicit representation of $\pi_1(K) \to GL_5(\mathbb{Z})$. To do so, we use the computer program Knottwister written by S. Friedl. For more information about the program and its documentation, see [4].

**Theorem 24** There exists a representation $\alpha : \pi_1(K) \to S_5 \to GL_5(\mathbb{Z})$ such that $\Delta_{K,\phi}^\alpha$ is not monic.

**Proof.** Knottwister takes the fundamental group of $K$ along with a cohomology class $\phi$ as the input data. For knots, $\phi$ can be chosen to be the abelianization map $\phi : \pi_1(K) \to \mathbb{Z}$. To identify explicitly the abelianization map $\phi$ we add the commutator relations to the fundamental group found in Lemma 23 to get the following group:
\[ \pi_1(K) = \langle x, y, n, s, t, a, b \vert xyx = yxy, aba = bab, ns = sn, nt = tn, x = n, s = x^{-1}yx^2yx^{-3}, st = a, n = a^{-1}ba^2ba^{-3}, xy = yx, xn = nx, xs = sx, xt = tx, xa = ax, xb = bx, yn = ny, ys = sy, yt = ty, ya = ay, yb = by, na = an, nb = bn, st = ts, sa = as, sb = bs, ta = at, tb = bt, ab = ba \rangle. \]

We can write the first relation as \( x^2y = xy^2 \). After the possible cancelations, this relation becomes \( x = y \). We simplify the rest of the relations to get the following new relations:

\[
\begin{align*}
x &= y, & st &= b, & s &= 1, & x &= b^2t^{-2}, & \text{and } x &= 1.
\end{align*}
\]

Considering these relations, if we switch to additive notation, \( \phi \) is the following map:

\[
\begin{align*}
\phi(x) &= \phi(y) = \phi(s) = 0 \text{ and } \phi(b) = \phi(t) = 1.
\end{align*}
\]

Let \( \alpha \) be a homomorphism from \( \pi_1(K) \) to the symmetric group \( S_5 \) such that:

\[
\begin{align*}
\alpha(a) &= (15234) \\
\alpha(b) &= (13524) \\
\alpha(n) &= (14523) \\
\alpha(s) &= (12345) \\
\alpha(t) &= (15234) \\
\alpha(x) &= (14523) \\
\alpha(y) &= (34125)
\end{align*}
\]
Here, one-line permutation notation is used. It can be easily checked that $\alpha$ is a homomorphism, meaning that it respects the relations of the fundamental group.

The ordinary Alexander polynomial is $t^2 - t + 1$, which is identical to that of the trefoil knot. However, the Knottwister gives the twisted Alexander polynomial $\Delta^{\alpha}_{K, \phi}$ with coefficients module $p$ for different prime numbers. The twisted Alexander polynomial given by the particular representation $\alpha$ over various fields equals:

$$0 \in F_5[t^{\pm 1}]$$
$$0 \in F_7[t^{\pm 1}]$$
$$0 \in F_11[t^{\pm 1}]$$
$$0 \in F_13[t^{\pm 1}]$$
$$0 \in F_17[t^{\pm 1}]$$
$$0 \in F_19[t^{\pm 1}]$$
$$0 \in F_23[t^{\pm 1}]$$
$$0 \in F_29[t^{\pm 1}]$$.

Since the twisted Alexander polynomial associated with this representation vanishes over these fields, it is not monic. ■

We can conclude from the previous theorem and Theorem 21 that the knot $K$ is not fibered. Clearly, having the polynomial vanish over any of the fields above would be sufficient to show it is not monic. However, the fact that it vanishes over all these fields
is a strong evidence that it is indeed 0. This observation is consistent with the enhanced version of Theorem 22 appearing in [7], since the genus of $K$ is 1 as we saw in Proposition 16.

3.2 The Link $L_\alpha$

3.2.1 Fundamental Group of the Exterior of $L_\alpha$

In this section, we use twisted Alexander polynomials to prove that the exterior of $L_\alpha$ is not fibered for two different cohomology classes. Again, we need to find its fundamental group first.

**Lemma 25** The fundamental group of the exterior of $L_\alpha$ is:

$$\pi_1(L_\alpha) = \langle c, d, e, f, g, h, i, j, k, l, o, p, q, r, u, v, w, a, x, y, n, s, t | x y x = y x y, n s = s n, n t = t n, s = x^{-1} y x^2 y x^{-3}, e = s t, g d = c g, v e = d v, c f = e c, p g = f p, v h = g v, w i = h w, a j = i a, e k = j e, r c = k r, e o = l e, r p = o r, g q = p g, v r = q v, c u = r c, p v = u p, h w = v h, i a = w i, j l = a j \rangle.$$  

**Proof.** Considering the splice diagram of $L_\alpha$, we look at its building blocks. As discussed for $K$, the node on the left is a copy of a right-handed trefoil knot, $T_L$, with fundamental group $\langle x, y | x y x = y x y \rangle$ (see Figure 3.4). We choose the meridian of this knot to be $x$, and hence the longitude is $x^{-1} y x^2 y x^{-3}$. The middle node gives the splice diagram of a 3-component necklace with fundamental group $\langle n, s, t | n s = s n, n t = t n \rangle$ (see Figure
3.2). The splicing on the left happens along the loop $N_1$. We notice that the meridian of this component is $s$ and its longitude is $n$. So the splicing relations on the left are:

$$s = x^{-1}yx^2yx^{-3}, \text{ and } x = n$$

as we identify $\mu(T_L)$ with $\lambda(N_1)$ and $\mu(N_1)$ with $\lambda(T_L)$.

What happens on the right is more complicated. For $L_\alpha$, the node on the right before splicing is the next figure.

![Figure 3.7: Splice Diagram of the Link $D$ on the Right](image)

The splice diagram in Figure 3.7 represents a 2-component link, as it has two arrowhead vertices. It is the $(2, 3)$ cable on the right-handed trefoil (see Proposition 7.3 in [3]). Hence each component is a copy of the right-handed trefoil knot, such that they have linking number 6. Figure 3.8 at the end of this subsection is a blackboard projection of this link. We call this 2-component link $D$.

Now we discuss splicing that happens on the right. As it happened for $K$, splicing on the right happens along the main loop of the necklace, $N_0$. If we choose to do the splicing on the outer trefoil of the $D$, and choose its meridian to be $\mu(D) = e$, the longitude will be
$\lambda(D) = cpvwxergve^{-3}$. Hence the splicing relations are:

$$n = cpvwxergve^{-3}, \text{ and } e = st.$$  

Therefore, considering the fundamental groups of the three building blocks of $L_{\alpha}$ and the relations that result from splicing, we see that the fundamental group of the exterior of $L_{\alpha}$ is:

$$\pi_1(L_{\alpha}) = \langle c, d, e, f, g, h, i, j, k, l, o, p, q, r, u, v, w, a, x, y, n, s, t |$$

$$xyx = yxy, ns = sn, nt = tn, s = x^{-1}yx^2yx^{-3}, e = st,$$

$$gd = cg, ve = dv, cf = ec, pg = fp, vh = gv, wi = hw, aj = ia, ek = je, rc = kr,$$

$$eo = le, rp = or, gq = pg, vr = qv, cu = rc, pv = up, hw = vh, ia = wi, jl = aj \rangle$$

as claimed. $\blacksquare$
Figure 3.8: The Link $D$
3.2.2 Finding Representations for $\pi_1(L_\alpha)$ in Two Cases

Since for all knots, the abelianization of their fundamental group is isomorphic to $\mathbb{Z}$, if one cohomology class is fibered, all are. However, for links, we might see that for the same link, some cohomology classes are fibered and others are not. Now we use twisted Alexander polynomials to show that two different cohomology classes for $L_\alpha$ are not fibered.

In the two following theorems, we will find explicit representations such that $\Delta^\alpha_{N,\phi}$ are not monic, when $N$ is the exterior of $L_\alpha$ and $\phi$ is one of the classes $(0,1)$ or $(1,-1)$. Consequently by Theorem 21, the pair $(N,\phi)$ is not fibered for either $\phi$.

**Theorem 26** If $N$ is the exterior of $L_\alpha$ and $\phi = (0,1)$, there is a representation $\alpha : \pi_1(N) \to S_5 \to GL(\mathbb{Z},5)$ such that $\Delta^\alpha_{N,\phi}$ is not monic.

**Proof.** First, we need to understand what $\phi$ does as a map. We add all the commutator relations to the fundamental group in Lemma 25. This will result in the following relations:

$$c = d = e = f = g = h = i = j = k = t$$

$$o = l = p = q = r = u = v = w = a$$

$$s = 1, x = y = n = v^6.$$  

As expected for a 2-component link, the abelianization of $\pi_1(L_\alpha)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. We can see from the splice diagram of this link that the two components that survive are one of the hanging loops of the necklace, $N_2$, and the trefoil knot inside the link $D$. These are the arrowhead vertices in the splice diagram. Hence $\phi$ is the homomorphism that sends $v$ to 0, and $t$ to 1. Again, the Knottwister takes the fundamental group of $L_\alpha$
from Lemma 25, along with the homomorphism $\phi$ as an input. In multiplicative notation, $\phi$ is the following map:

\[
\begin{align*}
\phi(c) &= \phi(d) = \phi(e) = \phi(f) = \phi(g) = \phi(h) = \phi(i) = \phi(j) = \phi(k) = \phi(t) = 1 \\
\phi(a) &= \phi(l) = \phi(o) = \phi(p) = \phi(q) = \phi(r) = \phi(u) = \phi(v) = \phi(w) = \phi(a) \\
&= \phi(s) = \phi(x) = \phi(y) = \phi(n) = 0.
\end{align*}
\]

The program gives us the following representation $\alpha : \pi_1(N) \to S_5 \to \text{GL}(\mathbb{Z},5)$, when the elements in $S_5$ are written in on-line permutation form:

\[
\begin{align*}
a &\mapsto (13245) & c &\mapsto (23415) & d &\mapsto (45321) & e &\mapsto (24351) \\
f &\mapsto (32514) & g &\mapsto (13524) & h &\mapsto (14532) & i &\mapsto (15234) \\
j &\mapsto (13524) & k &\mapsto (31425) & l &\mapsto (14325) & n &\mapsto (45312) \\
o &\mapsto (21345) & p &\mapsto (21345) & q &\mapsto (42315) & r &\mapsto (21345) \\
s &\mapsto (12345) & t &\mapsto (45312) & u &\mapsto (42315) & v &\mapsto (42513) \\
w &\mapsto (15342) & x &\mapsto (45312) & y &\mapsto (42513).
\end{align*}
\]

For this twist, we get the following twisted Alexander polynomials over various fields:

\[
\begin{align*}
0 &\in \mathbb{F}_7[t^{\pm 1}] \\
0 &\in \mathbb{F}_{11}[t^{\pm 1}] \\
0 &\in \mathbb{F}_{13}[t^{\pm 1}] \\
0 &\in \mathbb{F}_{17}[t^{\pm 1}] \\
0 &\in \mathbb{F}_{19}[t^{\pm 1}]
\end{align*}
\]
0 ∈ \mathbb{F}_{23}[t^{\pm 1}]

0 ∈ \mathbb{F}_{29}[t^{\pm 1}].

Since the twisted Alexander polynomial vanishes over these finite fields, it cannot be monic.

Again, by Theorem 21, the pair \((N, (0, 1))\) is not fibered. In the next theorem, we do the same for \(\phi = (1, -1)\).

**Theorem 27** If \(N\) is the exterior of \(L_\alpha\) and \(\phi = (1, -1)\), there is a representation \(\alpha : \pi_1(N) \to S_5 \to GL(\mathbb{Z}, 5)\) such that \(\Delta^\alpha_{N, \phi}\) is not monic.

**Proof.** Using multiplicative notation, \(\phi\) can be viewed as the map that does the following to the generators of \(\pi_1(L_\alpha)\):

\[
\begin{align*}
\phi(c) &= \phi(d) = \phi(e) = \phi(f) = \phi(g) = \phi(h) = \phi(i) = \phi(j) = \phi(k) = \phi(t) = -1 \\
\phi(a) &= \phi(l) = \phi(o) = \phi(p) = \phi(q) = \phi(r) = \phi(u) = \phi(v) = \phi(w) = \phi(a) = 1 \\
\phi(s) &= 0, \phi(x) = \phi(y) = \phi(n) = 6.
\end{align*}
\]

Given this information, Knottwister gives us the following representation \(\alpha\), (in one-line permutation form):

\[
\alpha : \pi_1(M) \to S_5 \to GL(\mathbb{Z}, 5)
\]
For this representation, the twisted Alexander polynomial over various fields equals:

\[0 \in \mathbb{F}_5[t^{\pm 1}]\]
\[0 \in \mathbb{F}_7[t^{\pm 1}]\]
\[0 \in \mathbb{F}_{11}[t^{\pm 1}]\]
\[0 \in \mathbb{F}_{13}[t^{\pm 1}]\]
\[0 \in \mathbb{F}_{17}[t^{\pm 1}]\]
\[0 \in \mathbb{F}_{19}[t^{\pm 1}]\]
\[0 \in \mathbb{F}_{23}[t^{\pm 1}]\]
\[0 \in \mathbb{F}_{29}[t^{\pm 1}]\]

Hence \(\Delta_{N,\phi}^x\) is not monic as claimed. ■

Therefore, that pair \((N, (1, -1))\) is not fibered by Theorem 21.

### 3.3 More on Links \(L_\beta\) and \(L_\gamma\)

For links \(L_\beta\) and \(L_\gamma\), the problem of fiberability if already solved by Theorem 13. However, as we saw in Chapter 2, the ordinary Alexander polynomials for these links vanish.
Consequently, they do not give a useful bound on the Thurston norm. In this section, we will introduce a new type of polynomials and we will use them to get meaningful bounds for the Thurston norm whenever possible.

### 3.3.1 Fundamental Groups of the Exteriors of $L_\beta$ and $L_\gamma$

In this section, we compute the fundamental groups of the exteriors of $L_\beta$ and $L_\gamma$.

**Proposition 28** The fundamental group of $L_\beta$ is the following:

$$\pi_1(L_\beta) = \langle x, y, a, b, s, r, t, n | aba = bab, xyx = yxy,$$

$$nr = rn, nt = tn, ns = sn, x = n, s = x^{-1}yx^2yx^{-3}, a = rst, n = a^{-1}ba^2ba^{-3} \rangle.$$  

**Proof.** For this 2-component link, the left node and the right node represent a copy of the right-handed trefoil each, similar to the knot $K$. However, the middle node represents a 4-component necklace. If we look at its splice diagram before splicing, we see the following.

*Figure 3.9: Splice Diagram of the 4-Component Necklace*
Figure 3.10 shows a blackboard projection of this necklace.

As before, the trefoil on the left has fundamental group \( \langle x, y | xyx = yxy \rangle \) with meridian \( x \) (by choice) and longitude \( x^{-1}yx^2yx^{-3} \). The trefoil on the right has fundamental group \( \langle a, b | aba = bab \rangle \) with meridian \( a \) and longitude \( a^{-1}ba^2ba^{-3} \). We use Wirtinger for links (see [12]) to calculate the fundamental group of the four-component necklace, which is represented by the node in the middle. This will give us the following group:

\[
\langle n, m, l, r, s, t | tn = mt, rl = nr, sm = ls, ls = sl, nr = rn, mt = tm \rangle.
\]

If we simplify this group, we get:

\[
\langle s, r, t, n | nr = rn, nt = tn, ns = sn \rangle.
\]

Splicing on the left happens along the hanging loop \( N_1 \). Hence the splicing relations are \( x = n \) and \( s = x^{-1}yx^2yx^{-3} \). Splicing on the right happens along the main loop \( N_0 \). We choose the meridian to be \( n \), so by the guidelines given in Remark 3.13 of [1], the longitude
is \(rst\). Therefore the splicing relations on the right are \(a = rst\) and \(n = a^{-1}ba^{2}ba^{-3}\). If follows that the fundamental group of \(L_{\beta}\) is:

\[
\langle x, y, a, b, s, r, t, n \mid aba = bab, xys = yxy, nr = rn, nt = tn, \\
ns = sn, x = n, s = x^{-1}yx^{2}yx^{-3}, a = rst, n = a^{-1}ba^{2}ba^{-3} \rangle
\]

as claimed. ■

In the next proposition, we calculate the fundamental group of \(L_{\gamma}\).

**Proposition 29** The fundamental group of the exterior of \(L_{\gamma}\) is the following group:

\[
\pi_{1}(L_{\gamma}) = \langle a, b, n, s, t, c, d, e, f, g, h, i, j, k, o, l, p, q, r, u, v, w \mid \\
gd = cg, ve = dv, cf = ec, pg = fp, vh = gv, wi = hw, xj = ix, ek = je, rc = kr, \\
eo = le, rp = or, gq = pg, vr = qv, cu = rc, pv = up, hw = vh, ix = wi, jl = xj, \\
aba = bab, ns = sn, nt = tn, a = st, n = a^{-1}ba^{2}ba^{-3}, e = n, s = cpvwxergve^{-3} \rangle
\]

**Proof.** For this link, the link \(D\) is presented by the left node. Not to overuse the letter \(a\), we call the arc that is labeled \(a\) in Figure 3.8, \(x\). In the middle, we have the same 3-component necklace as the one for \(K\), and a right-handed trefoil is presented by the right node. We splice on the left along the outer trefoil knot on \(D\) and the hanging loop \(N_{1}\). On the right, the splicing is identical to that of \(K\). Doing so, we get the fundamental group claimed in the proposition. ■
3.3.2 A “Secondary” Polynomial, $\tilde{\Delta}_1^\alpha(t)$

As we saw in the last chapter, since the ordinary Alexander polynomial is 0 for $L_\beta$ and $L_\gamma$, we may not use Theorem 13 to get a useful bound for the Thurston norm. From now on, we will only be concerned with the single-variable version of twisted Alexander polynomial for simplicity. Also, we replace $\mathbb{Z}[t^{\pm 1}]$ by $\mathbb{F}[t^{\pm 1}]$ in the definition of the Alexander module where $\mathbb{F} = \mathbb{F}_p$ is a field. The reason to do so is the fact that $\mathbb{F}[t^{\pm 1}]$ is a principal ideal domain. As a result, we have the following isomorphism:

$$H_1(N, \mathbb{F}^k[t^{\pm 1}]) \cong \mathbb{F}[t^{\pm 1}]^r \oplus \bigoplus_{j=1}^m \mathbb{F}[t^{\pm 1}]/(p_j(t))$$

for $p_1(t), ..., p_m(t) \in \mathbb{F}[t^{\pm 1}]$. The type of polynomials we will examine are defined to be:

$$\tilde{\Delta}_{N,\phi}^\alpha := \prod_{j=1}^m p_j(t)$$

regardless of the rank $r$. Not much is known about these polynomials.

S. Friedl and T. Kim have proved the following theorem that relates these polynomials to the Thurston norm in [5].

**Theorem 30** (Friedl-Kim, [5]) Let $L = L_1 \cup L_2 \cup \ldots \cup L_m$ be a link with $m$ components. Denote its meridian by $\mu_1, \ldots, \mu_m$. Let $\phi \in H^1(X(L); \mathbb{Z})$, be primitive and dual to a meridian $\mu_i$, when $X(L)$ denotes the exterior of $L$. Hence $\phi(\mu_i) = 1$ for some $i$ and $\phi(\mu_j) = 0$ for $j \neq i$. Then

$$\|\phi\|_T \geq \frac{1}{k} \text{deg}(\tilde{\Delta}_1^\alpha(t)) - 1.$$ 

Here, $k$ is the size of the representation $\alpha$. 

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Theorem 30 will help us improve the bound of the Thurston norm for the class 
(0, 1) for both $L_{\beta}$ and $L_{\gamma}$. Recall from Section 2.2 that for $L_{\beta}$, the Thurston norm of a 
general cohomology class $(p, q)$ is $|p + q|$. So for this link, $\|(0, 1)\|_T = 1$. In this case, 
Knottwister computes the $\tilde{\Delta}_1^\alpha(t)$ to be $1 - t + t^2$ over $\mathbb{F}_{13}$ when $\alpha$ is trivial (so $k = 1$). 
Therefore, for the pair $(L_{\beta}, (0, 1))$ we get 
$$\|(0, 1)\|_T \geq 2 - 1 = 1$$ 
which is a sharp bound.

Now, we consider the same cohomology classes on $L_{\gamma}$. We know from our calcu-
lations in section 2.2 that for this link, $\|\phi\|_T = \|(p, q)\|_T = 7|p| + |6p + q|$. So for this link 
$\|(0, 1)\|_T = 1$. The Knottwister yields the $\tilde{\Delta}_1^\alpha(t) = 1 - t + t^2$ over $\mathbb{F}_{13}$ again, when $\alpha$ is 
trivial, which is again a sharp bound.
Bibliography


