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Authors
Mariottini, F
Capolino, F
Maci, S
et al.

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High-Frequency Green's Function for a Rectangular Array of Dipoles with Weakly Varying Tapered Excitation

F. Mariottini¹, F. Capolino², S. Maci¹, and L. B. Felsen²

1) Dip. Ingegneria dell'Informazione, Università di Siena, Via Roma 56, 53100 Siena, Italy.
2) Dept. Aerospace and Mechanical Eng., and Dept. of Electrical and Comp. Eng., Boston University, 110 Cummington St., Boston, MA 02215, USA.

I. INTRODUCTION

The array Green's function (AGF) is the basic building block for the full-wave analysis of planar phased array antennas. Its representation in terms of element-by-element summation over the individual dipole radiations can be replaced by a more efficient global representation constructed via Poisson summation. The resulting Poisson-transformed integrals can be interpreted as the radiation from continuous equivalent Floquet wave (FW)-matched source distributions extending over the array aperture [1], [2]. Applying high-frequency asymptotics to each FW-matched array aperture casts the AGF in the format of a generalized Geometrical Theory of Diffraction (GTD) which includes conical wavefront edge diffracted rays as well as spherical wavefront vertex diffracted rays. In this paper, the results in [1], valid for equi-amplitude excitation, are extended to accommodate tapered illumination, which also includes dipole amplitudes tending to zero at the edges. This extension, which has been performed in [3] with a numerical technique based on the discrete Fourier transform (DFT), is herein carried out by a direct Poisson-transformed asymptotic evaluation of the strip-array GF with inclusion of asymptotically subdominant "slope" edge and vertex diffracted fields, in addition to the dominant edge and vertex diffracted fields for appreciable edge illumination. Numerical results are presented for illustration.

II. FORMULATION

Consider a rectangular periodic array of $N_1 \times N_2$ linearly phased dipoles located in the $z_1, z_2$-plane (Fig. 1a), with interelement spatial period along the $z_1$ and $z_2$ directions given by $d_1$ and $d_2$, and the interelement phase gradient by $\gamma_1$ and $\gamma_2$, respectively. All dipoles are oriented along the unit vector $\hat{J}_0$ (a bold character denotes a vector quantity, and a caret denotes a unit vector). Superimposed upon that background is a $z$-dependent amplitude-separable tapering function $f(z_1, z_2) = f_1(z_1)f_2(z_2)$, sampled at the dipole locations, $J(n_1d_1, n_2d_2) = J(n_1d_1, n_2d_2)e^{-j(\gamma_1n_1d_1 + \gamma_2n_2d_2)}$, with $J(z_1, z_2)$ denoting the dipole current amplitude, and $(z_1, z_2) = (n_1d_1, n_2d_2)$ denoting the location of the $(n_1, n_2)$th dipole. A time dependence $e^{j\omega t}$ is suppressed.

Without compromising practical utility, we assume $f(z_1, z_2)$ real and positive in the domain $z_1 \in [0, L_1]$, $z_2 \in [0, L_2]$ and zero elsewhere; here, $L_1 = (N_1 - 1)d_1$, and $L_2 = (N_2 - 1)d_2$. The electromagnetic vector field at any observation point can be derived from the vector potential $A(r)$ by summing over the individual $(n_1, n_2)$ dipole radiations $f(n_1d_1, n_2d_2)e^{j(\omega r - k_0 r)}/(4\pi \varepsilon_0 r)$, where $\varepsilon_0 = \frac{c}{\omega}$. Employing the $(k_1, k_2)$ spectral Fourier representation of the free space Green's function as shown in [1], yields

\[ A(r) = \frac{1}{\sqrt{s^2 + k^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(k_1) f_2(k_2) e^{j\gamma_1 k_1 d_1} e^{j\gamma_2 k_2 d_2} d_{k_1} d_{k_2}. \tag{1} \]

with $I(n_1d_1) = \sum_{n_2=0}^{N_2-1} e^{j\gamma_2 n_2 d_2} f(n_1d_1)$, $i = 1, 2$, $g_1 = k_1 z_1 + k_2 z_2$, and $k_0 = \sqrt{k_1^2 - k_2^2}$ with $\Im k_0 < 0$ for $k_1^2 - k_2^2 < k_1^2$, $\Im k_0 > 0$ for $k_1^2 - k_2^2 > k_1^2$ on the top Riemann sheet of the complex $k$-plane (see [2]). The $n_2$-sum in (1) is manipulated via the truncated Poisson sum formula into a $p, q$-sum of Fourier transformed functions $\tilde{f}(k_1') = \int_{k_1}^{k_1'} e^{j\gamma_1 k_1'} f(n_1d_1) d_{k_1}$, translated by the FW...
Fig. 1. (a) Geometry of the rectangular array, $p_1 = \sqrt{1^2 + z_1^2}$, $p_2 = \sqrt{1^2 + z_2^2}$, (b) $E_{y}$ component of the electric field radiated by a rectangular array with excitation function $f(x_1, x_2) = \prod_{i} \sin(\pi x_i / L_i)$, as a distance $R = 10x$ from the vertex.

wavenumbers $k_{i\mu} = \gamma + 2\pi x_i / d_i$, with $i = 1, 2$, and $\mu = q, p$, i.e.

$$I_i(k_{i\mu}) = \frac{f_i(0)}{2} + \frac{f_i(L_i)}{2} \cdot \delta(k_{i\mu} - \gamma) + \frac{1}{d_i} \int_{0}^{L_i} f_i(k_{i\mu} - k_{i\mu})$$  \hspace{1cm} (2)

III. HIGH-FREQUENCY SOLUTION

Henceforth, we assume (legitimately for actual tapering functions for large arrays) that $f_i(k_{i\mu})$ varies slowly with respect to the wavelength $\lambda$. For such weak variation, and since $f_i(k_{i\mu})$ is positive in the domain $z_i \in (0, L_i)$, its spectrum $f_i(k_{i\mu})$ is localized around $k_{i\mu} = 0$, thereby enhancing contributions to $I_i(k_{i\mu})$ from $k_{i\mu} = 0$. Thus, adiabatic methods can be applied, based on perturbation about $f_i(k_{i\mu}) = \text{const.}$. Consequently, the integral in (1) which defines $A$ is dominated asymptotically by: a) one $(k_{1\lambda}, k_{2\lambda})$ saddle-point that satisfies $(d/dk_{1\lambda})g(k_{1\lambda}, k_{2\lambda}) = 0$, yielding the vertex diffracted field; b) "quasi poles" at $k_{1\mu} = k_{1\mu u}$, that describe the same phenomenology and localization property as the spectral poles for the semi-infinite array [1], yielding FWs; c) critical points at $k_{1\mu}^c$ and $k_{2\mu}^c$ which annul $(d/dk_{1\mu})g(k_{1\mu}, k_{2\mu})$ and $(d/dk_{2\mu})g(k_{1\mu}, k_{2\mu})$ respectively, and lead to diffracted fields from edge 2 (located at $z_2 = 0$) and edge 1 (at $z_1 = 0$), respectively. Diffraction from the other two edges can be found similarly by including the appropriate phase reference in the second term in the right side of (3).

Floquet Wave Contributions. Inserting (2) into (1), the contributions due to the critical points at $(k_{1\lambda}, k_{2\lambda}) = (k_{1\mu u}, k_{2\mu u})$ are found by expanding the exponent of the integrand in Taylor series in a neighborhood of $(k_{1\mu u}, k_{2\mu u})$ (see [4]). Retaining only the dominant asymptotic term of the remainder which applies for observation points away from the array surface one finds

$$A(p) \sim \sum_{p} A_{pW} f(z_{1p}, z_{2p}) U_{pW} = \frac{e^{-j(k_{1\mu u}z_1 + k_{2\mu u}z_2)}}{2d_1d_2k_{2p}}, \hspace{1cm} (3)$$

where $z_{1p} = z_1 - pL_1 / L_{1p}$, and $z_{2p} = z_2 - pL_2 / L_{2p}$ (with branches chosen according to (1)) in real for propagating FW, and $L_{1p} = L(z_{1p}), U(z_{2p}) = U(L(z_{2p}))$ with $U(z) = 1$ or $0$ if $z > 0$ or $z < 0$ respectively. Criteria for the asymptotic validity of the expansion obtained in (3) will be given elsewhere. In (3), $A_{pW}$ is the $p$th FW for equi-amplitude excitation [1], which is multiplied in (3) by the tapering function $f(z_{1p}, z_{2p})$ evaluated at the footprint $(z_{1p}, z_{2p})$ of the $p$th FW. The stationary phase evaluation of the radiation integral associated with each $p$, $q$th equivalent FW-matched aperture distribution would provide the same result.
(see 2D case in [5]) since \((z_{1pq}, z_{2pq})\) is the stationary phase point of the \(p, q\)th spatial radiation integral. The function \(U_{pq}\) is unity or zero for \((z_{1pq}, z_{2pq})\) inside or outside the finite array dimensions. The discontinuity of the truncated FW at the Shadow boundary (SB) plane angle defined by \(\theta_{pq}\) is restored by the diffracted field that arises from the saddle point evaluation of (1).

**FW-Induced diffracted contributions.** Here, we show only the final result, derived asymptotically in [4], for the diffracted field arising from the truncation at \(z_1 \in [0, L_1]\) (edge 1) associated with the critical spectral points \((k_{x1}, k_{z})\). The total propagating diffracted field arising from edge 1 is thus represented as

\[
A^{\omega_1}(r) = \sum_{q} A_{q}^{\omega_1}(r) = \frac{e^{-i(k_{11}r + k_{12}z)}}{2\pi i} f_1(z_{1pq}) \left[ f_{l}(0) B_1(k_{12}) F(\delta_{1pq}^+) - f_{l}^+(0) B_1^+(k_{12}) F(\delta_{1pq}^+) \right]
\]

where \(A_{q}^{\omega_1}(r)\) is the \(q\)-th conical wave decaying along \(p_1\) and \(A_{q}^{\omega_1}(r) = \psi_q = \psi_{z1pq}\) locates the shadow boundary cone (SBC) centered at the vertex, which truncates the domain of existence of the \(z_1\)-edge diffracted waves. In (4) the tapering function \(f_1\) is evaluated at the diffraction points \(z_{1pq} = z_1 - \frac{y_{1pq}}{k_{12}}\), with \(y_{1pq} = \sqrt{\delta_{1pq}^2 - (k_{12})^2}\).

The discontinuity across the SBC of the edge-diffracted contribution is repaired by the diffraction from the vertex \((0,0,0)\) of the array. Analogous diffracted fields arise from the other edges.

**Vertex diffracted contributions.** Near the vertex at \((z_1, z_2) = (0,0),\) for example, the \(z_1\)-edge and \(z_2\)-edge plane FW-shadow boundaries interact with the vertex-induced conical SBCs with symmetry axes \(z_1\) and \(z_2\), respectively, that arise from the truncation of the corresponding edge diffracted fields. The confluence of these four SBs near the vertex defines the asymptotics pertaining to the vertex diffracted field \(A^{\omega_1}(r),\) obtained by the following steps. First, \(I_1\) in (2) is expanded asymptotically (integrating by parts up to the second order) as

\[
I_1(k_{12}) \sim f_1(0) B_1(k_{12}) - f_{l}^+(0) B_1^+(k_{12}) + O((k_{12} - k_{12,b})^{-3}), \quad i = 1, 2
\]

where \(B_1(k_{12}) = 1 - e^{i(k_{12} - k_{12b})k_{12}}\) and \(B_1^+(k_{12}) = j\int e^{i(k_{12} - k_{12b})k_{12}}/\left[1 - e^{i(k_{12} - k_{12b})k_{12}}\right]^2\) is its derivative. Next, the integration contours are locally deformed along a 45°-line through the \((k_{12}, k_{z})\) saddle point and then extended parallel the real axis (see [2]). Thus, inserting in to (1) the two terms of the asymptotic expansion of \(I_1(k_{12}),\) \(i = 1, 2,\) we have

\[
A^{\omega_1}(r) \sim \sum_{q} A_{q}^{\omega_1}(r)\text{ expressed as a sum of four terms. The asymptotics is then obtained by the Pauly-Clemmow method. Within this method, the asymptotic evaluation of integrals characterized by specific arrangements of critical points (saddle points (SPs) and singularities) is addressed by mapping the given integrand (both phase and amplitude) onto the simplest canonical integrand that accommodates the relevant critical point configuration. For the vertex problem, the critical parameters are tied to the \((k_{11}, k_{22}) = (k_{12}^*, k_{12}^*) = (k_{x1}, k_{z})\) first order SP, and to the \(k_{x1}\) and \(k_{z}\)-poles in (1). For details, see [2]. The final asymptotics for each \(h\)-indexed integral yields

\[
A_{q}^{\omega_1}(r) \sim \frac{(-1)^{m+i+1}}{4\pi r^{m+i+1/2}} e^{-ik_{12}r} B_1^+(k_{12}) B_1(k_{12}) Y_h(q, r, \theta_{pq})
\]
where

\[
T_{k}(a_{k}, b_{k}, w) = \frac{e^{i(a_{k}^{2} - 1) \frac{1}{2} \text{Li}_{2}(b_{k})}}{j^{n_{1} + 1} \pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\xi - \frac{b_{k}}{\sqrt{\xi^{2} - a_{k}^{2}}})^{n_{1} + 1} d\xi d\eta
\]

(7)

are the canonical transition functions. In (7), the parameters \(a_{k}, b_{k}\) and \(\xi, \eta\) are defined in [2]; the superscript \(a_{k}\) denotes the \(a\)-th derivative and, for \(h = 1, m = l = 0, c = (1 - w^{2})^{1/2};\) for \(h = 2, m = 0, l = 1, c = -(1 - w^{2});\) for \(h = 3, m = 1, l = 0, c = -(1 - w^{2})^{3/2};\) for \(h = 4, m = 1, l = 1, c = (1 - w^{2})^{3/2}.\) The numerical evaluation of the integrals \(T_{k}\) can be performed in terms of an algebraic sum of standard Generalized Fresnel Integrals and UTD transition functions. It is found (see [2]) that the vertex-diffracted contribution \(A_{k}\) contains the integral function \(T_{k}\) and accounts for the transition from a vertex-centered spherical wave to an edge-centered cylindrical wave and it compensates for the discontinuities across the SBCs. The other vertex diffracted terms \(A_{k}\) of higher asymptotic order and compensate for the discontinuities across the SBCs when the \(f_{i}^{0}(z_{i})\) is weakly tapered.

Note that when the excitation function tends to zero at the vertex, only the \(A_{k}\) contribution remains.

Numerical Example. The total high frequency solution is

\[
A = \sum_{j=1}^{N_{1}} \sum_{k=1}^{2} A_{k}^{jw} f_{j}(a_{j}, b_{j}, w) E_{j} + \sum_{i=1}^{4} \sum_{i=1}^{N_{2}} A_{k}^{i} f_{i}(a_{i}, b_{i}, w) E_{i} + \sum_{i=1}^{4} \sum_{i=1}^{N_{2}} A_{k}^{i} f_{i}(a_{i}, b_{i}, w) E_{i}
\]

(8)

with \(u - q\) for \(i = 1, 3, u - p\) for \(i = 2, 4.\) The preliminary numerical example in Fig. 1b shows the accuracy of the truncated Floquet wave (TFW) asymptotics in (8), including the transition region close to the SBC, when compared with a reference solution obtained by an element-by-element summation over the contributions from each dipole. The test array has \(N_{1} = N_{2} = 200\) elements. The dipoles are oriented along \(z_{2}\) with interelement spacings \(d_{1} = d_{2} = 0.53\); interelement phasings \(\gamma_{1} = \gamma_{2} = 1.52/\lambda\) and tapering \(f_{i}(a_{i}, b_{i}) = \frac{1}{2} \sin(\frac{\pi}{2} a_{i}/L_{i}).\) Electromagnetic quantities are evaluated from the vector potential as in [1], and only the \(E_{2}\) component is shown in Fig. 1b along a scan at \(R = 10\) in the \(y_{1}, z_{2}\) plane from the vertex of the array.

We observe that when the observation point \(P\) passes through \(d_{s} = 55\) (scan angle = 14° in Fig. 1b), the TFW without the vertex contribution vanishes, and the tip contribution provides the required continuity for the total field. In this particular case, \(f_{i}^{0}(0, \pi/2) = 0,\) thus the diffracted field and the vertex field involves only the terms \(f_{i}(0),\) therefore providing a good test case for the additional "slope edge- and vertex-diffracted fields".

REFERENCES


