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FUNDAMENTAL GROUPS OF LINKS OF ISOLATED SINGULARITIES

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Starting with Grothendieck’s proof of the local version of the Lefschetz hyperplane theorems [Gro68], it has been understood that there are strong parallels between the topology of smooth projective varieties and the topology of links of isolated singularities. This relationship was formulated as one of the guiding principles in the monograph [GM88, p.26]: “Philosophically, any statement about the projective variety or its embedding really comes from a statement about the singularity at the point of the cone. Theorems about projective varieties should be consequences of more general theorems about singularities which are no longer required to be conical”.

The aim of this note is to prove the following, which we consider to be a strong exception to this principle.

**Theorem 1.** For every finitely presented group $G$ there is an isolated, 3-dimensional, complex singularity $(0 \in X_G)$ with link $L_G$ such that $\pi_1(L_G) \cong G$.

By contrast, the fundamental groups of smooth projective varieties are rather special; see [ABC+96] for a survey. Even the fundamental groups of smooth quasi projective varieties are quite restricted [Mor78, KM98a, CS08, DPS09]. This shows that germs of singularities can also be quite different from quasi projective varieties.

We think of a complex singularity $(0 \in X)$ as a contractible Stein space sitting in some $\mathbb{C}^N$. Then its link is $\text{link}(X) := X \cap S^{2N-1}$, an intersection of $X$ with a small $(2N-1)$-sphere centered at $0 \in X$. Thus $\text{link}(X)$ is a deformation retract of $X \setminus \{0\}$.

There are at least three natural ways to attach a fundamental group to an isolated singularity $(0 \in X)$. Let $p : Y \to X$ be a resolution of the singularity with simple normal crossing exceptional divisor $E \subset Y$. (That is, the irreducible components of $E$ are smooth and they intersects transversally.) We may assume that $Y \setminus E \cong X \setminus \{0\}$. The following 3 groups are all independent of the resolution.

- $\pi_1(\text{link}(X)) = \pi_1(X \setminus \{0\}) = \pi_1(Y \setminus E)$.
- $\pi_1(Y) = \pi_1(E)$; we denote it by $\pi_1(\mathcal{R}(X))$ to emphasize its independence of $Y$. These groups were first studied in [Kol93, Tak03].
- $\pi_1(D(E))$ where $D(E)$ denotes the dual simplicial complex of $E$. (That is, the vertices of $D(E)$ are the irreducible components $\{E_i : i \in I\}$ of $E$ and for $J \subset I$ we attach a $|J|$-simplex for every irreducible component of $\bigcap_{i \in J} E_i$; see Definition [KS] for details.) We denote this group by $\pi_1(\mathcal{DR}(X))$; it is actually the main object of our interest.

There are natural surjections between these groups:

$$\pi_1(\text{link}(X)) = \pi_1(Y \setminus E) \twoheadrightarrow \pi_1(Y) = \pi_1(E) \twoheadrightarrow \pi_1(D(E)).$$

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Usually neither of these maps is an isomorphism. The kernel of \( \pi_1(Y \setminus E) \to \pi_1(Y) \) is generated by loops around the irreducible components of \( E \), but the relations between these loops are not well understood; see [Mum61] for some computations in the 2-dimensional case.

The kernel of \( \pi_1(E) \to \pi_1(D(E)) \) is generated by the images of \( \pi_1(E_i) \to \pi_1(E) \). In all our examples the \( E_i \) are simply connected, thus \( \pi_1(E) = \pi_1(D(E)) \). We do not investigate the difference between these two groups in general.

Our proof of Theorem 1 proceeds in two distinct steps.

Simpson showed in [Sim10, Theorem 12.1] that every finitely presented group \( G \) is the fundamental group of a singular projective variety. He posed the question if this variety can be chosen to have only simple normal crossing singularities. Our first result shows that this is indeed true:

**Theorem 2.** For every finitely-presented group \( G \) there is a complex, projective surface \( S_G \) with simple normal crossing singularities only such that \( \pi_1(S_G) \cong G \).

Then we take the cone over \( S_G \) to get an affine variety \( C(S_G) \) such that, essentially, \( \pi_1(R(C(S_G))) \cong G \). (These cones have very non-isolated singularities and therefore it is not clear that \( \pi_1(R(C(S_G))) \) really makes sense for them.) Then we use the method of [Kol11] to smooth the singularities outside the origin while keeping the fundamental group of the resolution unchanged. (The smoothing yields an isolated singularity only in low dimensions; we get codimension 5 singularities in general.) Thus we have a singularity \( (0 \in X_G) \) such that \( \pi_1(R(X_G)) \cong G \). For some choices of \( S_G \) one can control the other two groups as well, completing the proof of Theorem 1.

It is interesting to study the relationship between the algebro-geometric properties of a singularity \( (0 \in X) \) and the fundamental group of \( \text{link}(X) \). We prove the following results for rational singularities in Section 7:

- Let \( (0 \in X) \) be a rational singularity [37]. Then \( \pi_1(DR(X)) \) is \( \mathbb{Q} \)-superperfect, that is, \( H_i(\pi_1(DR(X)), \mathbb{Q}) = 0 \) for \( i = 1, 2 \).
- Conversely, for every finitely presented, \( \mathbb{Q} \)-superperfect group \( G \) there is a 6-dimensional rational singularity \( (0 \in X) \) such that \( \pi_1(DR(X)) = \pi_1(R(X)) = \pi_1(\text{link}(X)) \cong G \).
- Not every finite group \( G \) occurs as \( \pi_1(DR(X)) \) for a 3-dimensional rational singularity. (We do not know what happens in dimension 4 and 5.)

3 (Open problems) Theorem 1 and its proof raise many questions; here are just a few of them.

3.1 Our examples show that links of isolated singularities are more complicated than smooth projective varieties. It would be interesting to explore the difference in greater detail.

3.2 Several steps of the proof use general position arguments and it is probably impractical to follow it to get concrete examples. It would be nice to work out simpler versions in some key examples, for instance for Higman’s group (as in Example 3.1) and to understand the geometry of the resulting singularities completely.

3.3 We have been focusing only on the fundamental group of the dual simplicial complex \( \pi_1(DR(X)) \) associated to an isolated singularity \( (0 \in X) \). However, it is quite possible that for every finite simplicial complex \( C \) there is an isolated singularity \( (0 \in X) \) such that \( DR(X) \) is homotopy equivalent to \( C \). Our results indicate
that this may well be true but our construction yields non-isolated singularities and only partial resolutions in general.

(3.4) Given an \( n \)-dimensional manifold (possibly with boundary) \( M \) our constructions give a \( (2n + 1) \)-dimensional singularity \((0 \in X)\) such that \( DR(X) \) is homotopy-equivalent to \( M \). It is reasonable to expect that there is an \( (n + 1) \)-dimensional singularity \((0 \in Z)\) such that \( DR(Z) \) is homeomorphic to \( M \).

(3.5) For a complex algebraic variety \( X \), its algebraic fundamental group \( \pi_{1}^{\text{alg}}(X) \) is the profinite completion of its topological fundamental group \( \pi_{1}(X) \). There are examples where the natural map \( \pi_{1}(X) \to \pi_{1}^{\text{alg}}(X) \) is not injective [Tol93, CK92], but in all such known cases the image of \( \pi_{1}(X) \to \pi_{1}^{\text{alg}}(X) \) is infinite and very large.

We now have examples of isolated rational singularities such that \( \pi_{1}(\text{link}(X)) \) is infinite yet \( \pi_{1}^{\text{alg}}(\text{link}(X)) \) is the trivial group, see Corollary 48.

All the examples in Theorem 1 can be realized on varieties defined over \( \mathbb{Q} \). Thus they have an algebraic fundamental group \( \pi_{1}^{\text{alg}}(\text{link}(X_{\mathbb{Q}})) \) which is an extension of the above \( \pi_{1}^{\text{alg}}(\text{link}(X)) \) and of the absolute Galois group \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \). We did not investigate this extension, thus we do not have a complete description of all possible groups \( \pi_{1}^{\text{alg}}(\text{link}(X_{\mathbb{Q}})) \).

1. Polyhedral complexes

A (convex) Euclidean polyhedron is a subset \( P \) of \( \mathbb{R}^{n} \) given by a finite collection of linear inequalities (some of which may be strict and some not). The polyhedron \( P \) is rational if it can be given by linear inequalities with rational coefficients. The dimension of \( P \) is its topological dimension, which is the same as the dimension of its affine span \( \text{Span}(P) \). (Recall that the empty set has dimension \(-1\).) Note that we allow polyhedra which are unbounded and neither open nor closed. A face of \( P \) is a subset of \( P \) which is given by converting some of these non-strict inequalities to equalities. Define the set \( \text{Faces}(P) \) to be the set of faces of \( P \). The interior \( \text{Int}(P) \) of \( P \) is the topological interior of \( P \) in \( \text{Span}(P) \). Again, \( \text{Int}(P) \) is a Euclidean polyhedron. We will refer to \( \text{Int}(P) \) as an open polyhedron.

An (isometric) morphism of two polyhedra is an isometric map \( f : P \to Q \) so that \( f(P) \) is a face of \( Q \). A morphism is rational if it is a restriction of a rational affine map.

**Definition 4.** A (Euclidean) polyhedral complex is a small category \( C \) whose objects are convex polyhedra and morphisms are their isometric morphisms satisfying the following axioms:

1) For every \( c_{1} \in \text{Ob}(C) \) and every face \( c_{2} \) of \( c_{1}, c_{2} \in \text{Ob}(C) \), the inclusion map \( i : c_{1} \to c_{2} \) is a morphism of \( C \).

2) For every \( c_{1}, c_{2} \in \text{Ob}(C) \) there exists at most one morphism \( f = f_{c_{2},c_{1}} \in \text{Mor}(C) \) so that \( f(c_{1}) \subset c_{2} \).

A polyhedral complex is rational if all its objects and morphisms are rational.

Analogously, one defines spherical, hyperbolic, affine, projective, etc., polyhedral complexes, but we will not need these concepts. Thus, a polyhedral complex for us will always mean a Euclidean polyhedral complex.

Given a finite rational polyhedral complex \( C \), by scaling one obtains an integral polyhedral cell complex \( C' \), where the polyhedra and morphisms are integral.
Example 5. Every simplicial complex $Z$ corresponds canonically to a Euclidean polyhedral complex $Z$: Identify each $k$-simplex in $Z$ with the standard Euclidean simplex in $\mathbb{R}^{k+1}$.

Objects of a polyhedral complex $C$ are called faces of $C$ and the morphisms of $C$ are called incidence maps of $C$. A facet of $C$ is a face $P$ of $C$ so that for every morphism $f : P \rightarrow Q$ in $C$, $f(P) = Q$. A vertex of $C$ is a zero-dimensional face. The dimension $\dim(C)$ of $C$ is the supremum of dimensions of faces of $C$. A polyhedral complex $C$ is called pure if the dimension function is constant on the set of facets of $C$; the constant value in this case is the dimension of $C$. A subcomplex of $C$ is a full subcategory of $C$. If $c$ is a face of a complex $C$ then $\text{Res}_C(c)$, the residue of $c$ in $C$, is the minimal subcomplex of $C$ containing all faces $c'$ such that there exists an incidence map $c \rightarrow c'$. For instance, if $c$ is a vertex of $C$ then its residue is the same as the star of $c$ in $C$; however, in general these are different concepts.

We generate the equivalence relation $\sim$ on a polyhedral complex $C$ by declaring that $c \sim f(c)$, where $c \in \text{Ob}(C)$ and $f \in \text{Mor}(C)$. This equivalence relation also induces the equivalence relation $\sim$ on points of faces of $C$.

If $C$ is a polyhedral complex, its poset $\text{Pos}(C)$ is the partially ordered set $\text{Ob}(C)$ with the relation $c_1 \leq c_2$ iff $c_1 \sim c_2$ so that $\exists f \in \text{Mor}(C), f : c_0 \rightarrow c_2$.

We define the underlying space or amalgamation $|C|$ of a polyhedral complex $C$ as the topological space which is obtained from the disjoint union

$$
\coprod_{c \in \text{Ob}(C)} c
$$

by identifying points using the equivalence relation: $\sim$. We equip $|C|$ with the quotient topology.

Definition 6. If $C$ is a polyhedral complex and $B$ is its subcomplex. For $c \in \text{Ob}(C)$ define the polyhedron

$$
c' := c \setminus \bigcup_{b \leq c, b \in B} f(b), \text{ where } f : b \rightarrow c, f \in \text{Mor}(C).
$$

For a morphism $f \in \text{Mor}(C)$, $f : c_1 \rightarrow c_2$, we set $f' : c'_1 \rightarrow c'_2$ be the restriction of $f$. We define the difference complex $C - B$ as the following polyhedral complex:

$$
\text{Ob}(C - B) = \{ c' : c \in \text{Ob}(C) \},
$$

$$
\text{Mor}(C - B) = \{ f' : c'_1 \rightarrow c'_2, \text{ where } f \in \text{Mor}(C), f : c_1 \rightarrow c_2 \}.
$$

Note that $C - B$ need not be a subcomplex of $C$.

We will be exclusively interested in finite Euclidean polyhedral complexes (i.e., complexes of finite cardinality), where the underlying space $|C|$ is connected.

Example 7. For us the most important Euclidean polyhedral complexes are obtained by subdividing (a domain in) $\mathbb{R}^N$ into convex polyhedra. Let $U \subset \mathbb{R}^N$ be an open subset and $V$ be a partition of $U$ in convex Euclidean polyhedra so that:

1) For every $c_1 \in V$ and every face $c_2$ of $c_1$, $c_2 \in V$.
2) For every two polyhedra $c_1, c_2 \in V$, $c_1 \cap c_2 \in V$.

Then $V$ becomes the set of faces of a polyhedral complex (again denoted by $V$ by abusing the notation), where inclusions of faces are the incidence maps.

Example 8. Let $\Delta^m$ be the closed Euclidean $m$-simplex. The (simplicial) cell complex of faces of $\Delta^m$ will be denoted $\mathcal{C}(\Delta^m)$. 
Definition 9. Let $\mathcal{C}$ be a pure $n$-dimensional polyhedral complex. The nerve $\text{Nerve}(\mathcal{C})$ of $\mathcal{C}$ is the simplicial complex whose vertices are facets of $\mathcal{C}$ (the notation is $v = c^*$, where $c$ is a facet of $\mathcal{C}$); distinct vertices $v_0 = c_0^*, ..., v_k = c_k^*$ or $\text{Nerve}(\mathcal{C})$ span a $k$-simplex if there exists an $n-k$-face $c$ of $\mathcal{C}$ and incidence maps $c \rightarrow c_i, i = 0, ..., k$. The simplex $\sigma = [v_0, ..., v_k]$ then is said to be dual to the face $c$.

Lemma 10. If $\mathcal{C}$ is finite then $|\mathcal{C}|$ is homotopy-equivalent to $|\text{Nerve}(\mathcal{C})|$.

Proof. Notice that the image of each face of $\mathcal{C}$ in $|\mathcal{C}|$ is contractible. Therefore, one can thicken each facet $c$ of $\mathcal{C}$ to an open contractible subset $U(c) \subset |\mathcal{C}|$ so that:
1. The collection of open sets $U(c)$, $c$'s are facets of $\mathcal{C}$, is a covering of $|\mathcal{C}|$.
2. For every $k + 1$-tuple of facets $c_0, ..., c_k$, the intersection
$$U(c_0) \cap ... \cap U(c_k)$$
is nonempty iff $[c_0^*, ..., c_k^*]$ is a simplex in $\text{Nerve}(\mathcal{C})$.
3. Each intersection $U(c_0) \cap ... \cap U(c_k)$ as above is contractible.

Now, the assertion becomes the standard fact of algebraic topology, see e.g. [Hat02]. \hfill \Box

Note that, in general, a face can have more than one, or none, dual simplex, except that each facet is dual to the unique vertex.

Definition 11. A polyhedral complex $\mathcal{C}$ is simple if:
1. $\mathcal{C}$ is pure, $\dim(\mathcal{C}) = n$,
2. For $k = 0, ..., n$ and every $k$-face $c$ of $\mathcal{C}$, $\text{Nerve}(|\text{Res}_c(c)|)$ is isomorphic to the complex $\mathcal{C}(\Delta^{n-k})$.

It is easy to see that each face $c$ of a simple $n$-dimensional complex $\mathcal{C}$ is dual to a unique simplex $c^*$ in $\text{Nerve}(\mathcal{C})$. Moreover, $\dim(c) + \dim(c^*) = n$.

Lemma 12. If $\mathcal{A}$ is a simple polyhedral complex and $\mathcal{B}$ is its subcomplex, then the complex $\mathcal{C} := \mathcal{A} - \mathcal{B}$ is again simple.

Proof. It is easy to see that $\mathcal{C}$ is is pure of the same dimension as $\mathcal{A}$ and for each face $c$ of $\mathcal{C}$, the poset of $\text{Res}_c(c)$ is isomorphic to the poset $\text{Res}_A(a)$, where $c = a'$ (see Definition 9). \hfill \Box

2. Voronoi complexes in $\mathbb{R}^N$

In what follows, we will use the notation $d$ for the Euclidean metric on $\mathbb{R}^N$.

Definition 13. Let $Y \subset \mathbb{R}^N$ be a finite subset. The Voronoi tessellation $\mathcal{V}(Y)$ of $\mathbb{R}^N$ associated with $Y$ is defined by: For each $y \in Y$ take the Voronoi cell
$$\mathcal{V}(y) := \{ x \in \mathbb{R}^N : d(x, y) \leq d(x, y'), \forall y' \in Y \}.$$ 
Thus, each cell $\mathcal{V}(y)$ is given by the collection of non-strict linear inequalities
$$d(x, y) \leq d(x, y'),$$
and
$$2(y' - y) \cdot x \leq y' \cdot y' - y \cdot y.$$
Then each cell $\mathcal{V}(y)$ is a closed (possibly unbounded) polyhedron in $\mathbb{R}^N$. Every $\mathcal{V}(y)$ is rational provided that $Y \subset \mathbb{Q}^N$. The union of Voronoi cells is the entire $\mathbb{R}^N$. We thus obtain the polyhedral complex, called the Voronoi complex, $\mathcal{V}(Y)$ using the faces $\mathcal{V}(y)$ as in Example 7.
Not every Voronoi complex is simple, but most of them are. In order to make this precise, we consider ordered finite subsets of $\mathbb{R}^N$; thus, every $k$-element subset becomes a point in $\mathbb{R}^k$.

**Lemma 14.** $k$-element subsets $Y \subset \mathbb{R}^N$ (resp. $Y \subset \mathbb{Q}^N$) whose Voronoi complex $\mathcal{V}(Y)$ is simple are open and dense in $\mathbb{R}^k$ (resp. $\mathbb{Q}^k$).

**Proof.** For a subset $Y \subset \mathbb{R}^N$, failure of simplicity of $\mathcal{V}(Y)$ means that there exists an $m$-element subset $W \subset Y$ so that the set of affine hyperplanes $H_{y_i,y_j} = \{x|d(y_i,x) = d(y_j,x)\}, y_i,y_j \in W$ has non-transversal intersection in $\mathbb{R}^N$. The subset $\Sigma_{m,N} \subset \mathbb{R}^{mN}$ of such $W$’s is closed and has empty interior. The Lemma follows from density of rational $k$-element subsets.

**Delaunay triangulations.** Dually, one defines the Delaunay simplicial complex $\mathcal{D}(Y) = \text{Nerve}(\mathcal{V}(Y))$, i.e., vertices of this complex are points of $Y$, vertices $y_0,\ldots,y_k$ span a $k$-simplex in $\mathcal{D}(Y)$ iff $\cap_{i=0}^{k} D(y_i) \neq \emptyset$. We have the canonical affine map $\eta : \mathcal{D}(Y) \to \mathbb{R}^N$ which is the identity on $Y$.

The proof of the following theorem can be found in [For97, Thm.2.1]:

**Theorem 15.** 1. If $\mathcal{V}(Y)$ is simple then the map $\eta : |\mathcal{D}(Y)| \to \mathbb{R}^N$ injective.

2. The image of the latter map is the closed convex hull $\text{Hull}(Y)$ of the set $Y$.

The Euclidean simplicial complex $\eta(\mathcal{D}(Y))$ is called the Delaunay triangulation of $\text{Hull}(Y)$ with the vertex set $Y$.

**Voronoi complexes associated with smooth submanifolds in $\mathbb{R}^N$.** Let $M$ be a subset of $\mathbb{R}^N$ and $\epsilon > 0$. A set $Y \subset \mathbb{R}^N$ is said to be $\epsilon$-dense in $M$ if every point $x \in M$ is within distance $<\epsilon$ from a point of $Y$. (Note that $Y$ need not be contained in $M$.) By compactness and Lemma 14 for every bounded subset of $\mathbb{R}^N$, $\epsilon > 0$, there exists a finite simple rational subset $Y \subset \mathbb{R}^N$ which is $\epsilon$-dense in $M$.

**Theorem 16.** (Cairns [Cai61]) Let $M$ be a $C^2$-smooth closed submanifold of $\mathbb{R}^N$. Then there exists $\epsilon_M > 0$ so that for every $\epsilon \in (0,\epsilon_M]$ the following holds:

Let $Y$ be a finite subset of $\mathbb{R}^N$ which is $\epsilon$-dense in $M$. For every face $c \in \mathcal{V}(Y)$ define the set $c_M := c \cap M$.

Then each $c_M$ is a (topological) cell in $M$ and the collection of cells $c_M$, $c \in \mathcal{V}(Y)$, is a cellulation of $M$.

Note that for a generic choice of $Y$ the intersections $c_M$ are transversal and, hence,

$$\dim(c_M) = \dim(c) + \dim(M) - N.$$  

Examination of Cairns’s proof of this theorem shows that it can be repeated verbatim to prove the following

**Theorem 17.** Let $S$ be a compact codimension 0 submanifold of $\mathbb{R}^N$ with $C^2$-smooth boundary $M$. Then for $\epsilon_M > 0$ as above, and for every $\epsilon \in (0,\epsilon_M]$ the following holds:

Let $Y$ be a finite subset of $\mathbb{R}^N$ which is $\epsilon$-dense in $S$. For every face $c \in \mathcal{V}(Y)$ define the set $c_S := c \cap S$. Then each $c_S$ is a (topological) cell in $S$ and the collection of cells $c_S$, $c \in \mathcal{V}(Y)$, is a regular cellulation $\mathcal{V}_S$ of $S$.

Again, for a generic choice of $Y$, $\dim(c_M) = \dim(c)$ unless $c_M = \emptyset$.  

We observe that \( \{ c \in \mathcal{V}(Y) : c_S = \emptyset \} \) is the face set of a subcomplex \( \mathcal{W}(Y) \) of \( \mathcal{V}(Y) \). We then let \( C_S := \mathcal{V}(Y) - \mathcal{W}(Y) \).

Clearly, \( C_S \) is pure, \( N \)-dimensional and \( \text{Nerve}(C_S) \) is isomorphic to \( \text{Nerve}(\mathcal{V}_S) \). Therefore, \( |\text{Nerve}(C_S)| \) is homotopy-equivalent to \( |\text{Nerve}(\mathcal{V}_S)| \cong S \).

By Lemma 14 we can assume that \( Y \) is simple, rational and generic; then \( C_S \) is again simple, rational and \( |C_S| \) is still is homotopy-equivalent to \( S \).

**Corollary 18.** Given \( S \subset \mathbb{R}^N \) as above, there exists a simple rational polyhedral complex \( C = C_S \) with simple parasites so that \( |C| \) is homotopy-equivalent to \( S \).

Faces of \( C_S \), in general, are not closed polyhedra.

### 3. Euclidean thickening of simplicial complexes

We are grateful to Frank Quinn for leading us to the following reference:

**Theorem 19.** *(Hirsch [Hir62]*) Let \( Z \) be a finite simplicial complex in a smooth manifold \( X \). Then there exists a codimension 0 compact submanifold \( S \subset X \) with smooth boundary which is homotopy-equivalent to \( |Z| \).

**Corollary 20.** For every \( n \)-dimensional finite simplicial complex \( Z \) there exists a codimension 0 compact submanifold \( M \subset \mathbb{R}^{2n+1} \) with smooth boundary, homotopy-equivalent to \( |Z| \).

**Proof.** Below is a simple and self-contained proof of this corollary communicated to us by Kevin Walker via Mike Freedman. We will think of \( Z \) as a cell complex and will construct \( M \) by induction on skeleta of \( Z \). Let \( B_v \) denote pairwise disjoint intervals in \( \mathbb{R} \), where \( v \in Z^{(0)} \) and set

\[
S_0 := \Pi_v B_v.
\]

Then the map \( Z^{(0)} \to S_0 \) sending each \( v \) to \( B_v \) is a homotopy-equivalence. Suppose that we constructed a codimension 0 submanifold \( M_k \subset \mathbb{R}^{2k+1} \) with smooth boundary and a homotopy-equivalence \( h : Z^{(k)} \to M_k \). Let \( f_i \) define the attaching maps \( \partial B^{k+1} \to Z \) of \( (k+1) \)-cells of \( Z \). The maps \( f_i \) define elements of \( \pi_k(M_k) \). Since the dimension of the manifold \( M_k \) is \( 2k + 1 \), these homotopy classes can be realized by smoothly embedded \( k \)-sphere \( s_i \) with trivial normal bundle in \( \mathbb{R}^{2k+1} \), see [KM63] §6. A priori, the spheres \( s_i \) may not be even homotopic to spheres contained in the boundary of \( M_k \). However, we replace \( M_k \) with \( M'_k \subset \mathbb{R}^{2k+3} \), obtained from \( M_k \times B^2 \subset \mathbb{R}^{2k+3} \) by “smoothing the corners.” Then \( s_i \) can be chosen in the boundary of \( M'_k \) and, for the dimension reasons, it bounds a smoothly embedded \( (k+1) \)-disk in \( \mathbb{R}^{2k+3} \setminus M'_k \). Then we attach the handle \( H_i \cong D^{k+1} \times D^{k+2} \subset \mathbb{R}^{2k+3} \to M'_k \) along \( s_i \), so that this handle intersects \( M'_k \) only along a tubular neighborhood of \( s_i \) in \( \partial M' \). Moreover, we can assume that distinct handles \( H_i \) are pairwise disjoint. We again smooth the corners after the handles are attached. Let \( M_{k+1} \subset \mathbb{R}^{2k+3} \) be the codimension 0 submanifold with smooth boundary resulting from attaching these handles and smoothing the corners. Then, clearly, the homotopy-equivalence \( Z^{(k)} \to M_k \) extends to a homotopy-equivalence \( Z^{(k+1)} \to M_{k+1} \). \( \square \)

By combining Hirsch’s theorem with Corollary 18 we obtain:

**Corollary 21.** Given a finite \( n \)-dimensional simplicial complex \( Z \), there exists a finite simple \((2n+1)\)-dimensional rational Euclidean polyhedral complex \( C \) so that \( |C| \) is homotopy-equivalent to \( |Z| \).
We note that the dimension of $\mathcal{C}$ in this corollary can be easily reduced to $N = 2n$. For instance, by a theorem of Stallings (see [DR93]), $Z$ is homotopy-equivalent to a finite simplicial complex $W$ which embeds in $\mathbb{R}^N$. One can improve on this estimate even further as follows. Suppose that $Z$ is a finite simplicial complex which admits an immersion $j : |Z| \to \mathbb{R}^N$. Then taking pull-back of an open regular neighborhood of $j(|Z|)$ via $j$ one obtains an open smooth locally-Euclidean $N$-dimensional manifold $X$ which is homotopy-equivalent to $|Z|$.

**Definition 22.** If $Z$ is a simplicial complex, then a locally-Euclidean Riemannian manifold $X$ is called a Euclidean thickening of $Z$ if there exists an embedding $|Z| \to X$ which is a homotopy-equivalence. We say that $X$ is rational, resp. integral if there exists a smooth atlas on $X$ with transition maps that belong to $\mathbb{Q}^n \rtimes GL(n, \mathbb{Q})$, resp. $\mathbb{Z}^n \rtimes GL(n, \mathbb{Z})$, where $n = \dim(X)$.

Note that if $X$ is an $n$-dimensional locally-Euclidean manifold which admits an isometric immersion in $\mathbb{R}^n$, then $X$ is integral.

Suppose that $X$ is a Euclidean thickening of $Z$. Applying Hirsch’s theorem above to the embedding $|Z| \subset X$, we obtain an $N$-dimensional smooth manifold with boundary $S \subset X$ homotopy-equivalent to $|Z|$. Even though such $X$ is not isometric to $\mathbb{R}^N$ (it is typically incomplete and, moreover, need not embed in $\mathbb{R}^N$ isometrically), the arguments in the proof of Cairns’s theorem [17] are local and go through if we replace $\mathbb{R}^N$ with $X$. We thus obtain

**Corollary 23.** Suppose that $Z$ is a finite simplicial complex and $X$ is an $N$-dimensional Euclidean thickening of $Z$. Then there exists a simple $N$-dimensional Voronoi complex $C$ so that $|C|$ is homotopy-equivalent to $Z$. Moreover, if $X$ is rational, resp. integral, the complex $C$ can be taken rational, resp. integral.

### 4. Complexfication of Euclidean polyhedral complexes

**Definition 24.** Let $V$ denote either the category of varieties (over a fixed field $k$) or the category of topological spaces.

Let $\mathcal{C}$ be a finite polyhedral complex. A $V$-complex based on $\mathcal{C}$ is a functor $\Phi$ from $\mathcal{C}$ to $V$ so that morphisms $c_i \to c_j$ go to closed embeddings $\phi_{ij} : \Phi(c_i) \hookrightarrow \Phi(c_j)$. By abuse of terminology, we will sometimes refer to the image category $\text{im}(\Phi)$ as a $V$-complex based on $\mathcal{C}$.

We call the functor $\Phi$ strictly faithful if the following holds:

If $x_i \in \Phi(c_i)$, $x_j \in \Phi(c_j)$ and $\phi_{ik}(x_i) = \phi_{jk}(x_j)$ for some $k$ then there is an $\ell$ and $x_\ell \in \Phi(c_\ell)$ such that $\phi_{i\ell}(x_\ell) = x_i$ and $\phi_{j\ell}(x_\ell) = x_j$.

The relation $x_i \sim \phi_{ij}(x_i)$ for every $i, j$ and $x_i \in X_i$ generates an equivalence relation on the points of $\amalg_{i \in I} \Phi(c_i)$, also denoted by $\sim$.

In the category of topological spaces, the direct limit $\lim \Phi(\mathcal{C})$ of the diagram $\Phi(\mathcal{C})$ exists and its points are identified with $(\amalg_{i \in I} \Phi(c_i)) / \sim$.

For example, suppose that $\Phi_{\text{taut}}$ is the tautological functor which identifies each face of $\mathcal{C}$ with the corresponding underlying topological space. Then $\lim \Phi_{\text{taut}}(\mathcal{C})$ is nothing but $|\mathcal{C}|$.

In general, Proposition 3.1 in [Cor92] proves the following.

**Lemma 25.** Suppose that $\Phi$ is strictly faithful and $\Phi(\mathcal{C})$ consists of cell complexes and cellular maps of such complexes. Then $\pi_1(\lim \Phi(\mathcal{C})) \cong \pi_1(|\mathcal{C}|)$ provided that each $\Phi(c), c \in \text{Ob}(\mathcal{C})$ is 1-connected.
In the category of varieties direct limits usually do not exist; we deal with this question in Section 5. Thus for now assume that $\Phi(\mathcal{C})$ has a direct limit $\lim \Phi(\mathcal{C})$ in the category of varieties. There is a natural surjection

$$(\Pi_{i \in I} \Phi(c_i))/\sim \to \lim \Phi(\mathcal{C}).$$

If this map is a bijection, we say that $\lim \Phi(\mathcal{C})$ is an algebraic realization of $|\mathcal{C}|$.

Our next goal is to describe two constructions of complexes of varieties based on polyhedral complexes.

**Projectivization.** Suppose that $\mathcal{C}$ is a Euclidean polyhedral complex.

We first construct a projectivization $\mathcal{P} = \mathcal{P}(\mathcal{C})$ of the Euclidean polyhedral complex $\mathcal{C}$. We regard each face $c$ of $\mathcal{C}$ as a polyhedron in $\mathbb{R}^N$. The complex affine span, $\text{Span}(c)$, of $c$ is a linear subspace of $\mathbb{C}^N$. Let $\mathcal{P}_c$ denote its projective completion in $\mathbb{P}^N$. Note that, technically speaking, different faces $c$ can yield the same space $\mathcal{P}_c$ if their affine spans are the same. To avoid these issues, we set $\mathcal{P}_c := \mathcal{P}_c \times \{c\}$.

For every morphism of $\mathcal{C}$, $f_{c_2,c_1} : c_1 \to c_2$, we have a unique linear embedding $F_{c_2,c_1} : \mathcal{P}_{c_1} \to \mathcal{P}_{c_2}$. Thus, we obtain the functor

$$\mathcal{P} : \mathcal{C} \to \text{Varieties}$$

which sends each $c \in \text{Ob}(\mathcal{C})$ to $\mathcal{P}_c$ and each morphism $f_{c_2,c_1}$ to $F_{c_2,c_1}$.

We refer to $\mathcal{P}$ as a complex of projective spaces based on the complex $\mathcal{C}$.

Observe, however, that $\text{im}(\mathcal{P})$ does not (in general) accurately capture the combinatorics of the complex $\mathcal{C}$, i.e., the corresponding functor $\Psi_{\text{var}} = \mathcal{P}$ need not be strictly faithful. For instance, in complex projective space any two hyperplanes intersect but a polyhedral complex usually has disjoint codimension 1 faces. More generally, any intersection $\cap_{i} P_{c_i}$ that is not equal to the projective span $P_c$ of some face $c \in \text{Ob}(\mathcal{C})$ shows that $\Psi_{\text{var}} = \mathcal{P}$ is not strictly faithful.

**Definition 26** (Parasitic intersections). Let $\sigma := (c_1, c_2, ..., c_k)$ be a tuple of faces incident to a face $c$. Consider the intersections

$$I_{c,\sigma} := \cap_{i=1}^{k} F_{c,c_i}(\mathcal{P}_{c_i}) \subset \mathcal{P}_c$$

such that there is no face $c_0$ such that $I_{c,\sigma} = F_{c,c_0}(\mathcal{P}_{c_0})$ and $c_0$ is incident to all the $c_1, c_2, ..., c_k$. Then the subspace $I_{c,\sigma} \subset \mathcal{P}_c$ is called a parasitic intersection in $\mathcal{P}_c$.

Note, however, that this collection of parasitic intersections in spaces $\mathcal{P}_c, c \in \text{Ob}(\mathcal{C})$, is not stable under applying morphisms $F_{c',c}$ and taking preimages under these morphisms. We thus have to saturate the collection of parasitic intersections using the morphisms $F_{c,c'}$. This is done as follows. Let $T$ denote the pushout of the category $\text{im}(\mathcal{P}_{\text{top}})$, where we regard each $\mathcal{P}_b, b \in \text{Ob}(\mathcal{C})$, as a topological space, so the pushout exists. Then for each $a \in \text{Ob}(\mathcal{C})$ we have the (injective) projection map $\rho_a : \mathcal{P}_a \to T$. For each parasitic intersection $I_{c,\sigma} \subset \mathcal{P}_c$, we define

$$I_{c,\sigma,a} := \rho_a^{-1} \rho_c(I_{c,\sigma}).$$

We call such $I_{c,\sigma,a}$ a parasitic subspace in $\mathcal{P}_a$. It is immediate that each parasitic subspace in $\mathcal{P}_a$ is a projective space linearly embedded in $\mathcal{P}_a$. With this definition, the collection of parasitic subspaces $I_{c,\sigma,a}$ is stable under taking images and preimages of the morphisms $F_{c,c'}$.

Note that the maps $\rho_a$ induce an embedding $\rho : |\mathcal{C}| \to T$. Furthermore, convexity of the polyhedra $c \in \text{Ob}(\mathcal{C})$ implies that if $a, b \in \text{Ob}(\mathcal{C})$ and $\rho_a(a) \subset \rho_b(P_b)$, then $a \leq b$. 
Lemma 27. 1. All parasitic subspaces have dimension \( \leq N - 2 \).

2. If \( c \) is simple and the intersection \( I_{c,\sigma} \) contains \( P_{c'} \) for some face \( c' \) of \( P_c \), then \( I_{c,\sigma} \) is not parasitic.

3. Suppose that the complex \( \mathcal{C} \) is simple. Then no parasitic subspace \( I_{c,\sigma,b} \subset P_b \) contains a face \( a \) of \( b \).

Proof. (1) is clear.

(2) Without loss of generality we may assume that \( \sigma = (c_1, \ldots, c_k) \) is such that \( c_1, c_2, \ldots, c_k, c' \) are faces of \( c \). Hence, \( I_{c,\sigma} \) also contains \( P_{c'} \) and for each \( i = 1, \ldots, k \), \( P_{c_i} \) contains the face \( c' \). By convexity of \( c \) it follows that \( c' \subset c_i, i = 1, \ldots k \).

(3) Let \( I_{c,\sigma} = P_{c_{k+1}} \) and, hence, \( I_{c,\sigma} \) is not parasitic.

We next define a certain blow-up \( BP \) of \( P \) above which eliminates parasitic subspaces. In the process of the blow-up, the projective spaces \( P_c \) are replaced with smooth rational varieties \( P_{c'} \) so that the linear morphisms \( F_{c_1, c_1} \in \text{Mor}(\text{im}(P)) \) correspond to embeddings \( bF_{c_1, c_1} : BP_{c_1} \to BP_{c_2} \). The varieties \( BP \) are obtained by a sequence of blow-ups of parasitic subspaces.

Proposition 28. Let \( \mathcal{C} \) be a simple Euclidean complex. Then there exists a strictly faithful complex of varieties \( A : \mathcal{C} \to BP(\mathcal{C}) \) based on \( \mathcal{C} \) so that:

1. The direct limit of \( \text{im}(A) = BP(\mathcal{C}) \) exists and is a projective variety \( X \) with simple normal crossing singularities.

2. \( X \) is an algebraic realization of \( |\mathcal{C}| \).

3. \( \pi_1(X) \cong \pi_1(|\mathcal{C}|) \).

4. If \( \mathcal{C} \) is rational, then \( X \) is also defined over \( Q \).

Proof. The complex \( A \) is constructed by inductive blow-up of the complex \( P \) above.

We proceed by induction on dimension of parasitic intersections. First, for each face \( c \in \mathcal{C} \) we blow up all parasitic subspaces in \( P_c \), which are points. We will use the notation \( B_0P_c \) for the resulting smooth rational varieties, \( c \in \text{Faces}(\mathcal{C}) \). Observe that this blow-up is consistent with linear embeddings \( F_{c_1, c_1} \), which, therefore, extend to embeddings \( b_0F_{c_1, c_1} : B_0P_{c_1} \to B_0P_{c_2} \). We let \( B_0P \) denote the functor \( \mathcal{C} \to \text{Varieties}, \)

\[
c \mapsto B_0P_c, \quad f_{c_1, c_1} \mapsto b_0F_{c_1, c_1}.
\]

Observe also that after the 0-th blow-up, all 1-dimensional parasitic subspaces become pairwise disjoint (as we blew up their intersection points). We, thus, can now blow up each \( B_0P_c \) along every 1-dimensional blown-up parasitic subspaces \( B_0I_{c,\sigma} \). The result is a collection of smooth rational varieties \( B_I P_c, c \in \mathcal{C} \). Again, the projective embeddings \( b_1F_{c_1, c_1} \) respect the blow-up, so we also get a collection of injective morphisms \( b_1F_{c_1, c_1} : B_I P_{c_1} \to B_1P_{c_2} \). We, therefore, continue inductively on dimension of parasitic subspaces. After at most \( N - 1 \) steps we obtain a complex \( A = BP \), whose image \( \text{im}(A) \) has blown-up projective spaces \( BP \) as objects and embeddings \( bF_{c_1, c_1} \) as morphisms. Observe, that the subvarieties along which we do the blow-up have dimension \( \leq N - 2 \). Moreover, we never have to blow up the
there exists a rational finite simple polyhedral complex $C$ for any $c \in \text{Ob}(C)$ (see Lemma 27). Now, by the construction, the functor $A$ is strictly faithful.

It is easy to see that the conditions of Proposition 31 hold for $A$, the key is that the complex $C$ was simple and the normality condition was satisfied by the complex $P$. Therefore, the complex variety $X$ which is the direct limit of $\text{im}(A)$ exists and is an algebraic realization of $|C|$. We check in 32 that the variety $X$ is projective.

By the construction, since the complex $C$ is simple, the variety $X$ has only normal crossing singularities. Since each $BP_c$ is simply-connected, Lemma 25 implies that $\pi_1(X) \cong \pi_1(|C|)$. Lastly, if we start with a rational complex $C$, then all the blow-ups are defined over $\mathbb{Q}$ and so is the direct limit $X$.

Suppose now that $W$ is a finite, connected, simplicial complex. By Corollary 21 there exists a rational finite simple polyhedral complex $C$ so that $|C|$ is homotopy-equivalent to $|W|$. Thus, we conclude

**Theorem 29.** There exists a complex projective variety $Z = Z_C$ defined over $\mathbb{Q}$ whose only singularities are simple normal crossings, so that $\pi_1(Z) \cong \pi_1(|W|)$.

### 5. Direct limits of complexes of varieties

Example 33 below shows that direct limits need not exist in the category of varieties need not exist, not even if all objects are smooth and all morphisms are closed embeddings. By analyzing the example, we see that problems arise if some of the images $\phi_k(X_i) \subset X_k$ and $\phi_k(X_j) \subset X_k$ are tangent to each other but not if they are all transversal. The right condition seems to be the seminormality of the images.

**Definition 30.** Recall that a complex space $X$ is called *normal* if for every open subset $U \subset X$, every bounded meromorphic function is holomorphic.

As a slight weakening, a complex space $X$ is called *seminormal* if for every open subset $U \subset X$, every continuous meromorphic function is holomorphic.

The following are some key examples: $(x^2 = y^3) \subset \mathbb{C}^2$ and $(x^3 = y^3) \subset \mathbb{C}^2$ are not seminormal (as shown by $x/y$ and $x^2/y$) but $(x^2 = y^2) \subset \mathbb{C}^2$ is seminormal.

The key property that we use is the following.

Assume that $X$ is seminormal. Let $Y$ be any variety (or complex analytic space) and $p : Y \to X$ be any algebraic (or complex analytic morphism) that is a homeomorphism in the Euclidean topology. Then $p$ is an isomorphism of varieties (or of complex analytic spaces).

**Proposition 31.** Let $X := \{X_i : i \in I, \phi_{ij} : X_i \to X_j : (i, j) \in M\}$ be a complex of varieties based on a finite polyhedral complex $C$. Assume that for each $k$ and each $J \subset I$ the subvariety $\bigcup_{j \in J} \text{im}(\phi_{jk}) \subset X_k$ is seminormal. Then

1. the direct limit $X^\infty$ exists,
2. the points of $X^\infty$ are exactly the equivalence classes of points of $\prod_{i \in I} X_i$, in particular, $X^\infty$ is an algebraic realization of $|C|$, and
3. $\bigcup_{j \in J} \text{im}(\phi_{j\infty}) \subset X^\infty$ is seminormal for every $J \subset I$.

**Proof.** The proof is by induction on $|I|$.

If there is a unique final object $X_j$ then $X^\infty = X_j$.

If not, let $X_j$ be a final object. Removing $X_j$ and all maps to $X_j$, we get a smaller complex $Y_j$. By induction it has a direct limit $Y_j^\infty$. 

From $Y_j$ take away all the $X_k$ that do not map to $X_j$ and all maps to such an $X_k$. Again we get a smaller complex $Z_j$ whose direct limit is $Z^\infty_j$.

There are maps $Z^\infty_j \to X_j$ and $Z^\infty_j \to Y^\infty_j$. We claim that these are both closed embeddings.

Both maps are clearly injective and their image is seminormal. For the first this follows from our assumption and in the second case by induction and (3). As we noted in Definition 30, these imply that these maps are closed embeddings.

Now we claim that $X^\infty$ is the universal push-out

$$
\begin{array}{ccc}
Z^\infty_j & \to & X_j \\
\downarrow & & \downarrow \\
Y^\infty_j & \to & X^\infty
\end{array}
$$

The existence of the push-out as an algebraic space is proved in [Art70, Thm.3.1] and as a variety in [Fer03], see also [Kol08, Cor.48]. If a limit of a diagram of seminormal varieties exists, it is automatically seminormal.

Finally we need to check that (3) holds. Let $W^\infty \subset X^\infty$ be the union of the images of $\phi_i(\{X_i\})$ for $i \in I'$ for some $I' \subset I$. Then $W_j := W^\infty \cap X_j$, $W^Z := W^\infty \cap Z^\infty_j$ and $W^Y := W^\infty \cap Y^\infty_j$ and are all unions of images of some of the $X_i$, hence these are seminormal by induction. Note that $Z^\infty_j \cup W_j$ and $Z^\infty_j \cup W^Y_j$ are also unions of images of some of the $X_i$, hence seminormal.

To check seminormality, we may assume that all varieties are affine. Let $h$ be a continuous meromorphic function on $W^\infty$. The restriction of $h$ to $W^Z_j$ is again a continuous meromorphic function, hence it is holomorphic since $W^Z_j$ is seminormal. This restriction thus extends to a holomorphic function $h^Z$ on $Z^\infty_j$.

The restriction of $h$ to $W_j$ (resp. $W^Y_j$) is a continuous meromorphic function, hence it is holomorphic since $W^Z_j$ (resp. $W^Y_j$) is seminormal.

Thus $h^Z$ and $h|_{W_j}$ define a continuous meromorphic function on $Z^\infty_j \cup W_j$. It is thus holomorphic and extends to a holomorphic function $h_j$ on $X_j$. Similarly, $h^Z$ and $h|_{W^Y_j}$ extend to a holomorphic function $h^Y_j$ on $Y^\infty_j$.

Finally $h_j$ and $h^Y_j$ agree on $Z^\infty_j$, thus, by the universality of the push-out, they define a holomorphic function $h^\infty$ on $X^\infty$. Its restriction to $W$ is $h$, hence $h$ is also holomorphic.

32 (Projectivity). As the examples (34) or [Kol08, Example 15] show, even if the $X_i$ are all projective and the direct limit $X^\infty$ exists, the latter need not be projective. The main difficulty is the following.

Let $L^\infty$ be a line bundle on $X^\infty$. By pull-back we obtain line bundles $L_i$ and isomorphisms $L_i \cong \phi^*_i L_j$ with the expected compatibility conditions. $L^\infty$ is ample iff each $L_i$ is ample. Conversely, giving line bundles $L_i$ and isomorphisms $L_i \cong \phi^*_i L_j$ with the expected compatibility conditions determines a line bundle $L^\infty$ on $X^\infty$.

The practical difficulty is that we need to specify actual isomorphisms $L_i \cong \phi^*_i L_j$; it is not enough to assume that $L_i$ and $\phi^*_i L_j$ are isomorphic. For line bundles this is not natural to do since we usually specify them only up-to the fiberwise $C^*$-action.

However, once we work only with subsheaves of a fixed reference sheaf $F^\infty$, the isomorphisms are easy to specify.

More generally, let $X = \cup_i X_i$ be a scheme with irreducible components $X_i$. Let $F$ be a coherent sheaf on $Y$. Then specifying a coherent subsheaf $G \subset F$ is equivalent to specifying coherent subsheaves $G_i \subset F|_{X_i}$ such that $G_i|_{X_i \cap X_j} = G_j|_{X_i \cap X_j}$ for
every $i, j$. Furthermore, by the Nakayama lemma, $G$ is locally free iff every $G_i$ is locally free.

In our case, we start with each irreducible component identified with a $\mathbb{P}^n$ and then we blow up parasitic subvarieties. Thus each irreducible component $X_i$ comes with a natural morphism to $\mathbb{P}^n$. For $F$ we choose the pull-back of $\mathcal{O}_{\mathbb{P}^n}(m)$ for some $m \gg 1$. These pull-backs are not ample since they are trivial on the fibers of $p_i : X_i \to \mathbb{P}^n$.

In Section 3 we construct the $X_i$ as follows.

(1) Fix a smooth projective variety $P$ with an ample line bundle $L$ (in our case in fact $P \cong \mathbb{P}^n$ and $L \cong \mathcal{O}_{\mathbb{P}^n}(1)$). Set $X_0^0 := P$.

(2) If $X_i^j$ is already defined, we pick a smooth subvariety $Z_i^j \subset X_i^j$ of dimension $j$ and let $\pi_i^j : X_i^{j+1} \to X_i^j$ denote the blow-up of $Z_i^j$ with exceptional divisor $E_i^{j+1}$.

(3) Set $X_i := X_i^n$ with morphisms $\pi_i^j : X_i^n \to X_i^j$.

Claim. 32.4. For all $m_0 \gg m_1 \gg \cdots \gg m_n > 0$, the following line bundle is ample on $X_i$:

\[
\left( (\Pi_0^r)^*(H^{m_0}) \right) \left( -m_1(\Pi_1^r)^*(E_1^r) - \cdots - m_n(\Pi_n^r)^*(E_n^{r-1}) - m_nE_n^n \right)
\]

Proof. Let $Y$ be a smooth variety and $Z \subset Y$ a smooth subvariety. Let $p_Y : B_2Y \to Y$ denote the blow-up with exceptional divisor $E_Y$. Let $H$ be an ample invertible sheaf on $X$. Then $p_Y^*H^a(-b \cdot E_Y)$ is ample on $B_2Y$ for $a \gg b > 0$ (cf. [Har77, Prop.II.7.10]).

Applying this inductively to the blow-ups $\pi_i^j : X_i^{j+1} \to X_i^j$ we get our claim. □

For later use, also not the following. Assume that we have $Y_1 \subset Y$ smooth and $Z_1 := Z \cap Y_1$ also smooth. Let $E_{Y_1}$ be the exceptional divisor of $p_{Y_1} : B_2Y_1 \to Y_1$. Then there is an identity

\[
(p_i^rH^a(-b \cdot E_Y))|_{B_2Y_1} = p_{Y_1}^*(H|_{Y_1})^a(-b \cdot E_{Y_1}).
\]

(32.5) All the $X_i$ map to $P$ in a compatible manner, hence we have a fixed reference map $\Pi^\infty : X^\infty \to P$. For $m_0 \gg 1$ we get our reference sheaf $F^\infty := (\Pi^\infty)^*(H^{m_0})$.

(32.6) For each $i$ we have $F_i := F^\infty|_{X_i} = (\Pi_i^r)^*(H^{m_0})$. For fixed $m_0 \gg m_1 \gg \cdots \gg m_n > 0$ the formula (32.4) defines a subsheaf $G_i \subset F_i$ and $G_i$ is an ample line bundle on $X_i$. As we noted above, all that remains is to prove that

\[
F^\infty|_{X_i \cap X_j} \supset G_i|_{X_i \cap X_j} = G_j|_{X_i \cap X_j} \subset F^\infty|_{X_i \cap X_j} \quad \forall i, j.
\]

This follows from the compatibility of blow-ups with restrictions noted after the proof of (32.4).

The next example shows that direct limits need not exist in the category of varieties.

Example 33. Start with the polyhedral subcomplex of $\mathbb{R}^2$ whose objects are

\[
(0, 0), \ (x \leq 0, 0), \ (x \geq 0, 0), \ (x, y \leq 0), \ (x, y \geq 0).
\]

We try to build an algebraic realization with objects

\[
\mathbb{C}^1_x, \mathbb{C}^2_x, \mathbb{C}^2_{x,y}, \mathbb{C}^3_x, \mathbb{C}^3_{y}, \mathbb{C}^3_{x,y}
\]

(33.1)
These are all embeddings (even scheme theoretically).
Next we compute these 2 ways. First we pull \( \sum \) polynomials

\[
\phi \text{ back to } C^3_u \text{ and } C^3_v \text{ to get polynomials}
\]

\[
\sum_{i,j} a(i, j)x_i^i y_j^j \quad \text{and} \quad \sum_{i,j} b(i, j)x_i^i z_j^j.
\]

Next we compute these 2 ways. First we pull \( \phi \) back to \( C^3_u \). We get a polynomial

\[
f(u_1, u_2, u_3) = \sum_{i,j,k} c(i,j,k)u_i^1 u_j^2 v_3^k.
\]

Pull it back to \( C^2_{x,y} \) and \( C^2_{x,z} \) to get that

\[
a(i, j) = c(i, j, 0) \quad \text{and} \quad b(i, j) = c(i, j, 0) + c(i, j - 2, 1) + c(i, j - 4, 2) + \cdots.
\]

Thus we obtain that

\[
a(i, 0) = b(i, 0) \quad \text{and} \quad a(i, 1) = b(i, 1) \quad \forall \ i.
\]

Next we pull \( \phi \) back to \( C^3_u \). We get a polynomial

\[
g(v_1, v_2, v_3) = \sum_{i,j,k} d(i, j, k)v_i^1 v_j^2 v_3^k.
\]

Pull it back to \( C^2_{x,y} \) to get that \( a(i, j) = d(i, j, 0) \). The pull-back to \( C^2_{x,z} \) involves the binomial coefficients; we are interested in the first 2 terms only:

\[
b(i, 0) = d(i, 0, 0) \quad \text{and} \quad b(i, 1) = d(i, 1, 0) + (i + 1)d(i + 1, 0, 0).
\]

Thus we obtain that

\[
a(i, 0) = b(i, 0) \quad \text{and} \quad a(i, 1) = b(i, 1) - (i + 1)b(i + 1, 0) \quad \forall \ i.
\]

Comparing (33.3) and (33.4) we see that \( a(i + 1, 0) = b(i + 1, 0) = 0 \) for \( i \geq 0 \), that is \( \phi \) is constant on the image of \( C^2_u \).

Note that the same argument holds if \( f, g \) are power series, thus the problem is analytically local everywhere along \( C^1 \). In fact, the problem exists already if we work with \( C^2 \)-functions. (That is if \( X \subset \mathbb{R}^N \) and we require \( C^3_u \to X \subset \mathbb{R}^N \) and \( C^3_v \to X \subset \mathbb{R}^N \) to be at least \( C^2 \).)

**Example 34 (Triangular pillows).** Take 2 copies of \( \mathbb{P}^2 := \mathbb{P}^2(x_i : y_i : z_i) \) of \( \mathbb{C}\mathbb{P}^2 \) and the triangles \( C_i := (x_i y_i z_i = 0) \subset \mathbb{P}^2_i \). Given \( c_x, c_y, c_z \in \mathbb{C}^* \) define \( \phi(c_x, c_y, c_z) : C_1 \to C_2 \) by \( (0 : y_1 : z_1) \mapsto (0 : y_1 : c_z z_1), (x_1 : 0 : z_1) \mapsto (c_x x_1 : 0 : z_1) \) and \( (x_1 : y_1 : 0) \mapsto (x_1 : c_y y_1 : 0) \) and glue the 2 copies of \( \mathbb{P}^2 \) using \( \phi(c_x, c_y, c_z) \) to get the surface \( S(c_x, c_y, c_z) \).
We claim that $S(c_x, c_y, c_z)$ is projective iff the product $c_x c_y c_z$ is a root of unity.

To see this note that $\text{Pic}^3(C_3) \cong \mathbb{C}^*$ and $\text{Pic}^r(C_3)$ is a principal homogeneous space under $\mathbb{C}^*$ for every $r \in \mathbb{Z}$. We can identify $\text{Pic}^3(C_3)$ with $\mathbb{C}^*$ using the restriction of the ample generator $L_1$ of $\text{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$ as the base point.

The key observation is that $\phi(c_x, c_y, c_z) : \text{Pic}^3(C_3) \to \text{Pic}^3(C_1)$ is the multiplication by $c_x c_y c_z$. Thus if $c_x c_y c_z$ is an $r$th root of unity then $L_1^r$ and $L_2^r$ glue together to an ample line bundle but otherwise every line bundle on $S(c_x, c_y, c_z)$ is topologically trivial.

### 6. Proof of Theorem □

So far, for every finitely presented group $G$ we have constructed (Theorem □) a complex projective variety $Z$ with simple normal crossing singularities such that $\pi_1(Z) \cong G$. Using any such $Z$, we next construct a singularity. This relies on the following result which is a combination of [Kol11 Thm.8 and Prop.10].

**Theorem 35.** Let $Z$ be an $(n \geq 2)$-dimensional projective variety with simple normal crossing singularities only and $L$ an ample line bundle on $Z$. Then for $m \gg 1$ there are germs of normal singularities $\{0 \in X = X(Z, L, m)\}$ with a partial resolution

$$Z \subset Y \quad \downarrow \quad \downarrow \pi \quad \text{where } Y \setminus Z \cong X \setminus \{0\}$$

such that

1. $Z$ is a Cartier divisor in $Y$,
2. the normal bundle of $Z$ in $Y$ is $K_Z \otimes L^{-m}$,
3. $\pi_1(\mathcal{R}(X)) \cong \pi_1(Z)$,
4. The kernel of $\pi_1(\text{link}(X)) \to \pi_1(\mathcal{R}(X))$ is cyclic, central and generated by any loop around an irreducible component of $Z$,
5. if $\dim Z \leq 4$ then $(0 \in X)$ is an isolated singular point.

Note that the isomorphism $\pi_1(\mathcal{R}(X)) \cong \pi_1(Z)$ is not explicitly stated in [Kol11 Thm.8]. However, $Z$ is a deformation retract of $Y$, hence $\pi_1(Y) \cong \pi_1(Z)$. By [Kol11 8.3] $Y$ has terminal singularities, hence, by [Kol93 Tak03],

$$\pi_1(\mathcal{R}(X)) \cong \pi_1(\mathcal{R}(Y)) \cong \pi_1(Y) \cong \pi_1(Z).$$

Alternatively, if $\dim Z = 2$ (which is the only case that we need here), we have a complete description of the possible singularities of $Y$. By [Kol11 Claim 5.10] they are of the form $(x_1 x_2 = x_3 x_4) \subset \mathbb{C}^4$ with $x_3 = 0$ defining $Z$. There are 2 local irreducible components of $Z$ given by $x_1 = x_3 = 0$ and $x_2 = x_3 = 0$. We can resolve these singularities by a single blow-up. The exceptional divisor is simply connected, hence $\pi_1(\mathcal{R}(Y)) \cong \pi_1(Y)$.

Note also that under any such blow-up the dual simplicial complex changes by getting a new vertex on the edge connecting the two local irreducible components. Thus its homeomorphism type is unchanged.

It is clear that the kernel of $\pi_1(\text{link}(X)) \to \pi_1(\mathcal{R}(X))$ is generated by the loops around the irreducible components of $Z$. Being cyclic, central is a special property of the actual construction in [Kol11]. However, our method at the end of this Section does not use □4.)
In order to apply this to obtain an isolated singular point, we need a low dimensional variety $Z$. We use the following singular version of the Lefschetz hyperplane theorem; see [GM88, Sec.II.1.2] for a stronger result and references.

**Theorem 36.** Let $X$ be a projective variety of dimension $\geq 3$ with local complete intersection singularities and $H \subset X$ a general hyperplane section. Then $\pi_1(H) \cong \pi_1(X)$.

Thus we can apply Theorem 35 with $\dim Z = 2$. The remaining issue is to deal with the kernel of $\pi_1(\text{link}(X)) \to \pi_1(\mathcal{R}(X))$. For this we make a more careful choice of $Z$.

Pick a smooth point $z \in Z$. We blow up $Z$ and then we blow up a point on the exceptional curve to get $Z_2$. There are two exceptional curves $E_1, E_2$ on $Z_2$ and $(K_{Z_2} \cdot E_1) = 0$ and $(K_{Z_2} \cdot E_2) = -1$. Since $\pi_1(Z_2) \cong \pi_1(Z)$, we can use this $Z_2$ in Theorem 35 to get $Z_2 \subset Y$. Let $N$ be the normal bundle of $Z_2 \subset Y_2$.

From (35.2) and the above computation we conclude that $c_1(N) \cap [E_1]$ and $c_1(N) \cap [E_2]$ are relatively prime.

Since $Z_2$ and $Y_2$ are both smooth along $E_i$, the boundary of the normal disc bundle restricted to $E_i$ is a lens space $L_i$ with $|\pi_1(L_i)| = c_1(N) \cap [E_i]$. Thus, if $\gamma_z$ denotes a small circle in $Y$ around $Z$ centered at $z$ then the order of $\gamma_z$ in $\pi_1(\text{link}(X))$ divides both $c_1(N) \cap [E_1]$ and $c_1(N) \cap [E_2]$. Hence the $\gamma_z$ is trivial in $\pi_1(\text{link}(X))$.

We can perform these blow-ups for points in every irreducible component of $Z$. Then every generator of the kernel of $\pi_1(\text{link}(X)) \to \pi_1(\mathcal{R}(X))$ is trivial, hence $\pi_1(\text{link}(X)) \cong \pi_1(\mathcal{R}(X))$. This completes the proof of Theorem 1. \hfill $\square$

7. Rational singularities and superperfect groups

**Definition 37.** A quasi projective variety $X$ has rational singularities if for one (equivalently every) resolution of singularities $p : Y \to X$ and for every algebraic (or holomorphic) vector bundle $F$ on $X$, the natural maps $H^i(Y, F) \to H^i(Y, p^*F)$ are isomorphisms. That is, for purposes of computing cohomology of vector bundles, $X$ behaves like a smooth variety. See [KM98, Sec.5.1] for details.

**Definition 38.** Let $(0 \in X)$ be a not-necessarily isolated singularity and choose a resolution of singularities $p : Y \to X$ such that $E := p^{-1}(0)$ is a simple normal crossing divisor. Let $\{E_i : i \in I\}$ be the irreducible components and for $J \subset I$ set $E_J := \cap_{i \in J} E_i$.

The dual simplicial complex of $E$ has vertices $\{v_i : i \in I\}$ indexed by the irreducible components of $E$. For $J \subset I$ we attach a $|J|$-simplex for every irreducible component of $\cap_{i \in J} E_i$. Thus $D(E)$ is a simplicial complex of dimension $\leq \dim X - 1$.

The dual simplicial complex of a singularity seems to have been known to several people but not explicitly studied until recently. The dual graph of a normal surface singularity has a long history. Higher dimensional versions appear in [Knu77, Per77, Gor80] but systematic investigations were started only recently; see [Thu07, Ste08, Pay09, Pay11].

It is proved in [Thu07, Ste08] that the homotopy type of $D(E)$ is independent of the resolution $Y \to X$. As before, we denote it by $D\mathcal{R}(0 \in X)$.

A possible argument runs as follows. Let $F \subset E_J$ be an irreducible component. If we blow up $F$, the dual simplicial complex changes by a barycentric subdivision of the $|J|$-simplex corresponding to $F$. If we blow up a smooth subvariety $Z \subset F$
that is not contained in any smaller $E_j$, then the dual simplicial complex changes
by attaching the cone over the star of the $|J|$-simplex corresponding to $F$. Thus in
both cases, the homotopy type of $D(E)$ is unchanged and by [Wlo03] this implies
the general case.

If $X$ has rational singularities then $H^i(E, \mathcal{O}_E) = 0$ for $i > 0$ by [Ste83, 2.14].
By Part 1 of Lemma 39 below we conclude that $H^i(DR(0 \in X), \mathbb{Q}) = 0$ for $i > 0$.
That is, $DR(0 \in X)$ is $\mathbb{Q}$-acyclic.

**Lemma 39.** Let $X$ be a simple normal crossing variety over $\mathbb{C}$ with irreducible
components $\{X_i : i \in I\}$. Let $T = D(X)$ be the dual simplicial complex of $X$. Then

1. There are natural injections $H^r(T, \mathbb{C}) \hookrightarrow H^r(X, \mathcal{O}_X)$ for every $r$ and
2. For $J \subset I$ set $X_J := \cap_{i \in J} X_i$ and assume that $H^r(X_J, \mathcal{O}_{X_J}) = 0$ for every
   $r > 0$ and for every $J \subset I$. Then $H^r(X, \mathcal{O}_X) = H^r(T, \mathbb{C})$ for every $r$.

**Proof.** The following is a combination of various arguments in [GS75, pp.68–72]
and [FM83, pp.26–27].

Fix an ordering of $I$. It is not hard to check that there is an exact complex

$$0 \rightarrow \mathbb{C}_X \rightarrow \sum_i \mathbb{C}_{X_i} \rightarrow \sum_{i<j} \mathbb{C}_{X_{ij}} \rightarrow \cdots$$

where the $k$th term is $\sum_{|J|=k} \mathbb{C}_{X_J}$ and $\mathbb{C}_{X_J}$ is the constant sheaf with support $X_J$.
If $i \in J$ then the map $\mathbb{C}_{X_{ij}} \rightarrow \mathbb{C}_{X_J}$ is the natural restriction with a plus (resp. minus) sign if $i$ is in odd (resp. even) position in the ordering of $J$.

Thus the cohomology of $\mathbb{C}_X$ is also the hypercohomology of the rest of the complex

$$\sum_i \mathbb{C}_{X_i} \rightarrow \sum_{i<j} \mathbb{C}_{X_{ij}} \rightarrow \cdots$$

This is computed by a spectral sequence whose $E_1$ term is

$$\sum_{|J|=q} H^p(X_J, \mathbb{C}) \Rightarrow H^{p+q}(X, \mathbb{C}). \tag{39.3}$$

The key observation is that this spectral sequence degenerates at $E_2$ [GS75, p.??].
The reason is that $H^p(X_J, \mathbb{C})$ carries a Hodge structure of weight $p$ and there are no maps between Hodge structures of weights.

Note also that the bottom (that is $p = 0$) row of (39.3) is

$$0 \rightarrow \sum_i H^0(T_i, \mathbb{C}) \rightarrow \sum_{i<j} H^0(T_{ij}, \mathbb{C}) \rightarrow \cdots$$

where $T_i \subset T$ denotes the open star of the vertex corresponding to $i \in I$ and $T_{ij} = \cap_{i \in J} T_i$. The homology groups of this complex are exactly the $H^j(T, \mathbb{C})$.
Thus we have injections

$$H^j(T, \mathbb{C}) \hookrightarrow H^j(X, \mathbb{C}_X). \tag{39.4}$$

Similarly, there is an exact complex

$$0 \rightarrow \mathcal{O}_X \rightarrow \sum_i \mathcal{O}_{X_i} \rightarrow \sum_{i<j} \mathcal{O}_{X_{ij}} \rightarrow \cdots$$

which gives a spectral sequence whose $E_1$ term is

$$\sum_{|J|=q} H^p(X_J, \mathcal{O}_{X_J}) \Rightarrow H^{p+q}(X, \mathcal{O}_X). \tag{39.5}$$

By Hodge theory, the natural map from the spectral sequence (39.5) to the spectral sequence (39.3) is a split surjection, hence (39.5) also degenerates at $E_2$ and so

$$H^j(T, \mathbb{C}) \hookrightarrow H^j(X, \mathcal{O}_X). \tag{39.6}$$

is an injection. Under the assumptions of (2) only the bottom row of (39.5) is nonzero, hence, in this case, $H^j(T, \mathbb{C}) = H^j(X, \mathcal{O}_X)$. \qed

In order to understand fundamental groups of links of rational singularities we
need the following definition:
Definition 40. Recall that a group $G$ is called perfect if it has trivial abelianization, equivalently, if $H_1(G, \mathbb{Z}) = 0$. Similarly, $G$ is called superperfect (see [Ben52]) if $\tilde{H}_i(G, \mathbb{Z}) = 0$ for $i \leq 2$. We generalize this notion to homology with coefficients in other commutative rings $R$: A group $G$ is $R$-perfect if $H_1(G, R) = 0$; $G$ is $R$-superperfect if $\tilde{H}_i(G, R) = 0$ for $i \leq 2$. (We will be interested only in the cases $R = \mathbb{Z}$ and $R = \mathbb{Q}$.)

Let $W$ be a cell complex. Recall that by a theorem of Hopf [Hop42] the natural homomorphism $H_2(W, R) \to H_2(\pi_1(W), R)$ is surjective and its kernel (in the case $R = \mathbb{Z}$) is the image of $\pi_2(W)$ under the Hurewicz homomorphism.

Therefore, if $\tilde{H}_i(W, R) = 0$ for $i \leq 2$ then $\tilde{H}_i(\pi_1(W), R) = 0$ for $i \leq 2$.

To see surjectivity in Hopf’s theorem observe the following: For $G = \pi_1(W)$ we let $f : W \to V = K(G, 1)$ be the map inducing the isomorphism of fundamental groups. Then there exists a map of the 2-skeleta $h : V^{(2)} \to W^{(2)}$ which is a homotopy–right inverse to $f$. Hence, $H_2(f) : H_2(W, R) \to H_2(V, R) = H_2(G, R)$ is onto for every commutative ring $R$.

Example 41. Higman’s group $G = \langle x_i | x_i x_{i+1} x_i, i \in \mathbb{Z}/4\mathbb{Z} \rangle$ is perfect, infinite and contains no proper finite index subgroups [Hig51]. If $W$ is the (2-dimensional) presentation complex of $G$ then, clearly, $\chi(W) = 1$. Thus, $\tilde{H}_i(W, \mathbb{Z}) = 0$, $i \leq 2$. In particular, $G$ is superperfect by Hopf’s theorem. Moreover, $W$ is $K(G, 1)$, see e.g. [BG04]. Thus, $\tilde{H}_i(G, \mathbb{Z}) = 0$ for all $i$.

Theorem 42. Let $(0 \in X)$ be a rational singularity. Then $\pi_1(\text{DR}(X))$ is $\mathbb{Q}$-superperfect and finitely presented. Conversely, for every finitely presented $\mathbb{Q}$-superperfect group $G$ there is a 6-dimensional rational singularity $(0 \in X)$ such that

$$\pi_1(\text{DR}(X)) = \pi_1(\mathcal{R}(X)) = \pi_1(\text{link}(X)) \cong G.$$ 

Remark 43. (1) The singularities constructed in Theorem 42 are not isolated. Their singular locus is 1-dimensional. Away from the origin it is the simplest possible non-isolated singularity, locally given by the equation

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 0) \subset \mathbb{C}^7.$$ 

We do not know if in Theorem 42 one can get isolated singularities or not.

(2) For an arbitrary rational singularity $(0 \in X)$, the three groups $\pi_1(\text{DR}(X))$, $\pi_1(\mathcal{R}(X))$ and $\pi_1(\text{link}(X))$ need not be isomorphic. For example, if $\dim X = 2$ then $\pi_1(\text{DR}(X)) = \pi_1(\mathcal{R}(X)) = 1$ yet $\pi_1(\text{link}(X))$ can be infinite [Mum61].

As another example, let $S$ be a fake projective plane, that is, $H_i(S, \mathbb{Z}) \cong H_i(\mathbb{C}P^2, \mathbb{Z})$ for every $i$ yet $\pi_1(S)$ is infinite. Such surfaces were classified in [PY07]. Let $(0 \in C(S))$ denote a cone over $S$. Then $\pi_1(\text{DR}(C(S))) = 1$ yet $\pi_1(\mathcal{R}(C(S))) = \pi_1(S)$ is infinite.

Proof of Theorem 42. The first claim of Theorem 42 follows from the above cited results of [Ste83] and [Hop42]. In order to see the converse, for every finitely presented $\mathbb{Q}$-superperfect group $G$ we construct below (Theorem 40) a simple 5-dimensional, $\mathbb{Q}$-acyclic, Euclidean polyhedral complex $\mathcal{C}$ whose fundamental group is isomorphic to $G$. Once this is done, we obtain a 5-dimensional projective variety $Z$ with simple normal crossing singularities such that $\pi_1(Z) \cong G$ and $H^i(Z, \mathcal{O}_Z) = 0$ for $i > 0$ by Lemma 23.
We now apply Theorem [35]. The proof of [Kol11 Prop.9.1] shows that for \( m \gg 1 \), the resulting \( X \) is a rational singularity. As we noted, \( X \) does not have isolated singularities but they are completely described by [Kol11 Claim.5.10].

Our next goal is to construct polyhedral complexes \( C \) used in the proof of Theorem [42]. The following theorem was proven by Kervaire for \( R = \mathbb{Z} \), but examination of the proofs in [Ker69] and [KM63] shows that they also apply to \( R = \mathbb{Q} \).

**Theorem 44.** Every finitely-presented \( R \)-superperfect group is isomorphic to the fundamental group of a smooth \( R \)-homology \( k \)-sphere \( M^k \) for every \( k \geq 5 \); here \( R = \mathbb{Z} \) or \( R = \mathbb{Q} \).

**Corollary 45.** Let \( R = \mathbb{Z} \) or \( R = \mathbb{Q} \). Then a finitely-presented group \( G \) is \( R \)-superperfect if and only if there exists a 5-dimensional finite simplicial complex \( W \) so that \( \pi_1(W) \cong G \) and \( |W| \) is \( R \)-acyclic, that is, \( \tilde{H}_i(W,R) = 0 \).

**Proof.** One direction of this corollary follows from Hopf’s result above. Suppose that \( G \) is \( R \)-superperfect and finitely-presented. Take the 5-dimensional homology sphere \( M \) as in Theorem [44]. Since \( M \) is smooth, we can assume that it is triangulated. Remove from \( M \) the interior of a closed simplex. The result is the desired simplicial complex \( W \).

We now estimate the dimension of Euclidean thickening of \( Z \) in Corollary [45]. A rough estimate is that \( Z \) immerses in \( \mathbb{R}^5 \), since \( Z \) is 5-dimensional. One can do much better as follows. Due to the results of [KM63 §6], the 5-dimensional manifold \( M^5 \) constructed in Theorem [44] can be chosen to be almost parallelizable, i.e., the complement to a point \( p \) in \( M^5 \) is parallelizable. Therefore, \( M^5 \setminus \{ p \} \) admits an immersion in \( \mathbb{R}^5 \), see [Phi67]. Hence, \( W \) admits a 5-dimensional thickening \( Y \), see Section 3. If \( R = \mathbb{Z} \), then one can do even better and obtain a thickening \( Y \) of \( W \) which is an open subset of \( \mathbb{R}^5 \), see [Liv05].

In order to reduce the dimension of \( X \) from 6 to 5 in Theorem [42] (and, thus, obtain isolated singularities) we have to impose further restrictions on the fundamental group \( G \). Recall that a finite presentation of a group is called balanced if it has equal number of generators and relators. A group \( G \) is called balanced if it admits a balanced presentation. Suppose that \( G \) is an \( R \)-superperfect group which is the fundamental group of a 2-dimensional \( R \)-acyclic cell complex \( W \). Without loss of generality, \( W \) has exactly one vertex, i.e., \( W \) is a presentation complex of \( G \). Then \( H_i(W,R) \cong H_i(G,R) = 0, i = 1, 2 \). In particular, \( \chi(W) = 1 \). It then follows that \( W \) has same number of edges and 2-cells. Hence, \( G \) is balanced (with the balanced presentation complex \( W \)). Hausmann and Weinberger in [HW85] constructed examples of finite superperfect groups which are not balanced, see [CHRR04] for more examples and a survey. Examples of finite \( \mathbb{Q} \)-superperfect groups which are not balanced are easier to construct: Take, for instance, the \( k \)-fold direct product \( A_{p,k} = \mathbb{Z}/p\mathbb{Z} \times \cdots \times \mathbb{Z}/p\mathbb{Z} \) where \( k \geq 2 \). In particular, such groups do not admit \( \mathbb{Q} \)-acyclic presentation complexes and they do not occur as \( \pi_1(\partial \mathcal{R}(X)) \) for a 3-dimensional rational singularity.

Suppose that \( G \) is balanced and \( R \)-superperfect (\( R = \mathbb{Z} \) or \( R = \mathbb{Q} \)); then there exists a smooth 4-dimensional \( R \)-homology sphere \( M^4 \) with the fundamental group \( G \), see [Ker69]. Moreover, in Kervaire’s construction one can assume that \( M^4 \) is almost parallelizable (i.e., it is a 4-dimensional Spin-manifold), see [Kap04]. Thus, for such \( G \) there is a 4-dimensional Euclidean thickening of its 2-dimensional (balanced) presentation complex \( W \). More explicitly, in view of Stallings’ theorem
we can assume that $W$ embeds in $\mathbb{R}^4$. Since $W$ is a balanced presentation complex of a perfect group, $\chi(W) = 0$, and, hence, $b_1(W) = b_2(W) = 0$. Thus we obtain a $\mathbb{Q}$-acyclic 4-dimensional Euclidean thickening of $W$.

Note that our methods cannot produce a 5-dimensional variety in Theorem 42 without the balancing condition. Specifically, given a $\mathbb{Q}$-superperfect group $G$ we would need a 4-dimensional $\mathbb{Q}$-acyclic manifold with the fundamental group $G$. However, one can show, repeating the arguments of [HW85], that for all but finitely many finite groups $G$ constructed in [HW85] such 4-dimensional manifold does not exist.

Reducing the thickening dimension to 3 is, of course, very seldom possible since it amounts to assuming that $G$ is a 3-manifold group, which are quite rare among finitely-presented groups.

By combining these observations with Corollary 21 we conclude

**Theorem 46.** Let $G$ be an $R$-superperfect finitely-presented group ($R = \mathbb{Z}$ or $R = \mathbb{Q}$). Then there exists a finite simple 5-dimensional Euclidean polyhedral complex $C$ so that $|C|$ is $R$-acyclic and has fundamental group isomorphic to $G$. Moreover, if $G$ admits a balanced presentation then we can take such $C$ to be 4-dimensional.

**Corollary 47.** Suppose that $G$ is a finitely-presented $\mathbb{Q}$-superperfect group which admits a balanced presentation. Then in Theorem 42 one can take $X$ which is 5-dimensional and $(0 \in X)$ an isolated singularity.

**Corollary 48.** There exists 5-dimensional, isolated, rational singularities $(0 \in X)$ so that the group $\pi_1^{alg}(\text{link}(X))$ is trivial yet $\pi_1(\text{link}(X))$ is infinite.

**Proof.** Take Higman’s group $G$, see Example 41. Then $G$ clearly has balanced presentation (its presentation has four generators and four relators), the group $G$ is also superperfect, infinite and has no nontrivial finite quotients. Now, the assertion follows from Theorem 42 and Corollary 47.

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