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May 1974

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RESONANCE AMBIGUITIES, BARRELET ZEROS, 
AND DUALITY IN K⁻p → Λπ⁺ᵀ

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Abstract: The method of Barrelet zeros (1) has been used to study the 
ambiguities of our partial wave solutions for the reaction K⁻p → Λπ⁺ from 1550 
to 2170 MeV (2). Details of this two body analysis technique are presented and 
ambiguous sets of partial waves corresponding to the same dσ/dΩ and dσ/dΩ P are 
analysed. Seven ambiguous solutions have been found which maintain the dominant 
Σ resonant structure (D3(1660), D5(1765), F5(1920), F7(2040)) but differ in the 
resonant couplings (t = √τ F P /τ) for these states, as well as in the resonant 
structure in the lower partial waves. Evidence is given for the possible 
existence of new resonant states in the F5, D5 and F7 partial waves and for 
additional resonances in lower waves. The resonant structure suggested by the 
additional solutions is more prolific than the structure previously indicated 
for this reaction (2). Several solutions contain resonant "daughter" states at 
the same mass but with different JP in qualitative agreement with Veneziano 
models and suggestive of the Harari-Freund conjecture of purely resonant scattering 
in non-diffractive, inelastic channels. The ambiguities are experimentally 
distinguishable only by measurement of Σ and Λ, which are predicted from the 
solutions. Each solution has been shown to be consistent with semi-local duality 
from observed averaging about zero exhibited by Im(A'(s,t)), the invariant, 
t channel helicity non-flip amplitude calculated at t = 0 and at t = m². Local 

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averaging was not seen in $B(s,t)$, the $t$ channel helicity flip amplitude. The variation of resonant couplings among these solutions for the dominant resonances suggests that SU(3) tests involving the $\Lambda\pi$ channel should be reconsidered (3). The results indicate that partial wave ambiguities should be more carefully studied and that the possible $\Sigma$ resonances should be looked for in other $\bar{K}N$ reactions.
1. INTRODUCTION

Many strong interaction resonances have been revealed through partial wave analysis. However, the determination of partial waves is subject to ambiguity because available experimental measurements of angular distributions, \( \frac{d\sigma}{dn} \), and polarizations, \( \frac{d\sigma}{d\Omega} \bar{P} \), are quadratic functions of the complex scattering amplitudes. Reference to these ambiguities may be found in the literature (1,4).

In a previous paper we have presented a partial wave analysis of the reaction \( \bar{K}^- + p \rightarrow \Lambda + \pi^0 \) for center-of-mass energies ranging from 1550 to 2170 MeV (2). Three solutions were found by the traditional \( \chi^2 \) minimization search techniques. In principle the number of ambiguous solutions may be much larger than 3 and the purpose of this paper is to explore the ambiguities systematically using the technique of Barrelet zeros (1). This procedure, which is outlined in Section 2, allows one to determine analytically all sets of scattering amplitudes which lead to identical cross sections, \( \frac{d\sigma}{d\Omega} \) and polarization \( \frac{d\sigma}{d\Omega} \bar{P} \). When applied to the data, the procedure must be used at each energy separately. Statistics available for this reaction are not sufficient to give good results using energy independent techniques. Therefore, we have explored the ambiguities associated with our energy dependent partial wave solutions.

Of the many ambiguities associated with our original three solutions, there are seven that maintain the dominant, well-established \( \Sigma \) resonant structure (D3(1660), D5(1765), F5(1920), F7(2040)). These differ in the resonant couplings \( t = \sqrt{e^{r} / r} \) for these states, as well as in the resonant structure in the lower partial waves. The resonant structure suggested by the additional solutions is more prolific than the structure previously indicated for this reaction (2). In general the ambiguous solutions contain "daughter" states at nearly the same mass as the resonances in the original solutions but with different \( J^P \), suggestive of the Veneziano model and of the Harari-Freund conjecture that non-diffractive inelastic scattering (where no pomeron exchange is possible) should consist entirely of resonant states (5). In principle these
ambiguities can be distinguished experimentally only by a measurement of the \( R \)
and \( A \) parameters for which we present predictions.

These solutions have been shown to be consistent with semi-local duality
from observed averaging about zero exhibited by \( \text{Im}(A'(s,t)) \), the invariant,
t channel helicity non-flip amplitude. This amplitude has been calculated for
each solution at \( t = 0 \) and at \( t = \frac{m_\pi^2}{k_\pi^2} \), where duality is expected to hold most
strongly (6). Local averaging about zero is not seen in \( B(s,t) \), the t channel
helicity flip amplitude.

Because of the variation of the resonant amplitudes found in the seven
ambiguous solutions, SU(3) tests involving the \( \Lambda \pi \) channel should be re-evaluated (3).
The Barrelet ambiguities should be investigated in other channels, and the numerous
\( \Sigma \) resonance candidates should be tested in other reactions. Consideration should
also be given to future measurement of the \( R \) and \( A \) parameters for this reaction.

2. THE METHOD OF BARRELET ZEROS

In this section we describe the methods and establish the conventions used.
The theoretical expressions for the angular distributions and polarizations for
the reaction
\[
K^- + p \rightarrow \Lambda + \pi^0
\]
can be expressed in eigenfunctions of orbital angular momentum as
\[
d\sigma/d\Omega = \frac{k_\pi}{k} (|f|^2 + |g|^2) \tag{2a}
\]
and
\[
d\sigma/d\Omega \frac{\hat{z}}{k} = -2 \frac{k_\pi}{k} \hat{n} \text{Im}(f^*g) \tag{2b}
\]
where \( k \) and \( k_\pi \) are the momenta of the \( k \) meson and the \( \pi \) meson in the centre of
mass, \( \hat{n} \) is a unit vector along the production normal \( \hat{k} \times \hat{k}_\pi \), \( z = \hat{k} \times \hat{k}_\pi \) and
\[
f = \frac{1}{r_k k_\pi} \sum_{l} ((l+1) T_{l+} + l T_{l-}) P_l(z) \tag{3a}
\]
and
\[
g = \frac{1}{\sqrt{k_k^* k^*}} \sum_{\ell} (T^+_{\ell} - T^-_{\ell}) P_{\ell}^1(z) \quad (3b)
\]

Clearly eqs (2a) and (2b) are quadratic in the complex amplitudes, \( T^+_{\ell} \) and \( T^-_{\ell} \).

On the other hand, to any desired degree of accuracy, the two body experimental distributions can be represented by the finite Legendre polynomial series:
\[
d\sigma/d\Omega = \frac{k^2}{L} \sum_{n=0}^{L} A_n P_n(z) \quad \text{(4a)}
\]
\[
d\sigma/d\Omega \hat{F} = \frac{k^2}{L} \sum_{n=1}^{L} B_n P_n^1(z) \quad \text{(4b)}
\]

where \( k = \frac{1}{k} \). \( L \) may be interpreted as twice the maximum orbital angular momentum generated in the scattering, \( L = 2L_{\text{max}} \).

The analysis problem is to determine the partial wave amplitudes, \( T_{\ell} \), from the coefficients \( A_n \) and \( B_n \). This correspondence is not unique and it has been difficult in the past to show that all important solutions have been examined. Barrelet has shown us a procedure for determining all sets of partial waves \( T_{\ell} \) which satisfy eqs (4).

He notes that the linear combinations
\[
\sigma^\pm = d\sigma/d\Omega (1 \pm P) \quad \text{(5)}
\]

may be expanded as a power series in the complex variable \( w = pe^{i\theta} \), where \( z = \frac{w + w^{-1}}{2} \), thus eliminating the square root branch points in eq (4b) at \( z = \pm 1 \).

The physical region in the complex \( z \) plane is mapped onto the unit circle in the \( w \) plane. By the Schwartz reflection principle and the relation \( \sigma^+(w) = \sigma^-(w^{-1}) \), both linear combinations are represented by the single function \( s(w) = \sigma^+(w) \) which can be continued into the entire complex \( w \) plane. The Laurent series
\[
s(w) = \frac{1}{w} \sum_{n=0}^{2L} C_n w^n \quad \text{(6)}
\]
must then have $2L$ unique roots. Because $s(w)$ is real on the unit circle, these roots, known as Barrelet zeros, occur in "mirror pairs", such that if $w_n$ is a root, then $w_n^{-M} \in (w_n^*)^{-1}$ is also a root, and $s(w)$ can be written

$$s(w) = \frac{C_{2L}}{w^L} \prod_{n=1}^{L} (w - w_n)(w - w_n^{-M})$$

$$= A_{2L}^2 \prod_{n=1}^{L} \frac{(w - w_n)(w - w_n^{-M})}{w(1 - w_n)(1 - w_n^{-M})}$$

$$= F(w, w_n) F(w, w_n^{-M})$$

where

$$F = (C_{2L})^{1/2} \prod_{n=1}^{L} \frac{(w - w_n)}{(w w_n^*)^{1/2}} e^{i\phi} = A_{2L} \prod_{n=1}^{L} \frac{(w - w_n)}{(1 - w_n)} e^{i\psi}$$

and one root, $w_n$, has been selected from each mirror pair to construct $F(w, w_n)$. Equation (7) is manifestly real on the unit circle if $A_{2L}$ is real.

The problem is to represent eqs (4a) and (4b) as the square of a scattering amplitude. This is accomplished by noting that $F(w, w_k^{-M}) = F^*(w, w_k)$ when $|w| = 1$, so that $s^*(w) = F F^*$ is identical to $s(w)$ in the physical region, and $s^+ = F(w) F^*(w)$ and $s^- = F(w^{-1}) F^*(w^{-1})$ when $|w| = 1$. In general, $s(w) = F(w) F^*(w^{-M}) = s^*(w^{-M})$.

By comparison with eqs (2) and (3), the function $F(w)$, called the transversity amplitude, is readily identified as

$$F(w) = \sqrt{\frac{k^*}{k}} (f + ig)$$

and

$$F(w^{-1}) = \sqrt{\frac{k^*}{k}} (f - ig)$$

The transversity amplitude is clearly not unique. There are two ways to select one root from each "mirror" pair and hence $2^L$ distinct amplitudes, $F(w)$, all of which lead to identical $s^\pm$. In addition to the $2^L$ ambiguities arising from the permutation of the roots, the cross sections and polarizations are invariant under the transformations.
\( F(w) + F(w^{-1}) \) \hspace{1cm} (9a)

and
\[ F(w) + w^n F(w) \] \hspace{1cm} (9b)

where \( n \) is an integer. The first of these is the usual Yang ambiguity. The second is the result of successive applications of the Yang and Minami ambiguities, which changes the number of partial waves implied. Complex conjugation combined with the Yang transformation, which corresponds to simultaneous replacement of all roots and \( A_{2L} + A_{2L}^* \), may be written
\[ F(w) + F^*(w^L) \] \hspace{1cm} (9c)

For any given transversity amplitude a unique set of partial waves, \( T_{J,P} \), may then be projected by a simple matrix inversion from the relation
\[ F(w) = \sum_{J,P} (\text{max} = L/2) T_{J,P} R_{J,P}(w) \] \hspace{1cm} (10)

The \( R_{J,P} \) are a set of orthogonal polynomials on the physical region in the \( w \) plane which can be defined in terms of the Legendre polynomials
\[ R_{J,P}(w) = (J + \frac{1}{2}) P_{\frac{1}{2}}(z) + m(w - w^{-1}) P'_{\frac{1}{2}}(z) \]

where \( \ell \) is the orbital angular momentum, \( P'_{\frac{1}{2}}(z) = d/dz(P_{\frac{1}{2}}(z)) \), \( J \equiv \ell \pm \frac{1}{2} \) is the total spin, the subscript \( P = (-1)^{\ell+1} \) is the parity and
\[ m = -1 \text{ if } J = \ell + \frac{1}{2} \]
\[ +1 \text{ if } J = \ell - \frac{1}{2} \]

These polynomials obey the recursion relation
\[ (2J - 1) R_{J,P}(w) = R_{J-1,P}(w) + 2Jw^m R_{J-1,-P}(w) \]

The cross sections may also be expressed in terms of these polynomials:
\[ \sigma(w) = \frac{\lambda^2}{L} \sum_{J,P} \beta_{J,P} R_{J,P}(w) \] \hspace{1cm} (11)

with \( \beta_{J,P} \) related to the \( A_{\frac{1}{2}} \) and \( B_{\frac{1}{2}} \) of eqs (4a) and (4b),
\[ \beta_{J,P} = \frac{1}{(2\ell + 1)} \left\{ A_{\frac{1}{2}} + \text{im} \left[ \frac{\ell(\ell+1)}{2 + \left(\frac{1-m}{2}\right)} B_{\frac{1}{2}} \right] \right\} \] \hspace{1cm} (12)
The Barrelet zeros may be calculated from existing partial wave solutions using eq (10) or from published Legendre coefficients using eq (11). Experimental and statistical difficulties notwithstanding, the zeros may best be fit directly to the experimental distributions of eq (5). High statistics experiments with good polarization data are therefore crucial for improved partial wave analyses. After the zeros have been determined, the full set of ambiguities may be studied. Some of these ambiguities may be resolved through physical constraints on the transformed waves. For example unitarity or the Wigner causality condition (7) may be violated, or a priori known resonant structures may be eliminated. In different charge states of the same reaction, isospin may not be conserved. In general, however, it is necessary to use additional information in the form of $R$ and $A$ parameters to distinguish between solutions.

In addition to defining the complete set of ambiguous solutions, the advantage of the Barrelet zero expansion over other product representations of $\sigma^+$ (e.g. ref (4a)) is the stability of the zeros as $L$ increases, due to the orthogonality of the $R_{J,P}$ polynomials. The number of stable zeros, i.e. zeros independent of $L$ that lie inside the Lehmann ellipse through the nearest $t$ channel singularity, limits the number of partial waves which may be experimentally determined (1). Partial wave solutions will, in general, contain small amplitude waves of high spin that lead to zeros lying outside the boundary of the convergence ellipse. Therefore, as long as their contribution is within the experimental errors, these higher waves can only be determined from a model. This gives rise to the "continuum ambiguity" which may be removed in practice by truncation of the partial wave expansion at order $L/2$, thereby permitting only discrete Barrelet ambiguities. Moreover, even for a specific value of $L$, experimental uncertainties do not usually permit the calculation of a unique set of stable zeros. Nevertheless, once the zeros of a particular distribution have been found, calculation of the partial waves is straightforward.
Once the zeros are determined, the behavior of the partial waves depends on the energy connection of the zeros and on the absolute phase at each energy. The analyticity of $F(w)$ requires the stable zeros to vary "regularly" from one energy to the next, allowing the zeros to be connected into trajectories. This corresponds to selecting the sign of the imaginary part of each zero in the complex $z$ plane. In cases where the connection of the zeros is not obvious, "least path" or "minimum surface" methods could be applied to the zeros in the complex $w$ or $z$ planes. The success of least path techniques used in $\pi N$ elastic scattering is largely due to the fact that most of the intrinsic ambiguities violate partial wave unitarity. This is not the case in inelastic reactions, where unitarity is not a powerful constraint. Because of the energy dependence built into our solutions we have not needed to apply such techniques to connect the zeros.

We have been able to correlate the regular behavior of our zero trajectories with the dominant resonant structure when the zeros are plotted as a function of energy in the Mandelstam plane ($\tau, \eta$). This correlation is expected from eq (10). In addition, the trajectories of the zeros with $|Re z| > 1$ are consistent with crossing through the intersections of the leading $s$ and $t$ channel poles, as required to cancel double poles in the scattering amplitude (9). Indeed, if the behavior of the zero trajectories outside the physical region were prescribed by a model such as that proposed by Odorico in ref (9), this would usefully predict the unmeasurable high spin waves.

From eq (8) we see that the phase of $F(w)$ must be fixed by physical considerations. In elastic channels the optical theorem may be applied. In inelastic channels the usual procedure is to let the dominant resonance in an energy interval define the overall phase. For our energy dependent solutions we have fixed the phase of the $D5(1765)$ at the resonant energy, letting the energy dependent parameterization determine the absolute phase at other energies. In order to compare solutions generated by Barrelet transformations, the invariant
amplitude $A'(s,t)$ was required to have the same phase as the energy dependent solution at $t = m^2_{K^*}$. This provided continuity to all partial waves over the interval 1550 to 2170 MeV, and, more importantly, preserved the Breit-Wigner energy dependence of the established resonances, in agreement with the usual procedure in inelastic channels. Additional details may be found in Reference 8.

3. THE AMBIGUOUS SOLUTIONS

The ambiguity transformations have been applied to our best energy dependent fits (2,8). For practical reasons, and because $G7$ and $G9$ are small in magnitude we have set $l_{max} = 3$. We have investigated the ambiguities associated with the transformation of each zero trajectory in turn, selecting trajectories with least change of direction when crossing the physical region. Despite choosing the absolute phase at each energy in order to reproduce the established $D5(1765)$ and $F7(2040)$ resonances, most of the ambiguity solutions did not exhibit "resonant-like" behaviour in these waves. Only those solutions which preserved a resonant structure at both $D5(1765)$ and $F7(2040)$ were selected for further consideration. It should be mentioned that all ambiguities satisfied unitarity.

One of the purposes of such a selection was to discover whether an ambiguity existed which preserved the 1765 and 2040 MeV states, but which did not contain an $F5(1920)$ resonance. Seven Barrelet ambiguities, including the original solution, met the selection criteria, and all seven ambiguities contained both the $D3(1660)$ and $F5(1920)$ resonances.

In figs. 1-4, Argand diagrams for the seven ambiguities are presented from 1550 to 2170 MeV, along with speed plots for the individual partial waves. The algorithm used to calculate the speed was the following:

$$\text{Speed}(i) = \frac{1}{2} \left[ \frac{|T(i+1) - T(i)|}{E(i+1) - E(i)} + \frac{|T(i) - T(i-1)|}{E(i) - E(i-1)} \right]$$

Points were calculated every 10 MeV with the symbol being placed every 50 MeV. Each partial wave may be compared individually with the same partial wave
generated by an ambiguity transformation. The original solution is labelled 0; other ambiguities are labelled 1, 4, 8, 16, 17, and 20. The different energy dependent fits had similar Barrelet ambiguities. Therefore only ambiguities of fit AH are shown.

A study of the Argand diagrams, combined with speed plots, indicates a proliferation of resonant structure via the ambiguity transformation. A characteristic of these solutions is the presence of several resonances at nearly the same mass with similar widths but in different $J^P$ states. The dominant resonances are present in each ambiguity, albeit with varying couplings depending on the particular solution. Among ambiguities the range of the D5(1765) amplitude at resonance, $t = \sqrt{\frac{\Gamma}{\Gamma}}$, extends from -0.12 to -0.29; for D3(1660) from +0.06 to +0.16; for F5(1920) from -0.06 to -0.12; for F7(2040) from +0.06 to +0.19. This indicates that SU(3) tests using resonant couplings to the $\Lambda\pi$ channel should be reconsidered in light of the ambiguities (3). These solutions cannot be resolved by present $\Lambda\pi$ data; only model dependent assumptions or the $\hat{R}$ and $\hat{T}$ parameters will permit a preferred solution.

It should be noted that the Barrelet transformation does not identically propagate the Breit-Wigner line shape. This is, however, not critical, because of our incomplete understanding of the line shape of multi-channel hadron resonances. Subsequent energy dependent fits have shown that the $\Lambda\pi$ data will accommodate such resonant structures, parameterized with explicit Breit-Wigner shapes. However, present energy dependent fitting techniques would not have generated these solutions due to the complicated structure required in the lower waves. "Unphysical" kinks in the Argand plots may be smoothed by allowing a slightly different fit within the experimental errors.

In table I we give estimated resonance parameters for each ambiguity. The relative sign of $t$ for each resonance is also estimated where it is reasonably clear in the Argand diagram, but this sign is well-determined only for the original solution 0. Several of these ambiguity solutions have been used to
provide starting values for further energy dependent fits. Such fits reached improved $\chi^2$ values, while retaining parameters similar to table 1 for resonances with $t \geq 0.05$. Not all the resonances suggested in table 1 were required to re-fit the data, especially since energy-dependent techniques were not able to reproduce the complicated behaviour of the lower waves. The small size of the background amplitudes in these fits qualitatively supports the Harari-Freund conjecture that non-resonant background should not exist in reactions which do not couple to the Pomeron Regge trajectory in the $t$ channel. The precision of the data does not justify a more definitive test of this hypothesis.

It is difficult to classify the numerous I resonance candidates which are presented in table 1. The most reliable parameters are for states indicated with $t > 0.04$. These states require study in the context of particular models. Several authors have suggested that additional F5 states should exist in this energy range (10). In four solutions another F5 state is indicated in the interval 1800-1850 MeV. The presence of this state does not appreciably affect the Argand diagrams but shows up most strongly in the speed plots. The sign of its coupling is not well defined, so it might be either an octet or decuplet state. In the context of SU(6) @ O(3) a state could be assigned to either a $(56, 4^+)$ or possibly a $(70, 2^+)$ super-multiplet. The presence of a possible F7 resonance near 1940 MeV is indicated in three solutions. Quark models (10) and duality arguments (11) have also suggested the possibility of such a state. If we regard states with amplitude at resonance $t < 0.04$, which is near the limit of experimental resolution, as not required to fit the data, and therefore unproven to exist, the number of states in any one solution is in accord with predictions of SU(6) @ O(3) symmetry models (10).

It is evident that the ambiguities are in disagreement with the usual assumption of "minimum structure". For energy dependent solutions, minimum structure usually implies maximum coupling of the dominant resonances and the minimum number of states required to fit the data in addition to background.
Because the data themselves do not require a unique solution, these Σ resonance candidates need to be substantiated in other ΛN reactions. Due to the experimental uncertainty it is also likely that not all the resonances are experimentally required. Nonetheless, all symmetry models of the strong interaction require many additional Σ states, and the new states revealed by this analysis are entirely compatible with the ΛN + Λπ data.

Since each of these seven solutions preserves do/dΩ and do/dΩ P identically, these ambiguities can be experimentally distinguished only by measurement of the Λ and Λ̅ parameters. In our notation these observables are defined:

\[
do/dΩ = \frac{k}{k} (|f|^2 + |g|^2)
\]

\[
P do/dΩ = \frac{2k\pi}{k} \text{Im} (fg^*)
\]

\[
R do/dΩ = \frac{2k\pi}{k} \text{Re} (fg^*)
\]

\[
Λ do/dΩ = \frac{k\pi}{k} (|f|^2 - |g|^2)
\]

These parameters are predicted from our solutions and shown at two representative energies in fig. 5. It can be seen that measurements of Λ and Λ̅ at several energies would be capable of resolving these solutions. Further analyses of the Λπ channel using only P and do/dΩ measurements are likely to find variants of these solutions, resulting from choosing a slightly different fit allowed by the experimental errors or from different assumptions about the order of truncation of the partial wave expansion and the energy connection of the zero trajectories. Without Λ and Λ̅ analyses of this channel will continue to be model dependent.
4. A TEST OF DUALITY

Duality diagrams and the absence of SU(3) exotic states imply that the s-t duality diagram for $\bar{K}N \rightarrow \Lambda\pi$ is "illegal" or non-planar (12). Consequently, the invariant amplitude $A'(s,t)$, defined in ref (13), would be expected to become purely real in some high energy limit. At lower energies we might expect the amplitude to be real in an average or "semi-local" sense, i.e. Im($A'(s,t)$) would oscillate about zero. Using only the s channel resonance parameters Schmid and Storrow (6) qualitatively verify the predictions of duality in the positive t region (at $t = m_{K^*}^2$). These authors give justification for testing duality in the region $t > 0$, which is experimentally inaccessible.

Kernan et al (14) have performed a test of semi-local duality at $t = t_{\min}$ for $K^-p \rightarrow \Lambda\pi^0$. We have performed a similar test by calculating $A'(s,t)$ and $B(s,t)$ at both $t = 0$ and at $t = m_{K^*}^2$. We have extrapolated to $t = m_{K^*}^2$ by using the zeros of the transversity amplitude. This extrapolation is subject to uncertainty near the convergence boundary of the expansion, but is the most reasonable extrapolation we can make. In fig. 7 $A'(s,t = m_{K^*}^2)$ and $B(s,t)$ are shown for the original solution. The energy range, 1550 to 2170 MeV, is broad enough to observe the expected averaging of Im($A'(s,t)$). Qualitatively $A'(s,t)$ is similar for each ambiguity while the behaviour of $B(s,t)$ differs (8). We have tried imposing exact duality, i.e. Im($A'(s,t)$) = 0, by varying the absolute phase at each energy. However, this condition caused the established resonant behaviour to disappear in these solutions.

The resonant structure in the ambiguities is in general agreement with Veneziano models, which are explicitly dual (15). The Veneziano formula implies that when $\alpha(s) - \frac{1}{2} = n$, there should be a tower of $(n+1)$ resonant states degenerate in mass with spins from $J = \frac{1}{2}$ to $J = \alpha(s)$. For example, the finite width Veneziano model of Lovelace (16) generates resonances with equal widths in all partial waves. This is nearly the behaviour observed in table 1.
5. CONCLUSIONS

The method of Barrelet zeros has proven to be a useful technique in the analysis of the reaction $K^- p \rightarrow \Lambda \pi^0$. Seven ambiguous solutions have been generated which preserve the established resonances, $(D3(1660), D5(1765), F5(1920), F7(2040))$, but differ in the resonant couplings for these states. This indicates that SU(3) tests involving the $\Lambda \pi$ channel should be considered in view of the variation of these couplings, and that the uniqueness and energy continuation of other partial wave solutions should be tested.

Many new $I$ resonance candidates have been shown to be compatible with $\bar{K}N \rightarrow \Lambda \pi$ data. These states must await confirmation in other reactions. The seven solutions have been shown to be in agreement with semi-local duality by the observed averaging of $\text{Im}(A'(s,t))$ about zero. The implied resonant states with coupling $t > 0.04$ for several of these solutions are suggestive of Veneziano models and of the Harari-Freund conjecture that amplitudes may be purely resonant in non-diffractive, inelastic scattering. These solutions are capable of being distinguished by measurement of the $\bar{R}$ and $\bar{A}$ parameters.

6. ACKNOWLEDGEMENTS

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REFERENCES


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Table 1. Suggested Resonance Parameters for Each Ambiguity

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*Estimated from Speed Plots and Argand Diagrams. Parameters for states with +<0.04 are less reliable.

$$T = \frac{\sqrt{\chi^2}}{\varepsilon-1}, \quad \varepsilon = (E_R - E)/(\Gamma/2); \quad \left| \frac{dT}{dE} \right|_{\text{max}} = \text{Speed} (E_R) = \frac{\sqrt{\chi^2}}{\Gamma/2}.$$  

$$\text{Speed} (i) = \frac{1}{2} \frac{|T(i+1)-T(i)|}{E(i+1)-E(i)} + \frac{|T(i)-T(i-1)|}{E(i)-E(i-1)}$$

The sign of $t$ is estimated from the Argand diagram.
FIGURE CAPTIONS

Fig. 1. Argand diagrams and speed plots for the seven ambiguity solutions from 1550 to 2170 MeV. $S_{11}$ partial wave. Points are calculated every 10 MeV with symbols placed every 50 MeV.
(a) Ambiguities 0, 1, 4.
(b) Ambiguities 8, 16, 17, 20.

Fig. 2. Argand diagrams and speed plots for the seven ambiguity solutions from 1550 to 2170 MeV. $P_{11}$ and $P_{13}$ partial waves. Points are calculated every 10 MeV with symbols placed every 50 MeV.
(a) Ambiguities 0, 1, 4.
(b) Ambiguities 8, 16, 17, 20.

Fig. 3. Argand diagrams and speed plots for the seven ambiguity solutions from 1550 to 2170 MeV. $D_{13}$ and $D_{15}$ partial waves. Points are calculated every 10 MeV with symbols placed every 50 MeV.
(a) Ambiguities 0, 1, 4.
(b) Ambiguities 8, 16, 17, 20.

Fig. 4. Argand diagrams and speed plots for the seven ambiguity solutions from 1550 to 2170 MeV. $F_{15}$ and $F_{17}$ partial waves. Points are calculated every 10 MeV with symbols placed every 50 MeV.
(a) Ambiguities 0, 1, 4.
(b) Ambiguities 8, 16, 17, 20.

Fig. 5. $\bar{P}$, $\bar{R}$, $\bar{A}$ parameters at two representative energies for the seven partial wave ambiguity solutions. Ambiguities are numbered: 0, 1, 4, 8, 16, 17, 20.
(a) 1670 MeV; (b) 1790 MeV.

Fig. 6. $A'(s,t)$ and $B(s,t)$ invariant amplitudes vs $\sqrt{s}$ for ambiguity solution 0. $A'(s,t)$ is similar for each ambiguity while $B(s,t)$ varies considerably among solutions.
Fig. 1a.
Fig. 1b.
Fig. 2a.
Fig. 2b.
Fig. 3a.
Fig. 3b.
Fig. 4a.
Fig. 4b.
Fig. 5a.

P, R, A Parameters
E_{c.m.} = 1670 (MeV)
(a)
Fig. 5b.

\[
\begin{align*}
\text{cos } \theta_{\text{c.m.}} \\
\bar{P}, \bar{R}, \bar{A} \text{ Parameters} \\
E_{\text{c.m.}} &= 1790 \text{ (MeV)} \\
(b)
\end{align*}
\]
Fig. 6.
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