Title
Mixed structured and unstructured uncertainty modeling method with application to Linear Tape-Open drives

Permalink
https://escholarship.org/uc/item/13b4j8cm

Author
Wang, Longhao

Publication Date
2012

Peer reviewed|Thesis/dissertation
Mixed Structured and Unstructured Uncertainty Modeling Method with Application to Linear Tape-Open Drives

A thesis submitted in partial satisfaction of the requirements for the degree  
Master of Science  
in  
Engineering Science (Mechanical Engineering)  

by  
Longhao Wang

Committee in charge:  
Professor Raymond de Callafon, Chair  
Professor Robert Bitmead  
Professor Sonia Martinez  

2012
The thesis of Longhao Wang is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2012
DEDICATION

To my beloved parents
To the best girl in the world Qingzhu
This is not the end. It is not even the beginning of the end.
But it is, perhaps, the end of the beginning.
—Winston Churchill
So, never be lazy.
— Longhao Wang
# TABLE OF CONTENTS

Signature Page ................................................................. iii
Dedication ................................................................. iv
Epigraph ................................................................. v
Table of Contents ........................................................ vi
List of Figures ........................................................ viii
Acknowledgements ......................................................... ix
Vita ................................................................. x
Abstract of the Thesis ..................................................... xi

Chapter 1 Introduction and Motivation ................................ 1

Chapter 2 Mathematic Background ........................................ 3
  2.1 Linear Fractional Transformation[1] .............................. 3
  2.2 Principle Component Analysis [2, 10] ............................ 4
  2.3 Structured Singular Value Analysis [1] ........................... 6

Chapter 3 Experimental Data of LTO Servo Actuator ............... 9
  3.1 Experiment Setup ................................................... 9
  3.2 Frequency Responses .............................................. 10

Chapter 4 Modeling Variability of Servo Actuator ................. 12
  4.1 Structured Parametric Uncertainty Characterization .......... 12
    4.1.1 Linear parametric uncertainty model ........................ 12
    4.1.2 Reduce parameter independence ............................. 14
    4.1.3 Independence reduced linear parametric uncertainty model ........................................... 18
    4.1.4 Represent in the LFT form ................................... 18
    4.1.5 Application to LTO tape data ................................. 19
  4.2 Unstructured Uncertainty Characterization ...................... 21
    4.2.1 Dual Youla Parameterization ................................. 21
    4.2.2 LFT form of dual-Youla parameterization ................ 22
  4.3 Mixed Structured and Unstructured Uncertainty in LFT Form .................................................. 23
  4.4 MSU Model Application to LTO Drives ......................... 24
<table>
<thead>
<tr>
<th>Chapter 5</th>
<th>Robust Stability and Performance Tests</th>
<th>26</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>Robust Stability Test</td>
<td>26</td>
</tr>
<tr>
<td>5.2</td>
<td>Robust Performance Test</td>
<td>27</td>
</tr>
<tr>
<td>5.3</td>
<td>Application to LTO Drives</td>
<td>28</td>
</tr>
<tr>
<td>5.3.1</td>
<td>Robust stability test</td>
<td>30</td>
</tr>
<tr>
<td>5.3.2</td>
<td>Robust performance test</td>
<td>30</td>
</tr>
<tr>
<td>Chapter 6</td>
<td>Conclusion</td>
<td>33</td>
</tr>
<tr>
<td>Bibliography</td>
<td></td>
<td>34</td>
</tr>
</tbody>
</table>
LIST OF FIGURES

Figure 3.1: Block diagram of the system ........................................ 10
Figure 3.2: Magnitude plot of 15 frequency responses of LTO-drive servo actuators .................................................. 11

Figure 4.1: Magnitude plot of 15 fitted models from the frequency responses in Figure 3.2 ................................................. 13
Figure 4.2: Magnitude plot of 50 randomly chosen models of $G_{\hat{y}}(j\omega)$ in (4.8) .................................................. 15
Figure 4.3: LFT block of linear parametric uncertainty model ................ 19
Figure 4.4: Magnitude plot of 50 randomly chosen models $|G_{\delta}(j\omega)|$ from the linear parametric perturbation model of (4.25) ..................... 20
Figure 4.5: LFT block of unstructured uncertainty model ..................... 23
Figure 4.6: LFT block of mixed structured and unstructured uncertainty model ............................................................... 24
Figure 4.7: Comparison uncertainty models by MSU, Additive and DY methods ................................................................. 24
Figure 4.8: Comparison of the magnitude plots of unstructured uncertainties ............................................................... 25

Figure 5.1: LFT block of mixed structured and unstructured uncertainty model with feedback controller ........................................ 27
Figure 5.2: LFT block of MSU model with respect to performance channel and fictitious uncertainty ............................................. 28
Figure 5.3: Bode plot of $C_{int}$ and $C$ ................................................ 29
Figure 5.4: Robust stability test for MSU, DY and Additive methods .......... 30
Figure 5.5: Robust performance test for MSU, DY and Additive methods .......... 31
Figure 5.6: Magnitude plot of 50 randomly sampled sensitivity functions and $W_s^{-1}$ .................................................. 31
ACKNOWLEDGEMENTS

I would like to acknowledge Professor Raymond A de Callafon for his support as my advisor and Chair of my committee. His guidance have proved to be invaluable to me.

Thanks to my dear friends and your supports in my most difficult time.

Thanks to the Starbucks nearby. The latte there is really delicious and inspiring.
<table>
<thead>
<tr>
<th>Year</th>
<th>Education</th>
</tr>
</thead>
<tbody>
<tr>
<td>2010</td>
<td>B. S. in Mechanical Engineering, Shanghai Jiao Tong University, China</td>
</tr>
<tr>
<td>2010-2012</td>
<td>Graduate Student, University of California at San Diego</td>
</tr>
<tr>
<td>2012</td>
<td>M. S. in Engineering Science (Mechanical Engineering), University of California at San Diego</td>
</tr>
</tbody>
</table>
ABSTRACT OF THE THESIS

Mixed Structured and Unstructured Uncertainty Modeling Method
with Application to Linear Tape-Open Drives

by

Longhao Wang

Master of Science in Engineering Science (Mechanical Engineering)

University of California, San Diego, 2012

Professor Raymond de Callafon, Chair

Starting from multiple frequency domain measurements, this paper presents a procedure to formulate a dynamic model of a servo actuator that consists of a nominal model and an allowable model perturbation in the form of a parametric and unstructured uncertainty. A separation between parametric and unstructured uncertainty is achieved by first estimating low order linear parameter models via frequency domain curve fitting followed by a linear Principle Component Analysis (PCA) to bound the parametric variations on the estimated parameters. Remaining differences between the low order parametric models and the measured frequency responses are captured by a bounded unstructured uncertainty on a frequency dependent dual-Youla parameter that uses prior information on a stabilizing feedback
controller. The resulting perturbation model is written in a standard Linear Fractional Transformation (LFT) form and the procedure is applied to experimental data obtained from several mechanically equivalent servo actuators in a Linear Tape-Open (LTO) drive.
Chapter 1

Introduction and Motivation

Variations in servo actuator dynamics is commonly observed and it is caused by factors such as manufacturing variability, temperature and position dependency. Modern robust control design approaches [1, 3, 4] could potentially compensate for such variations. However, for guaranteeing stability and performance robustness a so-called perturbation or uncertainty model is needed to model and bound the variations in the dynamic behavior of a servo actuator.

The uncertainty modeling method introduced in this thesis aims at separating structured and unstructured variations in the dynamics [5]. In particular for servo actuators, structured variations are used to capture real-valued parametric variations in gain, location and damping of resonance modes. Complex unstructured variations are used to bound non-structural variations measured in the frequency response. This separation is even more important for high performance control of servo actuators in Linear Tape-Open (LTO) drives [6, 7] where frequency domain measurements are readily available for modeling purposes [8]. In LTO drives structural variations are mainly due to variations in manufacturing, while unstructured variations occur due to the exchange of different tape cartridges and the inherent tape/head interaction.

Starting from multiple frequency domain measurements obtained from LTO servo actuators [9], this paper presents a modeling procedure to formulate a perturbation model that consists of a nominal model and bounds on real-valued structured and complex unstructured variations. Separation between parametric and
unstructured uncertainty is achieved by first estimating low order dynamic models via frequency domain curve fitting followed by a linear Principle Component Analysis (PCA) [10]. The proposed linear PCA is a simplification of the nonlinear PCA used in [11, 12] solved with a non-convex optimization. However, the linear PCA allows to find the minimum number of independent perturbations in which the model parameters are varying with a straightforward singular value decomposition.

In addition, remaining differences between the low order parametric uncertainty model and the measured frequency responses are captured by a bounded unstructured uncertainty on a frequency dependent dual-Youla parameter. An unstructured dual-Youla uncertainty model can use prior information on a stabilizing feedback controller, creating an uncertainty model that is guaranteed to be stabilized by the feedback controller [13, 14]. The unstructured dual-Youla uncertainty model is known to be less conservative [15] in describing unstructured model uncertainty compared to standard additive or multiplicative uncertainty models.

The remainder of this thesis is organized as follows. Chapter 2 gives an brief introduction to the math that will be used in the following chapters. Chapter 3 introduces the experiment setup and how the frequency domain data of LTO servo actuators are measured. Chapter 4 fully illustrates the procedure of obtaining the mixed structured and unstructured uncertainty modeling method. Chapter 5 formulate the robust stability and robust performance test for mixed structured and unstructured uncertainty modeling method along with the comparison of the robust stability and performance analysis with other unstructured uncertainty modeling methods. All the related applications to LTO servo actuator is given accordingly at the end of each chapter.
Chapter 2

Mathematic Background

2.1 Linear Fractional Transformation[1]

Linear fractional transformation (LFT) in this thesis is focus on the matrix case.

Definition 2.1 (LFT). Let \( M \) be a complex matrix partitioned as

\[
M = \begin{bmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{bmatrix} \in \mathbb{C}^{(p_1+p_2) \times (q_1+q_2)},
\]

and let \( \Delta_l \in \mathbb{C}^{q_1 \times q_2} \) and \( \Delta_u \in \mathbb{C}^{q_1 \times p_1} \) be two other complex matrices. Then we can formally define a lower LFT with respect to \( \Delta_l \) as the map \( F_l(M, \bullet) : \mathbb{C}^{q_2 \times p_2} \to \mathbb{C}^{p_1 \times q_1} \) with

\[
F_l(M, \Delta_l) := M_{11} + M_{12}\Delta_l(I - M_{22}\Delta_l)^{-1}M_{21}
\]

provided \((I - M_{22}\Delta_l)\) exists. We can also define an upper LFT with respect to \( \Delta_u \) as

\[
F_u(\bullet, M) : \mathbb{C}^{q_1 \times p_1} \to \mathbb{C}^{p_2 \times q_2}
\]

\[
F_u(M, \Delta_u) := M_{22} + M_{21}\Delta_u(I - M_{11}\Delta_u)^{-1}M_{12}
\]

provided \((I - M_{22}\Delta_u)\) exists.

Based on the above definition, we have two very important property which will be used in getting the matrix \( P \) in (4.26) and the matrix \( M \) in (5.3).
Property 2.1. Let $G(s)$ be an parameter perturbed linear transfer function $P$ is the
\[ G(s) = (B(s) + \mathcal{V}_b(s)\delta)(A(s) + \mathcal{V}_a(s)\delta)^{-1} \]
where $B(s), A(s) \in \mathbb{C}$, $\mathcal{V}_b(s), \mathcal{V}_a(s) \in \mathbb{C}^{1 \times p}$ and $\delta \in \mathbb{R}^{p \times 1}$
Let $P(s)$ be the $2 \times 2$ LFT transfer matrix of $G(s)$ with respect to perturbation vector $\delta$. We have
\[
\begin{align*}
P_{11}(s) &= -\bar{A}^{-1}(s)\mathcal{V}_a(s) & P_{12}(s) &= \bar{A}^{-1}(s) \\
P_{21}(s) &= \mathcal{V}_b(s) - \bar{B}(s)\bar{A}^{-1}(s)\mathcal{V}_a(s) & P_{22}(s) &= \bar{B}(s)\bar{A}^{-1}(s)
\end{align*}
\]

Property 2.2. Let $M$ be suitably partitioned matrix shown in Definition 2.1
\[
M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}
\]
Suppose $\mathcal{F}_u(M, \Delta)$ is square and well-defined and $M_{22}$ is nonsingular. Then the inverse of $\mathcal{F}_u(M, \Delta)$ exists and is also an LFT with respect to $\Delta$:
\[
(\mathcal{F}_u(M, \Delta))^{-1} = \mathcal{F}_u(N, \Delta)
\]
with $N$ given by
\[
N = \begin{bmatrix} M_{11} - M_{12}M_{22}^{-1}M_{21} & -M_{12}M_{22}^{-1} \\ M_{22}^{-1}M_{21} & M_{22}^{-1} \end{bmatrix}
\]

2.2 Principle Component Analysis [2, 10]

Principle component analysis (PCA) is a very powerful technique from applied linear algebra. PCA transfers a cluster of data into a principle coordinate system. A linear PCA can be done by the Eigen decomposition.

Define a $p$ dimensional space $\mathcal{X}$. Given we have the original $N$ data vectors
\[
x_j = [x_j(1), x_j(2), ..., x_j(p)]^T \in \mathbb{R}^{p \times 1}, \ j = 1, 2, ..., N
\]
in space $\mathcal{X}$, where $x_j(i)$ means the $j^{th}$ vector’s projection on $i^{th}$ direction in $\mathcal{X}$ space.
We define the matrix $X = [x_1, x_2, ..., x_N] \in \mathbb{R}^{p \times N}$, we obtain the covariance matrix of $X$
\[
C_X = \frac{1}{N}XX^T
\]
Apparently, the $i^{th}$ row in $X$, defined as
\[ l_i^{(X)} = [x_1(i), x_2(i), \ldots, x_N(i)] \in \mathbb{R}^{1 \times N} \]
is a collection of projections in the $i^{th}$ direction. The $ij^{th}$ element in $C_X$
\[ c_{ij}^{(X)} = l_i^{(X)} l_j^{(X)T} / N \]
represents the covariances of all projections in $i^{th}$ and $j^{th}$ directions. If $c_{ij}^{(X)} = 0$, that means the $i^{th}$ and $j^{th}$ directions are perfectly uncorrelated. Unfortunately, the clustered raw data almost always have correlations between directions. Also, the directions will unavoidably slightly correlate because of the existence of noise in data measurement.

However, based on the goal of PCA, the desired principle space should contain directions which are perfectly uncorrelated.

We now define the $r (r \leq p)$ dimensional principle components space $Y$, the projection of data vector $x_j$ in space $Y$ as
\[ y_j = [y_j(1), y_j(2), \ldots, y_j(r)]^T, \quad j = 1, 2, \ldots N \]
and matrix $Y = [y_1, y_2, \ldots, y_N]$. So covariance matrix of $Y$
\[ C_Y = \frac{1}{N} YY^T \]
Also, the $i^{th}$ row in $Y$, defined as
\[ l_i^{(Y)} = [y_1(i), y_2(i), \ldots, y_N(i)] \in \mathbb{R}^{1 \times N} \]
is a collection of projections in the $i^{th}$ direction. The $ij^{th}$ element in $C_Y$
\[ c_{ij}^{(Y)} = l_i^{(Y)} l_j^{(Y)T} / N \]

**Property 2.3.** In the $r$ dimensional principle component space $Y$, $c_{ij}^{(Y)} = 0$ for all $i \neq j$ and $c_{ii}^{(Y)}$ the variances in each directions, where $Y$, $c_{ij}^{(Y)}$ is defined above and, $j = 1, 2, \ldots, r$.

Based on Property 2.3, we can see that an Eigen decomposition will transfer the covariance matrix in space $X$ to its principle component space $Y$ with the following relationship
\[ C_X = TC_Y T^T \]
where \( C_Y \) is a diagonal matrix as expected and \( T \in \mathbb{R}^{p \times r} \) is an projection operation matrix which project \( X \) in \( X \) into \( Y \) in \( Y \) with relationship

\[
X = TY
\]

Here, we have done a linear PCA on a group of data \( X \) by getting \( T \) and \( Y \). Also, we notice that a Singular value decomposition(SVD) on \( X \) will also achieve this according to Low-Rank Approximation Theorem.

**Theorem 2.1** (Low-Rank Approximation Theorem). Let \( X = U \Sigma V^* \) be the SVD of \( A \) and having the following partition

\[
X = U \Sigma V^* = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix}
\]

where \( X, U, \Sigma, V \in \mathbb{R}^{p \times p}, \ U_1, V_1 \in \mathbb{R}^{p \times r}, \ U_2, V_2 \in \mathbb{R}^{p \times (p-r)} \), \( \Sigma_1 \in \mathbb{R}^{r \times r} \) and \( \Sigma_2 \in \mathbb{R}^{(p-r) \times (p-r)} \) with \( r \leq p \). Then,

\[
\hat{X} = U_1 \Sigma_1 V_1^*
\]

satisfies

\[
\hat{X} = \arg \min_{X_r} \|X - X_r\|_F \text{ subject to rank}(X_r) = r \leq p
\]

Here, \( U_1 \in \mathbb{R}^{p \times r} \) is the transferring matrix. The multiplication \( \Sigma_1 V_1 \in \mathbb{R}^{r \times r} \) is the corresponding matrix in the low-rank \( r \) dimensional principle component space.

If we write \( \Sigma_1 V_1 = Y \), we have the same linear transformation relation:

\[
\hat{X} = U_1 Y
\]

### 2.3 Structured Singular Value Analysis [1]

Singular value are normally used as indicator to analyze robust stability of closed-loop system with unstructured perturbations. The advantage of singular value analysis is that we need little knowledge about the perturbations. However, this advantage also lead to its major disadvantage of conservatism. The uncertainty model is much larger than necessary and the uncertainties in the system is usually of obvious structure. For this reason, Doyle introduced the structured singular value
analysis with consideration of the structure of system perturbations. This gives us flexible of building uncertainty models by introducing the parametric perturbations and makes the uncertainty models much less conservative. In this section, we will get a glimpse of structured singular value and some result will be used in Chapter 5 for formulating robust stability and robust performance test.

Consider the mixed real and complex uncertainty involves three types of blocks: repeated real scalar, repeated complex scalar and full blocks. Three nonnegative integers, \(S_r, S_c\) and \(F\), represent the number of repeated real scalar blocks, the number of repeated complex scalar blocks and the number of full blocks and they satisfy

\[
\sum_{i=1}^{S_r} k_i + \sum_{j=1}^{S_c} r_j + \sum_{l=1}^{S_r} m_l = n
\]

The \(i^{th}\) repeated real scalar block is \(k_i \times k_i\), the \(j^{th}\) repeated complex scalar block is \(r_j \times r_j\), and the \(l^{th}\) full block is \(m_l \times m_l\). Define the structured uncertainty \(\Delta \subset \mathbb{C}^{n \times n}\) set to be

\[
\Delta = \{ \text{diag}[\delta_1 I_{k_1}, \ldots, \delta_{S_r} I_{k_{S_r}}, \phi_1 I_{r_1}, \ldots, \phi_{S_c} I_{r_{S_c}}, \\
\Delta_1, \ldots, \Delta_F] : \delta_i \in \mathbb{R}, \phi_j \in \mathbb{C}, \Delta_l \in \mathbb{C}^{m_l \times m_l} \}
\]  

(2.1)

With this uncertainty set, we can give the definition of mixed structured singular value (mixed \(\mu\)).

**Definition 2.2.** Let \(M \in \mathbb{C}\); then

\[
\mu_{\Delta}(M) := (\min \{ \bar{\sigma}(\Delta) : \Delta \in \Delta, \det(I - M\Delta) = 0 \})^{-1}
\]

unless no \(\Delta \in \Delta\) makes \(I - M\Delta\) singular, in which case \(\mu_{\Delta}(M) := 0\).

\[
Q = \{ \Delta \in \Delta : \delta_i \in [-1, 1], |\phi_j| = 1, \Delta_l \Delta_l^* = I_{m_l} \}
\]

The computation of \(\mu\) is a NP hard problem, which means that it may not be computable in a polynomial time. We usually use the upper bound as a conservative indicator of \(\mu\). To find the upper bound of \(\mu\), we first define another two set:

\[
\mathcal{D} = \{ \text{diag}[\hat{D}_1, \ldots, \hat{D}_{S_r}, D_1, \ldots, D_{S_c}, d_1 I_{m_1}, \ldots, d_{F-1} I_{m_{F-1}}, I_{m_F}] : \\
\hat{D}_i \in \mathbb{C}^{k_i \times k_i}, \hat{D}_i = \hat{D}_i^* > 0, d_j \in \mathbb{R}, d_j > 0 \}
\]

Then, the upper bound of \(\mu\) is evaluated by
Theorem 2.2. Let $M \in \mathbb{C}^{n \times n}$ and $\Delta \in \Delta$. Then

$$
\mu_{\Delta} \leq \inf_{D \in D, G \in G} \min_{\beta} \{ \beta : M^* D M + j(G M - M^* G) - \beta^2 D \leq 0 \}
$$

$\mu$ is useful in robustness and performance test and the theorems are formulated particularly for the mixed structured and unstructured uncertainty modeling method in Chapter 5. And we use the upper bound as the indicator of $\mu$ for the numerical evaluation. For more details about $\mu$, you can refer to [1, 16].
Chapter 3

Experimental Data of LTO Servo Actuator

The motivation for the mixed structured and unstructured uncertainty modeling method introduced in this thesis comes from measured frequency domain data from several servo actuators used in data track following in LTO drives.

3.1 Experiment Setup

The blockdiagram in Figure 3.1 shows the experimental setup of the tape drive. It is the same experiment setup used in [9].

In track following for an LTO drive[7], a magnetic flexible tape runs at variable speed along a magnetic read/write head and a digital PES is decoded from dedicated servo tracks on a flexible tape using a timing-based servo pattern. The digital PES is fed back to an digital embedded servo controller to generate control signals for an LTO servo actuator via Zero Hold Digital to Analog Converter amplifier to move the read/write head and follow the dedicated servo track despite Lateral Tape Movement(LTM).

In the block diagram, \( G(j\omega), C_{int}(j\omega), C(j\omega) \) are the actuator, the internal controller and the external controller. \( K_{loop} \) is the adjustable feedback gain. ADC and DAC are the digital-to-analog converter and the analog-to-digital converter. \( \alpha, \beta, \gamma \) are programmable gain refer to external control signal, current amplifier gain.
and amplification of $I_{\text{sense}}$. $\mu$ is converter from voltage to micrometer. $\tau_{\text{REF}}, \tau_{\text{PES}}$ are delays in REF-signal and PES computation. $\Delta T$ is the sampling time.

Four signals can be identified in the block diagram, namely the injected reference signal $REF$, Position Error Signal($PES$), the input current of the voice coils $I_{\text{sense}}$ and the disturbance signal $DST$. By injecting a disturbance signal into the REF-port, two frequency responses can be measured.

$$REF \rightarrow PES : PS(j\omega) = \frac{\alpha \beta \mu e^{-j\omega(\tau_{\text{REF}}+\Delta T+\tau_{\text{PES}})} P(j\omega)}{1 + \beta K_{\text{loop}} e^{-j\omega(\tau_{\text{REF}}+\Delta T/2)} C_{\text{int}}(j\omega) P(j\omega)}$$  \hspace{1cm} (3.1)$$

$$REF \rightarrow I_{\text{sense}} : S(j\omega) = \frac{\alpha \beta \gamma e^{-j\omega(\tau_{\text{REF}}+\Delta T/2)} P(j\omega)}{1 + \beta K_{\text{loop}} e^{-j\omega(\tau_{\text{REF}}+\Delta T/2)} C_{\text{int}}(j\omega) P(j\omega)}$$  \hspace{1cm} (3.2)$$

Dividing these two responses a frequency response of the actuator can be derived:

$$G(j\omega) = \frac{\gamma}{\mu} \frac{PS(j\omega)}{S(j\omega)} e^{j\omega(\tau_{\text{PES}}+\Delta T/2)}$$  \hspace{1cm} (3.3)$$

3.2 Frequency Responses

Fifteen frequency responses are obtained and shown in Figure 3.2.

The experiment data in Figure 3.2 is computed based on experimental data from several (mechanically equivalent) servo actuators mounted in different LTO drives reflecting manufacturing tolerances. Due to contact between the servo head
Figure 3.2: Magnitude plot of 15 frequency responses of LTO-drive servo actuators and the flexible tape, actuator dynamics varies depending on the tape manufacturing and flexibility. For operational condition variations, LTO drives were placed in a temperature controlled chamber where the temperature is varied from 15 to 50 degree Celsius to account for changes in tape and actuator flexibility. We can see from Figure 3.2 that there are perturbations in the two main resonance modes around 150Hz and 2.5kHz and changes in the frequency range 1-2kHz. These variations will be modeled via structured and unstructured model perturbations.
Chapter 4

Modeling Variability of Servo Actuator

In this chapter, we present servo dynamic modeling method of combining both structured parametric perturbation and unstructured perturbation. Further, we present this method into the form of Linear Fractional Transformation (LFT).

4.1 Structured Parametric Uncertainty Characterization

4.1.1 Linear parametric uncertainty model

To capture the major variabilities observed in the resonant models of the servo actuator, we use a low order parameter model

\[
\hat{G}(j\omega, \hat{\theta}_i) = \frac{b^{(i)}_0 + b^{(i)}_1 j\omega + \cdots + b^{(i)}_m (j\omega)^m}{1 + a^{(i)}_1 j\omega + \cdots + a^{(i)}_n (j\omega)^n}
\]

via MATLAB toolbox Itsfit 1.7 designed by Raymond de Callafon. The main idea of fitting such a model is based on stander frequency domain curve fitting

\[
\hat{\theta}_i = \arg\min_{\theta_i} \| (G_i(j\omega) - \hat{G}(j\omega, \theta_i)) W_i(\omega) \|_2
\]
using the corresponding frequency domain data $G_i(j \omega)$ and a frequency dependent weighting $W_i(\omega)$ that emphasizes the observed resonance frequencies in the data $G_i(j \omega)$. The minimization in (4.2) is solved via iterative least-squares optimization [17] to find the parametric variations on the estimated parameter

$$\hat{\theta}_i = [b_0^{(i)} \ b_1^{(i)} \ \cdots \ b_m^{(i)} \ a_1^{(i)} \ \cdots \ a_n^{(i)}]^T \in \mathbb{R}^{p \times 1}$$ (4.3)

**Figure 4.1:** Magnitude plot of 15 fitted models from the frequency responses in Figure 3.2

Figure 4.1 shows the frequency responses of fitted parametric model for 15 sets of frequency data of LTO read/write head servo actuators. To characterize the variations, we capture the parametric perturbations around a nominal set of parameters. Define the geometric mean $\bar{\theta}$ as this nominal parameter vector

$$\bar{\theta} = [\bar{b}_0 \ \bar{b}_1 \ \cdots \ \bar{b}_m \ \bar{a}_1 \ \cdots \ \bar{a}_n]^T \in \mathbb{R}^{p \times 1}$$ (4.4)

where

$$\bar{b}_l = \frac{\max_i b_l^{(i)} + \min_i b_l^{(i)}}{2}, \quad \bar{a}_k = \frac{\max_i a_k^{(i)} + \min_i a_k^{(i)}}{2}$$

$\forall i = 1, 2, \ldots, N, \ l = 0, 1, \ldots, m, \ k = 1, 2, \ldots, n$
Then, we define a parameter perturbation $\tilde{\theta}$ as

$$\tilde{\theta}_i = \hat{\theta}_i - \bar{\theta}$$  \hspace{1cm} (4.5)

Define $\hat{\theta}_i(k)$ to be the $k^{th}$ element in the parameter vector $\hat{\theta}_i$, then the bound $\gamma_k = \max |\hat{\theta}_i(k)|$ is defined to bring in a parameter perturbation set

$$S_{\tilde{\theta}} = \{ \tilde{\theta} : |\hat{\theta}_i(k)| \leq \gamma_k \forall k = 1, 2, ..., p \}$$  \hspace{1cm} (4.6)

And the set of $\theta$ is defined by

$$S_{\tilde{\theta}}^1 = \{ \tilde{\theta} : \theta = \bar{\theta} + \tilde{\theta}, \tilde{\theta} \in S_{\tilde{\theta}} \}$$  \hspace{1cm} (4.7)

Thus, a linear parametric model $P_{\tilde{\theta}}$ capture all the dynamics of 15 LTO read/write head servo actuators is given

$$P_{\tilde{\theta}} = \left\{ G_{\tilde{\theta}} : G_{\tilde{\theta}} = \frac{\bar{B}(s) + \bar{E}(s)\tilde{\theta}}{\bar{A}(s) + \bar{A}(s)\tilde{\theta}}, \tilde{\theta} \in S_{\tilde{\theta}} \right\}$$  \hspace{1cm} (4.8)

where $S_{\tilde{\theta}}$ is given in (4.6) and

$$\bar{B}(s) = \bar{b}_0 + \bar{b}_1 s + \cdots + \bar{b}_m s^m$$

$$\bar{A}(s) = 1 + \bar{a}_1 s + \cdots + \bar{a}_n s^n$$

$$\bar{E}(s) = [1 \hspace{0.2cm} s \hspace{0.2cm} \cdots \hspace{0.2cm} s^m \hspace{0.2cm} 0 \hspace{0.2cm} \cdots \hspace{0.2cm} 0] \in \mathbb{C}^{1 \times p}$$

$$\bar{A}(s) = [0 \hspace{0.2cm} 0 \hspace{0.2cm} \cdots \hspace{0.2cm} s \hspace{0.2cm} \cdots \hspace{0.2cm} s^n] \in \mathbb{C}^{1 \times p}$$  \hspace{1cm} (4.9)

From Figure 4.2, we can see models have variations in the full frequency region. However, what we desired is that variations is only obvious in two resonant modes. It is because although (4.8) would model the parametric variations in the measured frequency responses $G_i(j\omega)$, the bound $\gamma_k$ in (4.6) allows each element $\hat{\theta}_i(k)$ of the vector $\hat{\theta}$ to vary independently. The variations of $\hat{\theta}_i$ in (4.5) might be structured, especially when parameters vary jointly.

### 4.1.2 Reduce parameter independence

A linear parametric model has been given in last subsection. However, assuming all the parameters vary independently might be too conservative. In this
subsection, a linear Principle Component Analysis (PCA) is used to reduce the independence in parameters and find least possible principle parameters.

An introduction to PCA is given in Section 2.2. A linear PCA can be solved with a straightforward Singular Value Decomposition (SVD) or Eigenvalue Decomposition. Furthermore, additional differences between the frequency response of the \( G_i(j\omega) \) will be bounded by unstructured uncertainty.

For setting up PCA, we define the parameter perturbation matrix

\[
\hat{\Theta} = [\hat{\theta}_1 \ \hat{\theta}_2 \ \cdots \ \hat{\theta}_N] \in \mathbb{R}^{p \times N}
\]

(4.10)

from \( \hat{\theta}_i \in \mathbb{R}^{p \times 1} \) which is defined in (4.5). \( N \) here is the number of estimated parameter vectors. We have two steps to perform here: 1) Determine the number of principle components; 2) Get the linear transformation between original parameter perturbation matrix and the principle or reduced parameter perturbation matrix.
1) Determination the number of principle components

First of all, we need to define a scaled parameter perturbation matrix with regardless of the size of relative perturbation size in each direction

\[
\tilde{\Theta}^{(s)} = [\tilde{\theta}_1^{(s)} \quad \tilde{\theta}_2^{(s)} \quad \ldots \quad \tilde{\theta}_N^{(s)}] \in \mathbb{R}^{p \times N} \quad (4.11)
\]

where \(\tilde{\theta}_i^{(s)}(k) = \tilde{\theta}_i(k)/\bar{\theta}_i(k); \forall i = 1, 2, \ldots, N; k = 1, 2, \ldots, p\) which means each parameter is scaled by its own nominal value. So the relative difference between the observed variations in \(\hat{\theta}_i \in \mathbb{R}^{p \times 1}\) is normalized. Performing a SVD on \(\tilde{\Theta}^{(s)}\), we can rewrite it as

\[
\tilde{\Theta}^{(s)} = \begin{bmatrix} T^{(s)} & T_s^{(s)} \end{bmatrix} \begin{bmatrix} \Sigma^{(s)}_\sigma & 0 \\ 0 & \Sigma^{(s)}_s \end{bmatrix} \begin{bmatrix} V^{(s)*} \\ V_s^{(s)*} \end{bmatrix} \quad (4.12)
\]

where the singular values of \(\tilde{\Theta}^{(s)}\) are separated into \(r\) large singular values in \(\Sigma^{(s)}_\sigma\) and \(p - r\) small singular values in \(\Sigma^{(s)}_s\). With the separation of singular values we have

\[
\arg \min_C \|\tilde{\Theta}^{(s)} - C\|_F = T^{(s)} \Sigma^{(s)}_\sigma V^{(s)*} \quad (4.13)
\]

where \(C\) is a symmetric rank \(r\) matrix. The direct relation between Frobenius-norm minimization in (4.13) and the truncation of the SVD of the scaled covariance matrix \(\tilde{\Theta}^{(s)}\) makes the choice \(r \leq p\) a well-motivated choice for the number of independent principle directions for the parameter perturbations.

And one way to determine the value of \(r\) is to guarantee

\[
\frac{\text{trace}(\Sigma^{(s)}_\sigma)}{\text{trace}(\Sigma^{(s)}_s)} > \text{Const} \quad (4.14)
\]

A suitable value for Const can be 9.

2) Linear transformation

As we have determined the size of principle component, we move back to the unscaled systems to determine the principle components. For parameter perturbation matrix \(\tilde{\Theta}\) in (4.10), we can rewrite in with a SVD partition similar to (4.12)
with \( r \leq p \) determined by (4.14) and given by

\[
\tilde{\Theta} = \begin{bmatrix} T & T_s \end{bmatrix} \begin{bmatrix} \Sigma_{\sigma} & 0 \\ 0 & \Sigma_{\lambda} \end{bmatrix} \begin{bmatrix} V^* \\ V_{s*}^* \end{bmatrix}
\] (4.15)

where \( T \in \mathbb{R}^{p \times r} \). An similar approximation of \( \tilde{\Theta} \) can be obtained

\[
\tilde{\Theta} \approx T \Sigma_{\sigma} V^*
\] (4.16)

where all principle or deduced parameter perturbation matrices satisfy (4.16) is defined as

\[
\hat{\Sigma} := \Sigma_{\sigma} V^* := [\sigma_1 \sigma_2 \cdots \sigma_N] \in \mathbb{R}^{r \times N}
\] (4.17)

A relationship between original parameter perturbation matrix \( \tilde{\Theta} \) and the principle parameter perturbation matrix \( \hat{\Sigma} \) can be represented by

\[
\tilde{\Theta} = T \hat{\Sigma} + E
\] (4.18)

where

\[
E = [e_1 \ e_2 \ \cdots \ e_N] \in \mathbb{R}^{p \times N}
\] (4.19)

is the error matrix coming from the principle components approximation. And we know by Low-Rank Approximation Theorem, \( \Sigma \) is the optimal \( \hat{\Sigma} \) minimizing \( \|E\|_F \)

\[
\hat{\Sigma} = \arg \min_{\Sigma} \|\tilde{\Theta} - T \Sigma\|_F
\]

Defining

\[
\hat{\Sigma} = [\hat{\sigma}_1 \ \hat{\sigma}_2 \ \cdots \ \hat{\sigma}_N] \in \mathbb{R}^{r \times N}
\] (4.20)

allows the parameter perturbations \( \hat{\theta}_i \) to be written as

\[
\hat{\theta}_i = T \hat{\sigma}_i + e_i; \quad i = 1, 2, ..., N
\] (4.21)

where \( T \in \mathbb{R}^{p \times r} \) and \( \hat{\sigma}_i \) is a reduced size independent parameter perturbation of \( r \times 1 \) where \( r \leq p \).
4.1.3 Independence reduced linear parametric uncertainty model

Obtaining the optimal solution $\hat{\delta}_i$ in (4.21), an approximation of parameter can be defined as

$$\theta_i = \bar{\theta} + T\hat{\sigma}_i, \ i = 1, 2, \ldots, N$$  \hspace{1cm} (4.22)

where $\bar{\theta} \in \mathbb{R}^{p \times 1}$ is the geometric mean given in (4.4), $T \in \mathbb{R}^{p \times r}$ is the transferring matrix found from the SVD in (4.15) and $\hat{\sigma}_i \in \mathbb{R}^{r \times 1}$ is defined in (4.20). To write this in a standard structured parametric uncertainty model, consider the scaling of $\hat{\sigma}_i$ by the scaling matrix

$$S = diag(s_1, s_2, \ldots, s_r) \in \mathbb{R}^{r \times r},
\hspace{1cm} (4.23)$$

where $s_k = \max_i |\hat{\sigma}_i(k)|$, $k = 1, 2, \ldots, r$.

This allows a reduced size $r \leq p$ parameter perturbation

$$\theta = \bar{\theta} + TS\delta, \ \delta \in S_{\delta}$$

where the reduced size linear parameter perturbation set $S_{\delta}$ is defined as

$$S_{\delta} = \{\delta : |\delta(k)| < 1, \ \forall k = 1, 2, \ldots, r\}$$  \hspace{1cm} (4.24)

in which $\delta(k)$ again denotes the $k$th element of $\delta \in \mathbb{R}^{r \times 1}$. The final result is a reduced size $r \leq p$ linear parametric perturbation model

$$\mathcal{P}_{\delta} = \left\{G_{\delta} : G_{\delta} = \frac{\bar{B}(s) + \mathbb{V}_b(s)\delta}{\bar{A}(s) + \mathbb{V}_a(s)\delta}, \ \delta \in S_{\delta}\right\}$$  \hspace{1cm} (4.25)

where $S_{\delta}$ is given in (4.24) and

$$\mathbb{V}_b(s) = \mathbb{B}(s)TS \in \mathbb{C}^{1 \times r}$$
$$\mathbb{V}_a(s) = \mathbb{A}(s)TS \in \mathbb{C}^{1 \times r}$$

and $\mathbb{B}(s), \mathbb{A}(s)$ were defined previously in (4.9).

4.1.4 Represent in the LFT form

A convenient aspect of using model in (4.25) is that it is easy to be written into a LFT form. (You can refer to Section 2.1 for details.)
Figure 4.3 shows the LFT form of (4.25) where

\[
\begin{bmatrix}
z_n \\
y_n 
\end{bmatrix} = P \begin{bmatrix}
w_n \\
u_n 
\end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w_n \\
u_n \end{bmatrix}
\] (4.26)

\[
P_{11} = \begin{bmatrix} -\bar{A}^{-1}\varphi_a \\
\vdots \\
-\bar{A}^{-1}\varphi_a 
\end{bmatrix} \quad P_{12} = \begin{bmatrix} \bar{A}^{-1} \\
\vdots 
\end{bmatrix}
\] (4.27)

\[
P_{21} = \varphi_b - \bar{B}\bar{A}^{-1}\varphi_a \quad P_{22} = \bar{B}\bar{A}^{-1}
\]

and it can be written as

\[
G_\delta = \frac{y_n}{u_n} := \mathcal{F}_u(P, \Delta_s) = P_{22} + P_{21}\Delta_s(I - \Delta_s M_{11})^{-1}P_{12}
\] (4.28)

where \(\Delta_s\) is the structured real parametric perturbations

\[
\Delta_s = diag(\delta) \in \mathbb{R}^{r \times r}
\] (4.29)

### 4.1.5 Application to LTO tape data

Based on the 15 measured frequency responses depicted in Figure 3.2, fourth order continuous-time linear parametric models

\[
G(s, \theta_i) = \frac{b_0^{(i)} + b_1^{(i)} s + b_2^{(i)} s^2 + b_3^{(i)} s^3}{1 + a_1^{(i)} s + a_2^{(i)} s^2 + a_3^{(i)} s^3 + a_4^{(i)} s^4}
\]
are fitted to capture the structural variations in the main resonance modes around 150Hz and 2.5kHz. The fourth order models lead to parameter estimates \( \hat{\theta}_i \in \mathbb{R}^{p \times 1} \) with \( p = 8 \) and application of the linear PCA allows the structural parameter variations in \( \hat{\theta}_i \) to be approximated by (4.22) using \( T \in \mathbb{R}^{p \times r} \) where \( r = 4 \). One of the main reasons why the structural parameter variations can be reduced to a smaller size \( r \leq p \) is that variations in the main resonance modes around 150Hz and 2.5kHz are due to changes in resonance frequency only, while little change in damping is observed.

![Magnitude plot of 50 randomly chosen models](image)

Figure 4.4: Magnitude plot of 50 randomly chosen models \( |G_\delta(j\omega)| \) from the linear parametric perturbation model of (4.25)

Varying the \( r \times 1 \) perturbation \( \delta \) within the normalized bounds \( |\delta(k)| < 1 \) in the linear parametric perturbation model of (4.25) now allows the structural variations in the servo actuators to be modeled. This has been demonstrated in Figure 4.4, where the amplitude Bode plot of 50 randomly chosen models from the linear parametric perturbation model of (4.25) has been plotted. It can be observed that the structural variations in the resonance modes have been captured by the model defined in (4.25) for \( |\delta(k)| < 1 \).
4.2 Unstructured Uncertainty Characterization

Since we use only a low order linear parametric model $G(j\omega, \theta_i)$ to capture the major structure of raw frequency response $G_i(j\omega)$, there will inevitably exist some differences. Again, we have the parameter independence reduced model $G(j\omega, TS\delta_i)$ which is an best possible approximation of $G(j\omega, \theta_i)$ with the reduced parameter size of $r \leq p$. The gap between $G_i(j\omega)$ and $G(j\omega, \theta_i)$ is not negligible. The magnitude of the gap is much smaller than that of the frequency response itself. So it is reasonable for us to use unstructured uncertainty to bound this unmodeled dynamics.

4.2.1 Dual Youla Parameterization

In this thesis, we use dual-Youla parameter to bound the rest unmodeled dynamics. The dual-Youla parameterization uses the prior information on a stabilizing feedback controller and a model built according to a stable dual-Youla parameter are stabilized by this stabilizing controller itself [13, 14]. So here dual-Youla parameterization is used in bounding model perturbations to formulate closed-loop unstructured uncertainty models which is known to be stabilizable by the internal controller. Also, the dual-Youla parameterization is known to be less conservative than standard open-loop uncertainty models [15].

With $C_{int}$ stabilize all measured servo actuators $G_i(\omega), i = 1, 2, ..., N$ and verified that it stabilize all models $G(j\omega, TS\delta_i), i = 1, 2, ..., N$. Then, according to dual-Youla parameterization, $\exists \Delta_i(\omega) \in RH_\infty$ satisfies

\[
\begin{align*}
N_i(\omega) &= N_i(j\omega) + \Delta_i(j\omega)D_C(j\omega) \\
D_i(\omega) &= D_i(j\omega) + \Delta_i(j\omega)N_C(j\omega)
\end{align*}
\]

(4.30)

where $(N_i, D_i)$ is the unknown right co-prime factor of $G_i,(N_i, D_i)$ is the known right co-prime factor of $G(j\omega, \delta_i)$ and $(N_C, D_C)$ is known right co-prime factor of $C_{int}$. Knowing $(N_i, D_i), (N_C, D_C)$ and $G_i$ one can compute $\Delta_i$ explicitly via

\[
\Delta_i = D_C^{-1}(1 + G_iC_{int})^{-1}(G_i - G(\delta_i))D_i
\]

For a stable controller $C_{int}$ we may choose $N_C = C_{int}$ and $D_C = I$. Similarly, for a stable model $G(\delta_i)$ we may choose $N_i = G(\delta_i)$ and $D_i = I$, simplifying the explicit
expression for $\Delta_i$ to

$$\Delta_i = (1 + G_iC_{\text{int}})^{-1}(G_i - G(\delta_i))$$

It should be pointed out that $\Delta_i \in RH_\infty$ due to the dual-Youla parametrization. With frequency domain measurements $G_i(j\omega)$ we can formulate an upper bound for the unknown, but stable unstructured uncertainty

$$\Delta_i(j\omega) = (1 + G_i(j\omega)C_{\text{int}}(j\omega))^{-1}(G_i(j\omega) - G(j\omega, TS\delta_i))$$

Defining

$$\Delta_u(\omega) = \max_i |\Delta_i(\omega)| \forall \omega, i = 1, 2, \ldots, N$$

(4.31)

an unstructured dual-Youla uncertainty model set can be formulated via

$$\mathcal{P}_\Delta = \{G_\Delta : G_\Delta = (G(\delta_i) + \Delta W)(I - \Delta WC_{\text{int}})^{-1}, |\Delta| < 1\}$$

(4.32)

where $W(j\omega)$ is a stable and stable invertible filter that overbounds $\Delta_u(\omega)$ in (4.31) via

$$|\Delta_u(\omega) W^{-1}(j\omega)|_\infty < 1$$

(4.33)

### 4.2.2 LFT form of dual-Youla parameterization

It is also easy to represent dual-Youla parameterization in (4.32) into an upper LFT $\mathcal{F}_u(Q, \Delta) = Q_{22} + Q_{21}\Delta(1 - Q_{11}\Delta)^{-1}Q_{12}$ where the entries of the $2 \times 2$ block transfer matrix $Q$ are given by

$$
\begin{align*}
Q_{11} &= WC_{\text{int}} \\
Q_{12} &= W \\
Q_{21} &= I + G_\delta C_{\text{int}} \\
Q_{22} &= G_\delta
\end{align*}
$$

where $G_\delta$ is defined in (4.28).

The LFT block is shown in Figure 4.5 and the input-output relationship is given in (4.34)

$$
\begin{bmatrix} z \\ y \end{bmatrix} = Q(\delta) \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} WC_{\text{int}} & W \\ I + G_\delta & G_\delta \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}
$$

(4.34)
4.3 Mixed Structured and Unstructured Uncertainty in LFT Form

As we have obtained structured perturbations model LFT $2 \times 2$ block in (4.27) with structured uncertainty $\Delta_s$ defined (4.29) in and unstructured LFT $2 \times 2$ block in (4.34) with unstructured uncertainty $|\Delta| < 1$, a mixed structured and unstructured model can be defined in an upper LFT model set defined as

$$\mathcal{P}_{\delta,\Delta} = \{ G_{\delta,\Delta} : G_{\delta,\Delta} = \mathcal{F}_u(\hat{P}, \text{diag}(\delta, \Delta)) \}$$

$$|\delta(k)| < 1, \|\Delta(\omega)\|_{\infty} < 1, \ k = 1, 2, ..., r \}$$

(4.35)

and the Mixed Structured and Unstructured (MSU) uncertainty upper LFT $2 \times 2$ block transfer matrix $\hat{P}$ is given by

$$\hat{P}_{11} = \begin{bmatrix} P_{11} & P_{12}C_{\text{int}} \\ 0 & WC_{\text{int}} \end{bmatrix}, \quad \hat{P}_{12} = \begin{bmatrix} P_{12} \\ W \end{bmatrix}$$

$$\hat{P}_{21} = \begin{bmatrix} P_{21} & I + P_{22}C_{\text{int}} \end{bmatrix}, \quad \hat{P}_{22} = P_{22}$$

(4.36)

A block diagram of MSU model is shown in Fig 4.6 and the input and output relationship is given by

$$\begin{bmatrix} z_n \\ z \\ y \end{bmatrix} = \begin{bmatrix} \hat{P}_{11} & \hat{P}_{12} \\ \hat{P}_{21} & \hat{P}_{22} \end{bmatrix} \begin{bmatrix} w_n \\ w \\ u \end{bmatrix}$$

(4.37)
4.4 MSU Model Application to LTO Drives

(a) 15 Raw LTO servo actuator frequency responses

(b) 50 MSU constructed uncertainty models

(c) 50 Additive constructed uncertainty models

(d) 50 DY constructed uncertainty models

Figure 4.7: Comparison uncertainty models by MSU, Additive and DY methods
With the MSU method set up in last section, an uncertainty model constructed by MSU method is shown in Figure 4.7b with comparison of uncertainty models constructed by standard Additive unstructured (Additive) uncertainty modeling method (Figure 4.7c) and dual-Youla (DY) parameterization (Figure 4.7d). The Raw frequency responses is shown again here in Figure 4.7a.

We can see from Figure 4.7, the MSU constructed uncertainty models assemble the raw frequency responses much less conservative than the Additive method and DY methods. Especially in the frequency region at resonance modes, the benefit of involving structured parameter uncertainty is obvious. We can see it more directly by observing the unstructured uncertainties.

![Figure 4.8: Comparison of the magnitude plots of unstructured uncertainties](image)

In Figure 4.8, we can see that the additional step of extracting structural uncertainty via curve fitting and a PCA reduces the remaining unstructured uncertainty.
Chapter 5

Robust Stability and Performance Tests

For the system with structured uncertainty, the robust and performance analysis is conducted by structured singular analysis (µ analysis) which is introduced by Doyle.

5.1 Robust Stability Test

The uncertainty structure in the framework of MSU combines both real uncertainties and complex uncertainties. By defining the set of uncertainty here as

\[ \Delta = \{ \text{diag}(\delta(1), \delta(2), \ldots, \delta(r), \Delta) : \delta(k) \in \mathbb{R}, \Delta \in \mathbb{C} \} \]  

we are interested in the stability of the closed loop connection of \( G_{\delta,\Delta} \) and negative feedback controller \( C \) with respect to the bounded mixed uncertainty \( \Delta \).

Theorem 5.1 (Robust Stability[1]). Let \( \beta > 0 \). The closed-loop connection shown in Figure 5.1a is internal stable for all \( \|\Delta\|_\infty < \beta \), if and only if

\[ \mu_{\text{diag}(\delta,\Delta)}(M_{11}) < 1/\beta \]
5.2 Robust Performance Test

For formulating a test on performance robustness, first a definition on (nominal) performance must be given. To facilitate the use of the main loop theorem [1], nominal performance of the servo actuators in an LTO drive is specified as an weighted $H_\infty$ criterion on the disturbance rejection function $(I + CG)^{-1}$. Defining an error signal $e = W_s(d + y)$ as it is shown in Figure 5.2a and augmenting the LFT $y = F_u(\tilde{P}, \text{diag}(\delta, \Delta))u$ with a feedback connection $u = -C(d + y)$ leads to an LFT $e = F_u(M, \text{diag}(\delta, \Delta))d$ for the relation between the error signal $e$ and disturbance signal $d$. Performance robustness can now be verified with the main loop theorem and using the computation of a structured singular value $\mu_{\Delta}(-)$ with respect to the perturbation structure

$$\Delta_p = \{\text{diag}(\delta(1), \delta(2), \ldots, \delta(r), \Delta, \Delta_f) : \delta(k) \in \mathbb{R}, \Delta, \Delta_f \in \mathbb{C}\} \quad (5.2)$$

of the mixed $r$ dimensional (real) $\text{diag}(\delta)$ and a 2 dimensional complex uncertainty structure $\text{diag}(\Delta, \Delta_f)$.

**Theorem 5.2** (Robust Performance[1]). Let $\beta > 0$. Consider $\tilde{P}$ given in 4.36 and
the matrix \( M \) in Figure 5.2b is given by

\[
M = \begin{bmatrix}
M_{11} & -\bar{P}_{12}M_{22} \\
W_sM_{22}C\bar{P}_{21} & W_sM_{22}
\end{bmatrix}
\]  \hspace{1cm} (5.3)

where \( M_{11} = \mathcal{F}_1(\bar{P}, -C) \), \( M_{22} = (I + CP_{22})^{-1} \) and consider models \( G_{\delta,\Delta} \in \mathcal{P}_{\delta,\Delta} \) given in 4.35. The negative feedback connection of \( C \) and \( G_{\delta,\Delta} \) in Figure 5.2a with respect to the perturbation structure \( \|\Delta_p\|_\infty < \beta \) is robustly stable and the \( H_\infty \) performance specification \( \|\mathcal{F}_u(M, \text{diag}(\delta, \Delta))\|_\infty \leq 1/\beta \) is satisfied all \( G_{\delta,\Delta} \in \mathcal{P}_{\delta,\Delta} \) if and only if

\[
\mu_\Delta(M) < 1/\beta
\]

As computation of \( \mu_\Delta(M) \) is in general NP-hard, overbounds can only be computed by a frequency point wise evaluation of \( \mu_\Delta(M(j\omega)) \) over a frequency grid \( \omega \in \Omega \) \cite{1}. Such frequency dependent overbounds can still be used to check if \( \mu_\Delta(M(j\omega)) < 1 \forall \omega \in \Omega \) and robust performance can be verified provided the frequency grid \( \Omega \) is chosen to be dense.

\section{5.3 Application to LTO Drives}

Implementing a new controller \( C \), we conduct the robust stability test and robust performance test on the uncertainty model constructed by MSU method.
In order to see that MSU method is a less conservative method, we also show the result of robust stability test on the uncertainty model constructed by Additive Method and dual-Youla (DY) parameterization implementing the same controller $C$. Since Additive and DY methods only contains unstructured uncertainties, robust stability test for Additive and DY method is conducted by checking the infinity norm of the nominal transfer matrix. Without loss of generalization, we have chosen scaling matrix $S$ in (4.23) for parametric perturbation, weighting function $W$ for unstructured perturbation and performance filter $W_s$ for performance requirement. These matrix, weighing function and filter have bounded the infinity norm of $\delta$, $\Delta$ and $\Delta_f$ to be smaller than 1. That means the $\beta$ in Theorem 5.1 and Theorem 5.2 is 1. So robust stability and robust performance tests is to check if $\mu_{\Delta}(M_{11}) < 1$ and $\mu_{\Delta_p}(M) < 1$.

The bode plot of newly designed controller $C$ is given in Figure 5.3 together with the internal controller $C_{int}$ which is used for the closed-loop unstructured uncertainty modeling.

![Figure 5.3: Bode plot of $C_{int}$ and $C$](image-url)
5.3.1 Robust stability test

We can see from Figure 5.4, the additive method (blue line) fails the robust stability test by overpassing 1 in frequency region of the 110Hz 180Hz and around 2500Hz where is the region of two obvious resonant modes. It is reasonable because the large perturbations around the regions of resonant modes by modeling solely unstructured uncertainty. This can be seen from comparison of models Figure 4.7 and comparison of unstructured uncertainties Figure 4.8. The DY method and MSU passes the robust stability test and inspiring.

5.3.2 Robust performance test

The robust performance test summarized in Theorem 5.2 is a much stronger requirement than only robust stability. For testing performance robustness based on the mixed structured and unstructured perturbation(MSU) model $P_{\delta,\Delta}$ given in (4.35) and determined from the 15 frequency responses given in 3.2, a performance weighting function $W_s$ on the disturbance rejection function $(I + CG)^{-1}$ was chosen. Randomly selecting 50 different models $G_{\delta,\Delta} \in P_{\delta,\Delta}$ and computing the amplitude of $|(I + CG_{\delta,\Delta})^{-1}|$ leads to the amplitude Bode plot depicted in 5.6.
Figure 5.5: Robust performance test for MSU, DY and Additive methods

Figure 5.6: Magnitude plot of 50 randomly sampled sensitivity functions and $W_s^{-1}$
The result indicate that all chosen models satisfy the $H_{\infty}$-norm based performance specification due to $W_s^{-1}$ overbounding all 50 error rejection functions. The results is formally proven by the computation of the (upper bound) of $\mu_\Delta(M)$ in 5.5. As a frame of reference, in 5.5 also the robust performance results are plotted in case only an additive uncertainty or dual-Youla unstructured uncertainty (DY) is used without modeling the structural perturbations and indicate that $\mu > 1$ for those uncertainty descriptions. Clearly, the Mixed Structured and Unstructured (MSU) perturbation model $\mathcal{P}_{\delta,\Delta}$ given in (4.35) yields less conservative results when checking performance robustness.
Chapter 6

Conclusion

Starting from multiple frequency domain measurements, this paper presents a procedure to formulate a mixed structured and unstructured perturbation model. A separation between parametric and unstructured uncertainty is achieved by first estimating low order linear parameter models via frequency domain curve fitting followed by a linear Principle Component Analysis (PCA). Remaining differences are bounded by unstructured uncertainty on a dual-Youla parameter that uses prior information on a stabilizing feedback controller. The favorable properties of the perturbation model is demonstrated via a performance robustness test applied to data from servo actuators in Linear Tape-Open (LTO) drives.
Bibliography


