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MACSYMA AND MINIMAL SURFACES

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MACSYMA AND MINIMAL SURFACES

Paul Concus and Mario Miranda

The first proof of the existence of minimal singular surfaces of codimension one was given in 1969 by E. Bombieri, E. DeGiorgi, and E. Giusti [1]. They proved that the cone

\[
\{(x,y) | x \in \mathbb{R}^{k+1}, y \in \mathbb{R}^{k+1}, |x|^2 = |y|^2\},
\]

is a \((2k+1)\)-dimensional minimal surface if \(k \geq 3\). In 1972, B.H. Lawson proved that all cones

\[
\{(x,y) | x \in \mathbb{R}^{k+1}, y \in \mathbb{R}^{h+1}, h |x|^2 = k |y|^2\}
\]

are \((h+k+1)\)-dimensional minimal surfaces if \(h+k \geq 7\) or \(h = k = 3\) [6]. In 1983, U. Massari and M. Miranda presented an elementary proof of the Bombieri-DeGiorgi-Giusti result [8].

The aim of this note is to apply Massari and Miranda's method to the cones considered by Lawson. Such an application turns out to be successful with the additional assumptions

\[ h < 5k, k < 5h. \]

In Section 1 we sketch briefly the mathematical framework of the problem. In Section 2 we show how the use of symbolic algebraic manipulation, as provided to us by VAXIMA, the Berkeley VAX/UNIX version of the MACSYMA computer programming system [7], makes possible the application of Massari and Miranda's method to the nonsymmetric cones.

1. MINIMAL SURFACES OF CODIMENSION ONE

Throughout this paper we use the following definition of a codimension one surface in \((n+1)\)-Euclidian space.
DEFINITION 1. An \textit{n-dimensional surface} is the boundary of any measurable subset of \( \mathbb{R}^{n+1} \).

Definition 1 attaches the label \textit{n-dimensional surface} to closed subsets of \( \mathbb{R}^{n+1} \), which may be singular and have non-zero \((n+1)\)-dimensional measure. Nevertheless, Definition 1 together with a convenient definition of \( n \)-dimensional measure has been successfully used for the study of codimension one surfaces since 1954, when it was first introduced by E. DeGiorgi [2],[3]. The appropriate definition of \( n \)-measure for our surfaces is possible through use of the divergence theorem.

DEFINITION 2. If \( \Omega \) is a measurable subset of \( \mathbb{R}^{n+1} \) and \( A \) is an open subset of \( \mathbb{R}^{n+1} \), we take as \textit{n-dimensional measure of the portion of} \( \partial \Omega \) \textit{contained in} \( A \), the quantity

\[
\sup \left\{ \int_\Omega \text{div} \varphi(x) \, dx \mid \varphi \in [C^0_0(A)]^{n+1}, |\varphi(x)| \leq 1 \text{ for all } x \right\},
\]

which we denote by \( \mathcal{M}_n(\partial \Omega \cap A) \).

The following remark indicates the basis for Definition 2.

\textit{Remark 1.} If \( \partial \Omega \) is sufficiently smooth, then the divergence theorem

\[
\int_\Omega \text{div} \varphi(x) \, dx = \int_{\partial \Omega} \varphi(x) \cdot \nu(x) \, dH_n
\]

holds for any vector function \( \varphi \in [C^0_0(\mathbb{R}^{n+1})]^{n+1} \). In (1) \( \nu(x) \) denotes the outward unit vector normal to \( \partial \Omega \) at \( x \) and \( H_n \) is the Hausdorff \( n \)-dimensional measure in \( \mathbb{R}^{n+1} \). The validity of (1) obviously implies

\[
\mathcal{M}_n(\partial \Omega \cap A) = H_n(\partial \Omega \cap A), \text{ for all open } A
\]

We give finally the following definition of a minimal surface.
DEFINITION 3. \( \partial \Omega \) is a minimal surface in \( \mathbb{R}^{n+1} \) if the following inequalities hold

\[
M_n(\partial \Omega \cap A) < +\infty, \quad M_n(\partial \Omega \cap A) \leq M_n(\partial \Omega^* \cap A),
\]

for all bounded open sets \( A \) and any measurable set \( \Omega^* \) satisfying

\[
(\Omega - \Omega^*) \cup (\Omega^* - \Omega) \subseteq A.
\]

A theory based on Definitions 1-3 and rich in remarkable results was developed between 1960 and 1970. The cones considered by Bombieri-DeGiorgi-Giusti and Lawson belong to this theory, providing examples of singular minimal surfaces. Their lowest dimension is 7. That no lower dimension is possible follows from Theorem 1.

THEOREM 1 (W.H. Fleming [5], E. DeGiorgi, et al. [4], J. Simons [10]). The only minimal six-dimensional surfaces in \( \mathbb{R}^7 \) are planes.

Two more basic definitions of this paper are the following.

DEFINITION 4. A measurable set \( F \) of \( \mathbb{R}^{n+1} \) is \( M \)-concave if its boundary is minimal with respect to modifications contained in \( F \) itself.

In other words in order to be called \( M \)-concave the set \( F \) must satisfy the inequalities

\[
M_n(\partial F \cap A) < +\infty, \quad M_n(\partial F \cap A) \leq M_n(\partial F^* \cap A),
\]

for all bounded open sets \( A \) and all measurable sets \( F^* \subset F \) such that

\[
(F - F^*) \subseteq A.
\]

DEFINITION 5. The measurable set \( F \) of \( \mathbb{R}^{n+1} \) is said to be \( M \)-convex if the measurable set \( \mathbb{R}^{n+1} - F \) is \( M \)-concave.
The interest of Definitions 4 and 5 is underlined by the following obvious remark.

Remark 2. The measurable set $F$ has minimal boundary if and only if it is $M$-concave and $M$-convex.

A sufficient condition for $M$-concavity is provided by the following theorem.

**THEOREM 2.** If $\{f_j\}$ is a sequence of subsolutions to the minimal surface equation (m.s.e.) in the open set $F$ of $\mathbb{R}^{n+1}$, i.e.,

$$
\text{div} \left( \frac{\text{grad } f_j(x)}{\sqrt{1 + |\text{grad } f_j(x)|^2}} \right) \geq 0, \text{ for } x \in F, \text{ for all } j,
$$

(7)

if

$$
M_n(\partial F \cap A) < +\infty, \text{ for all open bounded sets } A,
$$

if

$$
f_j(x) = 0, \text{ for } x \in \partial F, \text{ for all } j,
$$

(8)

and if

$$
\lim_{j \to \infty} f_j(x) = +\infty, \text{ for } x \in F,
$$

(9)

then $F$ is $M$-concave.

**Proof.** Since $f_j$ is a subsolution to the m.s.e., the open set

$$
\left\{(x,z) \mid x \in F, z \in \mathbb{R}, z < f_j(x) \right\},
$$

is $M$-concave with respect to compact modifications contained in $F \times \mathbb{R}$. Conditions (8) and (9) imply that the cylinder $F \times \mathbb{R}$ itself if $M$-concave with respect to compact modifications contained in $F \times \mathbb{R}_+$, where $\mathbb{R}_+ = \{z \in \mathbb{R} \mid z > 0\}$. Therefore $F$ must be $M$-concave. □
Theorem 2 will be particularly useful in the special case of cones, as shown by the following corollaries.

**COROLLARY 1.** If $F$ is an open cone of $\mathbb{R}^{n+1}$, i.e., if $F$ satisfies

$$x \in F, \rho > 0 \Rightarrow \rho x \in F,$$

(10)

and if

$$\mathcal{M}_n(\partial F \cap A) < +\infty, \text{ for all open bounded sets } A,$$

then for $F$ to be $M$-concave it is sufficient that there exist one subsolution $f$ to the m.s.e. satisfying

$$f(x) = 0, \text{ for } x \in \partial F,$$

(11)

$$f(x) > 0, \text{ for } x \in F,$$

(12)

and

$$f(\rho x) = \rho^\alpha f(x), \text{ for } x \in F, \text{ for all } \rho > 0,$$

(13)

with $\alpha \neq 1$.

**Proof.** For any $\rho > 0$, the function

$$\rho^{-1}f(\rho x) = \rho^{-\alpha}f(x),$$

(14)

is a subsolution to the m.s.e. in $F$. Since $\alpha \neq 1$ and $\rho > 0$ is arbitrary, (14) implies that any multiple of $f$ is a subsolution. Therefore the sequence

$$\{f_j\} = \{jf\}$$

satisfies the hypotheses of Theorem 2, which implies that $F$ is $M$-concave.

**COROLLARY 2.** If $P: \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ is a homogeneous non-constant polynomial satisfying
\[ P(x) \text{ div } \left( \frac{\text{grad } P(x)}{\sqrt{1 + |\text{grad } P(x)|^2}} \right) \geq 0, \text{ for } x \in \mathbb{R}^{n+1}, \quad (15) \]

then the cone

\[ \{ x \in \mathbb{R}^{n+1} | P(x) > 0 \} \quad (16) \]

has minimal boundary.

Proof. If \( P \) is homogeneous of degree 1 the cone (16) is a plane; therefore, it is obviously minimal. We can thus assume that \( P \) is homogeneous of degree different from one.

The open set

\[ F = \{ x \in \mathbb{R}^{n+1} | P(x) > 0 \} \]

and the function

\[ f(x) = P(x), \]

satisfy the hypotheses of Corollary 1; therefore, \( F \) is \( M \)-concave. Again, the open set

\[ F^c = \{ x \in \mathbb{R}^{n+1} | P(x) < 0 \} \]

and the function

\[ f'(x) = -P(x), \]

satisfy the hypotheses of Corollary 1; therefore, \( F^c \) is \( M \)-concave. Since

\[ F^c = \mathbb{R}^{n+1} - (F \cup \partial F) \]

is equivalent to \( \mathbb{R}^{n+1} - F \), we get that \( F \) is \( M \)-convex, thus \( F \) has minimal boundary. \( \blacksquare \)
2. MACSYMA

MACSYMA is a symbolic algebraic manipulation computer programming system that has been developed by the Mathlab Group at the Massachusetts Institute of Technology Laboratory for Computer Science [7]. With it, a computer user can interactively carry out many operations on symbolic expressions, including those particularly useful for our study, such as algebraic manipulation, differentiation, factorization, and re-grouping of polynomials, as well as solving polynomial equations. A system like MACSYMA permits lengthy algebraic manipulation to be carried out easily and efficiently that would normally be almost impossible by hand, and without the inherent probability of human error to which lengthy hand calculations are prone.

In order to prove (with the aid of MACSYMA) that Lawson's cones are minimal, we consider the following polynomial

\[ P(x,y) = (hx^2 - ky^2)(ax^2 + by^2), \]

where \( x \in \mathbb{R}^{k+1}, y \in \mathbb{R}^{h+1}, \alpha, \beta \) are real numbers, and

\[ x^2 = \sum_{i=1}^{k+1} x_i^2, \quad y^2 = \sum_{j=1}^{h+1} y_j^2. \]

Let us denote by \( MP \) the polynomial

\[ MP = (1 + |\text{grad } P|)^{3/2} \cdot \text{div} \left( \frac{\text{grad } P}{\sqrt{1 + |\text{grad } P|^2}} \right). \]

We obtain that the remainder after dividing \( MP \) by \( hx^2 - ky^2 \) is

\[ \frac{2}{k} x^2 \{ \alpha(5h - k)k - \beta(5k - h)h \}. \]

This implies that the polynomial

\[ P^*(x,y) = (hx^2 - ky^2) \{ (5k - h)hx^2 + (5h - k)ky^2 \} \]

divides \( MP^* \) exactly.
We have then the relationship
\[ P^* M^* = (hx^2 - ky^2)^2 \{ (5k - h)hx^2 + (5h - k)ky^2 \} Q, \]
where \( Q \) is the polynomial
\[ Q = MP^*/(hx^2 - ky^2). \]

If we assume
\[ k < 5h, \quad h < 5k, \quad (17) \]
we have
\[ \{(x,y)| P^*(x,y) = 0\} = \{(x,y)| hx^2 = ky^2\}, \]
and for \( P^* \neq 0 \)
\[ P^* M^* \geq 0 \iff Q \geq 0. \]

Our problem is thereby reduced to the study of the sign of \( Q \). First we compute
\[ Q_0 = Q(0,0) = (8h - 12)k^2 + (8k - 12)h^2 + 72hk, \]
which is easily seen to be positive for all integers \( h,k \) satisfying (17). Let
\[ q = (Q - Q_0)/(64hk), \]
and consider \( q \) as function of \( t = hx^2 \) and \( s = ky^2 \). The computation of \( q \) at \( s = t \) gives
\[ q \bigg|_{s=t} = (h + k - 6)(h + k)^3(2t)^3, \]
which is nonnegative for \( h + k \geq 6. \)

Assuming
\[ h + k \geq 6, \quad (18) \]
we have only to study the behavior of \( q \) as a function of \( s \) for \( s \geq t \), and the behavior of \( q \) as a function of \( t \) for \( t \geq s \). The computation of \( q \) as a function of
$s, t$ gives a homogeneous polynomial of degree 3

$$q(s, t) = as^3 + 3bs^2t + 3cst^2 + dt^3,$$

where $a, b, c, d$ depend on $h, k$. Computation of $a$ and $d$ gives

$$a = (5h - k)^2(2k^2 + 2hk + 3k - 3h), \quad d = (5k - h)^2(2h^2 + 2hk + 3h - 3k).$$

Assumptions (17) imply

$$a > 0, \quad d > 0. \quad (19)$$

We have

$$\frac{d^2q}{ds^2} = 6(as + bt).$$

Therefore

$$\frac{d^2q}{ds^2} \bigg|_{s=t} \geq 0 \iff a + b \geq 0.$$ 

Computation of $a + b$ yields

$$a + b = -2(k + h)(k^3 - 5hk^2 + 2k^2 - 9hk^2 - 16hk - 3h^3 + 30h^2),$$

which shows that $a + b \geq 0$ if

$$h \leq k \leq 5h. \quad (20)$$

Assuming (20), we get that $q$ is a convex function of $s$ for $s \geq t$; therefore, it will also be nondecreasing if

$$\frac{dq}{ds} \bigg|_{s=t} = 3t^2(a + 2b + c) \geq 0.$$ 

Computing $a + 2b + c$, we obtain

$$a + 2b + c = 4(h + k)^2(2hk + k + 2h^2 - 13h).$$

Therefore $a + 2b + c \geq 0$ if (20) holds. Our first conclusion is thus

$$q \geq 0 \text{ for } s \geq t.$$
if (18) and (20) hold.

The proof that $q \geq 0$ for all $t \geq s$ proceeds similarly. We obtain

$$\frac{d^2 q}{dt^2} \bigg|_{t=s} = 6(d + c)s, \quad \frac{dq}{dt} \bigg|_{t=s} = 3(d + 2c + b)s^2$$

and the expressions

$$d + c = 2(k + h)(3k^3 + 9hk^2 - 30k^2 + 5hk^2 + 16hk - h^3 - 2h^2),$$
$$d + 2c + b = 4(k + h)^2(2k^2 + 2hk - 13k + h).$$

Assuming (18),(20) and $(h,k) \neq (1.5), (h,k) \neq (2,4)$, there follows

$$d + c \geq 0, \quad d + 2c + b \geq 0.$$  

We conclude that

$$q \geq 0 \text{ for } t \geq s.$$  

Therefore,

$$q \geq 0 \text{ for } s \geq 0, \quad t \geq 0$$

if (18),(20) hold and $(h,k) \neq (1.5), (h,k) \neq (2,4)$. In other words, Lawson's cones corresponding to $(h,k)$ satisfying (18),(20) and different from (1.5), (2,4) are minimal. Since the interchange of $x$ and $y$ does not change the cone, we can conclude that our method confirms Lawson's results under the additional restriction (17).
REFERENCES


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