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"Nomograms"

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1. Introduction

Since the invention of analytic geometry, the use of graphical methods for the representation, use and solution of equations has been of great importance. The field of mathematics which deals with the construction of charts representing mathematical laws is known as Nomography. Quite often it is necessary to repeatedly solve mathematical formulae, but with different values for the symbols involved. The manipulation of the formulae, or repeated constructions of graphs, may require considerable work, as well as carry with it the possibility of errors invalidating the results. In such a situation, the construction of a single chart which would serve for all the solutions would obviously be of considerable value.

One type of chart, which requires only the use of a straight edge to determine the solution, is known as an alignment chart. In an alignment chart, scales are constructed for the various elements so that, by joining values by straight lines (radix lines), solutions may be determined. Mackey defines the alignment chart as: "The graphical representation of a function of three variables, \( f(u,v,w) = 0 \), by means of three graphical
scales (not necessarily straight) arranged in such a manner that any straight line, called an index line, cuts the scales in values of u, v, and w satisfying the equation. He later extends this to include charts which are combinations of three-variable alignment charts and which require several lines for their solutions. In this paper nomograms will refer to general diagrams of mathematical formulae, while alignment charts will refer to the specific type already defined.

2. Methods and Definitions.

It is possible to approach the study of alignment charts by two methods. Given general types of formulae, it can be assumed that two variables may be represented by certain scales. Then by geometrical considerations the third scale may be located. If there are at least two linear scales, this method is useful, but it does not lend itself to a general theoretical development. Several authors use this technique, since it does not require the use of elementary determinant theory. The necessary knowledge of determinants involved in the development of alignment chart theory is not great, and their use does lead to a general comprehensive theory, so in this paper, only the method of construction based on determinants shall be presented.

Alignment charts may be classified according to the number of scales present. This number is defined as the class of alignment chart and is equal to the number of variables present. In rare cases it may be necessary to use two separate scales for variable and construct a compound alignment chart.

The scale factor is defined as the number of units of length of the scale which represent one unit of the function. It is represented throughout by the symbol \( \mu \), and is constant for a given scale. If the desired length of a scale is \( L \), and the extreme values of the function are \( f(u_1) \) and \( f(u_2) \), then \( \mu \) could be found by the equation \( L = \mu [f(u_2) - f(u_1)] \).
Usually a convenient value for $a$ close to the calculated one is used, altering the length slightly in order to facilitate computations.

**The limiting quadrilateral** is defined as that quadrilateral formed by joining the points representing the extreme values on the outermost scales of the chart by straight lines.

### 3. Nomographs of the First and Second Class

Properly speaking, there are no alignment charts of the first and second class. For the sake of continuity of discussion, this section has been included.

**Nomograms of the first class.** Obviously such a diagram represents a formula involving only one variable. It would be a fixed gauge and would have limited applications.

**Nomograms of the second class.** Formulae represented by nomograms of the second class contain two variables. A function, $f(u,v) = k$, can usually be written as $g(u) = g(v)$.

Let $y = g(u) = g(v)$. Any point on the $y$ axis of a diagram would correspond to values of $u$ and $v$ satisfying the original equation. The chart consists of a double sided scale, values of one variable on each side. A common example of this type of nomograph is the D and S scales of the "trig" slide rule, in which the values of the sines of angles, read on the S scale, are given on the D scale. Any conversion scale is a nomograph of the second class.

### 4. Alignment Charts of Third Class

The definition of an alignment chart requires that points representing values of the variables which satisfy the equation represented be collinear. Obviously, if there are more than three variables, they cannot all be
collinear as then the knowledge of only two would determine the values of all the others.

If \( f(u,v,w) = 0 \) is represented by an alignment chart, let points \((x_u, y_u), (x_v, y_v), \) and \((x_w, y_w)\) represent values of \(u,v,\) and \(w\) satisfying the equation. The condition that these points be collinear is

\[
\begin{vmatrix}
1 & x_u & y_u \\
1 & x_v & y_v \\
1 & x_w & y_w \\
\end{vmatrix} = 0
\]

If the \(u,v,\) and \(w\) scales be represented parametrically by

\[
\begin{align*}
x_u &= f(u) \\
x_v &= f(v) \\
x_w &= f(w) \\
y_u &= g(u) \\
y_v &= g(v) \\
y_w &= g(w)
\end{align*}
\]

The determinant

\[
\begin{vmatrix}
f(u) & g(u) & 1 \\
f(v) & g(v) & 1 \\
f(w) & g(w) & 1 \\
\end{vmatrix}
\]

must vanish identically if any three collinear points, one on each scale, represent values of \(u,v,\) and \(w\) satisfying the equation.

The problem of constructing an alignment chart for a function of three variables becomes a problem in expressing the function as a determinant containing only one variable in each row. Then elementary operations on the columns can be used to force the last column into ones.

Each row of this determinant represents a scale. The position and form of the scale can be determined by inspection of the row of the determinant.

Consider the row

\[
f(v) \quad g(v) \quad 1
\]
There are four possible forms which this row can take:

1. \[ k \quad g(v) \quad 1 \]
2. \[ f(v) \quad k \quad 1 \]
3. \[ k_1 f(v) \quad k_2 f(v) \quad 1 \]
4. \[ f(v) \quad g(v) \quad 1 \]
   \[ f(v) \] not linear function of \( g(v) \).

If the row is in the form of (1), the scale is parallel to the \( x \) axis and located \( k \) units from it. If the row is in the form of (2), the scale is parallel to the \( y \) axis and located \( k \) units from it. If the row is in the form of (3), the scale is a straight line inclined to the \( x \) axis at an angle \( \tan^{-1} \frac{k_1}{k_2} \). If the row is in the form of (4), the scale is curvilinear.

The classification of alignment charts according to type of scale can be developed here. If a variable occurs in only one function, the scale will be linear. If a variable occurs in two functions, the scale will be a curve of second order at least. When a variable occurs in more than two functions, it will be necessary to introduce an extra scale in order to represent the function.

The genus of a chart is defined to be the number of curved scales present in the chart. It may be found by writing the equation to be represented in the form containing the smallest number of functions of the variables and then subtracting the number of variables present from the number of functions present. Different functions are those, only, which are not linear combinations of each other.

In addition, it is possible to have various types of charts of each genus. The discussion of class three alignment charts in this paper will follow the following classification:
Table I

A. Genus Zero
   a. Three parallel straight lines.
   b. Two parallel straight lines and any third straight line.
   c. Three concurrent straight lines.
   d. Any three non-concurrent, non-parallel straight lines.

B. Genus I
   a. Two parallel straight lines and a curve.
   b. Any two straight lines and a curve.

C. Genus II
   a. Two curves and a straight line.

D. Genus III
   a. Three curves.

5. Compound Alignment Charts.

If there are more than three variables, it may be possible to construct a compound alignment chart by introducing a new variable. Let \( R \) equal a portion of the expression involving two variables; it will also equal the negative of the remaining terms in the expression. If this part involves only two variables, it may be possible to construct two alignment charts each involving \( R \). The scale factors and equations of the \( R \) scales are made equal and the two charts are constructed so that the \( R \) scales for each coincide.

Suppose the two charts are for the expression \( R = f(u, v) \), \( R = f(x, w) \). Then knowledge of \( u \) and \( v \), or \( x \) and \( w \), determines a point on the \( R \) scale. A straight line through this point, and the point determined by the value of the other known variable, locates the value of the fourth variable. It is
not necessary to graduate the \( R \) scale, since it is only necessary to use the point determined as a turning point.

This method can be extended to any number of variables. Since only those scales present in the original function need be graduated, they are the only ones used to determine the class of the chart.


In constructing an alignment chart for a given functional relation involving three variables, the first step is to express the function as a third order determinant having functions of only one variable in each row. Then by operations on the columns make the elements of the last row a one. The determinant will then be in the form of the basic determinant on page 4.

A chart may be constructed from this determinant but in many cases, the chart will be impractical for some reason. The major difficulties are of the following types

a. A curvilinear scale may have an infinite discontinuity.

b. The range of values desired may occur on widely separated portions of the scales.

Discontinuous scale. A scale which has an infinite discontinuity will occur in the determinant in the form

\[
\frac{f(u)}{h(u) - 1} \frac{g(v)}{h(u) - 1} 1
\]

If that row be multiplied through by the denominator, the determinant will have the form

\[
\begin{vmatrix}
  f(u) & g(u) & h(u) - 1 \\
  f(v) & g(v) & 1 \\
  f(u) & g(w) & 1 \\
\end{vmatrix} = 0
\]
It is possible to shift the position of the discontinuity by addition and subtraction of columns. This operation may introduce discontinuities in other places on the scale; but all of these may be located outside the range of desired values.

There are occasions when the discontinuity may be removed altogether by combining the u and v scales and adding an auxiliary scale.

Alignment charts having widely separated scales. There are three ways of handling such a problem depending on the actual position of the scales. The first consists of translation of the axes and sliding of the values along their scales; the second involves translation and rotation of axes; while the third is a "projective transformation". They will be discussed in this order.

Use of Oblique Axes. A problem arises in the construction of an alignment chart when the portions of the scales required occupy a parallelogram that is not a rectangle, as in Figure la.

In this case none of the required segments of the scales begin at the origin. Multiplying the third column by $x_0$ and subtracting it from the second column gives

$$\begin{vmatrix} x_u & f(u) - y_0 & 1 \\ 0 & f(v) - y_0 & 1 \\ x_w & f(w) - y_0 & 1 \end{vmatrix} = 0$$

This translates the x axis to the $x'$ axis in Figure lb. It is now desired to make line AB parallel to the x axis. It can be seen that

$$\frac{y'_u}{x_u} = \frac{y'_w}{x_w}$$

$$y'_u = x_u \frac{y'_w}{x_w}$$
If the elements of the first column are multiplied by \( \frac{y}{x} \) and subtracted from the second column, the determinant becomes

\[
\begin{vmatrix}
 x_u & f(u) - y_o' - y_u' & 1 \\
 0 & f(v) - y_o' & 1 \\
 x_w & f(w) - y_o' - y_w' & 1
\end{vmatrix} = 0
\]

The alignment chart represented by this determinant is shown in Figure 1b.

It is not necessary to actually perform the calculations from the above determinant since the scale factors are not altered; it is only necessary to plot the scales from the original determinant in rectangular form.

**Rotation of the axes.** If the "limiting quadrilateral" of a third order alignment chart has no side parallel to an axis, it may be desirable to translate the origin to the limiting value of the range on one of the scales, and then to rotate the axes.

If the chart assumes this position, (see Figure 2), the determinant will be

\[
\begin{vmatrix}
 f(u) & g(u) & 1 \\
 f(v) & g(v) & 1 \\
 f(w) & g(w) & 1
\end{vmatrix} = 0
\]

Translating the axes to the point P gives

\[
\begin{vmatrix}
 f(u) - x_o & g(u) - y_o & 1 \\
 f(v) - x_o & g(v) - y_o & 1 \\
 f(w) - x_o & g(w) - y_o & 1
\end{vmatrix} = 0
\]

Before proceeding further, a procedure for rotating axes will be established. From analytic geometry the relationship between coordinates in a purely rotational transformation is

\[
\begin{align*}
x &= x' \cos \theta - y' \sin \theta \\
y &= x' \sin \theta - y' \cos \theta
\end{align*}
\]
Let
\[
\cos \theta = \frac{a}{\sqrt{a^2 + b^2}}, \quad \sin \theta = \frac{b}{\sqrt{a^2 + b^2}}.
\]

then
\[
x = \frac{x'a}{\sqrt{a^2 + b^2}} - \frac{y'b}{\sqrt{a^2 + b^2}}, \quad y = \frac{x'b}{\sqrt{a^2 + b^2}} + \frac{y'b}{\sqrt{a^2 + b^2}}
\]

If in the determinant
\[
\begin{vmatrix}
x_u & y_u & 1 \\
x_v & y_v & 1 \\
x_w & y_w & 1
\end{vmatrix}
\]

it is desired to rotate the axes; substitution of the above values of \(x\) and \(y\) gives
\[
\begin{vmatrix}
x_u\frac{a}{\sqrt{a^2 + b^2}} - y_u\frac{b}{\sqrt{a^2 + b^2}} & x_u\frac{b}{\sqrt{a^2 + b^2}} + y_u\frac{a}{\sqrt{a^2 + b^2}} & 1 \\
x_v\frac{a}{\sqrt{a^2 + b^2}} - y_v\frac{b}{\sqrt{a^2 + b^2}} & x_v\frac{b}{\sqrt{a^2 + b^2}} + y_v\frac{a}{\sqrt{a^2 + b^2}} & 1 \\
x_w\frac{a}{\sqrt{a^2 + b^2}} - y_w\frac{b}{\sqrt{a^2 + b^2}} & x_w\frac{b}{\sqrt{a^2 + b^2}} + y_w\frac{a}{\sqrt{a^2 + b^2}} & 1
\end{vmatrix}
\]

Multiplying the elements in the first and second columns by \(\frac{a^2 + b^2}{a^2 + b^2}\) gives
\[
\begin{vmatrix}
x_u - y_u & x_u + y_u & 1 \\
x_v - y_v & x_v + y_v & 1 \\
x_w - y_w & x_w + y_w & 1
\end{vmatrix}
\]
FIGURE 2
The constant \( a \) may be the abscissa of any point through which the new axis passes, but \( b \) is the negative of the ordinate of the same point.

In Figure 2, to rotate the axes so that \( x \) axis passes through the point \( P \), substitute the values of the coordinates of the point into the last determinant, together with the functions for \( x \) and \( y \) obtained from \( x' \) and \( y' \).

**Projective transformation.** If the scales diverge badly or vary extremely in length, it may be necessary to make a "projective" transformation. It is necessary for our purposes that all straight lines remain straight lines, and that all lines intersecting in a common point, intersect in a common point after transformation. The only geometric projection under which this would be true, would be that of projection from a point outside the plane onto another plane. This will not change the degree of a curve. The four corners of the quadrilateral are to be transformed into the four corners of a rectangle. This may be done by construction, or mathematically using a "projective" transformation.

If the determinant

\[
\begin{vmatrix}
  f(u) & g(u) & h(u) \\
  f(v) & g(v) & h(v) \\
  f(w) & g(w) & h(w)
\end{vmatrix} = 0
\]

be multiplied by the determinant

\[
\begin{vmatrix}
  p_1 & q_1 & r_1 \\
  p_2 & q_2 & r_2 \\
  p_3 & q_3 & r_3
\end{vmatrix} = D \neq 0
\]

The result will be equivalent to multiplying the formula by a constant whose value is \( D \). Therefore, the original formula is unchanged. The resulting determinant itself will be
\[
\begin{align*}
\begin{array}{ccc}
  p_1 f(u) + q_1 g(u) + r_1 h(u) & p_2 f(u) + q_2 g(u) + r_2 h(u) & p_3 f(u) + q_3 g(u) + r_3 h(u) \\
p_1 f(v) + q_1 g(v) + r_1 h(v) & p_2 f(v) + q_2 g(v) + r_2 h(v) & p_3 f(v) + q_3 g(v) + r_3 h(v) \\
p_1 f(w) + q_1 g(w) + r_1 h(w) & p_2 f(w) + q_2 g(w) + r_2 h(w) & p_3 f(w) + q_3 g(w) + r_3 h(w)
\end{array}
\end{align*}
\]

or

\[
\begin{align*}
\begin{array}{ccc}
  \frac{p_1 f(u) + q_1 g(u) + r_1 h(u)}{p_3 f(u) + q_3 g(u) + r_3 h(u)} & \frac{p_2 f(u) + q_2 g(u) + r_2 h(u)}{p_3 f(u) + q_3 g(u) + r_3 h(u)} & 1 \\
  \frac{p_1 f(v) + q_1 g(v) + r_1 h(v)}{p_3 f(v) + q_3 g(v) + r_3 h(v)} & \frac{p_2 f(v) + q_2 g(v) + r_2 h(v)}{p_3 f(v) + q_3 g(v) + r_3 h(v)} & 1 \\
  \frac{p_1 f(w) + q_1 g(w) + r_1 h(w)}{p_3 f(w) + q_3 g(w) + r_3 h(w)} & \frac{p_2 f(w) + q_2 g(w) + r_2 h(w)}{p_3 f(w) + q_3 g(w) + r_3 h(w)} & 1
\end{array}
\end{align*}
\]
Choosing the corners of the limiting rectangle such that it is placed with sides parallel to the axes and corners \((x_1, y_1), (x_2, y_2), (x_1, y_2),\) and \((x_2, y_1)\). Corresponding to these values are values of \(u\) and \(w, u_1, u_2, w_1,\) and \(w_2\). Substituting these values in the determinant and expressing the result as parametric equations gives

\[
\frac{p_1 f(u_1) + q_1 g(u_1) + r_1 h(u_1)}{p_3 f(u_1) + q_3 g(u_1) + r_3 h(u_1)} = x_1
\]

\[
\frac{p_2 f(u_1) + q_2 g(u_1) + r_2 h(u_1)}{p_3 f(u_1) + q_3 g(u_1) + r_3 h(u_1)} = y_1
\]

\[
\frac{p_1 f(u_2) + q_1 g(u_2) + r_1 h(u_2)}{p_3 f(u_2) + q_3 g(u_2) + r_3 h(u_2)} = x_1
\]

\[
\frac{p_2 f(u_2) + q_2 g(u_2) + r_2 h(u_2)}{p_3 f(u_2) + q_3 g(u_2) + r_3 h(u_2)} = y_2
\]

\[
\frac{p_1 r(w_1) + q_1 g(w_1) + r_1 h(w_1)}{p_3 f(w_1) + q_3 g(w_1) + r_3 h(w_1)} = x_2
\]

\[
\frac{p_2 r(w_1) + q_2 g(w_1) + r_2 h(w_1)}{p_3 f(w_1) + q_3 g(w_1) + r_3 h(w_1)} = y_2
\]

Selecting the values \(x_1 = 0, x_2 = 1, y_1 = -1, y_2 = 1\), these eight equations can be solved for the constants in terms of one of them. A convenient value may be selected for this one in order to establish definite values.
for the others. The scale factors are then determined to make the chart proper size. Fortunately it is only necessary to use this method when curved scales are present, and usually only when at least two of the scales are curved.

7. Examples.

7.1 Genus zero, three non-concurrent, non-parallel scales. The function is \( a - \sqrt{1 + b^2} + ac - c - \sqrt{1 + b^2} = 0 \); \( b \) varies between 0 and 4, \( a \) varies between -1 and 2.2, \( c \) varies between 0 and 5. As a determinant, this function is represented by

\[
\begin{vmatrix}
a & a+1 & 1 \\
0 & \sqrt{1+b^2} & 1 \\
c & 0 & 1 \\
\end{vmatrix} = 0
\]

If the dimensions of the completed chart are to be 10 centimeters by 15 centimeters, the scale factor of the \( c \) scale will be \( \frac{5}{3} \) and the \( b \) scale will be 5. The determinant becomes

\[
\begin{vmatrix}
\frac{5}{3} & 2.5a & 1 \\
0 & 2.5\sqrt{1+b^2} & 1 \\
\frac{5}{3} & 0 & 1 \\
\end{vmatrix} = 0
\]

For the finished chart see Figure 3.
The equation shown in the graph is:

\[ \sqrt{l + b^2} + ac + c - c\sqrt{l + b^2} = 0 \]
7.2 Genus one, two parallel straight lines and one curve. The function is \( x^2 + bx + c \). Consider values of \( x \) between -1 and 5, \( b \) between -5 and 5, and \( c \) between -5 and 5. Expressed as a determinant this is

\[
\begin{vmatrix}
  x^2 & x & 1 \\
  -b & 1 & 0 \\
  -c & 0 & 1 \\
\end{vmatrix} = \begin{vmatrix}
  x^2 & x & 1 \\
  -b & 1 & 1 \\
  -c & 0 & 1 \\
\end{vmatrix} = 0
\]

The \( x \) scale has an infinite discontinuity at \( x = -1 \). However the first determinant can be rewritten as

\[
\begin{vmatrix}
  x^2 & x & 4 \\
  -b & 1 & 0 \\
  -c & 0 & 4 \\
\end{vmatrix} = \begin{vmatrix}
  x & 1 & 1 \\
  -b & 1 & 1 \\
  -c & 1 & 1 \\
\end{vmatrix} = 0
\]

If the \( b \) and \( c \) scales are to be 20 centimeters long, then their scale factors will be 2 and 8 respectively. The width is to be about 15 centimeters. To determine the scale factors consider the following determinant.

\[
\begin{vmatrix}
  0 & f(c) & 1 \\
  g(x), & f(x) & 1 \\
  1 & f(b) & 1 \\
\end{vmatrix} = 0
\]

Multiply the first column by \( a/d \); this gives

\[
\begin{vmatrix}
  0 & f(c) & 1 \\
  \frac{ag(x)}{d} & f(x) & 1 \\
  \frac{a}{d} & f(b) & 1 \\
\end{vmatrix} = \begin{vmatrix}
  0 & f(c) & d \\
  g(x) & f(x) & ag(x) + d \\
  1 & f(b) & a + d \\
\end{vmatrix}
\]
Let \( a + d \) be the scale factor of the c scale, scale factor of the b scale. Substituting this gives \( a = 4 \)

To make the width 15 centimeters multiply the first column.

This gives the determinant from which we construct the nomogram in F1.
\[ x^2 + bx + c = 0 \]

**FIGURE 4**
8. Bibliography

8.1 Robbins, E. S., The Theory of Alignment Charts, (thesis presented for the Master of Arts degree at the University of Wichita).


