Essays on Information in Labor Economics

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Economics

by

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Chapter 1: It is a commonly held view that the quality of unemployed workers varies countercyclically. During economic downturns, firms raise standards for retaining workers—better workers are fired, so it is natural to expect an accompanying rise in the unemployment pool’s quality. However, I present a model in which the reverse is true. Firms learn about employee quality over time and fire their lowest quality workers, but workers enter unemployment also via quitting. In equilibrium, the quality of the unemployment pool reflects a balance between flows of selective firings and of (relatively higher quality) quits. Although firms fire better workers during economic downturns, there are more of these firings at such times, and quits are no longer sufficient to balance the corresponding negative selection—the unemployment pool thus declines in quality. I use the model to explore the dynamic consequences of this. Firms limit hiring in response, and even after the economy rebounds otherwise, hiring will not resume until the unemployment pool’s quality recovers. This offers a possible contributing factor to jobless recoveries. Using CPS micro-data and JOLTS, I then provide empirical support for several testable implications of the model, focusing on
direct evidence for the mechanism driving the decline in unemployment pool quality. The model is consistent with other, previously-observed empirical patterns as well, and it provides a tractable framework for dynamic analysis of labor markets with private learning during employment.

Chapter 2: We consider a dynamic trading environment, where heterogeneous buyers and sellers are randomly paired in each period. Within each match, seller types become observable while buyer types remain private information, and sellers make take-it-or-leave-it offers. We first establish the existence of steady-state equilibrium where sellers offer prices that are continuous in their types. We then characterize properties of sorting under search frictions of varied strength, focusing on two extreme cases. With maximal search frictions—complete disregard for future payoffs—we demonstrate that log-supermodularity (log-submodularity) of the production function is a necessary and sufficient condition for positive (negative) assortative matching. Log-supermodularity (Log-submodularity) is stronger than the standard supermodularity (submodularity) sorting condition. The resistance to sorting comes from the fact that higher type sellers have stronger incentive to secure trade by lowering prices. At the other extreme, the incentive to secure trade grows inconsequential when search frictions vanish and hence the condition for positive (negative) sorting returns to supermodularity (submodularity).

Chapter 3: We study the dynamics of a market where agents trade assets that are heterogeneous in quality, but publicly indistinguishable. All agents begin with only public knowledge of the aggregate asset pool composition, but each owner learns privately about the quality of an asset in her possession. Ownership entails a constant choice between (i) the value of retaining the asset (and its corresponding payoffs) and (ii) that of selling it on the market for a uniform price that reflects the instantaneous average quality of assets being sold. In turn, the market composition reflects not only those owners who have opted for (ii), but also owners selling for reasons unrelated
Learning is gradual, so ownership can be understood as an optimal experimentation problem. Whereas an environment with public learning would entail symmetric timing of sales, private learning precludes this due to externalities sellers exert on other market participants. Instead, owners must spread sales over a broad time interval and, in turn, must experiment inefficiently.

Qualitatively, price dynamics resemble those found in speculation-fueled “panics” of the sort often invoked to explain market breakdowns. After an initial period without movement, prices enter a steady decline. Eventually, the stock—and corresponding flow—of owners looking to sell due to poor asset performance grows thin. Prices stop falling and finally rise as the presence of adverse selection fades from the market.
The dissertation of Kenneth Shangold Mirkin is approved.

Maria Casanova
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Chapter 1

The Informational Content of Unemployment and Labor Market Dynamics

1.1 Introduction

Several years after the official end of this past recession, the unemployment rate remains far above earlier levels (BLS - CPS Labor Force Statistics), and similar, albeit less extreme patterns have followed previous recessions in recent history. Consequently, unemployment should be not only a current policy concern, but perhaps also a recurring one in the context of future economic fluctuations. An effective policy targeting unemployment, however, requires that we understand the reasons for unemployed workers’ job-finding struggles. Possible explanations for these struggles drawn from previous research include human capital depreciation during unemployment (Pissarides, 1992; Ljungqvist and Sargent, 2008; Möller, 1990), negative sorting induced by selective hiring from
the unemployment pool (Lockwood, 1991), and even employer bias caused by the assumed presence of the previous two mechanisms (Jackman and Layard, 1991).

In this paper, I consider another factor affecting employment outcomes—one that may be of particular relevance in the current labor market and following recessions more generally: changes over time in the quality of workers entering the unemployment pool. In particular, workers of lower average productivity enter unemployment during recessions, and this compositional change may make firms less willing to hire. The analysis to follow explores this idea both theoretically and empirically. I present a dynamic model of the labor market in which these recession-induced declines in the quality of flows to unemployment arise endogenously. I use this model to explore the resulting impact on firms’ willingness to hire and, in turn, on aggregate unemployment dynamics. Further, I supplement the analysis with empirical evidence for these declining-quality flows to unemployment, which are a primary driver of the model’s key predictions.

In the model, firms hire selectively from a pool of heterogeneous, unemployed workers. This selective hiring is imperfect, and firms then learn privately during employment about worker productivity. Thus, we can parametrize both how much information about worker quality firms can obtain before hiring and how quickly firms obtain this information after hiring.

Workers can reach the unemployment pool either by quitting or by being targeted for firing. Firms fire workers due to low beliefs about productivity, so these workers are negatively selected, but this selection does not apply to job leavers. In equilibrium, the quality of those entering unemployment reflects a balance between low quality job losers and better quality job leavers.¹

¹Within the unemployment pool, better workers are more likely to be hired at each moment (due to the selective hiring of firms). As a result, the average quality of remaining unemployed workers decreases with unemployment duration. The model thus generates the (empirically-observed) pattern that hiring probabilities decline with unemployment duration.
Using this framework, I study the labor market dynamics induced by a recession, in which workers become less productive relative to their costs of employment. The analysis offers two main insights:

(1) Firms respond to this "shock" by raising standards for firing current workers and for employing new ones. Thus, consistent with the standard analysis in the literature (Nakamura, 2008; Kosovich, 2010; Lockwood, 1991), workers fired during recessions are of higher average quality than those fired under other economic conditions. However, a recession also throws off the preexisting balance between fires and quits—the flow of job losers overwhelms that of job leavers. As a result, a recession decreases the quality of the unemployment pool, reversing the standard conclusion.

(2) This drop in quality lowers each firm’s expected value of hiring a new worker from the unemployment pool, so firms will limit hiring. For hiring to return, the unemployment pool quality must rebound through inflows of workers who have not been targeted for low productivity, such as job leavers. The unemployment pool’s quality may take a long time to recover, and this may be further delayed if poor job-finding conditions reduce the natural turnover of quitting workers. In fact, this lower quality unemployment pool may continue to suppress hiring even if there is a positive productivity shock and the economy otherwise recovers to pre-recession conditions. Thus, this may help explain the "jobless recoveries" that have followed recent recessions.2

2A related prediction of the model is that employees remaining with a firm after a recession are disproportionately more productive than those employed beforehand. This mitigates the productivity decline that accompanies a recession and may provide an explanation for the recently-observed acyclicality/countercyclicality of average labor productivity (see Gali and van Rens, 2010).
The above results highlight the importance of explicitly modeling economic dynamics in this context, as conclusions drawn from comparing the predictions of two static models may be misleading. In the current setting, the unemployment pool in the new steady-state equilibrium associated with a recession is of higher average quality than the pool in the pre-recession steady-state, but the unemployment pool during the transition to the recession is of lower quality than that found in either steady-state. The model developed in this paper allows us to address such issues and to take dynamics seriously—it offers a tractable equilibrium framework that can be used to study the evolution of employment and/or wages under changing economic conditions.

To support the model, I document three empirical patterns which together highlight the precise mechanism driving the model’s dynamics. The first two of these are unsurprising—I first show that more job losers and fewer job leavers enter unemployment during recessions. Second, I provide evidence that job leavers are of higher quality than job losers. These findings suggest that the unemployment pool quality declines during recessions, but this may not hold if, for instance, the quality of job losers rises sufficiently during recessions. The third pattern I present addresses such concerns—I provide direct evidence that shifts from quits toward firings lower the quality of workers entering unemployment.

This evidence draws upon an empirical strategy based on the following intuition: Because firms hire selectively, better quality workers are more likely to be hired. This implies that worker reemployment probabilities correlate with worker quality. If we divide the unemployed into short-term unemployed workers (STU) and long-term unemployed workers (LTU), then workers will be counted among STU when they first enter the unemployment pool. Thus, we can proxy for changes in the quality of flows to unemployment with changes in the reemployment probabilities of STU relative to those of LTU.
Using CPS micro-data, I find that the reemployment probabilities of STU decline relative to those of LTU when flows to unemployment contain more job losers and fewer job leavers. This suggests that these shifts from job leavers toward job losers are associated with lower quality workers entering unemployment. The result persists regardless of whether the numbers of job losers and leavers come from the employer-reported Job Openings and Labor Turnover Survey (JOLTS) or from the employee-reported CPS monthly sample. In addition, I qualitatively examine the latest recession using this approach, and I observe that the surge in firings relative to quits was accompanied by a stark decline in STU reemployment probabilities relative to those of LTU. Thus, there is consistent evidence regarding both the different compositions of job leavers and job losers and the magnified role of these differences during economic downturns.

These empirical patterns are especially informative because they cannot be generated by human capital depreciation, selection among the remaining unemployed, or other explanations for persistent unemployment emphasized in the literature. While such alternative mechanisms are important for understanding unemployment, this analysis identifies a role for compositional change that is independent of these existing explanations.\(^3\)

The paper proceeds as follows: Section 1 overviews the literature on related topics. The next several sections develop a dynamic model of the labor market. Section 2 sets forth the structure of the economy, and Section 3 characterizes its steady-state equilibrium. Section 4 studies the dynamics of employment in response to a negative productivity shock (a "recession"), highlighting the causes and implications of the unemployment pool’s changing composition. This section also

\(^3\)Further, the proportions of firings and quits entering unemployment are linked directly to hiring outcomes, which are a primary welfare objective of policy makers. Thus, even if these patterns arise from a mechanism different from that suggested here, such an alternative mechanism should have similar implications regarding policies to support hiring.
considers a transitory shock and demonstrates how a jobless recovery can follow. Next, Section 5 presents empirical evidence that the unemployment pool’s quality declines during recessions, demonstrating that a proper analysis of unemployment pool dynamics must consider both job leavers and job losers. The remainder of the paper assesses broader implications of the theoretical analysis. Section 6 explains how the model can be understood in the context of more traditional models based on frictional search, and it also provides a frictional characterization of employment dynamics that is analogous to the paper’s main "frictionless" results. Section 7 discusses other potential robustness concerns and considers the implications of several theoretical extensions. Finally, Section 8 concludes.

1.2 Related Literature

The contributions of this paper relate to several literatures:

**Employer Learning**

Learning has been a relevant consideration in labor market models since at least Jovanovic’s seminal 1979 paper. The importance of formally modeling learning has been demonstrated in a number of settings (see Farber and Gibbons 1996, and Altonji and Pierret 2001 for examples); simplified, *ad hoc* representations of learning (such as the sudden attainment of full information after a fixed period of uncertainty) may fail to capture important implications of the information structure.\(^4\)

\(^4\)This paper also contributes to a recent literature analyzing learning in dynamic equilibrium environments (Anderson and Smith, 2010; Eeckhout and Weng, 2010). The models in both of these existing papers are designed to analyze assortative matching and, as such, are less well-suited to the settings I study. In particular, Anderson and Smith allow individual-specific reputations to persist across matches, while Eeckhout and Weng allow sorting to take
Additionally, empirical evidence suggests that the asymmetric learning I model is relevant in labor markets. The findings of Kahn (2009) and Pinkston (2009) support the presence of asymmetric learning during employment; in particular, Pinkston (2009) suggests that asymmetric learning is at least as important in this context as the public learning modeled by Anderson and Smith. This type of asymmetric learning warrants further investigation, and my analysis contributes to this cause.

Adverse Selection in Labor Markets

An assortment of research has studied aspects of adverse selection in labor markets. Gibbons and Katz (1991) began an empirical literature investigating differences between workers who lost jobs due to plant closings and those who lost jobs at plants which remained open ("layoffs"). They find that layoffs experience longer unemployment spells and lower wages upon reemployment. This is evidence that layoffs are of lower quality, but two underlying causes could drive this result. (1) place through wage offers. In contrast, I consider an environment in which firms cannot separate heterogeneous workers from an unemployment pool, and in which firms have only aggregate information about this pool.

For modeling purposes in this setting, the Poisson learning I use is somewhat more tractable than the Brownian motion-based learning used in other continuous-time models (such as Jovanovic 1979, or Eeckhout and Weng 2010). In some ways, the model developed here allows more precise characterization of equilibrium outcomes than was possible in such other models. Thus, this framework may offer insights beyond those directly mentioned in this paper, some of which may involve taking the model directly to data.

On the theoretical side, several papers have studied adverse selection between firms in more basic settings than the dynamic equilibrium structure of my model. Laing (1994) considers how adverse selection in turnover between firms affects contracts within the firm. Because firms release their lowest quality workers, outside firms infer that those remaining are better. As a result of this positive signal, remaining workers must be compensated, and this distorts the optimal contract. Waldman (1984) and Greenwald (1986) show that the negative signal sent by workers leaving jobs can inhibit turnover between firms.
Firms may stigmatize these workers based specifically on information about why they lost jobs. As a result, there may be explicit statistical discrimination against these workers. (2) If firms observe other worker traits—not including reason for unemployment—that correlate with quality, then these firms will hire selectively and offer reemployment wages based on these traits. If some of these firm-observed traits are unobservable to the econometrician, then higher quality groups (like non-layoffs) will have better reemployment outcomes empirically. In my analysis, both of these mechanisms impact employment dynamics similarly, so I do not distinguish between them.\textsuperscript{6}

More closely related to my analysis are Lockwood (1991) and Boone and Watson (2007), who investigate employer incentives to test prospective workers and how these incentives affect the equilibrium unemployment pool. Testing is costly to firms, and it is associated with labor market externalities through hiring decisions—this "filtering" process can worsen the unemployment pool from which other firms hire. As in my model, worker quality declines with unemployment duration due to selective hiring.\textsuperscript{7}

Both of these models assume that frictions arise from a search/matching process. In both cases, these frictions contribute to unemployment. In contrast, my model has no search frictions (though it can also be understood as the frictionless limit of a more standard, frictional matching model—I discuss this in some detail in Section 6 and Appendix B). Workers will be hired immediately whenever firms can obtain \textit{ex ante} profits from doing so. Perhaps surprisingly, my model generates

\textsuperscript{6}My main analysis holds independently of which of these mechanisms is responsible for this result. Further, my findings are completely unchanged as long as the second cause plays some role in the different outcomes between fired workers and quitting workers (as long as these differences do not result exclusively from the first cause).

\textsuperscript{7}I do not explicitly model firm investment toward more precise testing. This has important consequences for steady-state outcomes—it can even result in multiple equilibria with different levels of testing and different qualities of the unemployment pool. In the context of employment dynamics and economic shocks, however, this would not substantively change my analysis, so I omit it for simplicity.
equilibrium unemployment in spite of this. I show that selective hiring worsens the unemployment pool, and that this is sufficient to limit employment by reducing the *ex ante* value of hiring.

Additionally, these models study the consequences of selective hiring only in a steady-state equilibrium; extending them to analyze a dynamic environment would be problematic. Boone and Watson assume that firms immediately learn worker types after hiring, but that these firms must wait a fixed duration before firing these workers. As a result, firings would not increase during a recession, but hiring would decrease, so there could be less negative selection among the remaining unemployed. Thus, recessions could actually raise unemployment pool quality in their model. In Lockwood’s model, recessions could raise unemployment pool quality for a much simpler reason—there is no voluntary quitting.

**Human Capital Depreciation**

Human capital depreciation is a standard mechanism used to explain the persistence of unemployment following recessions. A seminal example of this is Pissarides (1992), whose mechanism relies on a "thin market externality." To understand this explanation, note first that a temporary negative shock which reduces hiring will raise unemployment durations. Because worker skills deteriorate during unemployment, this shock lowers the future returns to firms of searching for unemployed workers to hire. As a result, fewer firms enter the market in the period after the shock, and this further increases the durations of unemployed workers. In turn, worker skills deteriorate further, and the temporary shock is amplified.

Thus, via human capital depreciation, a recession can generate a mass of LTU who remain

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8Berger (2012) offers another explanation for this, though this explanation is unrelated to the unemployment pool (in fact, there is no unemployment in Berger’s model). He argues that selective firing during recessions increases firm efficiency, so firms enter recoveries better able to meet growing demand without hiring additional workers.
jobless long after other aspects of the economy have recovered. This result is often used to explain the “jobless recoveries” that have followed recent recessions (Ljungqvist and Sargent, 2008).

In this mechanism, workers who have been unemployed for a given duration do not change in quality over the business cycle. Instead, the distribution of unemployment durations changes—LTU are less productive, and there are more LTU in the aftermath of a recession. Thus, this standard human capital analysis ignores the impact of changes in the sources of unemployment (and, in turn, in the productivity of the unemployed) over the business cycle. The empirical evidence I present in Section 5 demonstrates the presence of these compositional changes in unemployment flows. The model that follows shows that these can cause persistent unemployment independently of the human capital depreciation mechanism.

Unobserved Heterogeneity vs. True Duration Dependence

Empirical reemployment probabilities decline with unemployment duration, and a significant literature has sought to understand the mechanism behind this. This pattern is consistent with the theoretical predictions of the selective hiring models described above, but human capital depreciation could also play a role.

More generally, this literature separates possible mechanisms into two categories: (1) true duration dependence (caused by human capital depreciation, the stigma of long-term unemployment, etc.) and (2) unobserved heterogeneity between the LTU and STU (which might be caused by selective hiring and negative sorting during unemployment). See Heckman (1991) and Machin and Manning (1999) for surveys of such analyses.

Several empirical papers focus more directly on the unemployment dynamics emphasized in my analysis; generally, these studies characterize variations over the business cycle in unobserved
heterogeneity and in true duration dependence (Baker, 1992; Dynarski and Shefrin, 1990; Kalwij, 2001; Imbens and Lynch, 2006). Of these, Baker’s analysis is most applicable to mine. He notes that unemployment durations increase overall during recessions, and he investigates how much of this rise results from changes in the composition of those entering unemployment. Baker finds that a significant part of the increase in durations is caused by changes specifically in the firings/ quits composition of flows to unemployment. This is certainly consistent with the evidence I present in Section 5, and this further supports the model I develop in the rest of the paper.

**Recessions and the Quality of Flows to Unemployment**

"If you think about it, people who were laid off recently may be, on average, worse candidates than people who were laid off a while ago. After all, people who have been out of work for two years or longer are people who were laid off during the recession. That means many of them were workers whose jobs were eliminated simply because their businesses were doing badly, not because they were personally incompetent."

- Catherine Rampell, New York Times *Economix* Blog (July 26, 2011)

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9 Several somewhat related papers (such as Elsby, Michaels, and Solon, 2009 and Bachmann and Sinning, 2011) conduct more macro-oriented analyses of compositional changes in unemployment inflows and outflows. In particular, Bachmann and Sinning find that the compositional effects begin to decrease outflows from unemployment toward the end of a recession. As I will show in Section 5, this time period coincides with the shift from job leavers toward job losers, suggesting that this shift lowers the quality of flows to unemployment.

10 Baker’s main argument is that compositional changes cannot explain all of the cyclical variation in unemployment durations, but he acknowledges that reasons for unemployment have a significant effect.

Nakamura (2008) considers how and why flows to unemployment change during recessions, focusing specifically on changes in the quality of these flows. In contrast to the analysis here, she argues that these flows (and the unemployment pool itself) increase in quality during recessions. Other studies have made this point as well (Lockwood, 1991; Kosovich, 2010).

This view of unemployment pool dynamics (also common in the popular press, as the above quote suggests) is based on the following intuition: Workers generally have less value to their firms during a recession. As such, firms must raise standards for hiring new workers and for continuing to employ existing workers. Therefore, workers fired during recessions are of higher "quality" on average than those fired under other economic conditions. Extending this logic to the unemployment pool yields the conclusion that the quality of the unemployed rises during recessions.

Considering the most recent recession’s aftermath, this view is problematic. Unemployment has remained high long after the recession’s end, and recent firm hiring has benefited mainly STU. In fact, 2010 witnessed many firms begin discriminating explicitly against LTU by requiring that job applicants be either currently employed or recently laid off. If those laid off during the recession are better workers, they should have disproportionate job-finding success. Instead, it seems almost as if firms have been trying to avoid specifically these workers.

Typically, the reasoning behind this common conclusion fails to properly account for the dy-


namic role of job leavers. The model developed in the following sections explores the dynamically evolving flows of job leavers and job losers into the unemployment pool. In this setting, the unemployment pool’s quality can fall in response to a recession—the negative effect on quality of the higher share of job losers entering unemployment flows may overwhelm the positive effect of higher firing standards. Further, the empirical findings in Section 5 are consistent with this model and suggest that flows to unemployment may, in fact, decline in quality in a recession.

1.3 The Model

The economy consists of a unit measure of workers and a mass of firms determined by free entry. Time is continuous and infinite, and firms discount the future at rate $r > 0$. All firms are identical, but workers are distinguished by a type $\theta \in \{H, L\}$. Of the unit measure of workers in the economy, the proportion $P \in (0, 1)$ are type $H$.

Employment, Payoffs, and Wages

Firms make hiring and firing decisions—each can employ at most 1 worker at a time and must pay the instantaneous flow cost $w$ to do so. Firms receive a payoff $Y > 0$ with Poisson intensity $\lambda > 0$ from each type $H$ worker, and they receive no payoffs from type $L$ workers. In turn, workers face a binary choice between working for wage rate $w$ and unemployment, from which they obtain a flow value of $\tilde{w} \in (0, \lambda Y)$. The value of unemployment is known to both workers and firms.\footnote{The model admits several other interpretations of $\tilde{w}$. Viewed as a reservation wage, $\tilde{w}$ could represent the flow effort cost of labor or even a government-imposed minimum wage. Alternatively, we need not even consider $\tilde{w}$ to be compensation for workers—it could represent a flow cost of operation for the firm. (In this case, the worker’s actual wage would be $w - \tilde{w}$). Yet another possibility, in the spirit of Ramey and Watson (1997), is for $\tilde{w}$ to reflect a payoff the worker can obtain by misbehaving (such as stealing or destroying property) and leaving the job immediately. In}
At Poisson intensity $\pi > 0$, workers quit their jobs and return to the unemployment pool (in search of new jobs). This simple, exogenous shock is meant to represent a personal reason for wanting to leave, such as a need to move geographically for family reasons or a developing distaste for the tasks of the current job. We assume that this motivation for leaving is strong enough that the firm cannot profitably retain the worker at a renegotiated, higher wage.

**Hiring and Contracts**

When out of the market, firms must pay cost $c > 0$ to hire a worker from the unemployment pool. This cost $c$ can be interpreted as including the search/interview/hiring costs associated with obtaining a new employee. Firms cannot observe specific worker types before hiring. However, because all firms are ex ante identical (and thus have identical equilibrium strategies), firms can infer the fraction of the unemployed at time $t$ with $\theta = H$.

Firms can use a filtering technology to refine the pool of potential hires, where the cost of filtering is included in the hiring cost $c$. For each firm, this screening technology instantly filters the unemployment pool into a hiring pool—type $H$ workers pass through this filter with probability 1, while type $L$ workers pass through with probability $\alpha \in [0, 1]$. Thus, if the unemployment pool has proportion $q_U$ of type $H$ workers, the hiring pool after screening will have the type $H$ proportion $q_H (q_U) = \frac{q_U}{q_U + (1-q_U)\alpha}$.

this case, $\tilde{w}$ is the wage at which the worker’s present value of the employment relationship is equal to the value he can obtain by misbehaving and quitting.

The only difference between these cases is the source of the firm’s production cost; from the firm’s perspective, they are otherwise identical. While these distinctions may matter in welfare analysis, the interpretation of $\tilde{w}$ will not impact equilibrium labor market outcomes and dynamics. For consistency, the remainder of the paper will simply refer to $\tilde{w}$ as the reservation wage.
It is worth emphasizing here that this screening technology reflects all individual characteristics observable to the firm. Thus, within the model, it does not make sense for firms to condition explicitly on worker characteristics, such as unemployment duration, reason for unemployment, etc. Insofar as these attributes are observed, firms condition on them implicitly—they are already used in the screening technology to improve the chances of hiring a type $H$ worker.\footnote{This differs from Lockwood (1991), for instance, where hiring firms are able to condition explicitly on employment histories in addition to screening.} This does not preclude discrimination by unemployment duration, but such discrimination must take place through the screening technology.

Firms offer workers a fixed-wage contract, which will pay the flow value $w$ at each instant while the worker is employed—firms reserve the right to terminate employment. As we will focus on employment (rather than on wages and contracting), we will assume that the realization of output $Y$ is not observable to workers and is not contractible regardless. The wage level $w$ is determined competitively; firms will attempt to outbid each other until it is no longer \textit{ex ante} profitable to do so.

\section*{Learning and Employment Termination}

Over time, firms learn about worker quality through payoff realizations. Suppose that, at time $t$, a firm has belief $p_t$ about the probability that its employee is type $H$. Firms update according to Bayes’ rule; if the firm receives a payoff at time $t$, it updates discretely to $p_{t+dt} = 1$, as this could not have happened with a type $L$ worker. In the absence of a payoff at time $t$, the firm shifts its belief infinitesimally downward by $dp_t = -\lambda p_t (1 - p_t) dt$.

Matches end either when workers quit or when firms decide to fire them. To understand when firing is optimal for the firm, consider the three options it faces at each point in time (taking the
wage rate as given): (1) It can retain its current employee, providing value $V(p_t)$. (2) It can fire its current employee and leave the market, obtaining a flow value of 0 and an option value of hiring in the future. (3) It can fire its current employee and pay cost $c$ to hire a new worker by applying the screening technology to the unemployment pool. This provides value $V(q_H(q_{U,t})) - c$, where $q_{U,t}$ is the proportion of type $H$ workers among the unemployed at time $t$.\footnote{Note: after a match has been terminated, firms must pay $c$ to hire a new worker regardless of whether the termination was targeted or exogenous. In this sense, we exclude firing costs from those represented by $c$. Adding a parameter to capture these firing costs has no substantive impact on the model’s main qualitative predictions.}

Obviously, the current match’s value is increasing in the belief $p_t$, so the firm will make termination decisions according to a threshold rule. It will prefer to end the relationship via option (2) or (3) when its belief about its worker’s type falls to some $p^*$, at which point it can obtain equal value either from leaving the market or from hiring a replacement.

To characterize the firm’s optimal firing decisions (and hiring decisions) in more detail, we must determine how the outside option evolves in equilibrium. This outside option depends on the value of hiring, which is affected by aggregate labor market conditions. The next section will characterize these conditions, so a more precise discussion of firm behavior will be included at that point.

\subsection*{1.4 Steady-State Equilibrium}

\subsubsection*{1.4.1 Full-Information Outcome}

As a benchmark for comparison, we will first consider equilibrium in a labor market with full information. Suppose that employers can perfectly observe worker types before hiring, so there is no role for learning during employment.\footnote{Note that this full-information setting can be viewed as the extreme case in which the screening technology’s effectiveness parameter $\alpha = 0$.} Assume for convenience that $V(0) < 0$ and $\lambda Y - \bar{w} \geq 0$,
so that firms will either employ type $H$ workers or leave the market. Type $H$ workers will never be fired intentionally, but they can still reach the unemployment pool by quitting. Free entry will force wages for type $H$ workers up to $w = \lambda Y - (r + \pi) c$, so that the value of hiring a type $H$ worker is 0. As such, high type workers who reach the unemployment pool will be instantly hired at this wage. Thus, employment (which we denote by $E$) will be $E = P$.

In the remainder of this section, we will show how the above outcome changes in a setting with \textit{ex ante} uncertainty and learning. We will see that these forces can hinder employment even in the absence of search frictions. In order for hiring to occur, wages must be significantly lower to compensate firms for the risk of hiring a low type worker. Depending on the reservation value $\tilde{w}$, the steady-state employment level can be lower or even higher than $P$. Of course, it may be misleading to compare these two cases based on employment levels alone; under \textit{ex ante} uncertainty, firms will sustain employment only at wages well below those paid under full information.

1.4.2 Equilibrium with Learning

The remainder of this section will characterize properties of equilibrium when firms cannot perfectly separate worker types before hiring. This will serve as a conceptual starting point for the following section’s analysis of employment dynamics during and after recessions. We will therefore focus on equilibria that are economically meaningful and well-suited to this application, and these equilibria will satisfy three restrictions. \textit{We will assume the following conditions are satisfied throughout the remaining analysis.}

First, in order to highlight the implications of learning during employment, we will limit the amount of \textit{ex ante} information that firms can acquire before hiring.\footnote{Intuitively, we want to distinguish this analysis from the full information benchmark, which corresponds to $\alpha = 0$.} Translating this to the model,
we will rule out α that are too close to 0. Formally, define \( q_{H}^{*} \) to be the belief level at which the firm’s value function is equal to the hiring cost \( c \) when wages are at the minimum level \( \tilde{w} \).\(^{20}\) Then this assumption can be written as a restriction on parameter values:

\[
\alpha > \left( \frac{r + \pi}{r + \pi + \lambda} \right) \left( \frac{\tilde{w}}{\lambda Y - \tilde{w}} \right) \left( 1 - \frac{q_{H}^{*}}{q_{H}} \right) \tag{A1}
\]

Often, no steady-state equilibrium exists for parameter combinations that violate this condition. If equilibria do exist, they will have unrealistic properties for our applications, such as prohibitively high wages and rapid turnover. Most importantly, such equilibria will require a type \( H \) proportion among fired workers (\( p^{*} \)) that is higher than that in the unemployment pool (\( q_{U} \));\(^{21}\) if firms could hire exclusively from workers fired by other firms, they would prefer to do this. Firings must improve the unemployment pool if this assumption does not hold—this conflicts both with basic intuition and with the results that will be presented in Section 5.

Our second restriction will rule out parameter values that lead to zero employment in equilibrium. We will do this by placing a lower bound on \( P \), the proportion of type \( H \) workers in the labor force. There can be no positive selection into unemployment, so the unemployment pool’s quality is bounded above by \( P \). Given values for the subset of parameters \( \{\alpha, r, \pi, \lambda, Y, \tilde{w}\} \), if \( P \) is too low, then the firm will negatively value a worker at belief \( q_{H} (P) \). In equilibrium, firms will never be willing to hire, even at a hiring cost of 0. Formally, for \( \alpha \) values that satisfy (A1), we require a sufficiently high labor force quality:\(^{22}\)

\[
P > \frac{\alpha (r + \pi) \tilde{w}}{(\lambda Y - \tilde{w})(r + \pi + \lambda) + \alpha (r + \pi) \tilde{w}} \tag{A2}
\]

\(^{20}\)Note that \( q_{H}^{*} \) depends only on the parameter combination \( \{r, \pi, \lambda, \tilde{w}, c\} \). Thus, \( q_{H}^{*} \) should be viewed as a reflection of parameter values, rather than as an endogenous equilibrium property.

\(^{21}\)This is demonstrated in Appendix A in the proof of Proposition 2 (which will be presented in the next section).

\(^{22}\)Note that we could also express this as a lower bound on \( Y \) or as an upper bound on \( \tilde{w} \), among other things.
Lastly, we will restrict our focus to "nontrivial steady-state employment equilibria," which must satisfy the following definition:

**Definition 1:** A steady-state employment equilibrium \( \{w^*_t, p^*_t, E_t, q_{U,t}, q_{E,t}\}_{t=0}^{\infty} \) consists of market wage rates \( w^*_t \), threshold rules \( p^*_t \in [0, 1] \), employment levels \( E_t \in [0, 1] \), and type H proportions of unemployed workers \( q_{U,t} \in [0, 1] \) and employed workers \( q_{E,t} \in [0, 1] \) such that:

1. Firm firing decisions are optimal,
2. Firm hiring decisions are optimal (free entry: \( V(q_{H}(q_{U,t})) \leq c \)),
3. The size and quality of the unemployment pool are consistent with the size and quality of the total labor force, and
4. The employment level is constant (\( E_t = E_{ss}, \forall t \)).

Further, such an equilibrium is "nontrivial" if \( E_{ss} \notin (0, 1) \).

Proposition A.1 (in Appendix A) establishes the existence of a unique nontrivial steady-state employment equilibrium for a range of "reasonable" hiring costs \( c \). It is also shown in Appendix A that this result holds when (A1) and (A2) are satisfied.

Several aspects of this definition merit discussion. Practical reasons motivate our restriction to equilibria that are nontrivial and meet Condition (iv). Condition (iv) is for simplicity—by focusing on a steady-state, we can provide clear intuition for the results in this section and the next. Additionally, it is worth noting the existence of equilibria which fail this condition. In these cases, employment fluctuates even without exogenous shocks. Though these equilibria are not used in this section’s analysis, they highlight more general conceptual insights regarding the structure of information in labor markets. I discuss this briefly in Section 7.

In turn, nontrivial equilibria in this model are those that are economically relevant. In a steady-state, full employment equilibria have little insight to offer about unemployment. More importantly, to study unemployment’s dynamic response to economic shocks, we require a model
in which unemployment is present not only after the shock, but also in the pre-shock equilibrium.\footnote{It should become clear in this section that this is merely a constraint on extreme parameter values, rather than a conceptual restriction on the economy.}

The remaining aspects of this definition—the first three conditions—are general properties of the economy. Condition (i) requires value matching \((V(p_t^*) = 0)\) and smooth pasting \((V'(p_t^*) = 0)\), where the latter must hold because downward updating of firm beliefs \(p_t\) is differentiable. Condition (iii) is mainly one of accounting—the measure of type \(H\) workers in the labor force is constant, so this must equal the sum of the measures of type \(H\) workers among the employed and unemployed. Formally, this requires \(E_t q_{E,t} + (1 - E_t) q_{U,t} = \mathbb{P}\).

Condition (ii), free entry, simply implies that firms cannot make \textit{ex ante} profits from hiring. Like conditions (i) and (iii), this is also a standard property, but it is crucial to understand how exactly firms compete away these profits in equilibrium. From a given starting point in this economy, the dynamic labor market forces preserving free entry can manifest themselves in three ways:

\textit{Scarcity:} If there is full employment in the economy, and if the new workers entering unemployment are of sufficiently high quality, they will be rehired immediately. Firms will be unable to profit from employment because there are no workers to hire.

Scarcity can be relevant only if employment is full, so it will be ignored in the remainder of this analysis. The next two mechanisms, however, are also important for equilibria with unemployment.

\textit{Competitive Wage Bidding:} Suppose that the wage level is \(w\), and suppose also that the unemployment pool quality is sufficiently high that \(V(q_H(q_{U,t})) > c\) at these wages. Firms will compete to profit from workers, and in doing so, they will bid up wages until these profits have been eliminated (and \(V(q_H(q_{U,t})) = c\)).

\textit{Selective Hiring:} In using the screening technology, firms disproportionately remove type \(H\) workers
from the unemployment pool. When one firm hires a worker, it marginally lowers the quality of
the unemployment pool from which other firms must hire, and in turn, it lowers the expected value
of a new hire as well. Thus, in addition to raising wages, firms can compete away profits by hiring
more intensely.

In what follows, we will see that selective hiring is dominant among these forces in equilibria
relevant to the present analysis. Toward demonstrating this, it will be useful to establish several
properties of nontrivial steady-state employment equilibria. First, note that there is no intrinsic
heterogeneity among firms, so the equilibrium wage and threshold rules characterize the behavior
of all firms.

It is straightforward to show that wages must equal the reservation value:

**Lemma 1:** In any nontrivial steady-state employment equilibrium, \( w = \bar{w} \).

*(See Appendix A for proof)*

Intuitively, if firms were offering wages greater than \( \bar{w} \), unemployed workers could benefit from
undercutting these offers and working for less; thus, this cannot occur in equilibrium.

With this result, we can also show that all aggregate characteristics of the economy must be
fixed in equilibrium. More specifically, define \( \eta_t \) to be the intensity of hiring at time \( t \), so \( \eta_t dt \) is
the measure of unemployed workers hired at this instant. Then we can write our aggregate values
as \( \eta_t = \eta_{ss} \), \( q_{U,t} = q_{U,ss} \), and \( q_{E,t} = q_{E,ss} \).\(^{24}\)

Another crucial implication is that the free entry condition must hold with equality:
\( V (q_H (q_{U,ss})) = c \). Let us define \( q_U^* \) to be the minimum unemployment pool quality at which firms
are willing to hire at reservation level wages. \( q_U^* \) must satisfy \( V (q_H (q_U^*)) = c \) (and \( q_H^* = q_H (q_U^*) \)),
so the equilibrium unemployment pool quality is \( q_{U,ss} = q_U^* \). This is why wage bidding effects are

\(^{24}\)This is proven after Lemma 1 in Appendix A.
suppressed—firms cannot offer wages above $\bar{w}$ because the expected value of hiring is negative at such wages.

Because of selective hiring, the labor market is stable at this equilibrium. To see this, consider first an equilibrium without selective hiring in which there is positive unemployment. With the unemployment pool of quality $q_U^*$, firms are perfectly indifferent between hiring and not. There exists a constant intensity of firm hiring such that the unemployment pool quality and size are unchanged, but there is no reason for this intensity to be realized. These conditions can be sustained only if we assume that precisely the right measure of indifferent firms choose to hire at each instant.

In contrast, when hiring is selective, the aggregate intensity of hiring directly impacts the value of hiring. If too few firms were hiring, the unemployment pool quality would rise and more firms would find it profitable to hire. If too many firms were hiring, their screening process would deplete the quality of the unemployment pool, and it would no longer be profitable to hire. Thus, the economy will support precisely the intensity of hiring such that the unemployment pool remains at quality $q_U^*$.

We will discuss this in more detail at the end of this section. At that point, we will analytically characterize the equilibrium forces acting on the unemployment pool. First, though, we digress briefly to explain how these forces are generated by optimal firm decisions.

**Equilibrium Firm Behavior**

Free entry has another consequence in this setting—firms always face an outside option of value 0. In conjunction with the unvarying equilibrium wage level, this implies that the firm value function depends only on the belief $p_t$. We can thus use value-matching and smooth-pasting conditions to
solve *explicitly* for the firm’s value function.\textsuperscript{25}

**Proposition 1:** In a nontrivial steady-state employment equilibrium, the firm’s value function can be written analytically as:

\[ V_{ss}(p_t) = \begin{cases} 
\left( \frac{1}{r+\pi} \right) \left[ \lambda p_t Y - \tilde{w} + \lambda \left( \frac{\tilde{w}(1-p_t)}{r+\pi+\lambda} \right)^{\frac{r+\pi+\lambda}{\lambda}} \left( \frac{(\lambda Y - \tilde{w})p_t}{r+\pi} \right)^{-\frac{r+\pi}{\lambda}} \right] & \text{for } p_t \in [p_{ss}^*, 1] \\
0 & \text{for } p_t \leq p_{ss}^* 
\end{cases} \]

Further, this threshold level is given by

\[ p_{ss}^* = \frac{\tilde{w}(r+\pi)}{Y(Y+r+\lambda)-\tilde{w}} \]

*(See Appendix A for proof)*

As can be seen in the expression for \( p_{ss}^* \), firms will retain workers at lower beliefs when they face higher payoffs \( Y \), faster learning (and more frequent payoffs) \( \lambda \), less frequent worker quits \( \pi \), less firm "impatience" \( r \) (or lower interest rates), and lower wages \( \tilde{w} \).

Additionally, the value function itself illustrates the role of learning in this setting. The term \( \frac{\lambda p_t Y - \tilde{w}}{r+\pi} \) represents the employee’s expected output, as a function of the perceived likelihood that she is type \( H \). A "myopic" firm would consider this value alone. It would employ only workers offering an expected instantaneous profit, so it would terminate employment at the belief \( p_m^* = \frac{\tilde{w}}{\lambda Y} \).

However, each firm can terminate workers at its own discretion. As a result, the firm has an option value of eliminating an unproductive worker and either hiring a new worker or leaving the market. In the value function above, this option value is captured in the additive term on the right, which is clearly decreasing in the belief \( p_t \). Intuitively, a decline in \( p_t \) decreases the value of the current employee and increases the likelihood that the firm will want to use the outside option. Since this outside option is fixed in value, its relative value increases.

\textsuperscript{25}Insights from Bellman and Cooke (1963), Presman (1990), and Keller, Rady, and Cripps (2005) are used in the derivation of this value function.
Due to the option to drop unproductive employees, optimal firms are willing to retain employees with negative expected flow values; this is why \( p_{ss}^* < p_m^* \). Figure 1 below reflects this intuition, comparing myopic and optimal value functions over the range of beliefs \( p_t \in [p_{ss}^*, 1] \).

![Figure 1: Firm value as a function of belief \( p_t \)](image)

Extending this optimal behavior throughout the labor market, we can characterize the equilibrium distribution of firm beliefs, which will appear as below:

![Equilibrium Distribution of Beliefs](image)

**Figure 2**: Distribution of Firm Beliefs

Because firms apply the screening technology to the unemployment pool, they can expect a new hire to be type \( H \) with probability \( q_H(q_{U,ss}) \). While this individual is employed, the firm’s belief
$p_t$ declines gradually unless the firm receives a payoff $Y$—if this occurs, the firm knows its worker must be of type $H$, so it updates to $p_{t+dt} = 1$ and employs the worker until random termination of the match (via $\pi$). Thus, the mass at $p_t = 1$ corresponds to firms who know they have type $H$ employees. Without a payoff, the firm will employ its worker until its belief falls to $p^*_s$, at which point the firm will terminate the match intentionally and either hire a new worker or leave the market.

We can think of the time after hiring required for firm beliefs to reach $p^*_s$ as a reflection of how "patient" firms will be with unproductive workers. This time (which we denote by $t^*_s$) can be written as:

$$t^*_s = \frac{1}{\lambda} \ln \left( \frac{1 - p^*_s}{p^*_s} \right) \left( \frac{q_{U,s}}{\alpha (1 - q_{U,s})} \right)$$

Proposition A.2 (relegated to Appendix A) provides this, as well as expressions for the equilibrium hiring intensity $\eta_s$ and employment level $E_s$.

**Steady-State Unemployment Inflows and Outflows**

We can use the above belief distribution to characterize the forces acting on the unemployment pool’s quality in equilibrium. With constant hiring and employment, the quality of this equilibrium unemployment pool will evolve according to:

$$q_{U,t+dt} = \frac{[1 - E_s] q_{U,t} - \eta_s dt q_H (q_{U,t}) + \eta_s dt q_H (q_{U,t-t^*_s}) e^{-(\lambda + \pi)t^*_s} + \pi dt E_s q_{E,t}}{1 - E_s - \eta_s dt + \eta_s dt q_H (q_{U,t-t^*_s}) e^{-\pi t^*_s} + q_H (q_{U,t-t^*_s}) e^{-(\lambda + \pi)t^*_s} + \pi dt E_s}$$

where clearly we require $q_{U,t+dt} = q_{U,t} = q_{U,s}$ (and $q_{E,t} = q_{E,s}$, $\forall t$) since the quality is constant in this equilibrium. To focus on the specific sources of unemployment flows in the equation above,
we will continue writing \( q_{U,t} \) and \( q_{E,t} \) as time-specific qualities in this explanation. Intuitively, the numerator tracks the measure of unemployed type \( H \) workers, while the denominator tracks the total measure of unemployed workers. To see this, consider these terms pairwise. The left-most of these account for already-unemployed workers—the size of this pool is \( 1 - E_{ss} \), of which proportion \( q_{U,t} \) are type \( H \) at time \( t \).

The right three terms correspond to flows. \( \eta_{ss} dt \) workers are removed instantaneously from the pool by hiring, and because firms do this selectively, the proportion \( q_H (q_{U,t}) > q_{U,t} \) of these are type \( H \). This reflects the previously-discussed fact that firms remove type \( H \) workers disproportionately, so hiring exerts downward pressure on the unemployment pool’s quality.

In turn, quits draw randomly from employed workers, so high types make up \( \pi dt E_{ss} q_{E,t} \) of the \( \pi dt E_{ss} \) job leavers at time \( t \). There is no mechanism in this environment for unemployed workers to be negatively selected into employment, so \( q_{E,t} \) must be greater than \( q_{U,t} \) in equilibrium. Quits therefore push the unemployment pool quality upward.

Note that workers are fired after duration \( t_{ss}^* \) of unproductive employment, so the firing intensity at time \( t \) depends on two things: (1) the intensity and quality of hirings at time \( t - t_{ss}^* \) and (2) how many of these workers neither quit nor reveal themselves to be type \( H \) before \( t \). Obviously, type \( L \) workers will never generate output, so the proportion \( e^{-\pi t_{ss}^*} \) of these workers (those who do not quit) will be fired at time \( t \). In contrast, the proportion \( e^{-(\lambda+\pi)t_{ss}^*} < e^{-\pi t_{ss}^*} \) of type \( H \) workers will reach this threshold, so low types will be disproportionately represented among those fired.

Thus, the steady-state equilibrium requires a precise balance between hirings, quits, and firings. In the absence of economic volatility, selective hiring ensures that this balance is stable. The next section, however, will show how even small shocks can disrupt this stability.
1.5 Shocks and Employment Dynamics

In this section, we will explore the dynamics of employment in response to economic shocks. Specifically, we will consider negative shocks to $Y$, which we will analyze in two cases.\textsuperscript{26} To provide clear and simple intuition for labor market dynamics, we will first consider an unanticipated permanent shock. We will then analyze dynamics when this shock is known to be transitory; the implications for jobless recoveries will be considered in this context.

1.5.1 Simplest Case: Permanent Shock

Suppose that, at time $t = \hat{t}$, the payoff $Y$ falls unexpectedly from $Y_{ss}$ to $Y_{ss} - z$. Suppose also that, before this shock, the economy was settled at its steady state employment equilibrium. The initial impact of this shock on firm behavior occurs through the threshold rule $p^*$, which rises from

$$p^*_{ss} = \frac{\tilde{w} (r + \pi)}{\lambda [Y (r + \lambda + \pi) - \tilde{w}]} \text{ to } p^*_t = \frac{\tilde{w} (r + \pi)}{\lambda [(Y - z) (r + \lambda + \pi) - \tilde{w}]} > p^*_{ss}$$

The decrease in $Y$ makes firms less patient in learning about worker types—the payoff associated with type $H$ workers has decreased (and also decreased relative to the payoff associated with type $L$ workers, which is 0), so the value of learning about worker type has also decreased. Hence, the threshold belief for terminating workers rises.

For notational purposes, let us define $t_z$ to be the new time after hiring associated with the economic shock. This $t_z$ applies to firms who hired workers before the shock occurred. Such firms will now be willing to wait $t_z$ after their initial hires without a payoff before cutting ties—if they have already waited for some time $t$ in the range $[t_z, t_{ss}]$, they will fire their workers immediately.\textsuperscript{27}

\textsuperscript{26}In a broader economic context, this may reflect a reduction in aggregate demand. Alternatively, to avoid focusing on the causes of the change in $Y$, we could interpret this as a generic decline in productivity.
As with $t^*_{ss}$ in the previous section, we can express this $t_z$ in terms of $p^*_t$ and $q_{U,ss}$:

$$t_z = \frac{1}{\lambda} \ln \left[ \left( \frac{1 - p^*_t}{p^*_t} \right) \left( \frac{q_{U,ss}}{\alpha (1 - q_{U,ss})} \right) \right]$$

With this notation established, we can begin to characterize the impact of this shock on the economy. The mass of firms who fire their workers immediately after the shock is visually depicted in Figure 3 (below).

![Impact of a Fall in Y](image)

**Figure 3:** Firings after the shock

These workers fired in response to the shock (in the belief region $[p^*_ss, p^*_f]$) are of better average quality than those fired in the preexisting steady-state. Note that this is perfectly in line with the standard intuition promoted in Nakamura (2008) and Lockwood (1991), among others. The standards for termination rise, so the quality of those terminated rises as well—this remains true in this model. The departure from this standard result is based not only on fired workers, but also on the mixture of fired workers and those who quit. This will be developed later in this section, when we characterize the evolving quality of the unemployment pool (see Proposition 2).

Next, consider the rise in the unemployed population—the unemployment pool will expand to include these newly fired workers. Formally, the measure of unemployed workers will rise from
1 - $E_{ss}$ to

\[ 1 - E_{ss} + \eta_{ss} \int_{t_2}^{t^*_s} \left[ q_H(q_{U,ss}) e^{-(\pi+\lambda)s} + (1 - q_H(q_{U,ss})) e^{-\pi s} \right] ds \]

where the second component in this expression describes the mass of workers fired in immediate response to the shock. To see the intuition for this, consider the path of beliefs followed by a firm hiring a worker at time 0 in the steady state, where the firm neither realizes output nor has its worker quit before time $t^*_s$. Of the workers hired in the steady state, proportion $q_H(q_{U,ss})$ of these are type $H$. Because firms with type $H$ workers can leave this belief path either through a worker quitting or through a realization of output, proportion $e^{-(\pi+\lambda)t}$ of the firms hiring type $H$ workers remain on this belief path after time $t$. In turn, firms with type $L$ workers can leave this path only if this worker quits, so proportion $e^{-\pi t}$ of these firms remain on the path at time $t$. Thus, this expression corresponds precisely to the mass of new fires depicted in Figure 3.

Of course, there is information in these firing decisions—the firms who fire workers in response to the shock have beliefs in the range $[p^*_{ss}, p^*_L]$. While these newly fired workers are better on average than those fired in the steady-state (at belief $p^*_{ss}$), they still represent the lowest belief range of previously employed workers. As such, relative to the set of employed workers, this group has disproportionately few type $H$ workers. Yet, this group’s quality alone is insufficient to lower the unemployment pool quality—the size of this group also plays a crucial role in the evolution of the unemployment pool.

In the steady-state, there was a precise balance in the flow to unemployment between quitting workers (of which proportion $q_{E,ss} > P$ were type $H$) and fired workers (of which proportion $p^*_{ss}$ were type $H$)—this balance helped maintain the quality of the unemployment pool. After the shock, the negative pressure on unemployment pool quality from the mass of directed firings overwhelms the positive pressure from the flow of quitting workers (which is always of order $dt$). Thus, even
though the workers fired in response to this shock are better (on average) than those fired in the steady-state, the unemployment pool decreases in quality.

**Proposition 2:** \( \exists \overline{z} > 0 \) such that for \( z \in (0, \overline{z}) \), the proportion of type \( H \) workers in the unemployment pool immediately following the shock \( (Y \rightarrow Y - z) \) falls to

\[
q_{U,t} = \frac{(1 - E_{ss}) q_{U,ss} + \eta_{ss} \int_{t_z}^{t_{ss}} q_H (q_{U,ss}) e^{-(\pi+\lambda)s} ds}{1 - E_{ss} + \eta_{ss} \int_{t_z}^{t_{ss}} \left[ q_H (q_{U,ss}) e^{-(\pi+\lambda)s} + (1 - q_H (q_{U,ss})) e^{-\pi s} \right] ds} < q_{U,ss}
\]

(See Appendix A for proof)

To see why we must bound values of \( z \) above for this to hold, consider the extreme case of \( z > Y - \frac{w}{\lambda} \). Such a shock would be so large that firms would fire even the workers they know to be type \( H \). We would have full unemployment, and \( q_{U,t} \) would rise to \( P > q_{U,ss} \). Clearly then, for \( z \) sufficiently close to this range, so many workers will enter unemployment that the unemployment pool quality will rise. We can sensibly ignore such cases as disconnected from reasonable applications of this analysis.

Of course, this contamination of the unemployment pool is not the only response to this shock. As in the previous section, let \( q_{U}^* \) represent the unemployment pool quality at which firms are indifferent between hiring and not. For our purposes here, we write this as a function of \( Y \):

\( q_{U}^* \equiv q_{U}^* (Y) \) (so \( q_{U}^* (Y) \) reflects firm hiring standards at the payoff level \( Y \)). Then we can show that hiring standards rise along with firing standards.

**Lemma 2:** \( q_{U}^* (Y) \) is strictly decreasing in \( Y \).

(See Appendix A for proof)

Since the economy was at its steady-state before the shock (and since there was free entry in this steady state), the value of hiring a worker from the pool of unemployed at \( Y_{ss} \) was 0 (meaning
V(q_{U,ss}) = c). Obviously, conditional on employing a worker with belief \( p_t > p^* \), the firm’s value is monotonically increasing in \( Y \). Thus, with \( Y \) now at \( Y - z \), \( V(q_H(q_{U,ss})) < c \) and firms will require a higher initial belief \( q_H(q^*_U(Y - z)) \) (and, in turn, a higher unemployment pool quality) to justify hiring a worker.

As we have seen, though, this increased standard is compounded by a drop in the unemployment pool quality—the rising standards for hiring and firing have forced a wedge between actual market conditions and those necessary for sustained hiring. As a result, hiring will cease completely for a positive span of time.

**Proposition 3:** After the output shock \( Y \rightarrow Y - z \), hiring will cease for the duration \( \hat{t}_H > 0 \). If \( q^*_U(Y - z) < P \), \( \hat{t}_H \) is finite.

(See Appendix A for proof)

Because there can be no positive selection into unemployment, \( q_{U,t} < P, \forall t \). Thus, if \( z \) is so large that \( q^*_U(Y - z) \geq P \), then it is impossible for the unemployment pool quality to reach a level at which hiring can resume, and employment will converge to 0 over time.\(^\text{27}\)

For \( q^*_U(Y - z) < P \), hiring will resume only when the unemployment pool quality has recovered sufficiently to match the new (higher) hiring standards. Employee quits, which are not negatively selected, are the channel through which this will occur. Firms will begin to hire again only after sufficiently many remaining employees have left jobs to enter the unemployment pool.

Assuming that employees continue to quit at a constant rate following the shock, we can analytically characterize the duration without hiring \( \hat{t}_H \). An expression that \( \hat{t}_H \) must satisfy is provided in the proof of Proposition 3 in Appendix A. This expression is analytically detailed, but the intuition behind it is a simple accounting exercise. We must keep track of the measure of type \( H \) workers

\(^{27}\)In the case where \( q^*_U(Y - z) = P \), \( q_{U,t} \) will converge to \( P \) as \( t \rightarrow \infty \), but it will not reach \( P \) in finite time.
and the total measure of workers in unemployment from time $\hat{t}$ until hiring resumes. Since there is no hiring during this period, we must merely keep track of workers already in unemployment before the shock, those sent to unemployment at the time of the shock, and those who enter afterward. In characterizing $\hat{t}_H$, Appendix A formally details the type $H$ proportions and total measures of these groups of workers.

To illustrate this further, the evolution of the employment level and the unemployment pool quality after the shock are depicted below.\(^{28}\)

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\(^{28}\) As seen in Figure 4, employment fluctuates after the return of hiring. To understand why, notice that unproductive workers are fired a certain duration after being hired. Because hiring ceased for a period, a corresponding period without firing must occur in the future. Because firing exerts downward pressure on the unemployment pool’s quality, this pressure will be reduced during this future period. The labor market can therefore support more hiring (which also lowers the unemployment pool’s quality) at this time, so employment will rise. However, this period with more intense hiring will also lead to a future period with more intense firing and less intense hiring, during which employment will fall. Thus, current fluctuations will bring about future fluctuations.
It is crucial to note that the duration $t_H$ characterized in Appendix A and depicted above in Figures 4 and 5 is computed assuming that workers continue to quit jobs at the same rate after the shock as they did in the previous steady-state. In reality, voluntary quits plummet during a recession (as can be seen in Figure 6 in the following section). If this were to happen in the model, the duration without hiring would be magnified drastically. As voluntary quits are the main source of upward pressure on the unemployment pool quality, reducing the volume of these quits would slow the recovery of this quality. As such, we can view this duration $t_H$ as a lower bound on the duration for which hiring should cease.

Figures 4 and 5 make clear that the unemployment pool will grow after the shock—unmitigated by hiring—until the quality of this pool has risen to the new threshold required for hiring to resume. It is easy to see that the new, lower output equilibrium will have more unemployment and a higher unemployment pool quality. This is in line with the standard view about the composition of the unemployment pool in recessions—more people are unemployed, and hiring/firing standards are higher, so the pool must be better. Of course, this model highlights the flaws with this logic: First, the transition from the first equilibrium to the second involves a significant period with a
lower quality unemployment pool. Second, the second equilibrium may not be a permanent state of the economy. A recession is commonly viewed as a temporary productivity shock—if productivity rebounds during the transitional period, this equilibrium with a better unemployment pool may never be reached in the first place. We address this issue below.

1.5.2 Transitory Shocks and Jobless Recoveries

To better understand the dynamics induced by a temporary economic downturn, consider one further variation on this economic shock structure. Suppose that, as before, there is an unanticipated output shock $Y \rightarrow Y - z$ at time $\hat{t}$ to the nontrivial steady-state employment equilibrium, and that firms respond to this immediately. Rather than permanently remaining at this level, however, suppose that output will permanently return from $Y - z$ to $Y$ at some point in the future, and suppose that firms know this. In particular, suppose it is known that the "recovery date" follows a Poisson distribution with parameter $\gamma$, so that at each point in time during the "recession," the economy recovers with probability $\gamma dt$. Then we can show that, for a range of recovery dates $t \in [\hat{t}, \hat{t} + t_{Y-z,Y}]$, there will be no hiring for some time even after the economy has recovered to the previous level.\(^{29}\)

Toward formalizing this result, define $p^*(Y - z, Y)$ to be the termination belief level after the shock but before the recovery, and define $t^*(Y - z, Y)$ to be the associated time firms will wait without output before terminating a worker. Further, define $q^{t^*}_{U}(Y - z, Y)$ to be the corresponding hiring threshold during this "recession," and again define $q^{t^*}_{U}(Y)$ to be the hiring threshold after the recovery. We can then establish the following:

\(^{29}\)An expression for determining the duration $t_{Y-z,Y}$ will be provided in the proof of Proposition 4 in Appendix A.
Proposition 4: There exists $t_{Y-z,Y}$ such that, after the unanticipated transitory output shock, if the recovery occurs before time $\hat{t} + t_{Y-z,Y}$, then the economy will remain without hiring for a positive duration of time even after the recovery. Further, $\exists \tau > 0$ such that for $z \in (0, \bar{z})$, $t_{Y-z,Y} > 0$ and both the likelihood and expected duration of a jobless recovery are increasing in the magnitude of the shock $z$.

*(See Appendix A for proof)*

Because the unemployment pool has been contaminated by the equilibrium response to the initial shock, the labor market may be unable to sustain hiring even after the recovery. In a sense, this result analyzes the impact of temporarily increased firing standards without the impact of increased hiring standards (as the hiring threshold returns to its steady-state level after the recovery). After the recovery, hiring will return sooner in this case than in the scenario considered in Proposition 3, but the key insight is the fact that stagnant hiring can persist even after other economic indicators have rebounded. Further, as the result above indicates, larger shocks may make these jobless recoveries more likely and longer-lasting.

### 1.6 Empirical Evidence

This section presents evidence consistent with the compositional shifts which drive the unemployment pool’s changing quality in the previous model. As mentioned earlier, three patterns comprise this evidence:

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30 In Appendix A, I also include an alternate formulation of this result (called Proposition 4.a) in which the initial unanticipated output shock $Y \rightarrow Y - z$ at time $\hat{t}$ is followed almost immediately by another unanticipated output shock $Y - z \rightarrow Y$ at time $\hat{t} + dt$, where firms have already responded to the first shock before the second occurs. Of course, a setting with two unanticipated shocks that are both expected to be permanent is farther from reality than the environment of Proposition 4, but it yields a similar result with simple, clear intuition.
1.6.1 Shifts From Quits Toward Firings During Recessions

First, the fractions of workers entering unemployment who are job leavers and job losers change during recessions. At these times, the newly unemployed consist increasingly of job losers. Figure 6 shows this pattern during the most recent recession.

![Figure 6: Evolution of firings vs. quits in the latest recession (Source: JOLTS)](image)

The red and blue lines represent monthly quits and firings, respectively, according to JOLTS data. The shaded region indicates the recession according to official NBER dates. Toward the end of the recession, there was a stark shift from job leavers toward job losers in the flows to unemployment.

1.6.2 Superior Job-Finding Prospects of Job Leavers

Second, job losers take longer than job leavers to regain employment after entering the unemployment pool. This is supported in Figure 7 below, which separately plots the cumulative distribution functions of unemployment durations for job leavers (in red) and job losers (in blue).

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If these intensities are instead computed using CPS employment data, the patterns are virtually identical to those displayed here in the JOLTS data (which are based on firm responses). Hence, these trends are robust to the exclusion of job-to-job transitions from the data.
using CPS employment data. Clearly, unemployment durations for job losers first-order stochastically dominate those for job leavers.

![Unemployment Duration Distribution](image)

**Figure 7:** Unemployment Duration Distribution—Job Leavers vs. Job Losers (Source: CPS)

One might worry that the unemployment duration distribution for quits is skewed by those who quit with future employment already in place (and thus enter unemployment only for a brief time). Two factors address this concern. First, the duration of unemployment for fires first-order stochastically dominates the duration for quits even when both distributions are truncated below at various dividing points between 1 and 25 weeks of unemployment. Job leavers appear to have better reemployment prospects than job losers even among those who have already been unemployed for some time, so unemployed workers with future jobs in place cannot explain these distributional differences alone. Second, the data generating these distributions include only workers who were unemployed during the monthly census sampling date, so most workers who quit with another job already in place would not have remained unemployed long enough to enter these data.\(^{32}\)

\(^{32}\)Another concern about the implications of this pattern is the possibility that, relative to fired workers, quitting workers disproportionately leave the unemployment pool by leaving the labor force, rather than by actually finding new jobs. Even if this is the case, though, unless there is positive selection among those who leave the labor force, this
The superior reemployment prospects of job leavers suggest that these workers are more desirable to employers—that they are of better "quality." Given this, the shift toward job losers during recessions could lower the quality of those entering unemployment. Of course, this quality depends not only on the relative shares of job losers and job leavers, but also on the qualities of these two groups. For instance, a shift from job leavers to job losers may not lower the quality of those entering unemployment if it is accompanied by a drastic rise in the quality of fired workers. In what follows, I document a third pattern that addresses such concerns.

1.6.3 Connecting Flows of Firings/Quits to Worker Quality

Here, I present direct evidence that shifts from job leavers toward job losers result in lower quality workers entering the unemployment pool. For the reader’s understanding, the main findings are preceded by an explanation of the empirical approach used. This explanation—below—provides general intuition, followed by the formal empirical framework. Omitted details of this set-up (and of the results that follow) can be found in Appendix C.

Intuition

Our goal is to determine empirically how the quality of workers entering unemployment changes when firings increase and quits decrease. For the purposes of our analysis, we will separate the

will not lower the perceived quality to firms of this pool of unemployed workers. In reality, there are many reasons to believe there is negative selection among those who leave the labor force (e.g. - if those who exit the labor force have received the most negative signals in the job market thus far, or if the perseverance required to continue job search also lends itself to performance on the job). If this selection is negative, then from an employer’s perspective, this selection is actually improving the pool of job applicants who quit previous jobs. As such, this would be consistent with firms preferring workers who quit previous jobs to those who were fired.
unemployment pool into two groups according to durations of unemployment. These groups will be divided at duration $T \in \mathbb{R}_+$—workers who have been unemployed for durations $\tau \in [0, T)$ are defined as STU, while those unemployed for durations $\tau \in [T, \infty)$ are LTU. For clarity, this division is represented in Figure 8 below.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{unemployment_duration.png}
\caption{Unemployment Durations: Short- and Long-term Unemployed}
\end{figure}

To assess the quality of workers entering unemployment, we first must recognize that flows to unemployment are grouped initially with the STU. If workers entering unemployment fall in quality, this decrease in quality should first affect the STU; the LTU should see no change in quality until these workers have been unemployed for long enough to be classified in this group. Thus, the short-term unemployed should initially worsen relative to the long-term unemployed. Because hiring is selective, the reemployment probability of a given group correlates with the quality of that group. Therefore, the reemployment probabilities of STU should also worsen relative to those of LTU.

Given this reasoning, if increases in firings and decreases in quits lower the quality of flows

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33In the empirical analysis that follows, I divide these duration groups between 12 and 13 weeks $\approx 3$ months; individuals with unemployment durations of 1-12 weeks are STU, while those with durations of at least 13 weeks are LTU. The empirical results are substantively unchanged when this cutoff unemployment duration is varied.

34This assumes that the quality of flows to unemployment at time $t$ does not correlate perfectly with the quality of flows at time $t - T$. In other words, a drop in quality of new flows into the STU must not be perfectly canceled by a simultaneous drop in the quality of flows from the STU to the LTU.
to unemployment, then these changes will decrease the reemployment probabilities of STU relative to those of LTU.\textsuperscript{35} Using this, we can test whether shifts from firings toward quits lower the unemployment pool’s quality.

Note that this result is informative specifically because it predicts different effects across durations of unemployment. For example, if we found that shifts from quits to firings lowered reemployment probabilities for the unemployment pool as a whole, this might simply reflect the fact that these shifts occur during periods when hiring decreases.\textsuperscript{36} This would not, however, explain why the job-finding probabilities of LTU improve relative to those of STU.

**Empirical Framework**

Empirically, we want to characterize how the reemployment probabilities of the short- and long-term unemployed vary in response to changes in the fires/ quits composition of flows to unemployment. Toward formalizing this, define $H_t^S$ to be the probability that STU in period $t$ are reemployed by period $t + 1$, and define $H_t^L$ to be the corresponding probability for LTU. In turn, let $H_t$ represent this probability for the set of all unemployed across both groups. Further, define $Q_t$ to be the number of quits at time $t$, scaled by the size of the unemployment pool, and define $F_t$ to be the corresponding scaled value for firings at time $t$. Thus, these can be written as

$$H_t^S = \frac{\text{# of STU in period } t \text{ hired by period } t + 1}{\text{# of STU in period } t}$$

$$H_t^L = \frac{\text{# of LTU in period } t \text{ hired by period } t + 1}{\text{# of LTU in period } t}$$

$$Q_t = \frac{\text{# of quits in period } t}{\text{# of unemployed in period } t}$$

\textsuperscript{35}Using a mechanical model of the unemployment pool’s evolution, it is straightforward to show formally that this prediction holds under very weak assumptions.

\textsuperscript{36}In the empirical analysis to follow, we also control for changes in hiring intensity.
In terms of our newly established notation, our aim is to determine how $H^S_t$ and $H^L_t$ respond to $Q_t$ and $F_t$. We can characterize these relationships in terms of four elasticities: 

$$\frac{\partial \ln(H^S_t)}{\partial \ln(F_{t-1})}, \frac{\partial \ln(H^L_t)}{\partial \ln(F_{t-1})}, \frac{\partial \ln(H^S_t)}{\partial \ln(Q_{t-1})}, \text{ and } \frac{\partial \ln(H^L_t)}{\partial \ln(Q_{t-1})}.$$ 

Time lags are included between firings/quits and hiring because, as will be explained below and in Appendix C, hiring outcomes are drawn from monthly CPS data. These lags ensure that flows to unemployment are counted among the STU when determining reemployment probabilities. An individual’s first appearance in unemployment (in the data) might correspond—in reality—to his first or second week of joblessness. In such cases, high quality unemployed individuals have little time to distinguish themselves by reclaiming employment, so the effects of changes in the quality of flows to unemployment might be muted empirically. Because I group individuals among the STU during their first three months of joblessness, we can better detect changes in quality by introducing 1-2 months of lag to this estimation.

Then we can write the prediction that increases in firings should lower the reemployment probabilities of STU relative to those of LTU as

$$\frac{\partial \ln(H^S_t)}{\partial \ln(F_{t-1})} - \frac{\partial \ln(H^L_t)}{\partial \ln(F_{t-1})} < 0$$

and we can write the prediction that increases in quits should have the opposite effect as

$$\frac{\partial \ln(H^S_t)}{\partial \ln(Q_{t-1})} - \frac{\partial \ln(H^L_t)}{\partial \ln(Q_{t-1})} > 0$$

**Data and Results**

Here, I will briefly describe the data used to assess these predictions—for a complete description of the data and empirical methodology (including the precise specifications estimated), see Appendix C. Hiring outcomes of the unemployed are drawn from individual-level CPS monthly
employment data. For robustness of the results, flows of firings and quits come from two independent sources: (1) I calculate these values directly using the monthly CPS data and (2) I use monthly JOLTS aggregate data to obtain a second set of these values.

Both sources are imperfect with regard to this specific analysis: JOLTS data reflect firm reports of labor turnover, and these figures may include job-to-job transitions, which do not involve the unemployment pool itself. If the intensity of these transitions fluctuates more or less than the inflows to and outflows from unemployment, then this may be a noisy (or even biased) representation of unemployment flows. In contrast, the CPS surveys individual workers, so it allows exclusion of those who never enter unemployment. Unfortunately, we can detect those in the unemployment pool only if they are in this pool at the time of the monthly survey, so the CPS may underestimate the relevant flows. To deal with these concerns, I obtain the results that follow using each of these data sources separately.

The sample used is restricted to men with no more than a high school education—this is merely to focus on the population groups where the mechanism has the most consistent effects. The results of interest remain when women and other education groups are included, but the corresponding estimates are less precise.

Specifically, women are excluded to avoid the complications caused by weak labor market attachment (such as a greater willingness to respond to adverse shocks by substituting effort from the labor market toward family investment). In turn, those without higher education faced the steep-

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37 I measure flows of firings and quits by totaling unemployed job leavers and losers with unemployment durations in the range of 1-4 weeks. I calculate the intensity of hiring as the fraction of unemployed workers who are successfully linked to the next month who gain employment in this period. (The fraction of the sample unable to be matched was extremely small—see Appendix C and Rothstein (2011) for more detailed discussions of the process of matching consecutive months in the CPS).
est increases in unemployment incidence during the recession, so the causes of this group’s rising unemployment are crucial to understanding aggregate employment dynamics. Further—regarding the mechanism suggested in this paper—education is a tool for signaling competence to prospective employers, so those with lower educational attainment may be less able to distinguish themselves from the unemployment pool. Therefore, increases in targeted firings may impact this group’s reemployment probabilities more severely than others. Consistent with this, the unemployment rate among those without a high school diploma is more than twice as great as that among those with greater educational attainment. Additionally, during the recession, this unemployment rate for non-high school graduates increased more than twice as much as that for higher educational attainment groups.

Because firings and quits are central to this compositional change mechanism, the sample used includes only job losers and job leavers. However, the additional inclusion of re-entrants and new entrants to the labor force has no discernable effects on the results.

Using the data described above, Table 1 confirms the aforementioned predictions:

<table>
<thead>
<tr>
<th>Source of unemployment</th>
<th>(i)</th>
<th>(ii)</th>
<th>(iii)</th>
<th>(iv)</th>
</tr>
</thead>
<tbody>
<tr>
<td>CPS</td>
<td>0.056</td>
<td>-0.071</td>
<td>-0.062</td>
<td>-0.077</td>
</tr>
<tr>
<td>JOLTS</td>
<td>0.072</td>
<td>0.097</td>
<td>0.079</td>
<td>0.102</td>
</tr>
<tr>
<td>CPS</td>
<td>0.137</td>
<td>0.268</td>
<td>0.141</td>
<td>0.233</td>
</tr>
<tr>
<td>JOLTS</td>
<td>0.051</td>
<td>0.183</td>
<td>0.052</td>
<td>0.184</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Probability under null</th>
<th>0.007</th>
<th>0.041</th>
<th>0.005</th>
<th>0.069</th>
<th>0.009</th>
<th>0.048</th>
<th>0.022</th>
<th>0.133</th>
</tr>
</thead>
</table>

Control for $\ln(1/H_1)$?
Y Y N N Y Y Y Y
Use sampling weights?
Y Y Y Y N N Y Y
Condition on individual observables?
Y Y Y Y Y Y N N

N 54,059 54,059 54,059 54,059 54,059 54,059 54,059 54,059
Clustered standard errors (by year-month) in parentheses
Data used above include unemployed men with no education beyond a HS diploma over the period Jan 2001 to Aug 2008

43
In all specifications, increases in firings worsen the reemployment probabilities of the STU relative to those of LTU. Similarly, increases in quits have the opposite effect in all specifications. For detailed descriptions of how these specifications differ, see Appendix C.

Additionally, the highlighted section of the table displays the results of tests of the aforementioned predictions. For simplicity, these predictions have been summarized into a single hypothesis test regarding the effect of shifts from quits toward firings. Specifically, we test the hypothesis that these shifts improve the relative reemployment probabilities of the STU:

\[
\frac{\partial \ln (H_{ST})}{\partial \ln (Q_{t-1})} - \frac{\partial \ln (H_{LT})}{\partial \ln (Q_{t-1})} > \frac{\partial \ln (H_{ST})}{\partial \ln (F_{t-1})} - \frac{\partial \ln (H_{LT})}{\partial \ln (F_{t-1})}
\]

against the null that

\[
\frac{\partial \ln (H_{ST})}{\partial \ln (Q_{t-1})} - \frac{\partial \ln (H_{LT})}{\partial \ln (Q_{t-1})} = \frac{\partial \ln (H_{ST})}{\partial \ln (F_{t-1})} - \frac{\partial \ln (H_{LT})}{\partial \ln (F_{t-1})}
\]

There is consistent support for rejecting this null hypothesis across specifications and data sources—we can reject the null at the 5% level (or lower) in all but one of the cases displayed. In the one exception to this (specification IV using JOLTS data), the estimated effects of firings and quits are still consistent with the given predictions. The higher p-value is obtained because individual-level covariates are not used, so the resulting estimates are less precise.

Thus, we have robust evidence that shifts from quits to firings result in lower quality workers entering unemployment.

**Implications for Recessions**

The analysis thus far was intended to detect the compositional changes that accompany moderate economic fluctuations. However, the theoretical analysis in the following sections characterizes the consequences of these changes in response to a significant economic downturn, so it will be
useful to see whether these results extend to recessions. For this purpose, we now investigate the
evolution of differences between the hiring outcomes of STU and LTU during the recent recession.\textsuperscript{38}

Note that the data used in Table 1 are restricted to the period January 2001 - August 2008. For September 2008 -
August 2011, the corresponding estimates are similar, but magnified. This is because changes in the flows of firings and quits are serially correlated during sustained economic fluctuations—during a downturn, firings will rise and quits will fall in consecutive months. Because I group individuals among the short-term unemployed during their first 3 months of joblessness, these sustained shifts will compound the effects of several months of compositional changes, and estimated reemployment disparities between the short- and long-term unemployed will be larger. In this sense, the estimates in Table 1 are weakened by the short-term unemployed who remain from the previous month; the effects are generated only by those new to the unemployment pool.

Corresponding estimates for the periods January 2001 - August 2011 and September 2008 - August 2011 are given in Table 2 (in Appendix C).\textsuperscript{39} The logic given above suggests that the estimated effects should be stronger in the full sample than in the pre-August 2008 data. In turn, these estimated effects should be stronger still in the post-August 2008 sample. This is precisely what we observe; for the full time period, we estimate $\frac{\partial \ln (H_{ST})}{\partial \ln (F_{t-1})} - \frac{\partial \ln (H_{LT})}{\partial \ln (F_{t-1})}$ to be -0.101 and $\frac{\partial \ln (H_{ST})}{\partial \ln (Q_{t-1})} - \frac{\partial \ln (H_{LT})}{\partial \ln (Q_{t-1})}$ to be 0.811. For the September 2008 - August 2011 period, these estimates grow in magnitude to -0.471 and 1.242.

These time periods are omitted from Table 1 for precision—because the serial correlation in

\textsuperscript{38}To study this recession, we must restrict ourselves to a smaller, noisier data sample. As such, this section focuses on characterizing patterns over time and checking that these are consistent with the mechanism’s predictions (rather than formal hypothesis testing).

\textsuperscript{39}Having already established consistency between the CPS-based and JOLTS-based estimates for the earlier sample, I report the (more precise) estimates using CPS-based firings/quits in this case.
flows is not present in all months of the data, the measured effects will vary across months. As a result, estimates obtained using data for the entire time period will be noisier than those in Table 1.40 Accordingly, the (more precise) estimates in Table 1 can be viewed as lower bounds for the effects of compositional changes on hiring outcomes.

For additional evidence, note that there was a sustained shift toward firings during late 2008 and early 2009; the model developed in this section suggests that the reemployment probabilities of the LTU should have risen significantly in comparison to those of the STU. This relative improvement would be brief—the lower quality workers entering unemployment would initially lower the relative outcomes of STU, but they would become LTU after 13 weeks and would depress the outcomes of this group thereafter. Figure 9 below displays the probabilistic hiring advantage of STU over six month intervals through the recession (the monthly flows of firings and quits appear as well).

![Figure 9](image)

**Figure 9:** Hiring likelihood advantage of STU through the recession (Sources: CPS, JOLTS)

It is clear that this sustained shift toward firings was accompanied by a sharp, relative decline

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40 One might worry about similar problems being caused by the firings/quits dynamics around the 2001 recession. The findings in both Table 1 and Table 2 (where applicable) are robust to the exclusion of this time period. The results persist—both in magnitude and in precision—for arbitrary sample start dates in the 2000-2004 range.
in reemployment likelihood for STU. Indeed, this decline had disappeared by the following six-month interval (and many of those who lost jobs during the surge of firings had joined the group of LTU by the following six-month period). Further, this disparity is statistically robust. During the 6 month interval when the advantage of STU is smallest (14.3%, which is 3.5% to 6% below the corresponding advantages for other intervals shown), the standard error of this advantage is 0.771%, so it is unlikely that this decline can be explained by empirical noise.

1.7 Further Discussion

1.7.1 Robustness of Section 5 Results

"Weakening" the Results

Propositions 3 and 4 show that negative economic shocks can stop hiring completely both during and after recessions. In reality, hiring does not halt completely at such times; it merely slows (although it can slow significantly). The model presented is not intended to be quantitatively precise,\textsuperscript{41} but the severity of the above results should provide more reason, not less, to take the analysis seriously. If these forces can weaken employment so greatly in the model, then even a small analog of this mechanism in reality may play a large role in the interplay between economic fluctuations and labor market dynamics.

Alternate Structure of Shocks

The previous section represented a recession as an unanticipated, discrete, one-time shock to productivity (which can be either permanent or transitory). This structure is used for intuitive and

\textsuperscript{41}At the cost of analytical tractability, we could generate a decrease (rather than a halt) in hiring by introducing firm heterogeneity. This complication adds little to the current analysis, so it is omitted.
analytic simplicity, but the results can be shown to hold when shocks take different forms. Firstly, the shocks need not be discrete—continuous declines in productivity, in which $Y$ falls proportionally to $d\ell$ at each time increment $d\ell$, are sufficient to stop hiring in this model.

In addition, the shocks need not be exogenous, unanticipated changes in the model. At some cost of analytical tractibility, this labor market can be modeled in an environment where productivity varies according to general Markov transitions. The model’s dynamic predictions apply to this setting as well.

**Match-Specific Productivity**

A potential criticism of the model concerns the productivity of type $L$ workers. Because employment imposes a flow cost of at least $\bar{w}$ on the firm, it is inefficient for type $L$ workers to be employed. As discussed in Section 2, $\bar{w}$ need not represent the cost of effort, so this inefficiency may not be the fault of type $L$ workers. Independently of the interpretation of $\bar{w}$, however, the results of Section 4 will persist even if match-specific productivity is incorporated and type $L$ workers can be profitably employed.

Suppose the present model is modified so that type $H$ workers are productive at a proportion $\mu_H$ of firms, while type $L$ workers are productive at a proportion $\mu_L < \mu_H$ of firms (choosing $\mu_H = 1$ and $\mu_L = 0$ yields our original model). If individual worker productivity is i.i.d. across firms, the previous results will extend to this setting. Yet, type $L$ workers are no longer inherently unemployable. If firms can observe not only workers’ types, but also their firm-specific productivities, then all workers can be profitably employed.

**More General Structures for Information and Learning**

48
The employment dynamics predicted in Section 4 will be substantively unchanged in a variety of alternative informational structures. For the present study, I have assumed that information arrives in the form of positive, perfectly informative (perfect good news) signals. The analysis will be extremely similar if these signals are imperfect. Departing from Poisson learning, the results will extend to the case where firms learn about worker output based on Brownian motion with unknown drift $\mu_\theta \in \{\mu_H, \mu_L\}$ (as in Bolton and Harris, 1999). Predictions will also be similar if instead $\mu_\theta$ can take on a continuum of values and firm priors are normally distributed (as in Jovanovic, 1979). In each of these cases, firms fire their least productive workers. Firing standards rise during recessions, and the resulting increase in firing causes a decline in unemployment pool quality.

1.7.2 Dynamic Equilibria: Aggregate Fluctuations without Shocks

The forces that stabilize the employment level—discussed in Section 3—will help preserve equilibrium even if the labor market is not at a steady-state. Consider selective hiring, which lowers the unemployment pool’s quality. Firms will refuse to hire if this quality is too low, so free entry dictates how much hiring the labor market can sustain. In a steady-state, hiring is fixed at an intensity to preserve the unemployment pool’s quality, but free entry and hiring can also be used to sustain equilibria in which employment varies. In these cases, the aggregate intensity of hiring

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42 In the spirit of Milgrom (1981) and the information economics literature more generally, consider a signal to be "good news" if it indicates high quality and "bad news" if it indicates low quality. The good or bad news is "perfect" if it fully reveals quality and "imperfect" otherwise.

43 These results may be weakened under bad news learning. If the recession-induced shock is small, firms will fire their workers only if they have beliefs sufficiently close to the steady-state termination threshold. With imperfect bad news signals, downward updating is discrete, so it is possible that no firms will have beliefs in this range. Under perfect bad news learning, each signal will result in termination, so no workers will be employed below the initial post-hiring belief.

44 In this setting, the termination threshold will depend on employment duration.
adjusts to balance its own effects on the unemployment pool with those of firings and quits. When firings intensify, hirings must decrease to prevent a fall in quality; conversely, when firings slow, hirings must increase to prevent a rise in quality.

This dynamic rebalancing can generate lasting fluctuations. Consider a simple example—suppose that the intensity of firing is high at time $t$, which implies that the time $t$ hiring intensity must be low. Recall from Section 3 that, because of the learning process, the firing intensity at time $t$ depends on the hiring intensity at $t - t^*$ (when the fired workers were hired). At time $t + t^*$, then, there will be limited firing and much hiring, and the pattern will continue. Thus, the labor market can sustain equilibria with evolving employment levels; some of these equilibria are cyclical. Additionally, output itself will vary, rising and falling with employment. I discuss this in more detail in Appendix B.

The permanence of this cyclical pattern is a consequence of the Poisson learning structure and its fixed time-to-firing. Other settings might rule out these perpetual cycles, but the short-term effects of hiring and firing fluctuations would persist under more general information structures. In fact, the intuition for permanent cycles extends not only to these general short-term effects, but also to actual labor market applications.

To see this, consider a labor market in which firms learn about worker productivity during employment. In this labor market, suppose that there are no variations in demand or worker productivity, but that the aggregate intensities of hiring and firing can vary. Take these variations

45If learning were based on Brownian motion, for instance, firm firing threshold beliefs would be reached at a continuous distribution of employment durations (in contrast to the pre-firing delay $t^*$ we saw here). As a result, the impact of a hiring intensity change on firings would not occur at a fixed future time, but it would instead be spread over a distribution of future times. This effect would therefore be muted at each time in the distribution. Fluctuations would dissipate over time, but not immediately. Though smaller, short-term effects would remain.
as given. In periods with more aggregate hiring, firms accumulate new workers, and these workers have more uncertain productivity. Not all of this uncertainty will be resolved at a single moment in the future (as in the good news learning structure modeled here), but it will all be resolved eventually. Thus, it is reasonable that aggregate hiring increases can lead to future firing increases. In turn, periods with more aggregate firing involve the (negative) resolution of uncertainty. Such periods will generate a lower quality, more negatively-selected unemployment pool. Firms then face lower expected returns to hiring workers, so hiring slows.

As this application illustrates, these dynamic equilibria provide two insights beyond the recessions mechanism highlighted throughout the paper:

(1) Independently of the state of the economy, there must be an aggregate balance in firm restructuring decisions that impact the labor market. This market can support only so much combined hiring and firing at once. For example, a sector in which many firms are firing unproductive workers may experience a lull in hiring due exclusively to the changing composition of the unemployment pool. This suggests that there can be aggregate fluctuations in employment and output even in the absence of economic shocks (to demand, to production costs, etc.).

(2) Further, fluctuations in hiring and firing can contribute to future fluctuations. Because these fluctuations change the aggregate productive uncertainty that firms face, their most significant consequences may be realized well-beyond the immediate future. Thus, serial correlation may be unable to characterize the dependence of these fluctuations across time.
1.7.3 Extensions

For theoretical applications beyond the scope of this paper, the framework developed here should be viewed merely as a starting point for modeling labor markets. The model can be enriched in a number of ways while remaining analytically tractable. I discuss two simple examples below.

Endogenous Quitting

The simplifying assumption of a constant quit rate $\pi$ is standard in many models of labor markets. In addition, though, I use this structure because it weakens my results. If worker quit rates varied in the model as they do in the data, far fewer workers would quit during recessions. The unemployment pool’s quality would fall even more than the model predicts, and the duration of jobless recoveries would be magnified. In this sense, the constant quit rate assumption demonstrates the robustness of the model’s predictions.

Of course, we may want to use this framework more generally to study how learning and private information impact the labor market equilibrium. In this case, it may be useful to endogenize quits. This can be done in many ways; a simple extension to the current model would have workers receive (still at intensity $\pi$) a stochastic cost, $x$, of continued employment at the current firm. If $x$ is always infinite, this is equivalent to the current model. More generally, $x$ could be distributed according to the c.d.f. $F(x)$ over $\mathbb{R}$. We might even want this cost to correlate with worker types, so that job leavers are selected (positively or negatively) on productivity.

For simplicity, suppose these costs are observed jointly by the worker and firm. Then costs greater than the firm’s remaining surplus would result in the worker quitting. In turn, realized costs smaller than this surplus would prompt wage renegotiation.\textsuperscript{46} Additionally, such wage renegotiation

\textsuperscript{46} Wage decreases can occur if $x$ is allowed to take on negative values.
would change the firm’s firing standard. Workers with higher wages would face more frequent dismissals; Schmieder and von Wachter (2010) document this pattern precisely.

To take quitting seriously, we would obviously require a more developed and better-justified theory of job leaving decisions. Such a theory could be incredibly powerful when embedded in the dynamic equilibrium framework developed in this paper. The example above demonstrates this power—this simple addition to the model impacts both the evolution of wages and the relationship between these wages and job turnover. Incorporating a more thorough treatment of quitting could yield a variety of more nuanced, empirically-testable predictions.

**Poaching and Job-to-Job Transitions**

This paper’s analysis has focused on unemployment, so we have assumed throughout that firms can hire only from the unemployment pool. This may be a reasonable approximation for the low-wage, low-human capital jobs that are responsible for much of the recent surge in unemployment. However, we can incorporate job-to-job transitions in this model and demonstrate how these change equilibrium labor market conditions.

Consider a simple modification to the model allowing firms to hire currently employed workers at cost \( c_E \). In attempting to hire these workers, firms cannot observe the worker’s output at her current job—this remains the private information of her current employer. The "poaching" firm can, however, view the worker’s duration of employment at her current firm, and all firms are identical, so equilibrium hiring/firing behavior is commonly known. In a steady-state equilibrium,

\[ 47 \text{ An interesting case is that in which only the worker can observe this cost. The worker would have an incentive to overreport this cost, but this incentive would be tempered by the reduced time to firing that accompanies higher wages. If some overreporting occurs in equilibrium, it will come disproportionately from type } H \text{ workers, who are more confident that they will be retained at the firing threshold.} \]
tenure at the current firm will be informative about worker types.

Only type H workers will be retained after a duration $t_{ss}^*$ of employment, so hires from this group are preferred to those from the unemployment pool. The current employer must therefore raise wages to defend these workers from the bidding of outside firms. Free entry drives the offers of these firms, so they will offer wages at which they expect value 0 from the type H workers. Further, only outside firms face the poaching cost $c_E$, so the current employer will retain precisely this expected value after raising wages to defend this worker from other firms. The current employer can always defend successfully, and no poaching will occur in equilibrium. We can use this setting to explore this "testing period" and its impact on wage dynamics.

Also note that this pooled-hiring equilibrium cannot exist unless each worker cannot retain private information about her type when reentering the unemployment pool. To see this, consider two unemployed workers—one knows she is type H and another knows she is type L. The type H worker would be willing to accept a lower initial wage because of the higher wages she might get by surviving through duration $t_{ss}^*$. The type L worker expects to be fired at time $t_{ss}^*$ and is thus unwilling to accept this lower wage. The employer may therefore be able to separate worker types even without being able to commit to an incentive contract—implicit incentives are provided by the wage response to poaching.

1.8 Conclusions

In this study, I have considered a new mechanism, in which changes in the quality of those entering unemployment can generate both a long post-recession period with limited hiring and large numbers of individuals reaching long-term unemployment. If employers have private information about worker ability, then periods in which many firms make firing decisions will involve many low-
ability workers entering the unemployment pool. Of course, previous research argued that these low-ability workers should still be better than those fired at other times (because firing standards are higher during recessions). What these studies have overlooked is the other impact of increasing firing standards—a significant increase in the number of workers fired. Workers fired before the recession were balanced in the unemployment pool by workers voluntarily quitting jobs, and these quitting workers need not have disproportionately low ability. During the recession, this balance was lost, and the unemployment pool may have worsened as a result. If true, this would at least partly explain the continued hesitancy of firms to hire, and this could grease the path to long-term unemployment for these disproportionately low-quality workers who would struggle to find work regardless.

To assess the impact of this compositional change on employment, I have provided an empirical strategy for detecting the role of changing flows to unemployment in firm hiring decisions. I have implemented this using CPS monthly employment data, and I have provided evidence linking these compositional changes to hiring decisions. The empirical patterns provided cannot be explained by human capital depreciation, negative selection of LTU, or other common explanations of persistent long-term unemployment, so this constitutes evidence that the compositional change mechanism impacts hiring independently of these other factors. As such, more investigation is warranted regarding the role of these changing flows on employment dynamics.

Building on this empirical motivation, I have formalized this mechanism in a dynamic framework that integrates employer learning and private information into a labor market equilibrium. The framework itself is a tool that merits further development and application. As I have demonstrated the danger of drawing conclusions about dynamic environments from comparisons of static models, the availability of a tractable dynamic framework for analysis of the labor market is an opportunity
to investigate whether other analyses of this setting have been flawed.

In my present analysis, I show that an economic shock which raises firing standards can not only generate significant unemployment, but also discourage firms from hiring for a sustained period of time. The conditions amplify each other, and this will be worsened significantly if the stagnant labor market discourages workers who want to quit their jobs from doing so. (In generating the results, I have assumed that quits will continue regardless of labor market conditions, so I may be understating the potential impact of this mechanism). The severity of the results suggests that, even if this mechanism plays a small role in actual labor markets, its impact may be large during economic downturns.

In addition to this, I show that the stagnant labor market can persist even after the economy has otherwise recovered from the shock, so this may offer insight regarding the "jobless recoveries" that have followed the past several recessions. Further, I show that the model does not even require a shock to generate employment/output fluctuations. In hiring and firing, firms worsen the labor unemployment pool faced by other firms, and an increase in firing decreases the amount of hiring the economy can support. Hiring and firing have opposing effects on aggregate output, so even in the absence of external shocks to the labor market, variation in the relative intensities of these flows can generate volatility throughout the economy.

These empirical and theoretical results offer strong motivation for future work investigating the role of this mechanism in recessions and in labor markets more generally.
APPENDIX A: Proofs of Theoretical Results

Lemma 1: In any nontrivial steady-state employment equilibrium, \( w = \bar{w} \).

Proof: Obviously, we need only consider cases in which \( E_{ss} \in (0, 1) \). Clearly, no one would work at lower wages, so \( w \geq \bar{w} \). Suppose, then, that firms offer equilibrium wages \( w > \bar{w} \). There is positive unemployment, so the unemployment pool is of sufficiently low quality that firms are unwilling to hire more at the wage \( w \). These unemployed workers, however, could benefit by undercutting the market wage and accepting some offer \( w' \in (\bar{w}, w) \), so this cannot be an equilibrium, and we have the result. ■

In addition, we can show that the above result implies a constant hiring intensity and constant qualities of the employed and unemployed.

Corollary: In any nontrivial steady-state employment equilibrium, \( \eta_t = \eta_{ss}, q_{E,t} = q_{E,ss}, \) and \( q_{U,t} = q_{U,ss}, \forall t \).

Proof: Because \( E_{ss} \in (0, 1) \), free entry implies that \( V(q_H(q_{U,t})) \leq c \) at the wage \( \bar{w} \). For \( \pi > 0 \) (assumed), there must be hiring in equilibrium—otherwise the employment level would decline due to the flow of quitting workers. Given that there is hiring, we can establish \( V(q_H(q_{U,t})) \geq c \) also, so \( V(q_H(q_{U,t})) = c \). Since \( w = \bar{w} \) and \( V(q_H(q_{U,t})) = c \), we know that \( q_{U,t} \) must be fixed at \( q_{U,ss} \), \( \forall t \). In turn, the accounting identity \( P = E_{ss} q_{E,t} + (1 - E_{ss}) q_{U,ss} \) tells us that \( q_{E,t} = q_{E,ss} \) must also be fixed.

Since \( w \) is fixed, \( p_{ss}^* = \frac{\bar{w}(r+\pi)}{\lambda[Y(r+\pi+\lambda)-\bar{w}]} \) must also be fixed. The constant values of \( p_{ss}^* \) and \( q_{U,ss} \) together require \( t_{ss}^* = \frac{1}{\lambda} \ln \left[ \frac{1-p_{ss}^*}{p_{ss}^*} \left( \frac{q_{U,ss}}{\alpha(1-q_{U,ss})} \right) \right] \) to be fixed as well. By solving for the hiring intensity to equate inflows and outflows to unemployment (or simply of type H workers), we verify that constant values of \( E_{ss}, q_{U,ss} \), and \( t_{ss}^* \) imply a constant hiring intensity \( \eta_{ss} \). ■

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We now proceed with the existence/uniqueness result. First, define $\bar{c} = V\left(\frac{P}{\pi - (1 - P)\alpha}\right)$ to be the maximum hiring cost at which there can be any employment in equilibrium. (Selection into employment must always be positive, so we must always have $q_{U, ss} < P$. With zero employment, $q_{U, ss} = P$.)

**Proposition A.1:** For any combination of parameters $\{Y, \bar{w}, \lambda, \pi, r, P, \alpha\}$, $\exists \bar{c} \in (0, \bar{c})$ such that for hiring costs $c \in (\underline{c}, \bar{c})$, there exists a unique nontrivial steady-state employment equilibrium.

**Proof:** Consider first the conditions this equilibrium must satisfy. Firm optimality (value-matching, smooth-pasting) provides us with our termination belief $p^*_{ss} = \frac{\bar{w}(r + \pi)}{\lambda Y (r + \pi + \lambda - \bar{w})}$ and the firm value function. Free entry must hold with equality, so the combination of free entry and the firm optimality conditions pin down the firm belief at which it can recover exactly the hiring cost $c$. Let us define $q^*_H$ to be this belief, so this must satisfy $V(q^*_H) = c$ and $q^*_H = \frac{q_{U, ss}}{q_{U, ss} + \alpha (1 - q_{U, ss})}$.

Further, the constant employment level (along with constant qualities in employment and unemployment) means that the steady-state hiring intensity must satisfy two conditions: (1) inflows to and outflows from unemployment must be equal and (2) type $H$ workers among inflows to and outflows from unemployment must be equal. Condition (1) requires

$$\eta_{ss} dt = \eta_{ss} dt \left[ q_H^* e^{-(\pi + \lambda)t^*_{ss}} + (1 - q_H^*) e^{-\pi t^*_{ss}} \right] + \pi E_{ss} dt$$

while condition (2) requires

$$\eta_{ss} q_H^* dt = \eta_{ss} q_H^* dt e^{-(\pi + \lambda)t^*_{ss}} + \pi E_{ss} q_{E, ss} dt$$

By solving each of these equations for $\eta_{ss}$, equating them, and by using the labor force condition that $P = E_{ss} q_{E, ss} + (1 - E_{ss}) q_{U, ss}$ to substitute for $q_{E, ss} = \frac{P - (1 - E_{ss}) q_{U, ss}}{E_{ss}}$, we can obtain an
expression for $E_{ss}$ in terms of $P$, $q_H^*$, $\alpha$, $\pi$, $\lambda$, and $t_{ss}^*$:

$$E_{ss} = \left( \frac{P - q_H^* (\alpha + P (1 - \alpha))}{1 - q_H^*} \right) \left( \frac{1 - e^{-\pi t_{ss}^*} + q_H^* e^{-\pi t_{ss}^*} (1 - e^{-\lambda t_{ss}^*})}{1 - \alpha + e^{-\pi t_{ss}^*} [\alpha - e^{-\lambda t_{ss}^*}]} \right)$$

This equation has used all of the equilibrium conditions, and if this expression yields an employment level $E_{ss} \in (0, 1)$, we have established existence. Toward this end, note that substituting our upper bound on the equilibrium quality of new hires, $\frac{P}{P+1-P\alpha}$, for $q_H^*$ in the expression for $E_{ss}$ above yields an employment level of 0. (Recall that $\pi = \frac{P}{P+1-P\alpha}$ is the upper bound on our desired range of hiring costs.) We thus need to show that $E_{ss}$ is decreasing in $c$ at $c = \bar{c}$ (so that $E_{ss} \in (0, 1)$ for $c$ less than, but sufficiently close to $\bar{c}$).

In $E_{ss}$ above, we can see that changes in $c$ affect the equilibrium employment level through only $q_H^*$ and $t_{ss}^* = \frac{1}{\lambda} \ln \left[ \frac{(1-p^{s}_{ss})}{p^{ss}} \right] (\frac{q_H}{1-q_H})$. Because $p^{s}_{ss}$ does not depend on $c$, substituting for $t_{ss}^*$ will provide us with an expression for $E_{ss}$ which $c$ affects through only $q_H^*$. Making this substitution yields:

$$E_{ss} = \frac{q_H^* (P - q_H^* [\alpha + P (1 - \alpha)])}{(1 - q_H^*)} \left[ (1 - p^{s}_{ss})^{\frac{\pi + \lambda}{\lambda}} (q_H^*)^{\frac{\pi}{\lambda}} - (1 - q_H^*)^{\frac{\pi + \lambda}{\lambda}} (p^{s}_{ss})^{\frac{\pi}{\lambda}} \right]$$

$$\frac{\partial E_{ss}}{\partial q_H^*} \bigg|_{q_H^*=\frac{P}{P+1-P\alpha}} = \frac{-P \left[ (1-p^{s}_{ss})^{\frac{\pi + \lambda}{\lambda}} (q_H^*)^{\frac{\pi}{\lambda}} - (1 - q_H^*)^{\frac{\pi + \lambda}{\lambda}} (p^{s}_{ss})^{\frac{\pi}{\lambda}} \right]}{(1 - q_H^*) \left[ (1 - p^{s}_{ss}) (q_H^*)^{\frac{\pi}{\lambda}} + [p^{s}_{ss} (1 - q_H^*)] \frac{\pi}{\lambda} (q_H^*)^{\frac{\pi}{\lambda}} \right] - p^{s}_{ss}}$$

< 0

To check the inequality, notice that the denominator is positive. Next, note that $(1-p^{s}_{ss})^{\frac{\pi}{\lambda}} (q_H^*)^{\frac{\pi}{\lambda}} > (1 - q_H^*)^{\frac{\pi}{\lambda}} (p^{s}_{ss})^{\frac{\pi}{\lambda}}$ whenever $q_H^* > p^{s}_{ss}$. Recall that we have assumed throughout the analysis that
\[ \tilde{w} < \frac{\gamma Y (r + \pi + \lambda) P}{(r + \pi + \lambda P + \alpha (r + \pi) (1 - P)} \] (because otherwise no combination of parameters could allow employment greater than 0 in equilibrium), so indeed \( q_H^s > p_{ss}^s \) must hold for \( q_H^s \) sufficiently close to \( \frac{P}{P + (1 - P)\alpha} \). Thus, the numerator and the entire object are negative, so our equilibrium \( E_{ss} \) is decreasing in \( c \), and a nontrivial steady-state employment equilibrium exists for hiring costs sufficiently close to \( \bar{c} \).

Uniqueness follows immediately; to see this, notice that inflows/outflows condition (1) above implies

\[
\eta_{ss} = \frac{\pi E_{ss}}{1 - q_H^s e^{-(\pi + \lambda)t_{ss}^s} - (1 - q_H^s) e^{-\pi t_{ss}^s}}
\]

while condition (2) implies

\[
\eta_{ss} = \frac{E_{ss} q_{E, ss} \pi}{q_H^s \left[ 1 - e^{-(\lambda + \pi) t_{ss}^s} \right]} = \frac{\pi \left[ P - (1 - E_{ss}) \left( \frac{\alpha q_H^s}{1 - q_H^s (1 - \alpha)} \right) \right]}{q_H^s \left[ 1 - e^{-(\lambda + \pi) t_{ss}^s} \right]}
\]

All parameters but \( E_{ss} \) are determined in these. In response to varying \( E_{ss} \), these expressions for \( \eta_{ss} \) have different slopes and cross only once, so only one hiring intensity can satisfy both conditions. In turn, the equilibrium is unique (if these conditions for \( \eta_{ss} \) are satisfied at an employment level in the \((0, 1)\) range).

**Corollary:** For any combination of parameters \( \{Y, \tilde{w}, \lambda, \pi, r, P, \alpha\} \) satisfying (A1) and (A2), \( \exists c \in (0, \bar{c}) \) such that for hiring costs \( c \in (c, \bar{c}) \), there exists a unique nontrivial steady-state employment equilibrium.

**Proof:** (A2) simply ensures that \( \bar{c} > 0 \), and the result above holds regardless of whether or not (A1) is satisfied. We can thus take the same approach as in establishing Proposition A.1 above to show that \( c \) just below \( \bar{c} \) will yield a unique nontrivial steady-state employment equilibrium.
**Proposition 1:** In a nontrivial steady-state employment equilibrium, the firm’s value function can be written analytically as:

\[
V_{ss}(p_t) = \begin{cases} 
\left( \frac{1}{r + \pi} \right) \left[ \lambda p_t Y - \bar{w} + \lambda \left( \frac{\bar{w}(1 - p_t)}{r + \pi} \right) \left( \frac{\lambda Y - \bar{w} p_t}{r + \pi} \right)^{r + \pi} \right] & \text{for } p_t \in [p^*_ss, 1] \\
0 & \text{for } p_t \leq p^*_ss
\end{cases}
\]

Further, this threshold level is given by \( p^*_ss = \frac{\bar{w}(r + \pi)}{\lambda Y (r + \pi + \lambda) - \bar{w}} \).

**Proof:** The firm’s optimal strategy will incorporate a threshold rule—it will be optimal to retain the worker until the belief \( p_t \) falls to some \( p^* \) at which the outside option offers value equal to that of the current match. At this point, the firm will choose this outside option, which can be written as \( \max \{ V(q_H(q_{U,ss})) - c, 0 \} \). Free entry requires that \( V(q_H(q_{U,ss})) - c < 0 \), so this outside option must always be 0. Additionally, this implies that the threshold \( p^* \) will be fixed at its steady-state level, so we can denote this cutoff by \( p^*_ss \).

We can use the fixed outside option to write the firm’s value function. When the firm’s belief is above \( p^*_ss \) (so that it chooses to remain with its current employee), its value function can be written as

\[
rV(p_t) = [\lambda p_t Y - \bar{w}] - \pi V(p_t) + \lambda p_t [V(1) - V(p_t)] - \lambda p_t (1 - p_t) V'(p_t)
\]

where we have used the standard approximation that

\[
V(p_t - \lambda p_t (1 - p_t) dt) \approx V(p_t) - \lambda p_t (1 - p_t) V'(p_t) dt
\]

and canceled out higher order terms. Further, substituting for the value of employing a type \( H \) worker, \( V(1) = \frac{Y \lambda - \bar{w}}{r + \pi} \), yields the following first-order ODE:

\[
[r + \pi + \lambda p_t] V(p_t) = Y \lambda p_t \left( \frac{r + \pi + \lambda}{r + \pi} \right) - \bar{w} \left( \frac{r + \pi + \lambda p_t}{r + \pi} \right) - V'(p_t) p_t (1 - p_t) \lambda \tag{1.1}
\]

\[48\text{We have already established that } w = \bar{w} \text{ in this equilibrium.}\]
We can use this equation with the value matching condition \( V(p_{ss}^*) = 0 \) and the smooth-pasting condition \( V'(p_{ss}^*) = 0 \) to determine \( p_{ss}^* \) explicitly. We thus obtain

\[
p_{ss}^* = \frac{\bar{w}(r + \pi)}{\lambda[Y(r + \lambda + \pi) - \bar{w}]}
\]

Continuing toward solving the above ODE, a particular solution is the expected value of committing forever to the current employee:

\[
\frac{\lambda p_t Y - \bar{w}}{r + \pi}
\]

To capture the option value of being able to terminate the match, we must look to the solution of the homogeneous part of the ODE, which will have the form \((1 - p_t)^{1+\mu} p_t^{-\mu}\) for some \(\mu\) to be determined. Applying techniques drawn from Bellman and Cooke (1963), Presman (1990), and Keller, Rady, and Cripps (2005), we obtain a solution of the form:

\[
V(p_t) = \frac{\lambda p_t Y - \bar{w}}{r + \pi} + K (1 - p_t)^{\frac{\lambda + r + \pi}{\lambda}} p_t^{-\frac{r + \pi}{\lambda}}
\]

(1.2)

Here, \(K\) is a constant to be determined by our boundary conditions. We obtain \(K\) in terms of the threshold \(p_{ss}^*\) by substituting equation (2) into equation (1) above at the belief \(p_t = p_{ss}^*\). Obviously, value-matching and smooth-pasting must again be satisfied; using these, we obtain

\[
K = \left( \frac{\bar{w} - \lambda p_{ss}^* Y}{r + \pi} \right) \left( \frac{1}{1 - p_{ss}^*} \right) \left[ \left( \frac{p_{ss}^*}{1 - p_{ss}^*} \right) \right]^{\frac{r + \pi}{\lambda}}
\]

From this, we can obtain the end result simply by substituting for \(p_{ss}^* = \frac{\bar{w}(r + \pi)}{\lambda[Y(r + \lambda + \pi) - \bar{w}]}\) and simplifying. □

**Proposition A.2:** In a nontrivial steady-state employment equilibrium:

(i) Without receiving a payoff \(Y\), a firm will wait for time \(t_{ss}^*\) after hiring a worker before firing him, where

\[
t_{ss}^* = \frac{1}{\lambda} \ln \left[ \left( \frac{1 - p_{ss}^*}{p_{ss}^*} \right) \left( \frac{q_{U,ss}}{\alpha (1 - q_{U,ss})} \right) \right]
\]
(ii) Unemployed workers are hired at intensity:

\[ \eta_{ss} = \pi E_{ss} \left[ \frac{q_{U,ss}}{q_{U,ss} [1 - e^{-(\lambda + \pi)t_{ss}^*}]} + 1 - e^{-(\pi + \lambda)t_{ss}^*} \right] \]

(iii) The employment level can be written:

\[ E_{ss} = \left[ \frac{P - q_{U,ss}}{1 - q_{U,ss}} \right] \left[ 1 + \left( \frac{1}{q_{U,ss}} \right) \left( \frac{1 - e^{-(\pi + \lambda)t_{ss}^*}}{1 - e^{-(\lambda + \pi)t_{ss}^*} - \alpha [1 - e^{-(\pi + \lambda)t_{ss}^*}]} \right) \right] \]

Proof (i): If a firm receives no payoff at time \( t \), the updating rules imply that \( \frac{\partial (\ln(p_t))}{\partial t} = -\lambda (1 - p_t) \) and \( \frac{\partial (\ln(1-p_t))}{\partial t} = \lambda p_t \). Thus we can relate these two derivatives by

\[ \frac{\partial (\ln (1-p_t))}{\partial t} = \frac{\partial (\ln (p_t))}{\partial t} + \lambda \]

Integrating both sides from 0 to \( t_{ss}^* \) yields

\[ \ln(p_{t_{ss}^*}) - \ln(p_0) + \lambda t_{ss}^* = \ln(1 - p_{t_{ss}^*}) - \ln(1 - p_0) \]

where \( p_0 \) is the firm’s initial belief about its employee’s type. Note that (a) \( p_{t_{ss}^*} = p_{ss}^* \), (b) \( p_0 \) must equal \( q_H(q_{U,ss}) \) in equilibrium, and (c) \( \frac{q_H(q_{U,ss})}{1 - q_H(q_{U,ss})} = \frac{q_{U,ss}}{\alpha (1 - q_{U,ss})} \), and the result follows. ■

Proof (ii): In equilibrium, the instantaneous flow into employment among the unemployed \( (\eta_{ss} dt) \) must equal the instantaneous flow out of employment among the employed. This flow out of employment at time \( t \) consists of both the measure of workers who quit jobs \( (E_{ss} \pi dt) \) and the measure of workers hired at time \( t - t_{ss}^* \) whose employers received no payoff during that time:

\[ \eta_{ss} dt \left[ (1 - q_H(q_{U,ss})) e^{-\pi t_{ss}^*} + q_H(q_{U,ss}) e^{-(\lambda + \pi)t_{ss}^*} \right] \]

Equating these inflows and outflows yields

\[ \eta_{ss} = \frac{\pi E_{ss}}{1 - e^{-\pi t_{ss}^*} (1 - q_H(q_{U,ss})) - q_H(q_{U,ss}) e^{-(\pi + \lambda)t_{ss}^*}} \]

into which we can substitute \( q_H(q_{U,ss}) \equiv \frac{q_{U,ss}}{q_{U,ss} + \alpha (1 - q_{U,ss})} \) to obtain our desired result. ■
**Proof (iii):** In addition to the expression for $\eta_{ss}$ obtained above (by equating inflows to and outflows from employment), we can also obtain $\eta_{ss}$ by equating type $H$ inflows to and type $H$ outflows from employment (this must also hold in a steady-state employment equilibrium). From this, we obtain

$$\eta_{ss} = \frac{E_{ss} q_{E,ss} \pi}{q_H (q_{U,ss}) \left[ 1 - e^{-(\lambda+\pi)s_{ss}} \right]} \text{ where } q_{E,ss} = \frac{P - (1 - E_{ss}) q_{U,ss}}{E_{ss}}$$

Equating the two expressions for $\eta_{ss}$ and solving for $E_{ss}$ yields the result. ■

**Proposition 2:** $\exists \bar{z} > 0$ such that for $z \in (0, \bar{z})$, the proportion of type $H$ workers in the unemployment pool immediately following the shock ($Y \to Y - z$) falls to

$$q_{U,\tilde{t}} = \frac{(1 - E_{ss}) q_{U,ss} + \eta_{ss} \int_{t_{ss}}^{t_{ss}^*} q_H (q_{U,ss}) e^{-(\pi+\lambda)s} ds}{1 - E_{ss} + \eta_{ss} \int_{t_{ss}}^{t_{ss}^*} [q_H (q_{U,ss}) e^{-(\pi+\lambda)s} + (1 - q_H (q_{U,ss})) e^{-\pi s}] ds} < q_{U,ss}$$

**Proof:** This expression follows from the same intuition given for the mass of firings immediately following the shock (the numerator consists of only the type $H$ workers from this mass).

To show that the inequality holds for sufficiently small $z$, we can substitute for $E_{ss}$, $\eta_{ss}$, $t_{ss}^*$, $t_z$, $p_{ss}^*$, $p_{T}^*$, and $q_H (q_{U,ss})$, and we can evaluate the expression. For small $z$, it is straightforward to see that this inequality will be satisfied if and only if $p_{ss}^* < q_{U,ss}$. This condition is implied by our limit on firms’ ex ante information (A1):

$$\alpha > \left( \frac{r + \pi}{r + \pi + \lambda} \right) \left( \frac{\tilde{w}}{\lambda \bar{Y} - \tilde{w}} \right) \left( 1 - q_{ss}^* \right)$$

$$\implies \alpha > \left( \frac{r + \pi}{r + \pi + \lambda} \right) \left( \frac{\tilde{w}}{\lambda \bar{Y} - \tilde{w}} \right) \left( \frac{\alpha (1 - q_{U,ss})}{q_{U,ss}} \right)$$

$$\implies q_{U,ss} (r + \pi + \lambda) (\lambda \bar{Y} - \tilde{w}) > (r + \pi) \tilde{w} (1 - q_{U,ss})$$

$$\implies q_{U,ss} > \frac{\tilde{w} (r + \pi)}{\lambda ((r + \pi + \lambda) \bar{Y} - \tilde{w})} = p_{ss}^*$$

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so we have the result. ■

**Lemma 2:** $q_U^*(Y)$ is strictly decreasing in $Y$.

**Proof:** Clearly $p_{ss}^* = \frac{\bar{w}(r+\pi)}{\lambda[Y(r+\pi+\pi)-u]}$ is strictly decreasing in $Y$. Note that, for a given $p_t \in [p_{ss}^*, 1]$, $V(p_t)$ is strictly decreasing in the threshold $p_{ss}^*$. In turn, the $q_U^*(Y)$ satisfying $V(q_H(q_U^*(Y))) = c$ is strictly increasing in $p_{ss}^*$ (and $q_H(q)$ is of course increasing in $q$). Thus, $q_U^*(Y)$ must be strictly decreasing in $Y$. ■

**Proposition 3:** After the output shock $Y \rightarrow Y - z$, hiring will cease for the duration $\hat{t}_H > 0$. If $q_U^*(Y - z) < P$, $\hat{t}_H$ is finite.

**Proof:** First, note that if $q_U^*(Y - z) \geq P$, $\hat{t}_H = \infty$ (hiring can never resume with output at $Y - z$). As mentioned following Proposition 3 in the text, there can be no positive selection into unemployment, so $q_{U,t} < P$, $\forall t$. Clearly, then, the random inflows to unemployment from job quitters can—at most—bring the unemployment pool quality asymptotically toward $P$. It can never reach any quality level $q_{U,t} > P$, and it can never reach quality level $q_{U,t} = P$ in finite time.

Because the quality of the unemployment pool falls discretely, hiring standards rise discretely, and the unemployment pool quality evolves continuously after the shock (due to Poisson transitions), it is clear that the duration without hiring must be strictly positive. This can be confirmed and clarified by writing the expression to pin down $\hat{t}_H$ analytically. To do this, consider the flows to unemployment that continue after hiring has stopped. We must track the inflows to unemployment of type $H$ and $L$ workers, and we must use these inflows to continuously update the type $H$ proportion in the unemployment pool. We know that this proportion is below the required hiring level at time $\hat{t}$ and, because transitions are Poisson, we know that this proportion evolves continuously
after \( \hat{t} \). Further, we know that this proportion must surpass any \( q_U^* (Y - z) < P \) in finite time, so \( \hat{t}_H \) will be the smallest positive value such that \( q_{U, \hat{t} + \hat{t}_H} = q_U^* (Y - z) \).

Tracing the evolution of \( q_{U,t} \) from \( t \) to \( \hat{t}_H \) reduces to an accounting exercise. To express this, we must account for workers who:

(i) were already unemployed at \( \hat{t} \)

(ii) were fired immediately after the shock

(iii) were fired upon beliefs reaching the new threshold \( p^*_H \) at some time \( t \in [\hat{t}, \hat{t} + \hat{t}_H] \)

(iv) quit their jobs at some time \( t \in [\hat{t}, \hat{t} + \hat{t}_H] \) while at firms with beliefs in the range \( p \in \left[ p^*_H, q_H \left( q_{U,ss} \right) \right] \)

(v) revealed themselves to be type \( H \) workers before the shock and quit their jobs at some time \( t \in [\hat{t}, \hat{t} + \hat{t}_H] \)

(vi) revealed themselves to be type \( H \) workers after the shock and quit their jobs at some time \( t \in [\hat{t}, \hat{t} + \hat{t}_H] \)

These six groups are represented (in order) in both the numerator and denominator on the right side of the condition below—the numerator includes only type \( H \) workers. \( \hat{t}_H \) is then the smallest positive value that satisfies

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where $m_{H,ss} = \left( \frac{\lambda}{\pi} \right) \eta_{ss} q_H (q_{U,ss}) \int_0^{t^*_z} e^{-(\pi+\lambda)s} ds$ is the steady-state mass of employed workers who have already revealed themselves to be type $H$ and $\hat{t}_F \equiv \min \{ t_z, \hat{t}_H \}$ is the time after the shock during which previously hired workers were being fired at the new standard. Note in (iii) and (iv) that, if all remaining unproductive workers hired at the previous steady-state have been fired before hiring restarts, then $\hat{t}_F = t_z$. If hiring begins while unproductive workers are still reaching this threshold, then $\hat{t}_F = \hat{t}_H$.

To see that this pool will reach any $q_{U,t} < P$ in finite time, first recall that, after the shock,
targeted firings continue to occur at the new belief threshold $p_t^*$, for those workers who were hired in the previous steady state (before the shock). Suppose that we have reached time $\hat{t} + t_z$ (where $t_z$ again represents how long after being hired workers who have not revealed themselves to be type H will be fired after the shock) and that hiring has not yet begun. (Obviously, if hiring begins before this point, we have already reached $q_{U,t} = q_U^*(Y - z)$, so we are done). Then the only remaining employed workers must have provided their employers a payoff, and these must be type H workers. Then of course, the type H proportion among the inflow to unemployment (which comes entirely through voluntary quits) must be 1. Over time, this flow to unemployment will raise the unemployment pool quality asymptotically toward $P$, and by the structure of the Poisson distribution, it must surpass any $q < P$ in finite time. ■

**Proposition 4:** There exists $t_{Y - z, Y}$ such that, after the unanticipated transitory output shock, if the recovery occurs before time $\hat{t} + t_{Y - z, Y}$, then the economy will remain without hiring for a **positive duration of time even after the recovery.** Further, $\exists \tau > 0$ such that for $z \in (0, \tau)$, $t_{Y - z, Y} > 0$ and both the likelihood and expected duration of a jobless recovery are increasing in the magnitude of the shock $z$.

**Proof:** In this case, the explicit value function presented in Section 4 no longer applies due to the added uncertainty about the recovery. Though the environment has changed, this result holds for reasons similar to those in Proposition 3. As before, let $q_{U, \hat{t} + t}$ correspond to the proportion of type H workers in the unemployment pool at time $\hat{t} + t$. Then $t_{Y - z, Y}$ must satisfy $q_{U, \hat{t} + t_{Y - z, Y}} = q_U^*(Y)$.

The intuition for constructing $q_{U, \hat{t} + t_{Y - z, Y}}$ will involve the same six groups used to construct $q_{U, \hat{t} + t_{Y - z}}$ in Proposition 3. In fact, given the same steady-state conditions and post-shock (pre-recovery in this case) waiting time before firing $t_z$ as in Proposition 3, $t_{Y - z, Y}$ depends on the
post-recovery hiring threshold \( q^*_U(Y) \) in the same way as \( \hat{t}_H \) depended on the permanent hiring threshold \( q^*_U(Y-z) \) in Proposition 3. To understand this, notice that a recovery occurring before \( \hat{t} + \hat{t}_{Y-z,Y} \) will decrease both the firing threshold \( p^* \) and the hiring threshold \( q^* \) (from \( q^*_U(Y-z) \) to \( q^*_U(Y) \)). The drop in \( p^* \) will result in delayed firings, but no immediate firm response. In turn, if hiring would not begin at the new, lower \( q^* = q^*_U(Y) \), this drop in \( q^* \) would cause no immediate firm response either. Thus, for a recovery at a time \( t \in \left[ \hat{t}, \hat{t} + \hat{t}_{Y-z,Y} \right) \), the unemployment pool and its quality will not change discretely in response to the recovery. Given this, we know that \( \hat{t}_{Y-z,Y} \) must satisfy:

\[
q^*_U(Y) = \frac{(1 - E_{ss}) q_{U,ss} + \int_{t_z}^{t_{ss}^*} e^{-(\pi + \lambda)s} q_H(q_{U,ss}) \, ds + \eta_{ss} e^{-(\pi + \lambda)t_z} q_H(q_{U,ss}) \hat{t}_F}{(1 - E_{ss}) + \int_{t_z}^{t_{ss}^*} \left[ e^{-(\pi + \lambda)s} q_H(q_{U,ss}) + e^{-\pi s} (1 - q_H(q_{U,ss})) \right] \, ds + \eta_{ss} \left[ e^{-(\pi + \lambda)t_z} q_H(q_{U,ss}) + e^{-\pi t_z} (1 - q_H(q_{U,ss})) \right] \hat{t}_F}
\]

\[
+ \int_{0}^{t_{Y-z,Y}} \left[ 1 - e^{-\pi(t_{Y-z,Y}-s)} \right] \lambda \int_{s}^{t_z} e^{-(\pi + \lambda)x} \, dx \, ds
\]

where \( m_{H,ss} = \left( \frac{\lambda}{\pi} \right) \eta_{ss} q_H(q_{U,ss}) \int_{0}^{t_{ss}^*} e^{-(\pi + \lambda)s} \, ds \) is the steady-state mass of employed workers who have already revealed themselves to be type \( H \). As in the previous result, \( \hat{t}_F \equiv \min\{t_z, t_{Y-z,Y} \} \).

If this \( q_{U,\hat{t}+t} \) has not risen above \( q^*_U(Y) \) by the time the recovery occurs, this recovery will have
no discrete effect on the unemployment pool quality. Thus, if the economy recovers at time $\hat{t} + t$, but $q_{U,\hat{t}+t}$ remains below $q^*_U(Y)$, there will be a period without hiring even after the recovery.

Note that the distribution of recovery times depends only on $\gamma$. Since $t_{Y-z,Y} = 0$ at $z = 0$ and $t_{Y-z,Y} > 0$ for positive $z$, a larger value of $t_{Y-z,Y}$ implies a greater likelihood of a jobless recovery (note also that this moves continuously in $z$). Since the expected duration of a jobless recovery was 0 at $z = 0$, this is increasing in $t_{Y-z,Y}$ as well. Thus, we can show that a jobless recovery’s likelihood and expected duration are increasing in $z$ by establishing that $t_{Y-z,Y}$ is increasing in $z$.

To see that $t_{Y-z,Y}$ is increasing in $z$ for a range of $z$, notice first that the post-shock unemployment pool quality $q_{U,\hat{t}}$ is decreasing in $z$ for $z$ sufficiently small (since we consider only equilibria for which $p^*_{ss} < q_{U,ss}$). Increases in $z$ on this range therefore lead to lower $q_{U,\hat{t}}$, but we must consider also that the post-shock flow to unemployment will come from a combination of random job-quitters and workers fired at the higher threshold $p^*(Y - z,Y)$. Because of this, we might worry that a faster rate of recovery might overcome the lower starting quality. Of course, this higher threshold will also yield a more intense flow of directed firings into the unemployment pool after the shock, and this will mitigate improvement in the rate of recovery caused by the higher threshold.

In line with this reasoning, the post-shock time required to reach the unemployment quality threshold $q^*_U(Y)$ is increasing in $z$ for small $z$, which is our desired result. This can be verified by differentiating the above expression for $q_{U,\hat{t}+t_{Y-z,Y}}$ with respect to $z$ at the point $z = 0$. Using this, we can solve for $\frac{\partial t_{Y-z,Y}}{\partial z}$ and verify that it is positive at $z = 0$. I omit an expression for this because it is algebraically involved and economically uninsightful, but it is straightforward to see why this is true in the expression for $q_{U,\hat{t}+t_{Y-z,Y}}$ above: since $p^*_{ss} < q_{U,ss}$, increases in $z$ will lower $q_{U,\hat{t}}$ further in a neighborhood of $z = 0$ because the effect of increases in firings relative to quits is greater than the effect of rising firing standards in this region.
In addition to the above result, I will provide an alternate formulation of this in which the initial unanticipated output shock $Y \rightarrow Y - z$ at time $\hat{t}$ is followed almost immediately by another unanticipated output shock $Y - z \rightarrow Y$ at time $\hat{t} + dt$. (Firms have already responded to the first shock before the second occurs.) Of course, a setting with two unanticipated shocks that are both believed to be permanent is farther from reality than the environment in Proposition 4. The result, however, is similar, and the simplicity of this setting allows us to connect the result to clear intuition.

**Proposition 4.a:** After an unanticipated output shock $Y \rightarrow Y - z$ at time $\hat{t}$, firm responses to this output shock, and an unanticipated perfect reversal of this output shock $Y - z \rightarrow Y$ at time $\hat{t} + dt$, hiring will cease for duration $\hat{t}_{H} > 0$.\(^{49}\)

**Proof:** First, we can bound $\hat{t}_{H}$ above by $t_{ss}^* - t_z$. To see this, note that the mass of firings in response to the shock (in the belief range $[p_{ss}^*, p_{f}^*]$) will be of better quality than the directed firings that would have occurred at belief $p_{ss}^*$ without the shock. This is because some of those fired in response to the shock would have revealed themselves to be type $H$ workers during the subsequent period of length $t_{ss}^* - t_z$, but these type $H$ workers are instead included in the mass firings after the shock. Additionally, until hiring begins, this downward pressure on the unemployment pool quality (which would have been present without the shock) will be absent. These two facts imply that, after the elapsed time $t_{ss}^* - t_z$, without hiring beginning, the unemployment pool quality without hiring beginning must be strictly higher after the two-shock even than it would have been after no shock at all. The unemployment pool quality threshold for hiring ($q_{U}^*(Y)$) is identical in both cases (since $Y$ returns to the same level after the second shock), and the unemployment pool quality

\(^{49}\)Note that $\hat{t}_{H}$ satisfies $q_{U}^*(Y) = q_{U, \hat{t} + t_{H}}$. This is analogous to $q_{U}^*(Y) = q_{U, \hat{t} + t_{Y - z, Y}}$ in Proposition 4, but with terms adjusted to account for the firing and hiring thresholds immediately returning to their previous levels.
without any shock will remain at precisely \( q_U^* (Y) \). Thus, after the elapsed time period \( t_{ss}^* - t_z \) following the two-shock sequence, the unemployment pool quality would be greater than \( q_U^* (Y) \) without hiring beginning, so hiring must begin again before this time \( t_{ss}^* - t_z \) has elapsed.

Given this bound, we can provide an expression analogous to that from Proposition 4 for the time at which hiring will be renewed (note that group \((iii)\) is no longer included, since the threshold returns to \( p_{ss}^* \), and since hiring must begin again before any workers who aren’t fired in the initial response to the shock reach this threshold):

\[
q_U^* (Y) = \frac{(i) \left(1 - E_{ss}^* \right) q_{U,ss} + \eta_{ss} \int_{t_z}^{t_{ss}^*} e^{-}\pi + 1)s \lambda q_H (q_{U,ss}) ds}{(ii) + \eta_{ss} \int_{0}^{\pi} \int_{s}^{t_z} e^{-}\pi + 1)x q_H (q_{U,ss}) dx ds}
\]
APPENDIX B: Extensions

Mechanics of Equilibrium with Varying Employment (From Section 7.2)

Here I discuss in greater detail the equilibria mentioned in Section 7.2. Again, the free entry condition requires the quality of the unemployment pool to be constant. In order to maintain this constant quality, the combined negative pressure from selective hirings and targeted firings must precisely counter the positive pressure from quitting workers. In equilibrium, the economy preserves this balance by adjusting the intensity of hiring to account for the disparity between the positive pressure of quitting workers and the negative pressure of fired workers. For purposes of intuition, it is worth remembering that these conditions regarding the intensity of hiring reflect optimal firm behavior—firms will continue to hire as long as it is profitable to do so.

Of course, it is crucial to note that, in this model, the intensity of targeted firings at $t$ is determined by the intensity of selective hirings at $t - t_v^*$. Hence, the intensity of hiring necessary to preserve equilibrium at time $t$ is a function of the current employment level $E_t$ and the hiring intensity $t - t_v^*$ earlier ($\eta_{t-t_v^*}$).

Let us formalize this by defining $\eta_v \left( \eta_{t-t_v^*}, E_t \right): \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$. To preserve the type $H$ proportion of the unemployment pool, the hiring intensity must be that at which the net flow into/out of unemployment will have proportion $q_{U,v}$ of type $H$ workers—this is the same type $H$

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50Note that we still must satisfy the accounting condition $E_t q_{U,v} + (1 - E_t) q_{E,t} = P$, and that all equilibria we are considering involve the same unemployment pool quality level $q_{U,v}$. Hence, a given employment level $E_t$ necessarily implies a unique type $H$ proportion among the employed $q_{E,t}$. (Of course, if the level $E_t$ implies a $q_{E,t} \notin [0, 1]$, such an $E_t$ level cannot be consistent with equilibrium. Further, an economic state with an implied $q_{E,t} \in [0, P)$ could not be reached by any of the forces considered in this paper.)
proportion as the unemployment pool itself. Thus, \( \eta_v \left( \eta_{t-t_v^*}, E_t \right) \) must satisfy:

\[
q_{U,v} = \frac{\text{directed firings} \left[ \eta_{t-t_v^*} dtq_H (q_{U,v}) e^{-\left(\lambda+\pi\right)t_v^*} \right] + \text{random quits} \left[ E_t q_{E,t} \pi dt \right] - \text{selective hirings} \left[ \eta_v \left( \eta_{t-t_v^*}, E_t \right) dtq_H (q_{U,v}) \right]}{\eta_{t-t_v^*} dt \left[ 1 - q_H (q_{U,v}) \right] e^{-\pi t_v^*} + q_H (q_{U,v}) e^{-\left(\lambda+\pi\right)t_v^*}} + \frac{E_t \pi dt}{\text{directed firings}} - \eta_v \left( \eta_{t-t_v^*}, E_t \right) dt \]

Obviously, there cannot be a negative intensity of hiring, so if \( \eta_v \left( \eta_{t-t_v^*}, E_t \right) < 0 \), the unemployment pool quality will fall below \( q_{U,v} \) and hiring will stop for some amount of time until the pool quality again rises to \( q_{U,v} \). This can occur if filtering has limited effectiveness (\( \alpha \) is not too close to 0)—so much hiring takes place at some point in time that conditions will require hiring to stop \( t - t_v^* \) later.\(^{51}\)

**Example: Cyclical Equilibrium**

In the general description above, we required no repeated cyclical employment pattern—employment evolved simply to preserve the free entry condition. For certain parameter combinations, though, it is possible to construct an equilibrium in which there are self-sustaining cycles in hiring and firing. To see this, consider a simple example:

Imagine an equilibrium with 2 intensities of hiring, \( \eta_H \) and \( \eta_L \) (assume \( \eta_H > \eta_L \)). This equilibrium will consist of repeated cycles with an expansionary period of length \( t_v^* \) during which \( \eta_t = \eta_H \), followed by a contractionary period of length \( t_v^* \) with \( \eta_t = \eta_L \). Obviously, the employment level grows during the expansionary period and shrinks during the contractionary period.\(^{52}\) To preserve the cyclicality, the total growth while \( \eta_t = \eta_H \) must equal the total contraction while \( \eta_t = \eta_L \).

\(^{51}\)In the case where filtering is extremely precise (\( \alpha \) close to 0), it is possible that \( p_{ss^*} > q_{U,ss} \). As a result, \( \eta_v \left( \eta_{t-t_v^*}, E_t \right) > 0 \) and free entry always holds with equality.

\(^{52}\)In turn, \( q_{E,t} \) falls during expansions and rises during contractions. So, in line with standard intuition, the average quality of employed workers is greatest when employment is lowest—toward the end of a contractionary period.
To understand why such an equilibrium can preserve the unemployment pool quality level, notice that whenever $\eta_t = \eta_H$, it must also be true that $\eta_{t-t^*_v} = \eta_L$. Similarly, these intensities are reversed during contractions. Hence, the hiring and firing intensities during an expansionary period must support a net flow into unemployment with type $H$ proportion $q_{U,v}$, and these intensities during a contractionary period must support a net flow out of unemployment with type $H$ proportion $q_{U,v}$.

In the equation satisfied by $\eta_v \left( \eta_{t-t^*_v}, E_t \right)$ given above, the numerator and denominator are both positive during expansionary periods and both negative during contractionary periods.

Let us now consider the aggregate conditions that must be satisfied to sustain these cycles (taking optimal firm behavior and the corresponding thresholds $p^*_v$ and $q_{U,v} = q^*_U (Y)$ as given). For ease of notation, define:

- the net change in the employment level from $t$ to $t + dt$:

$$\Delta_E (t) \equiv E_{t+dt} - E_t$$

- the mass of type $H$ workers employed at time $t$:

$$E_{H, t} \equiv q_{E,t} E_t$$

- the net change in the mass of type $H$ workers employed from $t$ to $t + dt$:

$$\Delta_{E,H} (t) \equiv q_{E,t+dt} E_{t+dt} - q_{E,t} E_t$$

$$= E_{H,t+dt} - E_{H,t}$$

Further, index times in the cycle by $t \in [0, 2t^*_v]$, where $t \in [0, t^*_v)$ correspond to the expansionary part of the cycle, and $t \in [t^*_v, 2t^*_v)$ correspond to the contractionary part. With this notation established, this cyclical employment equilibrium must satisfy the following:
The labor force must always be of unit mass, and it must have the proportion $P$ of type $H$ workers, so the employment level and type $H$ proportion of those employed must reflect this at all times:

$$E_{H,t} + (1 - E_t) q_{U,v} = P$$ for $t \in [0, 2t^*_v]$

The employment level and the mass of type $H$ workers employed must evolve according to the net flows into each. This net flow should consist of hirings minus directed firings and quits. Thus

$$\Delta_E (t) = -E_t \pi dt + \eta_H dt - \eta_L dt \left[ (1 - q_H (q_{U,v})) e^{-\pi t^*_v} + q_H (q_{U,v}) e^{-(\lambda + \pi) t^*_v} \right]$$ for $t \in [0, t^*_v]$

$$\Delta_{E,H} (t) = -\pi E_{H,t} dt + q_H (q_{U,v}) \left[ \eta_H - \eta_L e^{-(\lambda + \pi) t^*_v} \right] dt$$ for $t \in [0, t^*_v]$

$$\Delta_E (t) = -E_t \pi dt + \eta_L dt - \eta_H dt \left[ (1 - q_H (q_{U,v})) e^{-\pi t^*_v} + q_H (q_{U,v}) e^{-(\lambda + \pi) t^*_v} \right]$$ for $t \in [t^*_v, 2t^*_v]$

$$\Delta_{E,H} (t) = -\pi E_{H,t} dt + q_H (q_{U,v}) \left[ \eta_L - \eta_H e^{-(\lambda + \pi) t^*_v} \right] dt$$ for $t \in [t^*_v, 2t^*_v]$

Further, as explained above, the net flow into employment must always have proportion $q_{U,v}$ of type $H$ workers (regardless of whether this net flow is positive or negative):

$$q_{U,v} = \frac{\Delta_E (t)}{\Delta_{E,H} (t)}$$ for $t \in [0, 2t^*_v]$

Finally, in order for the equilibrium to be truly cyclical, the net inflows to employment during the expansionary period must be exactly reversed by the net outflows from employment during the contractionary period:

$$\int_0^{t^*_v} \Delta_E (s) ds + \int_{t^*_v}^{2t^*_v} \Delta_E (s) ds = 0$$

Thus, for this "expansion/contraction" equilibrium to exist, the expansion/contraction periods must each last as long as any individual firm would wait without output before firing a worker, and the total growth of employment during this expansion must erode completely during the following contraction. Further, this must all occur with net flows to/from employment always having the same proportion of type $H$ workers as the unemployment pool itself ($q_{U,v}$).
APPENDIX C: Data, Empirical Methodology, and Supplemental Results

CPS Monthly Employment Data

The Current Population Survey (CPS) data used are a monthly survey administered at the household-level. In addition to individual and household characteristics, these data also contain information about labor market participation and outcomes. Of particular use to the current study, individual observations in the data can be linked across months. This linking allows us to observe whether unemployed workers in a given month found employment by the following month, and these hiring outcomes serve as a dependent variable in my empirical analysis.

Each housing unit in the survey is interviewed for four consecutive months, dropped from the sample for eight months, and then brought back for another four months. This sampling structure is evenly distributed across months and years. In each month of data, it is year 1 in the sample for half of households and year 2 in the sample for the other half; further, $\frac{1}{8}$ of households are completely new to the sample, and $\frac{3}{4}$ of households will be included in the following month’s sample as well. The success rate in linking individuals across months was quite high—among the $\frac{3}{4}$ of the sample that were expected to appear in the following sample, well over 90% were actually matched.

Note that this empirical approach does not require a longitudinal data structure—for this month’s unemployed, I need only to observe the following month’s employment status. The section below explains in more detail how these linked monthly data are used in the analysis.

Details of Results in Table 1 and Table 2

To obtain the values $\frac{\partial \ln(H^S_t)}{\partial \ln(Q_{t-1})}$, $\frac{\partial \ln(H^F_t)}{\partial \ln(Q_{t-1})}$, and $\frac{\partial \ln(H^S_t)}{\partial \ln(F_{t-1})}$ reported in Tables 1 and 2,
I estimate equations of the form

\[
\partial \ln (H_i^t) = \beta_0 + \beta_1 \partial \ln (H^t) + \beta_2 \partial \ln (F_{t-1}) + \beta_3 \mathbb{I}[i \in S] \partial \ln (F_{t-1}) \\
+ \beta_3 \partial \ln (Q_{t-1}) + \beta_{3S} \mathbb{I}[i \in S] \partial \ln (Q_{t-1}) + \epsilon_{it}
\]

where \( \mathbb{I}[i \in S] \) is an indicator for whether individual \( i \) is among the short-term unemployed. Clearly, our estimate for \( \widehat{\beta}_{3S} \) corresponds to \( \frac{\partial \ln (H^t)}{\partial \ln (Q_{t-1})} - \frac{\partial \ln (H_{t-1}^t)}{\partial \ln (Q_{t-1})} \), while \( \widehat{\beta}_{2S} \) corresponds to \( \frac{\partial \ln (H^t)}{\partial \ln (F_{t-1})} - \frac{\partial \ln (H_{t-1}^t)}{\partial \ln (F_{t-1})} \).

Recall that \( H_t \) is the aggregate probability that time \( t \) unemployed workers are hired by \( t+1 \):

\[
H_t \equiv \frac{\text{total \# of unemployed hired at time } t}{\text{total \# of unemployed at time } t}
\]

and note that we use the discrete-time approximations \( \partial \ln (H_i^t) \approx \frac{H_i^t - H_{i-1}^t}{H_{i-1}^t} \), \( \partial \ln (H^t) \approx \frac{H_t - H_{t-1}}{H_{t-1}} \), \( \partial \ln (F_t) \approx \frac{F_t - F_{t-1}}{F_{t-1}} \), and \( \partial \ln (Q_t) \approx \frac{Q_t - Q_{t-1}}{Q_{t-1}} \). Our dependent variable \( \partial \ln (H_i^t) \) is the only source of individual-level variation in this estimation (beyond short/long-term unemployment status). We exploit individual-specific covariates in the data through this term, which we obtain in the following way:

Let \( x_i \) denote the \( k \)-dimensional vector of covariates specific to individual \( i \), and note that these covariates are fixed over time for each individual. Using these covariates, we estimate equations of the form \( y_{it} = g_t(x_i) + \epsilon_{it} \) for each \( t \), where \( y_{it} \) is a binary indicator for whether or not individual \( i \) (who was unemployed in period \( t \)) became employed in period \( t+1 \).\(^{53}\) From this, we obtain the optimal functions \( \hat{g}_t(\cdot) \) for each period \( t \), and \( H_i^t \) is simply \( \hat{g}_t(x_i) \). We thus approximate our dependent variable with

\[
\partial \ln (H_i^t) \approx \frac{\hat{g}_t(x_i) - \hat{g}_{t-1}(x_i)}{\hat{g}_{t-1}(x_i)}
\]

\(^{53}\) For transparency, the analysis here imposes linearity on \( g_t(\cdot) \), so we can write \( \hat{g}_t(x_i) \equiv \beta x_i \).
Intuitively, $g_t(x_i)$ represents the time $t$ reemployment probability of a worker with observables $x_i$. In turn, $g_t(x_i) - g_{t-1}(x_i)$ is the change in reemployment probability (from time $t - 1$ to time $t$) of a worker with observables $x_i$, and $\partial \ln (H^t_i)$ is the corresponding percent change. If $g_t(x_i) > g_{t-1}(x_i)$, then a worker with observables $x_i$ is more likely to be hired in period $t$ than in period $t - 1$.

Suppose that firm hiring decisions are based on individual characteristics that correlate positively with quality and that at least some of these characteristics are not contained in $x_i$ (meaning that they are unobservable to the econometrician). Then conditioning on $x_i$, a worker’s probability of being hired at time $t$ should correlate with her quality. In other words, firm hiring decisions in period $t$ will inform us about the average quality of individuals with a given covariate vector $x_i$ at time $t$. Thus, $\partial \ln (H^t_i) > 0$ indicates that the average quality of individuals with covariates $x_i$ improved from $t - 1$ to $t$; our dependent variable reflects unobserved worker quality. We can then extend this to the pools of short- and long-term unemployed to assess how unobserved worker quality changes for these two groups.

Additionally, it is worth recognizing the distinctions between specifications $I - IV$ in Tables 1 and 2. In Table 1, the estimates for specification $I$ were obtained according to the process described in Section 5.2 and above. Individual-level observables used to compute $\partial \ln (H^t_i)$ were age, race, and unemployment duration (the results are similar if reasons for unemployment are also included). CPS-provided household sampling weights are used both in obtaining $\partial \ln (H^t_i)$ and in the regression to measure the effects of changes in firings/ quits on $\partial \ln (H^t_i)$. Specification $II$ differs only in that the latter regression does not control for aggregate changes in hiring probabilities. In turn, specification $III$ deviates from $I$ only in that sampling weights are not used in the estimation. Finally, specification $IV$ further restricts the observables used in obtaining $\partial \ln (H^t_i)$—in this case, $x_i$ consists only of individual $i$’s unemployment duration group, so this directly compares changes
in the overall quality of workers in short- and long-term unemployment..

In Table 2, all estimates provided have been obtained via the same process as those in specification I of Table 1. However, specifications I and III include data from January 2001 - August 2011, while specifications II and IV include only the September 2008 - August 2011 window. Additionally, specifications III and IV include standard errors that have been clustered within each year-month pair; all standard errors reported in Table 1 use this clustering. The standard errors reported in specifications I and II, however, are not clustered.

The main results in these tables appear quite robust to these variations.
Table 2: Responses of STU - LTU hiring probabilities to firings/quits, including flows during recession (Source: CPS)

<table>
<thead>
<tr>
<th></th>
<th>(I)</th>
<th>(II)</th>
<th>(III)</th>
<th>(IV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial \ln (H^S_T)}{\partial \ln (F_{t-1})} ) - ( \frac{\partial \ln (H^L_T)}{\partial \ln (Q_{t-1})} )</td>
<td>-0.101</td>
<td>-0.471</td>
<td>-0.101</td>
<td>-0.471</td>
</tr>
<tr>
<td>(0.038)</td>
<td>(0.184)</td>
<td>(0.185)</td>
<td>(0.460)</td>
<td></td>
</tr>
<tr>
<td>( \frac{\partial \ln (H^S_T)}{\partial \ln (Q_{t-1})} )</td>
<td>0.811</td>
<td>1.242</td>
<td>0.811</td>
<td>1.242</td>
</tr>
<tr>
<td>(0.465)</td>
<td>(0.746)</td>
<td>(0.792)</td>
<td>(1.305)</td>
<td></td>
</tr>
</tbody>
</table>

| Standard errors clustered at year-month level? | N       | N       | Y       | Y       |
| Control for \( \partial \ln (H_t) \)?       | Y       | Y       | Y       | Y       |
| Use sampling weights?                        | Y       | Y       | Y       | Y       |
| Condition on individual observables?        | Y       | Y       | Y       | Y       |
| \( N \)                                     | 87,518  | 33,459  | 87,518  | 33,459  |

CPS data on unemployment flows are used to obtain estimates.
Sample used is restricted to unemployed men with no education beyond a HS diploma.
Table 3: Summary Statistics for Unemployed Workers (Source: CPS)

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>N</td>
<td>% Employed Next Month</td>
<td>N</td>
</tr>
<tr>
<td>All</td>
<td>380,519</td>
<td>45.0%</td>
<td>248,823</td>
</tr>
<tr>
<td>All Reasons for Unemployment</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Male</td>
<td>205,940</td>
<td>42.7%</td>
<td>150,335</td>
</tr>
<tr>
<td>Male &lt; HS Diploma</td>
<td>126,093</td>
<td>42.3%</td>
<td>80,120</td>
</tr>
<tr>
<td>Job Leavers / Losers only</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>All</td>
<td>250,984</td>
<td>44.0%</td>
<td>160,204</td>
</tr>
<tr>
<td>Male</td>
<td>147,995</td>
<td>43.9%</td>
<td>91,792</td>
</tr>
<tr>
<td>Male &lt; HS Diploma</td>
<td>87,518</td>
<td>43.8%</td>
<td>54,059</td>
</tr>
<tr>
<td>Job Leavers / Losers only</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Duration &lt; 12 weeks)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>All</td>
<td>141,216</td>
<td>50.3%</td>
<td>102,270</td>
</tr>
<tr>
<td>Male</td>
<td>81,967</td>
<td>50.3%</td>
<td>57,751</td>
</tr>
<tr>
<td>Male &lt; HS Diploma</td>
<td>49,249</td>
<td>50.2%</td>
<td>44,519</td>
</tr>
<tr>
<td>Job Leavers / Losers only</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Duration &gt; 12 weeks)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>All</td>
<td>107,123</td>
<td>35.1%</td>
<td>55,289</td>
</tr>
<tr>
<td>Male</td>
<td>64,308</td>
<td>35.1%</td>
<td>32,221</td>
</tr>
<tr>
<td>Male &lt; HS Diploma</td>
<td>33,841</td>
<td>34.6%</td>
<td>23,075</td>
</tr>
</tbody>
</table>
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Chapter 2

Search with Private Information:
Sorting and Price Formation

2.1 Introduction

We investigate the sorting of heterogeneous agents in a two-sided market where the value of a traded good depends on both buyer and seller types, focusing on a setting in which trade is hindered by search frictions and private information. Becker (1973) serves as a benchmark characterization of sorting—in a "frictionless" world, Positive Assortative Matching (henceforth PAM) arises when output is supermodular in types, while Negative Assortative Matching (NAM) occurs with submodular output. We depart from Becker’s setting by restricting the interactions of agents in two ways. First, it is difficult to meet potential trading partners, as each buyer encounters only a random seller in each period (and vice versa). Second, in each meeting of potential trade partners, the buyer is privately informed about her type.

More precisely, our environment is a repeated, bilateral matching market. Buyers and sellers are
heterogeneous and have persistent types. These buyers and sellers are randomly matched pairwise in each period, and if the pair agree to trade, the buyer receives the joint output, which is an increasing function of both agents’ types. In each match, the seller’s type can be jointly observed, but only the buyer knows her own type (and thus knows the joint output). In turn, the seller has all the bargaining power and can make a take-it-or-leave-it offer.

This setting reflects a broad set of applications—it is straightforward to imagine a market for vertically-differentiated products, where consumers differ in preferences across goods. In the housing market, for instance, homes for sale vary in kitchen appliances, and sellers are visited by buyers with different marginal return to better appliances. Our model can be seen as the case in which the buyer’s preference is her private information. Similarly, this might represent the hiring process in a labor market where heterogeneous workers and firms produce joint output. For consistency, we describe the model’s agents as buyers and sellers throughout the analysis to follow, but where appropriate, it will be insightful to draw motivation from other such applications.

Our study has two primary objectives—we want to characterize: (i) the sorting patterns that arise in the environment described above and (ii) the patterns in the price distribution that give rise to this sorting. Both tasks require us to understand how each seller determines her reserve price and how this decision changes over seller types. At its most basic level, optimal seller behavior is governed by the following intuition: Because output rises in type, a higher type seller faces a choice of how best to use this extra output—she can simply keep it in each of the trades she would have had, or alternatively, she can give it to buyers to incentivize more of them to trade with her. If prices rise in seller type by the precise amount to generate no sorting at all (meaning that different sellers match with the same set of buyers), then these sellers are clearly choosing the former. If prices are constant across types, sellers are choosing the latter. Inbetween these two outcomes—if
sorting is positive, but prices are rising in seller type—sellers are choosing a combination of the two.

Unfortunately, the simplicity of this intuition masks the underlying complexity of equilibrium decisions and interactions; in contrast to related models in the existing literature (e.g. - Shimer and Smith, 2000; Smith, 2006; Atakan, 2006), both prices and matching sets arise endogenously in our model. As a result, neither of the aforementioned objectives is straightforward to achieve, so we will take an indirect approach. We will see that, given prices, matching sets are intimately related to continuation values, and we will attempt to use this relationship to overcome the difficulty of characterizing equilibrium. Specifically, we will investigate how the patterns of interest depend on the factor by which agents discount the future (\( \beta \)), focusing especially on the extreme cases in which \( \beta = 0 \) (the static case) and \( \beta \rightarrow 1 \) (search frictions become insignificant).

In the static case, our analysis is simplified because the discounted continuation value is zero. We find that the direction of sorting depends on the log-supermodularity of output—log-supermodular production functions give rise to PAM, while log-submodularity leads to NAM. This is a stronger condition than the supermodularity than governs sorting in a frictionless market. Intuitively, the buyer’s private information generates an additional tradeoff for the seller between terms of trade and probability of trade. Higher types have a greater incentive to take advantage of their higher type-specific output by increasing the probability of trade (i.e. - by relaxing the terms of trade to induce more types to participate). This incentive pushes equilibrium sorting towards negative

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1 We also have a characterization of sorting and prices for the general, dynamic setting with discount factors in \((0, 1)\), but this is quite messy analytically, and the conditions are difficult to link coherently to economic intuition. Of course, these findings are available from the authors upon request.

2 Throughout the analysis, the distribution of buyer types entering the market in each period is assumed to satisfy the common increasing hazard rate property. Output is assumed to be everywhere positive, log-concave in buyer type, and strictly increasing in both buyer and seller types.
assortative even with a supermodular production function. Positive sorting therefore requires an output function with stronger complementarity. It is also worth noting that stronger log-concavity in buyer type of output weakens sorting in either direction, because this decreases the benefits to the seller of trying to trade with more types (lowering the price will induce less movement in the marginal, indifferent buyer under this condition).

We also characterize how prices move with seller type: we find that prices increase in seller type if output exhibits a sufficiently strong combination of supermodularity and log-concavity in buyer type. For instance—if output is concave in the buyer’s type, then it need only be supermodular for prices to rise with type. If output is merely log-concave in buyer type, however, then it must satisfy the more stringent condition of log-supermodularity to induce prices to rise in type.

At the other extreme, we can think of the case with increasingly patient agents as a setting with disappearing search frictions. We find that, when $\beta \rightarrow 1$, the inefficiency associated with private information disappears, and matching sets reduce to the unique stable matching of a frictionless market (when it exists). Thus, the standard supermodularity (submodularity) reemerges as the condition for PAM (NAM). In this case, we can neatly characterize prices analytically, and in equilibrium, each agent obtains her marginal contribution to total surplus up to a constant.

This frictionless limit is particularly informative regarding sorting and how it depends on the interplay between private information and search. Our analysis suggests generally that, when buyers have private information, agents are less likely to capitalize on productive complementarities by sorting positively. This resistance to sorting disappears, however, when we reduce search frictions. By removing the time preference of agents, we weaken the monopolistic aspect of bilateral trade, which allows agents to appropriate their marginal contributions. In this sense, increasing market competition can help agents sort efficiently even in the presence of private information.
The paper proceeds as follows: Section 2 frames our findings in the context of related literature. Section 3 introduces the model and establishes the existence of a search equilibrium. Section 4 characterizes sorting and price formation when the model is effectively static ($\beta = 0$), and Section 5 offers results for the other extreme case as agents become patient. Finally, Section 6 draws connections between these limiting cases and concludes.

2.2 Related Literature

As we will discuss below, our analysis builds upon previous theoretical work related to assorative matching. Studies in this area often attempt to answer two primary questions: (1) What types of agents will match with each other? (2) What types of agents would match with each other to optimize overall welfare, and if these matching patterns differ, why? To properly place our analysis in the context of this literature, we will compare our setting to several that have been previously studied, both through objective sorting characterizations and through insights regarding the relationship between sorting, search, and asymmetric information.

The Frictionless Matching Benchmark

A standard benchmark used for comparison is the frictionless, "Walrasian" setting studied by Becker (1973) and Rosen (1974). In this environment, there is full information regarding prices and types, and meeting trade partners is fully costless for buyers and sellers. Becker famously demonstrated that supermodular production functions give rise to PAM in this environment.

Beyond this, though, recent studies have taken a renewed interest in sorting, trying to understand how it is impacted by departures from the frictionless benchmark. Among these frictional extensions, the setting we study is especially well-suited for comparison to those involving two
particular classes of frictions—the bilateral monopoly which arises in random search (Shimer and Smith, 2000; Smith, 2006; Atakan, 2006) and the coordination frictions in directed search (Eeckhout and Kircher, 2010). We elaborate upon these connections below.

**Random Search**

There are obvious connections between our study and a series of papers on sorting with random search (Shimer and Smith (2000), Smith (2006) and Atakan (2006)).

Our study differs from these primarily in its incorporation of buyer private information—in the studies mentioned above, there is full information and an exogenously given sharing rule (under transferable utility) in each meeting of potential partners.

These studies focus exclusively on the impact on sorting of random search frictions. Competition is modeled in a reduced form way (Nash Bargaining). In contrast, we focus on the discriminatory behavior that accompanies one-sided private information. Random search is not itself our ultimate, but rather the channel through which we vary the strength of bilateral monopoly power (via the discount factor).

In Shimer and Smith (2000), search is itself the reason that PAM requires more stringent conditions than in the frictionless setting. In our case, private information is behind our stricter conditions for PAM—log-supermodularity is required even in our static environment. Repeated search—specifically the future’s value—actually weakens the requirements for PAM, and the standard supermodularity condition again becomes sufficient in the frictionless limit. As the monopoly power in each match vanishes, the conditions for sorting (and the equilibrium itself) become identical regardless of whether the buyer’s type is private information.
Directed Search

Eeckhout and Kircher (2010) study a static, one-shot setting with buyer private information and directed search. Sellers in their model post prices, and buyers can then observe these prices before deciding which seller to visit. In this environment, they investigate the impact on sorting of coordination frictions, governed by the function which maps the buyer-seller composition at each location to a realized number/measure of bilateral matches. In contrast, we ignore coordination frictions entirely—every searching agent is matched bilaterally to a potential trade partner in each period. This enables us to focus on the interplay between private information and random search frictions, free of interference from coordination friction-based mechanisms.

Note that, when viewed through the lens of timing, the two settings appear to be much more closely related. Eeckhout and Kircher’s sellers use posted prices to sort agents prior to meeting, while our sellers sort only after the buyer has arrived. In simplest terms, our sellers use \textit{ex post} sorting, while Eeckhout and Kircher’s sellers sort \textit{ex ante}. Intuitively, we focus on settings in which the search process is too imprecise for agents to be guided toward specific trading partners.\footnotemark

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Coarse Matching

Our results can also be thought of as related to coarse matching (Chao and Wilson, 1987; McAfee, 2002; Hoppe et al., 2011; Shao, 2011), a type of sorting equilibrium in which agents can sort into groups/locations, but these groups cannot fully separate types, and matching is probabilistic within these groups. A key insight of this literature is that the failure of complete sorting need not induce large efficiency losses, and this can be interpreted as a justification for the

\footnotetext[3]{In some sense, our analysis can be viewed as a bridge between such studies of private information in directed search and the literature considering full information, random search.}
applied relevance of sorting, in light of the critique that the precision and complexity of theoretical sorting patterns precludes their occurrence in reality.

In some sense, our search equilibrium entails coarse matching—with any search frictions, matching will not be one-to-one, so each agents will have a nontrivial matching set. In the context of the coarse matching literature, our findings can be interpreted as demonstrating the role of prices as a potential mechanism through which coarse matching can arise.⁴

2.3 The Model

We consider a discrete-time, dynamic model with heterogeneous buyers and sellers. There are equal measures of buyers and sellers each period in the equilibrium of the steady state economy, where buyer types \( x \) are distributed according to cdf \( F_B(x) \), and seller types \( y \) are distributed according to cdf \( F_S(y) \).⁵ Each buyer is randomly matched with one seller (and vice versa) in each period. In each pair, seller types become observable, while buyer types remain private information. Sellers, however, have the power to make take-it-or-leave-it offers \( P(y) \) to buyers.

If trade occurs, output \( z(x, y) \) is produced and both parties leave the market permanently with utility \( z(x, y) - P(y) \) for the buyer and \( P(y) \) for the seller. Those who do not trade experience an exogenous exit shock with probability \( 1 - \beta \), in which case they leave the market. Otherwise, remaining buyers and sellers play the same game in the following period, along with exogenous measures of newly entering buyers and sellers, \( (1 - \beta) \Gamma_B \) and \( (1 - \beta) \Gamma_S \). These new entrants are drawn from fixed distributions \( \gamma_B(x) \) and \( \gamma_S(y) \) over the bounded intervals \( X = [\underline{x}, \bar{x}] \) and

⁴The connection to coarse matching may become stronger in future versions of this paper, where we hope to characterize the efficiency properties of equilibrium in our model and how the magnitude of search frictions impacts these properties.

⁵As will soon be obvious, these distributions have bounded supports in equilibrium.
\[ Y = [y, \bar{y}] \]. Agents discount future payoffs only due to the possibility of leaving the market without, so the relevant discount factor for all players is \( \beta \).

A type \( x \) buyer is willing to accept price \( P(y) \) if \( P(y) \leq z(x, y) - \beta V_0(x) \), where \( V_0(x) \) is the buyer’s equilibrium payoff. Unlike in a frictionless market, equilibrium will not entail a deterministic, one-to-one matching. Rather, each type will match probabilistically with one agent from a range of "acceptable" types on the opposite side of the market. We therefore denote a type \( x \) buyer’s surplus from trading with a type \( y \) seller as \( s(x, y) = z(x, y) - \beta V_0(x) - P(y) \). Given \( P(y) \), we call the set of sellers whose price will be accepted by a type \( x \) buyer as the buyer \( x \)'s matching set and denote the equilibrium matching set as \( \mathcal{M}_B(x) \).

\[
\mathcal{M}_B(x) = \{y : s(x, y) \geq 0\}
\]

We can therefore express \( V_0(x) \) as
\[
V_0(x) = \frac{\int_{\mathcal{M}_B(x)} (z(x, y) - P(y)) dF_S(y)}{1 - \beta + \beta \int_{\mathcal{M}_B(x)} dF_S(y)}
\]

\( \mathcal{M}_S(y) \) in turn defines a type \( y \) seller’s matching set \( \mathcal{M}_S(y) = \{x : s(x, y) \geq 0\} \). Obviously, \( y \in \mathcal{M}_B(x) \) if and only if \( x \in \mathcal{M}_S(y) \).

Given \( F_B(x) \) and \( V_0(x) \), seller \( y \) chooses \( P(y) \) to solve
\[
\Pi(y) = \max_P \left\{ P \int_{\mathcal{M}_S(y; P; V_0(x))} dF_B(x) + (1 - \int_{\mathcal{M}_S(y; P; V_0(x))} dF_B(x)) \beta \Pi(y) \right\}
\]

The last equilibrium condition is the steady-state condition: the measure of outflow of any type must equal the measure of inflow of the same type. The pdf of the type distribution \( f \) and pdf of
the entrant type distribution \( \gamma \) therefore must satisfy,

\[
\hat{f}_B(x) = \frac{\gamma_B(x)}{\beta + (1 - \beta) \int f_B(x) f_S(y) dy} \\
\hat{f}_S(y) = \frac{\gamma_S(y)}{\beta + (1 - \beta) \int f_S(y) f_B(x) dx}
\]

where \( f_i(i = S, B) \) satisfy \( f_B(x) = \frac{\hat{f}_B(x)}{\int \hat{f}_B(x) dx} \) and \( f_S(y) = \frac{\hat{f}_S(y)}{\int \hat{f}_S(y) dy} \)

For subsequent analysis, we impose the following assumptions on \( z(x, y) \):

**Assumption 1:** Over the domain \( X \times Y \), the output function \( z(x, y) \) is:

(i) nonnegative and bounded above

(ii) twice continuously differentiable

(iii) strictly increasing in both arguments with uniformly bounded first partial derivatives

(iv) log-concave in \( x \)

We devote the remainder of this section to technical preliminaries for the sorting analysis. First, we offer a fairly general existence proof for an equilibrium in which prices are continuous in seller type. Following Shimer and Smith (2000), we then define PAM and NAM with non-degenerate matching set. Finally, we close the section by providing conditions that will allow us to tractably characterize sorting. In particular, we provide sufficient conditions on \( z(\cdot, \cdot) \) to ensure that convex seller matching sets will arise.

**Existence**

We will focus on equilibrium where price is continuous in seller type. Therefore, when proving existence, we first assume that \( P(y, V_0) \) is continuous. At the end of the existence proof, we then verify that the equilibrium price is indeed continuous.
We use the Schauder fixed point theorem in proving existence. In particular, we will show that the mapping from the continuation payoff $V_0(x)$ to itself defined by equilibrium conditions is well-defined and continuous. We will approach the problem by first providing some preliminary results, which is lemma 1, 2 and 3 and we then show that the mapping is continuous in proposition 1.

Throughout the paper, we assume that the output function is either supermodular or submodular.

**Assumption 2:** (SUP) The output function $z(x, y)$ is supermodular.

(SUB) The output function $z(x, y)$ is submodular.

**Lemma 1:** Given (A1), a type $x$ buyer’s outside option function $V_0(x)$ satisfies

$$V_0(x) \geq \frac{1}{1 - \beta} \int_M (z(x, y) - P(y) - \beta V_0(x)) f_S(y)dy$$

for any $M \subseteq [y, \bar{y}]$. In addition, $V_0(x)$ is non-negative, increasing in $x$ and Lipschitz continuous in equilibrium. In addition, if price is continuous in seller type and either A2-Sup or A2-Sub holds, $V_0(x)$ is differentiable in equilibrium, with

$$V_0'(x) = \frac{\int_{M(x)} z_1(x, y) f_S(y)dy}{1 - \beta + \beta \int_{M(x)} f_S(y)dy}$$

Unless otherwise mentioned, all proofs are provided in the appendix. Define the indicator function $d(x, y)$. $d(x, y) = 1$ if and only if $s(x, y) \geq 0$ and $d(x, y) = 0$ otherwise.

**Lemma 2:** Given A1, A2-Sup or A2-Sub, any Borel measurable mapping $V_0 \to d(x, y)$ from outside option functions to match indicator functions is continuous.

For the existence proof, we also need to show that the endogenous distribution is continuous in the indicator function $d(x, y)$. To do this, we need to assume that the pdf of entrant distributions are bounded and that the measures of entrants are the same on both sides.
Assumption 3: $\gamma_B(x) \in (0, \infty)$ and $\gamma_S(y) \in (0, \infty)$ for any $x$ and $y$, and $\int \gamma_B(x) \, dx = \int \gamma_S(y) \, dy$.

Lemma 3: The mapping $d(x, y) \rightarrow (f_B(x), f_S(y))$ is well-defined and continuous.

With the above preliminary results, we are now ready to show the existence of the equilibrium.

Proposition 1: Given $A1$ and $A2$-Sup or $A2$-Sub, an equilibrium exists in which prices are continuous in seller types.

Definition of Sorting

As we can see, matching sets in an environment with search frictions are normally non-degenerate. A natural definition of sorting for such conditions is that provided in Shimer and Smith (2000)—for PAM, they require that the set of mutually agreeable matches form a lattice. More explicitly:

Definition 1: Take $x_1 < x_2$ and $y_1 < y_2$.

PAM: There is PAM if $y_1 \in M_B(x_1)$ and $y_2 \in M_B(x_2)$ whenever $y_1 \in M_B(x_2)$ and $y_2 \in M_B(x_1)$.

NAM: There is NAM if $y_1 \in M_B(x_2)$ and $y_2 \in M_B(x_1)$ whenever $y_1 \in M_B(x_1)$ and $y_2 \in M_B(x_2)$.

Note that convex matching sets for both buyers and sellers are necessary conditions for either PAM or NAM.

Convexity of Seller Matching Set

For the earlier existence result, it was unnecessary to place restrictions on agents’ matching sets. Obviously, though, the definitions of PAM and NAM above will be economically meaningful only if matching sets are convex. We therefore provide sufficient conditions on $z(x, y)$ for this to always be the case.
Assumption 4: $z_1(x, y)$ is log-supermodular.

Assumption 5: $z_{12}(x, y)$ is log-supermodular.

Proposition 2: Given A4 and A5, the seller’s matching set $M_S(y)$ is convex for any $y$.

2.4 Limits of Search Frictions: $\beta = 0$ (One-Shot Bilateral Monopoly)

In this section, we consider the case in which agents do not value the future at all ($\beta = 0$) – the one-period game. In other words, all agents experience death shock after one period.

If a seller $y$ chooses the price that equals the output with a type $x$ buyer, i.e, $P(y) = z(x, y)$, then any buyer with type above $x$ will accept the price. Therefore, choosing $P(y)$ is equivalent to selecting the marginal type $x^*(y)$ to maximize the expected profit. That is,

$$\Pi(y) = \max_{\hat{x}} \{z(\hat{x}, y)(1 - F_B(\hat{x}))\}$$

Theorem 1. When $\beta = 0$, $P(y) = z(x^*(y), y)$ where the marginal type $x^*(y)$ is determined by:

$$z_1(x^*(y), y)(1 - F_B(x^*(y))) = z(x^*(y), y)f_B(x^*(y))$$ (2.1)

Further, both $P(y)$ and $x^*(y)$ are unique for any $y$.

As usual, $x^*$ is chosen so that marginal revenue equals marginal cost. The left hand side of equation [2.1] is the marginal revenue of raising the marginal type: the price is raised by the amount $z_1(x^*(y), y)$ and the seller can collect this increment with probability $1 - F_B(x^*(y))$, which is essentially the trading probability. The right hand side is the marginal cost of raising the marginal
type: the seller can no longer sell to type \(x^*(y)\) buyers and therefore the loss equals the price \((z(x^*(y), y))\) times the probability of meeting a type \(x^*(y)\) buyer \((f_B(x^*(y)))\).

Let us now characterize sorting. Under the threshold rule and the assumptions that ensure differentiability, the definition of PAM (NAM) reduces to the condition that the derivative of the marginal type is positive (negative). That is, sorting is positive if \(\frac{\partial x^*(y)}{\partial y} \geq 0\) and is negative if \(\frac{\partial x^*(y)}{\partial y} \leq 0\). So we only need to do a comparative static exercise to find out under what conditions is the derivative positive.

**Theorem 2.** Sorting is positive (PAM) if the output function \(z(x, y)\) is log-supermodular and sorting is negative (NAM) if \(z(x, y)\) is log-submodular.

**Proof:** From the equilibrium condition that determines \(x^*(y)\), it is easy to see that

\[
\frac{\partial x^*}{\partial y} = \frac{z_{12}(1 - F(x^*) - z_2 f(x^*))}{2z_1 f(x^*) + zf''(x^*) - z_{11}[1 - F(x^*)]} = \frac{z_{12} z_1 z_2}{(z_1)^2} \times \frac{1}{2 + \frac{z f'}{z_1} - \frac{z_{11}}{(z_1)^2}}
\]  

(2.2)

From the first line to the second line, we plugged in the equilibrium condition and rearranged terms.

We also know that \(\frac{1 - F}{f}\) decreases in \(x\). This implies

\[
\frac{\partial}{\partial x} \left(\frac{1 - F}{f}\right) = \frac{-f^2 - (1 - F)f'}{f^2} < 0
\]

\[
\Rightarrow 1 + \frac{1 - F}{f} \frac{f'}{f} > 0 \Rightarrow \frac{z f'}{z_1 f} > -1
\]

Therefore,

\[
2 + \frac{z f'}{z_1} - \frac{z_{11}}{(z_1)^2} > 2 - 1 - 1 \geq 0
\]

The sorting is positive if \(z_{12} z_1 z_2 \geq 0\), i.e., \(z(x, y)\) is log-supermodular; It is negative if \(z_{12} z_1 z_2 \leq 0\), i.e., \(z(x, y)\) is log-submodular. ■
To avoid repetition, we will only focus only on the PAM case in the following discussion, since the intuition for NAM is symmetric. Notice that log-supermodularity is a stronger condition than supermodularity, because $z_{12}$ has to be larger than $\frac{z_1 z_2}{z}$, which is strictly positive. Hence with search and information frictions, we need stronger complementarity to ensure positive sorting.

To see the intuition behind this result, consider two sellers, one with a higher type $y_1$ and the other one with a lower type $y_2$. Suppose currently they choose the same marginal type and each of them is deciding whether or not to raise the marginal type, facing the trade-off between price and probability of trade.

If both raise the marginal type by one unit, the type $y_1$ seller would enjoy a greater corresponding price increase because of supermodularity. This means that a higher type seller has a stronger incentive to raise her marginal type. On the other hand, the loss of giving up the marginal type buyer is the current selling price times the probability of meeting the marginal type. Since the price of $y_1$ is strictly higher, she loses more from a reduced trading probability. Hence, a higher type seller also has stronger incentive to increase her trading probability, or equivalently, to lower her marginal type.

Recall that PAM requires a higher type seller to choose a higher type marginal buyer. Therefore, the first effect must be large enough to outweigh the second one. In other words, supermodularity is insufficient to ensure positive sorting here. Rather, positive sorting in this case requires an output function with stronger complementarity—specifically, log-supermodularity.
In Figure 1, we plot the marginal type function $x^*(y)$ with parameter specifications as shown beneath the figure. From this example, we can easily verify two conclusions we had. First of all, the log-supermodular condition is stronger than supermodular: the output function $z(x,y)$ is always supermodular, but it is log-supermodular if and only if $\kappa > \eta^2$. Secondly, the sorting is positive if and only if $z(x,y)$ is log-supermodular.

When sorting is positive, higher type sellers are more willing to sacrifice trade likelihood for a higher price.

**Lemma 4:** If sorting is positive, $P(y)$ is increasing in $y$ and the trading probability decreases in $y$.

**Proof:** Because the trading probability equals $1 - F_B(x^*(y))$, as $x^*(y)$ increases in $y$, the trading probability decreases. To show $P(y)$ increases in $y$, notice that,

$$\frac{\partial P(y)}{\partial y} = z_1 \frac{\partial x^*}{\partial y} + z_2 > 0$$
2.5 Limits of Search Frictions: $\beta \to 1$ (Frictionless Limit)

In this section, we consider what happens when the time between periods shrinks to zero—that is, as the actual discount factor $\beta \to 1$.

**Theorem 3.** Let either A2-SUP or A2-SUB hold. For any $\xi > 0$, there exists $\epsilon > 0$ such that for any $\beta > 1 - \epsilon$,

1. $d(x, y) = 1$ if and only if $s(x, y) \in [0, \xi]$;

2. $\mu(M_B(x)) \in [0, \xi]$ and $\mu(M_S(y)) \in [0, \xi]$;

3. Matching sets converge to perfect positive sorting if $z(x, y)$ is supermodular, i.e., there exists a strictly increasing function $m(x)$ defined on $[x, \bar{x}]$ such that (i) $m(x) = y$ and $m(\bar{x}) = \bar{y}$, and (ii) for any $(x, y)$ with $d(x, y) = 1$, $|x - m^{-1}(y)| < \xi$ and $|y - m(x)| < \xi$.

4. Matching sets converge to perfect negative sorting if $z(x, y)$ is submodular, i.e., there exists a strictly decreasing function $m(x)$ defined on $[x, \bar{x}]$ such that (i) $m(x) = \bar{y}$ and $m(\bar{x}) = y$, and (ii) for any $(x, y)$ with $d(x, y) = 1$, $|x - m^{-1}(y)| < \xi$ and $|y - m(x)| < \xi$.

As search frictions vanish, we find that Becker’s result can be restored even if information frictions remain, that is, supermodularity (submodularity) is sufficient to ensure positive (negative) sorting in the limit. To understand this result, note that although sellers still face the trade-off between price and trading probability per period, they care less and less about the latter as they meet buyers more and more frequently. This is because, however small their matching sets are, sellers can sell almost surely before experiencing the death shock. Thus, any seller with a non-empty matching set would prefer to raise her price.
Recall the intuition presented for the static case. A higher type seller faces stronger incentives both to secure trade and to raise the marginal type. The direction of sorting depends on the relative strength of these two incentives. As argued in the last paragraph, the first incentive grows inconsequential as we approach the frictionless limit. Therefore, supermodularity is sufficient to ensure positive sorting.

Based on the function $m(x)$, we can further derive the equilibrium price and thus the division of surplus in equilibrium.

**Theorem 4.** If $z(x, y)$ is supermodular or submodular, equilibrium prices converge pointwise to the price function $P^*(y)$, where

$$
P^*(y) = z(x, y) + \int_y^y z_2(m^{-1}(\tilde{y}), \tilde{y})d\tilde{y}
$$

The above theorem shows that besides the equilibrium matching set, the equilibrium price also approaches its Walrasian counterpart: each player gets her marginal contribution in the limit. Because the buyer in a match can meet another trading partner almost immediately and almost for sure, the seller faces competitions from other sellers. The price increment of a higher type seller thus equals the seller’s marginal contribution.

### 2.6 Conclusion

The presence of buyer private information does impede sorting, and we have highlighted the relationship between the strength of this effect and the degree of competition in the market. At one extreme, when there is bilateral monopoly power in each buyer-seller meeting, PAM requires a log-supermodular production function, which is of course a stronger condition than standard
supremodularity. Higher types also have higher opportunity costs of failing to trade, so the added incentives to ensure trade takes place are in conflict with sorting.

These incentives remain relevant in a dynamic frictional setting, but they grow inconsequential as we approach the frictionless limit. Thus, as search frictions vanish, the sorting consequences of private information do as well, and the standard supremodularity condition is sufficient to generate positive sorting.
APPENDIX

Proof of Lemma 1

Proof:

- **Inequality for $V_0(x)$ and non-negativity of $V_0(x)$**

  First of all,

  $$V_0(x) = \int_{\mathcal{M}_B(x)} (z(x,y) - P(y))f_S(y)dy + \beta \int_{\mathcal{M}_B(x)} f_S(y)dy V_0(x)$$

  $$\Rightarrow V_0(x) = \frac{1}{1 - \beta} \int_{\mathcal{M}_B(x)} (z(x,y) - P(y) - \beta V_0(x))f_S(y)dy$$

  Any $M \neq \mathcal{M}_B(x)$ either excludes $y \in \mathcal{M}_B(x)$, in which case $z(x,y) - P(y) - \beta V_0(x) > 0$, or includes $y \notin \mathcal{M}_B(x)$, in which case $z(x,y) - P(y) - \beta V_0(x) < 0$. The inequality thus follows.

  Clearly, $V_0(x)$ is non-negative.

- **$V_0(x)$ increasing in $x$**

  Consider $x_2 \geq x_1$,

  $$(1 - \beta)[V_0(x_2) - V_0(x_1)]$$

  $$= \int_{\mathcal{M}_B(x_2)} (z(x_2,y) - P(y) - \beta V_0(x_2))f_S(y)dy - \int_{\mathcal{M}_B(x_1)} (z(x_1,y) - P(y) - \beta V_0(x_1))f_S(y)dy$$

  $$\geq \int_{\mathcal{M}_B(x_1)} [z(x_2,y) - z(x_1,y) - \beta(V_0(x_2) - V_0(x_1))]f_S(y)dy$$

  $$\Rightarrow V_0(x_2) - V_0(x_1) \geq \frac{\int_{\mathcal{M}_B(x_1)} [z(x_2,y) - z(x_1,y)]f_S(y)dy}{1 - \beta + \beta \int_{\mathcal{M}_B(x_1)} f_S(y)dy} \geq 0$$

- **$V_0(x)$ Lipschitz-continuous**

  Following the same steps,

  $$V_0(x_2) - V_0(x_1) \leq \frac{\int_{\mathcal{M}_B(x_2)} [z(x_2,y) - z(x_1,y)]f_S(y)dy}{1 - \beta + \beta \int_{\mathcal{M}_B(x_2)} f_S(y)dy}$$
Combine the two inequalities and use the fact that $|z(x_2, y) - z(x_1, y)| \leq \kappa(x_2 - x_1)$,

$$\frac{-\kappa(x_2 - x_1) \int_{\mathbb{M}_B(x_1)} f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}_B(x_1)} f_S(y) dy} \leq V_0(x_2) - V_0(x_1) \leq \frac{\kappa(x_2 - x_1) \int_{\mathbb{M}_B(x_2)} f_S(y) dy}{1 - \beta + \beta \int_{\mathbb{M}_B(x_2)} f_S(y) dy}$$

$|V_0(x_2) - V_0(x_1)| \leq \kappa(x_2 - x_1)$ follows and thus $V_0(x)$ is L-continuous.

- **Differentiability of $V_0(x)$**

  **Step 1:** $\mathbb{M}_B(x)$ is u.h.c. and a.e. l.h.c.

  - First, we show $\mathbb{M}_B(x)$ is u.h.c. Take any sequence $(x_n, y_n) \to (x, y)$ with $y_n \in \mathbb{M}_B(x_n)$ for any $n$. Therefore $z(x_n, y_n) - \beta V_0(x_n) - P(y_n) \geq 0$ for all $n$. In the limit, $z(x, y) - \beta V_0(x) - P(y) \geq 0$ because $z(x, y)$, $V_0(x)$ and $P(y)$ are continuous. This implies $y \in \mathbb{M}(x)$.

  - Next, we show that surplus function is rarely constant in one variable.

  Define $N_s(x) = \{y : s(x, y) = 0\}$, $N_s(y) = \{x : s(x, y) = 0\}$ and $N_s = \{(x, y) : s(x, y) = 0\}$. Pick $x \neq x'$ and $y \neq y'$, such that $s(x, y) = s(x', y) = s(x, y') = 0$. If A2-Sup or A2-Sub is satisfied, it must be true that $s(x', y') \neq 0$. To see that, notice

  $$s(x, y) - s(x', y) = z(x, y) - z(x', y) - \beta(V_0(x) - V_0(x'))$$

  $$s(x, y') - s(x', y') = z(x, y') - z(x', y') - \beta(V_0(x) - V_0(x'))$$

  Because $z(x, y) - z(x', y) \neq z(x, y') - z(x', y')$, $0 = s(x, y) - s(x', y) \neq s(x, y') - s(x', y')$, which implies $s(x', y') \neq 0$.

  Following a similar approach to that in Appendix B of Shimer and Smith (2000), we show that the following measures are zero almost everywhere: $\mu(N_s(x)) = 0$ for a.e. $x$, $\mu(N_s(y)) = 0$ for a.e. $y$ and $\mu(N_s) = 0$ a.e.
To show a.e. l.h.c. take any sequence $x_n \to x$ and any $y \in \mathbb{M}_B(x)$. First consider the case where there exists a subsequence $x_m$ of $x_n$, such that for any $x_m$,

$$\sup_{\hat{y}} \{ z(x_m, \hat{y}) - \beta V_0(x_m) - P(\hat{y}) \} \geq z(x, y) - \beta V_0(x) - P(y) \geq \inf_{\hat{y}} \{ z(x_m, \hat{y}) - \beta V_0(x_m) - P(\hat{y}) \}$$

By the continuity of $z(x, y)$ and $P(y)$, there exists at least one $y_m$ that satisfies,

$$z(x_m, y_m) - \beta V_0(x_m) - P(y_m) = z(x, y) - \beta V_0(x) - P(y)$$

If there are multiple solutions, pick the one that is the closest to $y$. This defines a sequence $\{ y_m \}$. Clearly, $y_m \in \mathbb{M}_B(x_m)$. We need to show that for almost all $x$, there exists a subsequence $y_k$ of $y_m$, such that $y_k \to y$.

First of all, a convergent subsequence always exists because $\{ y_m \}$ is bounded. Suppose $y_k \to \tilde{y}$.

**Case 1.** We first consider the case where there exist $\epsilon > 0$ and $K > 0$ such that for any $k > K$ and any $\hat{y} \in [y - \epsilon, y + \epsilon]$, $\hat{y} \notin \mathbb{M}_B(x_k)$. In this scenario, it is possible that $\tilde{y} \neq y$.

However, the measure of $\{ x, y \}$ that satisfies this condition is 0.

**Case 2.** Next, we consider all complementary scenarios, that is, for any $\epsilon > 0$ and $K > 0$, there exist $k > K$ and $\hat{y} \in [y - \epsilon, y + \epsilon]$, such that $\hat{y} \in \mathbb{M}_B(x_k)$.

Suppose $y_k \to \tilde{y} \neq y$. Since we have excluded the two situations in the previous two steps, there exists $\epsilon > 0$ and $K_1 > 0$, such that for any $k > K_1$,

$$| z(x_k, y) - z(x, y) - \beta(V_0(x_k) - V_0(x)) | > 2\epsilon$$

On the other hand, by continuity of $z$ and $V_0$, there exist $K_2$, such that for all $k > K_2$,

$$| z(x_k, y) - z(x, y) - \beta(V_0(x_k) - V_0(x)) |$$

$$\leq | z(x_k, y) - z(x, y) | + \beta | V_0(x_k) - V_0(x) | < 2\epsilon$$

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Finally, pick \( K = \max\{K_1, K_2\} \). The above two inequalities hold at the same time. This is a contradiction.

Finally, consider the case where subsequence \( x_m \) does not exist, i.e., for any subsequence \( x_m \), either \( \sup_{\hat{y}} \{ z(x_m, \hat{y}) - \beta V_0(x_m) - P(\hat{y}) \} < z(x, y) - \beta V_0(x) - P(y) \) or \( \inf_{\hat{y}} \{ z(x_m, \hat{y}) - \beta V_0(x_m) - P(\hat{y}) \} > z(x, y) - \beta V_0(x) - P(y) \). Here we show the proof just for the first case since the second case is similar.

Define \( y_m \in \text{argmax}\{ z(x_m, \hat{y}) - \beta V_0(x_m) - P(\hat{y}) \} \). If there is more than one argmax, pick the one that is closest to \( y \). By constructing \( y_m \) this way, we can then follow the same proof as in the first case to show that any convergent subsequence of \( y_m \) must converge to \( y \).

**Step 2: Decomposition of the slope of outside option**

- Take any sequence \( x_n \to x \), for each \( n \),

\[
(1 - \beta) \frac{V_0(x_n) - V_0(x)}{x_n - x} = \int_{M_B(x_n) - M_B(x)} \frac{z(x_n, y) - P(y) - \beta V_0(x_n)}{x_n - x} f_S(y)dy + \int_{M_B(x)} \frac{z(x_n, y) - z(x, y)}{x_n - x} - \beta \frac{V_0(x_n) - V_0(x)}{x_n - x} f_S(y)dy
\]

Take the limit as \( n \to \infty \); the first integral vanishes because 1) \( M_B(x) \) is continuous a.e. and when it is not continuous, the measures of the limiting set and the set in the limit are the same, and 2) buyers participate optimally. Rearranging terms, we get the proposed derivative.

[Proof of lemma 2]
Proof: We’ve proved that the surplus function is rarely constant in one variable in lemma 1. Define the set \( s(\eta) = \{(x, y) : |s(x, y)| \in [0, \eta]\} \). This set shrinks monotonically to \( \bigcap_{k=1}^{\infty} s(1/k) = N_s \).

\[
\lim_{\eta \to 0} (\mu \times \mu)(\sum s(\eta)) = (\mu \times \mu)(\bigcap_{k=1}^{\infty} s(1/k)) = (\mu \times \mu)(N_s) = 0
\]

Let \( V_0^1 \) and \( V_0^2 \) be two outside option functions, and \( d^1 \) and \( d^2 \) be the corresponding match indicator functions.

Since \( P(y, V_0) \) is continuous in \( V_0 \), for any \( \epsilon > 0 \), there exist \( \eta' > 0 \), such that

\[
\beta \| V_0^1(x) - V_0^2(x) \| < \eta' \Rightarrow |P(y, V_0^1) - P(y, V_0^2)| < \epsilon, \text{ for any } y
\]

In words, we can always pick close enough outside option functions such that the price functions are close. Let \( \eta = 2 \max\{\eta', \epsilon\} \).

If \( s^1(x, y) = z(x, y) - \beta V_0^1(x) - P(y, V_0^1) > \eta \), \( s^2(x, y) = z(x, y) - \beta V_0^2(x) - P(y, V_0^2) > \eta \). So \( d^1(x, y) = d^2(x, y) = 1 \). By the same logic, If \( s^1(x, y) < -\eta \), \( s^1(x, y) < 0 \). So \( d^1(x, y) = d^2(x, y) = 0 \).

As a result, \( \{(x, y) : d^1(x, y) \neq d^2(x, y)\} \subseteq s^1(\eta) \). The Lebesgue measure of \( s^1(\eta) \) vanishes as \( \eta \to 0 \). The continuity is thus established \( \lim_{\|V_0^1(x) - V_0^2(x)\| \to 0} \| d^1(x, y) - d^2(x, y) \| = 0 \).

Proof of Lemma 3

Proof:

- **Step 1: The mapping is well-defined.**

Given entrant densities \( \gamma_B(x) \) and \( \gamma_S(y) \), the mapping is well-defined if there exist unique
functions \( f_B \) and \( f_S \) which solve the following system of equations,

\[
\begin{align*}
    f_B(x) &= \frac{\gamma_B(x)}{\beta + (1 - \beta) \int d(x,y) dy / f_S(y) dy} \\
    f_S(y) &= \frac{\gamma_S(y)}{\beta + (1 - \beta) \int d(x,y) dx / f_B(x) dx}
\end{align*}
\]

From those conditions, we know that \( f_B(x) \in [\gamma_B(x), \gamma_B(x)/\beta] \) and \( f_S(y) \in [\gamma_S(y), \gamma_S(y)/\beta] \).

One can apply a log transformation method similar to that used in Shimer and Smith (2000) and reformulate the problem as a fixed-point problem.

\[
\begin{align*}
    \Phi_B(h) &= \log \frac{\gamma_B(x)}{\beta + (1 - \beta) \int d(x,y) e^{h_S(y)} dy / e^{h_B(y)} dy} \\
    \Phi_S(h) &= \log \frac{\gamma_S(y)}{\beta + (1 - \beta) \int d(x,y) e^{h_B(x)} dx / e^{h_S(x)} dx}
\end{align*}
\]

where \( h_B(x) = \log f_B(x) \), \( h_S(y) = \log f_S(y) \), \( h = (h_B, h_S) \). The mapping is well defined if \( \Phi(h) = h \) has a unique fixed point. We prove this using the Contraction Mapping Theorem.

Consider \( h^1 \) and \( h^2 \),

\[
\begin{align*}
    \Phi_B(h^2) - \Phi_B(h^1) &= \log \frac{\gamma_B(x)}{\beta + (1 - \beta) \int d(x,y) e^{h^1_S(y)} dy / e^{h^1_B(y)} dy} \\
    &\leq \log \frac{\beta + (1 - \beta) e^{h^2_S(y) - h^1_S(y)} \int d(x,y) e^{h^2_S(y)} dy / e^{h^1_S(y)} dy}{\beta + (1 - \beta) \int d(x,y) e^{h^2_B(x)} dx / e^{h^1_B(x)} dx} \\
    &\leq \log \frac{\beta + (1 - \beta) e^{\|h^1_S - h^2_S\|}}{\beta + (1 - \beta)} \\
    &= \log [\beta + (1 - \beta) e^{\|h^1_S - h^2_S\|}]
\end{align*}
\]

The first inequality follows because \( e^{\|h^1_S - h^2_S\|} > e^{h^1_S(y) - h^2_S(y)} \) for any \( y \). We thus have

\[
\frac{\Phi_B(h^2) - \Phi_B(h^1)}{\|h^1_S - h^2_S\|} \leq \frac{\log [\beta + (1 - \beta) e^{\|h^1_S - h^2_S\|}]}{\|h^1_S - h^2_S\|}
\]

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In addition, we know that \( h_S(y) \in [\log(\gamma_S(y)), \log(\gamma_S(y)) - \log \beta] \), which implies \( \| h_S^1 - h_S^2 \| \in [0, -\log \beta] \). Since the right hand side of the above inequality increases in \( \| h_S^1 - h_S^2 \| \),

\[
\frac{\Phi_B(h^2) - \Phi_B(h^1)}{\| h_S^1 - h_S^2 \|} \leq \frac{\log[\beta + \frac{1-\beta}{\log \beta}]}{\log \frac{1}{\beta}} = \chi \in (0, 1)
\]

The same argument applies in the other direction and we thus obtain

\[
\frac{\| \Phi_B(h^1) - \Phi_B(h^2) \|}{\| h_S^1 - h_S^2 \|} \leq \chi
\]

We have the symmetric inequality for \( y \). Denote \( \Phi(h) = (\Phi_B(h), \Phi_S(h))^t \), combining the two inequalities,

\[
\| \Phi(h^1) - \Phi(h^2) \| \leq A \| h^1 - h^2 \|
\]

where \( A \) is a matrix with \( |A| = -\chi^2 \in (-1, 1) \). We have thus proven that it is a contraction mapping.

- **Step 2: Continuity**

Define \( G_B(d, f)(x) = \phi_B(x)[\beta + (1 - \beta) \int d(x, y) \frac{f_S(y)}{f_S(y) dy} dy] - \gamma_B(x) \) and \( G_S(d, f)(y) = \phi_S(y)[\beta + (1 - \beta) \int d(x, y) \frac{f_B(x)}{f_B(x) dx} dx] - \gamma_S(y) \), \( G(d, f) = (G_B(d, f), G_S(d, f)) \). In equilibrium, \( G(d, f) = 0 \)

Suppose that there exist \( d^1 \) and \( d^2 \) with \( \| d^1 - d^2 \| \rightarrow 0 \), such that \( \| f^1 - f^2 \| \rightarrow 0 \). Then \( \| G(d^1, f^2) \| \rightarrow \varepsilon \). WLOG, assume \( \| G_B(d^1, f^2) \| > \varepsilon \).

On the other hand,

\[
\| G_B(d^1, f^2) \| = \| G_B(d^1, f^2) - G_B(d^2, f^2) \| \leq \varepsilon
\]

\[
= \| f^2_B(x)(1 - \beta) \int (d^1(x, y) - d^2(x, y)) \frac{f^2_S(y)}{f^2_S(y) dy} dy \| < \varepsilon
\]
The last line follows since \( \frac{f_B^2(y)}{f_S^2(y) dy} \) and \( f_B^2(x) \) are bounded for any \( x \) and \( y \). This leads to a contradiction.

**Proof of Proposition 1**

**Proof:**

- **Step 1:** Equilibrium exists if \( T(V_0) = V_0 \) has unique fixed point, where,

  \[
  T(V_0) = \int \max \{ z(x) - P(y, V_0), \beta V_0(x) \} f_S^{V_0}(y) dy
  \]

- **Step 2:** Following the Schauder Fixed Point Theorem, we need a nonempty, closed, bounded and convex domain \( \psi \) such that,

  1. \( T : \psi \rightarrow \psi \).

  2. \( T(\psi) \) is an equicontinuous family.

  3. \( T \) is a continuous operator.

  Let \( \psi \) be the space of L-continuous functions \( V_0 \) on \([x, \bar{x}]\), with lower bound 0 and upper bound \( \sup_{x,y} z(x, y) \). Clearly, \( \psi \) is nonempty, closed, bounded and convex.

- **Step 3:** \( T : \psi \rightarrow \psi \text{ and } T(\psi) \text{ is an equicontinuous} \). Take \( x_2 \neq x_1 \),

  \[
  | TV_0(x_2) - TV_0(x_1) | \\
  \leq \int | \max \{ z(x_2, y) - P(y, V_0), \beta V_0(x_2) \} - \max \{ z(x_1, y) - P(y, V_0), \beta V_0(x_1) \} | f_S^{V_0}(y) dy \\
  \leq \int | \max \{ z(x_2, y) - z(x_1, y), \beta(V_0(x_2) - V_0(x_1)) \} | f_S^{V_0}(y) dy
  \]

  Since both \( z(x, y) \) and \( V_0(x) \) are L-continuous, \( T(\psi) \) is L-continuous, which implies equicontinuous. This also establishes \( T \) is a mapping from \( \psi \) to \( \psi \).
Step 4: \textbf{T is continuous.} Take $V_0^2 \neq V_0^1$ in $\psi$, for any $x$,

$$\left| TV_0^2(x) - TV_0^1(x) \right|$$

$$= \left| \int \max\{z(x,y) - P(y,V_0^2), \beta V_0^2(x)\} f_S^{V_0^2}(y)dy - \int \max\{z(x,y) - P(y,V_0^1), \beta V_0^1(x)\} f_S^{V_0^1}(y)dy \right|$$

$$\leq \left| \int \max\{z(x,y) - P(y,V_0^2), \beta V_0^2(x)\} f_S^{V_0^2}(y)dy - \int \max\{z(x,y) - P(y,V_0^1), \beta V_0^1(x)\} f_S^{V_0^1}(y)dy \right|$$

$$+ \left| \int (\max\{z(x,y) - P(y,V_0^2), \beta V_0^2(x)\} - \max\{z(x,y) - P(y,V_0^1), \beta V_0^1(x)\}) f_S^{V_0^2}(y)dy \right|$$

$$= D_1(x) + D_2(x)$$

For $D_1(x)$

$$D_1(x) \leq \int \max\{z(x,y) - P(y,V_0^2), \beta V_0^2(x)\} \left| f_S^{V_0^2}(y) - f_S^{V_0^1}(y) \right| dy$$

$$\leq \sup_{x,y} \left[ \max\{z(x,y) - P(y,V_0^2), \beta V_0^2(x)\} \right] \int \left| f_S^{V_0^2}(y) - f_S^{V_0^1}(y) \right| dy$$

Since $f_S(y)$ is continuous in $V_0$, as $\| V_0^2 - V_0^1 \| \to 0$, $D_1(x) \to 0$.

For $D_2(x)$,

$$D_2(x) \leq \int \left| \max\{z(x,y) - P(y,V_0^2), \beta V_0^2(x)\} - \max\{z(x,y) - P(y,V_0^1), \beta V_0^1(x)\} \right| f_S^{V_0^1}(y)dy$$

$$\leq \int \left| \max\{P(y,V_0^1) - P(y,V_0^2), \beta V_0^2(x) - \beta V_0^1(x)\} \right| f_S^{V_0^1}(y)dy$$

Since $P(y,V_0)$ is continuous in $V_0$, as $\| V_0^2 - V_0^1 \| \to 0$, $D_2(x) \to 0$.

Step 5: \textbf{Verify that there exists at least one price function that is continuous in $y$ and $V_0$ in equilibrium.}

If it does not cause any confusion, we will abuse the notation and use $P(y,V_0)$ to also denote the set of optimal prices of a seller with type $y$ given $V_0$. Seller’s problem is $\max_p \Omega(y,p,V_0)$. $\Omega(y,p,V_0)$ is continuous in those three arguments because the matching set $M_S(p; y, V_0)$ is
almost everywhere continuous\(^6\). In addition, \( p \in [0, \sup_{x,y} \{ z(x,y) \}] \), which is compact valued.

By the Maximum Theorem, \( P(y, V_0) \) is u.h.c. in \( y \) and \( V_0 \).

Next, we can show that \( P(y, V_0) \) is also l.h.c. in \( y \) and \( V_0 \). If we decompose the slope of \( \Omega \) along price,

\[
(1 - \beta) \frac{\Omega(y, p_n, V_0) - \Omega(y, p, V_0)}{p_n - p} = [p_n - \beta \Omega(y, p_n, V_0)] \frac{\int_{M(y, p_n, V_0)} f_B(x) dx}{p_n - p} + [1 - \beta \frac{\Omega(y, p_n, V_0) - \Omega(y, p, V_0)}{p_n - p}] \frac{\int_{M(y, p, V_0)} f_B(x) dx}{p_n - p}
\]

\[= [p_n - \beta \Omega(y, p_n, V_0)] \sum_{i=1}^{K_1} f_{x_i}(y, p_n, V_0) f_B(x_i) dx - \sum_{j=1}^{K_2} \int_{z_j(y, p, V_0)} f_B(x) dx + [1 - \beta \frac{\Omega(y, p_n, V_0) - \Omega(y, p, V_0)}{p_n - p}] \frac{\int_{M(y, p, V_0)} f_B(x) dx}{p_n - p}
\]

\[= [p_n - \beta \Omega(y, p_n, V_0)] \sum_{i=1}^{K_1} f_{x_i}(y, p, V_0) f_B(x_i) dx - \sum_{j=1}^{K_2} \int_{z_j(y, p, V_0)} f_B(x) dx + [1 - \beta \frac{\Omega(y, p_n, V_0) - \Omega(y, p, V_0)}{p_n - p}] \frac{\int_{M(y, p, V_0)} f_B(x) dx}{p_n - p}
\]

\( K_1 \) and \( K_2 \) in the first term might be infinite, but there are always countable number of bounds in the matching set. Take \( p_n \to p \), we get the derivative,

\[
\Omega_2(y, p, V_0) = \frac{\int_{M(y, p, V_0)} f_B(x) dx + D(y, p, V_0)}{1 - \beta + \beta \int_{M(y, p, V_0)} f_B(x) dx}
\]

where, \( D(y, p, V_0) = \sum_{i} [f_B(\tilde{x}_i(y, p, V_0)) \frac{\partial \tilde{x}_i(y, p, V_0)}{\partial p} - f_B(\tilde{x}_i(y, p, V_0)) \frac{\partial \tilde{x}_i(y, p, V_0)}{\partial p}] \)

\[
\frac{\partial \tilde{x}_i(y, p, V_0)}{\partial p} = \frac{1}{z_1(\tilde{x}_i(y, p, V_0), y) - \beta V_0'(\tilde{x}_i(y, p, V_0))}
\]

\[
\frac{\partial x_i(y, p, V_0)}{\partial p} = \frac{1}{z_1(\tilde{x}_i(y, p, V_0), y) - \beta V_0'(\tilde{x}_i(y, p, V_0))}
\]

The last two lines follow from the boundary condition and Implicit Function Theorem.

\(^6\)The proof of the a.e. continuity of \( M(p, y, V_0) \) is similar to the proof of the a.e. continuity of \( M(x) \) and thus is skipped here. The proof is available upon request.
Take any sequence \((y^n, V_0^n) \to (y, V_0)\). Consider any \(p \in \text{argmax}_p \Omega(\hat{y}, \hat{p}, V_0)\). For interior \(p\), \(p\) solves the first order condition \(\Omega_2(y, p, V_0) = 0\). To show \(P(y, V_0)\) is l.h.c., we need to construct a sequence \(p_n\), such that

1. \(p_n \in P(y^n, V_0^n)\) and,
2. \(p_n \to p\).

Pick sequence \(p_n\) such that it solves \(\Omega_2(y^n, p_n, V_0^n) = 0\) and it is the solution that is closest to \(p\). (Without loss of generality, assume interior solutions exist along the sequence.) The first condition is satisfies by construction. Since the sequence is bounded, there must exist a convergent sequence \(p_k\). Suppose \(p_k \to p' \neq p\). Then there exists \(\epsilon > 0\) and \(K_1 > 0\), such that for any \(k > K_1\),

\[
| \Omega_2(y^k, p, V_0) | > \epsilon
\]

On the other hand, if we can show that \(\Omega_2(y, p, V_0)\) is continuous in \(y\) and \(V_0\), then for any \(\epsilon > 0\), there exist \(K_2 > 0\), such that for any \(k > K_2\),

\[
| \Omega_2(y^k, p, V_0^k) - \Omega_2(y, p, V_0) | = | \Omega_2(y^k, p, V_0^k) | < \epsilon
\]

Define \(K = \max\{K_1, K_2\}\), the above two inequalities hold simultaneously for \(\epsilon\) and any \(k > K\). It is a contradiction.

The only step left is to show the continuity of \(\Omega_2(y, p, V_0)\), which is equivalent to showing the continuity of \(D(y, p, V_0)\).
First we claim that \( f_B(x) \) is continuous in \( x \) for any given \( V_0 \). Notice that

\[
0 \leq | \log f_B(x_n) - \log f_B(x) | = | \log \gamma_B(x_n) - \log \gamma_B(x) + \log \frac{\beta + (1 - \beta) \int_{M_B(x)} f_S(y) dy}{\beta + (1 - \beta) \int_{M_B(x_n)} f_S(y) dy} | \leq | \log \gamma_B(x_n) - \log \gamma_B(x) | + | \log \frac{\beta + (1 - \beta) \int_{M_B(x)} f_S(y) dy}{\beta + (1 - \beta) \int_{M_B(x_n)} f_S(y) dy} |
\]

Both absolute values go to zero when \( x_n \to x \) because of the continuity of \( \gamma_B \) and \( M_B(x) \). Therefore, \( \log f_B(x) \) is continuous and so is \( f_B(x) \).

Next we show that \( \| (V_0^n)' - V_0' \| \to 0 \) as \( V_0^n - V_0 \| \to 0 \).

\[
(1 - \beta)(V_0^n)' = \int_{M_B(x,V_0^n)} (z_1(x,y) - \beta(V_0^n)') f_S^V_0(y) dy
\]

\[
\Rightarrow (V_0^n)' - V_0' = \frac{1}{1 - \beta + \beta \int_{M_B(x,V_0^n)} f_S^V_0(y) dy} \int_{M_B(x,V_0^n) - M_B(x,V_0)} [z_1(x,y) - \beta(V_0^n)' f_S^V_0(y) dy
\]

\[
+ \int_{M_B(x,V_0)} [z_1(x,y) - \beta V_0' f_S^V_0(y) - f_S^V_0(y)] dy
\]

\[
\Rightarrow \| (V_0^n)' - V_0' \| \to 0 \text{ as } \| V_0^n - V_0 \| \to 0
\]

As a result, \( \frac{\partial x_i(y,p,V_0)}{\partial p} \) and \( \frac{\partial x_i(y,p,V_0)}{\partial p} \) are continuous in \( y \) and \( V_0 \). Combining with the continuity of \( \bar{x}_i, \bar{x}_i \) and \( f_B(x) \), \( D(y,p,V_0) \) is continuous in \( y \) and \( V_0 \).

In sum, \( P(y,V_0) \) is u.h.c. and l.h.c., and thus continuous. Therefore, there exists a price function \( P(y,V_0) \) that is continuous in \( y \) and \( V_0 \).

Proof of Proposition 2

Proof: A sufficient condition for \( M(y) \) to be convex is \( s(x,y) \) being quasi-concave in \( x \). Since \( s(x,y) \) is continuous and differentiable in \( x \), by definition, we only need show that \( s(x_1,y) < s(x_2,y) \) for \( x < x_1 < x_2 \) implies \( s(x_1,y) \geq 0 \).
Fix any \( \underline{x} < x_1 < x_2 < \bar{x} \).

1. Step 1: \( s_1(x, y) \geq 0 \) for all \( y \geq \hat{y} \) (where \( \hat{y} \) is defined below).

   In the spirit of Diamond and Stiglitz (1974), Shimer and Smith (2000) adopt a Single Crossing Property (SCP) for gambles in their analysis, and the usefulness of this property extends to our setting. Since \( z_{12}(x, y) \) is log-supermodular, there exists \( \hat{y} \) such that

   \[
   z_1(x_1, \hat{y}) = \frac{\int_{M(x_1)} z_1(x_1, y) f_S(y) dy}{\int_{M(x_1)} f_S(y) dy}
   \]

   Therefore,

   \[
   \beta V_0'(x_1) = \frac{\beta \int_{M(x_1)} z_1(x_1, y) f_S(y) dy}{1 - \beta + \beta \int_{M(x_1)} f_S(y) dy} = \frac{\beta z_1(x_1, \hat{y}) \int_{M(x_1)} f_S(y) dy}{1 - \beta + \beta \int_{M(x_1)} f_S(y) dy} \leq z_1(x_1, \hat{y})
   \]

   This implies \( s_1(x_1, \hat{y}) = z_1(x_1, \hat{y}) - \beta V_0'(x_1) \geq 0 \). By the supermodularity of \( z(x, y) \), for any \( y \geq \hat{y} \), \( s_1(x_1, y) \geq s_1(x_1, \hat{y}) \geq 0 \).

2. Step 2: \( V_0(x_1) < V_0(x_2) \) and \( z(x_1, y) < z(x_2, y) \) whenever \( s(x_1, y) < s(x_2, y) \) and \( y < \hat{y} \).

   If \( s(x_1, y) \geq s(x_2, y) \) at \( x_2 \), there is nothing to verify.

   If \( s(x_1, y) < s(x_2, y) \) at \( x_2 \),

   \[
   z(x_2, y) - z(x_1, y) > \beta V_0(x_2) - \beta V_0(x_1)
   = \frac{\beta \int_{M(x_2)} z(x_2, y) f_S(y) dy}{1 - \beta + \beta \int_{M(x_2)} f_S(y) dy} - \frac{\beta \int_{M(x_1)} z(x_1, y) f_S(y) dy}{1 - \beta + \beta \int_{M(x_1)} f_S(y) dy}
   \geq \frac{\beta \int_{M(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy}{1 - \beta + \beta \int_{M(x_1)} f_S(y) dy}
   \]

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By the SCP for gambles, for all \( x' > x_1 \),

\[
\frac{\int_{M(x_1)} z_1(x', y) f_S(y) dy}{\int_{M(x_1)} f_S(y) dy} \geq z_1(x', \hat{y})
\]

Integrate over \( x' \in [x_1, x_2] \),

\[
\frac{\int_{M(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy}{\int_{M(x_1)} f_S(y) dy} \geq z(x_2, \hat{y}) - z(x_1, \hat{y})
\]

By strict supermodularity of \( z(x, y) \), \( z(x_2, \hat{y}) - z(x_1, \hat{y}) > z(x_2, y) - z(x_1, y) \). Combining all inequalities,

\[
\frac{\int_{M(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy}{\int_{M(x_1)} f_S(y) dy} > \frac{\beta \int_{M(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy}{1 - \beta + \beta \int_{M(x_1)} f_S(y) dy}
\]

\[
\Rightarrow \int_{M(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy > 0
\]

\[
\Rightarrow z(x_2, y) > z(x_1, y) \text{ and } V_0(x_2) > V_0(x_1)
\]

- Step 3: \( s_1(x_1, y) \geq 0 \) whenever \( s(x_1, y) < s(x_2, y) \) and \( y < \hat{y} \).

If \( s(x_1, y) \geq s(x_2, y) \) at \( x_2 \), there is nothing to verify.

If \( s(x_1, y) < s(x_2, y) \) at \( x_2 \), we know

\[
\begin{align*}
V'_0(x_1) &= \frac{\int_{M(x_1)} z_1(x_1, y) f_S(y) dy}{1 - \beta + \beta \int_{M(x_1)} f_S(y) dy} \\
V_0(x_2) - V_0(x_1) &\geq \frac{\int_{M(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy}{1 - \beta + \beta \int_{M(x_1)} f_S(y) dy}
\end{align*}
\]

\[
\Rightarrow \frac{V'_0(x_1)}{V_0(x_2) - V_0(x_1)} \leq \frac{\int_{M(x_1)} z_1(x_1, y) f_S(y) dy}{\int_{M(x_1)} [z(x_2, y) - z(x_1, y)] f_S(y) dy} \leq \frac{z_1(x_1, \hat{y})}{z(x_2, \hat{y}) - z(x_1, \hat{y})}
\]
By the log-supermodularity of $z_1(x, y)$, for $y < \tilde{y}$,

$$z_1(x_1, \tilde{y})[z(x_2, y) - z(x_1, y)] \leq z_1(x_1, y)[z(x_2, \tilde{y}) - z(x_1, \tilde{y})]$$

$$\Rightarrow \frac{z_1(x_1, \tilde{y})}{z(x_2, \tilde{y}) - z(x_1, \tilde{y})} \leq \frac{z_1(x_1, y)}{z(x_2, y) - z(x_1, y)}$$

$$\Rightarrow \frac{V_0'(x_1)}{V_0(x_2) - V_0(x_1)} \leq \frac{z_1(x_1, y)}{z(x_2, y) - z(x_1, y)}$$

$$\Rightarrow \beta V_0'(x_1) \leq z_1(x_1, y) \frac{\beta V_0(x_2) - \beta V_0(x_1)}{z(x_2, y) - z(x_1, y)} < z_1(x_1, y)$$

The second inequality in the last line follows because $z(x_2, y) - z(x_1, y) > \beta V_0(x_2) - \beta V_0(x_1)$.

Therefore, $s_1(x_1, y) = z_1(x_1, y) - \beta V_0'(x_1) > 0$.

---

**Proof of Theorem**

**Proof:** It is easy to see that seller $y$'s profit $\Pi(y) \rightarrow P(y)1\{M(y) \neq \emptyset\}$ when $\beta \rightarrow 1$.

1. We first show $d(x, y) = 1 \iff s(x, y) = z(x, y) - P(y) - \beta V_0(x) \in [0, \xi)$ for any $\xi > 0$.

   Direction "$\Leftarrow$" follows from the construction of function $d(x, y)$.

   To see the other direction, notice that $d(x, y) = 1$ implies $z(x, y) - P(y) - \beta V_0(x) \geq 0$. Suppose there exist $\tilde{\xi} > 0$, $x$ and $y$ such that $z(x, y) - P(y) - \beta V_0(x) > \tilde{\xi}$ in equilibrium. Then the seller $y$ can raise $\Pi(y)$ by increasing $P(y)$. This leads to a contradiction.

2. Suppose there exist $\tilde{\xi} > 0$ and $y$ such that $\mu(M_S(y)) > \tilde{\xi}$. In the proof of lemma 4, we have proved that the surplus function $s(x, y)$ is rarely constant. Therefore, $\mu(M_S(y)) > \tilde{\xi}$ implies $s(x, y) > \tilde{\xi}$ for some $x$ and $\tilde{\xi}$, contradicting the conclusion of the previous step.

Following the same reasoning, $\mu(M_B(x)) \in [0, \xi)$ for any $\xi > 0$ a.e.
Next, suppose \( s(x, y) \) is non-negative and constant when \( x = \tilde{x} \) and \( y \in [\tilde{y}_1, \tilde{y}_2] \). We claim that \( \mu([\tilde{y}_1, \tilde{y}_2]) \) converges to 0 when \( \beta \to 1 \). Otherwise, type \( \tilde{x} \) buyers will trade and exit the market faster than other types, which makes type \( y \) sellers’ (\( y \in [\tilde{y}_1, \tilde{y}_2] \)) expected profit converge to zero. This is a contradiction.

Finally, suppose \( s(x, y) \) is non-negative and constant when \( y = \tilde{y} \) and \( x \in [\tilde{x}_1, \tilde{x}_2] \). We can again show that \( \mu([\tilde{x}_1, \tilde{x}_2]) \) converges to 0 when \( \beta \to 1 \). Suppose otherwise. Then type \( \tilde{y} \) sellers will trade and exit the market faster than other types, which makes a type \( x \) buyer’s (\( x \in [\tilde{x}_1, \tilde{x}_2] \)) payoff \( V_0(x) \) converge to 0. Then \( s(x, \tilde{y}) \) converges to \( z(x, \tilde{y}) - P(\tilde{y}) \). Since \( z(x, y) \) strictly increases in \( x \), this contradicts \( s(x, \tilde{y}) \) being constant.

The above two paragraphs have shown that, even for those \( x, y \) where \( s(x, y) \) is constant, \( \mu(M_B(x)) \) and \( \mu(M_S(y)) \) converge to 0 when \( \beta \) approaches 1.

3. Pick any \( x, y, x' \) such that \( d(x, y) = 1 \) and \( x > x' \). We first show that there must exist a \( y' \in M_B(x') \), such that \( y' \leq y \), under the assumption of supermodularity. Suppose this is not the case. This implies that if we define \( \tilde{y} \) as \( \inf \{ M_B(x') \} \), then \( \tilde{y} > y \). Using supermodularity, this implies

\[
\begin{align*}
    z(x', y) + z(x, \tilde{y}) - P(y) - P(\tilde{y}) - \beta V_0(x) - \beta V_0(x') > \\
    z(x, y) + z(x', \tilde{y}) - P(y) - P(\tilde{y}) - \beta V_0(x) - \beta V_0(x') = 0 \\
    \Rightarrow s(x', y) + s(x, \tilde{y}) > 0
\end{align*}
\]

On the other hand, we know that \( s(x', y) \leq 0 \) and \( s(x, \tilde{y}) \leq 0 \) for sufficiently large \( \beta \) from the last step. This leads to a contradiction and the claim is proved. Similarly, \( y > y' \) implies that there exists \( x' \in M_S(y') \), such that \( x \geq x' \) is supermodular.
Next, we claim that the matching set of any type of buyer or seller is non-empty if there exist at least one lower type with a non-empty matching set. Suppose otherwise, say \( \mathbb{M}(\hat{x}) = \emptyset \). Define \( \hat{x}^* \) as any element in the set \( \{ x : x < \hat{x} \text{ and } \mathbb{M}(x) \neq \emptyset \} \) and pick any \( \hat{y}^* \in \mathbb{M}(\hat{x}^*) \). It follows that \( V_0(\hat{x}) = 0 \) and \( V_0(\hat{x}^*) \geq 0 \), a contradiction.

Finally, suppose \( \bar{x} \) is the highest buyer type such that \( \mathbb{M}(\bar{x}) = \emptyset \). We will show that for any \( \xi > 0 \), there exists \( \epsilon \), such that \( \bar{x} - \bar{x} < \xi \) for any \( \beta > 1 - \epsilon \). Suppose otherwise. Then it must be the case that \( f_B(x_1)/f_B(x_2) = \infty \) for any \( x_1 \in [\bar{x}, \bar{x}] \) and any \( x_2 \in [\bar{x}, \bar{x}] \). Type \( y \) sellers then have incentives to lower prices slightly in order to increase their trading probabilities.

Summarizing, matching approaches perfect positive sorting under the assumption of super-modularity.

The proof of perfect negative assortative matching under submodularity is essentially the same and thus is skipped here.

\[ \square \]

**Proof of Theorem 4**

**Proof:** We will show the proof only for supermodular output functions \( z(x, y) \). The proof with a submodular \( z(x, y) \) is essentially the same and is available upon request. When \( \beta \to 1 \), for any \( x \), the derivative of buyer’s value function

\[ V_0'(x) \to z_1(x, m(x)) \]

This implies

\[ V_0(x) \to V_0(\bar{x}) + \int_{\bar{x}}^{\bar{x}} z_1(\bar{x}, m(\bar{x})) d\bar{x} \]

\[ = V_0(\bar{x}) + \int_{\bar{x}}^{\bar{x}} dz(\bar{x}, m(\bar{x})) - \int_{\bar{y}}^{m(x)} z_2(m^{-1}(\bar{y}), \bar{y}) d\bar{y} \]
Since \( z(x, m(x)) - P(m(x)) - \beta V_0(x) \to 0 \), the price of \( y = m(x) \) type seller can be computed,

\[
P(y) \to z(m^{-1}(y), y) - V_0(m^{-1}(y))
\]

\[
= z(x, y) + \int_x^{x'} dz(\tilde{x}, m(\tilde{x})) - [(V_0(x) + \int_x^{x'} dz(\tilde{x}, m(\tilde{x})) - \int_{y}^{m(x)} z_2(m^{-1}(\tilde{y}), \tilde{y})d\tilde{y}]
\]

\[
= z(x, y) - V_0(x) + \int_{y}^{y'} z_2(m^{-1}(\tilde{y}), \tilde{y})d\tilde{y}
\]

Here, \( V_0(\bar{x}) = 0 \) following the same argument as in Diamond (1971).
Bibliography


Chapter 3

Optimal Stopping in a Dynamic Lemons Market

3.1 Introduction

In many settings (including markets for residential and commercial real estate, patent sales, markets for used automobiles and airplanes, and firm acquisitions to name a few), asset ownership produces information which may accrue privately to the owner. In general, this information reflects both an underlying asset quality and an idiosyncratic owner specific value. Using oilfields of uncertain yield as an example, information that a particular site has low reserves may depend either on the physical presence of oil or on the specific method of exploration employed by the owner. In response to poor performance, an owner therefore updates slowly over time that her asset has low personal value. When performance is not publicly observed and/or owner-match-specific values are relevant, owner values and buyer values may diverge as a result of this information, leading some owners to sell strategically in response to accrued negative information.
In this paper, we study a setting in which such forces are present. Assets are heterogeneous in quality, and all agents initially have information about only the aggregate asset pool composition, but each owner then learns privately over time about her own asset. As time passes, each owner must continually decide either to retain the asset (which entails an option value of future sales) or to sell it in a market composed of both owners selling their assets based on private information and those selling for reasons unrelated to asset quality.

With private learning, assets are traded at a single price, which reflects the average quality of all assets being sold in the market. The equilibrium relationship between quality and price prevents owners of poorly performing assets from all selling at the same time (as is optimal), since their assets are on average of lower quality. Instead, the equilibrium selling decision of owners is characterized by a broad interval of strategic sales as owners of poorly performing assets attempt to hide behind the sales of higher quality sellers who sell for reasons unrelated to their private information.

The path of prices implied by the selling decisions of owners with private information is characterized by an initial interval of stable prices when owners of poorly performing assets still hold relatively high beliefs regarding the value of their assets. Eventually however, these owners become sufficiently pessimistic, and some owners sell optimally. Since poorly performing assets are on average of lower quality, price falls in response to the increase in information-based selling. Importantly, all owners of poorly performing assets do not sell at this point, preventing the price from falling too far. Instead, price falls gradually as more negative information is revealed to some owners and as the intensity of information-based sales slowly increases. As time passes, the number of remaining owners looking to sell due to poor asset performance grows thin. In response, prices stop falling and finally rise as the presence of adverse selection fades from the market.

Interestingly, the path of prices in the aggregate economy is qualitatively similar to that implied
by models of underreaction and overreaction to news as in Hong and Stein (1999) and Daniel et
al. (1998). In particular, prices appear to reflect an initial underreaction to bad news (during the
phase when the price falls slowly) followed by an overreaction and subsequent rebound. In markets
such as those for commodities, real estate, farms, and capital assets where assets are indivisible
and private learning is relevant, our model therefore offers an alternative explanation in a fully
rational environment for this observed path of prices. In this regard, our model need not be viewed
in isolation, in which case we may expect these price patterns to be most prominent in markets
affected both by the optimal experimentation of privately informed owners and by behavioral biases
affecting investor learning (as in Hong and Stein and Daniel et al).

Our model also contributes to a recent literature on dynamic markets with adverse selection.
Janssen and Roy (2002) show that in a competitive market which opens periodically, equilibrium
prices increase over time as owners of higher type assets delay trade. Camargo and Lester (2014) find
similar price dynamics in a decentralized market. Importantly these papers assume that owners are
endowed with fixed asymmetric information. Our paper therefore differs by explicitly modelling
owners information aggregation. As a result, the dynamic price path in our setting features a
gradual fall in prices which results from marginal owners’ accrued negative information. After this
decline, prices increase (as in the fixed information setting) as the share of poorly performing assets
fades from the market.

We share with these models the observation that dynamic trading leads to a broad interval of
strategic sales whereas efficient trading is characterized by a single optimal selling time. In partic-
ular, with fixed asymmetric information, optimal reallocation involves the immediate transaction
of all unproductive assets. As a result, adverse selection in this setting is often framed in the
context of inefficient delay. Fuchs et al. (2014) consider asymmetric information in a dynamic
economy with sectoral productivity shocks. Adverse selection leads to delays in capital reallocation and thus slow recovery from shocks. In our setting with private learning, the owner’s problem - whether to sell at a point in time or continue to hold the asset and potentially learn in response to performance - is better viewed as one of optimal experimentation.\footnote{In this light, our modelling environment is also related to a literature on experimentation with informational externalities. Keller et al (2005) examine the private provision of public signals and show that free-riding leads to an inefficiently low level of experimentation. Moscarini and Squintani (2010) and Murto and Valimaki (2013), in contrast, show that when agents’ actions (but not signals) are observable, agents continue to experiment for too long and face regret in response to other agents’ actions. In our model with adverse selection, some sellers sell too soon, and when owner-match-specific values are sufficiently important, some owners sell too late as well.} Since buyers pay prices based on average quality assets, equilibrium necessarily entails inefficient experimentation. However in this case, inefficiencies potentially arise both from owners who delay trade - due to higher prices in the future - and from those who rush to sell too soon - due to falling prices at the onset of adverse selection sales. That some agents sell too soon turns out to be a robust feature of equilibrium whereas delayed trading occurs only when owner-specific values are sufficiently important.

Finally, we consider optimal government policy in our dynamic trading environment and establish criteria for the least costly government intervention to achieve first best experimentation. The optimal policy involves a short window of government purchases over which price rises gradually. In particular, prices rises fast enough to encourage those agents selling strategically not to sell too soon, while slow enough to entice agents selling for reasons unrelated to asset quality to sell immediately. In contrast, in the fixed asymmetric information setting, efficiency requires the immediate sale of all unproductive assets, and the optimal government policy involves purchases only at time $t = 0$. Fuchs and Skyrpacz (2014) study this related setting with fixed information and establish a sufficient condition for the optimal government intervention to involve subsidized trade only at
The paper proceeds as follows: Section 2 sets forth the model, and Section 3 characterizes dynamic equilibrium in this setting. Section 4 then considers a benchmark case with fixed asymmetric information and compares outcomes between the two. Section 5 considers the implications of government purchases on aggregate welfare. Section 6 concludes.

### 3.2 The Model

#### 3.2.1 Agents and Assets

The game takes place in continuous time, starting from zero, and all agents discount the future at rate $r > 0$. The economy consists of a mass $N > 1$ of risk-neutral prospective buyers and a unit mass of risk-neutral owners, each of whom is endowed with one asset. For expositional purposes, let us identify each owner with a unique, permanent index $i \in [0, 1] = I$ and, in turn, each prospective buyer with an index $j \in [0, N] = J$.

Assets are heterogeneous—each is characterized by a type, either $H$ or $L$, and a fraction $Q$ of assets are type $H$. This type helps determine the asset’s value to a given owner via what we call “local productivity;” a locally productive asset is more valuable to its owner than a locally unproductive one. For each asset, local productivity is iid across owners, but the probability of local productivity is tied directly to the asset’s underlying type—type $H$ assets are locally productive for an owner with probability $\omega_H$, while type $L$ assets are locally productive with probability $\omega_L < \omega_H$. 

3.2.2 Payoffs and Information Arrivals

For its owner, an asset generates (i) a constant flow payoff of $\mu$ which is independent of productivity, and (ii) a lump-sum payoff at the Poisson arrival rate $\lambda_k$, where $k = G$ if the asset is locally productive and $k = U$ if not. These lump-sum payoffs are drawn from a time-invariant distribution on $\mathbb{R}$ with known mean $y$ (which is independent of asset type or local productivity). The analysis to follow will focus primarily on the case in which $\lambda_G = \lambda > 0$ and $\lambda_U = 0$. We also set $y = Y > 0$ and normalize $\mu = 0$. With this parametric setup, the lump sum arrival can be interpreted as the stochastic arrival of a dividend payment which has value $Y$ to the owner of the asset. In this case, locally productive assets make a payment to their owner at the Poisson arrival rate $\lambda$ while locally unproductive assets never produce a payment. With these lump sum arrivals, each asset provides for its owner not just a payoff value, but also information. She learns over time about her asset’s local productivity (and indirectly about her asset’s type, as well).

For illustrative purposes, consider the information of an owner, beginning at time 0. Initially, she has no private information regarding her asset—like every other agent, she knows only that it is type $H$ with probability $Q$, and in turn, that it is locally productive with probability $a(Q) \equiv Q \omega_H + (1 - Q) \omega_L$. She will update continuously according to Bayes’ Rule from this point forward, conditioning her beliefs on her asset’s realized output. Let us denote by $g(t)$ her time-$t$ belief regarding the probability that her asset is locally productive.

In the “Perfect Good News” case on which we focus here, the arrival of a lump-sum payoff fully informs the owner regarding her asset’s local productivity. In response to such an arrival at time $t$, her belief $g(t)$ thus undergoes an instantaneous, discrete update to 1. In the absence of such an arrival during the infinitesimal time interval $[t, t + dt)$, $g(t)$ drifts gradually downward. More formally,

\footnote{We discuss equilibrium outcomes and qualitative predications of other parametric cases in the Appendix.}
Bayes’ Rule specifies this instantaneous belief drift to be 
\[ g'(t) = -[\lambda_H - \lambda_L] g(t)(1 - g(t)) \, dt = -\lambda g(t)(1 - g(t)) \, dt. \]

### 3.2.3 The Selling Decision

In this study, we focus on market dynamics that accompany the gradual, private arrival of aggregate information; as such, our analysis hinges on the selling decisions of owners. To sharpen our focus on this aspect, we restrict each asset to be traded at most once (thus, in acquiring an asset, a buyer obtains the expected value of holding the asset permanently). Trade involves a transaction cost \( c \) paid by buyers. To avoid technicalities, we make the following assumption,\(^3\)

**Assumption 1:** The transaction cost \( c \) satisfies

\[ c \in \left(0, a \left( \frac{Q(1 - \omega_H)}{1 - a(Q)} \right) \frac{\lambda Y}{r} \right). \]  

(A1)

Assumption 1 guarantees that the expected value of an unproductive asset to a new owner is not less than the transaction cost. As a result, transferring an unproductive asset from its existing owner to a new buyer generates positive surplus in expectation.

Turning now to the determinants of sales, owners in this setting are driven to sell their assets in response to two forces:

(i) **Cash flow shocks:** At an exogenous Poisson intensity \( \pi > 0 \), an owner’s expected value of perpetual ownership falls to zero (as does the flow value of ownership). Suitable interpretations of this shock depend on the model’s application—for instance, if the assets represent homes, this might correspond to an impending need for the cash obtained from a sale, or perhaps a family/work-motivated need for geographic relocation that would prevent continued use of the house. In the

\(^3\)The characterization of equilibrium is unchanged for \( c \in \left[0, a \left( \frac{Q(1 - \omega_H)}{1 - a(Q)} \right) \right] \) although proofs in this case are algebraically opaque. Results are available from the authors upon request.
case of a financial asset, this could represent a lower personal use for the asset class in this market (i.e. - due to changing portfolio needs).

Within the model, this “cash flow shock” has two defining features: First, it occurs for reasons unrelated to local productivity (and, by extension, unrelated to underlying type). Second, the shock’s arrival is the private information of the owner and is unverifiable to other agents.

(ii) Endogenous sales decisions: An owner who has updated her belief about the local productivity of her asset sufficiently far downward will prefer the value from selling to the remaining value of continuing to own. Thus, she will endogenously choose to sell. We will loosely refer to such sellers as “marginal sellers.”

3.2.4 Efficiency

The (unconstrained) efficient allocation can be framed as the solution to a two-armed bandit problem for each owner with a one-time switching cost equal to $c$. The first arm represents the value of ownership by the initial owner, and can be either productive or unproductive. If the arm is productive, it yields a lump-sum payoff $Y$ at Poisson intensity $\lambda$ until some random time $\tau_\pi$ (which is exponentially distributed) at which point the owner faces the cash flow shock. If the arm is unproductive it yields 0 always. The second arm, like the first, can be either productive or unproductive and represents the value of ownership to a new buyer. If productive, it yields lump-sum payoffs forever; and if unproductive, it yields 0 always. Because the payoff of each asset is independent of payoffs of all other assets, we can consider the optimal allocation of each owner’s asset - in response to accumulated information on the asset-owner match and in response to the owner’s cash flow shock - separately.

Let $k_i^t$ denote the allocation strategy for owner-$i$’s asset at time $t$. The problem then is to choose
a strategy that solves

$$\max_{k^i \in [0,1]} E_{\tau^n} \left[ \int_0^{\tau^n} e^{-rt} \left[ (1 - k^i(t))g^i(t)\lambda Y + k^i(t)(g^i(t)\lambda Y - rc) \right] dt + \int_{\tau^n}^{\infty} e^{-rt}k^i(t)(g^i(t)\lambda Y - rc) dt \right]$$

s.t. $k^i_{t_2} \geq k^i_{t_1}$ for all $t_2 \geq t_1$

where $g^i(t)$ denotes the probability that the owner-arm is productive for owner $i$ and $g^i(t)$ the probability that the buyer-arm is productive for owner-$i$’s asset. At time $t$, the owner-arm produces an expected flow payoff $g^i(t)\lambda Y$ prior to the cash flow shock and a flow payoff 0 following the cash flow shock. Switching to the buyer-arm involves paying the transaction cost $c$ (which is equivalent to a perpetual flow cost $rc$) and produces an expected flow payoff $g^i(t)\lambda Y$.

Clearly, it is optimal to immediately transfer an asset following the cash flow shock, and it is never optimal to transact a known productive asset prior to the cash flow shock. With local productivity, experimentation with the owner-arm reveals only indirect information about the asset’s type. As a result, we are limited in how much we can learn about the value of an asset to a new a buyer even after a sufficiently long period of experimentation. In particular, if an asset is known to be productive for its current owner, the probability it is productive for a new buyer is $\tilde{g} \equiv a \left( \frac{Q(1-\omega\mu)}{a(Q)} \right) \leq 1$, while if an asset is known to be unproductive for its current owner, the probability it is productive for a new buyer is $\tilde{g} \equiv a \left( \frac{Q(1-\omega\mu)}{a(Q)} \right) \geq 0$. The probability the buyer-arm is productive thus takes values in the interval $g^i(t) \in [\tilde{g}, \tilde{g}]$ for all $i$ and for all $t$.

Given this bound on the information available regarding the productivity of the buyer-arm, Assumption A1 guarantees that the expected value from playing this arm exceeds the switching cost for any asset. Since the expected flow value of the first arm falls to zero following the cash flow shock, immediate transaction is clearly optimal. In the case of a known productive asset, the flow value from playing the buyer-arm is at best equal to the flow value of the owner-arm if the
asset is sure to be productive for its new owner. Switching owners and paying (flow) cost $rc > 0$ is therefore never optimal. Given the simple switching rules for a productive asset, we can define $W_G$ as the expected total payoff generated by the optimal allocation of an asset that is revealed as productive,

$$W_G = \left( \frac{r}{r + \pi} \right) \left( \frac{\lambda Y}{r} \right) + \left( \frac{\pi}{r + \pi} \right) \left( \frac{\beta \lambda Y}{r} - c \right).$$

The first term is the discounted perpetuity value of a productive asset accounting for the likelihood of the current owner receiving the cash flow shock. The second term captures the expected discounted value that a new owner would obtain, net of the transaction cost, for the same asset following the realization of the cash flow shock.

The problem now is to determine the optimal allocation rule for an as-of-yet unproductive asset prior to the cash flow shock. Over time, the probability the owner-arm is productive drifts down in the absence of a productive breakthrough, suggesting that it may be optimal to transfer the asset to a new buyer after a sufficiently long period of poor performance. To begin, assume that for an as-of-yet unproductive asset the optimal switching rule involves a discrete choice $k^i \in \{0, 1\}$. Since each asset may trade at most once, this problem can be formulated as the stopping problem

$$W^*_i(t) = \max_{\tau^i} \int_0^{\tau^i} e^{-(r+\pi)(s-t)} \left[ e^{-\lambda(s-t)} g^i(t) \lambda (Y + W_G) \right. + \left. \pi \left[ 1 - g^i(t)(1 - e^{-\lambda(s-t)}) \right] \left( \frac{g^i(s) \lambda Y}{r} - c \right) \right] ds (3.1)$$

The optimal stopping time $t^*$ is given by the first order condition

$$g(t^*) \lambda \left[ Y + W_G - \left( \frac{g(t^*) \lambda Y}{r} - c \right) \right] + \frac{g'(t^*) \lambda Y}{r} = r \left( \frac{g(t^*) \lambda Y}{r} - c \right) (3.2)$$

where for ease of notation we have dropped the owner superscript $i$. The left hand side of Equation 3.2 reflects the expected flow value of continued experimentation with the owner-arm. It includes
the expected net benefit from the productive breakthrough \( g(t) \lambda \left[ Y + W_G - \left( \frac{g(t) \lambda Y}{r} - c \right) \right] \) and

the expected loss from continued poor performance. At the optimum, the expected flow value of experimentation equals the expected flow value of the buyer-arm net the transaction cost. Some algebra shows that the value experimentation exceeds the value of the buyer-arm iff \( t < t^* \). Thus, the solution to the assignment problem is unchanged if the decision space is expanded to include probabilistic transfers (i.e. \( k \in (0, 1) \)).

The value of an as-of-yet unproductive asset under optimal stopping behavior can also be expressed - using time as our state variable - as an ordinary differential equation for \( t \leq t^* \),

\[
rW_M(t) = g(t) \lambda [Y + W_G - W_M(t)] + \pi \left[ \frac{g(t) \lambda Y}{r} - c - W_M(t) \right] + W'_M(t),
\]

with initial conditions given by value matching and smooth pasting

\[
rW_M(t^*) = g(t^*) \lambda Y - r c \\
rW'_M(t^*) = g'(t^*) \lambda Y.
\]

**Proposition 1:** The unconstrained Pareto efficient stopping time \( t^* \) for owners of as-of-yet unproductive assets is

\[
t^* = \frac{1}{\lambda} \ln \left[ \left( \frac{r + \pi + \lambda}{r + \pi} \right) \left( \frac{a(Q)[\lambda Y - (\beta \lambda Y - r c)]}{(1 - a(Q))\beta \lambda Y - r c} \right) \right] .
\]

**Proof:** See Appendix. ■

The term first term within the natural log in Equation (3.4) represents the additional experimentation a planner is willing to undertake relative to the hypothetical experimentation of a myopic planner, while the latter term captures the relative surplus generated from appropriate allocation of a productive versus unproductive asset. It compares the marginal social product of retaining a productive asset with the current owner weighted by the time \( t = 0 \) probability the asset is
productive $a(Q)[\lambda Y - (\bar{a}\lambda Y - rc)]$ to the marginal social product of an unproductive asset to a new owner incorporating the transaction cost weighted by the probability the asset is unproductive $(1 - a(Q))[\bar{a}\lambda Y - rc]$.

Proposition 1 completes the optimal allocation rule: an asset transacts immediately following the random cash flow shock, and at $t^*$ in the absence of both the cash flow shock and the productive breakthrough. However, only the transfer at $t^*$ is a marginal decision. When agents face the cash flow shock, the decision to transfer the asset immediately is a strict preference. Furthermore, since the optimal ‘selling’ time is the same for all owners of as-of-yet unproductive assets, efficient experimentation involves a mass of marginal ‘sales’ at $t^*$. In the following section, we solve for the equilibrium selling decision when owners’ information is private. In contrast to the efficient benchmark, equilibrium involves marginal sales over a broad interval of time.

### 3.3 Dynamic Market Equilibrium

In this section, we consider the trading decisions of asset owners when agents learn privately over time regarding the productivity of their own asset. In this context, we examine the aggregate supply of assets for sale across time and characterize the dynamic path of prices which results from trade.

With private learning, assets for sale are indistinguishable from buyers’ points of view, since owners of productive assets cannot profitably differentiate their assets from those of unproductive sellers. As such, buyers place the same value on all assets for sale at a single point in time. Ultimately, this value will reflect the average quality (type-$H$ fraction) of assets for sale, $q(t)$, though buyers do not observe this average quality directly. Rather, they observe only the time $t$ seller pool size and past trade history over the time interval $[0, t)$. From these, however, they can
perfectly identify the overall quality of assets for sale.

To understand this inference, note first that buyers know the remaining mass of owners, so they know the corresponding flow of cash flow shock-driven sellers. Given this, observing the size of the overall seller pool enables them to perfectly identify the proportion of marginal sellers in this pool. Additionally, buyers know the belief a marginal seller would have at time $t$, and, having observed the trade flows back to time 0, they can infer the proportion of remaining sellers who have realized local productivity. Thus, they know the type-$H$ fraction of marginal sales and the type-$H$ fraction of assets selling due to the cash flow shock, which together are sufficient to infer the overall market quality.

Since the initial measure of potential buyers ($N > 1$) exceeds the total supply of assets, buyers always represent the long side of the market. As a consequence, buyers bid up the market price at each instance until they are indifferent between buying and not. The price at time $t$ is therefore equal to the expected perpetuity value of an average quality asset for sale net of the transaction cost $c$,

$$p(t) = a(q(t)) \frac{\lambda Y}{r} - c. \quad (3.5)$$

The flow of assets for sale and the composition of this flow depends on the behavior of the mass of remaining owners. In particular, an owner chooses to sell her asset whenever the value of continued ownership is less than or equal to the price she receives for selling the same asset. As discussed previously, this may occur for either of two reasons: (i) the owner faces a cash flow shock which lowers her expected flow value of ownership below that of buyers, or (ii) the owner’s private information reveals her asset to have low enough personal value that it falls below the price a buyer will pay for an average quality asset at time $t$. Before analyzing in greater detail owners’ optimal selling decisions, we make the following restriction on agents’ off equilibrium beliefs
Assumption 2: *Buyers believe that at an unpopulated trade price, the type-\(H\) fraction of assets for sale would equal the average quality asset of owners willing to sell at that price.*

Assumptions 1 and 2 imply that an owner sells her asset immediately following a cash flow shock. In the event an owner experiences this shock, her expected flow value of continued ownership falls to zero. By comparison, the expected flow value of ownership of an as-of-yet unproductive asset is \(g(t)\lambda Y > 0\) for all \(t\). If at a point in time, the prevailing market price is high enough to encourage owners of poorly performing assets to sell strategically, all owners facing cash flow shocks must also find it worthwhile to sell.\(^4\) Now consider the possibility that no owners sell at some time \(t\). Assumption 2 states that for sufficiently low prices buyers believe market quality at time \(t\) would reflect the type-\(H\) proportion of assets sold due to cash flow shocks only. Furthermore by Assumption 1, the expected surplus generated from transacting an average quality asset is strictly positive. Thus in the event of no trade, buyers are always willing to deviate and offer a price equal to their value of an average quality asset held by an owner who has realized the cash flow shock. Since the pool of sellers at \(t_+\) potentially includes strategic sellers as well, market quality cannot be higher at \(t_+\). As such, in absence of strategic selling, \(p'(t) \leq 0\). Since the price is non-increasing, cash-shocked owners have a strict preference for immediate sale.

We now turn our attention to the information-based sale of poorly performing assets. Since price is a function of the average quality of assets for sale, a relevant variable is the aggregate intensity (or

\(^4\)Note that this prohibits the existence of a fully separating equilibrium where owners of productive and unproductive assets sell almost surely at non-overlapping times. Owners of known productive assets may wish to separate from owners of poorly performing assets in order to obtain a higher selling price; however, owners of as-of-yet unproductive asset can always replicate the selling strategy of productive owners in order to capture the high price as well. Since productive owners who face a cash flow shock have a strictly lower value from ownership, any trading strategy which is profitable for these owners will also be profitable for some owners of unproductive assets.
hazard rate) of “marginal” sellers $\gamma(t)$. Importantly, each owner of a poorly performing asset has the same belief regarding the probability that her asset is productive. As such, when the prevailing price at time $t$ is strictly less than the value of continued ownership of an as-of-yet unproductive asset, only those owners who experience the cash flow shock find it optimal to sell. Alternatively, when the market price is strictly larger than the value from owning a poorly performing asset, all as-of-yet unproductive owners wish to sell and the intensity of adverse selection sales is infinite. This discussion however does not capture the interdependence between market price and the aggregate intensity of adverse selection sales. As the intensity of these sales increases, the proportion of high quality assets on the market declines, pushing down the average value of assets for sale and lowering the price buyers are willing to pay. If too many unproductive owners try to sell at a specific point in time, the equilibrium price may fall so far that these owners no longer wish to sell at the low price. In this case, some unproductive owners find it profitable to forgo trade today in hopes that their asset eventually will pay a dividend or that they will obtain a higher selling price in the future.

Taking equilibrium actions of all other agents as given, an owner fully anticipates the equilibrium path of informed sales and market prices. As a result, an owner’s only potential sources of uncertainty are the realization of her asset payoff, the time when she receives the cash flow shock, and her trade decision at future selling times if playing a mixed strategy. In particular, an owner of a high-performance asset knows with certainty that her asset is productive and that the payoff arrives with intensity $\lambda$. Furthermore, since an owner of a known productive asset possesses a weakly greater value for her asset than buyers place on any asset, the equilibrium price can never be high enough that productive assets sell in the absence of a cash flow shock.

For ease of notation, we consider a symmetric equilibrium where agents with identical beliefs
regarding the payoff of their own asset play the same strategy.\textsuperscript{5} Since owners’ beliefs drift down at the same pace in response to poor performance, owners beliefs can be summarized as either having received the productive signal at time $t$ or not.

The value of owning a known productive asset at time $V_G(t)$ is equal to the expected value of owning a productive asset up until the realization of the first cash flow shock at time $s > t$, at which point the owner receives the prevailing price $p(s)$:

$$V_G(t) = \frac{\lambda Y}{\pi} + \int_t^\infty e^{-(r+\pi)(s-t)}\pi p(s)ds. \quad (3.6)$$

An owner of an as-of-yet unproductive asset, on the other hand, chooses to sell either as the result of a cash flow shock or strategically in response to accumulated negative information (again taking optimal actions of other owners as given). In a symmetric equilibrium, the hazard rate of aggregate adverse selection sales $\gamma(t)$ corresponds to the instantaneous probability that a currently unproductive owner sells at that time. The value of owning an as-of-yet unproductive asset at time $t$ is

$$V_M(t) = g(t) \int_t^\infty e^{-(r+\pi+\lambda)(s-t)-\int_t^s \gamma(u)du} \left[ (\pi + \gamma(s))p(s) + \lambda(Y + V_G(s)) \right] ds$$

$$+ (1 - g(t)) \int_t^\infty e^{-(r+\pi)(s-t)-\int_t^s \gamma(u)du} (\pi + \gamma(s))p(s)ds \quad (3.7)$$

With probability $g(t)$, the owner believes that her asset is productive and that it makes the dividend payment with intensity $\lambda$, the realization of which is accompanied by an instantaneous jump in the owner’s value function to $V_G(.)$. This increase in the value function reflects the change in the owner’s optimal selling decision as a result of the jump in her belief that her asset is productive.

Prior to realizing this payoff, productive assets are indistinguishable from unproductive assets and

\textsuperscript{5}With a large number of agents, the equilibrium depends only on the aggregate behavior of marginal owners. In this way, any profile of individually optimal strategies which aggregate to the same market-wide dynamics yield an equivalent equilibrium with ex-ante identical agent payoffs.
are sold when the agent first realizes the cash flow shock or strategically at some point in the future. The owner believes her asset is unproductive with probability $1 - g(t)$, in which case, the asset pays zero always and is eventually sold.

Let us now define an equilibrium for this setting. Let $\Gamma(t) = 1 - e^{\int_0^t \gamma(s) \text{d}s}$ denote the cumulative distribution of marginal sales by time $t$, and let $\Lambda(t) = \int_0^t \lambda e^{-\lambda s}[1 - \Gamma(s)] \text{d}s$ denote the cumulative mass of productive breakthroughs.

**Definition 1:** A competitive trading equilibrium is defined by states

$$\{\Gamma(t), \Lambda(t), V_M(t), V_G(t), p(t), q(t)\}_{t=0}^{\infty}$$

such that for all $t \in [0, \infty]$, \footnote{Note that, to maintain focus on key aspects of this model, we have taken shortcuts to simplify this equilibrium definition. In particular, we have appealed to the homogeneity of agents and to the argument above that there is a single trade price in each period. More generally, we could have specified outcomes as for a variation of a continuum assignment model, mapping each match-asset type combination to price/trade date combination $I \times J \times \{H, L\} \to \mathbb{R} \times [0, \infty)$.}

1. Owners sell optimally:

$$\Gamma'(t) = \begin{cases} 0 & \text{when } p(t) < V_M(t) \\ \geq 0 & \text{when } p(t) = V_M(t) \end{cases} \quad (3.8)$$

2. Buyers behave optimally:

$$p(t) \leq a(q(t)) \left( \frac{\Lambda'}{r} \right) - c \quad (3.9)$$

3. Market quality reflects the aggregate type-H proportion of assets for sale at time $t$:

$$q(t) = Q \left( \frac{[1 - \omega_H(1 - e^{-\lambda})][(1 - \Gamma(t))\pi + \Gamma'(t)] + \omega_H \Lambda(t) \pi}{[1 - a(Q)(1 - e^{-\lambda})][(1 - \Gamma(t))\pi + \Gamma'(t)] + a(Q)\Lambda(t)\pi} \right) \quad (3.10)$$

Conceptually, each owner makes optimal selling/ pricing decisions conditional on the current and future equilibrium actions of other agents. In turn, each buyer makes optimal buying/ pricing decisions conditional on the current and future equilibrium actions of other agents. In turn, each buyer makes optimal buying/ pricing decisions conditional on the current and future equilibrium actions of other agents.
decisions conditional on the current and future equilibrium actions of other agents and inferred proportions of type-\(H\) assets in each selling pool at each point in time. Specifically, the proportion of type-\(H\) assets on the market at time \(t\) reflects the masses and qualities of remaining marginal and total owners, as well as the corresponding intensities of sales for strategic and cash flow reasons.

To better understand the relationship between market quality and the mass and composition of current owners, denote by \(M(t)\) the time \(t\) mass of remaining marginal owners, by \(N(t)\) the total mass of all remaining owners, and by \(q_M(t)\) and \(q_N(t)\) the type-\(H\) fraction of marginal and total owners, respectively. Then, market quality at time \(t\) is

\[
q(t) = \frac{M(t)q_M(t)\gamma(t) + N(t)q_N(t)\pi}{M(t)\gamma(t) + N(t)\pi}.
\]  

(3.11)

The pool of remaining marginal owners at time \(t\) consists of those owners who have not yet received the productive breakthrough and who have not sold their asset previously. As such, the mass and type-\(H\) fraction of remaining marginal owners are

\[
M(t) = [1 - \Gamma(t)]e^{-\pi t}\left[1 - a(Q)\left(1 - e^{-\lambda t}\right)\right]
\]

\[
q_M(t) = Q\left(\frac{1 - \omega_H(1 - e^{-\lambda t})}{1 - a(Q)(1 - e^{-\lambda t})}\right).
\]

The total pool of remaining owners includes the pool of remaining marginal owners plus the pool of owners with revealed productive assets who have not yet sold their asset as a result of a cash flow shock. The total mass and type-\(H\) fraction of remaining owners are therefore given by

\[
N(t) = [1 - \Gamma(t)]e^{-\pi t}\left[1 - a(Q)\left(1 - e^{-\lambda t}\right)\right] + a(Q)e^{-\pi t}\Lambda(t)
\]

\[
q_N(t) = Q\left(\frac{1 - \omega_H(1 - e^{-\lambda t}) + \omega_H\Lambda(t)}{1 - a(Q)(1 - e^{-\lambda t}) + a(Q)\Lambda(t)}\right).
\]

The expression for \(q(t)\) in the definition of equilibrium can be obtained by substituting the preceding expressions for mass and quality into Equation (3.11) and by making the additional substitution \(\Gamma'(t) = [1 - \Gamma(t)]\gamma(t)\).
If the pool of marginal owners runs out, only those owners who have realized the productive breakthrough remain in the market, and market quality therefore reflects the type-$H$ proportion of productive assets. Let $T$ denote the time when marginal sales run out. Formally $T$ is defined as the minimum $t \in [0, \infty)$ such that $\Gamma(t) = 1$.

**Lemma 1:** Marginal sales run out in finite time ($T < \infty$).

**Proof:** See Appendix. □

Eventually, the stock and corresponding flow of owners looking to sell due to poor asset performance necessarily grows thin and runs out. As the presence of adverse selection fades from the market, prices rise to reflect the shift in market composition towards known productive assets. At time $T$, when only revealed productive assets remain, market quality reaches its upper bound $q(T) = \overline{q} \equiv \frac{Q \omega H}{\alpha(Q)}$.

**Lemma 2:** An equilibrium is characterized by times $\hat{t}$ and $T$ (with $0 < \hat{t} < T$) such that the following hold on each interval:

1. For $t \in [0, \hat{t})$: $\Gamma(t) = 0$ which implies $V_M(t) \geq p(t)$, $\Lambda(t) = 1 - e^{-\Lambda t}$, and $q(t) = Q$
2. For $t \in [\hat{t}, T]$: $V_M(t) = p(t)$
3. For $t \in (T, \infty)$: $\Gamma(t) = 1$ which implies $q(t) = \overline{q}$,

and $p(t) = a(q(t)) \left( \frac{\lambda V}{r} \right) - c$ for all $t \in [0, \infty]$.

**Proof:** See Appendix. □

The following proposition establishes the existence of such an equilibrium.

**Proposition 2:** $\exists \overline{r} > 0$ such that for $r \in [0, \overline{r}]$ and parameters otherwise satisfying Assumption $A1$, a unique competitive trading equilibrium exists.
**Proof:** See Appendix.

In contrast to the Pareto efficient allocation which features a single stopping time $t^*$ for all as-of-yet unproductive assets, the market equilibrium displays a broad interval of marginal selling times $[\hat{t}, T]$ over which owners of as-of-yet unproductive assets are indifferent between trading and not. Inefficiency arises from buyers’ inability to price assets according to their specific quality. As a result, owners exert externalities on other owners through their impact on market composition.

Prior to $\hat{t}$, no owners of as-of-yet unproductive assets sell strategically, since the expected value of continued ownership exceeds the purchase price of an average quality asset. At time 0, before private learning begins, all owners are homogenous and share the same common belief that their asset is productive. Buyers, too, share this belief, since any asset purchased at time 0 is just as likely to be productive for a buyer as it is for the population of original owners. However, due to the transaction cost and the imbedded option value of selling a poorly performing asset, owners have a strictly greater value for their asset than the price buyers can offer at time 0. As time progresses however, owners of poorly performing assets update their belief downward in the absence of the productive payoff. This drives down the owner’s value for such an asset. Eventually after enough time has elapsed without realizing the productive breakthrough, an agent’s value of continued ownership drops low enough and marginal sales begin.

The onset of strategic selling, however, increases the relative proportion of poorly performing assets on the market, and subsequently drives down the price buyers are willing to pay for an average quality asset. The downward price pressure limits the speed with which informed owners can exploit their informational advantage, and as a result, price falls gradually as the beliefs of the pool of marginal sellers continues to decline. Eventually however, the pool of unsold unproductive assets becomes so small in relation to the pool of still owned productive assets that cash flow sales
dominate and quality begins to rise in accordance with this shift in composition of remaining unsold assets. Eventually at time $T$, the mass of remaining as-of-yet unproductive assets runs out, and prices stop rising, reaching their upper bound.

Figure 1(a) depicts the equilibrium price path alongside the value functions from owning productive and as-of-yet unproductive assets over time. The value of owning an as-of-yet unproductive asset falls as the agent becomes more pessimistic regarding the probability her asset is productive. Eventually the value function increases when the effect of rising prices overcomes the continued decline in beliefs.

Figure 1(b) displays the flow of marginal sales in comparison with the flow of trade stemming from cash flow shocks.\footnote{Note that the state variable $\Gamma(t)$ is the ex ante probability distribution an unproductive asset owner assigns to selling over time conditional on remaining marginal. To calculate the flow of marginal sales as depicted in Figure 1(b), we must then net out those unproductive asset owners selling for reasons unrelated to private information and those owners receiving the productive breakthrough.}

Informed selling begins slowly at $\hat{t}$ and continues until $T$. The pace of

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informed sales increases as the wedge between the value of productive and as-of-yet unproductive assets increases. Eventually in response to the rapid intensity of past endogenous sales, the composition of assets for sale shifts toward known productive assets, and prices rise.

In principle, this pattern of a gradual fall in price followed by a rebound and subsequent rise is consistent with an initial underreaction to bad news and an eventual overreaction (as in Hong and Stein (1999) and Daniel et al. (1998)) which leads to a market correction. Although in this case, the decline and subsequent rise in price reflect the change in the composition of assets for sale which results from the gradual and rational learning of asset owners. In some settings, the model thus provides an alternative explanation for what looks like biased updating on the part of informed investors.

The preceding analysis highlights the gradual decline in prices that results from marginal owners’ initial effect on market composition. However in equilibrium, asset owners anticipate this decline and internalize the effect of falling prices on the value of continued ownership. To capture this relationship, consider a marginal owner’s value function at \( \hat{t} \), which can be expressed as

\[
rV_M(\hat{t}) = g(\hat{t})\lambda[Y + V_G(\hat{t}) - V_M(\hat{t})] + \partial_+ p(\hat{t}),
\]

where \( \partial_+ p(t) \) is the right time derivative of price and where we have subbed in the indifference condition \( V_M(t) = p(t) \) for \( t \in [\hat{t}, \hat{t} + dt] \). Roughly speaking, the value of asset ownership at \( \hat{t} \) can be decomposed into the instantaneous expected value from receiving the productive breakthrough plus the value of changing prices which affects the owner’s value from selling in the next instant. At \( \hat{t} \), the right derivative of price is negative which lowers the agents value of ownership, all else equal. Ultimately, this effect also feeds back into the owner’s optimal selling decision. By lowering the agent’s value from ownership there is an increased incentive to sell early, locking in the current price before prices fall too far.

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Lemma 1: For parameters satisfying (A1), some marginal owners sell too soon,

\[ \Gamma(t^*_\pi) > 0. \]  

(3.13)

Proof: See Appendix. ■

While any individual marginal owner is indifferent between selling at any \( t \in [\tilde{t}, T] \), the collective pool of marginal sales shifts forward initially as agents ‘rush’ to sell before prices fall too far.\(^8\) This pressure to sell early has an important impact on welfare, and in particular leads to too little experimentation by some marginal owners (for any parameter values satisfying Assumption A1).

In contrast, for a broad range of parameter values, no marginal owners sell too late. When the transaction cost is large relative to buyers’ values of an unproductive asset, the surplus from transferring a marginal asset is small. In this case, the amount of time required until it is optimal to transfer the asset to a new owner as a result of prolonged poor performance can become arbitrarily long. While \( t^* \) can be become arbitrarily large, \( T \) remains finite even when \( c \) approaches its upper bound, since marginal owners can hide behind the cash flow sales of revealed productive owners.

3.4 Comparison with Fixed Asymmetric Information

To help put in context the equilibrium evolution of trade flows and prices from the previous section, it is instructive to consider an alternative setting in which all information concerning the local productivity of a particular owner’s asset is known to the owner initially (rather than learned over time as a result of performance). This environment with fixed asymmetric information parallels a recent literature (including Janssen and Roy (2002), Camargo and Lester (2014), and

---

\(^8\)With a discrete number of agents, there is a strict preference for early selling, and those agents who are fortunate enough to sell first are best off. In our continuum economy, agents are indifferent between selling at any \( t \in [\tilde{t}, T] \), however the some sales are shifted forward relative to a counterfactual market with flat prices.
Fuchs et al. (2014)) which studies the relationship between adverse selection and delayed capital reallocation.\(^9\) As in the previous analysis, assets traded as a result of owner private information are indistinguishable from those sold in response to realized cash flow shocks. As result, buyers price assets based on their beliefs regarding the type-$H$ fraction of assets for sale at each moment in time. For direct comparison with the preceding section, we consider an equilibrium in which owners sell instantly following a cash flow shock.\(^10\) As a result, the dynamics of trade are determined by the marginal decision of owners of unproductive assets.

For a given path of prices $p(t)$, the expected utility of an owner of an unproductive asset is represented by the differential equation

$$V_M'(t) = rV_M(t) - (\pi + \gamma(t))[p(t) - V_M(t)].$$

Since an unproductive asset yields a flow value of 0 to its owner, the expected utility from continued ownership is simply the price an owner expects to receive at the time of sale. If at any point, the current price is greater than the discounted path of future prices, the owner has a strict preference to sell. However, the presence of additional unproductive assets for sale reduces the price buyers are willing to pay, since unproductive assets are on average of lower quality. Similar to the preceding section, equilibrium with fixed asymmetric information therefore involves a broad time interval of marginal selling times over which owners of unproductive assets are indifferent between selling today and waiting for future selling times. Let $T$ denote the time when marginal sales run out, then for all $t \in [0, T]$, $p(t) = V_M(t)$. Substituting this indifference condition into the marginal owner’s

\(^9\)See Section 1 for a detailed review of this literature

\(^{10}\)Formally, suppose that for this section owners experience a small negative flow payoff from continued asset ownership following a cash flow shock. Then as in the previous section, the expected utility of an unproductive asset owner is strictly greater prior to the cash flow shock. In this case, only unproductive asset owners are marginal while owners who have realized the cash flow shock have a strict preference for immediate sale.
value function yields the equilibrium price dynamics

\[ p'(t) = rp(t), \]

with initial boundary condition

\[ p(T) = a \left( \frac{Q \omega_H}{a(Q)} \right) \frac{\lambda Y}{r} - c. \]

With fixed asymmetric information, prices rise, beginning at time 0, at a rate \( rp(t) \) until all unproductive assets have been sold, and the price reaches its terminal value. This is in contrast to the equilibrium price path with private learning which features an initial period of flat and declining prices, before prices rebound and begin to climb. In the case with learning, price declines result from the growing pessimism of owners of as-of-yet unproductive assets, which for a time, overwhelms the value of future higher prices. If information is, instead, known initially, the value of ownership of unproductive assets is as low as possible at time 0 and rises only as a result of rising prices.

While the rate of price increases is determined directly by the marginal owner’s indifference condition, price levels depend, in addition, on the duration of marginal sales. As in the previous section, the path of marginal sales which is consistent with required price dynamics is unique, and in this case with fixed private information, involves information-based selling beginning at time 0. As such, trade of unproductive assets takes place over the entire interval \([0, T]\). However, it is clear that, under Assumption [AI], it is efficient to immediately transfer all unproductive assets at time 0, since asset payoffs are imperfectly correlated across owners. Equilibrium with fixed private information therefore involves only inefficient delay in the reallocation of unproductive assets, in contrast to the equilibrium with private learning which always involves some marginal owners selling too soon.
The analysis in this section of an economy with fixed asymmetric information is related to a recent literature which studies in a variety of contexts the effect of private information on the dynamics of trade. Since much of this work is set in an environment with endowed private information, a prominent feature in these studies is the relationship between adverse selection and selling delays. However, in a setting which generalizes the information structure to include a gradual resolution of uncertainty, it is clear that inefficient delay is at best part of the story. While there exists a broad parameter space where equilibrium with learning involves only early trading, the case in which all owners sell too late is a boundary case which occurs only when all information is known initially.\footnote{Further research will focus on the properties of equilibrium in the limiting case as $\lambda \to \infty$. We conjecture that, even in the limit, some marginal owners sell too soon, leading to a discontinuity in welfare as learning becomes immediate.}

### 3.5 Constrained Efficiency

With the above contrast in mind, let us now consider the implications of our gradual learning environment for welfare-improving policy. First, to clarify the precise environment for our analysis, we consider a government which can commit to a specific future market intervention, but we allow limited commitment on behalf of individual owners and buyers. Further, we assume that private sales are unobservable to the social planner. This restricts the government’s ability to penalize inefficient transactions, since agents can always trade secretly in the private market. Note that, since owners are homogeneous before private learning begins, any policy which improves aggregate welfare represents an ex ante Pareto improvement. Crucially, this implies that any mechanism can be implemented as a budget-neutral policy through the use of time-0 lump sum taxes/subsidies. Finally, we also preclude the government from inducing equilibrium behavior by committing to
policies it will never face on the equilibrium path; in other words, the government can commit only to policies for which it will incur the associated costs (or reap the associated revenues). In our setting, this has the practical consequence of restricting the government from committing to influence prices after the last of the marginal owners have traded.

Analysis of policy in the fixed information case (as Fuchs and Skyrpacz, 2014 study for a similar setting) typically advocates a time-0 government subsidy (or similarly, a tax at all times $t > 0$).\(^{12}\) Conceptually, the government is buying the assets of all willing sellers—the entire market—at the outset and itself reallocating them to new owners, in order to prevent inefficient delay in this process. More precisely, it is optimal for the government to transfer locally unproductive assets immediately, but to transfer locally productive assets only when held by an owner facing a cash flow shock. Once all locally unproductive assets have been sold, the market price at all future points in time will be $a (\bar{q}) \frac{\lambda Y}{r} - c$, the value of a known, locally productive asset. Prices must therefore be at this level already when the last unproductive assets are sold; otherwise, all owners would prefer to wait an instant to exploit the upcoming rise in prices (precluding this outcome in the first place). To transfer these unproductive assets at time 0, then, the government must then commit to subsidize purchase prices of $p (0) = a (\bar{q}) \frac{\lambda Y}{r} - c$ at this time.

When information arrives gradually, however, implementation of the social optimum requires a more elaborate policy, even under regularity conditions analogous to those in the fixed information setting. Some of the reasoning behind this is straightforward; whenever trade has associated costs—explicit or opportunity—there is a social benefit to information acquired by each asset’s current owner. Thus, immediate reallocation of every asset would be supoptimal. Rather, as described

\(^{12}\)One can alternatively conceptualize this as the institution of a price floor which is high enough to bind at time 0.
in Section 2, the social optimum is achieved by transferring as-of-yet unproductive assets to new owners at an optimal stopping time \( t^* \). Like in the fixed information setting, the social optimum may be implementable via a price floor scheme (government-subsidized purchases) when certain conditions are satisfied. Because the "unproductive" assets are to be transferred after some span of time, however, a policy scheme in this setting must address the pre-transfer period as well, and it must account for the incentives it provides owners over this period.

As with the fixed information setting, the policy must induce owners of as-of-yet unproductive assets to sell at the optimal time, but induce revealed productive asset-owners to sell only when facing a cash flow shock. With a pre-transfer period preceding the optimal selling time, though, two additional concerns become relevant—(i) the policy must prevent as-of-yet unproductive asset-owners from selling before this time, and (ii) it must ensure that owners facing cash flow shocks never have incentives to delay selling.\(^{13}\) We can use these two requirements to characterize policies which implement the social optimum.

To do this, first note that—as in the fixed information case—the last sale of an as-of-yet unproductive asset will be immediately followed by an eternal price of \( a(q) \frac{\lambda_Y}{r} - c \). Therefore, for the same reasons as with fixed information, prices must reach this level when the last as-of-yet unproductive assets are sold. Given this condition at \( t^* \), the two aftermentioned requirements can be used to obtain a range of policies/price paths over \( t \in [0, t^*] \) which are compatible with the optimal outcome. Specifically, we can identify a region at each time \( t < t^* \) that is compatible with a price path reaching \( p(t^*) = a(q) \frac{\lambda_Y}{r} - c \) at time \( t^* \), and within which, the price \( p(t^*) \) motivates as-of-yet-unproductive asset owners to delay selling until \( t^* \) but motivates liquidity owners to sell

\(^{13}\)We focus on equilibria in which cash flow-shocked owners sell immediately not just for simplicity, but primarily because departures from this requirement are tangential to this paper’s subject matter, and equilibria involving such departures would distract from our main insights.
immediately. Naturally, this region is bounded by two indifference conditions which meet at \( p(t^*) \) at time \( t^* \)—the upper bound is the price path at which a marginal owner is indifferent between selling and maintaining the asset, and the lower bound is that at which a liquidity-shocked owner is indifferent between these. This region is depicted conceptually in Figure 2 above.

Each type of owner prefers to sell immediately at prices above her corresponding indifference condition and prefers to delay selling at prices below it. Intuitively, the region where liquidity-shocked owners prefer immediate sales extends below that for marginal owners because the latter stand to benefit from learning, while learning is inconsequential for the former.

One further restriction bounds this range—note that, in implementing the social optimum, it is impossible to support prices below \( a(Q) \frac{\Delta Y}{r} - c \) at \( t < t^* \). If there were trade at such prices, the buyer would be outbid by other prospective buyers, who are willing to pay up to \( a(Q) \frac{\Delta Y}{r} - c \) for these assets (because the owner pool at these times has yet to be influenced by selection). Thus, the feasible set of price paths over \( t \in [0, t^*] \) which implement the social optimum are those between the two indifference conditions and above the price of a randomly selected asset from the initial owner pool, \( a(Q) \frac{\Delta Y}{r} - c \).

Within this range, the least costly policy entails subsidizing prices to match the indifference path of a liquidity seller that reaches \( p(t^*) \) at \( t^* \). For \( t < t^* \), this path will rise at a rate of \( rp(t) \), and thus there is a time—call it \( t_1 \)—at which \( p(t_1) = a(Q) \frac{\Delta Y}{r} - c \). In turn, for \( t < t_1 \), the indifference level of liquidity seller is below \( a(Q) \frac{\Delta Y}{r} - c \). We therefore are left with two possible cases in characterizing the minimum cost policy that implements the optimum: If \( t_1 \leq 0 \), then this policy entails the prices
\[ p(t) = \begin{cases} 
  e^{-r(t^*-t)} \left[ a(\bar{q}) \frac{N}{r} - c \right] & \text{for } t \in [0, t^*] \\
  a(\bar{q}) \frac{N}{r} - c & \text{for } t \geq t^*
\end{cases} \]

If \( t_1 > 0 \), however, then these prices follow

\[ p(t) = \begin{cases} 
  a(Q) \frac{N}{r} - c & \text{for } t \in [0, t_1] \\
  e^{r(t-t_1)} \left[ a(Q) \frac{N}{r} - c \right] & \text{for } t \in [t_1, t^*] \\
  a(\bar{q}) \frac{N}{r} - c & \text{for } t \geq t^*
\end{cases} \]

Figure 2: Optimal Policy

Additionally, we provide a sufficient condition for the social optimum to be implementable via a scheme of this form.

**Proposition 3:** The social optimum can be implemented at minimal cost through a subsidized price path of the form above if

\[ rp_0 \leq g(t_0) \lambda [Y + V_G(t_0) - p_0] \] (3.14)
where \( t_1 = t^* - \frac{1}{r} \ln \left[ \frac{p}{p_0} \right] \)

\[
V_G(t_0) = \frac{\lambda Y}{r+\pi} + \left( re^{-(r+\pi)(t_1-t_0)} \left( 1 - e^{-(r+\pi)(t_1-t_0)} \right) \right) p_0
\]

and \( t_0 \equiv \arg \min_{\tilde{t}} V_M(\tilde{t}) = \arg \min_{\tilde{t}} \left\{ g(\tilde{t}) \left[ 1 - e^{-(r+\pi+\lambda)(t^*-\tilde{t})} \right] \left( \frac{\lambda Y}{r+\pi} \right) + \left( re^{-(r+\pi)(t_1-\tilde{t})} \left[ 1 - g(\tilde{t}) e^{-\pi(t^*-t_1)} \left( 1 - e^{-\lambda(t^*-t_1)} \right) + \pi \right] \right) p_0 \right\}

Proof: See Appendix.

3.6 Concluding Discussion

One can study a variety of real economic contexts (including markets for residential and commercial real estate, markets for used capital assets, and markets for intellectual property and patents, among others) through the lens of our private learning environment. Toward this end, it may be useful to think of the model as a situation beginning with a public revelation of aggregate information. Although this is informative at the aggregate level, it brings uncertainty about how this aggregate information will be realized across individuals (for instance—we might learn the share of “good” assets in a market, but we cannot yet identify which specific assets these are). Thus, our setting is one in which this individual uncertainty is resolved gradually over time, but it is resolved privately. To illustrate this point in a concrete real world setting, consider the path of prices of mortgage backed securities in response to a sudden increase in the expected mortgage default rate. For example, suppose that prior to time 0, owners believe that the default rate on U.S. mortgages is 2%, but at \( t = 0 \), agents learn that due to falling house prices and poor underwriting standards, the true default rate is in fact 5%. Furthermore, with a default rate of 2%, owners believe all
AAA-rated MBS are high quality and will payoff in full, while with a default rate of 5%, fraction \(1 - Q\) of these securities achieve a lower expected cash flow due to substantial defaults.

While the market is aware that some MBS no longer resemble high quality assets, agents are unable to identify which particular assets are affected. At this point the game begins, and owners begin to learn privately whether their asset is high or low quality in response to the realized payments on the underlying mortgages. For simplicity, suppose that if a sufficiently large fraction of the underlying mortgage payments are made in a given month, then the MBS owner is able to infer that her asset is high quality. Thus, with the release of the time 0 revelation of public information, the market price declines instantaneously as buyers update their belief regarding the average quality of AAA-rated mortgage backed securities on the market. Following this drop at time 0, the price stabilizes momentarily as owners of as-of-yet unproductive assets wait for their belief to fall far enough to begin selling endogenously. When selective selling begins, the price falls in response to the increase in lower quality assets flooding the market. This continues, until the mass of unsold low quality securities falls far enough that price eventually recovers, and settles between the date 0 price and original price prior to the public news announcement.

While we do not claim this as a proof of causality, it is interesting to note that this story mimics the qualitative path of observed prices for mortgage backed securities during the financial crisis. In the summer of 2006, national home prices began to fall and soon after, mortgage delinquencies began to rise. Prices of AAA-rated MBS were flat until the summer of 2007 and then began to steadily decline through the end of 2008, at which point, prices rebounded, rising steadily into 2011. While many market participants have described this price path in terms of a market overreaction to bad news in the housing market, our model produces a qualitatively similar result in a rational setting.
If transaction histories and asset performance are observable ex post, our model makes strong predictions regarding the ex post performance of securities purchased at different points along the price curve. That is, if the observed path of prices truly reflects the changing composition of high and low quality assets for sale, the average ex post performance of assets sold at times of higher prices should be greater. On the other hand, realized returns are greatest for those buyers lucky enough to have purchased high quality assets at low prices. However, because lower prices must be consistent with fewer high quality assets, disproportionately many purchasers at this time must experience low returns to offset the high returns of the lucky few who end up with high quality assets.
APPENDIX

Before presenting proofs of the primary lemmas and propositions, it is useful to establish a preliminary result.

**Lemma A1:** The path of prices $p(t)$ is continuous.

**Proof:** Consider the optimal decision facing an owner of an as-of-yet unproductive asset at time $t$. She sells at $t$ only if the price is at least as large as her continuation value from owning the asset. In addition, she faces the same decision in the next instant when deciding to sell or not sell at time $t_+$. Thus, her continuation value of ownership at time $t$ is bounded below by the discounted price at $t_+$, since she can always sell at this price if it is desirable to do so. Now suppose that the equilibrium price path jumps up discontinuously after time $t$. At this point, the continuation value of all owners is strictly greater than the price at time $t$ due to the jump in price. As a result, no owner chooses to sell at time $t$. However if there is no trade at time $t$, buyers believe market quality at time $t$ would, in the event of trade, reflect the type-$H$ proportion of assets sold due to cash flows shocks only. Since the pool of sellers at $t_+$ potentially includes strategic sellers, market quality after the jump in price cannot be higher than quality at $t$. Thus, a discontinuous jump in price is not possible.

Now suppose that the equilibrium path of prices drops discontinuously at time $t$ and consider the decision of a marginal seller who sells at the instant following the drop in price. Importantly for the price to decline at a point in time, the flow of marginal sellers at that point cannot be zero. Since there is never positive selection into selling, the average quality of assets owned is weakly increasing over time. Thus, if only cash flow sales occur at a point in time, the quality of assets for sale is at least as great as the quality of assets owned the instant earlier which is itself at least as great as the quality of assets sold (due to potential negative selection into selling). As a result,
there must exist at least one marginal seller at the lower price when price declines. For this owner to sell optimally at \( t_+ \), her value of owing the asset at \( t_+ \) cannot be larger than the price at this point. However, for the same owner not to have sold the instant sooner at the previously higher price, her value function at \( t \) must have been at least as great as the price at \( t \). This however is a contradiction to the continuity of the value function of an as-of-yet unproductive asset.

**Proposition 1:** The unconstrained Pareto efficient stopping time \( t^* \) for owners of as-of-yet unproductive assets is

\[
t^* = \frac{1}{\lambda} \ln \left( \frac{r + \pi + \lambda}{r + \pi} \right) \left( \frac{a(Q)[\lambda Y - (\varphi \lambda Y - rc)]}{(1 - a(Q))[\varphi \lambda Y - rc]} \right).
\]

**Proof:** If at time \( t \) an initial owner has yet to receive the productive breakthrough, the probability her asset is productive for her is

\[
g(t) = \frac{a(Q)e^{-\lambda t}}{1 - a(Q)(1 - e^{-\lambda t})}, \tag{3.15}
\]

while the probability her asset is productive for a new owner is

\[
g(t) = a \left( Q \left( \frac{1 - \omega_H(1 - e^{-\lambda t})}{1 - a(Q)(1 - e^{-\lambda t})} \right) \right). \tag{3.16}
\]

Substituting Equations \(3.15\) and \(3.16\) (along with the derivative of \(3.16\)) into Equation \(3.2\) and rearranging in terms of \( t \) yields the result.

**Lemma 1:** Marginal sales run out in finite time \( (T < \infty) \).

**Proof:**

Suppose \( T \) is infinite and that there exists some time \( t_1 \) such that \( V_M(t) = p(t) \) for all \( t \geq t_1 \).

From this indifference condition, the dynamic price path must satisfy the following differential
system for $t \in [t_1, \infty)$

\[ p'(t) = rp(t) - g(t)\lambda[Y + V_G(t) - p(t)], \tag{3.17} \]

\[ V'_G(t) = rV_G(t) - \lambda Y - \pi[p(t) - V_G(t)]. \tag{3.18} \]

Taking the derivative of Equation (3.17) with respect to $t$ gives

\[ p''(t) = r^2 p(t) - g(t)\lambda[(r - 2(1 - g(t)))\lambda Y + (2r + \pi - (1 - 2g(t)\lambda))(V_G(t) - p(t))]. \]

Note that the difference $V_G(t) - p(t)$ is bounded above by the perpetuity value of productive ownership. Furthermore, $\lim_{t \to 1} g(t) = 0$. Since price is bounded away from zero for any parameters satisfying Assumption [A1], there exists $t_2 > t_1$ large enough such that $p'(t) > 0$ and $p''(t) > 0$ for all $t > t_2$. Together these imply that $\lim_{t \to \infty} p(t) = \infty$ which contradicts buyer optimality. Thus $T$ must be finite.

Now consider the possibility that there exists a time sufficiently far in the future when marginal owners have a strict preference for continued ownership (i.e. $V_M(t) > p(t)$). Here, the same argument applies. In the limit as $t \to \infty$, the expected flow value of ownership tends to zero, since $g(t) \to 0$. If at a time $t$, no marginal owners sell, then $p(t) \geq a(Q) \frac{\lambda Y}{r} - c > 0$ due to the potential for past negative selection into selling. Since the flow value of ownership (sufficiently far in the future) tends to zero, it must be less than the equivalent flow value an owner can obtain by selling her asset. Thus, if a marginal owner has a strict preference not to trade at a positive price, it must be the result of an increasing price path. As prices rise, the rate of increase in price must rise as well in order to keep all marginal owners from selling. But prices cannot continue rising forever in this fashion, since the value buyers place on any asset is bounded above. As such, there exists an upper bound on the length of time marginal owners remain in the market. ■
Lemma 2: An equilibrium is characterized by times \( \hat{t} \) and \( T \) (with \( 0 < \hat{t} < T \)) such that the following hold on each interval:

1. For \( t \in [0, \hat{t}) \): \( \Gamma(t) = 0 \) which implies \( V_M(t) \geq p(t), \Lambda(t) = 1 - e^{-\lambda t} \), and \( q(t) = Q \)

2. For \( t \in [\hat{t}, T] \): \( V_M(t) = p(t) \)

3. For \( t \in (T, \infty) \): \( \Gamma(t) = 1 \) which implies \( q(t) = \overline{q} \),

and \( p(t) = a(q(t)) \left( \frac{\lambda_Y}{r} \right) - c \) for all \( t \in [0, \infty] \).

**Proof:** Since the initial measure of potential buyers \( (N > 1) \) exceeds the total supply of assets, buyers represent the long side of the market, and the price at time \( t \) is equal to the expected perpetuity value of an average quality asset for sale net of the transaction cost \( c \),

\[
p(t) = a(q(t)) \left( \frac{\lambda_Y}{r} \right) - c.
\]

Define \( \hat{t} \) as the time when marginal sales begin. Then, \( \hat{t} = \inf t \) s.t. \( \Gamma(t) > 0 \). From Lemma 1, marginal sales run out at \( T \). Thus, marginal sales begin at some time \( t \leq T \), and \( \hat{t} \) exists. Furthermore, \( \Gamma(\hat{t}) = 0 \) from continuity of the marginal owner value function. We now show that \( \hat{t} > 0 \). At \( t = 0 \), before private learning begins, all owners are homogenous and share the same common belief that their asset is productive. Buyers, too, share this belief, since any asset purchased at time \( 0 \) is just as likely to be productive for a buyer as it is for the population of original owners. Due to the transaction cost and the imbedded option value of selling a poorly performing asset, initial owners have a strictly greater value for their asset than the price buyers can offer at time \( 0 \). Thus, \( V_M(0) > p(0) \) which implies \( \hat{t} > 0 \).

(1.) From Equation 3.10, \( q(t) \) is continuous iff \( \Gamma(t) \) is continuously differentiable. From Lemma A1, prices are continuous, which implies that \( q(t) \) too is continuous. Thus, \( \Gamma(t) \) is continuously
differentiable, and as a result, \( \Gamma'(t) = 0 \) for all \( t < \hat{t} \). Since no marginal sales take place prior to \( \hat{t} \), seller optimality implies \( V_M(t) \geq p(t) \). Substituting \( \Gamma(t) = 0 \) and \( \Gamma'(t) = 0 \) for all \( t < \hat{t} \) into the definition of \( \Lambda(t) \) and the equilibrium equation (3.10) for market quality equation (3.10) yields \( \Gamma(t) = 1 - e^{-\lambda t} \) and \( q(t) = Q \).

(2.) Since marginal sales begin at \( \hat{t} \), Lemma A0 establishes \( V_M(t) = p(t) \) for all \( t \in [\hat{t}, T] \).

(3.) At time \( T \) marginal sales run out, and the pool of remaining owners consists only of those with revealed productive assets. As a result, any assets sold after \( T \) are necessarily productive, and market quality is equal to the type-H fraction of productive assets, \( q(t) = \frac{Q_H}{a(Q)} \).

Finally, since \( q(t) \) is continuous, \( \hat{t} < T \). □

**Proposition 2:** \( \exists \, \bar{r} > 0 \) such that for \( r \in [0, \bar{r}] \) and parameters otherwise satisfying Assumption A1, a unique continuous trading equilibrium exists.

**Proof:** Lemma 3 establishes that an equilibrium must entail times \( \hat{t} \) and \( T \) such that \( V_M(t) = p(t) \) \( \forall t \in [\hat{t}, T] \). Expressing all prices in terms of market quality, we can use this indifference condition to reduce our characterization of equilibrium to a system of four states evolving from \( \hat{t} \) to \( T \):

\[
\dot{q}(t) = \left( \frac{1}{\omega_H - \omega_L} \right) \left[ r \left( a(q(t)) - \frac{rc}{\lambda Y} \right) + g(t) \left( [\lambda a(q(t)) - r] - \frac{r[V_G(t) + c]}{Y} \right) \right] \tag{3.19}
\]

\[
\dot{V}_G(t) = (r + \pi) V_G(t) - \left( \frac{r + a(q(t)) \pi}{r} \right) \lambda Y + \pi c \tag{3.20}
\]

\[
\dot{\Lambda}(t) = \lambda e^{-\lambda t} \left[ 1 - \Gamma(t) \right] \tag{3.21}
\]

\[
\dot{\Gamma}(t) = \gamma(t) \left[ 1 - \Gamma(t) \right]
\]

Additionally, we can further substitute for \( \gamma(t) \left[ 1 - \Gamma(t) \right] \) in the last of these by using the expression for time \( t \) market quality \( q(t) \) as a function of the masses and qualities of remaining marginal
and total owners, as well as the corresponding intensities/hazard rates of sales for marginal and liquidity reasons (γ(t) and π), respectively:

\[ q(t) = \frac{M(t) q_M(t) \gamma(t) + N(t) q_N(t) \pi}{M(t) \gamma(t) + N(t) \pi} \]

where \( M(t) \) (the mass of remaining marginal owners), \( N(t) \) (the mass of all remaining owners), \( q_M(t) \) (the type H proportion of remaining marginal owners), and \( q_N(t) \) (the overall type H proportion of remaining owners) can be written as

\[ M(t) = [1 - \Gamma(t)] e^{-\pi t} \left[ a(Q) e^{-\lambda t} + (1 - a(Q)) \right] \]

\[ N(t) = [1 - \Gamma(t)] e^{-\pi t} \left[ a(Q) e^{-\lambda t} + (1 - a(Q)) \right] + a(Q) e^{-\pi t} \left[ \int_0^t \lambda e^{-\lambda s} [1 - \Gamma(s)] ds \right] \]

\[ q_M(t) = Q \left( \frac{\omega_H e^{-\lambda t} + (1 - \omega_H)}{a(Q) e^{-\lambda t} + (1 - a(Q))} \right) \]

\[ q_N(t) = Q \left( \frac{[1 - \Gamma(t)] [\omega_H e^{-\lambda t} + (1 - \omega_H)] + \omega_H \Lambda(t)}{[1 - \Gamma(t)] [a(Q) e^{-\lambda t} + (1 - a(Q))] + a(Q) \Lambda(t)} \right) \]

Solving this for \( \gamma(t) [1 - \Gamma(t)] \), we can write \( \dot{\Gamma}(t) \) as a function of our states \( q(t) \), \( V_G(t) \), \( \Lambda(t) \), and \( \Gamma(t) \):

\[ \dot{\Gamma}(t) = \pi \left[ \left( \frac{\omega_H Q - a(Q) q(t)}{q(t) [a(Q) e^{-\lambda t} + (1 - a(Q))] - Q [\omega_H e^{-\lambda t} + (1 - \omega_H)]} \right) \Lambda(t) - [1 - \Gamma(t)] \right] \]  (3.22)

We thus have four differential equations which characterize the evolution of the system from time \( \tilde{t} \) (not time 0, but the time at which marginal selling begins) to our “final” time \( T \) (the time at which the last remaining marginal owners sell their assets strategically).

We can analytically express the values of the 3 states \( q(t) \), \( \Lambda(t) \), and \( \Gamma(t) \) for \( t \in [0, \tilde{t}] \):

\( q(t) = Q \), \( \Lambda(t) = 1 - e^{-\lambda t} \), and \( \Gamma(t) = 0 \). Additionally, we know the time \( T \) values for 3 states:
\( q(T) = \frac{\omega \mu Q}{a(Q)} \), \( \Gamma(T) = 1 \), and \( V_G(T) = \frac{\lambda Y + \pi \left[ a \left( \frac{\omega \mu Q}{a(Q)} \right) \lambda Y - c \right]}{r + \pi} \). Our task, then, is to determine 4 values, which we can think of as parameters for the purpose here—\( V_G(\hat{t}) \) (our remaining “initial” value), \( \Lambda(T) \) (our remaining “terminal” value), and our two times \( \hat{t} \) and \( T \). These 4 values must ensure that:

1. The above system of 4 differential equations governs the evolution of state variables over \( t \in [\hat{t}, T] \) and satisfies all of the specified conditions regarding the pre-marginal selling region \([0, \hat{t}]\) and terminal time \( T \)

2. The realized paths of these states over \( t \in (\hat{t}, T) \) are consistent with:
   
   (i) optimal individual behavior: \( V_G(t) > a(q(t)) \left( \frac{\lambda Y}{r} \right) - c \)

   (ii) feasibility given the model structure and Bayesian learning:

\[
q(t) \in \left[ \frac{Q(1-\omega \mu)}{1-a(Q)}, \frac{\omega \mu Q}{a(Q)} \right] \\
\Lambda(t) \in \left[ 1 - e^{-\lambda \hat{t}}, 1 - e^{-\lambda T} \right] \text{ and } \Lambda'(t) \geq 0 \\
\Gamma(t) \in [0, 1] \text{ and } \dot{\Gamma}(t) \geq 0
\]

In what follows, we “construct” an equilibrium—meaning, for our purposes, we find values for the 4 objects mentioned above which meet these requirements. Before proceeding, we provide a brief outline of the arguments that follow.

To start, consider the system for hypothetical pairs of values for our initial time and corresponding locally productive asset owner value: \( \{ \hat{t} = t_S, V_G(\hat{t}) = V_G^S \} \). Given such a pair, we have a complete set of initial conditions and can trace our entire system forward from \( \hat{t} \). At our equilibrium (given the true \( \hat{t} \) and \( V_G(\hat{t}) \)), the remaining 2 objects, \( T \) and \( \Lambda(T) \), can be obtained in this way. As such, our task reduces to identifying a pair of values \( (t_S, V_G^S) \) such that the aforementioned requirements are satisfied. Note that, although the terminal time \( T \) is not required to take on a
particular value, the existence of such a time requires that the states \( q(t) \), \( \Gamma(t) \), and \( V_G(t) \) reach their specified terminal conditions simultaneously—we will exploit this in obtaining the equilibrium.

Before proceeding, it is helpful to define several objects to be used along the way. For given values \((t_S, V^S_G)\), define \( T_q(t_S, V^S_G) \) to be the first time \( t > t_S \) at which \( q(t) = \frac{Q_{\omega_H}}{a(Q)} \) and define \( T_\Gamma(t_S, V^S_G) \) to be the first time \( t > t_S \) at which \( \Gamma(t) = 1 \). In turn, let us define \( \widetilde{T}_q(t_S, V^S_G) \equiv T_q(t_S, V^S_G) - t_S \) and \( \widetilde{T}_\Gamma(t_S, V^S_G) \equiv T_\Gamma(t_S, V^S_G) - t_S \) to be these corresponding values, normalized relative to the inception time of marginal sales. Further, for a given inception time \( t_S \), we define \( V^S_G(t_S) \) implicitly to be the value for \( V^S_G \) at which \( \widetilde{T}_q(t_S, V^S_G(t_S)) = \widetilde{T}_\Gamma(t_S, V^S_G(t_S)) \).

Note that whenever \( q(t) \) reaches its terminal value of \( \frac{Q_{\omega_H}}{a(Q)} \) (and remains there permanently), equation (4) becomes

\[
\hat{\Gamma}(t) = \pi \left[ \left( \frac{0}{Q(1-Q)(\omega_H-\omega_L)} \right) \right] \Lambda \left( \widetilde{T}_q(t_S, V^S_G) \right) - \left[ 1 - \Gamma \left( \widetilde{T}_q(t_S, V^S_G) \right) \right] < 0
\]

Even ignoring the fact that \( \hat{\Gamma}(t) < 0 \) violates our requirements, this would preclude \( \hat{\Gamma}(t) \) from being positive for any \( t > \widetilde{T}_q(t_S, V^S_G) \) (because \( \hat{\Gamma}(t) \) is increasing in \( \Gamma(t) \)), and in turn, it would preclude \( \Gamma(t) \) from reaching its own terminal value of 1. Thus, if \( q(t) \) reaches its terminal value before \( \Gamma(t) \) does, \( \widetilde{T}_\Gamma(t_S, V^S_G) \) cannot exist.

As such, it will be useful to alter the definition of \( \widetilde{T}_\Gamma(t_S, V^S_G) \) slightly in order to simplify the process of obtaining a \( V^S_G(t_S) \) to accompany \( t_S \). Formally, we will use the definition

\[
\widetilde{T}_\Gamma(t_S, V^S_G) = \begin{cases} 
\min_{t\in[t_S, T_q(t_S, V^S_G))] \{t\} \text{ such that } \Gamma(t) = 1 & \text{if } \max_{t\in[t_S, T_q(t_S, V^S_G))] \{\Gamma(t)\} = 1 \\
T_q(t_S, V^S_G) + \kappa \left( 1 - \Gamma \left( \widetilde{T}_q(t_S, V^S_G) \right) \right)^2 & \text{if } \max_{t\in[t_S, T_q(t_S, V^S_G))] \{\Gamma(t)\} < 1
\end{cases}
\]

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for some constant $\kappa > 0$. This definition will ensure that for a given $t_S > 0$, both $\tilde{T}_q(t_S, V_G^S)$ and $\tilde{T}_G(t_S, V_G^S)$ will be continuous in $V_G^S$ over an interval containing $V_G^S(t_S)$ in its interior.

We now proceed in 4 steps. We first demonstrate that $\tilde{T}_q(t_S, V_G^S)$ and $\tilde{T}_G(t_S, V_G^S)$ are positive and finite for appropriate ranges of $t_S$ and $V_G^S$. Second, we establish that each $t_S$ maps to a unique $V_G^S(t_S)$ such that $\tilde{T}_q(t_S, V_G^S(t_S)) = \tilde{T}_G(t_S, V_G^S(t_S))$. Having reduced our problem to a single dimension, we demonstrate the existence of a $\hat{t}$ such that, for $t > \hat{t}$, the law of motion for $V_G(t)$ leads to its specified terminal value at $t = \tilde{T}_q(\hat{t}, V_G^S(\hat{t})) = \tilde{T}_G(\hat{t}, V_G^S(\hat{t})) = T$. Finally, we verify that the system corresponding to $t_S = \hat{t}$ and $V_G^S = V_G^S(\hat{t})$ satisfies the previously specified requirements.

Note that the differential equations governing the evolution of this system are continuous over their relevant bounds, and further, that each state’s realization at some finite time $t > \hat{t}$ moves continuously as a function of the initial values $\hat{t}$ and $V_G^S = V_G^S(\hat{t})$. We use this fact implicitly in much of what follows.

Step 1: For any $t_S > 0$, there are $\underline{V}_G^S(t_S)$ and $\overline{V}_G^S(t_S) > V_G^S(t_S)$ such that, for $V_G^S \in [\underline{V}_G^S(t_S), \overline{V}_G^S(t_S)]$, $\exists$ finite-valued $\tilde{T}_q(t_S, V_G^S(t_S)), \tilde{T}_G(t_S, V_G^S(t_S)) \in (0, \infty)$.

Note that $g(t) \to 0$ as $t \to \infty$, and thus, provided that $V_G(t)$ doesn’t grow unboundedly (and faster than the decline of $g(t)$), the dynamics of quality must eventually reflect $\dot{q}(t) \to \left(\frac{1}{\omega_H - \omega_L}\right) \left[ r \left( a(q(t)) - \frac{r \gamma Y}{\Delta Y} \right) \right] = \left(\frac{(\omega_H - \omega_L)\gamma Y}{r}\right)^{-1} \left[ a(q(t)) \left( \frac{\Delta Y}{r} \right) - c \right]$. Assumption 1 ensures that this is strictly positive for any $q(t) > 0$, so $q(t)$ will necessarily reach its terminal value of $\frac{\omega_H Q}{a(q)}$ as long as it does not fall below 0. $\dot{q}(t)$ will be negative whenever $a(q(t)) \left( \frac{\Delta Y}{r} \right) - c < \left( \frac{\lambda g(t)}{r + \lambda g(t)} \right) [Y + V_G(t)]$, but obviously the speed of this decline (the magnitude of $\dot{q}(t)$) depends on $V_G(t)$. Then by choosing an appropriate upper bound $\overline{V}_G^S(t_S)$ for the range of $V_G^S$ (and, in turn, bounding the rate of growth
of $V^S_G(t)$, we can ensure not only that $q(t)$ must eventually rise when holding a nonnegative value, but further that the path of $q(t)$ will never fall below 0.

It is worth noting that we want to be even more restrictive than this in choosing our bound $\overline{V^S_G}(t_S)$—any $q(t) < Q \left( \frac{\omega H e^{-\lambda t + 1 - \omega H}}{a(Q)e^{-\lambda t + 1 - a(Q)}} \right)$ (or $q(t) < Q \left( \frac{1 - \omega H}{1 - a(Q)} \right)$ at any time $t$) would be inconsistent with rationality and Bayesian learning in our setting, so we will use a choice of $\overline{V^S_G}(t_S)$ such that $q(t)$ remains above the relevant level for each point in time.

Thus, $q(t)$ must exhibit strictly positive (bounded from zero) growth in finite time, and in turn, it must eventually rise to reach its terminal value. As such, for $V^S_G \leq \overline{V^S_G}(t_S)$, $\tilde{T}_q(t_S, V^S_G)$ exists and is finite.

Given the construction of our altered definition for $\overline{T}_T(t_S, V^S_G)$, it will exist (and be continuous in its arguments) provided that either (a) $\Gamma(t)$ reaches its terminal value of 1 before $q(t)$ reaches its own terminal value or (b) $q(t)$ reaches its terminal value first, and $\Gamma(t)$ is well-defined for all $t \in \left[ t_S, \tilde{T}_q(t_S, V^S_G) \right]$. With $\tilde{T}_q(t_S, V^S_G)$ established above, we need only check that $\Gamma(t)$ is defined.

Clearly, $\Gamma(t)$ will be undefined only where the denominator of the first term in equation (4) is zero, and it is straightforward to verify that this will happen only as $q(t)$ reaches its lowest possible value for a given time $Q \left( \frac{\omega H e^{-\lambda t + 1 - \omega H}}{a(Q)e^{-\lambda t + 1 - a(Q)}} \right)$. However, our restriction $\overline{V^S_G}(t_S)$ will again preclude $q(t)$ from falling to such levels, so $\Gamma(t)$ is indeed defined, and we thus have the existence of $\overline{T}_T(t_S, V^S_G)$ which is continuous in its arguments.

Step 2: For any $t_S \geq 0$ and corresponding interval $[\overline{V^S_G}(t_S), \overline{V^S_G}(t_S)]$ of $V^S_G$ values for which finite-valued $\tilde{T}_q(t_S, V^S_G)$ and $\overline{T}_T(t_S, V^S_G)$ exist, there is a unique $V^S_G(t_S) \in [\overline{V^S_G}(t_S), \overline{V^S_G}(t_S)]$ for which $\tilde{T}_q(t_S, V^S_G(t_S)) = \overline{T}_T(t_S, V^S_G(t_S))$.

We now consider how $\tilde{T}_q(t_S, V^S_G)$ and $\overline{T}_T(t_S, V^S_G)$ depend on $t_S$ and $V^S_G$. 
\( \bar{T}_q(t_S,V_G^S) \) is monotonically decreasing in \( t_S \):

Holding other states fixed, it is easy to verify that \( \dot{q}(t) \) is increasing in \( t \). (As \( t \) rises, the belief \( q(t) \) falls, weakening the downward pressure on \( q(t) \) due to learning.) Because \( \dot{q}(t) \) is increasing in its own state \( q(t) \), this effect is compounded over time. Thus, raising \( t_S \) has the direct effect of raising the subsequent path of \( q(t) \) accordingly.

Additionally, this effect is compounded indirectly via the influence of \( V_G(t) \). From equation (2), \( \dot{V}_G(t) \) is decreasing in \( q(t) \), so as \( q(t) \) rises, the ensuing path of \( V_G(t) \) falls (holding fixed the level of \( V_G(t) \)). Because \( \dot{q}(t) \) is decreasing in the level of \( V_G(t) \), a rise in the path of \( q(t) \) also has a secondary feedback effect on the rate of change \( \dot{q}(t) \)—it raises \( \dot{q}(t) \) via the falling path of \( V_G(t) \), indirectly raising the subsequent path of \( q(t) \) further. (There is a further feedback effect as well, since \( \dot{V}_G(t) \) is increasing in its own state \( V_G(t) \).

Thus, \( t_S \) has a clear positive effect on the level of the \( q(t) \) path which follows. In particular, for a given \( x \), \( q(t_S + x) \) is increasing in \( t_S \). As such, \( q(t) \) will take less time to rise to its terminal value, and so \( \bar{T}_q(t_S,V_G^S) \) will fall.

\( \bar{T}_q(t_S,V_G^S) \) is monotonically increasing in \( V_G^S \):

To summarize the key dynamics that are relevant here (clear from equations (1) and (2)), \( \dot{q}(t) \) is increasing in \( q(t) \) and decreasing in \( V_G(t) \), while \( \dot{V}_G(t) \) is increasing in \( V_G(t) \) and decreasing in \( q(t) \). Thus, raising \( V_G^S \) would first raise the future path of \( V_G(t) \) itself. In turn, this would indirectly lower the path of \( q(t) \), and thus \( q(t) \) would take more time to reach its terminal value (in the range of values where this happens in finite time). Accordingly \( \bar{T}_q(t_S,V_G^S) \) rises.

\( \bar{T}_q(t_S,V_G^S) \) is monotonically increasing in \( t_S \):

Note first that the time \( t \) has a direct effect of decreasing \( \Gamma(t) \)—the rate of change for \( \Gamma(t) \)—at
each point. This can be understood intuitively in the context of market composition, but it can be
verified explicitly as well in equation (4). Extending this effect across time, increasing $t_S$ will lower
the entire path $\Gamma (t)$ that follows $t_S$ (normalized relative to that time $t_S$).

In addition, though, this occurs through a second, indirect channel via $q(t)$. Holding other states
fixed, $\dot{q}(t)$ is increasing in $t_S$ (from equation (1)). Raising $t_S$, then, raises the following path of
$q(t)$—as stated above, for a given $x$, $q(t_S + x)$ is increasing in $t_S$.
(There is a further feedback
effect as well, since $\dot{q}(t)$ is increasing in $q(t)$). In turn, $\dot{\Gamma}(t)$ is decreasing in $q(t)$, and so this
indirect channel compounds the direct effect, lowering the path of $\Gamma (t)$ further.

Thus, as a consequence of these relationships, raising $t_S$ leads $\Gamma (t)$ to take longer after $t_S$ to
reach its terminal level.

$\tilde{T}_\Gamma (t_S,V^S_G)$ is monotonically decreasing in $V^S_G$:

$V_G(t)$ has no direct effect on $\dot{\Gamma}(t)$ (as in equation (4)), so the dominant force here is exerted
through the path of $q(t)$. $\dot{V}_G(t)$ is increasing in $V_G(t)$, so raising $V^S_G$ raises the path of $V_G(t)$
that follows (meaning for a given $x$, $V_G(t_S + x)$ is increasing in $t_S$). Because $\dot{q}(t)$ is decreasing in
$V_G(t)$, this leads to a corresponding fall in the path of $q(t)$. Of course, $\dot{\Gamma}(t)$ is decreasing in $q(t)$,
so the effects feed back to the path of $\Gamma (t)$. Since $\dot{\Gamma}(t)$ is higher for each $t$, $\Gamma (t)$ reaches its terminal
value of 1 in less time. Hence, $\tilde{T}_\Gamma (t_S,V^S_G)$ falls.

And of course, each of these relationships is continuous in the corresponding argument. Having
established these, it will be illustrative to consider the object $\Delta (t_S,V^S_G) \equiv \tilde{T}_\Gamma (t_S,V^S_G) -
\tilde{T}_q (t_S,V^S_G)$, which is increasing in $t_S$ and decreasing in $V^S_G$. Naturally, it should be the case that
$\Delta (t_S,V^S_G(t_S)) = 0$.

As is obvious given our law of motion for $q(t)$, for a given $t$, it is possible to find such a low
$V_G(t)$ to make $\dot{q}(t)$ arbitrarily high. Applying this logic to the choice of $V^S_G$ allows us to extend
this effect through the entire path of \( q(t) \), and thus, \( V_S^S \) sufficiently low can ensure that \( q(t) \) rises to \( q(T) \) as rapidly as desired. Because \( V_G(t) \) has no direct effect on the dynamics of \( \Gamma(t) \) (it has only an indirect effect via its impact on \( q(t) \)), there is then a range of low \( V_S^S \) for which \( q(t) \) will reach its terminal value before \( \Gamma(t) \) (and for which \( \Delta(t_S, V^S_G(t_S)) > 0) \).

Conversely, a sufficiently high choice of \( V_S^S \) could generate such strong downward pressure on \( q(t) \) that, as described earlier, it would never reach its endpoint (and \( T_q(t_S, V^S_G) \) would therefore be undefined). Of course, because \( T_q(t_S, V^S_G) \) responds continuously to \( V^S_G(t_S) \), \( T_q(t_S, V^S_G) \) must grow without bound as we lower \( V^S_G \) toward the region where \( T_q(t_S, V^S_G) \) does not exist. Note also that \( T_S(t_S, V^S_G) \) will move more quickly toward its terminal value in response to such a lower path of \( q(t) \). Thus, considering increasing choices of \( V^S_G \) for a given \( t_S \), this \( t_S \) must have a corresponding value for \( V^S_G(t_S) \), with \( \Delta(t_S, V^S_G(t_S)) \) crossing zero from above.

Step 3: \( \exists \, \bar{t} \in R_+ \) such that \( V_G(T) = V^S_G(\bar{t}) + \int_{t_S}^{T} \dot{V}_G(t) \, dt \)

where \( T = T_q(t_S, V^S_G(t_S)) = T_T(t_S, V^S_G(t_S)) \)

and \( V_G(T) = \frac{Y + \pi[a(\frac{\mu_G}{\alpha_G}) \frac{\lambda Y}{Y - 1}]}{r + \pi} \) is the terminal value specified earlier.

From above, we know that \( T_q(t_S, V^S_G) \) is decreasing in \( t_S \) and increasing in \( V^S_G \), and conversely, that \( T_T(t_S, V^S_G) \) is increasing in \( t_S \) and decreasing in \( V^S_G \). These, together with the definition of \( V^S_G(t_S) \), imply that \( V^S_G(t_S) \) must be increasing in \( t_S \). Because \( \dot{V}_G(t) \) is increasing in the level of \( V_G(t) \), the realized value of \( V_G(T) \) must rise monotonically as in response to raising \( V_G(t_S) \), the level realized at \( \bar{t} = t_S \). Because \( V^S_G(t_S) \) is increasing in \( t_S \), raising \( t_S \) ultimately corresponds to raising the eventual \( V_G(T) \). Thus as we continue to increase the starting time above its lower bound of zero, there exists a \( t_S \) in which \( V_G(T) \) reaches its terminal value.

Step 4: Given the pair \((\bar{t}, V^S_G)\), the resulting system satisfies conditions (1) and (2) discussed
That (1) is satisfied is an immediate consequence of the previous 3 steps. Regarding (2), it is clear that (i) should be satisfied, but this can be seen explicitly through the dynamics implied by equation (2). In particular, $\dot{V}_G(t) > 0$ iff $V_G(t) > \frac{\lambda V + \pi \alpha(q(t))}{r + \pi}$. It is easy to see, via substitution, that $V_G(t)$ taking on any value below $a \left( Q \right) \frac{\lambda V}{r} - c$ at some $t \in [0, \hat{t}]$ would require it to be falling rapidly not just before $\hat{t}$, but afterward as well (because $q(t) \geq \frac{Q \omega_H e^{-\lambda T} + (1 - \omega_H)}{a(Q) e^{-\lambda T} + (1 - a(Q))}$), and it would be impossible for $V_G(t)$ to rise to its (much) higher terminal value.

For (ii), $q(t) \in \left[ \frac{Q(1 - \omega_H)}{1 - a(Q)}, \frac{\omega_H Q}{a(Q)} \right]$ holds as a result of how the parameters and initial conditions were set in Steps 1-3 in order to restrict the path of $q(t)$. Substituting our known conditions for $t_S = \hat{t}$ yields $\dot{\Gamma}(\hat{t}) = 0$. Further, by totally differentiating $\dot{\Gamma}(t)$ with respect to $t$ at $t = \hat{t}$, we find

$$\dot{\Gamma} (\hat{t}) = \frac{\dot{q}(\hat{t})}{Q (1 - Q) (\omega_H - \omega_L) \left( 1 - e^{-\lambda T} \right)}$$

$$= - \left( \frac{\pi \dot{q}(\hat{t})}{Q (1 - Q) (\omega_H - \omega_L) \left( 1 - e^{-\lambda T} \right)} \right) > 0 \text{ for } \dot{q}(\hat{t}) < 0$$

The conditions of the path of $\Gamma(t)$ are satisfied so long as $\Gamma(t)$ does not tend toward negative infinity before $\hat{T}_q(t_S, V_G)$. However, given the established bound on $q(t)$, the first term in the expression for the evolution of $\Gamma(t)$ is bounded above zero. As such, the rate at which $\Gamma(t)$ declines cannot be too large, and thus the conditions for the path of $\Gamma(t)$ are satisfied. Finally, $\Delta(t) \in \left[ 1 - e^{-\lambda T}, 1 - e^{-\lambda T} \right]$ and $\dot{\Delta}(t) \geq 0$ are direct consequences of the initial/terminal conditions and the path of $\Gamma(t)$.

Thus, we have established the existence of $(t_S, V_G(\hat{t}) = V_G^S)$ and a corresponding $T$ that are consistent with the evolution of our underlying states and that satisfy the necessary boundary conditions.

**Uniqueness:** The argument above provides the existence of an equilibrium. We now establish (by contradiction) that the equilibrium is unique. Suppose the equilibrium is not unique. Then
there exist two distinct equilibria which satisfy Definition 1. Appealing to the analysis above, these
equililibria include a choice of \( t_j \) and \( T_j \) where \( j \in \{1, 2\} \) denotes the particular equilibrium. Fur-
thermore, for all \( t \in [t_j, T_j] \), the evolutions of \( q(t) \) and \( V_G(t) \) satisfies Equations 3.19 and 3.20,
albeit at potentially different levels. Equations 3.19 and 3.20 represent a linear system of first-order
differential equations with boundary conditions given in the table above. Since \( g(t) \) is continuous in
\( t \), the Picard-Lindelof theorem establishes a unique solution to the differential system for a given
\( T \). In order for our candidate equilibria to then be distinct, we must have \( T_1 \neq T_2 \).

For notational convenience, we drop the subscript \( s \) on the quality of sales in the following
argument. Since \( T_1 \neq T_2 \), we can, without loss of generality, suppose \( T_1 < T_2 \). Importantly,
\( q_i(t) < \frac{\omega H}{a(Q)} \) for all \( t < T_i \). Since the quality of assets for sale can never exceed the quality of assets
owned, quality cannot rise to \( \frac{\omega H}{a(Q)} \) until the remaining mass of marginal owners is equal to zero,
which by definition occurs at \( T_i \). Then \( q_1(t) > q_2(t) \) for all \( t \in [T_1, T_2) \) and \( q_1(t) = q_2(t) \) for all
\( t \geq T_2 \).

Recall that the value function for an owner of a known productive asset can be written explicitly
as
\[
V^i_G(t) = \frac{\lambda Y}{r + \pi} + \int_t^{T_i} e^{-(r+\pi)(s-t)} \left[ a(q_i(s)) \left( \frac{\lambda Y}{r} \right) - c \right] ds
+ e^{-(r+\pi)(T_i-t)} \left( \frac{a(\frac{\omega H}{a(Q)})}{r + \pi} \left( \frac{\lambda Y}{r} \right) - c \right). \tag{3.23}
\]
Then for all \( t \in [T_1, T_2) \), we have \( V^1_G(t) > V^2_G(t) \). In addition, since marginal owners are indifferent
between selling at any time \( t \in [\bar{t}, T] \), the value function for a marginal owner can be written as if
the owner sells at \( T_j \) (taking quality as given),
\[
V^j_M(t) = g(t) \left( 1 - e^{-(r+\pi+\lambda)(T_j-t)} \right) \left( \frac{\lambda Y}{r + \pi} \right) + \int_t^{T_j} e^{-(r+\pi)(s-t)} \left[ a(q_j(s)) \left( \frac{\lambda Y}{r} \right) - c \right] ds
+ e^{-(r+\pi)(T_j-t)} \left[ \pi + r \left( 1 - g(t) \left( 1 - e^{-\lambda(T_j-t)} \right) \right) \right] \left( \frac{a(\frac{\omega H}{a(Q)})}{r + \pi} \left( \frac{\lambda Y}{r} \right) - c \right). \tag{3.24}
\]
It follows that $V^1_M(T_1) > V^2_M(T_1)$. As such, if $V^1_M(t) = V^2_M(t)$ for any $t \in [0,T_1)$ there exists a ‘last time’ prior to $T_1$ when the value of owning an as-of-yet unproductive asset in each equilibrium is equal. We denote this time by $\bar{t} = \sup\{t : V^1_M(t) = V^2_M(t)\}$. Since these value functions are continuous, $V^1_M(\bar{t}) = V^2_M(\bar{t})$ and $\bar{t} < T_1$. Thus, $V^1_M(t) > V^2_M(t)$ for all $t \in (\bar{t}, T_1]$.

We now consider two cases: (i) $\bar{t} \geq \hat{t}_1$, and (ii) $\bar{t} < \hat{t}_1$. If $\bar{t} \geq \hat{t}_1$, then

$$a(q_1(t)) \left(\frac{\lambda Y}{r}\right) - c = V^1_M(t) > V^2_M(t) \geq a(q_2(t)) \left(\frac{\lambda Y}{r}\right) - c \quad \text{for all } t \in (\bar{t}, T_1]$$

$$\implies q_1(t) > q_2(t) \quad \text{for all } t \in (\bar{t}, T_1]$$

However from Equation 3.24 this implies $V^1_M(\bar{t}) > V^2_M(\bar{t})$ which contradicts the definition of $\bar{t}$.

Now consider the alternative case when $\bar{t} < \hat{t}_1$. This case can be further broken down into two subcases: (iia) $V^2_M(\hat{t}_1) \geq V^1_M(\hat{t}_1)$ or (iib) $V^2_M(\hat{t}_1) < V^1_M(\hat{t}_1)$. If $V^2_M(\hat{t}_1) \geq V^1_M(\hat{t}_1)$, then by the Intermediate Value Theorem, the value functions intersect at some time $t \geq \hat{t}_1$, since the value functions are continuous and $V^2_M(T_1) < V^1_M(T_1)$. This contradicts $\bar{t} < \hat{t}_1$.

If instead $V^2_M(\hat{t}_1) < V^1_M(\hat{t}_1)$, then $V^2_M(t) < V^1_M(t)$ for all $t \in (\hat{t}_2, T_1]$ and subsequently $q_2(t) \leq q_1(t)$ for all $t \in (\bar{t}, T_1)$ with strict inequality for $t \in [\hat{t}_1, T_1]$. From Equation 3.24 this implies that $V^2_M(\bar{t}) < V^1_M(\bar{t})$, which again contradicts the definition of $\bar{t}$. Thus there cannot exist any time $(t \leq T_1)$ where the value functions of a marginal owner in the two equilibria are equal.

We now show that if $V^1_M(t) \neq V^2_M(t)$ for all $t \leq T_1$, then it must be the case that $\hat{t}_1 > \hat{t}_2$. We again proceed by contradiction. Suppose $\hat{t}_2 \geq \hat{t}_1$. We have already established that $\hat{t}_2 \neq \hat{t}_1$, since the prices would necessarily be equal at this point. Now consider the case with $\hat{t}_2 > \hat{t}_1$. Then, $V^2_M(\hat{t}_1) \geq V^1_M(\hat{t}_1)$ and $V^2_M(T_1) < V^1_M(T_1)$. Since the value functions are continuous, the paths must cross at some point in the interval $[\hat{t}_2, T_1]$ which contradicts the original supposition.

Thus far we have shown that if there exist two equilibria, the value of owning an as-of-yet unproductive asset cannot be equal in the two equilibria at any point in time. Furthermore, this
implies that in one equilibrium, marginal selling must begin later and end sooner than in the other. That is, \( \hat{t}_1 > \hat{t}_2 \) and \( T_1 < T_2 \). Finally, we show that there cannot exist two equilibria which satisfy these conditions. From the characterization of equilibrium, it is clear that

\[
\Gamma_1(T_1) = 1 > \Gamma_2(T_1) \quad (3.25)
\]
\[
\Gamma_1(\hat{t}_1) = 0 < \Gamma_2(\hat{t}_1), \quad (3.26)
\]

which requires \( \Gamma'_1(t) > \Gamma'_2(t) \) for some \( t \in [\hat{t}_1, T_1] \). Furthermore, since \( \Gamma'_1(t) = 0 \leq \Gamma'_2(t) \) for all \( t \leq \hat{t}_1 \), we can define \( \tau \) as the ‘first time’ \( \Gamma'_1(t) > \Gamma'_2(t) \). That is, \( \tau = \inf \{ t : \Gamma'_1(t) > \Gamma'_2(t) \} \).

Since \( \Gamma_i(t) \) is continuously differentiable, \( \Gamma'_1(t) = \Gamma'_2(t) \). From Equation (3.10), market quality at \( t \) in candidate equilibrium \( i \) is

\[
q_i(t) = Q \left( \frac{[1 - \omega_H(1 - e^{-\Lambda_i})][[1 - \Gamma_i(t)] \pi + \Gamma'_i(t)] + \omega_H \Lambda_i(t) \pi}{[1 - a(Q)(1 - e^{-\Lambda_i})][[1 - \Gamma_i(t)] \pi + \Gamma'_i(t)] + a(Q) \Lambda_i(t) \pi} \right).
\]

Since \( \Gamma'_1(t) = \Gamma'_2(t) \) while \( \Gamma_1(t) \leq \Gamma_2(t) \), some algebra shows that \( q_1(t) \leq q_2(t) \). In addition, \( \tau > \hat{t}_1 > \hat{t}_2 \). Thus, as a result of the marginal owner’s indifference condition, we have \( V^1_M(t) \leq V^2_M(t) \). However, we have already shown that \( V^1_M(T_1) > V^2_M(T_1) \), which violates the non-intersection criterion derived above. Thus there cannot exist multiple equilibria which satisfy Definition 1.

**Lemma 3:** For

\[
c \in \left( \left( \frac{\pi \lambda (1 - a(Q))}{r(r + \pi + \lambda) + \pi \lambda (1 - a(Q))} \right) a \left( \frac{Q(1 - \omega_H)}{1 - a(Q)} \right) \frac{\lambda Y}{r}, a \left( \frac{Q(1 - \omega_H)}{1 - a(Q)} \right) \frac{\lambda Y}{r} \right),
\]

some marginal owners sell too soon, \( \hat{t} < t^* \).
Proof: Suppose \( t^* \leq \hat{t} \), then from Lemma 4, \( V_M(t^*) \geq a(Q) \frac{\lambda Y}{r} - c \). Furthermore, this implies

\[
\begin{align*}
    r \left[ a(Q) \frac{\lambda Y}{r} - c \right] & \leq r V_M(t^*) \\
    &= g(t^*) \left[ Y + V_G(t^*) - a(Q) \frac{\lambda Y}{r} + c \right] + \pi \left[ a(Q) \frac{\lambda Y}{r} - V_M(t^*) \right] + V'_M(t^*) \\
    &\leq g(t^*) \left[ Y + \frac{\lambda Y + \pi \left( a \frac{Q}{a(Q)} \frac{\lambda Y}{r} - c \right)}{r + \pi} - a(Q) \frac{\lambda Y}{r} + c \right], \\
    \end{align*}
\]

(3.27)

where the last inequality uses \( V_G(t) \leq V_G(T) \) for all \( t \) and \( V'_M(t) \leq 0 \) for \( t \leq \hat{t} \). Substituting for \( t^* \) (from Equation 3.4) in Equation 3.27 and rearranging terms yields

\[
c \leq \left( \frac{\pi \lambda (1 - a(Q))}{r(\pi + \lambda)} \right) \left( \frac{Q(1 - \omega_H)}{1 - a(Q)} \right) \frac{\lambda Y}{r}.
\]

which contradicts (A3). Thus for parameters satisfying (A3), \( V_M(t^*) < a(Q) \frac{\lambda Y}{r} - c \), and \( \hat{t} < t^* \).

Proposition 3: The social optimum can be implemented at minimal cost through a subsidized price path of the form above if

\[
    rp_0 \leq g(t_0) \lambda [Y + V_G(t_0) - p_0]
\]

where \( t_1 = t^* - \frac{1}{r} \ln \left[ \frac{p}{p_0} \right] \).

\[
    V_G(t_0) = \frac{\lambda Y}{r + \pi} + \left( \frac{e^{-(r+\pi)(t_1-t_0)}(1-e^{-\pi[t^*-t_1]}+\pi)}{r+\pi} \right) p_0
\]

and \( t_0 \equiv \arg \min \ V_M(\hat{t}) = \arg \min_i \left\{ \frac{g(\hat{t}) \left[ 1 - e^{-(r+\pi+\lambda)(t^* - \hat{t})} \right] \left( \frac{\lambda Y}{r+\pi} \right)}{\pi (r+\pi) \left[ 1 - g(\hat{t})e^{-\pi(t^*-t_1)} \left( 1 - e^{-\lambda(t^* - \hat{t})} \right) + \pi \right]} p_0 \right\}.

Proof: We consider the case when \( t_1 > 0 \). In this case, the least costly contract involves prices

\[
p(t) = \begin{cases} 
    p_0 & \text{for } t \in [0, t_1] \\
    e^{(t-t_1)} p_0 & \text{for } t \in [t_1, t^*] \\
    p & \text{for } t \geq t^*
\end{cases}
\]
Note further that from the definition of $t_1$, $\bar{p} = e^{r(t^*-t_1)}p_0$. Given the contracted path of prices, the value functions for an owner of a revealed productive asset can be expressed analytically at $t_0$ as

$$V_G(t_0) = \frac{\lambda Y}{r + \pi} + \int_{t_0}^{t_1} e^{-(r + \pi)(s-t_0)} \pi p_0 ds + \int_{t_1}^{t^*} e^{-(r + \pi)(s-t_1)} \pi p_0 ds + \int_{t^*}^{\infty} e^{-(r + \pi)(s-t_0)} \pi e^{r(t^*-t_1)} p_0 ds$$

$$= \frac{\lambda Y}{r + \pi} + \left( \frac{re^{-(r + \pi)[t_1-t_0]}(1 - e^{-\pi[t^*-t_1]}) + \pi}{r + \pi} \right) p_0.$$ 

Since the path of prices is flat at $t^*$ and the price level is greater than $p^*_R$, any owner of an as-of-yet unproductive asset sells at $t^*$. Thus the value function for a marginal owner can be expressed analytically for $t \leq t_1$ as

$$V_M(t) = g(t) \left( 1 - e^{-(r + \pi + \lambda)(t^*-t)} \right) \left( \frac{\lambda Y}{r + \pi} \right) + \int_{t}^{t_1} e^{-(r + \pi)(s-t)} \pi p_0 ds + \int_{t_1}^{t^*} e^{-(r + \pi)(s-t_1)} \pi e^{r(s-t_1)} p_0 ds$$

$$+ e^{-(r + \pi)(t^*-t)} \left[ \pi + r \left( 1 - g(t) \left( 1 - e^{-\lambda(t^*-t)} \right) \right) \right] e^{r(t^*-t_1)} p_0$$

$$= g(t) \left[ 1 - e^{-(r + \pi + \lambda)(t^*-t)} \right] \left( \frac{\lambda Y}{r + \pi} \right)$$

$$+ \left( \frac{re^{-(r + \pi)[t_1-t]}(1 - g(t)e^{-\pi[t^*-t_1]}(1-e^{-\lambda[t^*-t]})} {r + \pi} \right) p_0$$

Given this expression, we can define $t_0 \equiv \arg\min_{\tilde{t}} V_M(\tilde{t})$ to be the time when $V_M(t)$ achieves its minimum. Now suppose that at its minimum value $V_M(t_0) < p_0$, then

$$rp_0 > rV_M(t_0)$$

$$= g(t_0) \lambda [Y + V_G(t_0) - V_M(t_0)] + \pi [p_0 - V_M(t_0)] + V_M'(t_0)$$

$$> g(t_0) \lambda [Y + V_G(t_0) - p_0]$$

where the last inequality uses $V_M(t_0) < p_0$ and $V_M'(t) \geq 0$ at the minimum. Thus if Equation 3.14 is satisfied, $\min V_M(t) \geq p_0$, and the contract can be implemented.
Bibliography


