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A Mnemonic for Feigenbaum's Universal Number \( \delta \)

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A Mnemonic for Feigenbaum's Universal Number $\delta$

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Abstract: A mnemonic relation between Feigenbaum's universal number $\delta$ and the fine structure constant is described.

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1. **Feigenbaum's universal number \( \delta \)**

In a variety of physical processes, a transition from ordered (periodic) to chaotic behavior may be induced by gradually increasing a parameter (say \( \Lambda \)) entering into the specification of the physical situation. The transition often proceeds by a sequence of doublings of the period of the ordered motion, the doublings occurring when the parameter \( \Lambda \) reaches values \( \Lambda_1, \Lambda_2, \ldots, \Lambda_n, \ldots, \Lambda_\infty \), say, and chaos setting in beyond \( \Lambda_\infty \). M. Feigenbaum\(^{1-4} \) gave reasons to believe that, for many systems, this sequence tends to a universal geometric progression such that

\[
\lim_{n \to \infty} \left( \frac{\Lambda_{n+1} - \Lambda_n}{\Lambda_{n+2} - \Lambda_{n+1}} \right) = \delta,
\]

where \( \delta \) is a universal number, whose value is

\[
\delta = 4.6692016091029\ldots \quad (2)
\]

This is a fundamental constant governing the transition from order to chaos in a variety of idealized systems that bear a marked resemblance to a number of actual physical processes.

2. **A mnemonic**

The following relation reproduces Feigenbaum's number \( \delta \) to 6 parts in \( 10^5 \):

\[
\delta \approx \sqrt{\frac{137}{2\pi}} = 4.6694997
\]

\[
= (1.000064)(4.6692016) \quad (3)
\]

3. **A fairy tale**

A fairy tale helps one to remember eq. (3):

"According to quantum mechanics, the smallest resolvable area in two-dimensional phase space is \( \hbar/2 \) (i.e., \( \Delta p \Delta x \geq \hbar/2 \)). On the other hand, in dynamical theories involving charged elementary particles in relativistic space-time, the natural unit with the dimensions of a phase-space area (the natural unit of action) is \( e^2/c \), where \( e \) is the elementary charge and \( c \) is the constant characterizing space-time. Now, if the quantal limit on resolution in phase space were in some way related to the inception of chaos in elementary-particle dynamics (a very big
'if indeed!*), Feigenbaum's number \( \delta \) might be expected to appear in the equation determining the smallest resolvable area. Since one is looking for an area, \( \pi \delta^2 \) seems like a promising term to experiment with. What could be simpler than to try

\[ \hbar/2 \approx (e^2/c)\pi\delta^2 \]  

(4)

End of fairy tale."

Eq. (4) reduces to eq. (3) if the approximation \( \hbar c/e^2 \approx 137 \) is used. Conversely, eq. (4) would give for the inverse of the fine structure constant, \( \alpha \), the approximation

\[ \alpha^{-1} \equiv \hbar c/e^2 \approx 2\pi\delta^2 = 136.982511. \]  

(5)

The ratio of the measured value\(^7\) of \( \hbar c/e^2 \), 137.035963±0.000015, to the above number is 1.000390, a correspondence accurate to 4 parts in \( 10^4 \).

Independently of whether there is any substance in the above fairy tale, the following statement is true and provides a useful mnemonic:

"The smallest resolvable area in phase space allowed by quantum mechanics, when expressed in the natural electromagnetic unit of action, is equal to the area of a circle whose radius, to within 2 parts in \( 10^4 \), is Feigenbaum's universal number \( \delta \)."

Fig. 1 illustrates this relation.

4. What about the remaining discrepancy?

Since eq. (4) is based on a speculation, further speculation ought to be able to improve it. Here is an example: perhaps the relation between \( \hbar c/e^2 \) (= \( \alpha^{-1} \)) and \( 2\pi\delta^2 \) should be thought of as an expansion in the fine structure constant (or in \( (2\pi\delta^2)^{-1} \), which is almost the same thing). Try

\[ \hbar c/e^2 = \alpha^{-1} \approx \beta^{-1} + k_0 + k_1 \beta + k_2 \beta^2 + \ldots, \]  

(6)

where \( \beta = (2\pi\delta^2)^{-1} \).  

(7)

*But, on the other hand, quite unexpected developments sometimes take place: who would have anticipated that "the criterion for smoothness of invariant tori in certain dissipative dynamical systems approaching chaos can be formulated in terms of an eigenvalue problem--a discrete Schrödinger equation ..." in which "the parameter playing the role of \( \hbar \) is related inversely to the damping of the dynamical system"! (Ref. 5). In this connection ref. 6 might also be of interest.
The following relatively simple expression turns out to be satisfactory:

$$\alpha^{-1} = \beta^{-1} + e^\beta / 6\pi \quad (8)$$

It gives

$$\alpha^{-1} = 137.0359510, \quad (9)$$

which deviates from the experimental value by 0.000012. This is within the experimental uncertainty of ± 0.000015 and corresponds to a precision of 1 part in 10^7.

The resulting fairy-tale equation for the smallest resolvable area in phase space, i.e., the equation for \(\hbar/2\) expressed in terms of \(e^2/c\), now reads:

$$\hbar/2 = (e^2/c)(\pi\delta^2 + e^\beta / 12\pi). \quad (10)$$

The exponential term implies a slight increase of the radius of the circle whose area is supposed to represent \(\hbar/2\). This could be thought of as a kind of halo, of relative order \(\alpha\). Its justification is, however, as obscure as the association of Feigenbaum's \(\delta\) with the radius of the circle in the first place.

5. An accurate mnemonic

Because of the smallness of \(\delta\), the exponential in eq. (8) may be expanded and the first two or three terms give an excellent approximation. It also follows that an almost exact solution of eq. (8) for \(\beta\) in terms of \(\alpha\) is:

$$\beta^{-1} = \alpha^{-1} - e^\alpha / 6\pi. \quad (11)$$

(The numerical solution of this equation for \(\alpha\) gives \(\alpha^{-1} = 137.0359507\).) Equation (11) is a very accurate mnemonic relation for Feigenbaum's number \(\delta\). It gives

$$\delta = 4.66920182 \pm 0.00000026, \quad (12)$$

which is consistent with the exact value, eq. (2): the difference amounts to 5 parts in 10^8 and is within the uncertainty caused by the limited precision in the value of \(\alpha\).
References

4. P. Cvitanovic, Universality in Chaos (Hilger, Bristol, 1984)
7. J. K. Tuli, Nuclear Wallet Cards, January 1985, National Nuclear Data Center, Brookhaven National Laboratory, Upton, New York 11973, Appendix III-vi
Figure caption

Fig.1. Illustration of the approximate numerical relation between the smallest resolvable phase space area $\hbar/2$, the natural unit of phase space area $e^2/c$ and Feigenbaum's number $\delta$. 
Fig. 1

\[ \text{Area} = \frac{\pi}{12} \]

\[ \delta = 4.669... \]

\[ \text{area} = \frac{e^2}{c} \]
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