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MODEL OF THE INTERACTION BETWEEN TWO RESONANCES*

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ABSTRACT

We present a simple model to illuminate the dynamical interplay between two nearby resonances. The model assumes that each resonance is associated with a different channel in the limit when inter-channel coupling is absent. We study the changes induced in the real and imaginary parts of the resonance-pole positions when coupling is "turned on." The relevance of this model to the $A_2$ mesons is discussed.
I. INTRODUCTION

Recently a remarkable splitting of the $A_2$ meson has been observed in certain experiments. The mass distribution can be fitted in three (essentially two) different ways corresponding to a pair of resonance-poles in three different configurations. We shall here study certain interesting dynamical aspects of overlapping resonances suggested by one of the fittings—the combination of one broad and one narrow resonance.

We shall study these by a simple model. In the model, we begin with two resonance poles located in the complex momentum plane, each pole being associated with a separate channel. Then we calculate the shifts in real parts and imaginary parts of these poles when we introduce a small coupling between these two channels. We shall pay particular attention, within the context of our model, to the familiar statement that when we couple two poles together, their imaginary parts "attract" each other, but their real parts "repel." We shall also study the variation of shapes of the different matrix elements of the scattering amplitude near the resonance region.

Some speculations concerning the $A_2$ mesons are presented in the last section.
II. SOME SIMPLE ARGUMENTS

It is well known that two levels in a quantum system usually become separated further if we introduce coupling between them. This can be seen from the second-order perturbation formula in nonrelativistic quantum mechanics if the two levels correspond to bound states. In the N/D equations for the two-channel partial-wave amplitude, if we have \( q_1 = \xi_1 - i\eta_1, \ i = 1, 2 \) as the locations of the resonance-poles before coupling, then we can write the D functions for each channel,

\[
D_{11}^0 \propto (q - q_1), \quad D_{22}^0 \propto (q - q_2). \quad (II-1)
\]

After coupling, a plausible guess for the determinant of the D matrix is

\[
\mathcal{D} \propto (q - q_1)(q - q_2) - g^2; \quad (II-2)
\]

where the parameter \( g \) measures the strength of coupling. This gives the new pole positions, up to order \( g^2 \), as

\[
q'_i \approx q_i + \frac{g^2}{q_2 - q_1}
= \left[ \xi_i + \frac{g^2(\xi_2 - \xi_1)}{(\xi_2 - \xi_1)^2 + (\eta_2 - \eta_1)^2} \right] - i \left[ \eta_i \pm \frac{g^2(\eta_2 - \eta_1)}{(\xi_2 - \xi_1)^2 + (\eta_2 - \eta_1)^2} \right],
\]

(II-3)

with upper signs for \( i = 1 \) and lower signs for \( i = 2 \).

We see here that there is always "repulsion" in the real direction (except when \( \xi_2 - \xi_1 = 0 \)) and "attraction" in the imaginary
direction (except when $\eta_2 - \eta_1 = 0$) as a result of the coupling. Notice that for the case $(\eta_2 - \eta_1)^2 \gg (\xi_2 - \xi_1)^2$, the repulsion in real parts is reduced by a factor of $\left[ (\xi_2 - \xi_1)/(\eta_2 - \eta_1) \right]^2$ in comparison with the case for two real poles.

The $D$ functions given by the above equations, however, are not real analytic and do not have the unitary cut. So this simple argument has likely oversimplified the pole interaction phenomena. In fact, from the model we are going to study, as we take account of Hermitian analyticity and unitary, we get corrections to the result given above.
III. TWO RESONANCES IN TWO COUPLED CHANNELS

We shall study the coupling of two channels by a simple model in the framework of the N/D formalism. In our model, we have two resonances, each belonging to a different channel before we "turn on" the coupling. Then we calculate the resonance-pole positions after we "turn on" the coupling.

Specification of the model:

1. We shall study the simplest case, the s-wave elastic scattering of two spinless equal mass particles. We take the two channels to have the same threshold. The left-hand discontinuity of the partial-wave amplitude is assumed to have the form

\[
f(v) = \left( \frac{\pi a_1 \delta(v + \nu_1) + \pi a_2 \delta(v + \nu_2)}{\pi g \delta(v + \nu_2)}, \frac{\pi g \delta(v + \nu_2)}{\pi b_1 \delta(v + \nu_1) + \pi b_2 \delta(v + \nu_2)} \right), \tag{III-1}
\]

where \( \nu = q^2 \), \( q \) being the magnitude of the momentum in the c.m. frame. We are thus replacing the left-hand discontinuity in each channel by two "effective poles," and we require \( a_1 b_1 > 0; a_2 b_2 < 0; \nu_2 > \nu_1 > 0 \) in order to produce a resonance pole in the \( q \) plane, \( q_1 = \xi_1 - i \eta_1 \) with \( \xi_1 > 0, \eta_1 > 0 \). In potential language such a requirement corresponds to long-range repulsion plus short-range attraction. We have made two additional arbitrary specifications, for simplicity: (i) The two channels have the same "range of forces" \( (\nu_1, \nu_2) \) but different "strengths" \( (a_1, a_2, b_1, b_2) \); by adjusting the strengths we can choose the initial resonance-pole positions as we
like (that this is possible is shown in Appendix B). (ii) The inter-channel coupling "range" is the same as that of the "short range" part of the diagonal coupling.

2. Models such as that defined by Eq. (III-1) are at best realistic only over a limited energy region. We can lose little physics by a nonrelativistic approximation and we gain greatly in mathematical simplification. Thus we shall use the nonrelativistic phase space factor

\[ \rho(v) = \begin{pmatrix} \frac{1}{v^2} & 0 \\ 0 & \frac{1}{v^2} \end{pmatrix}, \]

where we take the mass \( m = 1 \). Our model is now realized by the \( 2 \times 2 \) scattering amplitude \( T(v) = N(v) D^{-1}(v) \), with

\[ N(v) = \frac{1}{\pi} \int_{-\infty}^{V_L} dv' \frac{f(v') D(v')}{v' - v}, \]

\[ D(v) = 1 - \frac{1}{\pi} \int_{0}^{\infty} dv' \frac{\rho(v') N(v')}{v' - v}. \]

Later we shall further specify the particular initial pole positions which interest us.

Procedure and approximations in solving for the pole positions:

1. We put Eq. (III-1) into Eq. (III-3) to evaluate \( N(v) \); then we put \( N(v) \) and Eq. (III-2) into Eq. (III-4) to evaluate \( D(v) \). In the latter step, we first evaluate the integral in the region \( v < 0 \) so
that the factor \((-v)^{\frac{1}{2}}\), which appears there, is positive real. We thus get expressions for \(N(v)\) and \(D(v)\) in terms of constants \(D_{ij}(-v_k)\), where \(i,j = 1,2\) are the channel indices and \(k = 1,2\).

2. Next we substitute \(v = -v_1, -v_2\) into the expressions for \(D(v)\), to achieve sets of simultaneous linear algebraic equations for the constants \(D_{ij}(-v_k)\). Solving these equations, we obtain expressions for these constants in terms of the seven parameters of the model: 

\((a_1, a_2, b_1, b_2, v_1, v_2, \text{ and } g)\).

3. We then continue the expressions for \(N(v)\) and \(D(v)\) to the vicinity of the physical region according to the well-known +ic prescription, as required by causality, so that \((-v)^{\frac{1}{2}} = -i\eta\), where \(q = \xi - i\eta\).

4. The poles of the scattering amplitude are located at the zeros of the equation

\[ \mathcal{D} = \det D = D_{11}D_{22} - D_{12}D_{21} = 0. \]  \hspace{1cm} (III-5)

Putting \(g^2 = 0\) for the moment, we get

\[ \mathcal{D} = D_{11}^0D_{22}^0 = 0, \]  \hspace{1cm} (III-6)

where \(D_{11}^0, D_{22}^0\) are the respective \(D\) functions for the separated channels. The equation \(D_{11}^0 = 0\) evidently gives the pole position in channel 1 before coupling. It is found that \(D_{11}^0 \propto (q - q_1)(q + q_1^*)\), so \(q_1 = \xi_1 - i\eta_1\) is the resonance-pole position of the uncoupled channel 1. The expressions for \(\xi_1\) and \(\eta_1\) are given in terms of \((a_1, a_2, v_1, \text{ and } v_2)\) in Appendix A. Similar results are of course obtained for channel 2. As it turns out, Eq. (III-5) is a biquadratic
equation in $q$, so we can obtain the zeros in closed form without assuming $g^2$ small. However, in order to get a simple and transparent solution, we confine attention to the case when $g$ is a small perturbation, and we calculate the pole positions after coupling up to order $g^2$. In this approximation, Eq. (III-5) can be rewritten as

$$Q = D_{11}^0 D_{22}^0 - g^2 M,$$  \hspace{1cm} (III-7)

where $M$ is a complicated expression shown in Appendix A. The pole positions after coupling are obtained from

$$q_1' = q_1 + g^2 \left[ \frac{M}{D_{11}^0 \frac{\partial D_{11}^0}{\partial q}} \right]_{q=q_1} = q_1 + \delta q_1,$$  \hspace{1cm} (III-8)

with a similar expression for $q_2'$.

6. Let us define $\Delta \xi = \xi_2 - \xi_1$, $\Delta \eta = \eta_2 - \eta_1$ as the pole separations before coupling. We can then use the relations Eq. (A-9,10) and Eq. (A-5,6) to express the parameters of the model $(a_1, a_2, b_1, b_2, v_1, v_2, g^2)$ in terms of $(\xi_1, \eta_1, \Delta \xi, \Delta \eta, v_1, v_2, g^2)$. This step can be carried out without any approximation. Using the latter set of parameters to express our final result, the shifts $\delta q_1$ are related to the pole position before coupling.

7. So far our scheme can be used to study the interaction of two poles in any "pre-coupling" configuration. Now we aim at the presumed $A_2$ situation which stimulates this study, namely, the interaction of one broad and one narrow resonance which overlap. We thus
specify the initial pole positions such that,

(i) \( q_1 \) is a broad pole, \( q_2 \) is a narrow pole; with

\[
\frac{\eta_2}{\eta_1} \ll 1 \quad \text{(e.g., } \mathcal{O}(\frac{1}{10} \text{ to } \frac{1}{100})\text{), so } \Delta \eta = \eta_2 - \eta_1 \approx -\eta_1.\]

(ii) \( \frac{\eta_1}{\xi_1} \ll 1 \) and \( \frac{\Delta \xi}{\xi_1} \ll 1 \), [e.g., \( \mathcal{O}(\frac{1}{10}) \)]. The conditions that \( \frac{\eta_1}{\xi_1} \) be small guarantee that the poles be sufficiently close to the physical region as to be identified as resonances. That \( \frac{\Delta \xi}{\xi_1} \) is small means that the two poles are close to each other. We do not specify at this stage the relative magnitude of \( \eta_1 \) and \( \Delta \xi \), but keep in mind that \( \Delta \xi \) should not be many times larger than \( \eta_1 \), i.e., the initial real separation should not be many times larger than the width of the broad pole. We anticipate that two widely separated resonances do not significantly influence each other when the coupling is small and so concentrate our study on situations where the two resonances are overlapping.

Beyond the above, the range parameters \( \nu_1 = \mu_1^2 \) and \( \nu_2 = \mu_2^2 \) remain unspecified. To simplify the final expressions and get a result as easy as possible to interpret, we now assume, without much loss of generality, that

(iii) \( \nu_1 \) is sufficiently small (recall \( \nu_1^{-1} \) is related to the "long range" of the diagonal coupling) that the expression \([ (\mu_1^2 + \xi_1^2)/(\xi_1^2) ] \) is \( \mathcal{O}(1) \), while \( \nu_2 \) is not so large (recall \( \nu_2^{-1} \) is related to the "short range" of the diagonal coupling that we
must keep terms like $(\mu_2 \eta_1^2)$ in comparison with $\xi_1^2$. The latter
simplification is not strictly necessary for the result below, but
makes certain coefficients simpler.

Under the approximations described in 4, 6, and 7 above, we
obtain the following shifts of the two poles due to the coupling:

$$
\delta \xi_1 = g^2k_1 \frac{\xi_1^2}{4[(\Delta \xi)^2 + \eta_1^2]} \left\{ \left( \frac{\mu_1^2 + \xi_1^2}{\xi_1^2} \right) \left( \frac{2\Delta \xi}{\xi_1^2} \right) + 8 \left( \frac{\eta_1}{\xi_1} \right) \left( \frac{\eta_1}{\xi_1} \right) \right. \\
- \left( \frac{\mu_1^2 + 5\xi_1^2}{\xi_1^2} \right) \left[ \left( \frac{\Delta \xi}{\xi_1} \right)^2 + \left( \frac{\eta_1}{\xi_1} \right)^2 \right] + \mathcal{O} \left[ \left( \frac{\eta_1}{\xi_1} \right)^2 \left( \frac{\Delta \xi}{\xi_1} \right), \left( \frac{\Delta \xi}{\xi_1} \right)^3, \ldots \right] \right\}, \tag{III-9}
$$

$$
\delta \eta_1 = -g^2k_1 \frac{\xi_1^2}{4[(\Delta \xi)^2 + \eta_1^2]} \left\{ \left( \frac{\mu_1^2 + \xi_1^2}{\xi_1^2} \right) \left( \frac{2\eta_1}{\xi_1} \right) \right. \\
+ \mathcal{O} \left[ \left( \frac{\eta_1}{\xi_1} \right) \left( \Delta \xi \xi_1 \right), \ldots \right] \right\} + g^2k_{\Pi} \left\{ \frac{2\eta_1}{\xi_1} \right\}, \tag{III-9'}
$$

$$
\delta \xi_2 = g^2k_1 \frac{\xi_1^2}{4[(\Delta \xi)^2 + \eta_1^2]} \left\{ \left( \frac{\mu_1^2 + \xi_1^2}{\xi_1^2} \right) \left( \frac{2\Delta \xi}{\xi_1^2} \right) - 8 \left( \frac{\eta_1}{\xi_1} \right) \left( \frac{\eta_2}{\xi_1} \right) \right. \\
- \left( \frac{\mu_1^2 + 5\xi_1^2}{\xi_1^2} \right) \left[ \left( \frac{\Delta \xi}{\xi_1} \right)^2 + \left( \frac{\eta_1}{\xi_1} \right)^2 \right] + \mathcal{O} \left[ \left( \frac{\eta_1}{\xi_1} \right)^2 \left( \frac{\Delta \xi}{\xi_1} \right), \left( \frac{\Delta \xi}{\xi_1} \right)^3, \ldots \right] \right\} \\
+ g^2k_{\Pi} \left\{ \left( \frac{\mu_1^2 + \xi_1^2}{\xi_1^2} \right) \right. + \mathcal{O} \left[ \left( \frac{\eta_1}{\xi_1} \right)^2 \right] \right\}, \tag{III-10}
$$
\[ \delta \eta_2 = + \varepsilon^2 k_I \frac{\xi_1^2}{4[\varepsilon^2 + \eta_1^2]} \left[ \left( \frac{\mu_1^2 + \xi_1^2}{\xi_1^2} \right) \left( \frac{2\eta_1}{\xi_1^2} \right) \right] \]

\[ + \mathcal{O} \left[ \left( \frac{\eta_1}{\xi_1} \right) \left( \frac{\Delta \xi}{\xi_1} \right), \ldots \right] \left[ \frac{2\eta_2}{\xi_1} \right] \]  

where

\[ k_I = \left[ \frac{\mu_2^2 + \xi_1^2}{2\mu_2(\mu_2 + \mu_1)} \right]^4 \left[ \frac{\mu_1^2 + \xi_1^2}{(2\xi_1^2)(2\eta_1)(2\eta_2)} \right] \]  

\[ k_{II} = \frac{\xi_1^2}{\xi_1^2 + \mu_2^2} k_I. \]

Note that \( k_I > k_{II} \) and recall that
\[ q'_1 = q_1 + \delta q_1 = (\xi_1 + \delta \xi_1) - i(\eta_1 + \delta \eta_1). \]

Discussion of the result:

1. The imaginary parts: Examine \( \delta \eta_1 \) and \( \delta \eta_2 \) according to Eq. (III-9') and Eq. (III-10'). The terms with the coefficient \( k_{II} \) are not equal, both tending to make the respective poles broader, but they are small in comparison with the terms having the coefficient \( k_I \). The important contribution comes from the latter. We see therefore that the broad pole No. 1 becomes narrower, and the narrow pole No. 2 becomes broader: the two poles move equal but opposite amounts in the imaginary direction; thus \( \eta'_1 + \eta'_2 = \eta_1 + \eta_2 \), the sum of the widths being conserved.

2. The real parts: Examine \( \delta \xi_1 \) and \( \delta \xi_2 \) according to Eq. (III-9) and Eq. (III-10). The leading terms with coefficient \( k_{II} \)
give an increase in the real direction by the same amount, i.e., a common shift. They can be much smaller than, or of the same order of magnitude as the terms with coefficient $k_1$. The third terms within the brackets [ ] with coefficient $k_1$ give again a common shift, but the first and second terms give the relative shifts which are most interesting to us.

Let us keep only those terms with leading contributions and look at only relative shifts; we can then recast our result above as

$$\left(\frac{\delta \eta_2}{\eta_2}\right) = \left(\frac{-\delta \eta_1}{\eta_1}\right) \left(\frac{\eta_1}{\eta_2}\right),$$  \hspace{1cm} \text{(III-13)}

where $\delta \eta_2 = -\delta \eta_1$ is positive. At the same time,

$$(\delta s_1)_r = \left(-\frac{\delta \eta_1}{\eta_1}\right) \Delta \xi + 4 \left(\frac{\xi_1^2}{\mu_1^2 + \xi_1^2}\right) \left(-\frac{\delta \eta_1}{\eta_1}\right) \left(\frac{\eta_1}{\xi_1}\right) \eta_1$$ \hspace{1cm} \text{(III-14)}

$$(\delta s_2)_r = + \left(-\frac{\delta \eta_1}{\eta_1}\right) \Delta \xi.$$ \hspace{1cm} \text{(III-15)}

Thus the real separation after coupling is

$$\Delta \xi' = \xi_2' - \xi_1' = \Delta \xi + \left(-\frac{\delta \eta_1}{\eta_1}\right) \left[2 \Delta \xi - 4 \left(\frac{\xi_1^2}{\mu_1^2 + \xi_1^2}\right) \left(\frac{\eta_1}{\xi_1}\right) \eta_1\right].$$ \hspace{1cm} \text{(III-16)}

The first terms in Eq. (III-14) and Eq. (III-15) give the usual "repulsion," but we have now picked up the leading correction: the second term in Eq. (III-14), [the corresponding term in Eq. (III-15)]
was dropped because we assumed $\eta_2$ negligible in comparison with $\eta_1$. If $\Delta \xi < 0$, it follows that $|\Delta \xi'| > |\Delta \xi|$ always, and we may characterize the initial configuration as "unstable." In the following we consider only $\Delta \xi > 0$. If $\Delta \xi > \eta_1$, the correction term is negligibly smaller than the first term (by a factor of $\eta_1 \xi_1^2$), and thus the familiar picture of "repulsion" again appears. However, if $\Delta \xi < \eta_1$, i.e., the two resonances have already overlapped before coupling, it can happen that the correction term becomes large enough to counterbalance the repulsion (see Fig. 1), changing from less overlapping to more overlapping. We can have $|\Delta \xi'| < |\Delta \xi|$ if

$$\Delta \xi < 2 \left( \frac{\xi_1^2}{\mu_1^2 + \xi_1^2} \right) \left( \frac{\eta_1}{\xi_1} \right) \eta_1 < \Delta \xi \left[ \left( \frac{\eta_1}{-8 \eta_1} \right) + 1 \right], \quad (III-17)$$

in this case the real parts can be said to "attract" each other.

Imagine an example where $\eta_1 = 250$ (in MeV units, say),

$$\left( \frac{-5 \eta_1}{\eta_1} \right) = \frac{2}{10}, \quad \left( \frac{\eta_1}{\xi_1} \right) = \frac{1}{10}, \quad \Delta \xi = 15$$

then the correction term in Eq. (III-14) is about 20, while the repulsion term is about 3. Thus we end up with $\Delta \xi' \approx 1 << \Delta \xi$. This example, of course, corresponds to a high degree of pre-coupling overlap $\frac{\Delta \xi}{\eta_1} = \frac{15}{250}$, in order to achieve maximal overlap as a result of the coupling. The example is ad hoc, but it shows that this kind of perturbation can in principle enhance the degeneracy, instead of breaking or widening it. We shall below give a rough estimation of the first inequality in Eq. (III-17) on the basis of observed cases of overlapping resonances, namely, the $K_S-K_L$ mesons, $\rho-\omega$ mesons, and $A_2$ mesons.
IV. BEHAVIOR OF $T$ AROUND THE RESONANCE REGION

Let us discuss briefly the behavior of the different matrix elements of the transition amplitude in the physical region near the resonances. Recalling that $T_{11} = \mathcal{N}_{11}/\mathcal{O}$ with $\mathcal{N}_{11} = N_{11}D_{22} - N_{12}D_{21}$ etc., where $N_{ij}, D_{ij}$ are obtained from Eqs. (III-3,4), we evaluate these $\mathcal{N}_{ij}$'s at the physical region $q = \xi$ for $\xi$ near $\xi_1, \xi_2$ and use the same approximations as in the previous section. We get

$$\mathcal{N}_{11} \propto -2\eta_1(\xi - \xi_2) + (\mu_2 + \mu_1) \left[ \frac{2k_1}{\mu_2 \mu_1 - \xi_2} \right] \left[ \frac{4\eta_1}{\mu_2 + \mu_1} \right],$$

(IV-1)

$$\mathcal{N}_{22} \propto -2\eta_2(\xi - \xi_1) + (\mu_2 + \mu_1) \left[ \frac{2k_1}{\mu_2 \mu_1 - \xi_1} \right] \left[ \frac{4\eta_1}{\mu_2 + \mu_1} \right],$$

(IV-2)

$$\mathcal{N}_{12} =$$

$$\mathcal{N}_{21} \propto -g \left( \frac{\mu_2^2 + \xi_1^2}{2\mu_2(\mu_2 + \mu_1)} \right)^2 \left( \frac{\mu_1^2 + \xi_1^2}{2\xi_1} \right) \left[ 1 + \frac{(\mu_2 + \mu_1)\xi_1}{(\mu_2 \mu_1 - \xi_1^2)} \right],$$

(IV-3)

where $k_1$ is given by Eq. (III-11). The omitted common factor of proportionality in Eqs. (IV-1-3) is a slowly varying function of $\xi$, and the second factors in the second terms of Eqs. (IV-1,2) are of the same order magnitude as the expressions for $\delta\xi_1$ and $\delta\xi_2$ given in Sec. III. Thus, when we recall the behavior of $\mathcal{O}$, we see that $T_{12} = T_{21}$ is single peaked. Also, $T_{22}$ is single peaked in the limit $\eta_2 \to 0$, with $\eta_2 \ll \delta\eta_2$. On the other hand, $T_{11}$ will normally be double peaked for the range of parameters we are considering. The half
width of the single sharp peak in \( T_{12} = T_{21}, T_{22} \) is of the order of magnitude of \( \eta_2' \), while the half width of the envelope of the two peaks in \( T_{ll} \) is of the order of magnitude of \( \eta_1' \). Thus, a measurement of \( T_{ll} \) will reveal the presence of two poles rather than one only if the resolution is high. Variation of the peak structure for different matrix elements of a multi-channel S matrix dominated by close-lying poles is not new. This was studied by S. Coleman\(^5\) and C. Rebbi and R. Slansky,\(^6\) mainly from considerations of unitarity.
V. CONCLUSION AND SOME SPECTULATION ON THE $A_2$ MESONS

We have worked out explicitly the effect of small interchannel coupling on two resonance-poles, one broad and one narrow, with a unitary Hermitian-analytic model. The result of the coupling is that the broad pole becomes narrower and the narrow pole becomes broader; both widths change by the same amount (in the leading order), so the sum of widths is conserved. The real parts of the pole positions are in general separated further. However, if the narrow pole initially sits slightly higher than the broad pole and the condition Eq. (III-17) is satisfied, the coupling may cause the poles to come closer together in the real direction as well as in the imaginary.

If the narrow pole were originally very narrow, and if there were no coupling, then it would correspond to such a narrow bump in the cross section that it might be missed in a low-resolution experiment. But now if there happens to be a communicating broad pole almost degenerate with this "hibernated" pole, then the latter may acquire a substantial width and become observable together with the broad pole, appearing as a split broad peak in the square of the scattering amplitude, as a function of c.m. momentum (or energy), when we look at the appropriate reactions (the reaction, channel 1 to channel 1, in our model).

Phenomena of coupled overlapping resonances are rarely seen in the hadron spectrum of baryon number of $|B| \leq 1$. One of the features of this spectrum, thus far observed, is that, if we look at those resonances with the same J but different squared masses in the Chew-Frautschi plot, then we find in general the level separations are considerably larger than the level widths, i.e., they do not overlap.
Or even in cases where some levels do overlap, they do not communicate with each other due to different conserved "internal" quantum numbers; i.e., selection rules enable us unambiguously to separate the levels. However, in a few cases, we do observe the phenomenon of overlapping resonances. We observe indirectly the overlapping of $K_S$ and $K_L$, they are coupled by the interaction which violates CP. We observe directly the overlapping of $\rho$ and $\omega$, they are coupled by the electromagnetic interaction, which violates G parity. And we observe the splitting of $J^P = 2^+$ $A_2$ mesons.1

In the resonance production experiments, $\pi^- p \rightarrow p A_2^- \rightarrow p (MM)^-$, [where $(MM)^-$ is about 80% $(\pi\rho)^-$ and 20% $(\pi\eta)^-$ together with $(KK)^-)]$, a pronounced, nearly symmetric deeply split peak is observed in the $(MM)^-$ histogram. One of the fits of such a distribution suggested that the amplitude has two poles: a broad one $A_{2B} = (1298) - i \frac{1}{2}(90)$ MeV, and a narrow one $A_{2N} = (1297) - i \frac{1}{2}(12)$ MeV. [The reasons for preferring this solution, and the implication of various experiments concerning $A_2$ mesons, will be discussed elsewhere.9] Let us assume here that the splitting is indeed due to the coupling of two such overlapping resonances. We now speculate that these two resonances may be in a situation similar to that described at the beginning of this section, and use the model developed above as a prototype to simulate the possible actuality.

As mentioned above, split distributions are observed in resonance production experiments, not in resonance formation experiments like $KK$ to $KK$ or $\pi\rho$ to $\pi\rho$. Nevertheless, we believe that
the structure of the denominators of the actual amplitudes are similar to those of our model as least insofar as they involve two zeros corresponding to two coupled resonances. So far we have worked in the nonrelativistic momentum plane, but we presumably have corresponding structure in the energy plane so long as we are close enough to the physical region and well away from threshold. We know that the decay channels of $A_2$, e.g., the $\pi\rho$ D wave, are highly relativistic. It is not clear in this respect how much the results drawn from our model will be changed by going to higher partial waves, to relativistic phase space, and to a different type of coupling. However, to illustrate our points of view and for order-of-magnitude estimation, we use the formulas obtained above by neglecting $\frac{1}{2}\sigma_1^2$, replacing $\eta$ by $\frac{1}{2}\Gamma$ = the half width, and $\xi$ by $T_R$ = the real part of the energy above the threshold. Note that $\Delta T_R = \Delta m$ = the mass difference. Thus, from Eq. (III-13), we obtain for the change in widths

$$\left(\frac{8\Gamma_2}{\Gamma_2}\right) = \left(\frac{-8\Gamma_1}{\Gamma_1}\right)\left(\frac{\Gamma_1}{\Gamma_2}\right).$$

(V-1)

That is $8\Gamma_2 = -8\Gamma_1 = 8\Gamma$, where $8\Gamma$ is positive. Thus $\Gamma_1 + \Gamma_2 = \Gamma_1 + \Gamma_2$. From the first inequality (III-17), we have the condition for "attraction" in real parts to occur (recalling that one of the factors $\eta_1$ there came from the replacement $\Delta\eta \rightarrow -\eta_1$),

$$T_{1R}(\Delta m) \leq 2 \cdot \frac{\Gamma_1}{2} \left(\Delta \frac{\Gamma}{2}\right),$$

(V-2)

where $\Delta m = m_2 - m_1$, $\Delta \frac{\Gamma}{2} = \frac{\Gamma_1}{2} - \frac{\Gamma_2}{2}$. 
Speculations concerning the $A_2$ mesons:

1. The quantum numbers $I^G = 1^{+}$, as inferred from decay channels must be associated with the broad pole $A_{2B}$ (pole No. 1). But $A_{2N}$ (pole No. 2) may or may not have these same quantum numbers. Thus the coupling between the two may be strong or electromagnetic in nature. It is possible that, say, $\Gamma_1 = 91$ MeV, $\Gamma_2 = 11$ MeV, and the coupling gives $\delta \Gamma = 1$ MeV, then we end up with $\Gamma'_1 = 90$ MeV, $\Gamma'_2 = 12$ MeV. There is also another extreme possibility, that $A_{2N}$ acquires its observed width (almost) entirely from the effect of the coupling. If, say, $\Gamma_1 = 100$ MeV, $\Gamma_2 = 2$ MeV, and the coupling gives $\delta \Gamma = 10$ MeV, then we end up with $\Gamma'_1 = 90$ MeV, $\Gamma'_2 = 12$ MeV. In the latter case, $A_{2N}$ is difficult to see in isolation, but if by chance it falls within a normally broad resonance $A_{2B}$ and become "de-hibernated," it may be revealed in an experiment with good resolution.

We can imagine a more realistic model, for example, where $A_{2B}$ primarily communicates with the open $(\pi \rho)$ channel, and $A_{2N}$ primarily with closed channels, such as baryon-antibaryon or $(\rho \rho)$.

2. It may be possible that, the observed almost exact mass degeneracy of $A_{2B}$ and $A_{2N}$ ($\Delta m \sim 1$ MeV, to be compared with the $A_{2B}$ width of 90 MeV), is due to the enhancement of an originally not quite so degenerated situation, as a by-product of the coupling. An estimation of the inequality (V-2) would be interesting, whether or not there is really such an "attracting" relative mass shift is hard to know, of course, since what we observe are the pole positions "after coupling." We do not know the pole positions "before coupling" unless we have a
reliable and accurate calculation of the separated channels. However, if there were such an "attraction," then we should have, after coupling, \( \Delta \alpha' < \Delta \alpha \), \( \Delta \frac{\Gamma'}{2} \sim \Delta \frac{\Gamma}{2} \). Therefore, the inequality (V-2) would imply that

\[
T'_{1R}(\Delta \alpha') \leq 2 \cdot \frac{\Gamma'}{2} \left( \Delta \frac{\Gamma'}{2} \right).
\]  

(V-3)

For \( A_2 \) 's with the observed masses and widths quoted above and with the \((\pi \rho)\) threshold. We have \( T'_{1R}(\Delta \alpha') \sim 400 \text{ MeV}^2 \),

\[
2 \cdot \frac{\Gamma'}{2} \left( \Delta \frac{\Gamma}{2} \right) \sim 3500 \text{ MeV}^2.
\]

Certainly the experimental fitting \( \Delta \alpha' = 1 \text{ MeV} \) (here actually \( \Delta \alpha' = -1 \text{ MeV} \)) need not be taken too literally, but even with an "uncertainty" in \( \Delta \alpha' \) equals to several MeV, the inequality (V-3) is still satisfied. The known facts thus do not contradict our picture, although they by no means confirm it.\(^{11}\)

From a rough argument we can see that this inequality is not satisfied by the other cases of two overlapping resonances mentioned above. Consider the \( \rho \) and \( \omega \) mesons. If we take the \((\pi \pi)\) threshold, then we get \( T'_{1R}(\Delta \alpha') \sim 9060 \text{ MeV}^2 \), while \( 2 \cdot \frac{\Gamma'}{2} \left( \Delta \frac{\Gamma}{2} \right) \sim 7060 \text{ MeV}^2 \). Consider also the \( K_S \) and \( K_L \) mesons. Using

\[
\Delta \alpha' = m'_L - m'_S - m'_S \approx \frac{1}{2} \Gamma'_S \quad \text{and taking the \((\pi \pi)\) threshold, we get}
\]

\[
T'_{1R}(\Delta \alpha') \sim 0.84 \times 10^{-9} \text{ MeV}^2, \quad \text{while} \quad 2 \cdot \frac{\Gamma'}{2} \left( \Delta \frac{\Gamma'}{2} \right) \sim 1.46 \times 10^{-3} \text{ MeV}^2.
\]

Thus while there might be attraction between \( A_2 B \) and \( A_2 N' \), such is not likely for \( \rho - \omega \) and \( K_S - K_L \).

To sum up, we emphasize that the speculations and arguments presented here about \( A_2 \) mesons are extremely tentative. We have worked out in some detail a special example of the dynamics of overlapping resonances, as an illustration of possible physical situations.
After the completion of the major part of this work, we saw the paper by P. W. Coulter and G. L. Shaw, Phys. Rev. 188, 2443 (1969), which gives a slightly different point of view about $A_2^+$ splitting. They studied the case of two poles associated primarily with two separate and closed channels and decay through a third open channel. Some of their numerical D-wave results are close to those expected by us above. Attraction in masses also occurs in their model. But their initial pole positions correspond to "unstable" configurations in our model.
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APPENDIX A. EXPLICIT EXPRESSIONS

The $D$ function for channel 1, uncoupled, is

$$D_{11}^0(x) = 1 + \frac{\hat{a}_1}{x + \mu_1} + \frac{\hat{a}_2}{x + \mu_2}, \quad (A-1)$$

where $x = (-\nu)^{\frac{1}{2}} = -i\nu = -i(\xi - i\eta)$, $\mu_1 = (\nu_1)^{\frac{1}{2}}$, and

$$\hat{a}_1 = a_1 \frac{A_1^1}{A_1}, \quad \hat{a}_2 = a_2 \frac{A_2^1}{A_1}, \quad (A-2)$$

with $A_1^1$, $A_2^1$, and $A_1^1$ shown below.

The solutions to $D_{11}^0 = 0$ are $x_1 = \pm i\xi_1 - \eta_1$ (the physical resonance-pole is $x_1 = -i\xi - \eta_1$), with

$$\xi_1 = [(\mu_2 \mu_1 + \hat{a}_2 \mu_1 + \hat{a}_1 \mu_2) - \frac{1}{4}(\mu_2 + \mu_1 + \hat{a}_2 + \hat{a}_1)^2]^\frac{1}{2}, \quad (A-3)$$

$$\eta_1 = \frac{1}{2}[\mu_2 + \mu_1 + \hat{a}_2 + \hat{a}_1]. \quad (A-4)$$

The inverse relations of (A-3) and (A-4) are

$$\hat{a}_1 = \frac{\mu_1^2 + \xi_1^2 + \eta_1^2 - 2\mu_1 \eta_1}{\mu_2 - \mu_1} \quad (A-5)$$

$$\hat{a}_2 = -\frac{\mu_2^2 + \xi_2^2 + \eta_1^2 - 2\mu_2 \eta_1}{\mu_2 - \mu_1}. \quad (A-6)$$

Similar equations for channel 2 are obtained by making the substitution $a \rightarrow b$, $\xi_1 \rightarrow \xi_2$, and $\eta_1 \rightarrow \eta_2$. 
The constants $D_{ij}(-v)$ are, up to order $g^2$,

$$D_{11}(-v_1) = \frac{A_1^1}{A^1} - g^2 \left(\frac{1}{a_2 b_2}\right) A_2^1 \left(\frac{a_2}{\mu_2 + \mu_1}\right) B^2,$$

$$D_{11}(-v_2) = \frac{A_2^1}{A^1} - g^2 \left(\frac{1}{a_2 b_2}\right) A_2^1 A^1 \left(\frac{1}{b_2 B^2}\right),$$

$$D_{21}(-v_1) = +g \frac{A_2^1}{A^1} \left(\frac{1}{\mu_2 + \mu_1}\right),$$

$$D_{21}(-v_2) = -g \frac{A_2^1}{A^1} \left(\frac{1}{b_2 B^2}\right),$$

where

$$A_1^1 = \begin{vmatrix} 1 - \frac{a_1}{2\mu_1} & -\frac{a_2}{\mu_2 + \mu_1} \\ -\frac{a_1}{\mu_2 + \mu_1} & 1 - \frac{a_2}{2\mu_2} \end{vmatrix}, \quad A_1^1 = \begin{vmatrix} 1 & -\frac{a_2}{\mu_2 + \mu_1} \\ 1 & 1 - \frac{a_2}{2\mu_2} \end{vmatrix},$$

$$A_2^1 = \begin{vmatrix} 1 - \frac{a_1}{2\mu_1} & 1 \\ -\frac{a_1}{\mu_2 + \mu_1} & 1 \end{vmatrix}, \quad A_2^1 = \begin{vmatrix} 1 - \frac{a_1}{2\mu_2} & -\frac{a_2}{\mu_2 + \mu_1} \\ \frac{a_1}{\mu_2 + \mu_1} & -\frac{a_2}{2\mu_2} \end{vmatrix} = A^1 - \left(1 - \frac{a_1}{2\mu_1}\right),$$

$$A^1 = \begin{vmatrix} 1 & -\frac{a_2}{\mu_2 + \mu_1} \\ 1 & 1 - \frac{a_2}{2\mu_2} \end{vmatrix}.$$
with corresponding expressions for $B_1^1$, $B_1^2$, $B_2^1$, and $B_2^2$. The expressions for $D_{22}(-v_1)$, $D_{22}(-v_2)$, $D_{12}(-v_1)$, and $D_{12}(-v_2)$ are obtained by making the substitution $A \leftrightarrow B, a \leftrightarrow b$.

The inverse relations of Eqs. (A-2) are

$$\frac{1}{a_1} = \frac{1}{2\mu_1} + \frac{1}{\hat{a}_1} \left(1 + \frac{\hat{a}_2}{\mu_2 + \mu_1}\right), \quad (A-9)$$

$$\frac{1}{a_2} = \frac{1}{2\mu_2} + \frac{1}{\hat{a}_2} \left(1 + \frac{\hat{a}_1}{\mu_2 + \mu_1}\right). \quad (A-10)$$

The function $M$ in Eq. (III-7) is

$$M = \left[\left(\frac{1}{x + \mu_1}\right)\left(\frac{a_1}{\mu_2 + \mu_1}\right) + \left(\frac{1}{x + \mu_2}\right)\left(1 - \frac{a_1}{x + \mu_1}\right)\right] \frac{A_2^1}{(A^1)^2}$$

$$\times \left[\left(\frac{b_2}{(B^1)^2}\right)^2 + \left[\left(\frac{1}{x + \mu_1}\right)\left(\frac{b_1}{\mu_2 + \mu_1}\right) + \left(\frac{1}{x + \mu_2}\right)\left(1 - \frac{b_1}{2\mu_1}\right)\right]\right]$$

$$\times \left[\left(\frac{1}{x + \mu_1}\right)\left(\frac{a_1}{\mu_2 + \mu_1}\right) + \left(\frac{1}{x + \mu_2}\right)\left(1 - \frac{a_1}{2\mu_1}\right)\right] \frac{A_2^1}{(A^1)^2} \frac{B_2^1}{(B^1)^2}.$$  

(A-11)
APPENDIX B. PHYSICAL REGION OF THE PARAMETERS

Let

\[ \hat{a}_1 = \frac{x + y}{\sqrt{2}}, \quad \hat{a}_2 = \frac{x - y}{\sqrt{2}}. \]  \hspace{1cm} (B-1)

Then, we can rewrite Eq. (A-3) and Eq. (A-4) as

\[ \xi_0 = \left( \frac{1}{2} \right)^{\frac{1}{2}} \left\{ \frac{2}{\sqrt{2}} (\mu_2 - \mu_1) \left[ x - \frac{1}{2\sqrt{2}} (\mu_2 - \mu_1) \right] - y^2 \right\}^{\frac{1}{2}} \]

\[ \eta_0 = \left( \frac{1}{2} \right)^{\frac{1}{2}} \left[ \frac{1}{\sqrt{2}} (\mu_2 + \mu_1) + y \right], \]  \hspace{1cm} (B-2)

and

\[ \beta_0 = \xi_0^2 - \eta_0^2 = \frac{1}{\sqrt{2}} (\mu_2 - \mu_1) \left\{ x - \frac{1}{\sqrt{2} (\mu_2 - \mu_1)} \left[ \frac{1}{4} (\mu_2 + \mu_1)^2 + \frac{1}{2} (\mu_2 - \mu_1)^2 \right] \right\} - \]

\[ \chi \left[ y + \frac{1}{2\sqrt{2}} (\mu_2 + \mu_1) \right]^2 \]

\[ \equiv \frac{1}{\sqrt{2}} (\mu_2 - \mu_1) x'' - y^{n2} \]  \hspace{1cm} (B-3)

\[ \gamma_0 = 2\xi_0\eta_0 = \left[ \frac{2}{\sqrt{2}} (\mu_2 - \mu_1) x' - y'^2 \right]^{\frac{1}{2}} \left[ \frac{1}{\sqrt{2}} (\mu_2 + \mu_1) + y \right]. \]  \hspace{1cm} (B-5)
The region of the parameters which gives \( \xi_0^2 > 0 \) and \( \beta_0 > 0 \) is in the shaded area of Fig. 2. If we choose \( x' \) to be large enough, then we can make \( \xi_0 \gg \eta_0 \). If we choose, say, the two points marked by \( x \) in Fig. 2 to represent \( x_1 \) and \( x_2 \), then we can have \( \frac{\Delta \xi}{\xi_1} \) small and have different \( \eta \)'s with \( \eta_2 \approx 0 \).
FOOTNOTES AND REFERENCES

* This work was supported in part by the U.S. Atomic Energy Commission.

1. H. Benz et al., Phys. Letters 28B, 233 (1968); R. Baud et al.,
ments on A_2 mesons is given by W. Kienzle, in Meson Spectroscopy,
ed. by C. Baltay and A. H. Rosenfeld (W.A. Benjamin, Inc., New York,
1968).

B, they arrived at the same expression from quite a different
approach.

3. G. F. Chew and S. Mandelstam, Phys. Rev. 118, 467 (1960); G. F.
Chew, S-Matrix Theory of Strong Interaction (W. A. Benjamin, Inc.,

4. The second inequality is to avoid the situation where the correction
term is "too" large and makes |Δ'?| > |ΔΔ|.

5. S. Coleman, in Theory and Phenomenology in Particle Physics, Part B,


7. The usual phenomenological description of K_S and K_L was recently
reformulated in the form of overlapping resonances by L. Durand, III


9. C. F. Chan, Some Comments on the A_2 Mesons, Lawrence Radiation
Laboratory Report UCRL-20206, December 1970.

11. If, for some mysterious reason, the two masses are, say, exactly degenerate, at the beginning, then the coupling will make them "repel" each other by an amount up to ~3 MeV according to Eqs. (III-14,15). Then, the pole position after coupling still satisfies inequality (V-3), although there has actually been a repulsion in real parts.
FIGURE CAPTIONS

Fig. 1. The q plane and the relative movement of poles.

Fig. 2. Physical region of the parameters \( \hat{a}_1, \hat{a}_2 \) for given \( \mu_1 \) and \( \mu_2 \).
Fig. 1
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