UNIVERSITY OF CALIFORNIA, SAN DIEGO

Simulation of Stratified Turbulent Flows in Complex Geometry

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Engineering Sciences (Mechanical Engineering) by

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2015
The dissertation of Narsimha Reddy Rapaka is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California, San Diego

2015
Dedicated to my parents Lakshmi Rapaka and Ramachandra Reddy Rapaka ...
Neither money pays, nor name, nor fame, nor learning; 
It is CHARACTER that can cleave through adamantine walls of difficulties.

—Swami Vivekananda
TABLE OF CONTENTS

Signature Page .......................................................... iii
Dedication ................................................................. iv
Epigraph ................................................................. v
Table of Contents ....................................................... vi
List of Symbols ........................................................ viii
List of Figures ........................................................ ix
List of Tables ........................................................ xi
Acknowledgements ................................................... xii
Vita ................................................................. xiii
Abstract of the Dissertation ........................................ xiv

Chapter 1  **Introduction** ........................................ 1
  1.1 Internal waves .................................................. 1
  1.2 Internal Tide Generation at a Model Ridge .................. 4
  1.3 Immersed Boundary Method for Flow over Complex Bodies 6
  1.4 Thesis Outline ............................................... 10

Chapter 2  **Immersed Boundary Method on a Cartesian Grid** . . 11
  2.1 Governing Equations ........................................ 11
  2.2 Predictor-Corrector Algorithm ................................ 13
  2.3 Temporal Integration ......................................... 14
    2.3.1 RK3-ADI Step 1 ......................................... 16
    2.3.2 RK3-ADI Step 2 ......................................... 18
    2.3.3 RK3-ADI Step 3 ......................................... 19
  2.4 Immersed Boundary Method .................................. 20
    2.4.1 Geometric Preprocessing ................................ 21
    2.4.2 Reconstruction Procedure ................................ 22
  2.5 Near Wall Treatment for Large-Scale Flows ................. 29
    2.5.1 Drag Law ............................................... 29
    2.5.2 Boundary Conditions for Complex Topography: Slip
          Velocity ............................................... 30

Chapter 3  **Tidal Conversion and Turbulence at a Model Ridge: Di-
            rect and Large Eddy Simulations** ..................... 32
  3.1 Formulation of the problem .................................. 32
    3.1.1 Governing Equations .................................... 33
    3.1.2 Numerical Method ....................................... 34
    3.1.3 Selection of Simulated Cases ......................... 37
3.2 Results in the Laminar Flow Regime ........................................... 38
  3.2.1 Effect of Criticality on the Internal Wave Structure at \( Re_s = 30 \) ........................................ 38
  3.2.2 Frequency spectra in laminar flow cases ........................................ 38
  3.2.3 Modal structure in laminar flow cases ........................................ 40
  3.2.4 Radiative conversion ........................................ 41
3.3 Effect of forcing on the internal wave field in the critical slope case ........................................ 42
  3.3.1 Baroclinic Energy Budget ........................................ 44
3.4 Turbulence at the Ridge in Case 5 with Critical Slope ........................................ 47
3.5 Effect of Forcing in the Cases of Subcritical and Supercritical Slopes ........................................ 57

Chapter 4 An IBM for Direct and Large Eddy Simulation of Stratified Flows in Complex Geometry ............... 62
  4.1 Unstratified flow past a sphere at \( Re \) up to 300 ........................................ 62
  4.2 Direct Numerical Simulation (DNS) of turbulent channel flow at \( Re \tau =395 \) ........................................ 63
  4.3 DNS of stratified turbulent channel flow at \( Re \tau =180, Ri \tau =18 \) ........................................ 64
  4.4 Direct Numerical Simulations (DNS) of stratified flow past a laboratory scale model ridge at \( Re_s =177 \) ........................................ 65
    4.4.1 Mean velocity on the slope ........................................ 69
    4.4.2 Baroclinic response ........................................ 70
    4.4.3 Turbulence ........................................ 75
  4.5 Large eddy simulations (LES) of stratified flow past a large scale model ridge ........................................ 78
    4.5.1 Mean profiles ........................................ 78
    4.5.2 Baroclinic response ........................................ 78
    4.5.3 Turbulence ........................................ 79

Chapter 5 Summary ........................................ 80

Appendix A Decomposition of pressure and velocity ........................................ 84

Appendix B Methods for modal analysis and conversion factor ........................................ 85

Bibliography ........................................ 87
**LIST OF SYMBOLS**

\[ u \] Velocity in cartesian coordinate system
\[ U_0, U_{\infty} \] Free stream velocity
\[ u_r \] Friction velocity
\[ \tau_w \] Wall shear stress
\[ c_f \] Friction coefficient
\[ \rho \] Total density
\[ \rho^* \] Deviation of from the background density
\[ \rho^b \] Background density
\[ \rho_0 \] Reference density
\[ \nu \] Molecular viscosity
\[ \kappa \] Thermal diffusivity
\[ \beta \] Slope angle
\[ \theta \] Potential temperature
\[ Ri_g \] Gradient Richardson number
\[ B \] Buoyancy number
\[ Re \] Reynolds number
\[ Re_s \] Stokes Reynolds number
\[ Ex \] Excursion number
\[ \epsilon \] Criticality parameter
\[ \delta_s \] Stokes layer thickness
\[ Fr \] Froude number
\[ Pr \] Prandtl number
\[ \langle \cdot \rangle \] Reynolds average
\[ \langle \cdot \rangle \] Filtered quantity
\[ g \] Gravitational acceleration
\[ \Omega \] Wave frequency
\[ \Theta \] Wave characteristic angle
\[ N \] Buoyancy frequency
## LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Schematic of constant cone of $\Omega$ for three-dimensional internal waves. Here, $C_p$ denotes phase velocity and $C_g$ denotes group velocity.</td>
<td>2</td>
</tr>
<tr>
<td>2.1</td>
<td>Colocated velocity components ($u_i$), defined at the cell center P, and face velocity components ($U_i$), defined at the cell faces ($f = e, w, n, s, t, b$), are shown. Pressure ($p$) and density ($\rho^*$) are colocated with ($u_i$) at the cell center P.</td>
<td>12</td>
</tr>
<tr>
<td>2.2</td>
<td>(a) Solid cells, ghost cells (GC), immersed boundary (IB) cells, and intersection points (IP) on a Cartesian grid distinguished by the immersed body.</td>
<td>21</td>
</tr>
<tr>
<td>2.3</td>
<td>Stencil used in the trilinear interpolation for a GC.</td>
<td>22</td>
</tr>
<tr>
<td>2.4</td>
<td>Redefined Face Center (RFC), defined as the geometric centroid of a cut cell face lying in the fluid region indicated by thick green line.</td>
<td>27</td>
</tr>
<tr>
<td>2.5</td>
<td>Slip velocity ($u_w$) that provides the appropriate wall shear stress is imposed at the bottom wall ($w$). The non-linear drag law based on the tangential velocity ($u_t$) at point 1 gives the appropriate wall shear stress.</td>
<td>29</td>
</tr>
<tr>
<td>3.1</td>
<td>(a) Schematic of the problem: stratified fluid flows over a two dimensional topography as a response to oscillatory forcing, $F_0(t_d)$, in the streamwise direction. (b) Profiles of the smoothed triangular topography.</td>
<td>33</td>
</tr>
<tr>
<td>3.2</td>
<td>Streamwise velocity at $Re_s = 30$ in different flow regimes shown at time = 51.75 s, phase of the barotropic velocity, $\phi \approx \pi/2$.</td>
<td>39</td>
</tr>
<tr>
<td>3.3</td>
<td>Frequency spectra for (a) subcritical flow and (b) critical flow $Re_s = 30$.</td>
<td>40</td>
</tr>
<tr>
<td>3.4</td>
<td>(a) Example profile of the baroclinic vertical velocity profile in the laminar case, $Re_s = 30$. Profile shown at $x = 3$ m and time = 51.75 s, phase of the barotropic velocity, $\phi \approx \pi/2$. (b) Modal structure of the baroclinic field.</td>
<td>40</td>
</tr>
<tr>
<td>3.5</td>
<td>The effect of increasing barotropic forcing on: (a) the normalized radiated baroclinic flux and the energy conversion in critical (bottom two curves) and supercritical (top two curves) cases, and (b) the intensification of near-bottom velocity in critical cases.</td>
<td>43</td>
</tr>
<tr>
<td>3.6</td>
<td>Vertical profiles of the normalized values of: (a) baroclinic velocity amplitude, (b) baroclinic pressure amplitude, and (c) the product of pressure and velocity amplitudes. Profiles shown at $x = -3$ m, a location away from the topography.</td>
<td>43</td>
</tr>
<tr>
<td>3.7</td>
<td>The effect of forcing in the critical slope case: (a) Frequency spectra. Point A in the boundary layer shown in red and point B in the beam shown in green, and (b) Modal distribution at $x = 3$ m.</td>
<td>44</td>
</tr>
<tr>
<td>3.8</td>
<td>$\log_{10}(TKE)$ and isopycnals near the topography in case 5 are shown in (a)-(d) at time 42.5 s, 44 s, 44.8 s, and 46.3 s.</td>
<td>49</td>
</tr>
</tbody>
</table>
Figure 3.9: (a) \(Ri_t\) and isopycnals near the topography in case 5 are shown at time 44 s.

Figure 3.10: Flow and turbulence in case 5 at \(x = -0.77\ m\), the center of the left slope of ridge.

Figure 3.11: Flow and turbulence in case 5 at \(x = -0.4\ m\), end of the left slope of ridge.

Figure 3.12: Flow and turbulence in case 5 at \(x = -0.1\ m\), close to the center of the ridge.

Figure 3.13: Shaded regions A, B and C correspond to a region on the critical slope.

Figure 3.14: Evolution of TKE budget, integrated over three different areas shown in figure 3.13.

Figure 3.15: Supercritical case with \(Re_s = 177\). \(\log_{10}(TKE)\) and isopycnals near the topography in case 5SUP.

Figure 3.16: Supercritical case with \(Re_s = 177\). Evolution of TKE budget, integrated over three different areas.

Figure 3.17: The effect of increasing barotropic forcing on modal distribution at \(x = 3\ m\) in supercritical and subcritical cases.

Figure 4.1: Qualitative comparison of streamlines with Johnson & Patel (1999). (Left column) Data from Johnson & Patel (1999). (Right column) Data from present. From top to bottom: (a) \(Re = 50\). (b) \(Re = 100\). (c) \(Re = 150\). (d) \(Re = 200\).

Figure 4.2: Statistics of unstratified channel flow at \(Re_\tau = 395\), with channel walls simulated using IBM on a grid of 128x128x128, (a) Reynolds stress components, and (b) mean velocity normalized by wall units.

Figure 4.3: Statistics of stratified channel flow at \(Re_\tau = 180\) and \(Ri_\tau = 18\): (a) mean velocity, and (b) Reynolds shear stress normalized by wall units.

Figure 4.4: Topography of a smoothed triangular ridge used in laboratory scale simulations, shown by the shaded area.

Figure 4.5: Mean velocity on the slope. Vertical profiles of the streamwise mean velocity at the left mid slope \((x/l = -0.405)\).

Figure 4.6: Normalized vertical profiles of (left-right) the baroclinic velocity amplitude, the baroclinic pressure amplitude, and the product of pressure and velocity amplitudes.

Figure 4.7: Evolution of various terms in the baroclinic energy budget.

Figure 4.8: Spatial distribution of \(\log_{10}(TKE/U_0^2)\) and isopycnals near the topography in (a) case 1cri_Bf, (b) case 1cri, and (c) case 1cri_Ls. They correspond to a phase of peak upslope flow at the left midslope \((x/l = -0.405)\).

Figure 4.9: Spatial distribution of \(\log_{10}(TKE/U_0^2)\) and isopycnals near the topography, shown at time=6T, in (a) case 2cri_Bf, (b) case 2cri, and (c) case 2cri_Ls. Note that \(h_0\) is the height of the topography, and \(T\) is the barotropic cycle time period.
LIST OF TABLES

Table 2.1: Coefficients of mixed RK3-ADI Scheme ........................................... 15
Table 3.1: Parameters of the simulated cases. ...................................................... 36
Table 3.2: Conversion factor ($M$) at $Re_s = 30$. $^*$M increases from 0.59 to 0.77
abruptly as $\epsilon$ changes from 1 to 1.05. $^*$The value is not quoted
and the error band is associated with the digitization of figure 5(a)
in Pétrélis et al. (2006). ................................................................. 42
Table 3.3: Baroclinic energy budget, integrated over an area of the computational
domain from $x = -3$ m to $x = +3$ m. ................................................. 46
Table 3.4: Cycle averaged turbulent kinetic energy budget, integrated over areas A, B and C. ................................................................. 60
Table 4.1: Comparison with results of Johnson & Patel (1999). ....................... 64
Table 4.2: Nondimensional parameters and grid spacing of the cases of flow
past a ridge. ....................................................................................... 68
Table 4.3: Dimensional parameters of the simulated cases. ............................. 68
Table 4.4: Baroclinic energy budget: tendency ($\Delta E$), energy conversion to
waves ($C$), radiated wave flux ($M \equiv F_{bc}$), baroclinic energy dissipation ($\epsilon_{bc}$) ........................................................................... 74
Table 4.5: Normalized TKE and turbulent dissipation integrated over the area
between $x = -3$ to $+3$ m. ................................................................. 76
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ABSTRACT OF THE DISSERTATION

Simulation of Stratified Turbulent Flows in Complex Geometry

by

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Direct and Large Eddy Simulation approaches are used to study internal tide generation and turbulence at a model topography. A three dimensional finite difference code using an immersed boundary method is developed to simulate stratified turbulent flows over complex geometry on a Cartesian grid. The thesis is comprised of two phases.

In the first phase, a mixed spectral/finite difference code is used to solve the governing equations in generalized coordinates. Direct and large eddy simulations are performed using a body fitted grid (BFG) to study the internal waves generated by the oscillation of a barotropic tide over a model ridge of triangular shape. The objective is to assess the role of nonlinear interactions including turbulence in situations with low tidal excursion number. The criticality parameter, defined as the ratio of the topographic slope to the characteristic slope of the tidal rays, is varied from subcritical to supercritical values. The barotropic tidal forcing is also systematically increased. In laminar flow at low forcing, numerical results of the energy conversion agree well compared to linear theory in subcritical and supercritical cases but not at critical slope angle. In critical and supercritical cases with higher forcing, there are convective
overturns, turbulence and significant reduction (as much as 25%) of the radiated wave flux with respect to laminar flow results. The phase dependence of turbulence within a tidal cycle is examined and found to differ substantially between the ridge slope and the ridge top where the beams from the two sides cross.

In the second phase, a sharp-interface Immersed Boundary Method (IBM) is developed to simulate high Reynolds number density-stratified turbulent flows in complex geometry. The basic numerical scheme corresponds to a central second-order finite difference method, third-order Runge-Kutta integration for the advective terms and an alternating direction implicit (ADI) scheme for the viscous and diffusive terms. Both direct numerical simulation (DNS) and large eddy simulation (LES) approaches are considered. The focus is on accurate computation of the internal gravity wave field and turbulence near an underwater obstacle in a model problem where a tide oscillates over the obstacle. Methods to enhance the mass conservation and numerical stability of the solver to simulate high Reynolds number flows are discussed. The solver is validated using Direct Numerical Simulations (DNS) of channel flow with and without stratification, and tidal flow over a laboratory-scale (order of few meters) smoothed triangular ridge. The results including baroclinic energy flux, mean flow properties and turbulent kinetic energy agree reasonably well with our previous results obtained using a body-fitted grid (BFG). The deviation of IBM results from BFG results is found to increase with increasing steepness of the topography relative to the internal wave propagation angle. LES is performed on a large scale ridge, of the order of few kilometers in length, at significantly larger Reynolds number. A non-linear drag law is utilized to parameterize turbulent losses due to bottom friction. The large scale problem exhibits qualitatively similar behavior to the laboratory scale problem with some differences: slightly larger intensification of the boundary flow and somewhat higher nondimensional values for conversion, baroclinic wave flux and turbulent kinetic energy. The phasing of wave breaking and turbulence exhibits little difference between small and large scale obstacles. We conclude that IBM is a viable approach to the simulation of internal waves and turbulence in high Reynolds number stratified flows over topography.
Chapter 1

Introduction

1.1 Internal waves

The contents of the section 1.1 are taken from the PhD thesis of Gayen (2012).

The atmosphere and ocean are continuously stratified due to changes in temperature, pressure and composition. These changes can lead to significant variations of fluid density in the vertical direction. For example, fresh light water from rivers can rest on top of sea water, and due to the small diffusivity of the water, the density contrast remains for a long time. Density stratification supports oscillatory motion, i.e., internal waves. Here, gravity plays the restoring force for internal waves. Internal waves can transport energy in the ocean or atmosphere over long distances. Propagation, energy transfer and other basic properties of IW can be explained staring from the system of equations governing wave motion of an incompressible fluid with continuous density stratification.

The linearized, inviscid form of the NS equations for perturbation velocities, $u = [u \ v \ w]$, under the Boussinesq approximation (density is constant everywhere except in the gravity, or buoyancy term) are written as:

$$\nabla \cdot \mathbf{u} = 0$$  \hspace{1cm} (1.1)
$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_0} \nabla p^* - \frac{g}{\rho_0} \rho^* \mathbf{k} - f \times \mathbf{u}$$  \hspace{1cm} (1.2)
$$\frac{\partial \rho^*}{\partial t} = \frac{\rho_0 N_\infty^2(z) g}{w}.$$  \hspace{1cm} (1.3)

Here, $p^*$ and $\rho^*$ denote deviation from the background pressure and density, respectively. $N_\infty = \sqrt{(g/\rho_0)(d\rho_b/dz)}$ is the local buoyancy frequency based on the local
Here, \( C_p \) denotes phase velocity and \( C_g \) denotes group velocity. The density gradient \( d\rho_b/dz \), reference density \( \rho_0 \) and gravity, \( g \). \( f \) is the Coriolis parameter. Equations (1.1)-(1.3) can be simplified to wave equation

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \frac{\partial^2 w}{\partial t^2} + N_\infty^2(z) \left[ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right] + f \frac{\partial^2 w}{\partial z^2} = 0. \quad (1.4)
\]

We assume spatial and temporal periodic solution,

\[
w(x, z, t) = \text{Re} \left[ \tilde{W}_0(k, z)e^{i(kx + ly + mz - \Omega t)} \right], \quad (1.5)
\]

where \( \tilde{W}_0 \) is an arbitrary constant and \( k = [k \ l \ m] \) are the wave numbers in the streamwise (x), spanwise (y) and vertical (z) directions, respectively. Here, \( \Omega \) is the forcing frequency. Since the fluid motion is incompressible, \( \nabla \cdot \mathbf{u} = -i k \cdot \mathbf{u} = 0 \), i.e., the velocity is orthogonal to the wave number vector. These waves are also classified as shear waves. After substituting solution of the vertical velocity \( w \) into (1.4), the dispersion relation between wave number vector and frequency of waves follows as

\[
\Omega^2 = \frac{f^2 m^2 + (k^2 + l^2) N_\infty^2}{k^2 + l^2 + m^2}. \quad (1.6)
\]

For internal waves, surfaces of constant frequency in the wavenumber space are the cones with \( \Theta = \text{constant} \) as illustrated in figure 1.1 for a non-rotating environment \( (f = 0) \). Here, \( \Theta = \sin^{-1}(\Omega/N_\infty) \) is often known as phase propagation angle of IW. The phase lines of the internal waves oriented at the angle, \( \Theta \), with respect to the
horizontal, move with phase speed $C_p = \Omega |k|/|k|^2$. The energy of the internal waves travels with group speed $C_g = \nabla_k \Omega$ which can easily be shown to be orthogonal to the phase velocity. When the group velocity has an upward component, then the phase velocity has a downward component, and vice versa. Internal waves can be easily generated by oscillating a horizontal cylinder at a frequency smaller than the buoyancy frequency. This causes waves to be generated, so that energy moves away from the cylinder in four beams, with the crests and troughs (or phase) moving perpendicularly to the direction in which the beams carry energy away from the oscillating cylinder.

The slope of topography plays a crucial role in wave energetics. When the slope angle, $\beta$, becomes close to the characteristic angle of the internal wave, $\Theta$, a resonant response occurs during the generation or reflection, leading to formation of a localized IW beam. These internal waves further energize the boundary flow formed in the vicinity of the topography.
1.2 Internal Tide Generation at a Model Ridge

Internal tides are internal gravity waves generated by the interaction of an oscillating barotropic flow with bottom topography in a stratified fluid. Some of the energy converted from the barotropic tide to the baroclinic flow is locally trapped and dissipated due to small scale turbulence near the topography while the rest is radiated away from the topography. The tidal energy thus radiated is considered to have a significant contribution to the mixing of the stably stratified ocean (Polzin et al., 1997; Munk & Wunsch, 1998; Ledwell et al., 2000; Wunsch & Ferrari, 2004). Enhanced conversion to internal tides is found near sea-mounts (Lueck & Mudge, 1997; Kunze & Toole, 1997), submarine ridges (Rudnick et al., 2003; Klymak et al., 2006), submarine canyons (Polzin et al., 1996; Carter & Gregg, 2002), continental slope (Cacchione et al., 2002; Moum et al., 2002; Nash et al., 2004, 2007) and deep rough topography (Polzin et al., 1997; St. Laurent et al., 2001).

Inviscid linear estimates of the tidal energy radiated from various bottom topographies are well established in the literature. Linearization requires that the product $\epsilon Ex << 1$ (Balmforth et al., 2002) where the criticality parameter, $\epsilon = \tan \beta / \tan \theta$, is the ratio of the topographic slope $\tan \beta$ to the slope of internal wave characteristic $\tan \theta = \sqrt{(\Omega^2 - f^2)/(N_\infty^2 - \Omega^2)}$, and the excursion number, $Ex = U_0/\Omega l$, characterizes the ratio of the fluid excursion during a tidal cycle to the topographic length. Applications of linear theory (Bell, 1975a,b; Llewellyn Smith & Young, 2002; Balmforth et al., 2002; St. Laurent & Garrett, 2002; Khatiwala, 2003) invoked weak topography approximation (WTA) that requires shallow slope topography with $\epsilon << 1$ and topographic height much less than the vertical wavelength of the internal tide. Linear theory shows that the radiated internal wave energy flux is proportional to $\frac{\pi}{4} \rho_0 U^2 h^2 \sqrt{(N^2 - \Omega^2)}$ and provides analytical estimates of the proportionality coefficient, $M$. Supercritical slopes that occur at sites of steep topography, e.g. at the Hawaiian ridge, are also of interest and a key result of the analytical studies of Llewellyn Smith & Young (2003) and St. Laurent et al. (2003) is that the wave energy flux at steep topography can be substantially larger than at gentle slopes. Pétrélis et al. (2006) estimated the conversion rate for two different topographies: a triangular and a polynomial ridge and performed a parametric study of the effect of the topography height and width as well as water depth. For the triangular ridge to be examined in the present study, they showed that, for low-to-moderate values of the ratio of ridge height $h$ to water depth $H$, the normalized wave flux, $M$, is a weak
function of slope angle in the subcritical regime, increases abruptly at critical slope, and then increases slowly in supercritical topography. Echeverri et al. (2009) performed laboratory measurements of wave conversion at a two dimensional ridge with sub- to supercritical slopes and showed that, at low excursion number, results from experiment, computation and theory agreed well for the low modes. Small differences in velocity profiles were attributed to differences in higher modes.

Numerical process studies to study wave radiation and local losses have provided insights into nonlinear effects on wave conversion and frequency spectra. These simulations are at geophysical scales but employ large values of molecular viscosity to stabilize the algorithm so that turbulence is not permitted. Legg & Huijts (2006) employed two-dimensional simulations with the MIT model to quantify the effect of varying velocity amplitude over Gaussian topography of various aspect ratios and heights. They found that the numerical results showed good agreement in many respects with linear theory predictions, e.g., the proportionality of wave flux to $U_0^2$ and $h^2$, the increase of wave flux from subcritical slopes to supercritical slopes, and the appearance of high harmonics when the barotropic velocity increases. Narrow topography (strongly supercritical slope angle) had larger values of viscous dissipation (in laminar flow state) owing to high vertical wavenumber modes. Legg & Klymak (2008) further examined internal wave dynamics in the case of strongly supercritical topography with $\epsilon = 4$, low excursion number and low Froude number. Overturns and large values of viscous dissipation (laminar flow) were found near the top and in the lee, i.e., behind the obstacle with respect to the flow at the ridge top. It was proposed that breaking lee waves and transient hydraulic jumps that occur when $\epsilon \geq 3$ cause these overturns. In a following study, Klymak et al. (2010) parameterized tidal dissipation in supercritical topography from nonlinear breaking of lee waves and assessed its role in two-dimensional simulations.

Three-dimensional, high-resolution simulations that resolve turbulence are necessary for numerical studies to help understand the microstructure associated with wave breaking. The first turbulence-resolving simulations of internal tide generation were performed by Gayen & Sarkar (2010) for flow over an asymmetric sloping bottom, corresponding to a model continental slope, at critical slope angle, i.e., $\epsilon = 1$. These simulations showed strong near-bottom intensification of the velocity and a strong outgoing internal wave beam similar to the laboratory experiment of Zhang et al. (2008). However, unlike the laboratory experiments, the simulations
that were performed at higher Reynolds numbers showed transition to turbulence at $\text{Re}_s \equiv \frac{U_0 \sqrt{2\nu \Omega}}{\nu} \approx 100$. Both convective and shear instabilities were observed during different phases of the flow. Lim et al. (2010), using laboratory experiments, found beam formation, boundary-layer turbulence and upslope propagation of bores depending on the value of the Reynolds number. Gayen & Sarkar (2011b) employed DNS/LES to demonstrate that the beam width, beam velocity, and bottom turbulence energy and dissipation tend to increase with increasing length (height) of critical slope. Both, Gayen & Sarkar (2010) and Gayen & Sarkar (2011b), employed a streamwise inhomogeneous formulation to model internal wave generation in contrast to the streamwise periodic domain employed by Slinn & Riley (1998) who performed the first turbulence resolving simulation of internal wave reflection. The mechanisms of turbulence generation at various phases during the oscillating flow over a sloping bottom are explained in the later work of Gayen & Sarkar (2011a) using LES of a small patch of an internal tidal beam, scaled up to a width of 60m. Turbulent dissipation rate was found to peak when the near-bottom flow was near zero and reversed from downslope to upslope as in the observations of Aucan et al. (2006) at a bottom mooring on a deep flank at Kaena Ridge in Hawaii. Convective instability leading to overturns that span the internal wave beam was found to occur during flow reversal from down to up.

An outstanding question is how do nonlinear processes and bottom turbulence affect the internal wave generation and, in particular, what is the effect on the energy conversion to the baroclinic flow and on the wave flux radiated away from the topography? We address this question through turbulence-resolving simulations of oscillating flow over a smoothed triangular ridge, shown in figure 3.1(b). The symmetric triangular topography allows quantitative comparison with the linear theory results of Pétrélis et al. (2006). Another important goal is to determine the phasing and energetics of the turbulence.

### 1.3 Immersed Boundary Method for Flow over Complex Bodies

We discuss an immersed boundary method (IBM) that has been developed to simulate turbulent stratified flows at high Reynolds number over complex, three dimensional topography on a Cartesian grid using a finite difference method. The
motivating application is internal tides and associated turbulence near rough underwater topography, although the solver can be used for a variety of environmental and engineering flows that include an active scalar that influences the flow field through buoyancy. For problems of large computational size, the solver employs a domain decomposition method that utilizes the Message Passing Interface (MPI) for parallel processing on supercomputers (Brucker, 2009; de Stadler, 2013). At low Reynolds numbers, the flow can be simulated using direct numerical simulation (DNS) and large eddy simulation (LES) can be employed at high Reynolds number.

Internal tide generation in the ocean involves interaction of the flow with complex, three dimensional rough topography. Nonlinear processes and turbulence becomes important when the slope angle becomes equal to larger than the internal wave propagation angle. Our interest is such situations that involve strongly nonlinear internal waves, overturns and turbulence, requiring solution of the three-dimensional, Navier Stokes equations without making the hydrostatic assumption that is commonly made in large scale models. We have previously developed a finite difference solver for stratified turbulent flows at laboratory scale and lower Reynolds numbers (Gayen & Sarkar, 2011b) that employs a structured grid, generalized coordinates and a computational domain that conforms to two-dimensional topography. The solver has been applied to laboratory scale models of a continental slope by Gayen & Sarkar (2011b) and a ridge by Rapaka et al. (2013). However, the application of a finite difference method with a body-conforming structured grid to high Reynolds number oceanic flows over complex topography is scarce.

Stratified flows present the additional complexity of coupling between velocity and density fields that leads to internal gravity waves, new flow instabilities and buoyancy-affected turbulence. The angle of sloping regions of underwater topography need to be accurately represented in the simulation because the local conversion from the oscillating tidal energy to internal gravity wave energy depends substantially on whether the slope angle is less than (subcritical), equal to (critical), or greater than (critical) the wave propagation angle even in the linear case (Llewellyn Smith & Young, 2002; Pétrelis et al., 2006). Turbulence is found to be intensified at near-critical slopes in observations (Moum et al., 2002; Nash et al., 2007) and turbulence-resolving simulations (Gayen & Sarkar, 2010, 2011a), as well as near supercritical topography as shown by observations (Klymak et al., 2008; Alford et al., 2011) and suggested by two-dimensional simulations (Legg & Klymak, 2008; Klymak et al.,
Slope angle is also important when internal waves shoal on to the continental slope as recently reviewed by Lamb (2014).

The MIT GCM (Marshall et al., 1997) is a finite volume, non-hydrostatic model that handles topography with partially filled cells leading to a boundary region with piecewise constant cell thickness. SUNTANS (Fringer et al., 2006) is an unstructured finite volume, non-hydrostatic model. Topographic internal wave problems have been simulated with the MIT GCM (Legg & Klymak, 2008; Klymak et al., 2010), and with SUNTANS (Kang & Fringer, 2011; Zhang et al., 2011). More recently, SOMAR (Santilli & Scotti, 2015), a generalized-coordinate, structured grid solver with adaptive mesh resolution (AMR) has been developed for oceanic problems. None of these models, although non-hydrostatic, claim to tackle turbulence in the sense of large eddy simulation (LES) that resolves the energy containing turbulent motions.

Recent advances of the Immersed Boundary Method (IBM) show promise towards the goal of simulating flows in complex geometries while retaining a structured grid, and encourage us to develop the IBM for simulations of stratified flows over complex topography. Due to the nature of the linear systems arising from discretization on structured meshes, accuracy and efficiency can be higher compared to unstructured meshes. Two major categories exist in IBM: diffusive interface methods and sharp interface methods (see Mittal & Iaccarino (2005)). The diffusive interface methods employ forcing distributed over a few cells surrounding the immersed boundary while sharp interface methods (eg. Mittal et al. (2008)) enforce the no-slip boundary condition on the immersed boundary surface. The sharp interface methods have proved to be particularly well suited for wall-bounded turbulent flows where boundary layer turbulence plays an important role in the overall solution accuracy. There are many variants of immersed boundary methods available in the literature but the applicability of IBM to high-Reynolds number flows in a stratified medium with both topographically generated internal waves and turbulence remains an open question.

There are many ways to perform reconstruction of the flow near immersed boundary (IB). To compute the momentum forcing required to enforce no-slip conditions at the IB, Fadlun et al. (2000) used linear interpolation along Cartesian directions for cells near the immersed boundary in the fluid region. However, this ambiguity may arise in the direction of interpolation for highly curved boundaries. Balaras (2004) proposed unique interpolation along wall normal direction through
the use of an intermediate node. Roman et al. (2009) simplified method of obtaining the momentum forcing by removing the provisional step used in Balaras (2004) and directly imposing the reconstructed velocities on the cells close to the IB surface. Mittal et al. (2008) used ghost-cells inside the solid body to satisfy the boundary conditions on the IB.

Conservation of mass is very important to accurately predict flows at high Reynolds number. Mark & van Wachem (2008) excluded the face velocities inside the solid body when discretizing the continuity equation for the pressure to ensure no mass flux across the immersed boundary. Kang et al. (2009) defined new velocity components on the faces cut by the IB to accurately compute the mass fluxes using finite volume discretization. However, to reduce the complexity, the Laplacian of pressure is discretized using the conventional finite difference operator. Meyer et al. (2010) modified the finite volume discretization of the Navier-Stokes equations near the immersed boundary to conserve the mass and used a momentum exchange approach to enforce the boundary conditions on the IB.

One way to satisfy the geometric conservation law and local mass conservation accurately is through the use of a Cartesian cut-cell method (Ye et al., 1999; Udaykumar et al., 2001). In this method, the IB cells are reshaped into non-rectangular control-volumes and a finite-volume method is used to ensure strict satisfaction. However, the extension of these methods to 3D problems is very complicated. Seo & Mittal (2011) adopted a cut-cell based approach only for the discretization of Poisson and velocity correction equations to enforce geometric conservation and use the volume/face fraction of the fluid region in evaluating the mass fluxes.

As a part of the thesis, a computational fluid dynamics code that uses a finite difference discretization on a Cartesian grid to numerically solve the governing equations is developed. A Cartesian grid allows the use of efficient geometric multigrid method to solve the Poisson equation. An immersed boundary method (IBM) is developed to model the effect of the complex topography on the flow field. Large Eddy Simulation (LES) models are employed for tidal flows over large scale topographies. The present method is a direct forcing method that has a sharp fluid-solid interface and utilizes trilinear interpolation similar to that of Mittal et al. (2008). However, there are some differences in the identification of ghost cells and interpolation stencil used in the reconstruction procedure so as to improve the numerical stability, especially high Reynolds number flows. An accurate method to compute the mass fluxes
is utilized and applied to the momentum equations at no additional computational expense. In engineering applications, the near-wall treatment of high-\(Re\) boundary layer turbulence is critical to compute the drag. In the present application, the dominant contribution to turbulent losses is from convective and shear instabilities that are not controlled by viscous boundary layer dynamics. Therefore, a different near-wall treatment through a nonlinear drag law is not only sufficient but also essential to allow extension of simulation capability to larger scale (order kilometer) topography.

### 1.4 Thesis Outline

The governing equations, the numerical algorithm and the temporal integration scheme used in the finite difference code are described in chapter 2. In section 2.4, details of the IBM are presented. Near wall treatment to impose wall shear stress for high Reynolds number flows is described in section 2.5. Chapter 3 describes internal tide conversion and turbulence at a model ridge simulated using DNS and LES on a body fitted grid which is published in Rapaka et al. (2013). Sections 4.2 and 4.3 present validation of the IBM for unstratified and stratified turbulent flow in a channel, respectively. In section 4.4, internal tide generation and turbulence at a laboratory scale are considered using direct numerical simulation (DNS). The baroclinic energy budget and turbulence statistics are compared against the DNS studies of Rapaka et al. (2013) and Jalali et al. (2014) which utilize a body conforming grid. In section 4.5, internal tide generation is studied at a large scale topography using large eddy simulation (LES) along with a non-linear bottom drag. A summary of the thesis is presented in chapter 5.
Chapter 2

Immersed Boundary Method on a Cartesian Grid

As part of the thesis, a computational fluid dynamics code is developed to simulate stratified flows over complex topography on a Cartesian grid using finite difference discretization in all three directions. A sharp interface immersed boundary method (IBM) is developed to simulate complex bodies on a Cartesian grid. The governing equations, the numerical method and the details of the IBM are presented in this chapter. Near wall treatment used to impose bottom drag for large scale topographies is described in sec. 4.5.

2.1 Governing Equations

Three dimensional Navier-Stokes equations for incompressible flow with Boussinesq approximation are solved numerically using finite difference discretization over a Cartesian grid. Boussinesq approximation is used for applications with density stratified flows. A mixed RK3-ADI marching scheme is proposed to advance the solution in time. The diffusion terms are advanced implicitly using an ADI method in three dimensions proposed by Douglas Jr. (1962) (also see page 934 of Pozrikidis (2008)). All other terms are advanced using explicit Runge-Kutta-Wray3 (RKW3) time marching scheme. Second order central finite-difference scheme is used for spatial discretization on a colocated grid. The solver allows the use of inhomogeneous boundary conditions in all three directions. We do not include the effect of rotation in the present simulations. But, the numerical model is capable of adding rotation related physics.
Figure 2.1: Colocated velocity components \((u_i)\), defined at the cell center \(P\), and face velocity components \((U_i)\), defined at the cell faces \((f = e, w, n, s, t, b)\), are shown. Pressure \((p)\) and density \((\rho^*)\) are colocated with \((u_i)\) at the cell center \(P\).
Here, $C_M = 0.1$, $C_{\rho} = \frac{C_M}{Pr_T}$ with a turbulent Prandtl number ($Pr_T$) of unity, and the filter width for the cell ($\Delta$) is defined by $\Delta \equiv (\Delta_1 \Delta_2 \Delta_3)^{\frac{1}{3}}$. The filter width in each direction, $\Delta_i$, is taken to be equal to the grid size.

### 2.2 Predictor-Corrector Algorithm

The mixed RK-ADI temporal integration is detailed in sec. 2.3. In each RK-ADI substep, a predictor-corrector algorithm is used to advance the discretized momentum equation (Eq. 2.30) along with the continuity equation from step $n$ to $n+1$. Each sub-step has the discretized momentum equations of the form given below,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\delta(U_j u_i)^n}{\delta x_j} + \nu \frac{\delta \delta u_i^{n+1}}{\delta x_j \delta x_j} - \frac{1}{\rho_0} \frac{\delta p^{n+1}}{\delta x_i} - g \frac{\rho^*}{\rho_0} \delta i_3 \quad (2.6)$$

The velocity field is predicted ($u^*$) using a guess for the pressure ($p^*$), typically the pressure in the previous time step ($p^n$), by solving,

$$\frac{u_i^* - u_i^n}{\Delta t} = \frac{\delta(U_j u_i)^n}{\delta x_j} + \nu \frac{\delta \delta u_i^*}{\delta x_j \delta x_j} - \frac{1}{\rho_0} \frac{\delta p^*}{\delta x_i} - g \frac{\rho^*}{\rho_0} \delta i_3 \quad (2.7)$$

Here, $\frac{\delta}{\delta x_j}$ represents the second-order central differencing operator. The face velocity components, $U_i$, are predicted by using colocated velocity components at neighbouring cell centers ($nb = E,W,N,S,T,B$ in fig. 2.1) separated by the corresponding face ($f$) (see Zang et al. (1994)).

$$U_{i,f}^* = \bar{u}_{i,P}^* \frac{\Delta t}{\rho_0} \frac{\delta p^*}{\delta x_i} \Big|_f \quad (2.8)$$

$$\bar{u}_i = \bar{u}_{i,P}^* + \frac{\Delta t}{\rho_0} \frac{\delta p^*}{\delta x_i} \Big|_P \quad (2.9)$$

where $\bar{u}_i$ are the colocated velocity components predicted with zero pressure gradient and the overbar represents a linear interpolation between the velocity at $P$ and $nb$. The solver uses central difference spatial discretization schemes as opposed to the upwind based scheme used in Zang et al. (1994). The central difference schemes provide little numerical diffusion compared to the upwind based schemes and do not suppress checker board pressure associated with pressure-velocity decoupling. To address this issue, a modified pressure gradient, as described in Rhie & Chow (1983), is added to $U_i$ so that the $U_i$ at a face $f$ is computed based on the pressure difference between neighbor cell centers ($P, nb$).
The velocity is then made divergence free by solving the Poisson equation for pressure correction \( (p^\ast) \),
\[
\frac{\delta}{\delta x_i} \frac{\delta p^\ast}{\delta x_i} = \frac{\rho_0}{\Delta t} \frac{\delta U_i^\ast}{\delta x_i} \tag{2.10}
\]
and applying a correction to the predicted velocity and pressure using,
\[
u^{n+1}_i = \nu_i - \frac{\Delta t}{\rho_0} \frac{\delta p^\ast}{\delta x_i} \tag{2.11}
\]
\[
U_i^{n+1} = U_i^\ast - \frac{\Delta t}{\rho_0} \frac{\delta p^\ast}{\delta x_i} \tag{2.12}
\]
\[
p^{n+1} = p^\ast + \nu \Delta t \frac{\delta}{\delta x_j} \frac{\delta p^\ast}{\delta x_j} \tag{2.13}
\]

Note that the pressure gradient is calculated at cell center, \( P \), in Eq. [2.11]
and at face center, \( f \), in Eq. [2.12] so that the differencing schemes vary, which is the key to coupling between velocity and pressure.

### 2.3 Temporal Integration

The temporal integration is accomplished with a combination of RK3-ADI scheme. The explicit part is treated using explicit 3rd order Runge-Kutta method of Williamson (1980) and the implicit part is treated using an unconditionally stable ADI method in three dimensions with second-order accuracy in both time and space, see Douglas Jr. (1962) or page 934 of Pozrikidis (2008). The governing equations are given below,
\[
\frac{\partial \nu_i}{\partial t} = -\frac{\delta U_k u_i}{\delta x_k} - \frac{g \rho^\ast}{\rho_0} \delta_{ij3} + \frac{\delta}{\delta x_k} \left( \nu e \frac{\delta u_i}{\delta x_k} \right) + \frac{\delta}{\delta x_i} \left( \nu e \frac{\delta u_k}{\delta x_i} \right) - \frac{1}{\rho_0} \frac{\delta p}{\delta x_i} \tag{2.14}
\]
\[
= E(u_i, \rho^\ast) + I(u_i, p) \tag{2.15}
\]

where \( E(u_i, \rho^\ast) \) and \( I(u_i, p) \) represent the explicit and implicit parts of the RHS of Eq. 2.14 defined by,
\[
E(u_i, \rho^\ast) \equiv -\frac{\delta U_k u_i}{\delta x_k} - \frac{g \rho^\ast}{\rho_0} \delta_{ij3} + \frac{\delta}{\delta x_k} \left( \nu e \frac{\delta u_i}{\delta x_k} \right) \tag{2.16}
\]
\[
I(u_i, p) \equiv \frac{\delta}{\delta x_k} \left( \nu e \frac{\delta u_i}{\delta x_k} \right) - \frac{1}{\rho_0} \frac{\delta p}{\delta x_i} \tag{2.17}
\]

Note that the cross terms arising from \( \nu e \) in LES are added to the right hand side of Eq. 2.16. For DNS, these terms disappear due to continuity and \( \nu e \) is replaced
by the molecular viscosity in the diffusion terms. The temporal integration can be demonstrated using a model ordinary differential equation,

$$\dot{\Phi} = E(\Phi) + I(\Phi)$$

(2.18)

where $E$ and $I$ include explicit and implicit terms respectively.

| Table 2.1: Coefficients of mixed RK3-ADI Scheme |
|-------------------|---|---|---|
| $m$  | $C_1$ | $C_2$ | $C_3$ |
| 1    | 0    | 1/3  | 1/3  |
| 2    | -5/9 | 15/16| 5/12 |
| 3    | -153/128 | 8/15 | 1/4  |

$$q^m = C_1^m q^{m-1} + \Delta t \ E(\Phi^{m-1})$$

(2.19)

$$\Phi^m = \Phi^{m-1} + C_2^m q^m + C_3^m \Delta t \ I(\Phi^m)$$

(2.20)

Define $\Delta t_{rk} \equiv C_3^m \Delta t$, above equation can be written as,

$$\Phi^m = \Phi^{m-1} + C_2^m q^m + \Delta t_{rk} \ I(\Phi^m)$$

(2.21)

Note that above equation is implicit in $\Phi^m$, hence involves solving a system of linear equations.

The momentum equations 2.15 can be marched using above method. The details are given below. Temporal integration of the momentum equations along with the scalar equations is performed from step $n$ to $n+1$ in three sub-steps. The equation for the density follows similar procedure and is not included here.

$$q^1 = \Delta t \ E(u_i^n, \rho^n)$$

(2.22)

$$u_i^1 = u_i^n + C_1^1 q^1 + \Delta t_{rk1} \ I(u_i^1, \rho^1)$$

(2.23)

$$q^2 = C_1^2 q^1 + \Delta t \ E(u_i^1, \rho^1)$$

(2.24)

$$u_i^2 = u_i^1 + C_2^2 q^2 + \Delta t_{rk3} \ I(u_i^2, \rho^2)$$

(2.25)

$$q^3 = C_1^3 q^2 + \Delta t \ E(u_i^2, \rho^2)$$

(2.26)

$$u_i^{n+1} = u_i^n + C_2^3 q^3 + \Delta t_{rk3} \ I(u_i^{n+1}, \rho^{n+1})$$

(2.27)
2.3.1 RK3-ADI Step 1

This step includes RK3 sub-step 1 and simultaneous time integration of implicit terms using ADI method.

\[ q^1 = \Delta t \left[ -\frac{\delta U_k^n u^n_i}{\delta x_k} - g \frac{\rho^n}{\rho_0} \delta_{i3} + \frac{\delta}{\delta x_k} \left( \nu_c \frac{\delta u^n_i}{\delta x_i} \right) \right] \]  \hspace{1cm} (2.28)

\[ u^1_i = u^n_i + C_1 q^1 + \Delta t_{rk1} \frac{\delta}{\delta x_k} \left( \nu_c \frac{\delta u^n_i}{\delta x_k} \right) - \frac{\Delta t_{rk1}}{\rho_0} \frac{\delta p^1}{\delta x_i} \]  \hspace{1cm} (2.29)

After re-arranging above equation, we have:

\[ \left[ 1 - \Delta t_{rk1} \frac{\delta}{\delta x_k} \left( \nu_c \frac{\delta}{\delta x_k} \right) \right] u^1_i = u^n_i + C_2 q^1 - \frac{\Delta t_{rk1}}{\rho_0} \frac{\delta p^1}{\delta x_i} \]  \hspace{1cm} (2.30)

Details of the predictor-corrector algorithm and treatment of flow variables on a collocated grid arrangement are discussed separately in section 2.2. The equations solved are briefly presented below.

**Predictor Step**

Let \( p^1 = p^* + p' \) and \( u^1_i = u^*_i + u'_i \), and use \( p^* = p^n \)

\[ \left[ 1 - \Delta t_{rk1} \frac{\delta}{\delta x_k} \left( \nu_c \frac{\delta}{\delta x_k} \right) \right] u^*_i = u^n_i + C_2 q^1 - \frac{\Delta t_{rk1}}{\rho_0} \frac{\delta p^n}{\delta x_i} \]  \hspace{1cm} (2.31)

**ADI Method in 3D**

ADI method of Douglas (1962) which is unconditionally stable and second order accurate in time and space is used here to solve Eq. 2.31. Let us denote the right hand side by \( S \equiv u^n_i + C_2 q^1 - \frac{\Delta t_{rk1}}{\rho_0} \frac{\delta p^n}{\delta x_i} \). The solution is then marched from \( u^n_i \) to \( u^*_i \) in three steps

\[ \left[ 1 - \Delta t_{rk1} \frac{\delta}{\delta x_k} \left( \nu_c \frac{\delta}{\delta x_k} \right) \right] u^*_i = S \]  \hspace{1cm} (2.32)

**ADI Step 1**

\[ \left[ 1 - \Delta t_{rk1} \frac{\delta}{\delta x} \left( \nu_c \frac{\delta}{\delta x} \right) \right] u^1_i = S \]

\[ + \Delta t_{rk1} \left[ \frac{1}{2} \frac{\delta}{\delta x} \left( \nu_c \frac{\delta}{\delta x} \right) + \frac{\delta}{\delta y} \left( \nu_c \frac{\delta}{\delta y} \right) + \frac{\delta}{\delta z} \left( \nu_c \frac{\delta}{\delta z} \right) \right] u^n_i \]
ADI Step 2

\[
\left[ 1 - \frac{\Delta t_{rk} \delta}{2 \delta y} \left( \nu_e \frac{\delta}{\delta y} \right) \right] u_i^t = S \\
+ \Delta t_{rk} \left[ \frac{1}{2} \frac{\delta}{\delta x} \left( \nu_e \frac{\delta}{\delta x} \right) + \frac{1}{2} \frac{\delta}{\delta y} \left( \nu_e \frac{\delta}{\delta y} \right) + \frac{\delta}{\delta z} \left( \nu_e \frac{\delta}{\delta z} \right) \right] u_i^n \\
+ \Delta t_{rk} \left[ \frac{1}{2} \frac{\delta}{\delta x} \left( \nu_e \frac{\delta u_i^t}{\delta x} \right) \right] \\
\left[ 1 - \frac{\Delta t_{rk} \delta}{2 \delta y} \left( \nu_e \frac{\delta}{\delta y} \right) \right] u_i^t = u_i^t - \frac{\Delta t_{rk} \delta}{2 \delta y} \left( \nu_e \frac{\delta u_i^n}{\delta y} \right)
\] (2.33)

ADI Step 3

\[
\left[ 1 - \frac{\Delta t_{rk} \delta}{2 \delta z} \left( \nu_e \frac{\delta}{\delta z} \right) \right] u_i^t = S \\
+ \Delta t_{rk} \frac{1}{2} \left[ \frac{\delta}{\delta x} \left( \nu_e \frac{\delta}{\delta x} \right) + \frac{\delta}{\delta y} \left( \nu_e \frac{\delta}{\delta y} \right) + \frac{\delta}{\delta z} \left( \nu_e \frac{\delta}{\delta z} \right) \right] u_i^n \\
+ \Delta t_{rk} \frac{1}{2} \left[ \frac{\delta}{\delta x} \left( \nu_e \frac{\delta u_i^t}{\delta x} \right) + \frac{\delta}{\delta y} \left( \nu_e \frac{\delta u_i^t}{\delta y} \right) \right] \\
\left[ 1 - \frac{\Delta t_{rk} \delta}{2 \delta z} \left( \nu_e \frac{\delta}{\delta z} \right) \right] u_i^t = u_i^t - \frac{\Delta t_{rk} \delta}{2 \delta z} \left( \nu_e \frac{\delta u_i^n}{\delta z} \right)
\] (2.34)

Poisson Equation

\[
\frac{\delta}{\delta x_k} \frac{\delta p'}{\delta x_j} = \frac{\rho_0}{\Delta t_{rk} \delta x_i} \frac{\delta U_i^*}{\delta x_i}
\] (2.35)

Note that \( U_i^* \) is obtained by interpolating \( u_i^t \) onto the cell face and adding a modified pressure-gradient as described in section 2.1.

Corrector Step

\[
p^t = p^* + p' \\
u_i^t = u_i^* - \Delta t_{rk} \frac{\delta p'}{\delta x_i} \\
U_i^t = U_i^* - \Delta t_{rk} \frac{\delta p'}{\delta x_i}
\] (2.36, 2.37, 2.38)

The time marching of RK3-ADI sub-steps 2 and 3 follow in a similar procedure and is not discussed here.
2.3.2 RK3-ADI Step 2

This step includes RK3 sub-step 2 and simultaneous time integration of implicit terms using ADI method.

\[
q^2 = C_1^2 q^1 + \Delta t \left[ \frac{\delta U^1_k u^1_k}{\delta x_k} - \frac{p^1_i}{\rho_0} \delta \iota_3 + \frac{\delta}{\delta x_k} \left( \nu_c \frac{\delta u^1_k}{\delta x_k} \right) \right] \quad (2.39)
\]

\[
u^2_i = \nu^1_i + C_2^2 q^2 + \Delta t_{rk2} \frac{\delta}{\delta x_k} \left( \nu_c \frac{\delta u^2_k}{\delta x_k} \right) - \Delta t_{rk2} \frac{\delta p^2}{\delta x_i} \quad (2.40)
\]

After re-arranging above equation, we have:

\[
\left[ 1 - \Delta t_{rk2} \frac{\delta}{\delta x_k} \left( \nu_c \frac{\delta}{\delta x_k} \right) \right] u^2_i = u^1_i + C_2^2 q^2 - \Delta t_{rk2} \frac{\delta p^2}{\delta x_i} \quad (2.41)
\]

**Predictor Step**

Let \( p^2 = p^* + p^' \) and \( u^2_i = u^*_i + u^'_i \), and use \( p^* = p^1 \)

\[
\left[ 1 - \Delta t_{rk2} \frac{\delta}{\delta x_k} \left( \nu_c \frac{\delta}{\delta x_k} \right) \right] u^*_i = u^1_i + C_2^2 q^2 - \Delta t_{rk2} \frac{\delta p^1}{\delta x_i} \quad (2.42)
\]

Let us denote the right hand side by \( S \equiv u^1_i + C_2^2 q^2 - \Delta t_{rk2} \frac{\delta p^1}{\delta x_i} \), we march the solution from \( u^1_i \) to \( u^*_i \) by solving

\[
\left[ 1 - \Delta t_{rk2} \frac{\delta}{\delta x_k} \left( \nu_c \frac{\delta}{\delta x_k} \right) \right] u^*_i = S \quad (2.43)
\]

**ADI Step 1**

\[
\left[ 1 - \frac{\Delta t_{rk2}}{2} \frac{\delta}{\delta x} \left( \nu_c \frac{\delta}{\delta x} \right) \right] u^1_i = S + \Delta t_{rk2} \left[ \frac{1}{2} \frac{\delta}{\delta x} \left( \nu_c \frac{\delta}{\delta x} \right) + \frac{\delta}{\delta y} \left( \nu_c \frac{\delta}{\delta y} \right) + \frac{\delta}{\delta z} \left( \nu_c \frac{\delta}{\delta z} \right) \right] u^1_i \quad (2.44)
\]

**ADI Step 2**

\[
\left[ 1 - \frac{\Delta t_{rk2}}{2} \frac{\delta}{\delta y} \left( \nu_c \frac{\delta}{\delta y} \right) \right] u^1_i = u^\dagger_i - \frac{\Delta t_{rk2}}{2} \frac{\delta}{\delta y} \left( \nu_c \frac{\delta u^1_i}{\delta y} \right) \quad (2.45)
\]

**ADI Step 3**

\[
\left[ 1 - \frac{\Delta t_{rk2}}{2} \frac{\delta}{\delta z} \left( \nu_c \frac{\delta}{\delta z} \right) \right] u^1_i = u^\dagger_i - \frac{\Delta t_{rk2}}{2} \frac{\delta}{\delta z} \left( \nu_c \frac{\delta u^1_i}{\delta z} \right) \quad (2.45)
\]
Poisson Equation

\[
\frac{\delta}{\delta x_k} \frac{\delta p'}{\delta x_k} = \frac{\rho_0}{\Delta t_{rk^2}} \frac{\delta U_i^*}{\delta x_i}
\]  (2.46)

Note that \( U_i^* \) is obtained by interpolating \( u_i^* \) onto the cell face and adding a modified pressure-gradient as described in section 2.2.

Corrector Step

\[
p^2 = p^* + p'
\]  (2.47)

\[
u^2_i = u_i^* - \Delta t_{rk^2} \frac{\delta p'}{\delta x_i}
\]  (2.48)

\[
U^2_i = U_i^* - \Delta t_{rk^2} \frac{\delta p'}{\delta x_i}
\]  (2.49)

2.3.3 RK3-ADI Step 3

This step includes RK3 sub-step 3 and simultaneous time integration of implicit terms using ADI method.

\[
q^3 = C_1^3 q^2 + \Delta t \left[ -\frac{\delta U_i^2 u_i^2}{\delta x_k} - g \frac{p^2 \rho_0}{\delta x_i} + \frac{\delta}{\delta x_k} \left( \nu \frac{\delta u_i^2}{\delta x_i} \right) \right]
\]  (2.50)

\[
u_i^{n+1} = u_i^2 + C_2^3 q^3 + \Delta t_{rk^3} \frac{\delta}{\delta x_k} \left( \nu \frac{\delta u_i^{n+1}}{\delta x_k} \right) - \frac{\Delta t_{rk^3}}{\rho_0} \frac{\delta p^{n+1}}{\delta x_i}
\]  (2.51)

After re-arranging above equation, we have:

\[
\left[ 1 - \Delta t_{rk^3} \frac{\delta}{\delta x_k} \left( \nu \frac{\delta}{\delta x_k} \right) \right] u_i^{n+1} = u_i^2 + C_2^3 q^3 - \Delta t_{rk^3} \frac{\delta p^{n+1}}{\delta x_i}
\]  (2.52)

Predictor Step

Let \( p^{n+1} = p^* + p' \) and \( u_i^{n+1} = u_i^* + u_i' \), and use \( p^* = p^2 \)

\[
\left[ 1 - \Delta t_{rk^3} \frac{\delta}{\delta x_k} \left( \nu \frac{\delta}{\delta x_k} \right) \right] u_i^* = u_i^2 + C_2^3 q^3 - \Delta t_{rk^3} \frac{\delta p^2}{\delta x_i}
\]  (2.53)

Let us denote the right hand side by \( S \equiv u_i^2 + C_2^3 q^3 - \Delta t_{rk^3} \frac{\delta p^2}{\delta x_i} \), we march the solution from \( u_i^2 \) to \( u_i^* \) by solving

\[
\left[ 1 - \Delta t_{rk^3} \frac{\delta}{\delta x_k} \left( \nu \frac{\delta}{\delta x_k} \right) \frac{\delta}{\delta x_k} \frac{\delta}{\delta x_k} \right] u_i^* = S
\]  (2.54)
ADI Step 1
\[
\left[ 1 - \frac{\Delta t_{rk}^3}{2} \frac{\delta}{\delta x} \left( \nu_c \frac{\delta}{\delta x} \right) \right] u_i^1 = S + \frac{\Delta t_{rk}^i}{Re_o} \left[ \frac{1}{2} \frac{\delta}{\delta x} \left( \nu_c \frac{\delta}{\delta x} \right) + \frac{\delta}{\delta y} \left( \nu_c \frac{\delta}{\delta y} \right) + \frac{\delta}{\delta z} \left( \nu_c \frac{\delta}{\delta z} \right) \right] u_i^2
\]

ADI Step 2
\[
\left[ 1 - \frac{\Delta t_{rk}^3}{2} \frac{\delta}{\delta y} \left( \nu_c \frac{\delta}{\delta y} \right) \right] u_i^1 = u_i^1 - \frac{\Delta t_{rk}^3}{2} \frac{\delta}{\delta y} \left( \nu_c \frac{\delta u_i^2}{\delta y} \right)
\]

ADI Step 3
\[
\left[ 1 - \frac{\Delta t_{rk}^3}{2} \frac{\delta}{\delta z} \left( \nu_c \frac{\delta}{\delta z} \right) \right] u_i^{n+1} = u_i^1 - \frac{\Delta t_{rk}^3}{2} \frac{\delta}{\delta z} \left( \nu_c \frac{\delta u_i^2}{\delta z} \right)
\]

Poisson Equation
\[
\frac{\delta}{\delta x_k} \frac{\delta p^*}{\delta x_k} = \frac{\rho_0}{\Delta t_{rk}^3} \frac{\delta U_i^*}{\delta x_i}
\]

Note that \( U_i^* \) is obtained by interpolating \( u_i^* \) onto the cell face and adding a modified pressure-gradient as described in section 2.2.

Corrector Step
\[
p^{n+1} = p^* + p'
\]
\[
u_i^{n+1} = u_i^* - \Delta t_{rk} \frac{\delta p'}{\delta x_i}
\]
\[
U_i^{n+1} = U_i^* - \Delta t_{rk} \frac{\delta p'}{\delta x_i}
\]

2.4 Immersed Boundary Method

Immersed Boundary Method (IBM) is used to include the effect of solid objects with finite difference discretization on a Cartesian grid. In IBM, the fluid and the solid cells are distinguished by the immersed body surface. The governing equations are solved only for the fluid cells and the flow variables are reconstructed at the so-called ghost cells (GC) which are immediately adjacent to the immersed boundary. Thus, the flow is decoupled between the solid and the fluid regions.
Figure 2.2: (a) Solid cells, ghost cells (GC), immersed boundary (IB) cells, and intersection points (IP) on a Cartesian grid distinguished by the immersed body. The shaded area represents the immersed body and the empty boxes are fluid cells. (b) Modified IB cells tagged as GC. The modification leads to more uniform distance between IB cell nodes and the boundary.

2.4.1 Geometric Preprocessing

The immersed boundary surface information is provided in a triangulated mesh format and the fluid/solid cells are identified using a ray tracing procedure described in Roman et al. (2009). Figure 2.2(a) shows a shaded region representing the immersed body, ghost cells (GC), immersed boundary cells (IB) and intersection points (IP). Solid cells that have at least one neighbor, along the Cartesian directions, in the fluid region are tagged as GC. The fluid cells that have at least one GC as the neighbor along the Cartesian directions are tagged as IB. The intersection point (IP) is the intersection of the immersed boundary with the surface normal passing through the center of the corresponding GC. Note that the IP nodes associated with IB cells are located in a similar way and are not shown in fig. 2.2(a).

The governing equations are solved at all fluid cells, including the IB cells, while the solution is reconstructed at the GC using trilinear interpolation as described below. The solid cells are excluded from the computational stencil and all variables in the solid cell are explicitly assigned with values of zero except pressure. Some of the IB cells, for example highlighted by the two lined boxes in figure 2.2(a), may have cell centers that lie very close to the immersed boundary. We observed that the numerical stability of the solver improves if the flow is reconstructed at these...
cells instead of solving the governing equations especially for high Reynolds number flows. So, if an IB cell is closer to the immersed boundary than the neighbor GC, i.e., $|\mathbf{dx}|_{IB-IP} < |\mathbf{dx}|_{GC-IP}$ where $|\mathbf{dx}|_{IB-IP}$ and $|\mathbf{dx}|_{GC-IP}$ are the shortest distances from IB and GC cell centers to the immersed boundary, respectively, then the IB cell is tagged as a GC, as shown in figure 2.2(b). The corresponding neighbor GC is then tagged as a solid cell as shown by the black colored circles in figure 2.2(b). The new solid cells are not used in the discretization stencil of any fluid or IB cell, and hence no interpolation is required for these cells. Thus, our method has ghost cells on both sides of the immersed boundary and does not restrict to solid region only (Mittal et al. (2008)) or fluid region only (Roman et al. (2009)).

### 2.4.2 Reconstruction Procedure

Linear reconstruction is used at GC for all variables, except pressure, at the beginning of each RK3 sub-step. This can be performed in two different ways based on the number of steps used in the reconstruction.
Direct Reconstruction

The first method, which is described below, consists of a single-step reconstruction and uses trilinear interpolation directly for the GC. The trilinear interpolation is similar to that described in Mittal et al. (2008) but there are few differences in the details which are discussed later. A generic variable $\phi$ is expressed using trilinear interpolation as,

$$\phi(x, y, z) = C_1 x + C_2 y + C_3 z + C_4 xy + C_5 yz + C_6 xz + C_7 xyz + C_8$$  \hspace{1cm} (2.61)

The coefficients $C_i, i = 1, 2, ..., 8$ are determined using the values of $\phi$ from seven neighbor cells and the boundary condition at the IP node located on the immersed boundary. The neighbors are chosen in all three directions based on the boundary normal direction, as shown in fig. 2.3(a). The interpolation stencil for a given ghost cell may contain other ghost cells. For instance the interpolation stencil of GC inside the rectangular box shown in 2.3(a) includes point 4 which is also a ghost cell that lies in a plane parallel to the plane of view. This does not pose any inconsistency. However, for cells with large aspect ratio, eg. grids used in oceanic flows, more than one GC lie along one of the Cartesian directions as indicated by 1, 2, and 3 in fig. 2.3(b). If GC 1, which is close to the IP node of GC2, is included in the interpolation stencil of GC 2, the linear system formed by matrix $[V]$, defined in Eq. 2.65, may become ill-conditioned and result in spurious values at GC2. To circumvent this issue, the interpolation stencil for all the three GC (1, 2, and 3), include cells in the fluid region and the corresponding IP nodes as shown by the red colored rectangular box in fig. 2.3(b). This interpolation stencil is consistent with that used in the method 2 described later.

Eq. 2.61 is rearranged into matrix form as,

$$\{\phi\} = [V]\{C\}$$  \hspace{1cm} (2.62)

where,

$$\{C\}^T = \{C_1, C_2, ..., C_8\}$$  \hspace{1cm} (2.63)

$$\{\phi\}^T = \{\phi_1, \phi_2, ..., \phi_7, \alpha\}$$  \hspace{1cm} (2.64)
and

\[
[V] = \begin{bmatrix}
x|_1 & y|_1 & z|_1 & xy|_1 & yz|_1 & xz|_1 & xyz|_1 & 1 \\
x|_2 & y|_2 & z|_2 & xy|_2 & yz|_2 & xz|_2 & xyz|_2 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
x|_7 & y|_7 & z|_7 & xy|_7 & yz|_7 & xz|_7 & xyz|_7 & 1 \\
x|_{IP} & y|_{IP} & z|_{IP} & xy|_{IP} & yz|_{IP} & xz|_{IP} & xyz|_{IP} & 1
\end{bmatrix}
\] (2.65)

The last row in Eq. 2.65 assumes Dirichlet boundary condition at IP, that lies on the immersed boundary surface, given by,

\[
\alpha = \phi_{IP} = (C_1 x + C_2 y + C_3 z + C_4 xy + C_5 yz + C_6 xz + C_7 xyz + C_8)|_{IP}
\] (2.66)

Here, \( \alpha \) is the boundary value of \( \phi \) specified at IP.

The density (temperature) gradient often satisfies a Neuman boundary condition, e.g., the adiabatic boundary condition employed here. In the case of Neumann boundary condition on the IB surface, the last row of the matrix \([V]\) is modified using,

\[
\alpha = \frac{\partial \phi}{\partial n}|_{IP}
\]

\[
= \frac{\partial \phi}{\partial x} n_x + \frac{\partial \phi}{\partial y} n_y + \frac{\partial \phi}{\partial z} n_z
\] (2.68)

\[
= [C_1 n_x + C_2 n_y + C_3 n_z + C_4 (y n_x + x n_y) + C_5 (z n_y + y n_z) + C_6 (z n_x + x n_z) + C_7 (y z n_x + x z n_y + x y n_z)]_{IP}
\] (2.69)

\[
+ C_8 (z n_x + x n_z) + C_7 (y z n_x + x z n_y + x y n_z)]_{IP}
\] (2.70)

where \( n_x, n_y, n_z \) are the components of the unit vector normal to the boundary. The above expressions can be simplified when the origin for the interpolation stencil of Eq. 2.61 is moved to \( \vec{x}_{IP} \) so that \( \vec{x} = \vec{x} - \vec{x}_{IP} \) and

\[
\alpha = C_8 \quad \text{for Dirichlet BC} \quad (2.71)
\]

\[
\alpha = C_1 n_x + C_2 n_y + C_3 n_z \quad \text{for Neumann BC} \quad (2.72)
\]

The matrix \([V]\) is inverted using the Gauss-Jordan method with full pivoting at each step. The coefficients \( \{C\}^T \) are then obtained using,

\[
\{C\} = [V]^{-1}\{\phi\}
\] (2.73)

At the beginning of each RK3-ADI sub-step, ghost cell values are then evaluated using,

\[
\phi_{GC} = \sum_{i=1}^{8} \beta_i \phi_i
\] (2.74)
where
\[ \beta_i = v^T_{GC}[V]^{-1}, \]
and
\[ v^T_{GC} = \begin{bmatrix} x|_{GC} & y|_{GC} & z|_{GC} & xy|_{GC} & yz|_{GC} & xz|_{GC} & xyz|_{GC} & 1 \end{bmatrix}. \]

Note that \( \beta_i \) include only geometric variables. Hence, this procedure is carried out once at the beginning of the computation and the coefficients \( \beta_i \) are saved.

**Reconstruction using an Intermediate Node (PP)**

The second method is a two-step procedure for interpolation at GC. During the first step, the trilinear interpolation is carried out for PP node, which is a projection of the GC along the surface normal direction and placed inside a closest fluid or IB cell as shown in fig. 2.3(a). During the second step, linear interpolation is carried out for GC along the surface normal direction using the PP node and the boundary condition at the IP node as described by Mittal et al. (2008). The ghost cells 1, 2 and 3 shown in fig. 2.3(b) have their respective PP lie within the same fluid cell. So, PP nodes of all three ghost cells have the same interpolation stencil, indicated by red colored rectangular box in fig. 2.3(b). Thus, both the first and second methods have consistent interpolation stencil and give similar results.

In Mittal et al. (2008), PP is placed such that the IP is midway between GC and PP along the surface normal direction, i.e., \( |\overrightarrow{dx}|_{PP-IP} = |\overrightarrow{dx}|_{GC-IP} \). However, the PP located using this method reflects the non-uniform distribution of the GC. This would result to non-uniform errors in the vicinity of immersed boundary and, in our experience, can cause numerical instabilities (small-scale, spurious waves) in simulations of high-Reynolds number stratified flows where the density couples strongly to the velocity through the buoyancy term. In the present approach, PP is placed such that it must lie inside a fluid or an IB cell that is closest to the surface normal. This usually places the PP at a characteristic distance from IP (or the immersed boundary) of \( d_{IP-PP} \) such that \( d_{IP-PP} = (0.5 - 1.5) \min(\Delta x, \Delta y, \Delta z) \) as shown in fig. 2.3(a-b). Here, we assume that the wall normal grid spacing is small compared to the other directions parallel to the wall. Note that our present method in which the GC lies on both sides of the immersed boundary reduces the non-uniform distribution of the PP points.

In Mittal et al. (2008), the interpolation stencil for PP node, matrix \([V]\) in Eq. 2.65, may not necessarily include the boundary condition at IP, if it is surrounded
by all fluid cells. Also, the row position of the boundary condition, if included, in $[V]$ varies from cell to cell and requires a local search procedure. The first method described earlier allows straightforward identification of the interpolation stencil and includes the boundary condition at IP for each GC at a fixed row (number 8) of matrix $[V]$. Since the boundary condition is included in the matrix $[V]$, it can also be used to compute values of the variable at other locations such as redefined face center (RFC) introduced in section 2.4.2

In the present work, the first method in which the flow is reconstructed at GC directly is used. The solver additionally provides an option to use the second method involving a two-step procedure for reconstruction at GC. The PP nodes are used in the present work to impose bottom drag for large scale topography as described in section 2.5.

**Improved Mass Conservation**

Accurate prediction of mass flux is important in the simulation of high Reynolds number flows. Seo & Mittal (2011) adopted a cut-cell based discretization using fluid area fraction of the cut cell faces to enforce geometric conservation. They limit the method only to the pressure Poisson equation (Eq. 2.10) and the velocity correction equations (Eq. 2.11). Here, a method to compute the mass fluxes that is more accurate than the standard procedure is discussed and applied to the advective fluxes in the momentum equations as well at no additional computational expense and complexity.

Figure 2.4 shows the cells cut by the immersed boundary surface. The part of the cut cell faces lying in the fluid region is highlighted by thick green colored lines and corresponding fluid area fraction ($a_f$), $f=e,w,n,s,t,b$, is computed as described by Seo & Mittal (2011). The geometric centroid of the corresponding area is denoted as Redefined Face Center (RFC). In Seo & Mittal (2011), $U_i$ is obtained by interpolating the neighboring cell center values on to the cell face as described in Eq. 2.8. Fluid area fraction of the cut cell face is used to proportionately scale down the mass fluxes. However, $U_i$ obtained using this method does not provide the mass flux through the face accurately, even after scaling down the flux using $a_f$. For example, consider $U$ at the cuboid cell face marked by the black colored arrow in fig. 2.4. If the velocity in the fluid region is positive and a no slip boundary condition is applied at the solid boundary, a linear interpolation would result in negative velocity in the solid region.
Figure 2.4: Redefined Face Center (RFC), defined as the geometric centroid of a cut cell face lying in the fluid region indicated by thick green line. The mass fluxes are computed using $U_{i, RFC}$ and corresponding fluid area fraction $a_f, f = e, w, n, s, t, w$. The velocity $U_i$ at the original cuboid face center is also shown by the black arrow.

Thus, since the cell face center lies in the solid region, $U$ (calculated using interpolation of cell-center values) would become negative and results in a negative mass flux through that face, even after the flux is scaled by fluid area fraction. However, a positive velocity in the fluid region would require a net positive mass flux through the fluid region of the face marked by the thick green line in fig. 2.4.

In the present method, the mass flux through the fluid region of a cut cell face is evaluated using the face velocity defined at RFC, $U_{i, RFC}$. Since advective flux through the solid region of a cut cell face is zero, the total advective flux through a cut cell face is given by,

$$AU_i = A_f U_{i, RFC} \quad (2.77)$$

where $A$ and $A_f$ are the total area and the fluid area of each cut cell face and $U_{i, RFC}$ and $U_i$ are the velocities at the corresponding face centroids as shown in fig. 2.4.

Noting that the fluid area fraction is given by $a_f \equiv A_f / A$, the face velocity at the center of the original cuboid cell ($U_i$) is then given by,

$$U_i = a_f U_{i, RFC} \quad (2.78)$$

Here, the area fraction is used to accurately compute the mass flux used in both the momentum and the Poisson equations. The face velocities at RFC ($U_{i, RFC}$) are
evaluated using trilinear interpolation and the weightage coefficients are given by,

$$\beta_{RFC,i} = v^T_{RFC}[V]^{-1}$$  \hspace{1cm} (2.79)

where, $v^T_{RFC}$ is evaluated at the RFC location as,

$$v^T_{RFC} = \begin{bmatrix} x|_{RFC} & y|_{RFC} & z|_{RFC} & xy|_{RFC} & yz|_{RFC} & xz|_{RFC} & xyz|_{RFC} & 1 \end{bmatrix}$$  \hspace{1cm} (2.80)

During the predictor step, the face velocities ($U^*_i, RFC$) defined at RFC are computed using,

$$U^*_i, RFC = \sum_{k=1}^{8} \beta_{RFC,k} \tilde{u}_i,k - \frac{\Delta t}{\rho} \frac{\delta p^*}{\delta x_i}$$  \hspace{1cm} (2.81)

where $\tilde{u}_i,k$ are the intermediate colocated velocity components at the neighbor cell centers with pressure gradient subtracted. The predicted face velocity at the original cuboid cell face ($U_i$) is computed using $U^*_i = a_f U^*_i, RFC$ which is then used in the Poisson equation, Eq. 2.10, to solve for the pressure correction. Note that the right hand side of Eq. 2.10 becomes zero in the solid region as the solid cell faces have zero fluid area fraction. A geometric multi-grid method is used to solve the Poisson equation 2.10.

Note that the area fraction is not used in the discretized Laplace operator in Eq. 2.10 which requires computation of the area fraction at each multi-grid level. This poses additional complexity in the implementation of the multi-grid method. Thus, mass conservation is satisfied in an approximate sense similar to the approximate mass conservation described in Kang et al. (2009). Once Eq. 2.10 is solved in the entire domain, the boundary condition $\frac{\delta p^*}{\delta n} = 0$ is imposed on the immersed boundary, by updating the pressure correction at GC using trilinear interpolation, before the corrector step.

The face velocities for the cut cell faces are then corrected using,

$$U^{n+1}_{i, RFC} = U^*_i, RFC - \frac{\Delta t}{\rho} \frac{\delta p^*}{\delta x_i}$$  \hspace{1cm} (2.82)

$$U^{n+1}_i = a_f U^{n+1}_{i, RFC}$$  \hspace{1cm} (2.83)

which is then used in computing the advective fluxes in the next time step. The colocated velocity is corrected using,

$$u^{n+1}_i = u^*_i - \frac{\Delta t}{v_f \rho} \frac{\delta (a_f p^*)}{\delta x_i}$$  \hspace{1cm} (2.84)
Figure 2.5: Slip velocity ($u_w$) that provides the appropriate wall shear stress is imposed at the bottom wall ($w$). The non-linear drag law based on the tangential velocity ($u_t$) at point 1 gives the appropriate wall shear stress. Node 1 is the first cell center in a body fitted grid or a PP node in the case of IBM.

2.5 Near Wall Treatment for Large-Scale Flows

2.5.1 Drag Law

A nonlinear drag law is utilized to model the bottom friction so as to reduce the near wall resolution requirements for higher Reynolds number flows, specifically, over large scale topographies. Let us decompose the velocity into components that are parallel ($\vec{u}_t$) and normal ($\vec{u}_n$) to the wall so that

$$\vec{u} = \vec{u}_t + \vec{u}_n$$  \hspace{1cm} (2.85)

$$\vec{u}_n \equiv (\vec{u} \cdot \hat{n})\hat{n}$$  \hspace{1cm} (2.86)

$$= (u_n x + v_n y + w_n z)\hat{n}$$  \hspace{1cm} (2.87)

where $\hat{n} = n_x \hat{i} + n_y \hat{j} + n_z \hat{k}$ is the unit outward surface normal into the fluid.

A turbulent surface layer of constant stress is assumed and the first grid point (shown as node 1 in fig. 2.5 and, hereafter, indicated by subscript 1) is taken to lie in the surface layer. Therefore, the shear stress at the no-slip wall ($\vec{\tau}_w$) is taken to be equal to the shear stress parallel to the wall at point 1 ($\vec{\tau}_{t1}$) so that,

$$\vec{\tau}_w = \vec{\tau}_{t1}.$$  \hspace{1cm} (2.88)

Here, $\vec{\tau}_{t1}$ is evaluated at the first cell center above the wall using a non-linear drag
model,
\[ \vec{\tau} = \left. \left( \mu_e \frac{\partial \vec{u}_t}{\partial n} \right) \right|_{1} = \left( C_D \rho_0 |\vec{u}_t| \vec{u}_t \right)_{1} \]  
(2.89)

where, \( C_D \) is the drag coefficient, and \( \mu_e \) is the effective viscosity, \( \mu_e = \mu + \mu_{SGS} \), with \( \mu_{SGS} \) being the sub-grid scale stress used in LES. The choice of \( C_D = 0.002 - 0.0025 \) is common in oceanographic applications (Legg et al., 2006) and we use \( C_D = 0.0025 \) here. It is worth noting that LES of a stratified bottom boundary layer (Taylor & Sarkar, 2008) with stochastic forcing towards a log law yielded a value of \( C_D \simeq 0.0025 \).

Rearranging the above equation yields,
\[ \left( \frac{\partial \vec{u}_t}{\partial n} \right)_{1} = \left( \frac{C_D \rho_0 |\vec{u}_t| \vec{u}_t}{\mu_e} \right)_{1} = \alpha \vec{u}_{t1}, \quad \alpha = \frac{C_D \rho_0 |\vec{u}_t|}{\mu_e}. \]  
(2.90)

The boundary condition for the velocity components is then obtained using,
\[ \vec{\tau}_w = \left( \mu_e \frac{\partial \vec{u}_t}{\partial n} \right)_{w} = \left( \mu_e \frac{\partial \vec{u}_t}{\partial n} \right)_{1} \]  
(2.91)

so that
\[ \left( \frac{\partial \vec{u}_t}{\partial n} \right)_{w} = \left. \frac{\mu_e}{\mu_e} \right|_{1} \left( \frac{\partial \vec{u}_t}{\partial n} \right)_{1} \]  
(2.92)

At a no-slip wall \( \mu_{SGS} = 0 \) and \( \mu_e = \mu \), which would result in huge velocity gradients near the wall, \( \left( \frac{\partial \vec{u}_t}{\partial n} \right)_{w} \). Resolution of the large velocity gradient at the wall and the rapid variation in the inner layer of turbulence is not possible for large-scale oceanic flows. This lack of resolution would result in inappropriate velocities at the near boundary cells and may cause numerical instabilities. To circumvent this problem, Eq. 2.92 is modified as follows,
\[ \left( \frac{\partial \vec{u}_t}{\partial n} \right)_w = \left( \frac{\partial \vec{u}_t}{\partial n} \right)_1 = \alpha \vec{u}_{t1}, \]  
(2.93)

that follows from assuming that \( \mu_e \left|_{w} = \mu_e \right|_{1} \) or \( \frac{\partial \mu_e}{\partial n} \left|_{w} = 0 \right. \). Thus a linear discretization of the diffusion fluxes in the momentum equation yields a consistent shear stress and numerical stability. Note that the ghost cell value of \( \mu_{SGS} \) is updated using the boundary condition \( \frac{\partial \mu_{SGS}}{\partial n} = 0 \).

### 2.5.2 Boundary Conditions for Complex Topography: Slip Velocity

For three dimensional complex topography, the wall boundaries need not be aligned with Cartesian directions. Such cases are simulated using the IBM and they
require boundary conditions for the Cartesian velocity components that satisfy the imposed wall shear stress \( \left( \frac{\partial u_t}{\partial n} \right)_w \) and the impermeability condition \( (\vec{u}_n)_w = 0 \). We prescribe a slip velocity at the wall \( (\vec{u}_w) \) that satisfies the impermeability condition and yields an appropriate velocity gradient, thus shear stress, at the wall. Since \( \vec{u}_{tw} \) is parallel to \( \vec{u}_t \), the value of \( \vec{u}_1 \) can be expressed in terms of \( \vec{u}_{tw} \) using a Taylor series expansion as

\[
\vec{u}_1 = \vec{u}_{tw} + \left( \frac{\partial u_t}{\partial n} \right)_w (\Delta n) = \vec{u}_{tw} + (\alpha \vec{u}_1) (\Delta n),
\]

where \( \Delta n \) is the distance from the first cell center to the wall as shown in fig. 2.5. Rearrangement of Eq. 2.94 yields

\[
\vec{u}_{tw} = (1 - \alpha \Delta n) \vec{u}_1.
\]

Note that the impermeability boundary condition gives \( \vec{u}_{nw} = 0 \) and hence, \( \vec{u}_w = \vec{u}_{tw} \), allowing the computation of the slip velocity at the wall as

\[
\vec{u}_w = (1 - \alpha \Delta n) \vec{u}_1.
\]

Note that for inclined surfaces in the IBM, the PP node (an image of the solid region GC node) as shown in figure 2.5, is used as point 1 where the shear stress is modeled. \( \mu_{SGS_1} \), used in Eq. 2.90, is evaluated at PP nodes using trilinear interpolation including seven neighbor cells and the boundary condition \( \frac{\partial \mu_{SGS}}{\partial n} = 0 \) at the immersed boundary. Eq. 2.96 is used to prescribe boundary conditions for the Cartesian velocity components at the immersed boundary, which are then used in the reconstruction of the velocity at ghost cells (GC).
Chapter 3

Tidal Conversion and Turbulence at a Model Ridge: Direct and Large Eddy Simulations

The content of the present chapter is published in Rapaka et al. (2013). Three-dimensional direct and large eddy simulations are performed to investigate internal tides and turbulence generated over a smoothed triangular ridge. The simulations are performed on a body fitted grid using a mixed spectral/finite difference code Gayen (2012). The computational model and the cases selected are discussed in sec. 3.1. The results and analysis are presented in later sections.

3.1 Formulation of the problem

The near-bottom flow resulting from a current oscillating over a two dimensional ridge is illustrated in figure 3.1(a). The bottom is adiabatic while there is a background thermal stratification with constant buoyancy frequency, $N_{\infty}$. The flow is forced by an imposed pressure gradient oscillating in time ($t_d$),

$$F_0(t_d) = \rho_0 U_0 \Omega \cos(\Omega t_d),$$

(3.1)

in the horizontal direction that results in a background barotropic current, $U(x) \sin(\phi)$, where $\phi$ is the tidal phase. Coordinates x, y and z denote the streamwise, spanwise and vertical directions and $u$, $v$ and $w$ are the corresponding velocity components. A larger view of the model ridge is shown in figure 3.1(b). The triangular ridge, without
smoothing, can be described mathematically as:

\[ z(x) = \begin{cases} 
  h_0 \left(1 - \frac{|x|}{l_0}\right), & \text{if } |x| \leq l_0, \\
  0, & \text{otherwise,}
\end{cases} \]

(3.2)

where, \( h_0 = 0.4 \) and \( l_0 = 1.5 \). After smoothing, the ridge has a height of \( h = 0.328 \) and a half-length of \( l = 1.9 \). The constant slope portion of the ridge ranges from \( z = 0.13 \) to \( 0.28 \) on both sides which correspond to \( 0.45 \leq |x| \leq 1.01 \).

### 3.1.1 Governing Equations

The dimensional quantities in the problem are the freestream velocity amplitude \( U_0 \), tidal frequency \( \Omega \), background density gradient \( d\rho_b/dz\big|_\infty \), and the fluid properties: molecular viscosity, \( \nu \), thermal diffusivity, \( \kappa \), and density, \( \rho \). The variables in the problem are nondimensionalized as follows:

\[
\begin{align*}
  t &= t_d\Omega, \quad \mathbf{x} = (x, y, z) = \left(\frac{x_d, y_d, z_d}{U_0/\Omega}\right), \\
  \mathbf{u} &= (u, v, w) = \left(\frac{u_d, v_d, w_d}{U_0}\right), \\
  p^* &= \frac{p_d^*}{\rho_o U_0^2}, \\
  \rho^* &= \frac{\rho_d}{\rho_0 \frac{d\rho_b}{dz|_{\infty}}}. 
\end{align*}
\]

(3.3)
The resulting non-dimensional form of the governing equations is as follows:

\[
\nabla \cdot \mathbf{u} = 0 \quad (3.4a)
\]

\[
\frac{D\mathbf{u}}{Dt} = -\nabla p^* + \cos(t)i + \frac{1}{Re} \nabla^2 \mathbf{u} - B\rho^* \mathbf{k} - \nabla \cdot \mathbf{\tau} \quad (3.4b)
\]

\[
\frac{D\rho^*}{Dt} = \frac{1}{Re Pr} \nabla^2 \rho^* + w - \nabla \cdot \mathbf{\lambda} \quad (3.4c)
\]

Here, \( p^* \) denotes deviation from the background hydrostatic pressure and \( \rho^* \) denotes the deviation from the linear background state, \( \rho^b(z) \).

The governing equations have three nondimensional parameters: Reynolds number \( Re \), Buoyancy parameter \( B \), and Prandtl number \( Pr \), where

\[
Re \equiv \frac{l_{ex} U_0}{\nu} = \frac{U_0^2}{\Omega \nu}, \quad B \equiv -g \frac{d\rho^b}{dz}\Big|_\infty \frac{1}{\rho_b \Omega^2} = \frac{N^2_\infty}{\Omega^2}, \quad Pr \equiv \frac{\nu}{\kappa}. \quad (3.5)
\]

Here, \( l_{ex} = U_0/\Omega \) is the tidal excursion length and \( N_\infty \) is the background value of buoyancy frequency, assumed constant. The following Reynolds number,

\[
Re_s = \frac{U \delta}{\nu} = \sqrt{2Re}, \quad (3.6)
\]

based on the Stokes boundary layer thickness, \( \delta = \sqrt{2\nu/\Omega} \), is a commonly used alternative to \( Re \). The ridge geometry is given by the slope angle, \( \beta \), and the slope length in \( x \)-direction, \( l \). The angle of the internal wave phase lines with the horizontal is given in a non-rotating environment by \( \theta = \tan^{-1} \sqrt{\Omega^2/(N^2_\infty - \Omega^2)} \). Thus, in addition to those listed in (3.5), there are three other independent non-dimensional parameters: the excursion parameter \( Ex = U_0/(l\Omega) \), the slope angle \( \beta \) and the slope criticality parameter, \( \epsilon = \tan(\beta)/\tan(\theta) \). The topographic Froude number, \( Fr = U_0/(N_\infty h) \), although not independent of the six non-dimensional parameters listed above, is also of interest.

The governing equations (3.4) are written in the following coordinates (see Gayen & Sarkar (2011b) for details) and transformed to the strong conservation law as described by Fletcher (1991):

\[
\xi = \xi(x, z), \eta = \eta(x, z), \zeta = \zeta(y), \quad (3.7)
\]

where, at the bottom topography, \( \xi \) points parallel to and across the ridge while \( \eta \) is normal to the ridge.

3.1.2 Numerical Method

Transfinite interpolation (TFI) is used to generate the boundary conforming grid and the transformed governing equations are solved using a mixed spectral/finite
difference algorithm as described by Gayen & Sarkar (2011b). Variable time stepping with a fixed CFL number 0.8 is used. Time steps are the order of $10^{-3}$.

Periodicity is imposed in the spanwise ($\zeta = \zeta(y)$) direction on velocity, density $\rho^*$ and pressure, $p^*$.

The bottom boundary, $\eta = 0$, has zero velocity and zero density gradient. Grids are designed to be orthogonal near the boundary so that

$$\frac{\partial \rho}{\partial \eta} = 0 \Rightarrow \frac{\partial \rho^*}{\partial \eta} = -\frac{\partial \rho_b}{\partial \eta} \quad \text{at} \quad \eta = 0 .$$

At the top of the domain, $\partial u/\partial \eta = 0$; $v, w = 0$, and $\rho^* = 0$. At the left and right sides, $\partial u/\partial \xi = 0$; $v, w = 0$, and $\rho^* = 0$. To match the boundary condition for the density deviation, $\rho^*$, between the left and the bottom (similarly, the right and the bottom) boundaries, $\partial \rho^*/\partial \eta$ is set to 0 at both the left and right ends of the bottom boundary, then it gradually reaches to the value given by (3.8) within the width of the sponge layer from both the ends and it is fixed at this value for the remaining extent of the bottom boundary. The pressure boundary conditions are $\partial p^*/\partial \eta = 0$ at the bottom and top wall and $p^* = 0$ at the left and right of computational domain.

Rayleigh damping or a ‘sponge’ layer is used at the left and right boundaries of the computational domain as shown in figure 3.1(a) so as to minimize spurious reflections from the artificial boundary into the ‘test’ section of the computational domain. The velocity and scalar fields are relaxed towards the background state in the sponge region by adding damping functions $-\sigma(\xi, \eta) [u_i(x, t) - 0]$ (i=2,3) and $-\sigma(\xi, \eta) [\rho^*(x, t) - 0]$ to the right hand side of the momentum and scalar equations, respectively. The value of $\sigma(\xi, \eta)$ is zero everywhere except in a region close to left and right boundary where it increases quadratically and reaches a maximum value corresponding to $\sigma(\xi, \eta) \Delta t \sim O(0.1)$ where $\Delta t$ is the time step of the simulation. Since $\Delta t \sim O(10^{-3})$, it follows that $\sigma(\xi, \eta) \sim O(100)$.

The dynamic eddy viscosity model (Zang et al., 1993; Vreman et al., 1997) is used for the subgrid scale (SGS) stress tensor, $\tau$, when the simulation is performed in LES mode. The SGS heat flux, $\lambda$, is obtained using a dynamic eddy diffusivity model, Armenio & Sarkar (2002). The expressions for the SGS models are described by Gayen & Sarkar (2011b).
Table 3.1: Parameters of the simulated cases. In cases A-C, the criticality parameter is changed in the laminar flow regime by changing the stratification. In cases 1-7, the tidal forcing velocity, $U_0$, is progressively increased in the critical slope case. In cases 2SUP-5SUP, the tidal forcing is increased in the supercritical regime. For all cases: $L_x = 40 \text{ m}$ ($30 \text{ m}$ for cases 6 and 7), $L_y = 0.5 \text{ m}$ ($0.25 \text{ m}$ for cases 6 and 7), $L_z = H = 3.28 \text{ m}$, $l = 1.9 \text{ m}$, $h = 0.328 \text{ m}$, $\beta = 15^\circ$, $\Omega = 1 \text{ s}^{-1}$, $\nu = 10^{-6} \text{ ms}^{-2}$, $Pr = 1$. Note that the simulations are at laboratory scale, for example, $\Omega$ and $N_\infty$ are larger than those in the ocean but $\Omega/N_\infty$ takes values realized in the ocean. Similarly, the topographic length scales are small but the excursion number and Froude number correspond to deep oceanic wave generation sites of interest.

<table>
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<th>Case</th>
<th>$Re_x$</th>
<th>$U_0 \text{ m s}^{-1}$</th>
<th>$N_\infty^2 \text{ s}^{-2}$</th>
<th>$\epsilon$</th>
<th>$Ex$</th>
<th>$Fr$</th>
<th>$\theta ^\circ$</th>
<th>$N_x$</th>
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<tr>
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<tr>
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<td>1.0</td>
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<tr>
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<td>705</td>
<td>128</td>
<td>385</td>
<td>DNS, turbulent</td>
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### 3.1.3 Selection of Simulated Cases

Table 3.1 gives important parameters of the simulations. Cases $A - C$ are simulations performed at a fixed value of $Re_x$ in the laminar flow regime, and the buoyancy frequency ($N_\infty$) is varied to study the effect of the criticality parameter ($\epsilon$). Cases $A$, $B$ and $C$ correspond to sub-critical, critical and super-critical flow, respectively, with results to be compared to linear theory later. A different series of simulations, cases $1 - 7$, are performed to assess the effect of increasing forcing amplitude $U_0$ and therefore $Re_x$. The criticality parameter shows that cases $1$ to $7$ are near critical ($\epsilon \sim 1$) and have excursion number, $Ex \ll 1$, except cases $6, 7$ where $Ex$ is $O(0.1)$. In these cases, the buoyancy frequency is fixed and the barotropic free stream velocity $U_0$ is varied to study the effect of $Re_x$. Cases 4 and 5 are turbulent flow DNS while cases 6 & 7 correspond to a resolved-LES model with a dynamic eddy viscosity model. In order to assess the effect of forcing in the supercritical regime, cases $2^{SUP}$, $4^{SUP}$ and $5^{SUP}$ are simulated with values of $Re_x$ that match cases $2$, $4$, and $5$, respectively.

The flow is statistically homogeneous in the spanwise direction and a $y$-average is used to compute the time dependent mean, $\langle A \rangle_y(x, z, t)$, as follows:

$$\langle A \rangle_y(x, z, t) = \frac{1}{L_y} \int_0^{L_y} A(x, y, z, t) \, dy$$  \hspace{1cm} (3.9)

The turbulent fluctuations are inferred via departures of instantaneous velocity, pressure and density from the spanwise average. Statistics of turbulent quantities are a function of $x$ and $z$, and are computed by spanwise averaging.

The computational domain lengths in the horizontal directions, $L_x$ and $L_y$, and the vertical domain length, $L_z$ which is equal to $H$, are specified in table 3.1. The spanwise domain length $L_y$ is chosen so as to accommodate the largest possible spanwise vortical structures. Cases $A - C$ and $1 - 3$ are laminar flow simulations while cases $4 - 7$ are turbulent flow simulations. The laminar flow simulations are over-resolved for consistency among grids; an order of magnitude lower spanwise resolution leads to less than 3 \% change in the wave flux. The turbulent flow simulations require high resolution. Case 5 is a DNS with the distance to the first grid point from the wall $z_1^+ = 2.5$ in terms of the viscous wall unit $\nu/u_\tau$ and minimum grid resolution in the streamwise direction $\Delta x_{\min}^+ = 21$. Here, $u_\tau$ is the cycle average of the instantaneous friction velocity ($= (\sqrt{2\tau_w})$) based on the wall shear stress ($\tau_w$) at the mid-slope. The spanwise direction has spectral accuracy with uniform grid spacing in physical space,
$\Delta y^+ = 16$. Cases 6 – 7 correspond to a resolved-LES mode with a dynamic eddy-viscosity model.

3.2 Results in the Laminar Flow Regime

3.2.1 Effect of Criticality on the Internal Wave Structure at $Re_s = 30$

The criticality parameter, $\epsilon$, is an indicator of nonlinear response near the topography and determines the structure of the flow field such as formation of higher modes which combine to form an internal wave beam. For fixed geometry of the topography and forcing frequency, $\epsilon$ is varied by varying the level of stratification that determines the internal wave characteristic slope. At $Re_s = 30$, the flow remains laminar and two dimensional. Figure 3.2(a)-(c) shows the instantaneous streamwise velocity at $Re_s = 30$ in subcritical, critical and supercritical flow regimes, respectively. In subcritical flow, as shown in 3.2(a), internal waves originating near the topography radiate upward along the characteristic direction given by $\theta = \sin^{-1}(\Omega/N_\infty)$. The baroclinic response in subcritical flow is weaker compared to the barotropic forcing. In critical flow, as shown in 3.2(b), since the internal wave characteristic slope matches with the topographic slope, resonant interaction of the internal wave with the topography generates higher spatial modes which combine to form an intensified internal wave beam. In supercritical flow, as shown in 3.2(c), resonant beam intensification occurs in a small region near the crest of the topography where the topographic slope is equal to the internal wave characteristic slope, and the beams are directed both upwards and downwards. The baroclinic response is significantly stronger in both critical and supercritical flow.

3.2.2 Frequency spectra in laminar flow cases

Fast Fourier Transform (FFT) of the time series data of the stream-wise velocity component is performed at two different points, $A$ and $B$, shown in figure 3.2(b). The frequency spectra at locations $A$ and $B$ are shown in figure 3.3(a) for subcritical flow and in figure 3.3(b) for critical flow at $Re_s = 30$. In figure 3.3(a), at both locations $A$ and $B$, the frequency spectrum has a sharp peak at the fundamental frequency. In figure 3.3(b), in addition to the peak at the fundamental frequency, the
Figure 3.2: Streamwise velocity at $Re_s = 30$ in different flow regimes shown at time $t = 51.75 \, s$, phase of the barotropic velocity, $\phi \approx \pi/2$. Frequency spectra at point A (midslope and in the boundary layer) and B (above point A and in the wave beam) will be shown later. Topography shown in white.
Figure 3.3: Frequency spectra for (a) subcritical flow and (b) critical flow $Re_s = 30$. Point A is located in the boundary layer at midslope and point B is located vertically above A and in the beam as shown in figure 3.2(b).

Figure 3.4: (a) Example profile of the baroclinic vertical velocity profile in the laminar case, $Re_s = 30$. Profile shown at $x = 3 m$ and time $= 51.75 s$, phase of the barotropic velocity, $\phi \approx \pi/2$. (b) Modal structure of the baroclinic field. Spectra has discrete peaks at higher harmonics that are below the buoyancy frequency ($N_\infty/\Omega = 3.9$). More than 99% of the energy is contained at the fundamental frequency in cases with laminar flow. Hence, the modal decomposition (see Appendix B) is performed solely for the velocity field corresponding to the fundamental frequency.

3.2.3 Modal structure in laminar flow cases

Figure 3.4 (a) shows profiles of the baroclinic component of vertical velocity field, computed using the procedure described in appendix A, at location $x = R = 3m$, indicated in figure 3.2. In figure 3.4 (a), the subcritical flow profile is dominated by modes 3-5 whereas critical and supercritical flow profiles suggest superposition of
serveral additional normal modes to represent the intensified beam. The positive narrow peak of \( w(z) \) in critical flow at \( z \approx 1.2 \, m \) corresponds to the intensified beam at fundamental frequency followed by a broad region of negative peak. The local peaks in supercritical flow at \( z \approx 0.75 \, m \) and \( z \approx 0.02 \, m \) correspond to the primary generated beam with group velocity directed upwards and a secondary beam formed by bottom reflection of the downward generated beam, respectively.

Modal analysis is performed using the time series of the baroclinic component of the vertical velocity profiles at the location \( x = R = 3 \, m \) (see appendix B) and the corresponding modal structure is shown in figure 3.4 (b). The supercritical flow has a relatively wide range of active modes. The sinusoidal structure superimposed on the decaying profile of the modal distribution in supercritical flow is due to the presence of two beams which have vertical components of the group velocity in opposite directions at the generation region. The number of modes required to contain a given percentage of the total energy increases from the subcritical to supercritical flow: in subcritical flow, the first 9 modes are sufficient to represent 90% of the total energy whereas critical and supercritical flows require the first 30 and 50 modes, respectively.

### 3.2.4 Radiative conversion

The radiative conversion factor, \( M \) in Equation (B.3), is the value of the radiated wave energy flux integrated over the boundary of a domain enclosing the topography and normalized with \( \frac{2}{\pi} \rho_0 U^2 h^2 \sqrt{(N^2 - \Omega^2)} \). The quantity \( M \) is computed at \( Re_s = 30 \) for three different values of \( \epsilon \) and presented in table 3.2 along with the analytical estimate given by Pétrélis et al. (2006) using inviscid linear theory. In both theory and simulations, the conversion to radiated wave flux increases from sub-critical to super-critical flow. The simulations agree well with the theory in sub-critical and super-critical cases. However, the linear theory underestimates the radiated flux when \( \epsilon \sim O(1) \) in agreement with Khatiwala (2003). Note that, as shown in figure 3.1(b), the critical length on the slope of the smoothed triangular ridge is shorter than the triangular ridge used in the linear analysis by Pétrélis et al. (2006). This is due to the smoothing performed at top and bottom of the ridge to avoid numerical instabilities during the simulation. The value of \( M \) in critical flow DNS would be even higher than 0.67 if the level of smoothing was decreased.
Table 3.2: Conversion factor (M) at $Re_s = 30$. *M increases from 0.59 to 0.77 abruptly as $\epsilon$ changes from 1 to 1.05. **The value is not quoted and the error band is associated with the digitization of figure 5(a) in Pétrélis et al. (2006).

<table>
<thead>
<tr>
<th></th>
<th>M</th>
<th>sub-critical</th>
<th>critical</th>
<th>super-critical</th>
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<td>0.67 ± 0.01</td>
<td>0.92 ± 0.02</td>
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</tr>
<tr>
<td>Pétrélis et al. (2006)</td>
<td>0.56</td>
<td>0.59*</td>
<td>0.95 ± 0.02**</td>
<td></td>
</tr>
</tbody>
</table>

### 3.3 Effect of forcing on the internal wave field in the critical slope case

The decrease of radiative conversion factor, $M$, i.e., the normalized wave flux, with increasing forcing is a major result of this paper. As shown in figure 3.5 (a), with increasing forcing, thereby $Re_s$ and $Ex$, $M$ (the line with circles) in the critical slope cases 1-7 decreases in the laminar regime, has a sharper decrease when the flow transitions to turbulence at $Re_s \simeq 100$, and eventually exhibits a gradual drop at the higher values of forcing. The value of $M$ in case 7 with $Re_s = 400$, $Ex = 0.168$ is reduced by 25% of the peak to a value that is even lower than that in the corresponding subcritical case. The energy conversion factor, $C$, decreases and will be discussed in section 3.3.1 on the baroclinic energy budget. In the present series of simulations, when forcing is increased, extensive patches of three-dimensional turbulence fluctuations are seen at $Re_s \simeq 100$ and above. In figure 3.5 (b), intensification of the along slope velocity is shown as a function of $Re_s$ and $Ex$. The intensification drops owing to increased drag and mixing of momentum. It is also found (not shown) that the intensification during downslope flow is greater than that during upslope flow. It is of interest to compare the normalized wave flux in the case with $Re_s = 177$ in the present flow over a ridge with that for a single-slope topography with similar length and with $Re_s = 177$ given by Gayen & Sarkar (2011b). The present case with 2 slopes and, consequently, a beam on each side of the topography, has approximately twice the normalized flux, $M = 0.28$, of the single-sloped topography.

To compute $M$, the vertically integrated values of cycle-averaged outgoing energy flux, $p_{bc}u_{bc}$, were computed at $x = \pm 3$ m, summed, and then normalized. To further understand the decrease of $M$, the vertical profiles at $x = 3$ m of the amplitude of $p_{bc}$, computed as the root mean square of $p_{bc}(z,t)$, and the amplitude of $u_{bc}$, computed similarly, are plotted in figure 3.6 (a)-(b). The peak velocity amplitude
Figure 3.5: The effect of increasing barotropic forcing on: (a) the normalized radiated baroclinic flux and the energy conversion in critical (bottom two curves) and supercritical (top two curves) cases, and (b) the intensification of near-bottom velocity in critical cases.

Figure 3.6: Vertical profiles of the normalized values of: (a) baroclinic velocity amplitude, (b) baroclinic pressure amplitude, and (c) the product of pressure and velocity amplitudes. Profiles shown at $x = -3 \text{ m}$, a location away from the topography.

occurs in the internal wave beam and drops with increasing $Re_s$ when $Re_s$ exceeds 75. This decrease in velocity is the primary reason for the drop of peak wave energy flux seen in figure 3.6 (c).

Figure 3.7 (a) shows the effect of forcing, denoted by $Re_s$, on frequency spectra at locations $A$ and $B$, shown earlier in figure 3.2. The contribution of the higher harmonics relative to the fundamental increases with increasing forcing. In the case with $Re_s = 10$ without turbulence, the continuous spectrum at frequencies beyond $N_\infty$ is identical between A and B. The higher $Re_s$ cases show significantly higher energy beyond the buoyancy frequency at point A in the boundary layer and, although at a somewhat lower level, also at point B in the beam. In these higher $Re_s$ cases, the
energy at $\omega > N_\infty$ resides both at discrete peaks corresponding to evanescent internal waves and a broad-band continuous component that corresponds to turbulence. Most of the energy is carried by the fundamental frequency. In general, away from the generation region, more than 90% of the energy is carried by the fundamental frequency. Hence the modal analysis is performed for the fundamental frequency even for the turbulent flow simulations. The modal amplitudes are shown in figure 3.7 (b). When forcing ($Re_s, Ex$) increases, the peak modal amplitude decreases as can be anticipated from the systematic decrease of the peak of velocity profile with increasing forcing that was seen in figure 3.6(a). Interestingly, the amplitude of the higher modes decrease substantially in the simulations with $Re_s \geq 100$. We will show later that the conversion to turbulence (measured by turbulent production) also increases substantially when $Re_s \geq 100$.

### 3.3.1 Baroclinic Energy Budget

In the case of inviscid theory, the internal wave flux is equal to the conversion from the barotropic to the baroclinic wave field. DNS/LES allows the separation of effects of forcing on conversion from those on wave flux. The velocity is split into a mean field (computed by spanwise averaging) and a three-dimensional fluctuation field, e.g., $u(x, y, z, t) = \langle u \rangle (x, z, t) + u'(x, y, z, t)$. The mean field is then partitioned into a barotropic and a baroclinic component as follows and as further discussed in

---

**Figure 3.7**: The effect of forcing in the critical slope case: (a) Frequency spectra. Point A in the boundary layer shown in red and point B in the beam shown in green, and (b) Modal distribution at $x = 3 \, m$. 
Appendix A,

\[ \langle u \rangle = U + u_{bc}, \quad \langle w \rangle = W + w_{bc}, \quad \langle p^* \rangle = P^* + p_{bc}, \]

where \( p^* \) is the deviation from hydrostatic pressure. \( U, W, P^* \) are the barotropic components and \( u_{bc}, w_{bc}, p_{bc} \) are baroclinic components defined such that \( U, P^* \) are the depth average of \( \langle u \rangle, \langle p^* \rangle \) respectively, and \( W(z) = -\frac{\partial}{\partial x} \langle [z - h(x)] U \rangle \). The buoyancy is defined as \( b \equiv -g < \rho^* > / \rho_0 \) where \( \rho^* \) is the deviation from background density.

The equation for the baroclinic energy (see Carter et al. (2008) and Kang & Fringer (2011)) with advective and diffusive fluxes of wave energy neglected is

\[
\frac{\partial}{\partial t}(KE + PE) + \nabla \cdot \mathbf{F} = \bar{C} - \varepsilon_{bc} - \bar{P} \tag{3.10}
\]

where,

\[
KE = \frac{1}{2} (u_{bc}^2 + v_{bc}^2 + w_{bc}^2), \quad PE = \frac{1}{2} N^{-2} b^2, \quad \mathbf{F} = p_{bc} \mathbf{u}_{bc}, \quad C = \frac{\partial p^*}{\partial z} W,
\]

\[
\varepsilon_{bc} = \nu \frac{\partial (u_{bc})_i}{\partial x_j} \frac{\partial (u_{bc})_j}{\partial x_i}, \quad P \equiv -\langle u_i' u_j' \rangle_y \langle \tau_{ij} \rangle_y - \langle \tau_{ij} \rangle_y \langle S_{ij} \rangle_y .
\]

The overbar represents depth integration. \( \bar{C} \) represents conversion from the barotropic to baroclinic wave field, \( \varepsilon_{bc} \) represents viscous dissipation of the baroclinic energy, and \( \mathbf{F} \) represents the linear wave energy flux. The term, \( -\bar{P} \), is not present in Kang & Fringer (2011) but is required here to account for turbulence. In the TKE equation, \( P \) appears with a positive sign on the right hand side and is commonly referred to as turbulent production since \( P \) is generally (but not always) a source for TKE. In the absence of a density field and at \( Re_s = 177 \), the flow is laminar; therefore all the turbulence in the present case with density stratification is associated to the baroclinic field. Therefore, in the present context, \( P \) can be interpreted as local conversion from the internal tide to turbulence. \( \tau_{ij} \) is the SGS stress tensor discussed in section 3.1.2. The bottom drag term that appears in Kang & Fringer (2011) is not present in Eq. (4.2) since viscous effects at the bottom are resolved in the present study without recourse to any explicit drag parameterization in the momentum conservation equation.

At steady state, the conversion from barotropic to baroclinic tide, \( \bar{C} \), is balanced by the radiative conversion, \( \nabla \cdot \mathbf{F} \), in the linear inviscid approximation. In general, there are two additional terms, the viscous dissipation, \( \varepsilon_{bc} \), of the mean
Table 3.3: Baroclinic energy budget, integrated over an area of the computational domain from $x = -3 \, m$ to $x = +3 \, m$ and averaged over three tidal cycles, in the critical slope cases. All terms are normalized with $(\pi/4)\rho_0 U^2 h^2 \sqrt{(N^2 - \Omega^2)}$.

<table>
<thead>
<tr>
<th>Case</th>
<th>$Re_a$</th>
<th>Ex</th>
<th>Tendency</th>
<th>Conversion to waves</th>
<th>Wave flux</th>
<th>Baroclinic dissipation</th>
<th>Turbulent production</th>
<th>Residual</th>
</tr>
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<tr>
<td>1</td>
<td>10</td>
<td>0.004</td>
<td>0.009</td>
<td>0.695</td>
<td>0.676</td>
<td>0.030</td>
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<td>-0.020</td>
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<tr>
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<td>0.011</td>
<td>0.702</td>
<td>0.671</td>
<td>0.027</td>
<td>-</td>
<td>-0.007</td>
</tr>
<tr>
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<td>75</td>
<td>0.031</td>
<td>-0.005</td>
<td>0.669</td>
<td>0.658</td>
<td>0.020</td>
<td>-</td>
<td>-0.004</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
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<td>-0.008</td>
<td>0.629</td>
<td>0.622</td>
<td>0.018</td>
<td>0.022</td>
<td>-0.025</td>
</tr>
<tr>
<td>5</td>
<td>177</td>
<td>0.074</td>
<td>-0.008</td>
<td>0.572</td>
<td>0.551</td>
<td>0.013</td>
<td>0.025</td>
<td>-0.009</td>
</tr>
<tr>
<td>6</td>
<td>300</td>
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<td>-0.001</td>
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<td>0.515</td>
<td>0.011</td>
<td>0.061</td>
<td>-0.028</td>
</tr>
<tr>
<td>7</td>
<td>400</td>
<td>0.168</td>
<td>-0.009</td>
<td>0.547</td>
<td>0.504</td>
<td>0.010</td>
<td>0.051</td>
<td>-0.009</td>
</tr>
<tr>
<td>2SUP</td>
<td>30</td>
<td>0.013</td>
<td>-0.008</td>
<td>0.951</td>
<td>0.923</td>
<td>0.017</td>
<td>-</td>
<td>0.019</td>
</tr>
<tr>
<td>4SUP</td>
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<td>-0.005</td>
<td>0.898</td>
<td>0.895</td>
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<td>0.843</td>
<td>0.824</td>
<td>0.007</td>
<td>0.028</td>
<td>-0.008</td>
</tr>
</tbody>
</table>

Field and the conversion to turbulence, $\bar{P}$. The radiative conversion was shown earlier to decrease with increasing forcing. We now assess the behavior of the other terms in the baroclinic energy balance. Each term in Eq. (4.2) is integrated over $-3 < x < 3 \, m$ in the horizontal direction, averaged over three tidal cycles, normalized by $(\pi/4)\rho_0 U^2 h^2 \sqrt{(N^2 - \Omega^2)}$, and shown in table 3.3. The residual, computed as the sum of all terms in Eq. (4.2) with the tendency and flux terms taken to the right hand side, is generally small and ranges from 1 to 5% of the conversion.

The primary inference from Table 3.3 is that the initial drop of the wave flux from the laminar case, $Re_a = 30$, is primarily associated with a reduction in conversion. An additional contribution to the decrease in wave flux is the increasing value of conversion to turbulence, $\bar{P}$, which reaches approximately 10% of the energy conversion, $\bar{C}$, at higher values of $Re_a$. The viscous dissipation, $\bar{\varepsilon}_{bc}$, of the mean baroclinic component decreases substantially in the turbulent flow cases, owing to reduced baroclinic shear.
3.4 Turbulence at the Ridge in Case 5 with Critical Slope

In this section, turbulence near the ridge in case 5 is characterized by computing the statistics at three different locations: the center of the left slope of the ridge \( x = -0.77 \) m, a location where the left slope experiences a change in its slope \( x = -0.4 \) m and near the center of the ridge \( x = -0.1 \) m, represented by the dashed vertical lines in red, white and black in 3.8(a), respectively. Since the ridge is symmetric about \( x = 0 \), the dynamics on the right slope of the ridge are similar to those on the left but with a phase difference of approximately 180° and hence are not discussed.

The governing equation for turbulent kinetic energy, \( K = 1/2\langle u'\bar{u}'\rangle_y \) also denoted by TKE, is given below:

\[
\frac{\partial K}{\partial t} + \langle u \rangle y \frac{\partial K}{\partial x} + \langle w \rangle z \frac{\partial K}{\partial z} = P - \varepsilon + B - \frac{\partial T_x}{\partial x} - \frac{\partial T_z}{\partial z} \tag{3.11}
\]

Here, \( T \) is the transport of TKE including pressure transport, turbulent transport, viscous transport and subgrid scale (SGS) transport,

\[
T_x \equiv \frac{1}{\rho_0} \langle p' \bar{u}' \rangle_y + \frac{1}{2} \langle u_i' u_i' \rangle_y - \nu \frac{\partial K}{\partial x} + \langle \tau_{11} \rangle_y.
\]

\[
T_z \equiv \frac{1}{\rho_0} \langle p' \bar{w}' \rangle_y + \frac{1}{2} \langle u_i' w_i' \rangle_y - \nu \frac{\partial K}{\partial z} + \langle \tau_{33} \rangle_y.
\]

\( P \) is the production term, defined in equation (4.2). The turbulent dissipation rate, \( \varepsilon \), is defined as the sum of the resolved and SGS components:

\[
\varepsilon \equiv \nu \left[ \frac{\partial \bar{u}_i'}{\partial x_j} \frac{\partial \bar{u}_i'}{\partial x_j} \right] - \langle \tau_{ij} S_{ij} \rangle_y.
\]

Finally, \( B \) is the buoyancy flux defined as

\[
B \equiv -\frac{g}{\rho_0} \langle \bar{\rho}' \bar{w}' \rangle_y.
\]

Figures 3.8(a)-(d) display snapshots of TKE, isopycnals and velocity profiles near the topography at different times over a tidal cycle. Turbulence across the ridge is inhomogeneous in both the streamwise and vertical directions. TKE patches are spread across the topography with varying magnitude and thickness. Close to the center of the ridge, TKE shows structures of large vertical extent. The phasing of turbulence relative to the near-bottom velocity was discussed by Gayen & Sarkar.
(2011a) who assumed an internal wave beam profile with streamwise homogeneity that allowed LES of a thick beam with width of approximately 60 m and $M^2$ forcing of 12.4hrs. There is an important similarity (turbulence generation by convective instability during flow reversal from down to up) and an important difference (large shear production) in the present problem. The difference is because, by construction, the streamwise periodic boundary conditions employed by Gayen & Sarkar (2011a) to study the evolution of the stratified bottom jet that forms during critical slope generation did not allow the inhomogeneous wave propagation found here. Furthermore, there are substantial differences near the top of the ridge because of the crossing of the two beams at opposite sides, a phenomenon not present in either the turbulent beam simulation of Gayen & Sarkar (2011a) or the simulation of generation at a slope by Gayen & Sarkar (2011b).
Figure 3.8: $\log_{10}(TKE)$ and isopycnals near the topography in case 5 are shown in (a)-(d) at time 42.5 s, 44 s, 44.8 s, and 46.3 s. These times correspond to phase -90 (peak downslope velocity), 0, 90, 180 of the streamwise velocity at $x = -0.77$ m in the beam center, respectively, not with respect to the barotropic velocity. The dashed vertical lines in red, white and black in (a) indicates $x = -0.77$ m (midslope), $-0.4$ m and $-0.1$ m, respectively. Forcing time period is $T = 2\pi$ s and ridge height is $h = 0.328$ m.
The times shown in Figure 3.8 are chosen on the basis of near-bottom velocity at \( x = -0.77 \text{ m} \), a mid-location on the left slope of the ridge: part (a) corresponds to peak downward velocity (taken to be phase -90), part (b) to zero velocity (phase 0), part (c) to peak upward velocity (phase 90), and part (d) to zero velocity (phase 180). Figure 3.8 (a) shows downward flow over the entire left slope and a thin turbulent layer of thickness approximately equal to the beam width. Figure 3.8 (b) shows a thicker turbulence patch on the left slope. At this time and at midslope, \( x = -0.77 \text{ m} \), the near-bottom velocity is almost zero (taken to be phase of zero) and the near-bottom isopycnals show steepening to almost vertical. A convective instability associated with wave breaking is seen. The turbulent patch and the density front propagate upward as a bore. The vertical extent of the turbulence patch at phase 90 in figure 3.8(c) is largest at \( x \approx -0.4 \text{ m} \) and spans a region of reduced stratification as can be seen by the increased distance between isopycnals. At phase 180, shown in figure 3.8(d), there is turbulence at the top of the ridge.

The location of turbulence relative to the near-bottom velocity is of interest. The initiation of turbulence on the critical slope, shown in figure 3.8(b), corresponds to flow reversal from down to up, similar to (Gayen & Sarkar, 2011a), when the local near-bottom velocity passes through zero and there is a convective overturn. The convective overturns that are seen at this phase in the present critical slope case followed by bore like features are similar to those noted by Legg & Klymak (2008) in their study of a strongly supercritical (\( \epsilon = 4 \)) ridge. However, an important difference is that, unlike Legg & Klymak (2008) who find overturns in the lee (rearward with respect to the flow on the ridge) of the topography, we find overturns both on the windward side, parts (a) and (c) of figure 3.8, and on the leeward side, figure 3.8 (b) and (d).

The gradient Richardson number, \( \text{Ri}_g(x, z) = \frac{N^2(x, z)}{S^2(x, z)} \) where \( N \) is the mean buoyancy frequency and \( S \) is the mean shear, is an indicator of unstable regions. Figure 3.9(a) is an example corresponding to the time instant corresponding to Figure 3.8 (b). Convectively unstable regions \( (\text{Ri}_g < 0) \) and regions susceptible to shear instability \( (0 < \text{Ri}_g < 0.25) \) can be seen clearly. Consequently, the evolution of terms in the TKE balance to be discussed below show that both positive buoyancy flux and positive shear production lead to TKE generation. In figure 3.9(b), the density perturbations in the spanwise plane at the ridge center \( (x = 0) \) are shown at time 25.2 s corresponding to phase 0. It shows three dimensionality of the flow in
Figure 3.9: (a) $Ri_g$ and isopycnals near the topography in case 5 are shown at time 44 s corresponding to phase 0 of the streamwise velocity at $x = -0.77 \, \text{m}$ in the beam center respectively. (b) Density perturbations in the spanwise vertical plane at the center of the ridge ($x = 0 \, \text{m}$) shown at time 25.2 s corresponding to phase 0 of the streamwise velocity for case 5. Forcing time period is $T = 2\pi \, \text{s}$ and ridge height is $h = 0.328 \, \text{m}$.

case 5.

To understand, the phase dependence, the evolution of vertical profiles of turbulence statistics at the midslope $x = -0.77 \, \text{m}$ is shown in figure 3.10(b)-(e) over two tidal cycles after the simulation has reached a quasi-steady state. Note that the simulations are performed for 9 tidal cycles and the quasi-steady state is observed after 5 cycles. In figure 3.10(a) the evolution of the mean velocity and the density profiles at different $z$ locations ($z = 0.2 \, \text{m}$ to $z = 0.28 \, \text{m}$) are shown. Upper to lower $z$ locations are represented by lighter to darker lines, respectively. The time series data leads to the following results: (1) there is significant asymmetry between upslope and downslope flow, e.g., the upslope flow shows a rapid acceleration (corresponding to a upslope bore, also seen previously by Gayen & Sarkar (2011b)) and it occupies a shorter duration of the cycle; (2) there is significant asymmetry between acceleration and deceleration stages; (3) the density lags behind the velocity by a substantial amount, as much as 90° phase during some stages of the cycle; and (4) the stratification is significantly reduced during some portions of the cycle. The evolution of statistics shown begins with a phase corresponding to the peak downslope velocity.

In figure 3.10(b), four distinct phases can be observed in the periodic evolution of TKE, each indicated by a black circle in that figure and by a corresponding red circle in the streamwise velocity evolution of figure 3.10(a). The first black circle at
Figure 3.10: Flow and turbulence in case 5 at $x = -0.77$ m, the center of the left slope of ridge: Evolution of (a) streamwise velocity (solid lines) and density (dashed lines) at various locations, $z = 0.2$ m (the darkest) to $z = 0.28$ m (the lightest), (b) TKE, $m^2/s^2$, (c) production, $m^2/s^3$, (d) buoyancy flux, $m^2/s^3$, and (e) dissipation, $m^2/s^3$. The solid lines in black in (b) - (e) represent isopycnals. Four filled circles shown at time $42.5$ s, $44$ s, $44.8$ s and $46.3$ s are used to illustrate the phase dependence of the statistics. Forcing time period is $T = 2\pi$ s and ridge height is $h = 0.328$ m.
$t = 42.5\, s$ corresponds to the phase of maximum downslope velocity during which TKE is prominent in a small region slightly above the bottom wall, around $z = 0.2\, m$. At this phase, turbulence is shear driven as reflected by the the presence of significant production in Figure 3.10(c) and absence of positive buoyancy flux. The second circle at $t = 44\, s$ corresponds to down-to-upslope flow reversal during which a large TKE structure extending from the wall at $z = 0.2\, m$ to $z = 0.27\, m$ is observed. Notably, the buoyancy flux at this time, corresponding to the flow reversal, is positive. The reason is that continued downslope flow replaces the heavier fluid in the jet core region with lighter fluid from the top. The corresponding density profile has a positive gradient in the region above the peak velocity; an unstable configuration that results in turbulent overturns. Density inversions can also be seen in figure 3.10(a) where the heavier density lines shown in dark cross the lighter density ones. The third circle at $t = 44.8\, s$ corresponds to maximum upslope velocity during which TKE is prominent in a thin region with vertical extent of $z \approx 0.01\, m$, attached to the bottom wall. The buoyancy flux is negative and the shear production is positive signifying that the TKE is largely due to shear and not convective instability. The fourth circle at $t = 46.3\, s$ corresponds to up-to-downslope flow reversal during which TKE is somewhat elevated. The turbulent dissipation shown in figure 3.10(e) is similar to the TKE and shows patches associated with wave breaking that originate away from the boundary as well as boundary patches associated with boundary layer shear.

We now describe the behaviour at $x = -0.4\, m$, a location where there is a change in the slope angle from the critical value. Figure 3.11(a) shows the evolution of streamwise velocity and density at various locations, $z = 0.29\, m$ to $z = 0.39\, m$. There are some differences with respect to the previously shown mid-slope location since $x = -0.4\, m$ is closer to the center of the ridge and the outer portion of the profiles is influenced by the beam from the other side of the ridge leading to substantial distortion of the isopycnals. For instance, the first circle in figure 3.11(b) shows TKE at a larger distance away from the bottom relative to the corresponding phase at midslope. The cause is a density overturn that appears at $z \approx 0.45\, m$ due to the interaction with the outer beam and, correspondingly, a positive value of buoyancy flux. The TKE corresponding to the second circle at $t = 44.8\, s$ is significant up to larger heights (twice as much compared to that at $x = -0.77\, m$). At this time, the positive buoyancy flux is more detached from the boundary relative to similar phase at $x = -0.77\, m$. The outer beam contributes to development of a wider density overturn
Figure 3.11: Flow and turbulence in case 5 at \( x = -0.4 \, m \), end of the left slope of ridge: (a) Evolution of streamwise velocity (solid lines) and density (dashed lines) at various locations, \( z = 0.29 \, m \) (the darkest) to \( z = 0.39 \, m \) (the lightest) and (b) TKE, \( m^2/s^2 \). The solid lines in black represent isopycnals. Four filled circles shown at time 43 s, 44.8 s, 45.6 s and 47 s are used to illustrate the phase dependence of the statistics. Forcing time period is \( T = 2\pi \, s \) and ridge height is \( h = 0.328 \, m \).

and positive density deviation. The other two phases, one with peak bottom shear and other with restratification, indicated by third and fourth circles in figure 3.11(b) are qualitatively similar to those in figure 3.10(b). The evolution of the dissipation is similar to the TKE and is not shown.

We now turn to the evolution of turbulence near the top of the ridge, \( x = -0.1 \, m \) where the behavior is found to be qualitatively different from that at mid-slope as can be seen by comparing figure 3.12 with figure 3.10. The reason is that leftward and rightward beams originating from the two slopes cross, leading to a significant interaction at \( x = -0.1 \, m \). At the top of the ridge, there are two events of TKE with large vertical extent in a cycle in contrast to one such event per cycle at mid-slope. Each of the four TKE events in figure 3.12(b), that shows the evolution over two tidal cycles, originates away from the boundary in a region where the isopycnals (black lines in the figure) show a lower stratification than the background value. The region of lower stratification moves downward with increasing time in figure 3.12(b) and so does the TKE. The downward propagating phase is consistent with the upward propagating energy in the internal wave field.
Figure 3.12: Flow and turbulence in case 5 at $x = -0.1\ m$, close to the center of the ridge: Evolution of (a) streamwise velocity (solid lines) and density (dashed lines) at various locations, $z = 0.33\ m$ (the darkest) to $z = 0.43\ m$ (the lightest) and (b) TKE, $m^2/s^2$. The solid lines in black represent isopycnals. Four filled circles shown at time $43.5\ s$, $44.7\ s$, $46.1\ s$ and $47.6\ s$ are used to illustrate the phase dependence of the statistics. Forcing time period is $T = 2\pi\ s$ and ridge height is $h = 0.328\ m$.

Figure 3.13: Shaded regions A, B and C correspond to a region on the critical slope ($-1 < x < -0.6\ m$), a region where the slope changes from critical ($-0.5 < x < -0.3\ m$), and a region at the top of the ridge ($-0.25 < x < 0\ m$), respectively. These regions are used to analyze the evolution of the TKE budget terms shown in figures 3.14 and 3.16, and the cycle averaged TKE budget terms shown in table 3.4.
Figure 3.14: Evolution of TKE budget, integrated over three different areas shown in figure 3.13: (a) Area A, (b) Area B, and (c) Area C, as a function of tidal cycle. All terms are normalized by \((\pi/4)\rho_0 U^2 h^2 \sqrt{(N^2 - \Omega^2)}\). Note that \(-\varepsilon\) (dissipation) and \(-\partial K/\partial t\) (tendency) are plotted. The barotropic velocity is \(U_0 \sin \left(2\pi \frac{t}{\tau} \right)\).
The TKE at given spatial regions varies over a cycle. Figure 3.14 shows the cycle variation of terms in the area-integrated TKE budget at three different regions: A, B and C, shown in figure 3.13. Figure 3.14 (a), corresponding to the midslope region A, shows that, during $41 < t < 44.5$ s that spans peak downslope flow to flow reversal, primarily shear production as well as buoyancy flux and advection lead to accumulation ($-\frac{\partial K}{\partial t}$ is negative in the figure) of TKE and some dissipation. During the rest of the cycle, TKE decreases in time primarily by advection out of the region and also because of dissipation and mixing indicated by negative buoyancy flux. The behavior in the adjacent region B, where the slope angle decreases from critical, is shown in figure 3.14(b). Early in the cycle, TKE accumulates as a result of advection from the critical slope region. Later, during $43.5 < t < 45$ s, there is accumulation of TKE owing to shear production, buoyancy flux and advection. At $t = 45$ s, TKE starts to decrease despite positive production and buoyancy because of the advection term, which is a sink of TKE during $45 < t < 46.5$ s. During the same time period of $45 < t < 46.5$ s, advection acts as as source for TKE in the adjacent region at the top of the ridge as shown in Figure 3.14 (c). At the top of the ridge, region C, the primary balance is between advection and tendency terms. Temporal integration of the terms plotted in Figure 3.14, shown in table 3.4, lead to the following result for cycle-averaged values. Turbulent production acts as the main source of TKE and it acts primarily at the critical slope and the top end of the critical slope. At the top of the ridge, advection from depth acts as the primary source of TKE. Cycle-averaged turbulent dissipation does not vary significantly among the three regions.

3.5 Effect of Forcing in the Cases of Subcritical and Supercritical Slopes

In simulations performed up to $Re_s = 177$, there was little turbulence in the subcritical case in contrast to the supercritical case. The normalized values of radiative and energy conversion exhibit little change.

To illustrate the characteristics of turbulence and the effect of forcing on the energy conversion in supercritical flow, three cases were investigated: $Re_s = 30, 100$ and $177$, indicated by 2SUP, 4SUP and 5SUP in table 3.1, respectively. At $Re_s = 30$, the flow is laminar and, for $Re_s = 100$ and above, the flow is turbulent.

Similar to the critical case, the normalized energy conversion in the supercriti-
Figure 3.15: Supercritical case with $Re_s = 177$. $\log_{10}(TKE)$ and isopycnals near the topography in case 5SUP are shown in (a)-(b) at time 42.5 s, and 44 s, same as (a) and (b) of the corresponding figure 3.8 of the critical case. The dashed vertical lines in red, white and black in (a) indicates $x = -0.77$ m (midslope), $-0.4$ m and $-0.1$ m, respectively.

Critical cases decrease at higher forcing levels. The baroclinic energy budget for supercritical cases is included in table 3.3. The wave energy conversion and the wave radiative conversion decrease by 12.5% and 10.5%, respectively, from $Re_s = 30$ to $Re_s = 177$ as the flow becomes turbulent. The baroclinic dissipation also decreases from $Re_s = 30$ through $Re_s = 177$, similar to the critical case. The turbulent production and dissipation increase with $Re_s$, similar to the critical case. Overall, the percentage decrease in conversion from $Re_s = 30$ to 177 in supercritical slope is smaller (12.5%) compared to the critical slope (19%). This is due to the smaller area over which turbulence is significant in the supercritical case relative to the critical case.

We will discuss turbulence in case 5SUP and compare with the critical case 5 at the same value of $Re_s = 177$. Figure 3.15(a)-(b) display snapshots of TKE, isopycnals and velocity profiles near the topography corresponding to case 5SUP at the same phases as figure 3.8(a)-(b), respectively, for case 5. TKE patches are clustered in a small region near the top of the ridge where the internal wave beam generation occurs. In contrast to case 5, the constant slope region in case 5SUP is not critical and, therefore, does not have the significant level of TKE associated with breaking waves.

Figure 3.15(a) shows that TKE is present at both windward and leeward sides of the topography. Figure 3.15(b), corresponding to later phase, shows elevated levels of TKE primarily on the leeward side in a region around $x = -0.4$ m where the slope is near critical, and secondarily in the downward beam on the windward side.
Figure 3.16: Supercritical case with $Re_s = 177$. Evolution of TKE budget, integrated over three different areas shown in figure 3.13: (a) Area A, (b) Area B, and (c) Area C, as a function of tidal cycle. All terms are normalized by $(\pi/4) \rho_0 U^2 h^2 \sqrt{(N^2 - \Omega^2)}$. Note that $-\varepsilon$ (dissipation) and $-\partial K/\partial t$ (tendency) are plotted. The barotropic velocity is $U_0 \sin \left(2\pi \frac{t}{T}\right)$. 
Table 3.4: Cycle averaged turbulent kinetic energy budget, integrated over areas A, B and C (shown in figure 3.13), in the critical and supercritical slope cases. All terms are normalized with $(\pi/4)\rho_0 U^2 h^2 \sqrt{(N^2 - \Omega^2)}$.

<table>
<thead>
<tr>
<th>Case</th>
<th>Tendency</th>
<th>Advection</th>
<th>Production</th>
<th>Dissipation</th>
<th>Buoyancy</th>
<th>Transport</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area A</td>
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<td>-7.45 $\times$ 10^{-4}</td>
<td>4.69 $\times$ 10^{-3}</td>
<td>-2.05 $\times$ 10^{-3}</td>
<td>-1.48 $\times$ 10^{-5}</td>
<td>2.77 $\times$ 10^{-5}</td>
</tr>
<tr>
<td>Area B</td>
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<td>-9.51 $\times$ 10^{-4}</td>
<td>2.71 $\times$ 10^{-3}</td>
<td>-1.17 $\times$ 10^{-3}</td>
<td>5.78 $\times$ 10^{-4}</td>
<td>-2.25 $\times$ 10^{-4}</td>
</tr>
<tr>
<td>Area C</td>
<td>-5.38 $\times$ 10^{-5}</td>
<td>1.38 $\times$ 10^{-3}</td>
<td>1.44 $\times$ 10^{-3}</td>
<td>-1.61 $\times$ 10^{-3}</td>
<td>1.64 $\times$ 10^{-4}</td>
<td>-2.77 $\times$ 10^{-5}</td>
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<tr>
<td>Case 5SUP</td>
<td>Tendency</td>
<td>Advection</td>
<td>Production</td>
<td>Dissipation</td>
<td>Buoyancy</td>
<td>Transport</td>
</tr>
<tr>
<td>Area A</td>
<td>4.78 $\times$ 10^{-3}</td>
<td>-7.46 $\times$ 10^{-6}</td>
<td>9.46 $\times$ 10^{-3}</td>
<td>-2.21 $\times$ 10^{-4}</td>
<td>-4.85 $\times$ 10^{-5}</td>
<td>2.22 $\times$ 10^{-5}</td>
</tr>
<tr>
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<td>3.66 $\times$ 10^{-4}</td>
<td>9.82 $\times$ 10^{-3}</td>
<td>-3.52 $\times$ 10^{-3}</td>
<td>-1.92 $\times$ 10^{-3}</td>
<td>-4.06 $\times$ 10^{-4}</td>
</tr>
<tr>
<td>Area C</td>
<td>4.63 $\times$ 10^{-5}</td>
<td>-5.98 $\times$ 10^{-4}</td>
<td>1.32 $\times$ 10^{-3}</td>
<td>-1.60 $\times$ 10^{-3}</td>
<td>-1.73 $\times$ 10^{-4}</td>
<td>1.09 $\times$ 10^{-4}</td>
</tr>
</tbody>
</table>

large patch of turbulence around $x = -0.4$ m is associated with convective overturns and flow reversal from down to up discussed earlier in the critical case. However, the corresponding velocity profile at this time indicates stronger shear relative to the critical case 5 and, therefore, the turbulent production in this region is also significantly higher when compared to case 5 as will be shown.

TKE is generated primarily in a region (area B, shown in figure 3.13) above and adjacent to the constant slope region, where the slope angle transitions through the critical angle. Figure 3.16 shows evolution over a tidal cycle of the TKE budget integrated over three different areas, similar to case 5 shown in figure 3.14. Figure 3.16(a) shows that, in contrast to case 5, the budget terms at area A in the midslope region are an order of magnitude lower than the corresponding values in area B (figure 3.16(b)) and C (figure 3.16(c)). This behavior is consistent with the finding that TKE levels are not significant at the constant slope region in case 5SUP. In figure 3.16(b), area B exhibits strong shear production that is substantially larger than that at midslope or at the top of the ridge. The advection term indicates transport of the generated TKE from area B to area C between time 50.5 and 51.5 s. The shear production of TKE in area C near the top of the ridge, shown in figure 3.16(c), although smaller than that in area B, shown in figure 3.16(b), is also significant.

The cycle-averaged values of terms in the TKE budget are given in table 3.4. The largest production and dissipation of turbulence is over the critical slope region (area A of case 5 and area B of case 5SUP) of the model ridge.

For completeness, the influence of forcing on the modal distribution in the supercritical and subcritical regime has been examined and the results are plotted in figure 3.17. The supercritical cases, similar to the critical cases, show that high
Figure 3.17: The effect of increasing barotropic forcing on modal distribution at $x = 3$ m in supercritical and subcritical cases.

modes are progressively eliminated when the forcing increases. On the other hand, subcritical cases where there is little turbulence, show an enhancement of energy content at the high modes with increased forcing.

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Chapter 4

An IBM for Direct and Large Eddy Simulation of Stratified Flows in Complex Geometry

The content of the present chapter is under consideration for publication in a journal. The validation of the immersed boundary method (IBM) discussed in chapter 2 is presented in sections 4.1 - 4.3. DNS of stratified flow over a laboratory scale is presented in sec. 4.4 and comparison is made with the earlier results, presented in chapter 3, using a body fitted grid. Large eddy simulations of stratified flow over a large scale ridge are presented in sec. 4.5.

4.1 Unstratified flow past a sphere at \( Re \) up to 300

Here we consider unstratified flow past a sphere with increasing Reynolds number up to 300. The flow is axisymmetric up to \( Re \approx 200 \), loses axisymmetry and forms streamwise vortices behind the sphere from \( 210 < Re < 270 \) and then begins to shed vortices with a constant strength at a consistent frequency for \( 280 < Re < 375 \).

Data from Johnson & Patel (1999) is available for comparison for \( Re \leq 300 \). As shown in Figure 4.1 the flow evolves in a qualitatively similar way between the two cases with both cases showing symmetry about the vertical axis and similar shapes for the recirculating region. Flow statistics such as the separation angle, \( \Theta_s \), the separation length, \( L_s \), and the relative position of the vortex core with respect to the center of sphere \( (X_c \text{ and } Y_c) \) are calculated and shown in Table 4.1 for a quantitative comparison. The IBM results agree well with those of (Johnson & Patel, 1999).
Figure 4.1: Qualitative comparison of streamlines with Johnson & Patel (1999). (Left column) Data from Johnson & Patel (1999). (Right column) Data from present. From top to bottom: (a) $Re = 50$. (b) $Re = 100$. (c) $Re = 150$. (d) $Re = 200$.

The above results show that the IBM code agrees well at low Reynolds number when the flow is axisymmetric. In the unsteady planar symmetric regime the flow oscillates at a periodic shedding frequency. Integrated quantities such as the Strouhal frequency were found to agree as well, the present simulations obtained a Strouhal frequency of 0.133 which compares well with the range $St = 0.134 - 0.137$ observed by Johnson & Patel (1999).

### 4.2 Direct Numerical Simulation (DNS) of turbulent channel flow at $Re_\tau = 395$

Direct numerical simulation of turbulent channel flow is performed at a Reynolds number ($Re_\tau = \frac{u_\tau \delta}{\nu}$) of 395 based on half the channel height ($\delta$) and friction velocity ($u_\tau$). The IBM method is used to model the solid walls. Domain and grid sizes employed in the simulation are $2\pi\delta \times 2\pi\delta \times 2\delta$ and $256 \times 192 \times 192$ in the
<table>
<thead>
<tr>
<th>(Re)</th>
<th>(\Theta_s)</th>
<th>(L_s/D)</th>
<th>(X_c/D)</th>
<th>(Y_c/D)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>139</td>
<td>139</td>
<td>0.42</td>
<td>0.40</td>
</tr>
<tr>
<td>100</td>
<td>127</td>
<td>129</td>
<td>0.88</td>
<td>0.85</td>
</tr>
<tr>
<td>150</td>
<td>121</td>
<td>125</td>
<td>1.22</td>
<td>1.16</td>
</tr>
<tr>
<td>200</td>
<td>117</td>
<td>120</td>
<td>1.46</td>
<td>1.39</td>
</tr>
</tbody>
</table>

streamwise (x), the spanwise (y), and the vertical (z) directions, respectively. The grid is uniform in the streamwise and spanwise directions and stretched in the vertical direction from the wall towards the center with resolution in wall units of \(\Delta x^+ = 8.8, \Delta y^+ = 8.8, \Delta z_{min}^+ = 0.48, \Delta z_{max}^+ = 5.9\). Periodic boundary conditions are employed for all variables in the x and the y directions while the no-slip boundary condition \((u, v, w = 0, dp'/dz = 0)\) is employed on the channel walls \((z = \pm 1)\). The flow is driven by a constant pressure gradient. The initial flow field corresponds to the laminar fully developed flow in a channel superimposed with perturbations obtained using the method of Rogallo (1981).

The turbulence statistics, obtained by averaging along x, y directions and in time, including the Reynolds stress components and the mean velocity are shown in figure 4.2 (a) and (b), respectively. The profiles agree well with those of Moser et al. (1999).

### 4.3 DNS of stratified turbulent channel flow at \(Re_\tau=180, Ri_\tau=18\)

Direct numerical simulation of stratified channel flow is performed at a Reynolds number, \(Re_\tau = \frac{u_\tau \delta}{\nu} = 180\), based on half the channel height \((\delta)\) and friction velocity \((u_\tau)\) and a friction Richardson number of \(Ri_\tau = 18\). Here, \(Ri_\tau = \frac{\Delta \rho g \delta}{\rho_0 u_\tau^2}\) where \(\Delta \rho\) is the density difference between top and bottom walls. The channel walls are discretized using a triangulated surface mesh and IBM is used to simulate the effect of channel walls. Domain and grid sizes employed in the simulation are \(4\pi \delta \times 4\pi / 3\delta \times 2\delta\) and \(128 \times 128 \times 128\) in the streamwise (x), the spanwise (y), and the vertical (z) directions, respectively. The grid is uniform in the streamwise and spanwise directions and stretched in the vertical direction from the wall towards the center with
Figure 4.2: Statistics of unstratified channel flow at $Re_\tau = 395$, with channel walls simulated using IBM on a grid of 128x128x128, (a) Reynolds stress components, and (b) mean velocity normalized by wall units. Lines correspond to the present DNS results and circles to those of Moser et al. (1999).

$\Delta x^+ = 19.9, \Delta y^+ = 6.6, \Delta z_{min}^+ = 0.48, \Delta z_{max}^+ = 5.9$. Periodic boundary conditions are employed for all variables in the x and the y directions while the no-slip boundary condition ($u, v, w = 0, dp/dz = 0$) is employed on the channel walls ($z = \pm 1$). The flow is driven by a constant pressure gradient. Initial flow field corresponds to the laminar fully developed flow in a channel that is superimposed with perturbations obtained using the method of Rogallo (1981).

The turbulence statistics, obtained by averaging along x, y directions and in time, including the mean velocity and Reynolds shear stress, normalized by wall units, are shown in figure 4.3 (a) and (b), respectively. The profiles agree well with the numerical studies of Armenio & Sarkar (2002) and Garcia-Villalba & del Álamo (2011).

4.4 Direct Numerical Simulations (DNS) of stratified flow past a laboratory scale model ridge at $Re_s = 177$

Oscillating flow over a laboratory-scale smoothed triangular ridge, shown in fig. 4.4, in a stratified environment is studied numerically using the Immersed Boundary Method (IBM). The flow is driven by an oscillating pressure gradient,
$dp/dx = U_0 \Omega \cos(\Omega t)$, with frequency $\Omega$ that results in an oscillating barotropic velocity of magnitude $U_0$. This oscillatory tidal forcing leads to internal waves launched from the ridge as well as near-field turbulence. Internal wave and turbulence properties change qualitatively, depending on the choice of parameters. Therefore, several cases are simulated to explore the ability of IBM in different dynamical regimes. The laboratory scale model (length of order several meters) results are compared to earlier studies of Rapaka et al. (2013) and Jalali et al. (2014) using a body fitted grid to validate the IBM. After successful validation, large scale simulations (length of the order of several kilometers) are performed to illustrate the capabilities of IBM. Some important non-dimensional parameters that characterize the flow are Reynolds number, $Re_s = U_0 \delta_s/\nu$, based on the Stokes boundary layer thickness ($\delta_s = \sqrt{2\nu/\Omega}$) and the barotropic velocity ($U_0$); Excursion number, $Ex = U_\infty/\Omega l$, the ratio of fluid excursion length ($l_{ex} = U_0/\Omega$) to the topographic length scale ($l$); and Criticality parameter ($\epsilon = \tan(\beta)/\tan(\alpha)$), the ratio of the topographic slope ($\tan(\beta)$) to the internal wave characteristic slope ($\tan(\alpha)$). The Reynolds number, $Re = U_\infty l_{ex}/\nu$, based on the fluid excursion length is also of interest.

The cases simulated are listed along with the nondimensional and dimensional parameters in tables 4.2 and 4.3, respectively. The cases marked with IBM and BFG, in the remarks column of table 4.2, are simulations using IBM and body fitted grid (BFG), respectively. Cases corresponding to the DNS of a laboratory scale topography are marked by DNS. Cases corresponding to LES of flow past large scale topography are marked by LES and BD indicates that the bottom drag law is used for that
Two values of excursion number (a measure of nonlinearity of the oscillating tidal forcing) are considered: $Ex = 0.066$ (case numbers starting with 1) and $Ex = 0.4$ (case numbers starting with 2). The model ridge has a linear slope with angle $\beta = 15^\circ$ that is 20% of its streamwise extent. The stratification sets the buoyancy frequency, $N_\infty$, of the background and, thus, the wave propagation angle, $\alpha$. The value of $N_\infty$ is varied among the $Ex = 0.066$ cases to change $\alpha$ so that the linear slope is subcritical ($\beta < \alpha$ or $\epsilon < 1$), critical ($\beta = \alpha$ or $\epsilon = 1$), or supercritical ($\beta > \alpha$ or $\epsilon > 1$). The internal wave response is expected to be increasingly nonlinear when $\epsilon$ is near unity (the case of resonant response) and also in supercritical cases when $\epsilon$ is much larger than unity. Cases 1cri_bf and 1sup_bf correspond to cases 5 and 5sup of Rapaka et al. (2013), respectively, and case 2cri_bf corresponds to case CEX3 of Jalali et al. (2014) that has a higher $Ex$. The results of those recent BFG studies are compared with the results obtained here with IBM. Specifically, case 1sub is compared to case 1sub_bf , case 1sup to 1sup_bf , and case 2cri to case 2cri_bf .

Cases 1cri_ls and 2cri_ls correspond to near critical flows at low and high $Ex$, respectively, over a large scale model ridge with shape same as in fig. 4.4 but length in each direction is increased by 1000 times. Note that these two cases have the same nondimensional parameters (see table 4.2) as those of cases 1cri and 2cri, respectively.
Table 4.2: Nondimensional parameters and grid spacing of the cases of flow past a ridge. The selection of parameters is discussed in the accompanying text.

<table>
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<tr>
<th>case</th>
<th>$Re_s$</th>
<th>$Re_l$</th>
<th>$Ex$</th>
<th>$\epsilon$</th>
<th>$N_s$</th>
<th>$N_l$</th>
<th>$\Delta x$ (m)</th>
<th>$\Delta y$ (m)</th>
<th>$\Delta z$ (m)</th>
<th>Remark</th>
</tr>
</thead>
<tbody>
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<td>0.57</td>
<td>896</td>
<td>128</td>
<td>768</td>
<td>.006</td>
<td>.006</td>
<td>0.002</td>
</tr>
<tr>
<td>1cri</td>
<td>177</td>
<td>1.56 x 10^4</td>
<td>0.066</td>
<td>1</td>
<td>896</td>
<td>128</td>
<td>768</td>
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<tr>
<td>1sup</td>
<td>177</td>
<td>1.56 x 10^4</td>
<td>0.066</td>
<td>2.17</td>
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<td>321</td>
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<td>.0075</td>
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</table>

Table 4.3: Dimensional parameters of the simulated cases.

<table>
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<tr>
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<th>$N_x^2$ (s^{-2})</th>
<th>$\Omega$ (s^{-1})</th>
<th>$\nu$ (m^2 s^{-1})</th>
<th>$l$ (m)</th>
<th>$h$ (m)</th>
<th>$L_x$ (m)</th>
<th>$L_y$ (m)</th>
<th>$L_z$ (m)</th>
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</thead>
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<td>3.28</td>
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<tr>
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<td>10^{-6}</td>
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<td>0.328</td>
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<td>0.25</td>
<td>3.28</td>
</tr>
<tr>
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<td>10^{-6}</td>
<td>1.9</td>
<td>0.328</td>
<td>40</td>
<td>0.5</td>
<td>3.28</td>
</tr>
<tr>
<td>1criJs</td>
<td>0.0175</td>
<td>29.26 x 10^{-4}</td>
<td>1.4 x 10^{-4}</td>
<td>10^{-6}</td>
<td>1900</td>
<td>328</td>
<td>40000</td>
<td>250</td>
<td>3280</td>
</tr>
<tr>
<td>2cri</td>
<td>0.760</td>
<td>14.93</td>
<td>1</td>
<td>3.7 x 10^{-5}</td>
<td>1.9</td>
<td>0.328</td>
<td>40</td>
<td>0.25</td>
<td>3.28</td>
</tr>
<tr>
<td>2cribf</td>
<td>0.760</td>
<td>14.93</td>
<td>1</td>
<td>3.7 x 10^{-5}</td>
<td>1.9</td>
<td>0.328</td>
<td>40</td>
<td>0.25</td>
<td>3.28</td>
</tr>
<tr>
<td>2criJs</td>
<td>0.106</td>
<td>29.26 x 10^{-8}</td>
<td>1.4 x 10^{-4}</td>
<td>10^{-6}</td>
<td>1900</td>
<td>328</td>
<td>40000</td>
<td>250</td>
<td>3280</td>
</tr>
</tbody>
</table>

except that they have much larger Reynolds numbers ($Re_s$, $Re_l$). The results of these two cases are summarized later in section 4.5.

The domain is $L_x$ m x $L_y$ m x $L_z$ m in the streamwise (x), the spanwise (y), and the wall normal (z) directions, respectively. Sponge region is employed at both the boundaries in the streamwise (x) direction to absorb the outgoing internal waves. Periodic boundary conditions are employed for all variables in the spanwise (y) direction. The no-slip condition ($u, v, w = 0$) and $dp/dz = 0$ are applied at the bottom boundary ($z = h(x)$) while the rigid lid boundary condition ($w = 0, d/dz(u, v, p') = 0$) is employed at the top boundary ($z = H$).

The IBM cases use uniform grid over the topography in order to resolve the boundary layer throughout the height of the topography. The grid is stretched away from the topography with up to a maximum of 4% and 3% stretching in the horizontal and vertical directions, respectively. For a given number of grid points, the IBM has
Figure 4.5: Mean velocity on the slope. Vertical profiles of the streamwise mean velocity at the left mid slope \((x/l = -0.405)\): (a) case 1cri shown at a phase of peak downslope velocity; and (b) case 1sup shown at \(t/T = 7.59\). Here, \(T\) is the barotropic cycle time period. Note that the z-axis for case 1cri\_ls is shown on the right side of figure (a), and, if normalized by a factor of 1000, collapses onto that of case 1cri.

lower boundary-normal resolution compared to the body fitted grid, because some of the grid points lie inside the solid region. Additional points are used in the z direction for IBM cases to make the grid resolution comparable to the BFG. These additional points provide better resolution in the region away from the immersed body when compared to the BFG. This is useful in problems where important nonlinear flow features are present away from the boundary, e.g., internal tide generation at large \(Ex\) and wakes generated behind a bluff body. Also, note that the ghost cells used in the IBM have a distribution near the immersed boundary that is non-uniform relative to the BFG which also affects the flow resolution.

### 4.4.1 Mean velocity on the slope

Figure 4.5(a) shows the vertical profiles of the normalized streamwise mean velocity, \(<u>_y\), corresponding to the phase of peak downslope velocity at the left midslope \((x/l = -0.405)\) for cases 1cri, 1cri\_bf, and 1cri\_ls. The intensification of the near boundary velocity (almost a factor of 3 larger than the oscillatory forcing velocity) is present in the IBM simulation. The downward propagating boundary flow has thickness and amplitude in case 1cri, using IBM, that are similar to those in 1cri\_bf. However, there is a small difference in the boundary layer, likely because case 1cri has lower wall-normal resolution compared to case 1cri\_bf. Also, the ghost
cells (and the IB fluid cells adjacent to the boundary) have a distribution that is not as uniform as the near-wall cells in the BFG, which also leads to differences in the boundary layer velocity profiles at the slope.

Figure 4.5(b) compares cases 1sup, and 1sup_bf, corresponding to a non-dimensional time, \( t/T \), of 7.59. Case 1sup shows approximately the same magnitude and shape as that of case 1sup_bf in the core of the beam. Away from the topography, the IBM case has a coarser grid that results in some differences in the baroclinic response above \( z = 0.4m \). These profiles suggest that, given similar resolution, IBM yields similar velocity field as that using a body fitted grid.

### 4.4.2 Baroclinic response

The amount of energy converted from the oscillatory forcing (barotropic tide) to internal waves (the internal tide) and the resultant energy fluxed by the internal waves is a key result of key interest because this is the energy that could result in eventual turbulent mixing. To study the energetics of the baroclinic response, the spanwise mean velocity and pressure \( \langle u \rangle_y (x, z, t), \langle p \rangle_y (x, z, t) \) are decomposed into barotropic and baroclinic components. The barotropic components are computed as the depth average of the spanwise mean which, then, depend only on \( x \) and \( t \). The baroclinic component, which, by definition, has a zero mean along the depth and also over the tidal cycle, is computed from Eq. 4.1. See appendix A of Rapaka et al. (2013) for more details.

\[
\langle \phi \rangle_y (x, z, t) - \langle \phi \rangle_y T (x, z) = \phi_{bt}(x, t) + \phi_{bc}(x, z, t),
\]

where \( \phi = u, p \) and \( \langle \phi \rangle_y T (x, z) = \frac{1}{T} \int_0^T \langle \phi \rangle_y (x, z, t) dt \) which is usually very small after the initial transient.

The baroclinic response is quantified by the analysis of the baroclinic energy budget. The equation for the baroclinic energy with diffusive fluxes of wave energy neglected and averaged over complete tidal cycle is given by (also see Eq. 4.1 of Rapaka et al. (2013) and Kang & Fringer (2011).)

\[
\frac{\Delta E}{T} + \nabla H \cdot \begin{bmatrix} \langle F_{bc} + F_{adv} \rangle_T \\ \langle P \rangle_T \end{bmatrix} = \langle \mathcal{C} \rangle_T - \langle \mathcal{P}_{bc} \rangle_T - \langle \bar{P} \rangle_T
\]

where,

\[
E = KE + PE, \quad KE = \frac{1}{2} (w_{bc}^2 + v_{bc}^2 + w_{bc}^2), \quad PE = \frac{1}{2} N_\infty^{-2} \left( \frac{\rho^*}{\rho_0} \right)^2
\]
\( \mathbf{F}_{bc} = p_{bc} \mathbf{u}_{bc}, \quad \mathbf{F}_{adv} = \mathbf{u}^* (E + u_{bt}^* u_{bc}), \quad C = \frac{\partial p^*}{\partial z} W(z), \quad W(z) = -\frac{\partial ([z - h(x)]U)}{\partial x} \)

\[ \varepsilon_{bc} = \nu \frac{\partial (u_{bc})_i}{\partial x_j} \frac{\partial (u_{bc})_i}{\partial x_j}, \quad P \equiv -\langle u'_i u'_j \rangle_y \langle S_{ij} \rangle_y - \langle \tau_{ij} \rangle_y \langle S_{ij} \rangle_y. \]

The overbar represents depth integration. \( \bar{C} \) is the conversion from the barotropic to the baroclinic wave field, \( \tau_{bc} \) is the viscous dissipation of the baroclinic energy, \( \mathbf{F}_{bc} \) is the wave energy flux, \( \mathbf{F}_{adv} \) is the advective energy flux, and \( P \) is the turbulent production. The operator \( \nabla_H \) is the horizontal gradient which simplifies to \( \frac{\partial}{\partial z} \) in the present problem. Note that \( S_{ij} \) is the rate of strain tensor and \( \tau_{ij} \) is the subgrid scale tensor which is non-zero only when LES is used. Physically, the topographic energy conversion (at rate \( C \)) from the oscillatory forcing is transported away from the obstacle as internal waves with a energy flux (\( F_{bc} \)), and the rest of the terms in the energy balance act as a loss insofar as the internal wave energy field. Linear theory assumes that \( C \equiv F_{bc} \).

Figure 4.6 shows the normalized vertical profiles of the baroclinic velocity amplitude (left column), baroclinic pressure amplitude (central column) and baroclinic wave flux amplitude (right column) at a location away from the topography \((x/l = -1.58)\). The amplitude of a variable is computed as its root mean square over two complete tidal cycles. The IBM results are compared with the previously published BFG results. The primary quantity of interest, the baroclinic wave flux (right column), compares very well between IBM and BFG methods. The velocity amplitude also compares well and, although there are differences in the pressure amplitude, they do not degrade the prediction of the baroclinic wave flux. Differences between IBM and BFG results increase with increasing nonlinearity in the flow response to the oscillatory pressure gradient, i.e., with increasing slope criticality (\( \epsilon \)) and increasing Excursion number (\( Ex \)). We elaborate below.

Fig. 4.6(a-c) compares the baroclinic profiles of case 1sub with case 1sub_bf. The region around the peak near \( z = 1.8m \) indicates the location of the tidal beam where there is a local maximum in the baroclinic response. Case 1sub using IBM compares very well with case 1sub_bf using body fitted simulations. As the criticality parameter increases, the tidal beam becomes shallower as shown in fig. 4.6(d-f) for case 1crii and fig. 4.6(g-i) for case 1sup which show the location of the beam center around \( z = 1.2m \) and \( z = 0.65m \), respectively. Fig. 4.6(d-f) compares the baroclinic profiles of case 1cri with case 1cri_bf. The profiles of the baroclinic velocity and baroclinic wave flux for case 1cri are in close agreement with those for case 1cri_bf.
Figure 4.6: Normalized vertical profiles of (left-right) the baroclinic velocity amplitude, the baroclinic pressure amplitude, and the product of pressure and velocity amplitudes; for cases (top-bottom) 1sub, 1cri, 1sup, and 2cri. All the profiles are shown at $x/l = 1.58$, a location away from the topography.
Figure 4.7: Evolution of various terms in the baroclinic energy budget, shown in table 4.4, integrated over an area from $x = -3m$ to $x = +3m$ for case 1cri. The balance shown is the net difference between the baroclinic energy conversion and the sum of all other terms including tendency, wave flux, baroclinic dissipation, advection, and turbulence production.

obtained using a body fitted grid (Rapaka et al. (2013)). Figure 4.6(g-i) shows similar profiles for case 1sup. The supercritical case has beams propagating both downwards and upwards. The baroclinic intensification in case 1sup is slightly larger compared to case 1cri. Both the baroclinic velocity and the wave flux agree well with those of case 1sup, bf obtained using a body fitted grid (Rapaka et al. (2013)).

Figure 4.6(j-l) shows similar profiles for case 2cri with $Ex = 0.4$. Owing to large $Ex$, the beam like behavior is weaker in case 2cri. The baroclinic energy is spread into higher harmonics which can be seen by the secondary peak in fig. 4.6(j). The baroclinic wave flux is larger in case 2cri compared to case 2cri, bf that utilizes a body fitted grid (Jalali et al. (2014)). This may be due to larger velocities involved and the lower resolution in case 2cri compared to case 2cri, bf.

Figure 4.7 shows the evolution of various terms in the baroclinic energy budget integrated over an area from $x = -3m$ to $x = +3m$ enclosing the topography for case 1cri. The conversion of energy from barotropic to baroclinic flow is balanced, to a large extent, by baroclinic wave flux and tendency terms. There is substantial oscillatory modulation in $C$ that leads to a modulation in the tendency term. To quantify the energetics, the area integrated baroclinic energy budget is averaged over two complete tidal cycles and shown in table 4.4.

Table 4.4 shows the baroclinic energy budget including the conversion to internal waves ($C$), radiated wave flux ($M$ or $F_{bc}$), advective flux ($F_{adv}$), the baroclinic energy dissipation ($\varepsilon_{bc}$) and the turbulence production ($P$) which appear in the baro-
Table 4.4: Baroclinic energy budget: tendency ($\frac{\Delta E}{T}$), energy conversion to waves ($C$), radiated wave flux ($M \equiv F_{bc}$), baroclinic energy dissipation ($\varepsilon_{bc}$), baroclinic advective flux ($M_{adv} \equiv F_{adv}$), and turbulence production ($P$), integrated over the area between $x = -3$ to $+3$ m. All the quantities are normalized by $\frac{\pi}{4} \rho_0 U_0^2 h^2 \sqrt{N^2 - \Omega^2}$, the term appearing in linear analysis. The balance term, small compared to the conversion, is the net difference between the baroclinic energy conversion ($C$) and the sum of all other terms including tendency, wave flux, baroclinic dissipation, advection, and turbulence production.

<table>
<thead>
<tr>
<th>case</th>
<th>$Ex$</th>
<th>tendency</th>
<th>$C$</th>
<th>$F_{bc}$</th>
<th>$\varepsilon_{bc}$</th>
<th>$F_{adv}$</th>
<th>$P$</th>
<th>balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1sub</td>
<td>0.066</td>
<td>0.000</td>
<td>0.689</td>
<td>0.691</td>
<td>0.019</td>
<td>0.000</td>
<td>0.002</td>
<td>-0.022</td>
</tr>
<tr>
<td>1cri</td>
<td>0.066</td>
<td>0.005</td>
<td>0.665</td>
<td>0.617</td>
<td>0.029</td>
<td>0.006</td>
<td>0.016</td>
<td>-0.006</td>
</tr>
<tr>
<td>1sup</td>
<td>0.066</td>
<td>-0.005</td>
<td>1.009</td>
<td>0.962</td>
<td>0.021</td>
<td>-0.007</td>
<td>0.017</td>
<td>0.020</td>
</tr>
<tr>
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<td>0.011</td>
<td>0.689</td>
<td>0.681</td>
<td>0.012</td>
<td>0.002</td>
<td>0.003</td>
<td>-0.004</td>
</tr>
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<td>-0.008</td>
<td>0.572</td>
<td>0.551</td>
<td>0.013</td>
<td>0.002</td>
<td>0.025</td>
<td>-0.011</td>
</tr>
<tr>
<td>1sup_bf</td>
<td>0.066</td>
<td>-0.008</td>
<td>0.843</td>
<td>0.824</td>
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<td>-0.007</td>
<td>0.028</td>
<td>-0.008</td>
</tr>
<tr>
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<td>0.704</td>
<td>0.645</td>
<td>0.078</td>
<td>0.007</td>
<td>0.017</td>
<td>-0.052</td>
</tr>
<tr>
<td>2cri</td>
<td>0.400</td>
<td>0.013</td>
<td>0.726</td>
<td>0.609</td>
<td>0.075</td>
<td>0.081</td>
<td>0.025</td>
<td>-0.077</td>
</tr>
<tr>
<td>2cri_bf</td>
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<td>0.574</td>
<td>0.463</td>
<td>0.054</td>
<td>0.050</td>
<td>0.022</td>
<td>-0.023</td>
</tr>
<tr>
<td>2cri_ls</td>
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<td>-0.003</td>
<td>0.721</td>
<td>0.612</td>
<td>0.087</td>
<td>0.092</td>
<td>0.031</td>
<td>-0.098</td>
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</table>
Figure 4.8: Spatial distribution of $\log_{10}(TKE/U_0^2)$ and isopycnals near the topography in (a) case 1cri_bf, (b) case 1cri, and (c) case 1cri_ls. They correspond to a phase of peak upslope flow at the left midslope ($x/l = -0.405$).

clinic energy budget (see Eq. 4.2). For case 1sub, the near wall resolution is sufficient enough and both the conversion and the baroclinic flux agree well with case 1sub_bf. For case 1cri, conversion and wave flux are higher by about 12% compared to case 1cri_bf. For case 1sup, conversion and wave flux are higher by about 15% compared to case 1sup_bf. The differences with respect to the body fitted grid results are not large.

4.4.3 Turbulence

Figure 4.8 shows the spatial distribution of TKE for cases 1cri_bf, 1cri and 1cri_ls at phase of peak upslope flow at the left midslope ($x/l = -0.405$). The turbulence generated at the ridge under these conditions is mainly due to density
Table 4.5: Normalized TKE and turbulent dissipation integrated over the area between $x = -3$ to $+3 \, m$.

<table>
<thead>
<tr>
<th>case</th>
<th>$Ex$</th>
<th>$\frac{k}{U_0^2 h^2}$</th>
<th>$\frac{\varepsilon}{\frac{1}{4} \rho_0 U_0^2 h^2 \sqrt{N^2 - \Omega^2}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1sub</td>
<td>0.066</td>
<td>0.005</td>
<td>$9.00 \times 10^{-4}$</td>
</tr>
<tr>
<td>1cri</td>
<td>0.066</td>
<td>0.183</td>
<td>$1.10 \times 10^{-2}$</td>
</tr>
<tr>
<td>1sup</td>
<td>0.066</td>
<td>0.152</td>
<td>$6.80 \times 10^{-3}$</td>
</tr>
<tr>
<td>1sub_bf</td>
<td>0.066</td>
<td>0.001</td>
<td>$2.50 \times 10^{-3}$</td>
</tr>
<tr>
<td>1cri_bf</td>
<td>0.066</td>
<td>0.156</td>
<td>$1.36 \times 10^{-2}$</td>
</tr>
<tr>
<td>1sup_bf</td>
<td>0.066</td>
<td>0.072</td>
<td>$1.46 \times 10^{-2}$</td>
</tr>
<tr>
<td>1cri_ls</td>
<td>0.066</td>
<td>0.117</td>
<td>$6.90 \times 10^{-3}$</td>
</tr>
<tr>
<td>2cri</td>
<td>0.400</td>
<td>0.090</td>
<td>$2.22 \times 10^{-2}$</td>
</tr>
<tr>
<td>2cri_bf</td>
<td>0.400</td>
<td>0.064</td>
<td>$1.65 \times 10^{-2}$</td>
</tr>
<tr>
<td>2cri_ls</td>
<td>0.400</td>
<td>0.146</td>
<td>$1.44 \times 10^{-2}$</td>
</tr>
</tbody>
</table>

Overturns and shear instability in the stratified oscillating layer adjacent to the ridge slope that, while occurring in the boundary flow, are not primarily associated with the shear at the bottom boundary (Rapaka et al., 2013). The IBM captures these mechanisms well Figure 4.8(a) corresponds to case 1cri_bf and is adapted from figure 8(c) of Rapaka et al. (2013). It shows the inhomogeneous distribution of turbulent kinetic energy (TKE) including a large vertical patch of turbulence extending from the bottom that was generated due to density overturns when the flow reverses from downslope to upslope and is then advected upward. Figure 4.8(b) shows the spatial distribution of TKE for case 1cri which compares well with case 1cri_bf. However, some turbulent features very close to the bottom are not resolved as well in IBM due to the lower wall-normal resolution. For instance, the thin patch of TKE in the downslope attached boundary layer on the right side of the ridge in fig. 4.8(a) is absent in fig. 4.8(b).

Table 4.5 shows the normalized TKE and the rate of turbulence dissipation integrated over the area between $x = -3$ to $+3 \, m$ ($km$ for case 1cri_ls). The low-$Ex$ cases tend to have somewhat larger TKE and somewhat lower turbulent dissipation rate with the IBM relative to the BFG simulations. For instance, turbulence in the convective overturns that occur during upslope flow is somewhat stronger in the IBM case 1cri and the area integrated TKE averaged over two tidal cycles in table 4.5 is
Figure 4.9: Spatial distribution of $\log_{10}(TKE/U_0^2)$ and isopycnals near the topography, shown at time=6T, in (a) case 2cri$_{bf}$, (b) case 2cri, and (c) case 2cri$_{ls}$. Note that $h_0$ is the height of the topography, and $T$ is the barotropic cycle time period.

Figure 4.9 shows the spatial distribution of TKE for cases 2cri$_{bf}$, 2cri and 2cri$_{ls}$ at time 6T where T is tidal period. Case 2 has a high excursion, $Ex = 0.4$. Figure 4.9(a) corresponds to case 2cri$_{bf}$ and is adapted from figure 8(c) of Jalali et al. (2014). It corresponds to a phase of down-to-upslope barotropic flow reversal and shows a large vertical patch of turbulence extending from the bottom to approximately twice the topographic height, on the leeside of the topography. This a result of the breaking lee waves as described in Jalali et al. (2014). Figure 4.9(b) shows spatial distribution of TKE for case 2cri which also exhibits breaking lee waves similar to case 2cri$_{bf}$. The dissipation rate in case 1cri is slightly lower compared to case 1cri$_{bf}$. Case 2cri has both the TKE and dissipation rate somewhat higher compared to case 2cri$_{bf}$.

Figure 4.9(b) shows spatial distribution of TKE for case 2cri which also exhibits breaking lee waves similar to case 2cri$_{bf}$. The dissipation rate in case 1cri is slightly lower compared to case 1cri$_{bf}$. Case 2cri has both the TKE and dissipation rate somewhat higher compared to case 2cri$_{bf}$.
2cri_{bf}. The TKE magnitude is also similar but there is a quantitative difference: the area integrated TKE averaged over two tidal cycles in table 4.5 is 40% higher in case 2cri compared to case 2cri_{bf}.

4.5 Large eddy simulations (LES) of stratified flow past a large scale model ridge

Cases 1cri_{ls} and 2cri_{ls}, shown in tables 4.2 and 4.3 correspond to tidal flow over a large scale topography with same shape as shown in fig. 4.4 but with length increased by 1000 times in all three directions. As a result, the $Re_s$ in these two cases is much larger compared to the laboratory scale topography. Also, the disparity between the scale of the large turbulent overturn ($2h_0 = 600$ m in figure 4.9(c)) and the viscous boundary layer increases for the large scale ridge. This makes DNS, which requires resolution of the turbulence dissipative length scales, prohibitively expensive. To reduce the computational expense, Large Eddy Simulation (LES) is used. To further reduce the computational cost, the near-wall turbulence is not resolved, instead a nonlinear drag (Eq. 2.89) is used to parametrize the bottom friction as described in section 2.5. A drag coefficient of 0.0025, commonly used in oceanographic applications, is used.

4.5.1 Mean profiles

For case 1cri_{ls}, all the non-dimensional parameters, except the $Re_s$, are kept the same as in case 1cri by decreasing the buoyancy frequency and forcing frequencies. Note that the case 1cri_{ls} does not resolve the bottom boundary layer and employs a nonlinear bottom drag. Yet, the position of the peak velocity in the beam, as shown in figure 4.5, is very close compared to that of the case 1cri_{bf}, and the magnitude of the velocity is slightly higher.

4.5.2 Baroclinic response

In figure 4.6(d-f), despite of the lack of enough near wall resolution in case 1cri_{ls} with $Re_s = 2092$, the imposed bottom drag still yields reasonably well baroclinic response away from the wall. In fig. 4.6(j-l), the higher $Re_s = 12669$ case, however, has higher baroclinic response compared to case 2cri.
4.5.3 Turbulence

Figure 4.8(c) shows the spatial distribution of TKE for case 1cri_ls corresponding to the same phase as in figure 4.8(a) for case 1cri_bf. Approximately 100m thick patch of turbulence extends above the bottom boundary on the left slope and is qualitatively similar to that in the laboratory scale cases 1cri_bf and 1cri shown in figure 4.8(a) and (b). Note that the characteristic thickness of the beam shear determines the length scales of the turbulence patches near the bottom boundary. However, the area integrated TKE and dissipation, shown in table 4.5, for case 1cri_ls is about 23% lower compared to case 1cri_bf, although the $Re_s = 2092$ is larger in case 1cri_ls compared to case 1cri_bf ($Re = 177$).

Figure 4.9(c) shows spatial distribution of TKE for case 2cri_ls which also show the breaking lee waves similar to case 2cri. But, the breaking lee waves show a thicker patch of TKE extending to a height of twice the topographic height. The normalized TKE magnitude, shown in table 4.5, is more than twice that of case 2cri_bf. This may be due to much larger $Re_s$ of 12669 in case 1cri_ls while case 1cri_bf has $Re_s = 177$ only.
Chapter 5

Summary

A sharp interface immersed boundary method (IBM) is developed to simulate stratified turbulent flows over complex topography using a Cartesian grid. As part of the thesis, a computational fluid dynamics code (IBM code) is developed that uses central second order finite difference discretization and a mixed RK3-ADI time marching scheme. Chapter 2 explains the numerical method including the predictor-corrector algorithm to solve the discretized incompressible Navier-Stokes equations. The numerical solver uses domain decomposition with MPI application for parallel processing on large scale supercomputers (Brucker, 2009; de Stadler, 2013). In the IBM, mass fluxes are computed at the centroids of the cut cell faces in the fluid region to improve mass conservation properties of the solver. Ghost cells are placed on both the fluid and solid sides of the immersed boundary depending on their proximity to the immersed boundary to improve the numerical stability, especially for high Reynolds number flows. DNS is used for laboratory scale simulations that have enough grid resolution to resolve the skin friction and other boundary layer properties. For large scale topographies with high Reynolds number and marginal boundary layer resolution, LES is chosen with the option of imposing a non-linear drag model near the solid walls. Various LES models including Standard and Dynamic Smagorinsky Models are made available in the solver.

In chapter 3, three-dimensional DNS and LES approaches are used to examine the local flow as well as the radiated internal tide at a model ridge taken to have triangular topography. These simulations are performed on a body fitted grid using a mixed spectral/finite difference code (Gayen & Sarkar, 2010) that solves the governing equations in generalized coordinates. Nonlinear effects on the tidal energy conversion are examined by increasing the tidal forcing so that the excursion number increases,
while remaining significantly smaller than unity, and the Reynolds number, $Re_s$, based on the Stokes boundary layer thickness also increases. Implications of the present work for linear predictions of internal wave flux from sloping topography in the regime of excursion number less than $O(1)$ are as follows. Linear theory works well in subcritical cases where there is little turbulence for all values of tidal forcing examined here and for supercritical cases with low forcing where there is also little turbulence. In critical or supercritical cases with higher forcing, the energy conversion to the internal waves decreases with increasing forcing, as much as 25% in the present simulations compared to the laminar value.

Nonlinear effects on the tidal energy conversion are examined in the critical slope ridge by increasing the tidal forcing so that the excursion number increases from 0.004 to 0.168 and the Reynolds number, $Re_s$, increases from 10 to 400. The simulated cases with higher forcing exhibit wave breaking leading to a near-bottom layer of turbulence and upslope propagation of turbulent bores along with radiated internal wave beams. The internal wave energy flux is found to decrease substantially with the onset of turbulence. The radiative conversion (normalized wave flux) decreases to a value of $M \approx 0.50$, smaller than the corresponding subcritical case, corresponding to a substantial reduction from the laminar value. Evaluation of the baroclinic energy balance shows that the decrease in $M$ is associated with a decrease in normalized energy conversion, $C$, from the barotropic to baroclinic flow and additionally because of conversion to turbulence, i.e., the turbulent production, $P$. Modal analysis of the radiated wave field shows that, with increasing forcing, not only does the peak modal amplitude decrease but also the high modes are progressively eliminated. The contribution of higher temporal harmonics relative to the fundamental increases with increasing forcing. Turbulence varies over a cycle with a systematic dependence on tidal phase and is found at both leeward and windward sides of the ridge. Both convective and shear instability mechanisms are found to initiate transition to turbulence within a cycle. There is a substantial variation in turbulence properties when comparing three locations: at the middle of the critical slope, at the upper ridge where the slope angle changes from critical to smaller values, and at the top of the ridge where internal wave beams from opposite sides interact.

Supercritical ridges also exhibit decreases in energy conversion and radiated wave flux with increasing forcing. In contrast to the critical slope case, turbulence is insignificant at the constant supercritical slope portion and is limited to the region
between the critical portion and the top of the ridge. Because of the reduced area of turbulence, the decrease in energy conversion is less in the supercritical case compared to the critical case. Turbulence is present at both leeward and windward sides of the ridge. Subcritical ridges do not exhibit turbulence in the range of parameters studied here and, correspondingly, the energy conversion factor shows little change.

In chapter 4, validation of the IBM code is presented. Turbulence statistics including mean flow and Reynolds stresses in a turbulent channel flow agree well with those of Moser et al. (1999) for unstratified case and Armenio & Sarkar (2002); Garcia-Villalba & del Álamo (2011) for stratified case. DNS approach has been used to evaluate the IBM for tidal flow over a laboratory-scale (order of few meters) smoothed triangular ridge. The baroclinic energy budget and turbulence statistics have been examined at both low (0.066) and high (0.4) values of $Ex$ under the near critical flow conditions. In addition, for $Ex = 0.066$, criticality parameter ($\varepsilon$) is also varied to evaluate the performance in subcritical and supercritical cases. The energy conversion and baroclinic wave flux are in excellent agreement with those obtained using a body fitted grid for the subcritical case. For critical and supercritical flows, both the conversion and wave fluxes are higher by 10-15% in the IBM case. The difference can be attributed to a lower wall normal resolution in the IBM case and the distribution of the ghost cells along the immersed boundary.

The phasing of turbulence statistics and spatial distribution of TKE field are studied. The mechanisms of turbulence generation, including the large convective overturns after the down-to-up flow reversal and the TKE generated due to shear production during the peak down-slope flow, are observed as described by Rapaka et al. (2013) and Jalali et al. (2014) using a body fitted grid.

Large Eddy Simulations (LES) are performed for stratified flow over a large scale topography, of the order of few kilometers, with the same $Ex$ and the criticality parameter as in the small scale topography but significantly larger Reynolds number. Non-linear drag is imposed to parameterize the bottom friction. The intensification of the beam is slightly higher and so are the conversion and the baroclinic wave flux. Phasing of turbulence is qualitatively similar when compared to the small scale topography. Normalized turbulence kinetic energy and turbulence dissipation, integrated over an area enclosing the topography, are of the same order as in the DNS of small scale topography.

The IBM does not provide uniformly high resolution compared to the BFG.
However, for oceanic flows, the boundary layer resolution and skin friction are not as important as in engineering applications. So, it is no longer a disadvantage for oceanic flows and the IBM approach is a viable method to improve the capabilities of simulations of oceanic flows with topography. Also, the IBM provides an efficient approach to simulate complex wall bounded flows using Cartesian grids and provides a powerful platform to be used with local grid refinement techniques (such as those used in adaptive mesh refinement (AMR) methods) that are computationally economical.
Appendix A

Decomposition of pressure and velocity

Decomposition of the flow field into barotropic and baroclinic components is performed using the method described by Nash et al. (2004) except for pressure for which they impose hydrostatic balance. In the present work, since the pressure field is discretely available throughout the domain, we use the same procedure for pressure as that for the velocity field. The procedure for a generic spanwise-averaged variable $\phi(x, y, t) = <u>, <p>$ is summarized below:

The baroclinic component of $\phi(x, t)$ is defined as

$$\phi_{bc}(x, t) \equiv \phi(x, t) - \bar{\phi}(x) - \phi_b(x, y, t) \quad (A.1)$$

where $\bar{\phi}(x) = \frac{1}{T} \int_t^{T+t} \phi(x, t) dt$ is a cycle-averaged mean and $\phi_b(x, y, t)$ is calculated by enforcing baroclinicity:

$$\int_{h(x)}^{H} \phi_{bc}(x, t) dz = 0. \quad (A.2)$$

Here, $h(x)$ is height of the ridge topography with respect to the flat bottom.
Appendix B

Methods for modal analysis and conversion factor

The far-field vertical velocity that describes the linear baroclinic response to flow oscillating with frequency $\Omega$ over an isolated two-dimensional ridge in a linearly stratified finite-depth ocean is (see Pétrélis et al. (2006); Echeverri et al. (2009))

$$w_{bc}(X, Z, t) = \frac{U}{\mu} \text{Real} \left\{ \sum_{n=1}^{\infty} \gamma_n \sin(nZ) e^{i(nX-\Omega t+\pi/2)} \right\}, \quad (B.1)$$

where

$$Z = \frac{\pi z}{H}, \quad X = \frac{\pi x}{\mu H}, \quad \mu = \frac{\sqrt{N_\infty^2 - \Omega^2}}{\Omega}. \quad (B.2)$$

Here, $n$ is the mode number and $\gamma_n$ is the mode amplitude.

The nondimensional conversion factor, $M$, is given by (see Pétrélis et al. (2006)),

$$M \equiv \frac{2}{\pi} \frac{\int_0^h J(x \geq l, z) \cdot \hat{x} dz}{\frac{1}{2} \rho h^2 U^2 \sqrt{N^2 - \Omega^2}} \frac{2}{B^2} \sum_{n=1}^{\infty} \gamma_n \gamma_n^* \frac{n}{\mu} \quad (B.3)$$

where, $J$ denotes the phase average of the baroclinic energy flux, $(p_{bc} u_{bc}, p_{bc} w_{bc})$, $\hat{x}$ is the unit vector in the horizontal direction, and $B = \pi h/H$.

The mode amplitude $\gamma_n$, is calculated as below (see Echeverri et al. (2009)),

Define $\gamma_n = |\gamma_n| e^{i\phi_n}$, and project the baroclinic vertical velocity profile (the procedure used to extract baroclinic component from the simulated flow field is described in appendix A.) at a location away from the topography, $X$, onto the sinusoidal vertical basis modes of the linear stratification:

$$\infty_n (t) \equiv \int_0^{\infty} w_{bc} \sin(nZ) dZ = \left( \frac{\pi U}{2\mu} \right) |\gamma_n| \text{Real} \left\{ e^{i(\phi_n + nX - \Omega t + \pi/2)} \right\} \quad (B.4)$$
and the mode amplitude corresponding to the fundamental frequency is given by,

\[
|\gamma_n| = \frac{2}{\pi} \sqrt{\left(\int_0^T \gamma_n \cos(\Omega t) dt\right)^2 + \left(\int_0^T \gamma_n \sin(\Omega t) dt\right)^2} \left(\frac{\pi U}{2\mu}\right)
\]  

(B.5)

where \(T\) consists of a complete number of wave periods associated with the fundamental frequency \(\Omega\).
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