Quasi-Fuchsian surface subgroups of infinite covolume Kleinian groups

by

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A dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy in Mathematics in the Graduate Division of the University of California, Berkeley

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Abstract

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Given a complete hyperbolic 3-manifold $N$, one can ask whether its fundamental group $\Gamma = \pi_1 N$ contains any quasi-Fuchsian surface subgroups. Equivalently, given a pared 3-manifold $(M, P)$, one can ask whether there exists a closed immersed $\pi_1$-injective surface in $M$ that avoids the peripheral subgroups associated to $P$. This is known to be true for closed hyperbolic 3-manifolds, and more generally for finite volume hyperbolic 3-manifolds. We outline a strategy to solve the case of infinite volume hyperbolic 3-manifolds, that is, infinite covolume Kleinian groups. As a first step in this program, we give a characterization of books of $I$-bundles which contain quasi-Fuchsian surface subgroups.
To Mommy and Daddy
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Chapter 1

Introduction

1.1 Introduction

Let $N$ be a complete hyperbolic 3-manifold. $N$ can be realized as the quotient manifold of a Kleinian group $\Gamma = \pi_1 N$. This paper is devoted to the following general question:

Problem 1.1.1. Does $\pi_1 N$ contain any quasi-Fuchsian surface subgroups?

In this paper, we answer this question in the special case where $N$ is a book of $I$-bundles. We tackle this problem topologically, representing $N$ with a pared 3-manifold $(M, P)$. The pared structure $P \subseteq \partial M$ specifies the locus corresponding to cusps of $N$. Finding a quasi-Fuchsian surface subgroup of $\pi_1 N$ amounts to asking whether we can find a closed immersed $\pi_1$-injective surface in $M$ which avoids $P$ up to homotopy. That is, we’re trying to choose a surface subgroup of $\pi_1 M$ which has trivial intersection with each of the peripherial subgroups associated to components of $P$. We fully characterize which $(M, P)$ admit a surface subgroup in the book of $I$-bundles case.

To motivate this choice of problem and special case to solve, we briefly discuss the history of this problem and previous work on it. We begin with a straightforward, purely topological question:

Problem 1.1.2. Which 3-manifold groups contain surface subgroups? In particular, which 3-manifolds contain closed $\pi_1$-injective surfaces of negative Euler characteristic?

This problem arises quite directly when considering 3-manifold groups. The first explicit answer was proposed by Waldhausen [41] in the Surface Subgroup Conjecture: every closed irreducible 3-manifold with infinite $\pi_1$ contains a surface subgroup. As originally stated, the Conjecture only applies to closed 3-manifolds. By geometrization, we can consider two cases: Seifert fibered spaces or, more generally, graph manifolds, and manifolds with at least one hyperbolic piece. For graph manifolds, the problem was fully solved by Neumann [52]. After some preliminary efforts in the hyperbolic case by Cooper–Long [18] and Li [43], it was finally resolved in general by Kahn and Markovic in 2012 [39]. In fact, the surface subgroups
they find are quasi-Fuchsian, and they also obtained a density result stating that there are many such surfaces [38].

We can naturally ask this question for manifolds with boundary as well. Cooper, Long, and Reid showed [20] that a compact connected irreducible 3-manifold with non-empty incompressible boundary must contain an essential closed surface — that is, a $\pi_1$-injective non-peripheral closed surface — unless it’s covered by a product $F \times I$. In particular, this shows that every non-closed hyperbolic 3-manifold group with incompressible boundary contains a surface subgroup.

However, since the result of Kahn and Markovic provided us with surface subgroups which are quasi-Fuchsian, we might ask whether we can do this in the non-closed case as well. Note that the Cooper-Long-Reid result is insufficient, as the surfaces obtained may contain accidental parabolics, that is, overlap in $\pi_1$ with the pared locus $P$. If this occurs they will not be quasi-Fuchsian. This does commonly occur in practice — see other work of Cooper–Long–Reid [19] and Menasco–Reid [50] for examples. Of course, in the closed case there cannot be any parabolics to worry about, as the boundary is empty.

More recently, there have been efforts to find quasi-Fuchsian surface subgroups specifically. Masters and Zhang [47] showed that every hyperbolic knot complement contains a quasi-Fuchsian surface subgroup, and later extended this result to link complements [48]. This fully addresses the case of finite volume cusped hyperbolic 3-manifolds. An alternate proof was provided by Baker and Cooper [8]. The density result of Kahn–Markovic was also extended by work of Cooper and Futer [21].

The remaining case is infinite volume hyperbolic 3-manifolds. This paper presents a preliminary approach to the problem. We outline a general approach for all infinite volume hyperbolic 3-manifolds. We explicitly solve a special case, books of $I$-bundles, which we believe will generalize. We prove that all books of $I$-bundles admit quasi-Fuchsian surface subgroups, except for a few obvious negative cases where the pared locus is too large. We now present a brief outline of the topics we’ll cover.

In Chapter 2, we provide some brief background on relevant topics. We define quasi-Fuchsian subgroups and describe various criteria we can use to establish that a given surface subgroup is quasi-Fuchsian. We define pared 3-manifolds and explain how to translate these criteria to topological criteria for surfaces in pared 3-manifolds. We briefly discuss the case of geometrically infinite hyperbolic 3-manifolds. We restrict to the geometrically finite case, with reference to hyperbolization.

In Chapter 3, we outline a general procedure to either find a closed quasi-Fuchsian surface in a pared 3-manifold, or conclude that there is no such surface. We decompose the pared 3-manifold along disks and annuli in a JSJ hierarchy, and prove that any quasi-Fuchsian surface and pared structure behave nicely under this decomposition. We prove that the hierarchy is finite and terminates in balls and acylindrical 3-manifolds. We briefly describe strategies for the case of an acylindrical piece and the case of no acylindrical pieces. We explain how these might rely on the books of $I$-bundles theorem, motivating our study of that special case.

In Chapter 4, we prove preliminary facts we will need for the main theorem on books
CHAPTER 1. INTRODUCTION

of $I$-bundles. The quasi-Fuchsian surface construction relies on subgroup separability arguments to simplify the 3-manifold under consideration. We give a brief overview of subgroup separability, and prove key topological lemmas using it that we will need later. We also prove many topological properties of books of $I$-bundles, as well as quasi-Fuchsian surfaces and pared loci inside them. These results refine the general decomposition properties proven in Chapter 3. We use these properties to prove a criterion that establishes whether or not a given closed surface in a book of $I$-bundles is quasi-Fuchsian. Finally, we use this criterion to explicitly identify closed quasi-Fuchsian surfaces in an example book of $I$-bundles. We describe which pared structures on this example allow closed quasi-Fuchsian surfaces to be present. This example will motivate our main theorem.

In Chapter 5, we state and prove the main theorem on books of $I$-bundles. We first describe a reduction process that removes parts of the book of $I$-bundles that cannot intersect a closed quasi-Fuchsian surface. We state and prove the negative case of the main theorem — that is, which pared books of $I$-bundles do not admit quasi-Fuchsian surfaces.

We then state and prove the positive case, that all other pared books of $I$-bundles do admit quasi-Fuchsian surfaces. This proof relies on an elaborate construction. First, we use the separability properties from Chapter 4 to simplify the topology of the book of $I$-bundles $M$ and its pared locus $P$. We then construct a surface with boundary corresponding to each boundary component of $M$, with certain nice properties with respect to $P$. We then attach these surfaces with boundary together to form a closed surface. Our construction allows us to use the criterion from Chapter 4 to show that the resulting surface is quasi-Fuchsian. This will complete the main proof.

Note that all later chapters depend on Chapter 2. However, Chapters 4 and 5 may be read independently of Chapter 3. Chapter 4 is largely elementary, and readers familiar with subgroup separability and cut-and-paste arguments may wish to read only the definitions and the statement of the pared lifting criterion before moving to Chapter 5.
Chapter 2

Background

2.1 Quasi-Fuchsian surface subgroups

We assume familiarity with standard terminology and results of 3-manifold topology and Kleinian groups. For 3-manifold topology, see Hempel [32], Jaco [35], Thurston [62], or Thurston’s notes [59]. For a gentler introduction, see Schultens [55] or notes by Lackenby [42], Hatcher [31], or Calegari [12]. For general hyperbolic geometry, see Ratcliffe [54], Benedetti–Petronio [9], or Hubbard [34]. And for Kleinian groups, see Maskit [46], Marden [45], Morgan [54], Kapovich [40], or Thurston’s notes [59]. For a gentler introduction, see Matsuzaki–Taniguchi [49] or Calegari’s notes [13].

All Kleinian groups in this paper are assumed to be finitely generated and torsion-free. Unless stated otherwise, all hyperbolic manifolds are complete.

We first recall the following basic definitions.

Definition 2.1.1. A surface group is a group isomorphic to the fundamental group of a closed connected surface of negative Euler characteristic. Such a group can be written with the finite presentation \( \pi_1 \Sigma_g = \langle a_1, b_1, \ldots, a_g, b_g | a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle \) where \( g \geq 2 \).

Note that surface groups are not free. The fundamental group of a compact connected surface with non-empty boundary of negative Euler characteristic \( \Sigma_{g,b} \) is the free group on \( 2g + b - 1 \) generators. We wish to consider surface subgroups of Kleinian groups which are quasi-Fuchsian.

Definition 2.1.2. A quasi-Fuchsian group is a Kleinian group \( \Gamma < \text{PSL}_2 \mathbb{C} \) for which there exists a quasi-conformal homeomorphism \( f : S^2_\infty \to S^2_\infty \) which conjugates \( \Gamma \) to a Fuchsian group, that is, a discrete subgroup of \( \text{PSL}_2 \mathbb{R} \). In this paper we also require that a Fuchsian group have limit set \( S^1 \) — in some sources, this is called a Fuchsian group of the first kind.

As discussed in the introduction, our general problem statement is quite simple.

Problem 2.1.3. Let \( \Gamma \) be a Kleinian group of infinite covolume. Does \( \Gamma \) contain a surface subgroup which is quasi-Fuchsian?
We know that being quasi-Fuchsian is equivalent to a few other conditions. These include the condition that we actually use in this paper.

**Proposition 2.1.4.** Let $S$ be a compact surface (possibly with boundary), and $\phi: \pi_1 S \to \text{PSL}_2 \mathbb{C}$ be a discrete faithful representation. The following are equivalent.

1. $\phi(\pi_1 S)$ is quasi-Fuchsian.
2. $\phi(\pi_1 S)$ is geometrically finite, and the conjugates of peripheral subgroups of $\pi_1 S$ are precisely those elements of $\pi_1 S$ which are parabolic after applying $\phi$.
3. $\mathbb{H}^3 / \phi(\pi_1 S)$ has convex core homeomorphic to $S \times I$.
4. $\phi(\pi_1 S)$ has limit set a Jordan curve, and no element of $\phi(\pi_1 S)$ interchanges its complementary components.

**Proof.** See Morgan [51, Proposition 9.2].

As discussed in the introduction, we wish to identify quasi-Fuchsian surface groups — that is, quasi-Fuchsian groups as above where $S$ is a closed surface. Condition (2) above tells us that such a subgroup must be geometrically finite. Furthermore, it must not contain any parabolic elements at all.

We now make another observation. By condition (3), any quasi-Fuchsian surface group $\phi(\pi_1 S)$ will contain an embedded copy of the surface $S$ inside its convex core. This embedding $i: S \to \mathbb{H}^3 / \phi(\pi_1 S)$ is an isomorphism on $\pi_1$. Given a Kleinian group $\Gamma$, a quasi-Fuchsian surface subgroup $\phi(\pi_1 S) < \Gamma$ corresponds to a covering map $p: \mathbb{H}^3 / \phi(\pi_1 S) \to \mathbb{H}^3 / \Gamma$. This map is $\pi_1$-injective. Projecting $i(S)$ downward into $\mathbb{H}^3 / \Gamma$, we obtain a closed surface $p(i(S))$. $p \circ i: S \to \mathbb{H}^3 / \Gamma$ is $\pi_1$-injective. In fact, $(p \circ i) \ast (\pi_1 S) = \phi(\pi_1 S)$.

We can therefore re-state our general problem as follows.

**Problem 2.1.5.** Let $N$ be an infinite volume hyperbolic 3-manifold. That is, $\pi_1 N$ is a Kleinian group of infinite covolume. Does there exist a $\pi_1$-injective map $f: S \to N$, where $S$ is a closed surface, such that $f \ast (\pi_1 S)$ is geometrically finite and contains no parabolic elements?

In the remainder of this paper, we’ll refer to surfaces $S$ and maps $f$ which satisfy the problem condition as quasi-Fuchsian surfaces or (QF) surfaces. By abuse of notation, we’ll often simply refer to the image $f(S)$ as a (QF) surface if the choice of $S$ and $f$ is clear.

However, even though we are now looking for a topological surface map, solving this question as written still requires knowledge of the geometry of $N$. We need to know which elements of $\pi_1 N$ are parabolic, and whether or not a given subgroup is geometrically finite. We now discuss how to reduce these conditions to purely topological conditions on $S$ and $f$. 
2.2 Pared 3-manifolds and hyperbolization

Notation. We write $T^2$ for the standard 2-torus and $A^2$ for the compact annulus $S^1 \times I$.

**Definition 2.2.1.** A pared 3-manifold $(M, P)$ is a compact orientable irreducible 3-manifold $M$, together with a submanifold $P \subseteq \partial M$, such that the following conditions hold:

1. Every component of $P$ is a torus or annulus, incompressible in $M$.
2. Every noncyclic abelian subgroup of $\pi_1 M$ is peripheral with respect to $P$ — i.e., conjugate to the fundamental group of a component of $P$.
3. $(M, P)$ is "$A^2$-incompressible": every $\pi_1$-injective map $(A^2, \partial A^2) \to (M, P)$ is homotopic (as a map of pairs) to a map into $P$.

We call $P$ the pared locus or parabolic locus of the pared 3-manifold $(M, P)$.

Note that there are a few pared manifolds that are special cases, like with elementary Kleinian groups. In fact, these are precisely the pared manifolds that correspond to elementary Kleinian groups when we construct pared 3-manifolds from Kleinian groups below. This is a standard definition — for instance, see Canary–McCullough [17, pp88].

**Definition 2.2.2.** A pared manifold $(M, P)$ is elementary if it is homeomorphic (as a pair) to one of the following: $(T^2 \times I, T^2 \times \{0\})$, $(A^2 \times I, A^2 \times \{0\})$, $(A^2 \times I, \emptyset)$, or $(B^3, \emptyset)$.

Pared manifolds arise naturally in the study of Kleinian groups. Intuitively, given a geometrically finite Kleinian group $\Gamma$, the pared 3-manifold is a compact core, that is a compact submanifold $M \subseteq \mathbb{H}^3/\Gamma$ with $\pi_1 M = \Gamma$. The pared locus $P$ is the union of boundary components corresponding to cusps of $\mathbb{H}^3/\Gamma$, that is, parabolic subgroups of $\Gamma$ (up to conjugacy). Reversing this correspondence hyperbolizes a pared 3-manifold, giving it a geometrically finite geometric structure where the desired boundary regions become cusps.

To be precise, given a geometrically finite Kleinian group $\Gamma$, we construct a pared 3-manifold $(M, P) = (\mathcal{M}(\Gamma), P(\Gamma))$ as follows.

We first recall the definitions of the thick and thin parts of a convex core. Let $\mu: N \to \mathbb{R}^+$ be the map sending $x \in N$ to the length of the shortest nontrivial loop through $x$ in $N$. Now write $C(N)_{[\epsilon, \infty)} = C(N) \cap \mu^{-1}([\epsilon, \infty))$ and $C(N)_{(0, \epsilon]} = C(N) \cap \mu^{-1}((0, \epsilon])$. We refer to these as the thick part and thin part of $C(N)$, respectively.

We know that the convex core $C(\Gamma) \subseteq \mathbb{H}^3/\Gamma$ has finite volume. For $\epsilon$ sufficiently small, the thick part $C(\Gamma)_{[\epsilon, \infty)}$ is compact, and the thin part $C(\Gamma)_{(0, \epsilon]}$ is a union of small neighborhoods of the cusps. As long as $\epsilon$ is sufficiently small, the exact choice of $\epsilon$ is arbitrary, as the resulting thick and thin parts are homeomorphic. Now let $\mathcal{M}(\Gamma) = C(\Gamma)_{[\epsilon, \infty)}$, and $P(\Gamma) = C(\Gamma)_{[\epsilon]} = \partial C(\Gamma)_{[\epsilon, \infty)} \cap \partial C(\Gamma)_{(0, \epsilon]}$. That is, $P(\Gamma)$ is the boundary locus of the thick part along which we truncated to remove the thin part. We can rebuild the convex core from $(\mathcal{M}(\Gamma), P(\Gamma))$ by gluing cusp neighborhoods onto each component of $P(\Gamma)$.
Proposition 2.2.3. \((\mathcal{M}(\Gamma), \mathcal{P}(\Gamma))\) is a pared 3-manifold. \(\mathcal{M}(\Gamma) - \mathcal{P}(\Gamma)\) is homeomorphic to the convex core of \(\Gamma, C(\Gamma)\). There exists a deformation retraction from \(\mathcal{M}(\Gamma)\) to \(C(\Gamma)\), the nearest point retraction. The parabolic elements of \(\Gamma\) precisely correspond to conjugates of the peripheral subgroups \(\pi_1 \mathcal{P}(\Gamma) \subseteq \pi_1 \mathcal{M}(\Gamma)\) (really, the conjugates of the peripheral subgroups associated to each component of \(\mathcal{P}(\Gamma)\)).

Proof. See Morgan [51, Lemma 6.1 and Corollary 6.10].

We refer to \((\mathcal{M}(\Gamma), \mathcal{P}(\Gamma))\) as the pared 3-manifold associated to \(\Gamma\). Similarly, if \(N\) is a geometrically finite hyperbolic 3-manifold, we refer to \((\mathcal{M}(\pi_1 N), \mathcal{P}(\pi_1 N))\) as the pared 3-manifold associated to \(N\).

Conversely, there is Thurston’s famous hyperbolization theorem:

Theorem 2.2.4. Let \((M, P)\) be a pared 3-manifold with \(M\) Haken. Then there exists a geometrically finite hyperbolic 3-manifold \(N\) such that \((\mathcal{M}(\pi_1 N), \mathcal{P}(\pi_1 N))\) is homeomorphic to \((M, P)\).

Proof. See surveys by Morgan [51] and Scott [58] or Kapovich’s monograph [40] for a broad overview of the proof. Thurston first announced the theorem in [60]. Key details of the argument are presented in Thurston’s 3-part paper ([61], [63], and [64]).

For more details and examples of pared 3-manifolds, see Canary–McCullough [17], Morgan [51], or Thurston’s first paper on hyperbolization [61].

We now return to our original problem. We can convert a given geometrically finite Kleinian group \(\Gamma\) to a pared 3-manifold \((M, P)\). Intuitively, since \(P\) corresponds to cusps, i.e., parabolics, a surface subgroup of \(\Gamma\) avoids parabolic elements if and only if it avoids the peripheral subgroups associated to components of \(P\).

First, suppose we have a geometrically finite Kleinian group. We use the following theorem of Thurston.

Theorem 2.2.5. Let \(N\) be a geometrically finite hyperbolic manifold with convex core \(C(N)\). Suppose that \(N\) has infinite volume. Then every covering space \(N'\) of \(N\) with finitely generated nonelementary fundamental group is also geometrically finite. In fact, it suffices to assume \(N\) is geometrically finite and \(\partial C(N)\) is nonempty.

Proof. See Morgan [51, Proposition 7.1].
of a closed surface. In fact, because $\Gamma'$ is quasi-Fuchsian, we know that the convex core $C(N')$ is homeomorphic to $S \times I$, where $S$ is a closed surface such that $\pi_1 S = \Gamma'$. Map $S \to S \times \{1/2\} \subseteq S \times I$. This map is an isomorphism on $\pi_1$. Composing this with the covering map $N' \to N$, we obtain a $\pi_1$-injective map $\phi: S \to N$. This corresponds to a $\pi_1$-injective map to $(M, P)$ by applying the nearest point retraction. Applying the theorem above, no element of $\Gamma' = \phi(\pi_1 S)$ is parabolic. Because $(M, P)$ is a pared structure for $N$, this implies that no loop in $S$ is freely homotopic to a loop in $P$.

Conversely, suppose we are given a $\pi_1$-injective map $S \to M$ where $S$ is a closed surface. Suppose further that no loop in $S$ is freely homotopic to a loop in $P$. This implies that the induced subgroup $\pi_1 S < \pi_1 M$ is disjoint from all conjugates of $\pi_1 P$. Because $(M, P)$ is a pared structure for the hyperbolic 3-manifold $N$, this implies that no elements of $\pi_1 S$ are parabolic under the representation $\rho: \pi_1 S \to \text{PSL}_2 \mathbb{C}$ induced by $\pi_1 S < \pi_1 M = \pi_1 N$. Furthermore $\rho(\pi_1 S)$ is geometrically finite (by the first theorem), and $S$ is a surface without cusps. Applying the theorem, $\pi_1 S$ induces a quasi-Fuchsian surface subgroup of $\pi_1 N$.

We summarize our discussion of the geometrically finite infinite volume case as follows.

**Proposition 2.2.6.** Let $N$ be a geometrically finite hyperbolic 3-manifold of infinite volume, and $(M, P)$ a pared 3-manifold structure for $N$. (In fact, it is sufficient to assume that $N$ is noncompact.) Then quasi-Fuchsian surface subgroups of $\pi_1 N$ precisely correspond to $\pi_1$-injective maps of a closed surface $S \to M$ such that no loops in $P$ are freely homotopic to images of loops in $S$.

This allows us to reduce the main problem to the following:

**Problem 2.2.7.** Let $(M, P)$ be a pared 3-manifold which has an infinite volume hyperbolization. Does there exist a closed surface $S$ and a $\pi_1$-injective map $f: S \to M$ such that no loops in $P$ are freely homotopic to images of loops in $S$?

Similarly, we will refer to surfaces $S$ and maps $f$ which satisfy this condition for a pared 3-manifold $(M, P)$ as quasi-Fuchsian surfaces or (QF) surfaces. Again, by abuse of notation, we will simply refer to $f(S)$ as a (QF) surface if $f$ and $S$ are clear from context.

We now briefly discuss the geometrically infinite case. Because the arguments that follow in later sections are purely topological arguments applied to the pared structure, we can always use Thurston’s hyperbolization theorem to produce a geometrically finite Kleinian group to which our arguments apply. However, we would like to be able to address the geometrically infinite case also.

First note that any finitely generated Kleinian group, whether geometrically finite or geometrically infinite, is tame, that is, homeomorphic to the interior of a compact 3-manifold. This is a deep result due to Agol [11] and, independently, Calegari and Gabai [14].

We begin with the following theorem of Canary.

**Theorem 2.2.8.** Let $N = \mathbb{H}^3 / \Gamma$ be an infinite volume tame hyperbolic 3-manifold, and let $\Gamma' < \Gamma$ be a finitely generated subgroup. Then either
(1) $N' = \mathbb{H}^3/\Gamma'$ is geometrically finite, or

(2) $\Gamma'$ contains a (conjugate of a) finite index subgroup of a geometrically infinite peripheral subgroup of $\Gamma$.

Above, a geometrically infinite peripheral subgroup is a subgroup corresponding to the subsurface of a relative compact core boundary which cuts off a geometrically infinite relative end of $N$. Equivalently, since all geometrically infinite ends of tame manifolds are simply degenerate, a geometrically infinite peripheral subgroup is the subgroup associated to the boundary in the interior of $N$ of a geometrically infinite end.

Proof. See work of Canary [16]. Also see previous related work by Canary [15], Bonahon [11], and Thurston [59].

The above proposition in the geometrically finite case still works in the forward direction. The reverse direction will carry through if we can show that $N'$ (ie $\Gamma'$) is geometrically finite. That is, we want to avoid condition (2) in Canary’s theorem above. This requires an additional condition, which we incorporate in the following proposition. (2) holds precisely when our surface is freely homotopic to a finite-sheeted cover of the corresponding geometrically infinite end boundary surface. Since such an end is simply degenerate, that is, homeomorphic to a thickened surface, we can also apply the covering lemma (see 4.2.10) to obtain the following proposition.

**Proposition 2.2.9.** Let $N$ be a hyperbolic 3-manifold of infinite volume, possibly geometrically infinite, and $(M, P)$ a pared 3-manifold structure for $N$. Then quasi-Fuchsian surface subgroups of $N$ precisely correspond to $\pi_1$-injective maps of a closed surface $S \to M$ such that the following conditions hold.

(1) No loops in $P$ are freely homotopic to images of loops in $S$.

(2) $S$ is not freely homotopic into a geometrically infinite end of $M$, or, equivalently, not freely homotopic to a finite-sheeted cover of a geometrically infinite end boundary surface.

We will not elaborate on the geometrically infinite case in this paper. However, analogous constructions should provide results similar to our main result.
Chapter 3

General strategy

3.1 Decomposition of pared 3-manifolds

In the geometrically finite case, we’ve reduced the quasi-Fuchsian surface subgroup question to a topological problem about $\pi_1$-injective maps of a surface into a pared 3-manifold. However, infinite volume hyperbolic 3-manifolds may have quite complex topology, so it is not immediately clear how to find such surfaces in the resulting pared 3-manifolds. Instead of finding (QF) surfaces directly, we decompose the pared 3-manifold along disks and annuli. While the precise statements and proofs that follow are new, these are standard techniques in 3-manifold topology.

See monographs by Jaco and Shalen [36] or Johannson [37] for an overview of the JSJ theory we will use. Also see Jaco’s lectures [35]. Canary–McCullough [17] contains some discussion of JSJ theory for pared 3-manifolds. Note also that our choice of decomposition was inspired by the decomposition used by Thurston to prove geometrization of Haken manifolds. This is a decomposition into acylindrical pieces, $I$-bundles, and solid tori. See Morgan [51] or Thurston [64] for details.

First, a few simple propositions.

Proposition 3.1.1. Let $M$ be a compact orientable irreducible 3-manifold. Let $f : S \to M$ be a $\pi_1$-injective map, where $S$ is a closed surface. Let $D \subseteq M$ be a compression disk for $\partial M$. That is, $D$ is properly embedded, and $\partial D$ does not bound a disk in $\partial M$. Then $f$ is homotopic to a map such that $f(S) \cap D = \emptyset$. Given any finite collection of disjoint compression disks we can homotope $f$ away from all of them.

Proof. Intuitively, $f$ is $\pi_1$-injective, so any loop where $S$ intersects $D$ is contractible in $D$, hence contractible in $S$. Then we can use irreducibility and an innermost disk argument.

More precisely, we proceed as follows. Homotope $f$ locally so it’s transverse to $D$. $D$ is compact and properly embedded, and $S$ is compact, so $f^{-1}(D)$ is a properly embedded compact submanifold of $S$. That is, it’s a union of finitely many disjoint simple closed curves $\alpha_1, \ldots, \alpha_k$. For each $\alpha_i$, $f(\alpha_i) \subseteq D$, so $f(\alpha_i)$ is homotopically trivial in $M$. Since $f$ is $\pi_1$-injective, each $\alpha_i$ must be homotopically trivial in $S$. Since $S$ is a closed surface, this means
each $\alpha_i$ bounds a disk in $S$. Choose an innermost such disk $D' \subseteq S$. That is, $D'$ is bounded by some $\alpha_i$ and doesn’t contain any other $\alpha_j$ in its interior.

Now construct a map $\phi: S^2 \to M$ as follows. The closed upper hemisphere is homeomorphic to $D' \subseteq S$. Map it along $D'$. Then the equator maps into $f(\alpha)$, which is contractible, as it’s in $D$. Choose the lower hemisphere of $\phi$ to correspond to such a contraction of $f(\alpha)$ within $D$. $M$ is irreducible, hence aspherical, so $\phi$ must be homotopically trivial. Homotoping $\phi$ to a point and re-expanding, we can see that $f|_{D'}$ is homotopic to a map into $D$ (ie corresponding to a contracting homotopy of $\alpha$). Pushing $f$ slightly away from $D$ using a tubular neighborhood of $D$ in $M$, we can eliminate the intersection curve $f(\alpha)$ between $f(S)$ and $D$. Since this homotopy only affects a neighborhood of $D' \subseteq S$, it cannot add any additional intersections or affect any of the other intersection curves. Repeating this process, we can eliminate all intersection curves by homotopy.

We have a similar fact for the pared locus of a pared 3-manifold.

**Proposition 3.1.2.** Let $(M, P)$ be a pared 3-manifold where $M$ is compact orientable irreducible, and let $D \subseteq M$ be a compression disk for $\partial M$. Then we can isotope $P$ within $\partial M$ such that $P$ has minimal intersection with $\partial D$. In this minimal intersection, torus components of $P$ do not intersect $D$, and if we cut $M$ along $D$, all the resulting rectangles obtained by cutting annulus components of $P$ are essential — that is, they’re thickenings of essential curves in $\partial M - \partial D$. We can perform a similar operation for any finite collection of disjoint compression disks.

**Proof.** This is straightforward. Torus components of $P$ cannot intersect $\partial D$, as each torus component of $P$ is precisely a torus component of $\partial M$. This is because a torus cannot map $\pi_1$-injectively to any other genus surface, as it’d correspond to a rank 2 abelian subgroup. So if $\partial D$ intersects a torus component of $P$, it’d have to lie entirely inside it. Then $D$ would be a compression disk for this torus component, violating $\pi_1$-injectivity.

Now consider annulus components of $P$ that lie on the boundary component $F$ of $M$ that contains $\partial D$. It is sufficient to consider the core curves of these annuli, which are a disjoint set of simple closed curves in $F$. Our statement now follows from the elementary fact (about curves on surfaces) that we can put these in minimal position with respect to another simple closed curve, $\partial D$, such that the intersection contains only essential arcs. Briefly, if there are inessential arcs, they cobound disks, so begin with the innermost disks and isotope to remove any inessential arcs. Again, we can repeat this process for any disjoint collection of compression disks. This completes the proof.

We can reduce our main problem in the case where $M$ has compressible boundary. By Proposition 3.1.1, any closed $\pi_1$-injective surface $S$ can be made to avoid any compression disks. This motivates the idea that we should try to find (QF) surfaces by decomposing along
compression disks first, and then trying to find (QF) surfaces in the pieces of this decomposition. However, manifolds with incompressible boundary can still be quite complicated to study. In order to understand the problem better, we want another type of decomposition, along disjoint properly embedded essential annuli. We obtain the following similar results for a decomposition along annuli.

**Proposition 3.1.3.** Let $M$ be a compact orientable irreducible 3-manifold with incompressible boundary, and let $f: S \to M$ be a $\pi_1$-injective map of a closed surface. Let $A \subseteq M$ be a properly embedded annulus which is essential — that is, incompressible and boundary incompressible in $M$. Then $f$ is homotopic to a map such that $f(S) \cap A$ is a union of multiples of the core curve of $A$, each of which pulls back to an essential curve in $S$. For a disjoint union of annuli, we have the same result.

**Proof.** Homotope $f$ to be transverse to $A$. $A$ is an annulus, so any component of $f(S) \cap A$ is either homotopic to a multiple of the core curve of $A$ or to a contractible loop in $A$. We can locally homotope $f$ such that $f(S)$ intersects $A$ in curves of this form. Then each contractible curve must correspond to a curve that’s contractible in $S$ as well, by $\pi_1$-injectivity. These two contractions, each of which is a singular map of a disk into $M$, together induce a map $S^2 \to M$. Since $M$ is irreducible, it is aspherical. So this sphere is contractible, that is, there is a singular map $B^3 \to X$ filling it. Homotoping across this ball, we can remove this intersection between $f(S)$ and $A$. These homotopies are all local, so they don’t affect any other intersections between $f(S)$ and $A$, or between $f(S)$ and other annuli. Therefore we can do this for all intersections between $f(S)$ and a disjoint union of annuli. This completes the proof.

We also have a similar fact about the pared locus, but for annuli.

**Proposition 3.1.4.** Let $(M, P)$ be a pared 3-manifold where $M$ is compact oriented irreducible with incompressible boundary and $A \subseteq M$ is a properly embedded essential annulus. Then we can isotope $P$ within $\partial M$ such that $P$ has minimal intersection with $\partial A$. In this minimal intersection, if we cut $M$ along $A$, all the resulting rectangles obtained by cutting annulus components of $P$ are essential. We can perform a similar operation for any finite collection of disjoint properly embedded essential annuli.

**Proof.** This is proven identically to Proposition 3.1.2 above.

We will need the following definition.

**Definition 3.1.5.** Let $M$ be a compact 3-manifold with incompressible boundary which is not a 3-ball. $M$ is *acylindrical* if every proper $\pi_1$-injective map $A^2 \to M$ is homotopic rel boundary into $\partial M$. 

Note that strictly speaking, balls have incompressible boundary and satisfy the acylindrical condition, but we will want to consider them as a separate case later in this chapter. We have the following key fact from the theory of JSJ decompositions. Analogously to the loop and sphere theorems, it allows us to go from singular surfaces to embedded surfaces.

**Theorem 3.1.6 (Annulus theorem).** Let $M$ be a compact orientable irreducible 3-manifold with incompressible boundary. If $M$ is not acylindrical, then $M$ admits a properly embedded essential annulus.

**Proof.** See work by Jaco–Shalen [36] or Johannson [37]. Scott [57] has an algebraic proof. 

We now construct a hierarchy for a compact orientable irreducible 3-manifold $M$ with nonempty boundary. Let $M_0 = M$. Construct $M_{i+1}$ from $M_i$ as follows:

1. If $i$ is even, choose a maximal set of disjoint non-parallel compression disks $D_{ik}$ for the boundary of $M_i$. Cut $M_i$ along these disks to obtain $M_{i+1}$.

2. If $i$ is odd, choose a maximal set of disjoint pairwise non-parallel properly embedded essential annuli $A_{ik}$. Cut $M_i$ along these annuli to obtain $M_{i+1}$.

In both cases, if $M_i$ is not connected, we make these choices in all components.

**Definition 3.1.7.** We refer to this as an *JSJ hierarchy* for $M$.

**Lemma 3.1.8 (JSJ hierarchy lemma).** Let $M$ be a compact orientable irreducible 3-manifold with nonempty boundary. Then the JSJ hierarchy for $M$ has finite length $n$. Each component of $M_n$ is either a ball or an acylindrical 3-manifold with incompressible boundary.

**Proof.** It is immediately clear that the hierarchy terminates in a union of balls and acylindrical 3-manifolds with incompressible boundary. If any component did not have incompressible boundary, we could cut along a compression disk. Similarly, if any component was not acylindrical, we could apply the annulus theorem to find an annulus to cut along. It remains to show that the hierarchy is finite length.

By cutting out tubular neighborhoods of the disks and annuli we’re removing, we can embed each level of the hierarchy: $M_n \hookrightarrow M_{n-1} \hookrightarrow \ldots \hookrightarrow M_0$. Furthermore, by isotoping along a small tubular neighborhood of the boundary, we can in fact embed each $M_i$ into the interior of $M_{i-1}$. The hierarchy induces a hierarchy of nested disjoint embedded surfaces $\partial M_n, \partial M_{n-1}, \ldots, \partial M_0$. For odd $i$, each $\partial M_i$ is incompressible in $M_{i-1}$, since we constructed $M_i$ from a maximal set of compression disks for $\partial M_{i-1}$. We claim that each odd $\partial M_i$ is in fact incompressible in $M_{i-2}$. Suppose not, and let $D \subseteq M_{i-2}$ be a compression disk for $\partial M_i$. Let $A \subseteq M_{i-2}$ be one of the essential annuli that we cut along to obtain $M_{i-1}$. Both $A$ and $D$ are properly embedded in $M_{i-2}$, so after making them transverse, $A \cap D$ is a union of disjoint properly embedded arcs and curves in $D$. 
First, suppose the intersection contains at least one closed curve. Take an innermost such curve $\alpha \subseteq A \cap D$ — that is, one such that its interior disk $D' \subseteq D$ does not intersect any other curves in $A \cap D$. $\alpha$ cannot be essential in $A$, as otherwise $D'$ would induce a compression of $A$, contradicting the assumption that $A$ was essential in $M_{i-2}$. But $A$ is an annulus, so this implies $\alpha$ bounds a disk $D''$ in $A$. Since $M$ is irreducible, the $M_i$ are irreducible as well. So $D' \cup D''$ bounds a ball in $M_{i-2}$, and we can isotope $D$ across this ball to remove the intersection. Repeating this process, we can remove all closed curve intersections.

We now claim that $A$ and $D$ cannot intersect in any arcs. If they do, choose an outermost such arc — that is an arc $\alpha \subseteq A \cap D$ such that, for some $\beta \subseteq \partial D$ connecting the endpoints of $\alpha$, $\alpha \cup \beta$ bounds a disk $D'$ that does not intersect any other arcs in $A \cap D$. But $\partial D \subseteq \partial M_{i-2}$, so this $D'$ is a boundary compression disk for $A$, contradicting the assumption that $A$ was essential. Therefore after this isotopy we can guarantee that $D$ is disjoint from $A$.

Repeating this for all such cutting annuli $A, D$ passes to a properly embedded disk in $M_{i-1}$. But $\partial M_i$ is incompressible in $M_{i-1}$, so $\partial D$ must bound a disk $D''$ in $\partial M_{i-1}$. The cutting annuli in $M_{i-2}$ induce boundary annuli in $M_{i-1}$. As proven above, none of these annuli intersect $D$. No annuli can be contained in $D''$ as they wouldn’t be essential (as $D''$ is a boundary disk). So no annuli can intersect $D''$. Therefore the disk $D''$ induces a boundary disk in $M_{i-2}$. This proves that $\partial M_i$ is incompressible in $M_{i-2}$.

Furthermore, we now claim that $\partial M_i$ is incompressible in $M_0$. By construction $\partial M_1$ is incompressible in $M_0$. We proceed by induction. Assume $\partial M_{i-2}$ is incompressible in $M_0$. A compression disk $D$ for $\partial M_i$ intersects $\partial M_{i-2}$ in a union of disjoint simple closed curves in $D$. Any innermost disk $D'$ is a compression disk for $\partial M_{i-2}$ in $M_0$. Hence $\partial D'$ bounds a disk $D'' \subseteq \partial M_{i-2}$. By irreducibility, $D' \cup D''$ bounds a ball in $M_0$. Isotoping across this ball removes this intersection of $D$ with $\partial M_{i-2}$. Repeating this process we find that $D$ is disjoint from $\partial M_{i-2}$. But this implies $D \subseteq M_{i-2}$ (we know $\partial D \subseteq \partial M_i \subseteq M_{i-2}$. Since $\partial M_i$ is incompressible in $M_{i-2}$, $\partial D$ bounds a disk in $\partial M_i$. This proves the claim.

Furthermore, we claim that for odd $i$, unless the sequence has terminated (that is, $M_i = M_{i-1} = M_{i-2}$), $\partial M_i$ is not boundary parallel in $M_{i-2}$. That is, $\partial M_i$ and $\partial M_{i-2}$ are not parallel in $M_{i-2}$. Suppose that they are. Observe that there must be at least one properly embedded essential annulus $A \subseteq M_{i-2}$ that we cut along. $A$ corresponds to two embedded annuli $A_0, A_1 \subseteq \partial M_{i-1}$ after cutting. Note that the core curves of both $A_0$ and $A_1$ are essential in $\partial M_{i-1}$, as otherwise $A$ would not be essential in $M_{i-2}$. Look at the compression disks we cut along to reduce $M_{i-2}$ to $M_i$. Suppose that none of these disks essentially intersect $A_0$ (that is, we can isotope them so they don’t intersect $A_0$). Now simply observe that $A_0$ is itself boundary parallel in $M_{i-1}$ (it’s part of the boundary), and therefore boundary parallel in $M_i$ as well. But $\partial M_i$ is parallel to $\partial M_{i-2}$ by assumption, so $A_0$ is boundary parallel in $M_{i-2}$, contradicting the assumption that $A$ was essential in $M_{i-2}$.

Now suppose there are cutting disks that essentially intersect $A_0$. Then it is immediately clear that $\partial M_i$ cannot be parallel to $\partial M_{i-2}$, as a curve following along one of the boundary components of $A$ (which induces one of the boundary components of $A_0$) cannot possibly be parallel to a closed curve in $\partial M_{i-2}$, as it’d have to cross the compression. This proves that $\partial M_i$ is not boundary parallel in $\partial M_{i-2}$.
We now claim that in fact all of the $\partial M_i$ are pairwise nonparallel in $M_0$. Each is nonparallel to its neighbors in the sequence — $\partial M_{i-2}$ and $\partial M_{i+2}$. If any two were parallel, because of the nesting, any parallelism homotopy between them would have to cause the surfaces in between to be parallel as well. This is impossible as we know a surface is not parallel to its neighbors in the sequence.

This means that considering only the boundaries of odd terms in this JSJ hierarchy, we have a sequence of disjoint nonparallel incompressible surfaces in $M_0$. By Kneser–Haken finiteness, any such set of surfaces is finite. This completes the proof.

We now combine the earlier propositions with this lemma. Given an alternating hierarchy for a pared 3-manifold, at each stage, we can push the pared structure to avoid any decomposition disks and have essential intersections with the decomposition annuli. Note that our propositions above are not sufficient to push the pared structures repeatedly down the hierarchy, as after the first annular step in the decomposition the pared locus includes rectangles (intersecting the annulus cuts essentially) in addition to annuli and tori. This requires the notion of some kind of boundary pattern to keep track of where in the boundary we cut along annuli and disks at each inductive step. Therefore, to make this argument precise, we'll need to generalize the above propositions. This ought to be straightforward.

At the bottom of the hierarchy is a union of acylindrical 3-manifolds and balls. We make the following conjecture:

**Conjecture 3.1.9.** Let $(M, P)$ be a pared acylindrical 3-manifold. Then $M$ admits a (QF) surface.

Intuitively, this is because any acylindrical 3-manifold $M$ contains a great many proper essential $\pi_1$-injective surfaces $S$. If we combine $S$ with $\partial M$ where they meet, this induces a map from a book of $I$-bundles $M'$ into $M$. We believe that applying a combination theorem, for instance that of Baker–Cooper [7] or Agol–Groves–Manning [5], we can guarantee (under the right conditions) that this map is $\pi_1$-injective. This allows us to pull the problem back to $M'$, pulling the pared structure back and constructing a (QF) surface there. By $\pi_1$-injectivity, this will induce a (QF) surface in $M$.

We also believe that in most cases with no acylindrical pieces (that is, manifolds where the bottom of the hierarchy is a union of balls), there should still be (QF) surfaces. We believe that the same book of $I$-bundles mapping argument should work in most cases. More work is needed.

In the remainder of this paper, we address the case of books of $I$-bundles.
Chapter 4

Books of $I$-bundles — preliminary results

4.1 Separability properties of groups and spaces

We discuss some algebraic properties that will allow us to perform nice topological constructions. These are standard definitions. See surveys by Agol [3], Long–Reid [44], and Aschenbrenner–Friedl–Wilton [6] for an overview of recent work.

Definition 4.1.1. Let $G$ be a group. $G$ is residually finite, or $RF$, if for any $g \in G$, there exists a finite-index subgroup $H < G$ such that $g \notin H$.

Let $G$ be a group, and $H$ a subgroup of $G$. $H$ is separable in $G$ if for any $g \in G$, $g \notin H$, there exists a finite-index subgroup $H' < G$ such that $H' \geq H$, and $g \notin H'$.

$G$ is subgroup separable, or $LERF$, if all of its finitely generated subgroups are separable.

The following equivalences are well-known. We’ll use these facts later in our proofs of topological properties.

Proposition 4.1.2. $G$ is $RF$ if and only if for any $g \in G$ there exists a map $\phi: G \to F$, where $F$ is a finite group, such that $\phi(g) \neq id$.

$G$ is $LERF$ if and only if for any $g \in G$, $H < G$, $g \notin H$, there exists a map $\phi: G \to F$ such that $\phi(g) \notin \phi(H)$.


Corollary 4.1.3. $G$ is $LERF$ if and only if for any $g_1, \ldots, g_n \in G$, $H < G$, $g_1, \ldots, g_n \notin H$ there exists a map $\phi: G \to F$ to a finite group such that $\phi(g_i) \notin \phi(H)$ for all $i$.

Proof. Let $G$ be $LERF$. For each $g_i$, $g_i \notin H$, so there exists a map $\phi_i: G \to F_i$, a finite group, such that $\phi_i(g_i) \notin \phi_i(H)$. Now let $F = F_1 \times \cdots \times F_n$, and $\phi = \phi_1 \times \cdots \times \phi_n$. For
each $i$, $\phi(g_i) \notin \phi(H)(i)$, so $\phi(g_i) \notin \phi(H)$. This proves the forward direction. The converse is trivial.

A priori it is not obvious that any well-known groups are LERF. It is a classical theorem of Hall [30] that free groups are LERF. Scott [56] showed that surface groups are LERF as well. However, we will need stronger results in this paper. Deep work of Wise [65], building on work of Haglund–Wise [29] and Hsu–Wise [33] shows that every non-closed hyperbolic 3-manifold has LERF fundamental group. We note that the closed case has also been settled by Agol [2], incorporating work of Bergeron–Wise [10] and Kahn–Markovic [39]. Since we’re studying books of $I$-bundles, we will use the former result in this paper.

Note that Wise’s theorem is actually an extremely deep fact. Wise’s proof actually shows that hyperbolic 3-manifold with boundary groups satisfy a technical condition, namely that they are virtually compact special. Additional work by Haglund–Wise [28] demonstrates that a virtual compact special group is virtually a quasi-convex subgroup of a right-angled Artin group. Therefore later work by Haglund [27] applies in this case, showing that these groups are in fact LERF. See Agol’s survey [3] for a more detailed overview.

We now describe some elementary topological consequences of LERF. These exact statements are new, but most of the following propositions are simple translations of the conclusions of LERF into topological statements about loops and covers. In all of the following we assume $X$ is connected, path connected, and semi-locally simply connected. That is, we make the assumptions needed to apply elementary covering space theory. We also assume that $\pi_1 X$ is LERF.

See Figure 4.1 for the following propositions.

**Proposition 4.1.4.** Fix a basepoint $x_0 \in X$. Suppose that $Y \subseteq X$ is a homotopically nontrivial connected path-connected semi-locally simply connected subspace containing $x_0$, $\pi_1 Y$ is finitely generated, and $\alpha$ is a loop at $x_0$ which is not homotopic into $Y$ (relative to the basepoint $x_0$). Then there is a finite-sheeted regular cover $p: (X', x'_0) \rightarrow (X, x_0)$ with the property that the lift $\alpha'$ of $\alpha$ to $x'_0$ connects two different connected components of $p^{-1}(Y)$.

**Proof.** We apply Proposition 4.1.2. Let $g = \alpha$ and $H = \pi_1 Y \leq \pi_1 X$. By assumption, $H$ is finitely generated and $g \notin H$. Therefore there exists a map $\phi: G \rightarrow F$ such that $\phi(g) \notin \phi(H)$, where $F$ is a finite group. Without loss of generality we can assume this map is surjective (take its image). Let $H' = \ker \phi$. Now $H'$ is a finite-index normal subgroup, so it induces a finite-sheeted regular cover $p: X' \rightarrow X$. Let $x'_0$ be an arbitrary lift of $x_0$ to $X'$. We claim that $(X', x'_0)$ has the desired property. $X'$ is a regular cover, so it has covering transformation group $F$. $F$ acts on the fiber $p^{-1}(x_0)$. We know that any loop $\gamma \subseteq X$ based at $x_0$ lifts to a loop $\gamma'$ which starts at $x'_0$ and ends at $\phi(\gamma') \cdot x'_0$. Since $\phi(\alpha) \notin \phi(\pi_1 Y)$, $\alpha$ cannot end at any point that is connected to $x'_0$ by a lift of a loop in $\pi_1 Y$. Therefore its right endpoint must lie in a different connected component of $p^{-1}(Y)$.
Figure 4.1: Topological consequences of LERF
Proposition 4.1.5. Fix a basepoint \( x_0 \in X \). Suppose that \( \alpha \) and \( \beta \) are homotopically nontrivial loops at \( x_0 \) such that \( \alpha \) is not homotopic (relative to the basepoint \( x_0 \)) to a multiple of \( \beta \). Then there is a finite-sheeted regular cover \( p: (X', x'_0) \to (X, x_0) \) with the property that the lift \( \alpha' \) of \( \alpha \) to \( x'_0 \) has its right endpoint at a point not reachable by lifting multiples of \( \beta \) to \( x'_0 \).

Proof. Let \( g = \alpha \) and \( H = \langle \beta \rangle \). Since \( \alpha \) is not homotopic to a multiple of \( \beta \), \( \alpha \notin H \). Apply Proposition 4.1.2 to find a map \( \phi: G \to F \) such that \( \phi(g) \notin \phi(H) \), where \( F \) is a finite group. Let \( H' = \ker \phi \), and \( X' \) be the corresponding finite-sheeted regular cover of \( X \). By the same argument as above, a lift \( \alpha' \) of \( \alpha \) to a lifted basepoint \( x'_0 \) cannot end at any point that is connected to \( x'_0 \) by a lift of a multiple of \( \beta \). This completes the proof.

Proposition 4.1.6. Fix a basepoint \( x_0 \in X \). Suppose that we have a homotopically nontrivial loop \( \alpha \) at \( x_0 \) of infinite order in \( \pi_1 X \). Then given an integer \( k > 0 \), there is a finite-sheeted regular cover \( p: (X', x'_0) \to (X, x_0) \) with the following property. Consider the subset \( C \) of \( p^{-1}(\alpha) \) obtained by lifting multiples of \( \alpha \) to \( x'_0 \). Let \( d \) be the degree of the restricted covering map \( p|_C: C \to \alpha \). Then \( k \mid d \).

Proof. Let \( H = \langle \alpha^k \rangle \). Since \( \alpha \) has infinite order, \( \alpha, \alpha^2, \ldots, \alpha^{k-1} \notin H \). Apply Corollary 4.1.3 to find a map \( \phi: G \to F \) such that \( \phi(\alpha), \ldots, \phi(\alpha^{k-1}) \notin \phi(H) \). Let \( H' = \ker \phi \), and \( p: X' \to X \) be the associated finite-sheeted regular cover. We claim that \( X' \) has the desired property. To see this, observe that \( d \) is the smallest integer such that \( \alpha^d \) lifts to a closed curve in \( X' \). \( C \) is a finite-sheeted cover of the loop \( \alpha \), so it must be cyclic. Because \( X' \) is regular, this means \( d \) is the smallest integer such that \( \phi(\alpha)^d \) is trivial. That is, the deck transformation induced by \( \alpha \) has order \( d \).

We cannot have \( d < k \), as \( \phi(\alpha^d) = \phi(\alpha)^d = 1 \in \phi(H) \) contradicts the LERF assumption on \( \phi \). Write \( d = m_1k + m_2 \), for some \( m_1, m_2 \in \mathbb{Z}, 0 \leq m_2 < k \). \( \phi(\alpha^d) = 1 = \phi(\alpha^{m_1k})\phi(\alpha^{m_2}) \). That is, \( \phi(\alpha^{m_2}) = \phi(\alpha^k)^{-m_1} \). The right-hand side is in \( \phi(H) \). By the LERF assumption on \( \phi \), this forces \( m_2 = 0 \). So \( d \mid k \).

We also have the following propositions, which generalize the above propositions to multiple simultaneous basepoints and loops/subspaces at those basepoints. Note that the following three propositions are equivalent to saying that the three properties above are preserved under taking covers.

Proposition 4.1.7. Consider a finite collection of basepoints \( x_1, \ldots, x_n \) with corresponding loops \( \alpha_i \) and subspaces \( Y_i \) at each basepoint \( x_i \). Assume all the \( Y_i \) are homotopically nontrivial connected path-connected semi-locally simply connected subspaces with finitely generated fundamental group. Then there is a finite-sheeted regular cover \( X' \to X \) such that each pair \( \alpha_i, Y_i \) satisfies the conclusion of Proposition 4.1.4. Note that \( X' \) is regular, so the choice of lifted basepoints is arbitrary.
Proposition 4.1.8. Consider a finite collection of basepoints \( x_1, \ldots, x_n \) with corresponding loops \( \alpha_i, \beta_i \) at each basepoint \( x_i \). Then there is a finite-sheeted regular cover \( X' \to X \) such that each pair \( \alpha_i, \beta_i \) satisfies the conclusion of Proposition 4.1.5.

Proposition 4.1.9. Consider a finite collection of basepoints \( x_1, \ldots, x_n \) with corresponding loops \( \alpha_i \) at each basepoint \( x_i \). Then given integers \( k_1, \ldots, k_n > 0 \), there is a finite-sheeted regular cover \( p: X' \to X \) such that each pair \( \alpha_i, k_i \) satisfies the conclusion of Proposition 4.1.6.

Proof. All these lemmas are proved the same way. Fix a basepoint \( x_0 \) once and for all. Since \( X \) is connected, choose a path from \( x_0 \) to \( x_i \) for each \( x_i \), and use this to fix an isomorphism \( \sigma_i: \pi_1(X, x_0) \cong \pi_1(X, x_i) \). For each \( x_i \), we apply the appropriate proposition above (4.1.4, 4.1.5, or 4.1.6) to construct a map \( \phi_i: \pi_1(X, x_i) \to F_i \) with the appropriate property. Now let \( F = F_1 \times \cdots \times F_n \) and \( \phi = (\phi_1 \circ \sigma_1) \times \cdots \times (\phi_n \circ \sigma_n) \). By the same argument as Corollary 4.1.3, this cover will have the appropriate property for all \( n \) conditions.

\[ \square \]

4.2 Topology of books of I-bundles

In this section we establish basic definitions and facts about books of I-bundles and quasi-Fuchsian surfaces contained inside them. While the detailed arguments are new, these are basic applications of standard “cut-and-paste” techniques in 3-manifold topology. See Hempel [32] or Jaco [35]. Schultens [55] is a gentler treatment.

Recall that we are considering possible pared structures \( P \) on \( M \), a convex hyperbolic 3-manifold. Our goal is to find a surface map \( \phi: S \to M \) that is quasi-Fuchsian — i.e., such that \( S \) is a closed surface, \( \phi \) is \( \pi_1 \)-injective, and \( \pi_1 P_k \cap \phi_*(\pi_1 S) = 1 \) for each component \( P_k \) of \( P \).

Note that since we haven’t fixed a basepoint, these subgroups are really only defined up to conjugacy (that is, they’re sets of free homotopy classes) — what we’re saying is they fail to intersect for an arbitrary choice of conjugacy class for each subgroup. This corresponds to multiples of the parabolic curves not being freely homotopic into the immersed image of the given surface.

Definition 4.2.1. An I-bundle is a fiber bundle with fiber \( I = [0, 1] \), and base space a compact surface with boundary. We’ll refer to this as its base surface. In this paper we adopt the convention that all I-bundles are oriented. The boundary of an I-bundle \( I \to B \to \Sigma \) decomposes into the binding boundary \( \pi^{-1}(\partial \Sigma) \) and the side boundary \( \bigcup \partial \pi^{-1}(x) \). Note that the binding boundary is a collection of disjoint annuli. The reason for these terms is the following definition.

A book of I-bundles is a 3-manifold obtained from the following construction. Let \( B \) be a collection of I-bundles, and \( C \) a collection of solid tori. Attach the I-bundles to the solid tori by attaching some (or all) of the binding boundary components to disjoint nontrivial annuli in the boundaries of the solid tori. If we let \( A \) be the union of glued annuli in the
resulting manifold, our manifold can be written as \( M = B \cup_\mathcal{A} C \). Note that by construction each component of \( \mathcal{A} \) is properly embedded and 2-sided in \( M \). See Figure 4.2 for an example of this construction. Figure 4.3 illustrates the gluing map at the bottom of this diagram.

In this paper we additionally impose the following conditions on a book of \( I \)-bundles, to avoid elementary or degenerate cases.

1. The underlying manifold is a compact connected orientable irreducible hyperbolic 3-manifold with incompressible boundary.

2. Each page is an \( I \)-bundle over a surface of negative Euler characteristic.

This is a standard definition. See papers of Culler–Shalen [22, pp286] and Agol–Culler–Shalen [1, Definition 2.1] for the original definition and previous study of this special class of hyperbolic 3-manifolds.

Intuitively, if we take a physical book with thick pages and imagine bending it in a circle so its top and bottom are identified, the spine becomes a solid torus. If we also attach each
page along its top and bottom, the pages become thickened annuli, attached to the spine at one end. This manifold can be built from the above construction. Take $C$ to be a single solid torus, and $B$ to be a union of thickened annuli, one for each page of the book. Choose one binding annulus for each component of $B$, and glue all these to parallel longitudinal annuli in $\partial C$.

**Definition 4.2.2.** In keeping with the “book” terminology, we will generally refer to the components of $B$ as *pages*, the components of $C$ as *spines*, and the components of $A$ as *bindings* or *binding annuli*. Furthermore, we’ll say that $M$ is *fully bound* if all binding boundary components of all pages are glued, that is, correspond to binding annuli in $M$.

See Figures 4.4 and 4.5 for illustrations of a prototypical page and spine. For reasons we’ll see later, the books of $I$-bundles that we want to consider should all be fully bound.

Algebraically, a book of $I$-bundles corresponds to a particularly simple graph of groups. The graph is bipartite, and one side of the partition (the spines) are just copies of $\mathbb{Z}$. The
other side has noncyclic free groups, corresponding to surfaces of negative Euler characteristic. The edge groups are copies of $\mathbb{Z}$ as well.

Let us briefly explain why we impose the two conditions in Definition 4.2.1. Obviously we want $M$ to be compact connected orientable hyperbolic, as this is the general case we’re studying (Kleinian manifolds). If $M$ were nonorientable we could reduce to the orientable double cover. Hyperbolic 3-manifolds must be irreducible.

If $M$ had compressible boundary, let $D$ be a compressing disk for the boundary. Consider a $\pi_1$-injective closed surface map $\phi: S \to M$ whose image intersects $D$. The preimage $\phi^{-1}(D)$ is a union of simple closed curves in $S$, since $D$ is properly embedded. Every such closed curve $\alpha$ has homotopically trivial image in $M$, because it’s a curve in the disk $D$. But $S$ is $\pi_1$-injective, so $\alpha$ must be homotopically trivial in $S$ as well. Since $S$ is a surface, contracting the loop $\alpha$ yields a disk $D' \subseteq S$. Combining $\phi(D')$ and a contraction for $\phi(\alpha)$ in $D$ yields a sphere map $S^2 \to M$. $M$ is irreducible, hence aspherical, so this map is homotopically trivial. Using one hemisphere of this homotopy, we can homotope $\phi(D')$ across $D$ to remove the intersection curve $\phi(\alpha)$. Repeating this process, we can assume that $\phi(S)$ is disjoint from $D$. This proves that any quasi-Fuchsian surface is essentially disjoint from compression disks. Therefore, since we’re interested in quasi-Fuchsian surfaces, we might as well compress as much as possible before looking for surfaces. This also follows from the similar discussion in Chapter 3.

Ignoring for the moment the degenerate case of a single page and no spines, each page’s base surface must have at least one boundary component to glue to in order for $M$ to be connected. We don’t want to consider base surfaces which are disks. If a page were a disk, this would cause that page and its attached spine to form a 3-manifold with a finite-sheeted cover by a $k$-puctured ball — that is, a cover of a punctured lens space. This is because we can arrange things so that in the cover, the disk attaching map only traverses the longitude once, and we get a thickened disk. This means that we’ll have finite order summands in our group, which correspond to elliptic pieces, for instance lens spaces, in the JSJ decomposition.
These cases are not hyperbolic.

Similarly, allowing annulus or Moebius strip base surfaces means that many books of \(I\)-bundles we build will not be hyperbolic. For instance, by chaining annulus pages together in a loop, we can construct a book of \(I\)-bundles containing a closed non-peripheral immersed torus. In fact, in any case where there are no such loops, we can homotope away all the annulus pages (attaching the associated spines together, and possibly passing to a finite sheeted cover). The case of Moebius strips is identical once we pass to a double cover on each Moebius strip page. Therefore we will not consider these in the sequel.

Finally, our definition does technically allow for \(M\) to consist of a single page and no spines. Since \(M\) is connected, this is the only case where \(M\) can contain a page with closed base surface. In this case, the page’s base surface obviously must have negative Euler characteristic in order for \(M\) to be nonelementary hyperbolic. (Note that a thickened torus is realized by a Kleinian group, but this group must be elementary.) Combining all these cases, we only want to consider base surfaces with negative Euler characteristic.

We make the following simple observation.

**Proposition 4.2.3.** Let \(M\) be a book of \(I\)-bundles, and let \(M' \to M\) be a finite-sheeted covering space. Then \(M'\) is also a book of \(I\)-bundles.

**Proof.** This is straightforward. Lift the spines and pages of \(M\) to obtain spines and pages of \(M'\). Lift the binding annuli of \(M\) to binding annuli of \(M'\).

These facts are well-known basic results in 3-manifold topology. The following result follows immediately from the classification of \(I\)-bundles over surfaces.

**Proposition 4.2.4.** Let \(M\) be a book of \(I\)-bundles. Recall we require that \(M\) is oriented. To be orientable, every \(I\)-bundle in \(M\) is either a trivial bundle over an oriented surface, or a \(\mathbb{Z}/2\)-twisted bundle over a nonorientable surface. In the latter case, the homotopy classes in the base with nontrivial bundle twisting are precisely those without a consistent tubular neighborhood orientation.

Furthermore, every twisted \(I\)-bundle has a double cover which is a trivial \(I\)-bundle over a base surface which is an oriented double cover of the original nonorientable base surface.

**Proof.** See Hempel [32, Theorem 10.5].

This fact reflects the origin of the book of \(I\)-bundles construction from a relative JSJ decomposition. To analyze the boundary components of a book of \(I\)-bundles, we'll need a couple more simple definitions.

**Definition 4.2.5.** Let \(M\) be a book of \(I\)-bundles, and \(C\) a spine in \(M\). The **valence** \(v(C)\) is the number of binding annuli that intersect \(C\) or, equivalently, that are contained in \(\partial C\).
Let $A$ be a binding annulus in $M$. Note that $A$ lies in the boundary of exactly one spine $C_A$. The degree of $A$ is the geometric intersection number $i(A, D)$, where $D$ is a meridian disk of $C_A$. That is, the degree is the minimum number of components of $A \cap D$ under proper isotopy of $D \subseteq C_A$ and $A \subseteq \partial C_A$.

In fact, note that all binding annuli in a given spine $C$ must have the same degree. This is because their core curves are curves on a torus, and any nontrivial curves with different slopes must intersect. However, by definition binding annuli are disjoint. Since they have the same slope, they intersect a meridian curve (slope $\infty$) the same number of times. Therefore, we refer to this as the degree $d(C)$ of the spine $C$.

**Proposition 4.2.6.** Let $M$ be a book of $I$-bundles. Recall that we require that $M$ has incompressible boundary. Then $M$ cannot have any spines $C$ such that $v(C) = 0$, $d(C) = 0$, or $v(C) = d(C) = 1$. Furthermore, $M$ must be fully bound — that is, it cannot have any leftover binding annuli of pages that are not attached to spines.

**Proof.** Any spine $C$ with $v(C) = 0$ forces $C$ to be a solid torus component. This does not have incompressible boundary (it also cannot be nonelementary hyperbolic).

Any spine $C$ with $d(C) = 0$ has a meridian disk that properly embeds in the resulting book of $I$-bundles, since it’s disjoint from all the binding annuli. This is a boundary compressing disk.

Let $C$ be a spine with $v(C) = d(C) = 1$. Let $A$ be the sole binding annulus, $B$ the attached page, and $\Sigma$ its base surface. Let $\beta$ be the projection of $A$ down to $\Sigma$. $\beta$ is a boundary component of $\Sigma$. Let $\alpha$ be an essential arc in $\Sigma$ with both (distinct) endpoints in $\beta$. Note that such an $\alpha$ must exist as $\Sigma$ has negative Euler characteristic. The preimage of $\alpha$ under the projection is a rectangle $R \subseteq B$ with two edges $\gamma_1, \gamma_2$ in the the side boundary of $B$, and two edges $\delta_1, \delta_2$ that are parallel proper essential arcs in $A$. See Figure 4.6. Since $A$ intersects each meridian disk of $C$ exactly once, and does so in a single essential arc, we can construct two meridian disks $D_1, D_2$ such that $D_1 \cap R = \delta_1$, and $D_2 \cap R = \delta_2$. Let $D = D_1 \cup_{\delta_1} R \cup_{\delta_2} D_2$.

We claim that $D$ is a compression disk for $\partial M$. It is properly embedded: $D_1$ and $D_2$ are properly embedded in $C$, $R$ is properly embedded in $B$, and the union lines up the boundary components along $A$. It suffices to show that $\partial D$ does not bound a disk in $\partial M$. The options for such a disk $D'$ are very limited. $C \cap \partial M$ is an annulus, and $\partial D$ divides it into two disk regions. $D' \cap C$ must be one of these two regions, which implies it intersects $\partial A$ in two parallel boundary arcs, each with one endpoint in $\delta_1$ and one in $\delta_2$. Call these arcs $\epsilon_1$ and $\epsilon_2$. Now we can see that the only way to bound a disk in $\partial M$ is if $\gamma_1 \cup \epsilon_1$ and $\gamma_2 \cup \epsilon_2$ bounded a disk. But if this disk were inside $B$, either of these would project down to $\Sigma$ to contradict the assumption that $\alpha$ was essential. $\partial D \subseteq B \cup C$, so if $D'$ essentially intersected any other spine or page it would have to do so in an innermost disk. But $D' \subseteq \partial M$, so it can only essentially intersect a page in a surface of negative Euler characteristic and a spine in an annulus. So this is impossible, and $D'$ cannot intersect any spines or pages outside of $B$ and $C$. This proves that $D$ is a compression disk.
Any leftover (un-glued) binding annulus on a page induces a compression disk as well.
Let $B$ be a page with base surface $\Sigma$ and $A$ a binding annulus in $\partial B$ which is not glued. That is, $A \subseteq \partial M$. $A$ corresponds to a boundary curve $\beta \subseteq \Sigma$. Choose a proper essential arc $\alpha \subseteq \Sigma$ with both endpoints in $\beta$. The preimage of $\alpha$ under the bundle map is a properly embedded disk $D \subseteq M$. We claim that $D$ is a compression disk. To see this, observe that any boundary disk $D' \subseteq B \cap \partial M$ would contradict that $\alpha$ was essential, by the same argument as above. $D'$ cannot intersect any other pages or spines for the same reason as well. This proves that $D$ is a compression disk, completing the proof.

We make a few observations about the boundary of $M$. Each page contributes two (if it’s a trivial $I$-bundle) or one (if it’s a twisted $I$-bundle) side boundary pieces to the boundary of $M$. Each spine contributes a number of disjoint annuli equal to its valence, that is, the number of attached pages. These are the annuli in its boundary which lie in between the binding annuli. As these glue up to form the boundary components of $M$, each spine annulus connects two side boundaries of “adjacently glued” pages. This intuitive picture will be very important later as we construct quasi-Fuchsian surfaces.

The following basic property will be useful later.

**Proposition 4.2.7.** Let $X$ be a page or spine in a book of $I$-bundles $M$. Then $X$ is irreducible, and each binding annulus in $\partial X$ (ie component of $A \cap X$) is incompressible in $X$.

**Proof.** We first claim $\pi_2 X = 0$. A trivial $I$-bundle is a thickened surface of negative Euler characteristic, hence is aspherical. A twisted $I$-bundle has a double cover which is a trivial $I$-bundle over the oriented double cover of the base surface. Solid tori are aspherical as well. It follows from the sphere theorem that $X$ is irreducible.

Let $A \subseteq \partial X$ be a binding annulus. To prove $A$ is incompressible in $X$, we’ll show that it is $\pi_1$-injective. Let $X$ be a solid torus. Then since $A$ intersects a meridian disk at least once (as shown above), it must contain a multiple of a generator of $\pi_1 X$. So $A$ is $\pi_1$-injective. Let $X$ be an $I$-bundle. Again, we can pass to a double cover which is a trivial $I$-bundle. Now $A \subseteq X$ is $\pi_1$-injective because it’s a thickening of a boundary component of a compact surface. This completes the proof.

In fact, we can extend this to a few key facts for the entire book of $I$-bundles. First, we have the following proposition:

**Proposition 4.2.8.** Every book of $I$-bundles that we construct without violating Proposition 4.2.6 does in fact have incompressible boundary. (That is, our definitions are consistent.)

**Proof.** Let $M$ be a book of $I$-bundles constructed without violating Proposition 4.2.6. Suppose it has compressible boundary, and let $D$ be a compression disk. $D$ is properly embedded
in \( M \), so it intersects the binding annuli \( A \) in a disjoint union of properly embedded arcs and simple closed curves. We first claim that we can isotope \( D \) rel boundary to remove all intersections that are closed curves.

Let \( A \) be a binding annulus. Any closed curve in \( A \) is either homotopic to the core or bounds a disk in \( A \). If \( D \) intersects \( A \) in a curve homotopic to the core, that curve has to bound a subdisk \( D' \subseteq D \). \( D' \) is a compression disk of \( A \subseteq M \). We claim that such a disk must be trivial — that is, \( A \) is incompressible in \( M \). Look at \( D' \cap A \), which is a disjoint union of simple closed curves in \( D' \), and consider an innermost curve \( \alpha \) in \( D' \). \( \alpha \) lies in some annulus \( A' \subseteq A \). By Proposition 4.2.7, \( A' \) is incompressible in the page and spine on each side. But since \( \alpha \) is innermost, it bounds a subdisk \( D'_\alpha \) of \( D' \) that’s contained in a single page or spine \( X \). By incompressibility, \( \alpha \) must bound a disk \( D''_\alpha \) in \( A' \). By irreducibility, \( D'_\alpha \cup D''_\alpha \) bounds a ball in \( X \), and we can isotope across this ball to remove the intersection curve \( \alpha \). Repeating this process, \( D' \) cannot intersect \( A \) anywhere except on its boundary. But this implies that \( D' \) lies in a single page or spine. Therefore by Proposition 4.2.7 \( D' \) must be trivial. Hence \( A \) is incompressible in \( M \), and no core curve intersections are possible.

We have shown that \( D \) can only intersect a binding annulus in curves that bound disks in that annulus. We now isotope \( D \) rel boundary to remove all such intersections. Take an innermost curve \( \alpha \) in \( D \cap A \), and say it lies in some binding annulus \( A \). Since it’s innermost, \( \alpha \) bounds a subdisk \( D'_\alpha \) of \( D \) that is properly embedded in a single page or spine of \( M \). But \( \alpha \) also bounds a disk \( D''_\alpha \subseteq A \). By irreducibility, \( D'_\alpha \cup D''_\alpha \) bounds a ball in \( X \), and we can isotope across this ball to remove the intersection curve \( \alpha \). Repeating this process, \( D \cap A \) contains no closed curves.

Let \( A \) be a binding annulus. Any properly embedded arc in \( A \) is homotopic to a standard transverse arc. If \( D \) intersects \( A \) in such an arc, that arc cuts off a subdisk \( D' \subseteq D \). \( D' \) is a boundary compression disk of \( A \subseteq M \). As above, we claim that such a disk must be trivial — that is, \( A \) is boundary incompressible in \( M \). We make a similar argument. Look at \( D' \cap A \), which is a disjoint union of properly embedded arcs in \( D' \), and consider an outermost arc \( \alpha \) in \( D' \). \( \alpha \) is a transverse arc of some annulus \( A' \subseteq A \). \( \alpha \) cobounds a disk \( D'_\alpha \) with an arc \( \beta \subseteq \partial D' \). Since \( \alpha \) is outermost, \( D'_\alpha \) must be contained in a single page or spine \( X \) adjacent to \( A' \). Note that \( \beta \subseteq X \cap \partial M \), and the two endpoints of \( \beta \) lie on the two boundary curves of \( A' \).

But if \( X \) is a spine, it must have \( d(X) > 1 \) or \( v(X) > 1 \). Either way it is impossible to construct such a \( \beta \) connecting the endpoints of a transversal \( \alpha \) such that \( \alpha \cup \beta \) bounds a nontrivial disk. The only nontrivial embedded disk is a meridian disk, and if we try to bound that, \( \beta \) will cross a different binding annulus (if \( v(X) > 1 \)) or a different part of the same binding annulus (if \( d(X) > 1 \)).

If \( X \) is a page, it must be fully glued. Therefore, if \( X \) is a trivial \( I \)-bundle, \( X \cap \partial M \) has two components. The two endpoints of \( \alpha \) lie in two different components, so an arc \( \beta \subseteq X \cap \partial M \) connecting them obviously cannot exist. If \( X \) is a twisted \( I \)-bundle, the same argument applies after lifting to a trivial \( I \)-bundle double cover.

Hence any such disk \( D' \) is trivial, and \( A \) is boundary incompressible. Since any such transverse arc in \( D \cap A \) corresponds to a trivial boundary compressing disk, we can isotope
such that the intersection contains no transverse arcs as well. But this implies that $D$
 is disjoint from $A$. Hence $D$ is contained in a single page or spine $X$, and $\partial D$
 avoids all the binding annuli adjacent to $X$. Also, $\partial D$ is contained in a single
 component of $X \cap \partial M$. To prove $D$ is a trivial compressing disk, we show that the components of $X \cap \partial M$ map $\pi_1$-injectively to $X$.

If $X$ is a spine, the components of $X \cap \partial M$ are parallel to the binding annuli, so the
 proof is identical to the proof of Proposition 4.2.7. If $X$ is a page, since it’s fully glued,
 the components of $X \cap \partial M$ are either two disjoint copies of the base surface or an oriented
double cover of the base surface. In either case they immediately map in $\pi_1$-injectively. This
 proves that $X \cap \partial M$ is incompressible in $X$. Since $D$ is a compression disk of $X \cap \partial M$ in $X$, it must be trivial. This completes the proof.

We highlight the key facts we’ve proven about the binding annuli.

**Corollary 4.2.9.** Let $M$ be a book of $I$-bundles. Then in fact all the binding annuli of $M$
 are incompressible and boundary incompressible in $M$.

We now discuss the possible closed $\pi_1$-injective surfaces inside a book of $I$-bundles. These
 are the surfaces we’ll need to consider in order to find a closed quasi-Fuchsian surface —
 that is, a quasi-Fuchsian surface subgroup. Recall that we refer to these as (QF) surfaces.

Note that since every hyperbolic 3-manifold is aspherical, by elementary obstruction
 theory any injective map $\pi_1 S \to \pi_1 M$ is induced by a $\pi_1$-injective map $S \to M$. See
discussion in Long–Reid [44, Section 1.1].

Furthermore, by minimal surface theory, we can guarantee that any $\pi_1$-injective surface
 in a hyperbolic 3-manifold is homotopic to an immersed surface. A surface which is not
 immersed will contradict minimality. This is due to Freedman, Hass, and Scott [25]. See
discussion in Neumann [52, Section 1].

In what follows we’ll speak only of $\pi_1$-injective surfaces, or possibly surface subgroups.
 But it is important to note that in this situation those are the same thing, and can be chosen
to be immersed as well.

**Lemma 4.2.10** (Surface covering lemma). Let $\phi: S \to S'$ be a proper $\pi_1$-injective map
 between compact connected oriented surfaces with boundary, and suppose that $S$ is not a
 2-sphere or an annulus. Then $\phi$ is homotopic rel boundary to a finite-sheeted covering map.
 Note that the converse holds more generally — that is, any finite-sheeted covering is a proper
 $\pi_1$-injective map.

**Proof.** This is originally due to Nielsen [53]. See Gabai–Kazez [26] for a modern discussion.

We now decompose $\pi_1$-injective surfaces and pared structures on a book of $I$-bundles
 across the pages and spines. This is a refinement of the decomposition properties proved in
 Chapter 3.
Lemma 4.2.11 (Surface decomposition lemma). Let $M$ be a book of $I$-bundles, and $\phi: S \to M$ be a $\pi_1$-injective map, where $S$ is a (connected) closed orientable surface. Abusing notation, we will also refer to the image of this map as $S$. Then, we can place $S$ in minimal position with respect to $M$ such that:

1. For each page $B$, $S \cap B$ is a finite-sheeted cover of the page’s base surface (that is, to be more precise, the map $\phi^{-1}(B) \to B \to \Sigma$ is a finite-sheeted cover).

2. For each spine $C$, $S \cap C$ is a union of annuli parallel to multiples of the binding annuli in $\partial C$. The two boundary curves of each annulus lie in different binding annuli.

3. The page covers and spine annuli are attached along curves parallel to the multiples of the binding annuli at each spine. That is, for each binding annulus $A$, $S \cap A$ is a union of multiples of the core curve of $A$. All boundary components of the page and spine intersections are attached in this way to yield a closed surface.

Proof. Homotope $S$ to be transverse to $A$, the union of all binding annuli. Cutting $S$ along $A$ decomposes it into a union of properly immersed surfaces in each page or spine of $M$. We first move $S$ into minimal position with respect to $A$. For each page or spine $X \subseteq M$, $S \cap X$ is a union of properly immersed surfaces. Consider an arbitrary component $S' \subseteq S \cap X$.

If $S'$ is a disk, its boundary is an (immersed) curve in some binding annulus $A \subseteq \partial X$. $\partial S'$ is contractible in $X$ as it bounds an immersed disk. Since $A$ is incompressible, $\partial S'$ must be contractible in $A$. Combine these two contractions to form a map $S^2 \to X$ where the equator maps along $\partial S'$ and each hemisphere maps along one of the contractions. Since $X$ is aspherical, this map is homotopically trivial. This yields a homotopy of $S'$ into $A$. To keep $S$ transverse to $A$, we will actually homotope it slightly past $A$ into whatever page or spine is glued to the other side of $A$.

Now suppose that $S'$ is an annulus, and that both boundary curves of $S'$ lie in the same binding annulus $A$. We claim that $S'$ is homotopic rel boundary to an annulus in $A$. We know the two boundary components are conjugate in $\pi_1$ (they’re freely homotopic, following $S'$). Since they both lie in $A$, they must be parallel, as different multiples of the generator of $\pi_1 A$ cannot possibly be conjugate in $\pi_1 X$ (which is cyclic or free). It follows that they bound an annulus $S''$ in $A$. To prove $S'$ and $S''$ are homotopic, observe that together they describe a map $\psi: T^2 \to X$. But $\pi_1 X$ does not contain an abelian group of rank greater than 1, so $\psi$ cannot be $\pi_1$-injective. Choose a primitive $\beta$ for the kernel of this map. $\beta$ bounds an immersed disk $D$. Cutting $S \cup S''$ along $\beta$ and attaching 2 parallel copies of $D$ yields a map $\psi': S^2 \to X$. But $X$ is aspherical, so this map is homotopic to a point. Following along this homotopy (and reversing part of it) yields a homotopy between $S'$ and $S''$.

Suppose we have eliminated every piece of $S$ which is a disk or disallowed annulus in this way, by homotoping it across the binding annuli. We may need to repeat inductively, but at every stage we’re reducing the number of intersections with $A$ so this is guaranteed to terminate. We now claim that after no more such moves can be made, the result satisfies (1) and (2).
Again consider an arbitrary page or spine \( X \), and an arbitrary component \( S' \) of \( S \cap X \). We know \( \pi_1 S' \neq 1 \) by the above argument. We also know it cannot be an annulus if \( X \) is a page. Annuli with both boundary components on the same binding annulus were removed above, and no other annuli can exist because they’d project down to give a \( \pi_1 \) conjugacy between two boundary classes in a surface with boundary, which is impossible as \( \pi_1 X \) is a noncyclic free group. We claim it is \( \pi_1 \)-injective as a map \( S' \to X \) (abusing notation slightly — really this is asking about the preimage of \( S' \) on the left-hand abstract surface \( S \)). Suppose not. Then we have a nontrivial loop \( \alpha \subseteq S' \), which is homotopically trivial in \( X \). Therefore \( \alpha \) is homotopically trivial in \( M \), using that same homotopy. Since \( S \to M \) is \( \pi_1 \)-injective, \( \alpha \) must be homotopically trivial in \( S \). Since \( S \) is an oriented surface, the only way \( \alpha \) can be homotopically trivial in \( S \) but not \( S' \) is if one of the boundary components of \( S' \) cuts off a disk \( D \subseteq S \). If it cuts off a surface of any higher genus, we do not obtain any relations among elements of \( \pi_1 S' \). But if we cut \( D \) along \( A \), we can see that at least one innermost piece of \( D \) must be a disk, contradicting our minimality assumption. Hence each piece \( S' \) is \( \pi_1 \)-injective in \( X \).

Let \( B \) be a page. We know that each piece \( S' \) of \( S \cap B \) is \( \pi_1 \)-injective and proper. Projecting \( B \) to its base surface \( \Sigma \), we know this is a deformation retract (\( B \) is an \( I \)-bundle) so the composition \( S' \to \Sigma \) is also \( \pi_1 \)-injective and proper. Applying the covering lemma, we see that each piece \( S' \) is homotopic rel boundary to a finite-sheeted covering of \( \Sigma \). This proves (1).

Let \( C \) be a spine. Again we know that each piece \( S' \) of \( S \cap C \) is \( \pi_1 \)-injective and proper. Since \( C \) has cyclic fundamental group, and no piece \( S' \) is a disk, every piece \( S' \) must be an annulus. These annuli are \( \pi_1 \)-injective, and by construction must have boundary curves on different binding annuli. This proves (2).

To prove (3), look at the boundary curves of these annuli in the spines. Each boundary curve must be contained in a binding annulus \( A \subseteq \partial C \). Therefore it must be a multiple of the core curve of \( A \). This proves the first statement in (3). The remainder of (3) follows directly from the fact that \( S \) is a closed surface and that \( M \) is a union of pages and spines along binding annuli.

We now consider the possible pared structures \((M, P)\), where \( M \) is a book of \( I \)-bundles. We have a straightforward fact.

**Proposition 4.2.12.** Let \( M \) be a book of \( I \)-bundles, and \( P \) a pared structure on \( M \). \( P \) cannot contain any tori. That is, \( P \) consists entirely of annuli.

**Proof.** Each page of \( M \) has a base surface of negative Euler characteristic, so each component of a page boundary must also have negative Euler characteristic (it’s either a copy or an orientable double cover of the base surface, as discussed earlier). Since each boundary component of \( M \) consists of page boundaries glued together along annuli in spines, which contribute nothing to Euler characteristic, it immediately follows that each boundary com-
ponent of $M$ has negative Euler characteristic itself. So no boundary component admits a $\pi_1$-injective torus, and all pared locus components must be annuli.

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Lemma 4.2.13 (Pared decomposition lemma). Let $(M, P)$ be a pared 3-manifold, where $M$ is a book of $I$-bundles. Then we can isotope $P$ within $\partial M$ such that:

1. For each page $B$, $P \cap B$ is a union of disjoint rectangles and annuli, each of which projects to a thickened arc or curve in the base surface $\Sigma$ of $B$. Any such arc or curve is essential in $\Sigma$.

2. For each spine $C$, $P \cap C$ is a union of disjoint rectangles and at most one annulus. Each rectangle connects two adjacent binding annuli in $\partial C$ along a thickened arc that is essential in that component of $\partial M \cap C$ (the "intermediate annulus" between these two binding annuli). The annulus component, if any, is parallel to the binding annuli in $\partial C$.

3. For each binding annulus $A$, $P \cap A$ is a union of points in $\partial A$.

Proof. We can isotope $P$ locally, keeping all its components disjoint and embedded in $\partial M$, so that it’s transverse to $A$. This immediately guarantees (3).

Let $X$ be a page or spine in $M$. $P \cap X$ is a union of disjoint rectangles and annuli, which are thickenings of arcs and curves in $\partial M \cap X$. Suppose one of these is an arc $\alpha$ which is not essential in $\partial M \cap X$. Then $\alpha$ cobounds a disk with an arc in $\partial(\partial M \cap X)$. There may be other arcs in the interior of this disk, but with an innermost disk argument we can inductively push all such arcs out of $\partial M \cap X$ while keeping them disjoint. Note that the interior of this disk cannot contain any closed curves coming from $P$ because they would not be $\pi_1$-injective. This is an isotopy of $P$ inside $\partial M$. Repeat this inductively for all pages and spines until no such arcs remain at all. Since at each step we’re decreasing the number of intersections of $P$ with $A$, this process is guaranteed to terminate.

Let $B$ be a page. Let $\alpha$ be a closed curve in $\partial M \cap B$ which thickens to an annulus in $P \cap B$. $\alpha$ cannot bound a disk in $\partial M \cap B$, or that component of $P$ would fail to be $\pi_1$-injective in $\partial M \cap B$, hence in $M$. So either $\alpha$ is essential in $\partial M \cap B$, or $\alpha$ is parallel to a component of $\partial(\partial M \cap B)$. In the latter case, if this component corresponds to a binding annulus $A$, push the thickened $\alpha$ annulus across $A$ into the attached spine. There cannot be any annuli or rectangles in between, because components of $P$ can’t be parallel and we already eliminated all the non-essential rectangles.

Suppose we’ve performed the above isotopy on $P$. Since we can preserve transversality, (3) still holds. We claim the result satisfies (1) and (2).

To prove (1), let $B$ be a page. Observe that the rectangles and annuli in $P \cap B$ are thickenings of arcs and curves which are essential in $\partial M \cap B$. Let $\Sigma$ be the base surface of $B$. As discussed above, $\partial M \cap B$ is either two copies of $\Sigma$ (if $B$ is a trivial $I$-bundle) or an orientable double cover of $\Sigma$ (if $B$ is a twisted $I$-bundle). In either case the projection of each arc and curve must be essential in $\Sigma$, as otherwise we could lift a contracting or boundary-exiting isotopy to $\partial M \cap B$. This proves (1).

To prove (2), let $C$ be a spine. Observe that $\partial M \cap C$ is a union of parallel annuli, each of which is parallel to the binding annuli in $\partial C$. Rectangles in $C$ are essential, as guaranteed
earlier. Up to isotopy, the only embedded closed curve in an annulus is the core curve. So the only annuli in \( P \cap C \) are parallel to the binding annuli. Because the components of \( P \) are nonparallel in \( \pi_1 M \), there is at most one annulus component of \( P \cap C \). This completes the proof.

**Definition 4.2.14.** Let \((M, P)\) be a pared 3-manifold, where \( M \) is a book of \( I \)-bundles. We say that \( P \) is in minimal position if it satisfies the requirements of the pared decomposition lemma. Similarly, we say that a closed \( \pi_1 \)-injective immersed surface \( S \subseteq M \) is in minimal position if it satisfies the requirements of the surface decomposition lemma.

Let \( M \) be a book of \( I \)-bundles, and \( S \to M \) be a \( \pi_1 \)-injective map, where \( S \) is a connected closed surface. Place \( S \) and \( P \) in minimal position. We have a constructive topological criterion for when \( S \) is a (QF) surface.

Construct a “pared lifting pattern” on \( S \) as follows. We build a pattern of arcs and curves on each component of \( S \cap X \), where \( X \) is a spine or page. For each page \( B \), a component \( S' \) of \( S \cap B \) is a finite-sheeted cover of the base surface \( \Sigma \) of \( B \). The side boundary \( \partial M \cap B \) is either two copies of \( \Sigma \) or an oriented double cover. In either case, we have a canonical covering map from \( S' \) to each component of \( \partial M \cap B \). We define the pared lifting pattern on \( S' \) to be the preimages of the projection of \( P \cap B \) to \( \Sigma \) under these covering maps. Note that these images may intersect even though they are disjoint in \( P \cap B \), because pushing down to \( \Sigma \) may cause overlaps.

For each spine \( C \), a component \( S' \) of \( S \cap C \) is an immersed annulus parallel to a multiple of the binding annuli. The two components of \( \partial S' \) lie on two distinct binding annuli. Now there may or may not be a component \( A' \) of \( \partial M \cap C \) with boundary components in these same two binding annuli. (This is true when the two binding annuli are “adjacent” in \( C \)). If there is, we can homotope the boundary components of \( S' \) to be multiples of the boundary components of \( A' \), since they lie in the same binding annuli. This gives us a canonical covering map from \( S' \) to \( A' \). Begin by defining the pared lifting pattern on \( S' \) to be the preimage of \( P \cap A' \) under this map. If there is no such component \( A' \), define the pared lifting pattern to be empty. Now, if there are any annulus components of \( P \cap C \), these must be parallel to the binding annuli and contained in a component of \( \partial M \cap C \). Again, they have canonical preimages under the covering map from \( S' \) to that component of \( \partial M \cap C \). Add these to the pared lifting pattern for \( S' \).

We now construct the pared lifting pattern on \( S \) by gluing these components of \( S \cap X \) back together to form \( S \). Whenever we glue, we guarantee that for any given component of \( P \) which is represented in the pared lifting patterns on both sides, the lifted components will connect up across the gluing. Equivalently, if we look at a tubular neighborhood \( N \) of a gluing annulus \( A \), \( S \cap N \) is a union of covers of the components of \( N \cap \partial M \). This gives a local description of the pared lifting pattern near each gluing annulus — we want to ensure that the two sides match to give this description when we glue. This is easy to guarantee locally as the covering degree is the same on both sides of each gluing, so we just need to
perturb topologically so the endpoints line up. Another way of saying this is at each gluing, we make sure to attach lifts of pieces of the same component of $P$, and not to attach lifts of pieces of different components of $P$.

We generally describe the pared lifting pattern using curves and arcs, not annuli and rectangles. The topology is identical — we can thicken within $S$ to obtain the original pared lifting pattern.

**Definition 4.2.15.** We say that $S$ satisfies the pared lifting criterion or parabolic lifting criterion if the pared lifting pattern on $S$ contains no closed curves (equivalently, closed annuli, if we thicken them).

Note that in particular, if any page or spine of $M$ contains an annulus in $P$, any $S$ that essentially intersects this page or spine will automatically fail the pared lifting criterion, because that will yield a closed curve in any overlapping pieces of $S$.

**Theorem 4.2.16.** Let $M$ be a book of $I$-bundles, and $S \to M$ be a $\pi_1$-injective map, where $S$ is a connected closed surface. Place $S$ and $P$ in minimal position as above. $S$ is a (QF) surface if and only if it satisfies the pared lifting criterion.

**Proof.** One direction is straightforward. If the pared lifting pattern contains a closed curve $\alpha$, every arc making up $\alpha$ in a page or spine must be a lifted copy of an arc in a single component $P_0 \subseteq P$, because we only attached copies that came from the same component downstairs. Therefore the curve $\alpha$ is locally a cover a $P_0$, hence a cover of $P_0$. This demonstrates that $S$ contains a multiple of $P_0$ up to homotopy. So $\pi_1 S$ fails to avoid all conjugates of $\pi_1 P_0$, and $S$ is not a (QF) surface.

Now suppose that the pared lifting pattern contains no closed curves. We claim that $S$ is a (QF) surface. Suppose not. Let $\alpha$ be a multiple of the core curve of a component $P_0 \subseteq P$ such that $\alpha$ is freely homotopic into $S$. Homotope $\alpha$ into $S$. A priori $\alpha$ is no longer in minimal position with respect to the binding annuli. As we did with $P$, we can now put $\alpha$ into minimal position with respect to the binding annuli. We claim that it’s possible to homotope $\alpha$ to minimal position while keeping it within $S$. To prove this, follow the procedure in Lemma 4.2.13 to homotope $\alpha$ to minimal position. $\alpha$ begins in $S$. Since $S$ is transverse to $A$, we can obviously homotope $\alpha$ within $S$ to be transverse. Observe that every homotopy of a part of $\alpha$ across a disk or annulus (as in the proof of Lemma 4.2.13), that same disk or annulus must be part of the local component of $S$ that this part of $\alpha$ is contained in. We know this because we fully described the behavior of $S$ once we placed it in minimal position, in Lemma 4.2.11. $S$ will locally cover any of the disks or annuli that we can homotope across, so the respective parts of $\alpha$ will have available parts of $S$ to homotope within.

So we can homotope $\alpha$ to minimal position within $S$. We now claim that we can choose this minimal position curve in $S$ to be (up to homotopy within individual pages and spines) a multiple of the original minimal position of $P_0$ within $(M, P)$. We know that $P_0$ began in minimal position, so any multiple is also in minimal position. Now reverse the moves
we made to homotope \( \alpha \) into \( S \) originally. Because we’re dividing \( M \) along a union of \( \pi_1 \)-injective disjoint annuli, the only possible moves \( \alpha \) can make between different pages and spines are the disk and annulus moves described above. So the reverse of our homotopy into \( S \) is a homotopy toward minimal position of the form that we described in the previous paragraph. Hence we can perform it while keeping \( \alpha \) within \( S \).

This proves that \( \alpha \), the multiple of \( P_0 \), can be homotoped to lie in \( S \), while remaining in its original minimal position with respect to the binding annuli. This implies that locally on each page and spine, \( \alpha \) is (up to homotopy rel boundary) some collection of multiples of the pared lifting pattern components corresponding to \( P_0 \). Since any multiples connect up between pages and spines in the same way as the original pared lifting pattern, we find that \( \alpha \) locally covers components of the pared lifting pattern. Hence \( \alpha \) covers a component of the pared lifting pattern globally. \( \alpha \) is a closed curve, so the pared lifting pattern contains a closed curve. This completes the proof.

\[\square\]

### 4.3 Example

We construct a book of \( I \)-bundles \( M \) as follows. Let \( \Sigma_{1,1} \) be the compact surface of genus 1 with a single boundary component. Let \( B_1, B_2, \) and \( B_3 \) be 3 trivial \( I \)-bundles over \( \Sigma_{1,1} \). That is, \( B_1 = B_2 = B_3 = \Sigma_{1,1} \times I \). Each \( B_i \) has a single binding boundary and two side boundaries. Denote the side boundaries by \( \partial_+ B_i \) and \( \partial_- B_i \). Recall that each side boundary component of a trivial \( I \)-bundle is a homeomorphic copy of the base surface \( \Sigma_{1,1} \).

Let \( C = S^1 \times D^2 \) be a solid torus. Attach the \( B_i \) to \( C \) by gluing the binding boundaries to parallel annuli in \( \partial C \), each of which intersects a meridian disk exactly once. The result is a book of \( I \)-bundles \( M \). Let \( A_1, A_2, \) and \( A_3 \) be the respective gluing annuli. Then \( M = B \cup A \cup C \) where \( B = B_1 \cup B_2 \cup B_3, \) \( C = C, \) and \( A = A_1 \cup A_2 \cup A_3 \). See Figure 4.7 for an illustration of \( M \).

Now \( \partial M \cap C \) is a union of 3 annuli. Denote these annuli by \( A'_{12}, A'_{23}, \) and \( A'_{31} \), where each boundary annulus is labeled based on its adjacent binding annuli. We choose orientations and a cyclic order for the gluing such that the 3 boundary components of \( M \) are precisely \( \partial_+ B_1 \cup A'_{12} \cup \partial_- B_2, \) \( \partial_+ B_2 \cup A'_{23} \cup \partial_- B_3, \) and \( \partial_+ B_3 \cup A'_{31} \cup \partial_- B_1 \). We’ll refer to these as \( \partial_1 M, \partial_2 M, \) and \( \partial_3 M \), respectively. Each boundary component of \( M \) consists of two copies of \( \Sigma_{1,1} \) glued along an annulus. Topologically, this forms a genus two surface.

We know from Proposition 4.2.12 that any pared structure \( P \) on \( M \) consists entirely of annuli. Furthermore, we can apply the pared decomposition lemma to homotope \( P \) to minimal position with respect to \( A \). We then have the following consequences of the pared lifting criterion. (For all of the following, we assume \( P \) is in minimal position).

**Proposition 4.3.1.** Suppose \( P \) has a component \( P_0 \) such that \( P_0 \subseteq C \). Then \((M, P)\) does not contain a \((QF)\) surface.
Figure 4.7: Example of a book of $I$-bundles
Proof. Suppose it does. Let $S$ be a (QF) surface, and homotope $S$ to minimal position using the surface decomposition lemma. Applying the pared lifting criterion, any annulus component $S \cap C$ will cover the binding annuli at $C$ and allow $P_0$ to lift to a closed curve in the pared lifting pattern. This would show $S$ is not (QF), by the pared lifting criterion. So $S$ must be disjoint from $C$. The pages $B_i$ have free fundamental group, so none of them admit a $\pi_1$-injective map from a closed surface. Therefore such an $S$ cannot exist.

Proposition 4.3.2. Suppose $P$ has no components contained in $C$, but does have a component $P_0$ such that $P_0 \subseteq B_i$, for some $i$. Without loss of generality we can let $P_0 \subseteq \partial B_1$. Then $(M, P)$ contains a (QF) surface if and only if $P$ has no components that are contained in $\partial B_2 \sqcup \partial_{23} M \sqcup \partial_+ B_3$. Furthermore, in this case the only (QF) surfaces are covers of $\partial_{23} M$.

Proof. For the forward direction, first place $S$ in minimal position. We claim that $S \cap B_1 = \emptyset$. Suppose not. Applying the pared lifting criterion to $S \cap B_1$, the annulus $P_0 \subseteq P \cap B_1$ will yield a closed curve in $S \cap B_1$, which is a finite-sheeted cover of the core curve downstairs. Thus the surface $S$ won't be (QF) in this case.

So $S$ cannot intersect $B_1$. Using the surface decomposition theorem, we can see that the components of $S \cap B_2$ and $S \cap B_3$ are covers of the base surfaces — copies of $\Sigma_{1,1}$ — and the components of $S \cap C$ are annuli. Consider the annulus pieces in $S \cap C$. Since $S$ has no decomposition pieces in $B_1$, every annulus in $S \cap C$ must have one boundary component in $A_2$ and one boundary component in $A_3$ (otherwise it connects to $A_1$, which is impossible as $S$ is closed). Notice that these annuli locally cover $\partial_{23} M \cap C = A'_{23}$, and are attached at their boundaries to surfaces covering $\partial_{23} M \cap B_2$ and $\partial_{23} M \cap B_3$ (since these are both homeomorphic to the respective base surfaces by projecting down). Hence $S$ covers $\partial_{23} M$.

It follows that $P$ cannot contain any components in $\partial B_2$, $\partial_+ B_3$, or $\partial_{23} M$. Any such component corresponds to a closed curve in the pared lifting pattern for $\partial_{23} M \subseteq M$, which therefore lifts to a closed curve in the pared lifting pattern for $S$.

Conversely, take $S = \partial_{23} M$. $S$ meets $C$ and two of the spines, $B_2$ and $B_3$. By assumption, none of these intersects $P$ (in minimal position). Therefore, the pared lifting pattern is empty, and $S$ is (QF).

The proof above obviously generalizes to more complex books of $I$-bundles. Later we’ll use it to simplify any pared structures containing annuli that fit in a single $I$-bundle.

The hardest case to consider (and the one we will devote much of this paper to generalizing) is when none of these simplifying assumptions hold:

Theorem 4.3.3. Suppose $P$ has no components contained in $C$, or in any of the $B_i$. Then $(M, P)$ contains a (QF) surface.

Instead of giving a full proof, we’ll solve this example explicitly in a special case. This will give an idea of the flavor of the general construction. Theorem 4.3.3 will follow from the main theorem — see Chapter 5.
Theorem 4.3.4. Suppose that $P$ has no components contained in $C$, or in any of the $B_i$. Suppose further that every component of $P$ intersects each page $B_i$ in at most a single rectangle. Then $(M, P)$ contains a (QF) surface.

Choose an arbitrary page boundary component $F = \partial B_i \cong \Sigma_{1,1}$. Since $P$ is in minimal position, $F \cap P$ is a thickened set of disjoint proper essential arcs in $F$ (there are no closed curves or non-essential arcs). We now have the following elementary fact about curves on surfaces. The proof provided is new, but uses standard techniques. For an overview, see Farb–Margalit [23] or Fathi–Laudenbach–Poenaru [24].

Lemma 4.3.5. These arcs form at most 3 “bands” of parallel arcs. Furthermore, if we choose a representative arc from each band, there exists an automorphism of $\Sigma_{1,1}$ taking these arcs to a standard set of 3 disjoint non-parallel arcs. We can construct the standard set by drawing closed curves of slope 0, 1, and $\infty$ through a point on the torus, and then deleting a small open neighborhood of that point.

Proof. We first need a preliminary definition. After cutting a surface with boundary along arcs, we’ll obtain one or more connected surfaces, each with one or more boundary components. Each boundary component after cutting will have pieces from the original boundary as well as pieces from the arcs that we cut along. Given labels $\gamma_1, \ldots, \gamma_k$ for the boundary components and $\alpha_1, \ldots, \alpha_l$ for the arcs (on the original surface with boundary), we can describe each boundary component of the cut surfaces as a union of arcs, each labeled with $\gamma_1, \ldots, \gamma_k, \alpha_1, \ldots, \alpha_l$. We call this an arc pattern for that boundary component of the new surface.

We first show that $\Sigma_{1,1}$ admits at most 3 disjoint essential non-parallel arcs, and the possible surfaces and arc patterns obtained by cutting along these arcs are very restricted.

Label the boundary component of $\Sigma_{1,1}$ by $\gamma$. Consider a proper essential arc $\alpha_1 \subseteq \Sigma_{1,1}$. Fix an orientation for $\alpha_1$. Since $\Sigma_{1,1}$ is orientable, we can look at local neighborhoods of $\alpha_1$ and see that $\alpha_1$ has a well-defined “left side” and “right side” as we travel along it. Looking at the endpoints of $\alpha_1$ along $\gamma$, we are forced to connect certain endpoints in the cut-up $\gamma$ with $\alpha_1$, in order to preserve the parity. This tells us the (possibly disconnected) cut surface $S_1$ will have two boundary components. Each will have an arc pattern consisting of two arcs, one labeled $\gamma$ and one labeled $\alpha_1$.

Since we cut along a properly embedded arc, the Euler characteristic increases by one. $\chi(S_1) = 0$ and $S_1$ has two boundary components. By classification of surfaces, $S_1 = \Sigma_{0,2}$ or $\Sigma_{1,1} \sqcup \Sigma_{1,1}$. But if $S_1$ contained a disk with the arc pattern described above, embedding that disk back in $\Sigma_{1,1}$ would describe a homotopy of $\alpha_1$ into the boundary. So $S_1 = \Sigma_{0,2}$.

Now suppose we had a second proper essential arc $\alpha_2 \subseteq \Sigma_{1,1}$, disjoint from and non-parallel to $\alpha_1$. Since it’s disjoint from $\alpha_1$, $\alpha_2$ induces a proper arc in $S_1$ which connects two regions in the arc pattern labeled $\gamma$. $\alpha_2$ must have one endpoint on each boundary component of $S_1$. If both are on the same side, it’s either homotopic to the boundary of $\Sigma_{1,1}$ or parallel to $\alpha_1$. Cutting along $\alpha_2$ yields a new surface $S_2$. Topologically $S_2 = D^2$, with arc pattern consisting of 8 components in the cyclic order $(\gamma, \alpha_1, \gamma, \alpha_2, \gamma, \alpha_1, \gamma, \alpha_2)$. 
Finally, adding our 3rd proper essential arc $\alpha_3$, disjoint and non-parallel to the first two arcs, a similar argument shows that $\alpha_3$ must connect opposite $\gamma$ pieces in the arc pattern. Cutting along $\alpha_3$ yields two disks with the same arc pattern. Depending on the choice of $\alpha_3$, the arc pattern on these disks is either $(\gamma, \alpha_1, \gamma, \alpha_2, \gamma, \alpha_3)$ or $(\gamma, \alpha_1, \gamma, \alpha_3, \gamma, \alpha_2)$. So up to relabeling $\alpha_1, \alpha_2, \alpha_3$, cutting along 3 proper essential disjoint non-parallel arcs has only one possible choice of cut surfaces and arc patterns.

Observe that it is not possible to add any more disjoint non-parallel arcs. In particular, any arc we draw between $\gamma$ components of the arc pattern on either disk is homotopic to the boundary or parallel to an existing arc. Furthermore, if we add new disjoint arcs and allow them to be parallel, we can see that they must form “bands” around the existing 3 arcs in order to remain disjoint. That is, we can homotope all the arcs parallel to a given arc into a small neighborhood of that arc in the disk, without intersecting any of the non-parallel arcs.

We claim there exists an automorphism of $\Sigma_1, 1$ taking any set of 3 such arcs to any other set (in particular, to the standard set, as illustrated in Figure 4.8). Since there is only one topological result of cutting along the arcs, choose a homeomorphism of the cut surfaces. Up to relabeling the arcs, we can choose a homeomorphism that identifies matching arcs in the arc patterns (as shown above, there is only one possible arc pattern up to relabeling). Glue both sides along the arcs to obtain the desired automorphism of $\Sigma_1, 1$.

We consider connected double covers $\tilde{F} \to F \cong \Sigma_1, 1$. These covers have 2 boundary components. Any proper arc in $P \cap F$ lifts to 2 arcs in $\tilde{F}$. We say an arc in $F$ is returning for a given $\tilde{F}$ if any (or, equivalently, all) lifts of that arc have both endpoints in the same boundary component of $\tilde{F}$. Otherwise, we say it’s non-returning for that cover. See Figure 4.8.

**Lemma 4.3.6.** Given any 3 disjoint non-parallel proper essential arcs in $\Sigma_1, 1$, and any double cover of $\Sigma_1, 1$, 2 of the 3 arcs are non-returning, and the 3rd is returning. Furthermore, we can choose any 2 of the 3 we wish to be non-returning with an appropriate cover.

**Proof.** By Lemma 4.3.5 there exists an automorphism of $\Sigma_1, 1$ taking these arcs to the standard set of 3 disjoint non-parallel proper arcs. Now it suffices to observe, by looking at relative first homology or just by direct construction, that each of the three standard connected double covers (corresponding to nontrivial maps $\mathbb{Z}^2 \to \mathbb{Z}/2$) makes two of the three arcs non-returning and the third returning.

**Proof of Theorem 4.3.4.** We begin with $P$ in minimal position. By assumption, each $P \cap \partial_{\pm} B_i$ is a union of thickened arcs (that is, there are no annulus components in any $\partial_{\pm} B_i$). By Lemma 4.3.5 these can be grouped into bands. The problem is most constrained when there are exactly 3 bands, so we’ll consider that case (if there are fewer than 3, just draw some more disjoint non-parallel proper arcs on that component arbitrarily, and carry out the proof).
Figure 4.8: Lifting a standard set of 3 arcs in $\Sigma_{1,1}$
We build our surface $S$ as follows. Begin by taking copies of $A'_{12}$, $A'_{23}$, and $A'_{31}$ in $C$. Isotope each as a properly embedded submanifold of $(M, \mathcal{A})$ to bring the images into the interior of $C$ and make the boundary components on $\mathcal{A}$ disjoint. We'll refer to the homeomorphic copies we obtain from this as $F'_{12}$, $F'_{23}$, and $F'_{31}$. Let $S \cap C = F'_{12} \cup F'_{23} \cup F'_{31}$. Observe that $S \cap C$ intersects each binding annulus $A_i$ in two parallel embedded curves.

Choose each $S \cap B_i$ to be a closed immersed $\pi_1$-injective surface corresponding (via the covering lemma) to a connected double cover of the base surface. We describe the exact choice of cover below. Call these covers $F_1$, $F_2$, and $F_3$. Each of these has two boundary components. Construct $S$ by attaching the two boundary components of each $F_i$ to the two curves in the binding annulus $A_i$ described above. (Note that since every double cover we are considering is regular, it doesn't matter which boundary component of the two we connect to which curve.) See Figure 4.9. By the surface decomposition theorem and the covering lemma, $S$ is a closed immersed $\pi_1$-injective surface in $M$. We claim that we can choose the $F_i$ so that $S$ satisfies the pared lifting criterion.

First observe that $S \cap C$ consists of homeomorphic copies $F'_{12}$, $F'_{23}$, and $F'_{31}$ of each of $A'_{12}$, $A'_{23}$, and $A'_{31}$. By assumption, there are no annuli in $S \cap C$, so the pared lifting pattern on each $F'_{ij}$ is simply a homeomorphic copy of (the core of) $P \cap A'_{ij}$. This is because each is topologically a homeomorphic copy or 1-fold cover of the appropriate piece of the boundary, so the “lifting” is actually trivial. Each of these pared lifting patterns is a union of disjoint transverse arcs.

Choose an arbitrary connected double cover for $F_1$. By definition, the pared lifting pattern on $F_1$ consists of lifts of (thickened) arcs on $\partial_+ B_1$ and $\partial_- B_1$. The arcs on each form at most 3 bands, by Lemma 4.3.5 $F_1$ is a double cover of each. By Lemma 4.3.6 two of the bands on $\partial_+ B_1$ are non-returning, and the other is returning. The same holds for $\partial_- B_1$.

We cannot choose $F_2$ arbitrarily and necessarily satisfy the pared lifting criterion. We may accidentally connect a returning arc in $F_1$ to two arcs in $F'_{12}$ which connect to a returning arc in $F_2$ in $\partial_- B_2$. This would yield a closed curve in the pared lifting pattern. But by definition of the pared lifting pattern, in order for this to occur, the returning arc in $F_1$ would have to be lifted from $\partial_+ B_1$, and the returning arc in $F_2$ would have to be lifted from $\partial_- B_2$. This is because arcs that connect in the pared lifting pattern must come from pieces of the same component of $P$, and any component of $P$ with pared lifting pieces in $F_1$ and $F_2$ has to lie on $\partial_2 M = \partial_+ B_1 \cup A'_{12} \cup \partial_- B_2$.

Instead, we proceed as follows. Recall that the core arcs of components of $P \cap \partial_+ B_1$ and $P \cap \partial_- B_2$ must be contained in at most three “bands” of parallel arcs. Now let $\alpha$ and $\beta$ be core curves of two components of $P$ contained in $\partial_2 M$. By assumption, $\alpha_1 = \alpha \cap \partial_+ B_1$ is a single properly embedded arc in $\partial_+ B_1$. Similarly for $\beta_1 = \beta \cap \partial_+ B_1$. Suppose that $\alpha_1$ and $\beta_1$ are both returning arcs in $F_1$. By Lemma 4.3.5 and Lemma 4.3.6 they must lie in the same band of $\partial_+ B_1$. Write $\alpha_2 = \alpha \cap \partial_- B_2$ and $\beta_2 = \beta \cap \partial_- B_2$. We claim that $\alpha_2$ and $\beta_2$ lie in the same band of $\partial_- B_2$.

To see this, consider the standard set of 3 arcs again (see Figure 4.8). Any two distinct members of this standard set have endpoints which are interleaved. That is, if we look at a cyclic order on the boundary of this punctured torus, the endpoints of two distinct
Figure 4.9: Construction of $S$
components of the set will alternate. However, if we consider two parallel copies of a single arc in this set, the endpoints of each component will be adjacent in the cyclic order. Applying a homeomorphism, this holds in general. Since \( \alpha_1 \) and \( \beta_1 \) are parallel, their endpoints must be adjacent, not alternating. But these are connected to the endpoints of \( \alpha_2 \) and \( \beta_2 \) by transversals in the annulus \( A_{12}' \), which must be embedded. The cyclic order is preserved, so the endpoints of \( \alpha_2 \) and \( \beta_2 \) must be parallel as well. This proves the claim.

So, there is only one band on \( \partial_- B_2 \) with arcs that match with the band of \( \partial_+ B_1 \) that we chose to be returning. Therefore, we must choose \( F_2 \) such that this band is not returning.

Similarly, for \( F_3 \) we have two such constraints. We have a returning band on \( \partial_- B_1 \) from our choice of \( F_1 \), and a returning band on \( \partial_+ B_2 \) from our choice of \( F_2 \). Applying the same argument, there is at most one band on \( \partial_- B_3 \) and one band on \( \partial_+ B_3 \) that may connect across \( F_{23}' \) or \( F_{31}' \), respectively, to form closed curves. Choose \( S_3 \) such that both of these bands are non-returning. Note that this requires a slight modification to Lemma 4.3.6 — since these bands are on different boundary surfaces, they may be parallel or non-disjoint. If they aren’t disjoint, we can still find a cover that makes both non-returning by a similar relative homology argument as Lemma 4.3.6. If they are parallel, just ignore one set of bands.

Glue the \( F_i \) and \( F'_{ij} \) as described to obtain \( S \). We claim that \( S \) satisfies the pared lifting criterion. Suppose not, and let \( \alpha \) be a closed curve in the pared lifting pattern. By assumption, \( \alpha \) must intersect at least two pages of \( M \). Choose two such pieces of \( \alpha \) that are adjacent, that is, \( \alpha_i \subseteq F_i \), \( \alpha_j \subseteq F_j \) that are directly joined in the pared lifting pattern by a single arc \( \alpha_{ij} \subseteq F'_{ij} \). By the construction of \( S \), at least one of \( \alpha_i, \alpha_j \) is non-returning. We guaranteed above that two returning arcs were never directly connected.

Without loss of generality, let \( \alpha_i \) be non-returning. This means that the other endpoint of \( \alpha_i \) (the one not connected to \( \alpha_{ij} \)) lies in the other boundary component of \( F_i \). By construction of \( S \), this boundary component connects to a different annulus \( F'_{ik} \) in \( S \cap C \). This annulus cannot contain any lifts of components of \( \alpha \), as its two boundary components are parallel to a different boundary annulus \( F_{ik} \) of \( C \subseteq M \). This means that \( \alpha \) cannot continue past this gluing in the pared lifting pattern, and therefore cannot be a closed curve. Hence \( S \) satisfies the pared lifting criterion, and is (QF).

We briefly summarize the salient features of this proof which we will generalize. We began by describing which cases would be simplified or eliminated by certain closed loops, and solving those. Then, for a more general pared structure, we constructed covers of the page base surfaces which behaved nicely with respect to arcs in the pared lifting pattern. These properties will become more elaborate in the general case, but are analogous — we want arcs to connect different boundary components as the non-returning arcs do. We then derived a method for gluing that avoided creating closed loops, by taking advantage of the ability to branch between different annuli at each spine. This is the trickiest part of the general proof.
Chapter 5

Books of $I$-bundles — main theorem

5.1 The case of no quasi-Fuchsian surface

We prove a generalization of the above example. Basically, we want to first identify the cases that obviously can’t have a surface that is (QF). Then we’ll prove by construction that every other book of $I$-bundles does in fact contain such a surface. We first introduce an additional definition.

**Definition 5.1.1.** Let $(M, P)$ be a pared book of $I$-bundles, possibly with compressible boundary. We say $(M, P)$ is *reduced* if it satisfies the following criteria.

1. A meridian disk of each spine intersects the union of all binding annuli at least three times.

2. $M$ is fully bound — that is, there are no free binding boundaries of pages. Note that this would be a consequence of Proposition 4.2.6, but we are allowing $M$ to have compressible boundary.

3. Each component of $P$ (once placed in minimal position with respect to the gluing annuli) intersects at least one page $B$. Furthermore, if it intersects exactly one page, it traverses at least one spine $C$ attached to that page. In this case, either $C$ is of valence 1, or it’s of valence greater than 1 and $B$ is attached to $C$ along more than one boundary component. If valence 1, we require that $B$ is attached to $C$ along a gluing of degree at least 3.

Note that any spines of valence 1 and degree 1 would also be eliminated by Proposition 4.2.6. Also note that since it lies on a single boundary component of $M$, a parabolic cannot enter and exit the same binding annulus at a spine unless that spine has valence 1. Therefore, we can divide the cases with a parabolic that intersects a spine but only hits a single page into valence 1 spines, and multiple attached binding annuli from same page. The above criterion restricts the possible valence 1 spine cases to those which have degree at least 3.
This definition is motivated by the following important idea. If \((M, P)\) is not reduced, we can attempt to reduce it inductively. At each step, we remove pages or spines from our pared books of \(I\)-bundles to produce new pared books of \(I\)-bundles. Because we’re deleting pieces, we may end up with multiple connected components. Each component is either a pared book of \(I\)-bundles, possibly with compressible boundary, or an isolated page with no remaining boundary components — that is, a thickened closed surface. This is why we must consider \(M\) with compressible boundary in the above definition. However, if \(M\) is reduced, it will in fact have incompressible boundary by Proposition [4.2.8](#).

**Definition 5.1.2.** We *reduce* a book of \(I\)-bundles as follows. We can repeat this inductively on connected components of the result of a reduction.

(A) If \((M, P)\) has a spine \(C\) which violates (1):

   (A1) If \(C\) has 0 binding annuli, delete \(C\).

   (A2) If \(C\) has 1 binding annulus, note that the annulus may intersect a meridian disk once or twice. If it intersects once, delete \(C\). Note that the attached page will soon be deleted by criterion (2).

   (A3) If it has 1 binding annulus which intersects twice, the book of \(I\)-bundles is homeomorphic to one obtained from the following. Delete \(C\) and glue the attached page to a twisted \(I\)-bundle over a Mobius strip. This corresponds to attaching a Mobius strip to the page’s base surface.

   (A4) If \(C\) has 2 binding annuli, note that they must be parallel and each intersect a meridian disk only once. The book of \(I\)-bundles is homeomorphic to one obtained from the following. Delete \(C\) and glue two pages (or possibly two locations on the same page) along the now-free binding annuli. Notice that this also matches up parabolics.

   Note that (A3) and (A4) are the only steps that can produce a thickened closed surface (instead of a true book of \(I\)-bundles).

(B) If \((M, P)\) has a page \(B\) which violates (2), delete \(B\).

(C) If \((M, P)\) has a component \(P_0\) of \(P\) which violates (3), \(P_0\) intersects at most one page:

   (C1) If \(P_0\) intersects a single page \(B\), delete \(B\).

   (C2) If \(P_0\) intersects no pages, it lies entirely inside a spine \(C\). Delete \(C\).

In all cases, modify \(P\) by removing all components which have nonempty minimal intersection with the deleted pages or spines.
Notice that in all cases we have a pared embedding of our new manifold into the original. See Figure 5.1 for an example of this reduction algorithm. Note that in the figure, all pages are trivial $I$-bundles, and all binding annuli intersect a meridian disk of the spine exactly once. To simplify the picture, the annuli are omitted and the spines are drawn as their base surfaces.

**Theorem 5.1.3. (Reduction theorem)**

Let $(M, P)$ be a pared book of $I$-bundles. Repeatedly reducing terminates after finitely many reductions in some $(M_r, P_r)$, with an associated pared embedding $\iota: (M_r, P_r) \to (M, P)$. Each resulting connected component is either a reduced book of $I$-bundles or a thickened closed surface with empty pared locus.

Furthermore, any $(QF)$ surface in $(M, P)$ is contained in $\iota(M_r, P_r)$. In particular, if $(M_r, P_r) = \emptyset$, $(M, P)$ does not contain a $(QF)$ surface.

**Proof.** Write $M_0 = (M, P)$ and $M_i$ for the (possibly disconnected) manifold we obtain after reducing $i$ times.

Let $\text{size}(M_i) = \sum_{M_{ij}} \text{component of } M_i \text{ pages}(M_{ij}) + \text{spines}(M_{ij})$. We claim that any reduction must decrease the size. If we consider a thickened closed surface to be made out of one page and no spines, it’s immediately clear that $\text{size}(M_i) > 0$ if $M_i \neq \emptyset$. So this will imply that reducing repeatedly must terminate after finitely many reductions. Furthermore, to show that $(QF)$ surfaces are included, we must show that any reduction deletes a set disjoint from all $(QF)$ surfaces.

We consider each possible reduction.

Reduction (A1) removes spines without attached pages. No such spine can contain a $\pi_1$-injective closed surface, so it’s clearly disjoint from $(QF)$ surfaces.

Reduction (A2) removes spines attached to a single page along a binding annulus intersecting a meridian disk only once. Again, no $\pi_1$-injective closed surface can cross this spine, as by Proposition 4.2.6 the boundary admits a compression which our surface can be made disjoint from.

Reductions (A3) and (A4) do not actually delete anything. They produce a homeomorphic book of $I$-bundles (or possibly a thickened closed surface) with a homeomorphically corresponding pared structure. Therefore there’s nothing to prove in this case.

Reduction (B) removes a page with a boundary component that’s not glued to a spine. By the surface decomposition lemma, any $\pi_1$-injective surface that intersects the page at all must be a cover of the that page’s core surface. Therefore it must have boundary components covering all boundary components of the page under consideration. This includes the free boundary component, but then by the surface decomposition lemma again, these have nothing they can glue to. So it is impossible to produce a closed $\pi_1$-injective surface (without boundary) that intersects this page.

Alternatively, apply Proposition 4.2.6 to obtain a compression of the boundary, which we can make any $\pi_1$-injective surface disjoint from. By the surface decomposition lemma, this forces our surface to be disjoint from that page.
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Figure 5.1: Example of the reduction algorithm.

Steps 1-8: produce a closed trivalent surface without parallelograms.
Reduction (C1) removes a page $B$ which intersects a parabolic $P_0$. There are two cases to consider here.

1. If $P_0$ is contained entirely inside $B$, that is, it doesn’t intersect any spines at all, it’s easy to see why no (QF) surface can intersect $B$. By the surface decomposition lemma, any surface intersecting $B$ would have to be a cover of $B$. But then $P_0$ will lift to any such cover, violating the parabolic lifting criterion.

2. If $P_0$ intersects a spine, in order to satisfy the (C1) reduction criterion, it must be a valence 1 spine $C$ with $B$ attached by degree 2. $C$ cannot be valence 1 degree 1 because otherwise we’d violate (A2). Now it suffices to consider a surface that intersects the page $B$. Such a surface must be a cover of $B$. By the surface decomposition lemma, it must be glued to itself by annuli along the spine $C$. But since the gluing of $B$ to $C$ is degree 2, there can only be one topological choice of gluing through $C$, and this is the same choice that the parabolic $P_0$ makes. This is easy to see if we look at a double cover near where $B$ is glued to $C$. Since our parabolic only traverses spines of this form, our surface will necessarily remain parallel to the parabolic through each spine, and the resulting surface will violate the parabolic lifting criterion for the parabolic $P_0$. So no (QF) surface can intersect a page which satisfies the (C1) reduction criterion.

Reduction (C2) removes a spine $C$ which intersects a parabolic $P_0$. Again, by the surface decomposition lemma, any surface which traverses that spine will have to contain an annulus connecting two page gluings — that is, contain a multiple of that parabolic. This violates the pared lifting criterion.

Finally, observe that in all cases, the parabolic components we remove from $(M, P)$ as we decompose — i.e., those that intersect the deleted pieces — cannot possibly affect whether a surface outside the deleted pieces is (QF), as any surface that was otherwise (QF) but contained a multiple of any deleted parabolic would have to intersect a deleted piece. So as we inductively delete pieces of our book of $I$-bundles, we preserve the property that all (QF) surfaces and parabolics we need to determine which surfaces are (QF) remain outside the deleted parts of the book of $I$-bundles. Since our induction is guaranteed to terminate by the size measure above, and we know that every termination is a reduced book of $I$-bundles, a thickened closed surface with no parabolics, or empty, this completes the proof.

As motivation, also observe that our specific example in the earlier section is one of the simplest possible topological structures for a reduced book of $I$-bundles.

Note that the surface case is very easy.

**Proposition 5.1.4.** Let $(M, P) = (\Sigma \times I, \emptyset)$ be a thickened closed surface with empty pared locus. Then every cover of $\Sigma$ induces a (QF) surface in $(M, P)$.

**Proof.** $P = \emptyset$, so this follows directly from the covering lemma. Every closed $\pi_1$-injective surface is (QF).
5.2 The case of a quasi-Fuchsian surface

We are now ready to state the precise theorem in the positive case as well.

Theorem 5.2.1 (Main theorem, positive case). Every reduced book of I-bundles contains a (QF) surface.

Note that together with the earlier reduction theorem, we’ve fully covered the non-reduced case also. This is the main result of this paper.

Theorem 5.2.2 (Main theorem). Let $M$ be a book of I-bundles. If the reduction theorem yields a nonempty set of reduced books of I-bundles and thickened closed surfaces in $M$, then $M$ contains a (QF) surface. Otherwise, it does not.

5.3 Simplification of main theorem

We now make a number of topological simplifications by passing to an appropriate finite-sheeted cover of the book of I-bundles. Once we can construct a surface satisfying (QF) inside this cover, we will push it down and perturb to obtain a surface satisfying (QF) downstairs (since it $\pi_1$-injects into a subgroup, it will definitely still be $\pi_1$-injective).

Definition 5.3.1. A good book of I-bundles $M$ is a reduced book of I-bundles which satisfies the following additional conditions.

1. Each binding annulus on a spine intersects a meridian disk of that spine exactly once. That is, every spine has degree 1.

2. Each page is glued to a given spine at most once. That is, for any page $B$ and spine $C$, $B \cap C$ is at most a single component of $A$.

3. The two endpoints of a fiber in each page are in different boundary components of $M$. In particular, each page is a trivial I-bundle over an oriented surface.

4. Each spine intersects each boundary component of $M$ in at most a single annulus.

5. Each arc of $P$ on a page connects two different binding annuli. That is, there are no essential arcs that begin and end at the same binding annulus.

Theorem 5.3.2. Let $M$ be a reduced book of I-bundles. Then $M$ has a finite-sheeted regular cover which is good.
Proof. To prove this, we’ll use the fact that $\pi_1 M$ is LERF to construct finite sheeted covers with nice properties. Note that (1), (2), (3), and (4) are all properties that once true, remain true when we lift to finite-sheeted covers. So it suffices to guarantee each property one at a time by lifting to finite-sheeted covers, because we can lift repeatedly and the old properties will still hold. We also know that finite-sheeted covers of $M$ will be reduced.

We first take care of condition (1). For each spine $C$, let $k = d(C)$. Let $A_1, \ldots, A_n$ be the binding annuli on $C$. These annuli must be parallel. Notice that $\pi_1 C$ is infinite cyclic, as its only torsion could come from the identification with the binding annuli, but we’ve assumed that these are $\pi_1$-injective. Now if $\alpha$ is a generator for the infinite cyclic group $\pi_1 C$, $\alpha^k$ generates each $\pi_1 A_i \subseteq \pi_1 C$. Fix a point $x_0 \in C$, and apply Proposition 4.1.6 to $\alpha$ and $k$ to obtain a finite-sheeted regular cover $M' \to M$. For any spine $C'$ covering $C$, we claim that $C'$ has degree 1. Observe that $C' \to C$ is a degree $d$ cover with $k$ dividing $d$. Therefore $\alpha^d$, which lifts to traverse $C'$ exactly once, is a power of $\alpha^k$, the generator of each binding annulus. Hence each binding annulus has preimage a union of components, each of which traverses $C'$ exactly once. Hence $C'$ has degree 1.

We now produce a cover $M'$ that satisfies (1). Perform the above construction for each spine to produce generators $\alpha_1, \ldots, \alpha_n$ and degrees $k_1, \ldots, k_n$. Applying Proposition 4.1.9, we can follow the above argument locally on each spine in $M'$ to show that it has degree 1.

So now assume $M$ is reduced and satisfies (1). We claim $M$ has a finite-sheeted cover which satisfies (2).

Let $B$ be a page intersecting a spine $C$ more than once. For each pair of binding annuli $A_1, A_2$ which attach $B$ to $C$, we’ll draw a closed curve as follows. Fix a meridian disk $D$ of $C$. Choose an arbitrary point $x_1 \in A_1 \cap D$, $x_2 \in A_2 \cap D$. Choose an arbitrary arc $\alpha' \subseteq B$ connecting $x_1$ and $x_2$. Let $\alpha$ be the closed curve obtained by closing up $\alpha'$ with an arc along the interior of $D$. Denote this arc by $\alpha''$.

Fix $x_1$ to be our basepoint. We claim that $\alpha \notin \pi_1 B$. $\alpha$ intersects each of the binding annuli $A_1$ and $A_2$ in a single point. Any homotopy of $\alpha$ to a curve in $\pi_1 B$ would have to homotope $\alpha''$ rel boundary into $B$, but this is clearly impossible. So $\alpha \notin \pi_1 B$. $\pi_1 B$ is finitely generated. Apply Proposition 4.1.4 to $\alpha$ and $B$, obtaining a finite-sheeted cover $\pi: M' \to M$.

Any lift of $\alpha$ to $\alpha'$ in this cover must connect two different components of $\pi^{-1}(B)$. Any lift of $\alpha''$ still lies on a meridian disk of a spine covering $C$, and connects binding annuli covering $A_1$ and $A_2$. But $\alpha'$ lifts to arcs inside $\pi^{-1}(B)$. So in order for a lift of $\alpha$ to connect two different components of $\pi^{-1}(B)$, the lifts of $\alpha''$ must have endpoints in different components of $\pi^{-1}(B)$.

We now construct a cover $M'$ that satisfies (2). Repeat the above construction for each pair of binding annuli for each page with multiple attachments to a spine. This yields closed curves $\alpha_1, \ldots, \alpha_n$ and corresponding pages $B_1, \ldots, B_n$. Note that there may be duplicates in this list of pages, but the argument remains the same. Apply Proposition 4.1.7. By following the above argument locally, we see that any page upstairs with multiple gluings to the same spine would correspond to a lift of some $\alpha''$ with endpoints in the same component of $\pi^{-1}(B_i)$, a contradiction.
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So we have a cover which satisfies (1) and (2). We now show that $M$ satisfying (1) and (2) has a finite-sheeted cover satisfying (3).

$M$ is orientable by definition, so each page must be orientable as well. So the pages which are not trivial $I$-bundles over oriented pages must be twisted $I$-bundles over nonoriented pages (in order for the resulting page to be orientable). In particular, the endpoints of the fibers of a twisted $I$-bundle will connect globally to form a single side of the page. So if we look locally at an fiber, the two endpoints are guaranteed to be in the same boundary component of $M$ (since they’re in the same boundary component even if we just look at that page). This explains why ensuring distinct boundary components for each fiber guarantees trivial $I$-bundles.

This argument is very similar to (2). Let $B$ be a page such that fibers have both endpoints on the same boundary component $\partial_a M \subseteq \partial M$. Choose a binding annulus $A \subseteq \partial B$, and an arbitrary transversal $\alpha'' \subseteq A$. Let $\alpha' \subseteq \partial_a M$ be an arc in the boundary connecting the endpoints of $\alpha''$. Such an $\alpha'$ must exist as the two endpoints are part of the same boundary component. $\alpha'$ and $\alpha''$ combine to form a closed curve $\alpha$. Fix one endpoint of $\alpha'$ to be our basepoint. $\partial M$ is a compact surface, so $\pi_1 \partial_a M$ is finitely generated.

We claim that $\alpha \notin \pi_1 \partial_a M$. As above, any such homotopy would correspond to a homotopy rel boundary of $\partial''$ into $\partial_a M$. Since $\alpha''$ is a transversal of the binding annulus $A$, such a homotopy would correspond to a boundary compression of $A$, which is impossible as $A$ is boundary incompressible. Apply Proposition 4.1.4 to $\alpha$ and $\partial_a M$, obtaining a finite-sheeted cover. By the same argument as above with boundary compressions, $\alpha = \alpha' \cup \alpha'' \notin \pi_1 \partial_a M$, so following the same argument proves (4).

So we have a cover which satisfies (1), (2), (3), and (4). To satisfy (5), let $\alpha'$ be the core of a component of $P \cap B$ which has both endpoints in the same binding annulus $A$. Let $\alpha''$ be a segment in $\partial A$ which connects the two endpoints of $\alpha'$. Let $\alpha = \alpha' \cup \alpha''$ be the resulting closed curve. Since $\alpha'$ is essential, $\alpha \notin \pi_1 A$. Apply Proposition 4.1.4 to $\alpha$ and $A$, and follow the argument used in (2) and (3). This proves that $M$ has a cover satisfying (1)-(5), completing the proof.
5.4 Proof of main theorem

We’ve now reduced to the case of a good book of $I$-bundles. However, the remaining work is still quite involved. We proceed somewhat similarly to the first example. We take covers over the pages and glue them together cleverly to construct our surface. We check that the resulting surface satisfies the pared lifting criterion. As in the first example, this suffices to prove that our surface is (QF). We first need to take the correct cover over each page. However, the details are more complex. We first will need the following important notion.

**Definition 5.4.1.** Let $M$ be a book of $I$-bundles. A jigsaw surface $Q \subseteq M$ is obtained from a closed $\pi_1$-injective surface $S$ in minimal position (following the surface decomposition lemma, Lemma 4.2.11) by deleting any number of components of $Q \cap C$, for each spine $C$ of $M$. The result is a possibly disconnected surface with boundary, where each component is $\pi_1$-injective and in minimal position. If $(M, P)$ is a pared book of $I$-bundles, the pared lifting pattern on $Q$ is defined componentwise in the surface decomposition, just like for closed surfaces. We say that $Q$ satisfies the pared lifting criterion if its pared lifting pattern contains no closed curves.

Intuitively, a jigsaw surface looks like a partially completed jigsaw puzzle. The “pieces” are components of $Q \cap B$, where $B$ is a page in $M$. Some boundary components of these pieces are glued together (via annuli in $Q \cap C$), while others are exposed (as those annuli were deleted).

Note that at each spine $C$ of $M$, there must be an even number of boundary components of $Q$ which lie in $C$, by parity considerations with the deleted annuli. Also note that, assuming $M$ is reduced, it is trivial to build a jigsaw surface that satisfies the pared lifting criterion. To do so, simply take a closed $\pi_1$-injective surface and delete all the intersections with spines. No individual page contains a pared closed curve, so this trivial jigsaw surface (which is just a disjoint union of page covers) cannot either.

**Definition 5.4.2.** Let $Q$ be a jigsaw surface in a book of $I$-bundles $M$. The defect of $Q$ is the total number of boundary components, summing over all connected components. The local defect at a spine $C \subseteq M$ is the number of boundary components which intersect $C$. The degree of $Q$ is the covering degree on each page. (Note that since some spine components were deleted, the covering degree on a spine may be smaller.)

Our strategy is as follows. For each boundary component of $M$, we will build a jigsaw surface. This surface will be obtained by removing spine components from a cover of that boundary component. We will then align the number of boundary components of the different jigsaw surfaces where they meet at spines, and glue them all together with new annuli. This will produce a closed surface. As suggested by the above terminology, we want our jigsaw surface to have as small a defect as possible.

We will also need our jigsaw surface to satisfy the following technical conditions:
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**Definition 5.4.3.** Let $Q$ be a jigsaw surface in a book of $I$-bundles $M$. We say that $Q$ has no protoloops if it satisfies the following two conditions:

1. The pared lifting pattern on $Q$ contains no closed curves.
2. Every arc in the pared lifting pattern on $Q$ begins and ends at two different boundary components of $Q$.

Otherwise, we say that $Q$ has protoloops.

**Definition 5.4.4.** We say that $Q$ has sparse defect if, in the surface decomposition of $Q$, each component of $Q \cap B$ contains at most one component of $\partial Q$, where $B$ is an arbitrary page of $M$. That is, the “defective” (non-glued) boundary components are “spread out” among the components of $Q$ in each page.

**Lemma 5.4.5.** Let $M$ be a good book of $I$-bundles. Fix a boundary component $F$ of $M$. Then, for any sufficiently large $d$, there exists a jigsaw surface $Q \subseteq M$ with the following properties:

1. $Q$ is obtained by removing spine components from (a surface homotopic to) a finite-sheeted cover of $F$ of degree $d$.
2. $Q$ has no protoloops.
3. $Q$ has sparse defect.
4. $Q$ has defect bounded above by a constant $C$ depending only on $(M,P)$ (i.e., not depending on $d$).

**Proof.** Suppose that $F$ intersects the spines $C_1, \ldots, C_r$ and pages $B_1, \ldots, B_s$ of $M$. Since $M$ is good, $F$ intersects each of these in a single component. Denote the page intersections by $F_i = F \cap B_i$ and the spine intersectons by $F'_j = F \cap C_j$. See Figure 5.2.

Let $Q_0$ be the surface obtained by taking $d$ many disjoint copies of each of the $F_i$. $Q_0$ is obviously a jigsaw surface obtained by removing all spine components from the trivial $d$-fold cover of $F$ (that is, $d$ many disjoint copies of $F$), which is a closed $\pi_1$-injective surface in $M$. The pared lifting pattern on each component of $Q_0$ is a homeomorphic copy of (the core of) $P \cap F_i$. Since $M$ is reduced, this pattern has no closed curves. Since $M$ is good, every arc in the pattern connects two different boundary curves of its component. Hence $Q_0$ has no protoloops.

However, $Q_0$ obviously fails conditions (3) and (4) of the lemma. We need to add some annuli to ensure these conditions hold.

Fix a large integer $m > 0$. Now, reserve a subset $\mathcal{R} \subseteq \partial Q_0$. This subset should have the following properties:

1. $\mathcal{R}$ is a union of connected components of $\partial Q_0$. 

Figure 5.2: Intersections of $F$ with pages, spines, and binding annuli
(2) Each boundary component of each $F_i$ is covered by exactly $m$ elements of $\mathcal{R}$.

(3) Every connected component of $F_i$, that is, lifted copy of each page, contains at most one component of $\mathcal{R}$.

This is possible as long as $d \geq c_0 m$, where $c_0$ is the maximum number of boundary components of any $F_i$. $c_0$ is fixed and depends only on $(M,P)$. If $d \geq c_0 m$, each local cover $Q_0 \cap B_i$ has at least $c_0 m$ many disjoint copies above its respective boundary piece $F_i$. So we have enough copies to make our choices above.

We begin adding annuli that are copies of the annuli $F \cap C_j = F'_j$. Given any such annulus $F'_j$, we know there are exactly two pages $B_{i_1}$ and $B_{i_2}$ such that the two boundary components of $F'_j$ are attached to $F_{i_1}$ and $F_{i_2}$ in $F$. There are in fact two specific boundary components $\gamma_1$ of $F_{i_1}$ and $\gamma_2$ of $F_{i_2}$ that are attached in this way. Each time we add an annulus copy of $F'_j$ to $Q_0$, we must attach it to two curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ which are lifted copies of $\gamma_1$ and $\gamma_2$ in components of $Q_0 \cap B_{i_1}$ and $Q_0 \cap B_{i_2}$, respectively. This is necessary to ensure that $Q_0$ remains a jigsaw surface that can be extended to a cover of $F$. See Figure 5.3.

We add annuli one at a time. Each time we add an annulus, we make sure to add in such a way that the resulting surface still has no protoloops. We also require that we do not attach any annuli to the boundary components in $\mathcal{R}$. Aside from these requirements and the condition above (that our gluing extends to a cover of $F$), we repeatedly choose an arbitrary annulus. This process will terminate at some some surface $Q_1$, after there are no more possible gluings to make.

**Claim.** We claim that $\#(\partial Q_1 \setminus \mathcal{R})$ is bounded by $c$, a constant depending only on $(M,P)$. In particular, $c$ is independent of the choice of $d$ and $m$ made earlier.

**Proof of Claim.** Consider an arbitrary boundary component $\tilde{\gamma} \subseteq \partial Q_1 \setminus \mathcal{R}$. It covers some boundary component $\gamma \subseteq F_i$. Look at the number of arcs of $P \cap F_i$ which are incident to $\gamma$. This number depends only on $(M,P)$. Let $c_1$ be the maximum such number of incident arcs of $P$ for any boundary component $\gamma$ of any $F_i \subseteq F$. Let $c_2$ be the total number of boundary components of the $F_i$. Let $\gamma' \subseteq F'_i$ be the boundary component matching $\gamma$ — that is, the boundary component that is attached to $\gamma$ in $F$ by some spine annulus. Let $F'_j = F \cap C_j$ be the annulus in $F$ that connects them. Suppose we attempt to add an annulus to $Q_1$ that covers $F'_j$, with one of its boundary components attached to $\gamma$.

The cover extension condition allows us to attach the other boundary component to any component of $\partial(Q_1 \cap F'_i) \setminus \mathcal{R}$ above $\gamma'$. Because we always glue matching boundary components (and we started with the same number of each), $\partial Q_1 \setminus \mathcal{R}$ has the same number of remaining boundary components above $\gamma'$ as it does above $\gamma$. So there must be at least one to glue to. Since we stopped gluing at $Q_1$, this implies that any gluing we could make would force the resulting surface to have protoloops.

Let $\tilde{\gamma}' \subseteq \partial(Q_1 \cap F'_i) \setminus \mathcal{R}$ be an arbitrary available lift of $\gamma'$. Suppose gluing to $\tilde{\gamma}'$ produces a closed curve $\alpha$ in the pared lifting pattern. Obviously $\alpha$ intersects the newly added annulus $A$, otherwise $Q_1$ would already contain a closed curve. $P$ is in minimal position and $M$ is
Figure 5.3: Adding annuli to $Q_0$.
reduced, so within $A$ the pared lifting pattern is just a union of transverse arcs. Removing $A$ therefore divides $\alpha$ into a union of arcs with endpoints on either $\gamma$ or $\gamma'$. $Q_1$ has no protoloops, so none of these arcs can have both endpoints on the same boundary component. So there exists an arc in $Q_1$ connecting $\gamma$ to $\gamma'$. (This is the reason for the second condition in our definition of a “protoloop”). The pared lifting pattern in $Q_1$ near $\gamma$ is a homeomorphic copy of $(\text{the core of})$ $P \cap F_i$, so the number of arcs incident to $\gamma$ is bounded by $c_1$. Tracing these arcs through $Q_1$, they can hit at most $c_1$ other boundary components. These are the only boundary components we can attach our annulus to to produce a closed curve.

Similarly, suppose gluing to $\tilde{\gamma}'$ produces an arc $\alpha$ with endpoints on the same boundary component. Let $\delta$ be this boundary component. Again, $\alpha$ intersects the added annulus $A$, otherwise $Q_1$ would already have protoloops. Removing $A$ divides $\alpha$ into a union of arcs. Except for the two endpoints of $\alpha$ (which now lie on two different subarcs — call these subarcs $\alpha_0$ and $\alpha_1$), all other endpoints of these arcs must either lie on $\gamma$ or $\gamma'$. But no arc can have both endpoints on the same boundary component, or $Q_1$ would already have protoloops. So any subarcs except $\alpha_0$ and $\alpha_1$ must connect $\gamma$ to $\gamma'$. By parity we can see that $\alpha_0$ and $\alpha_1$ both have one endpoint on $\delta$, but their other endpoints must be different. That is, either $\alpha_0$ ends on $\tilde{\gamma}$ and $\alpha_1$ on $\tilde{\gamma}'$, or vice versa. This is the set of circumstances that leads to a returning arc.

Now, as above, the arcs incident to $\tilde{\gamma}$ hit at most $c_1$ other boundary components. These are our possible $\tilde{\delta}$. Each of these has at most $c_1$ incident arcs itself, one of which returns to $\tilde{\gamma}$, leaving $c_1 - 1$ that we need to care about. We have a total of at most $c_1(c_1 - 1)$ many “distance two” available boundary components. If we attach our annulus to one of these boundary components, we will produce a surface with protoloops.

By construction of $Q_1$, there are no more legal gluings. So every remaining boundary component must be disallowed for one of the above reasons. So $Q_1$ can have at most $c_1 + (c_1(c_1 - 1)) = c_1^2$ many leftover boundary components (that is, components of $\partial Q_1 \setminus R$) above $\gamma'$. Since our choice of $\tilde{\gamma}$ was arbitrary, this implies that there are at most $c = c_1^2 c_2$ many leftover boundary components, that is, components of $\partial Q_1 \setminus R$. This proves the claim.

\begin{proof}[Proof of Lemma 5.4.5 continued] We now attach more annuli to build a surface $Q$ which satisfies (3) and (4). Intuitively, the boundary components that make up the reserve are sparse. As long as our reserve is sufficiently large we can make use of it to attach enough annuli that only reserve boundary components remain. All the remaining non-reserve boundary components will be attached to reserve boundary components by annuli.

Construct $Q$ from $Q_1$ as follows. For each boundary component $\gamma$ downstairs, as discussed above, there are at most $c_1^2$ many components of $\partial Q_1 \setminus R$ above it. For each of these, there are at most $c_1^2$ we could glue to above $\gamma'$ that produce a surface with protoloops. We now allow ourselves to glue to the reserve boundary components. Assume that $m \geq 2c_1^2$. Then attach each leftover (non-reserve) lift $\tilde{\gamma}$ of $\gamma$ to a component of $\mathcal{R}$ above $\gamma'$ with a copy of the annulus $F'_j$, one at a time. At any point there will be at least $c_1^2 + 1$ components of $\mathcal{R}$ remaining above $\gamma'$. As in the proof of the claim, at most $c_1^2$ of these elements have arc
connections with $\tilde{\gamma}$ that will cause that gluing to yield protoloops. So we’ll always be able to choose one such that our surface still has no protoloops. Repeat this process for each $\gamma$ to construct $Q$. Since we glued all the leftovers, it immediately follows that $\partial Q \subseteq \mathcal{R}$, and therefore $Q$ satisfies (3).

Let’s analyze the possibilities for the defect of $Q$, that is, the number of boundary components. Above each $\gamma \subseteq \partial F_i$, $\mathcal{R}$ has $m$ elements. So $\partial Q$ has at most $m$ components above $\gamma$, or $c_2m$ many boundary components in total. The lower bound is determined by the maximum number of leftovers, since the only way we’ll remove reserve components from $\partial Q$ is by gluing them to non-reserve leftover components. There are at most $c_1^2$ leftover components above $\gamma$, and similarly above $\gamma'$, so there will be at least $m - c_1^2$ components remaining in $\partial Q$ above $\gamma$. We find that

$$c_2(m - c_1^2) \leq \text{defect}(Q) \leq c_2m$$

Any choice of $m \geq 2c_1^2$ works, as long as $d$ is sufficiently large relative to $m$. This proves (4).

**Proof of Main Theorem.** We use Lemma 5.4.5 to construct a jigsaw surface $Q_a$ for each boundary component $F_a$ of $M$. Now we want to glue these together, but they may have very different numbers of boundary components! So we’ll need to normalize them so they all have the same number of boundary components near each spine, so we can do a local construction with annuli in each spine.

Let $c_a$ be the constant $c_1$ we computed above for each jigsaw surface $Q_a$. Let $c_{\text{max}} = \max_a c_a$, that is, the maximum number of components of $P \cap B_i$ incident to a single boundary component of $\partial M \cap B_i$ for any spine $B_i$. By the inequality in Lemma 5.4.5, for each $\gamma \subseteq \partial F_i$,

$$\#\{\text{components of } \partial F_i \text{ above } \gamma\} \geq m - c_{\text{max}}^2.$$  

We want to make this an equality for each $Q_a$ by adding some more annuli to $Q_a$ in the same way that we added annuli in Lemma 5.4.5. Again, we use the size of the reserve $\mathcal{R}$ to our advantage. Assuming $m \geq c_{\text{max}}^2 + c_a^2$ for each $F_a$, by the same argument as the lemma, there will always be choices remaining from the reserve $\mathcal{R}$ to make the gluing. Combining these all we need is $m \geq 2c_{\text{max}}^2$.

By abuse of notation, we’ll call these normalized surfaces $Q_a$ also (the non-normalized ones will not come up again). Each has exactly $m' = m - c_{\text{max}}^2$ many boundary components above any $\gamma \subseteq \partial F_i$.

Build a single closed surface $S$ by attaching annuli to the $Q_a$ as follows.

Let $C$ be an arbitrary spine of $M$. There are exactly $d(C)$ many binding annuli in $\partial C$, and $C$ is attached to $d(C)$ distinct pages, since $M$ is good. Each page has two sides, so there are $2d(C)$ many “incident neighbor types” in the $Q_a$, corresponding to the $2d(C)$ boundary components of binding annuli in $\partial C$. These are attached to the $2d(C)$ side boundary components of pages adjacent to $C$. Because $M$ is good, we know that all these pages are distinct and all the corresponding binding annuli have degree 1.

Introduce notation as follows. A neighborhood $\mathcal{N}(C)$ consists of $d(C)$ neighborhoods inside pages, attached to $C$ by annuli. The meridian of $C$ gives a cyclic order to the attached
pages. Fix an orientation for the meridian and an (arbitrary) starting point along it, and label the pages $1, \ldots, d(C)$ under this ordering. The orientation of the meridian also gives an ordering of the boundary components of each binding annulus. Using this, label the $d(C)$ gluing annuli by $A_1, \ldots, A_{d(C)}$. Also label the $2d(C)$ binding annulus boundary components by $\gamma_1^-, \gamma_1^+, \ldots, \gamma_d^-, \gamma_d^+$, where $\gamma_i^-$ and $\gamma_i^+$ are the two boundary components of the binding annulus for the $i$th page. Extend these labels to the corresponding boundary components of the page boundaries attached to the binding annuli. Remember that all these labels are downstairs — that is, they’re labels of pieces of $\partial M$.

By construction of the $Q_a$, there are $2d(C)m'$ many components of $\partial(\bigsqcup_a Q_a) \cap C$. Attach them as follows. Let $A_{1,3}$ be an embedded valence 1 annulus in $C$ with one boundary component in $A_1$ and one in $A_3$. Define $A_{2,4}, A_{3,5}, \ldots, A_{(d(C)-1),1}, A_{d(C),2}$ similarly. Now construct the surface $S$ by taking $m'$ many copies of each $A_{i,i+2}$, and attaching their boundary components to lifts of $\gamma_i^+$ and $\gamma_{i+2}^-$. See Figure 5.4. By construction of the $Q_a$ there will be exactly $m'$ many available lifts of each to attach to. Therefore, when we repeat this for each spine $C$, the result is a closed surface $S$. Note that since $M$ is reduced, we know that each spine has at least 3 annuli. This means we’ll never attempt to attach an annulus with both boundary components in the same bounding annulus.

We claim that $S$ satisfies the pared lifting criterion.

Fix a boundary component $F_a \subseteq \partial M$. Consider an arbitrary parabolic core curve $\alpha \subseteq F_a$. We claim that the pared lifting pattern components of $\alpha$ in $S$ do not connect to form a closed curve $\tilde{\alpha}$. If we can prove this for every boundary component $F_a$ and parabolic curve $\alpha$, $S$ will satisfy the pared lifting criterion. We break down the possibilities for $\tilde{\alpha}$ as follows.

1. $\tilde{\alpha} \subseteq Q_a$.
2. $\tilde{\alpha} \subseteq Q_b$, for some $b \neq a$.
3. $\tilde{\alpha}$ intersects some $Q_b$, for $b \neq a$, in at least one arc (that is properly embedded in $Q_b$).

It is clear that these are all the possibilities, as if $\tilde{\alpha}$ is not contained in a single jigsaw surface of $S$, it must intersect at least two of them. Therefore at least one of the surfaces it intersects is not $Q_a$. We now claim that all of these are in fact impossible.

By construction of $Q_a$, the pared lifting pattern components of $\alpha$ in $Q_a$ form a union of arcs. Hence (1) is impossible.

We now claim that (2) is impossible as well. That is, the pared lifting pattern components of $\alpha$ in $Q_b$ ($b \neq a$) also form a union of arcs.

Intuitively, this is because the only way to get a pattern component of $\alpha$ in a page in $Q_b$ is to have a page $B$ such that $F_a$ and $F_b$ contain the two sides of $B$. But then, at every spine adjacent to $B$, $F_a$ and $F_b$ will diverge, so the $\alpha$ pattern cannot continue in $Q_b$ (since it must extend to a cover of $F_b$).

More precisely, suppose $\alpha'$ is a pattern component of $\alpha$ in a page component of the surface decomposition of $Q_b$. Let $B$ the corresponding page in $M$, and $C$ a spine adjacent to $B$. Since $\alpha' \subseteq F_a \neq F_b$, the only way to have $\alpha'$ components in a page component of
Figure 5.4: Attaching the $Q_a$ at $C$
$Q_b$ is to have $F_a \cap B$ and $F_b \cap B$ both be nonempty. That is, $F_a$ contains one side of $B$, and $F_b$ contains the other. Let $A$ be the binding annulus between $B$ and $C$. Since $C$ has valence at least 3, $\partial M \cap C$ has at least 3 components, and more importantly, no two of the components have both boundaries on the same two binding annuli. This means that $F_a \cap C$ and $F_b \cap C$ may have one binding annulus $A$ “in common” (that is, $F_a \cap C$ and $F_b \cap C$ both have nonempty intersection with $A$). However, they cannot possibly have two binding annuli in common. Since $\alpha \subseteq F_a$, it induces pattern components in $F_a \cap C$, and any surface parallel to those. However, it cannot induce pattern components in a surface parallel to $F_b \cap C$. Hence $\alpha$ cannot connect to a pattern arc in any spine adjacent to $B$. So (2) is impossible.

Finally, we claim that (3) is impossible. We know that $Q_b$ has sparse defect. This means that if $\alpha'$ is contained in a component of $B \cap Q_b$, at least one endpoint of $\alpha'$ must be attached to a component of $C \cap Q_b$, where $C$ is a spine adjacent to $B$. That is, the endpoints of $\alpha'$ cannot both be free to attach to the annuli we added in the final step. But as shown above, $\alpha$ has no pattern components in $C \cap Q_b$, so this $\alpha'$ cannot possibly extend far enough to have both endpoints in $\partial Q_b$. Hence $\alpha'$ cannot be properly embedded, so it cannot be part of a closed curve in $S$. So (3) also cannot occur.

Therefore $S$ satisfies the pared lifting criterion, and hence $S$ is (QF). This completes the proof.
Bibliography


