Title
A proof of the nonretractibility of a cell onto its boundary

Permalink
https://escholarship.org/uc/item/1n62r4ms

Journal
Proceedings of the American Mathematical Society, 14(2)

ISSN
0002-9939

Author
Hirsch, MW

Publication Date
1963-02-01

DOI
10.1090/S0002-9939-1963-0145502-8

Peer reviewed
SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

A PROOF OF THE NONRETRACTIBILITY OF A CELL ONTO ITS BOUNDARY

MORRIS W. HIRSCH

By appealing to the simplicial approximation theorem [2, p. 64] it suffices to prove that there is no simplicial retraction of a subdivision of a closed $n$-simplex $E$ onto its boundary $\partial E$.

Suppose $f: E \rightarrow \partial E$ is a simplicial retraction. Let $a$ be the barycenter of an $(n-1)$-simplex $A \subset \partial E$. The point is this: $f^{-1}(a)$ is a compact one-dimensional manifold whose boundary is contained in $\partial E$. The component of $f^{-1}(a)$ containing $a$ is thus a broken line segment with one endpoint at $a$; but the other endpoint cannot exist. It would have to be a point of $\partial E$ different from $a$ which maps onto $a$ under $f$, contradicting the assumption that $f|\partial E$ is the identity.

The proof that $f^{-1}(a)$ has the stated properties is simple and classical (cf. [3]). Any $n$-simplex $B$ mapping onto $A$ has precisely two faces mapping onto $A$, so that $B \cap f^{-1}(a)$ is the line segment in $B$ joining the barycenters of the two faces. These line segments fit together to form a manifold whose boundary is in $\partial E$ because every $(n-1)$-simplex $C$ of $E$ is incident to either one or two $n$-simplices, according to whether $C \subset \partial E$ or not.

The same proof applies if $E$ is a compact triangulated manifold with boundary $\partial E$. More generally, the proof works if $E$ is a finite $n$-dimensional complex such that each $(n-1)$-simplex is a face of not more than two $n$-simplices, and $\partial E$ is the union of those $(n-1)$-simplices incident to at most one $n$-simplex.

In the case where $E$ is a compact differentiable manifold, one may use the differentiable approximation theorem in place of the simplicial one, and take $a$ to be a regular value. This of course requires a theorem such as Brown’s [1, Theorem 3.III], Dubovitski’s [4, Theorem 4], or Sard’s [5] on the existence of regular values.

REFERENCES


Received by the editors October 8, 1962.


*University of California, Berkeley*