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ON FLECK QUOTIENTS

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Abstract. Let \( p \) be a prime, and let \( n \geq 1 \) and \( r \) be integers. In this paper we study Fleck’s quotient

\[ F_p(n, r) = (-p)^{-\lfloor (n-1)/(p-1) \rfloor} \sum_{k \equiv r \pmod{p}} \binom{n}{k} (-1)^k \in \mathbb{Z}. \]

We determine \( F_p(n, r) \mod p \) completely by certain number-theoretic and combinatorial methods; consequently, if \( 2 \leq n \leq p \) then

\[ \sum_{k=1}^{n} (-1)^{pk-1} \binom{pn-1}{pk-1} \equiv (n-1)! B_{p-\nu} p^n \pmod{p^{n+1}}, \]

where \( B_0, B_1, \ldots \) are Bernoulli numbers. We also establish the Kummer-type congruence \( F_p(n+p^a(p-1), r) \equiv F_p(n, r) \pmod{p^a} \) for \( a = 1, 2, 3, \ldots \), and reveal some connections between Fleck’s quotients and class numbers of the quadratic fields \( \mathbb{Q}(\sqrt{\pm p}) \) and the \( p \)-th cyclotomic field \( \mathbb{Q}(\zeta_p) \). In addition, generalized Fleck quotients are also studied in this paper.

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1. Introduction and main results

Let $m \in \mathbb{Z}^+ = \{1, 2, \ldots \}$, $n \in \mathbb{N} = \{0, 1, \ldots \}$ and $r \in \mathbb{Z}$, and define

$$C_m(n, r) = \sum_{k \equiv r \pmod{m}} \binom{n}{k} (-1)^k.$$  \hfill (1.0)

This sum has been studied by various authors and many applications have been found (cf. [S02] and its references). The following well-known observation is fundamental:

$$mC_m(n, r) = n \sum_{k=0}^n \binom{n}{k} (-1)^k \sum_{\gamma^m=1} \gamma^{k-r} = \sum_{\gamma^m=1} \gamma^{-r} (1-\gamma)^n.$$  

Note that $C_m(n+1, r) = C_m(n, r) - C_m(n, r-1)$ since $x^{n+1} = x^n - x^{n-1}(1-x)^n$.

Let $p$ be a prime, and let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. In 1913 A. Fleck (cf. [D, p. 274]) showed that

$$\text{ord}_p(C_p(n, r)) \geq \left\lfloor \frac{n-1}{p-1} \right\rfloor,$$

where $\text{ord}_p(\alpha)$ denotes the $p$-adic order of a $p$-adic number $\alpha$, and $\lfloor \cdot \rfloor$ is the well-known floor function. Fleck’s result is fundamental in the recent investigation of the $\psi$-operator related to Fontaine’s theory, Iwasawa’s theory, and $p$-adic Langlands correspondence (cf. [Co], [SW] and [W]); it also plays an indispensable role in Davis and Sun’s study of homotopy exponents of special unitary groups (cf. [DS] and [SD]). In this paper we are interested in the Fleck quotient

$$F_p(n, r) := (-p)^{-\lfloor (n-1)/(p-1) \rfloor} C_p(n, r) + \lfloor n = 0 \rfloor.$$  \hfill (1.1)

(Throughout this paper, for an assertion $A$ we let $[A]$ take 1 or 0 according as $A$ holds or not.)

For $a \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$, we use $\{a\}_m$ to denote the least nonnegative residue of $a \mod m$ (thus $\{a\}_m/m$ is the fractional part $\{a/m\}$ of $a/m$). For a prime $p$ and an integer $a$, we define $q_p(a) = (a^{p-1} - 1)/p$ which is an integer if $a \not\equiv 0 \pmod{p}$.

By a number-theoretic approach related to Gauss sums, we establish the following explicit result.

**Theorem 1.1.** Let $p$ be a prime, and let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Set $n_0 = \{n\}_p$ and $n_1 = \{n_0 - n\}_{p-1} = \{-[n/p]\}_{p-1}$. If $n_0 \leq n_1$, then

$$F_p(n, r) \equiv \frac{(-1)^{n_1}}{n_1!} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k - r)^{n_1} \pmod{p}.$$  \hfill (1.2)
If $n_0 > n_1 = 0$, then
\[ F_p(n, r) \equiv (-1)^{r} \binom{n_0}{r_p} \pmod{p}. \] (1.3)

If $n_0 > n_1 > 0$, then
\[ F_p(n, r) \equiv (-1)^{n_1 - 1} \sum_{k=0}^{n_0} \binom{n_0}{k}(-1)^{k-r} q_p(k-r) \pmod{p}. \] (1.4)

**Corollary 1.1.** Let $p$ be a prime and let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Then
\[ F_p(pn, r) \equiv \frac{r^{n^*}}{n^*!} \pmod{p} \] (1.5)
where $n^* = \{-n\}_{p-1}$. Consequently,
\[ F_p \left( \frac{p-1}{2}, r \right) \equiv \begin{cases} \frac{(-1)^{h(-p)+1/2}}{(\frac{p}{2})} \pmod{p} & \text{if } p \neq 3 \text{ & } 4 \mid p + 1, \\ \frac{(-1)^{h(p)-1/2}}{\frac{p}{2}} \pmod{p} & \text{if } 4 \mid p - 1, \end{cases} \] (1.6)
where $(\frac{\cdot}{p})$ is the Legendre symbol, and $h(-p)$ and $h(p)$ are the class numbers of the quadratic fields $\mathbb{Q}(\sqrt{-p})$ and $\mathbb{Q}(\sqrt{p})$ respectively, and for $p \equiv 1 \pmod{4}$ we write the fundamental unit of $\mathbb{Q}(\sqrt{p})$ in the form $(v+u\sqrt{p})/2$ with $u, v \in \mathbb{Z}$ and $u \equiv v \pmod{2}$.

**Proof.** Note that $\{pn\}_p = 0$. By Theorem 1.1,
\[ F_p(pn, r) \equiv \frac{(-1)^{n^*}}{n^*!} \sum_{k=0}^{0} \binom{0}{k}(-1)^{k-r} q_p(k-r) n^* = \frac{r^{n^*}}{n^*!} \pmod{p}. \]

When $p \neq 2$ and $n = (p-1)/2$, we have $n^* = (p-1)/2$ and hence
\[ F_p \left( \frac{p-1}{2}, r \right) \equiv r^{(p-1)/2}(-1)^{(p-1)/2} \prod_{k=1}^{(p-1)/2} \frac{k(p-k)}{(p-1)!} \] (by Euler’s criterion)
\[ \equiv (-1)^{(p+1)/2} \frac{r}{p} \pmod{p} \] (by Wilson’s theorem).

If $p > 3$ and $p \equiv 3 \pmod{4}$, then
\[ \frac{p-1}{2}! \equiv (-1)^{h(-p)+1/2} \pmod{p} \]
by a result of L. J. Mordell [M]. When \( p \equiv 1 \pmod{4} \) and \( \varepsilon_p = (v + u\sqrt{p})/2 > 1 \) is the fundamental unit of \( \mathbb{Q}(\sqrt{p}) \) with \( u, v \in \mathbb{Z} \) and \( u \equiv v \pmod{2} \), by S. Chowla [C] we have

\[
\frac{p-1}{2}! \equiv (-1)^{(h(p)+1)/2} \frac{v}{2} \pmod{p}.
\]

Combining the above we immediately obtain (1.6). \( \square \)

**Remark.** Let \( n \) be a positive integer and \( p > 2n + 1 \) be a prime. By the first part of Corollary 1.1 in the case \( r = 0 \), we have

\[
\binom{2pn}{pn} (-1)^n + 2 \sum_{k=0}^{n-1} \binom{2pn}{pk} (-1)^k = \sum_{k=0}^{2n} \binom{2pn}{pk} (-1)^{pk} \equiv 0 \pmod{p^{2n+1}}
\]

and hence

\[
\binom{2pn-1}{pn-1} = \frac{1}{2} \binom{2pn}{pn} \equiv \sum_{k=0}^{n-1} (-1)^{n-1-k} \binom{2pn}{pk} \pmod{p^{2n+1}}. \quad (1.7)
\]

When \( n = 1 \) and \( p > 3 \), this gives the Wolstenholme congruence

\[
\frac{1}{2} \binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.
\]

When \( n = 2 \) and \( p > 5 \), (1.7) yields the following new congruence

\[
\binom{4p-1}{2p-1} = \frac{1}{2} \binom{4p}{2p} \equiv \binom{4p}{p} - 1 \pmod{p^5}.
\]

Our second approach to Fleck quotients is of combinatorial nature. It involves Stirling numbers of the second kind as well as higher-order Bernoulli polynomials.

Let \( n \in \mathbb{N} \). The Stirling numbers \( S(n, k) \) \((k \in \mathbb{N})\) of the second kind are given by

\[
x^n = \sum_{k \in \mathbb{N}} S(n, k)(x)_k,
\]

where

\[(x)_0 = 1 \quad \text{and} \quad (x)_k = x(x-1)\cdots(x-k+1) \text{ for } k = 1, 2, \ldots.
\]

Clearly, \( S(n, n) = 1 \), and \( S(n, k) = 0 \) if \( k > n \). When \( n + k > 0 \), \( S(n, k) \) is actually the number of ways to partition a set of cardinality \( n \) into \( k \)
nonempty subsets. Here is an explicit formula (cf. [LW, p. 126]) for Stirling numbers of the second kind:

\[ S(n, k) = \frac{1}{k!} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^n. \]

As \( S(i, k) = 0 \) for all those \( i \in \mathbb{N} \) with \( i < k \), we have Euler’s identity

\[ \sum_{j=0}^{k} \binom{k}{j} (-1)^j P(j) = 0, \]

where \( P(x) \) is any polynomial with \( \text{deg } P < k \) having complex number coefficients. It is known (cf. [LW, p. 126]) that

\[ \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!} = \frac{(e^x - 1)^k}{k!}; \]

in other words,

\[ (e^x - 1)^k = \sum_{n=k}^{\infty} \bar{S}(n, k)x^n \quad \text{with} \quad \bar{S}(n, k) = \frac{k!}{n!} S(n, k). \]

For \( m = 0, 1, \ldots. \), the \( m \)-th order Bernoulli polynomials \( B_n^{(m)}(t) \) \( (n \in \mathbb{N}) \) are defined by

\[ \frac{x^m e^{tx}}{(e^x - 1)^m} = \sum_{n=0}^{\infty} B_n^{(m)}(t) \frac{x^n}{n!}, \quad (1.8) \]

and those \( B_n^{(m)} = B_n^{(m)}(0) \) are called the \( m \)-th order Bernoulli numbers.

The usual Bernoulli polynomials and numbers are \( B_n(t) = B_n^{(1)}(t) \) and \( B_n = B_n(0) = B_n^{(1)} \) respectively. (It is well known that \( B_0 = 1, B_1 = -1/2 \) and \( B_{2k+1} = 0 \) for \( k = 1, 2, \ldots; \) the reader may consult [IR, pp. 228–248] for the basic properties of Bernoulli numbers.) For a formal power series \( f(x) = \sum_{n=0}^{\infty} a_n x^n \), we use \([x^n]f(x)\) to denote the coefficient \( a_n \) of the monomial \( x^n \) in \( f(x) \). Thus

\[ B_n^{(m)}(t) = [x^n] n! \left( \frac{x}{e^x - 1} \right)^m e^{tx} \]

\[ = [x^n] n! \sum_{k=0}^{\infty} B_k^{(m)} \frac{x^k}{k!} \sum_{j=0}^{\infty} \frac{(tx)^j}{j!} = \sum_{k=0}^{n} \binom{n}{k} B_k^{(m)} t^{n-k}. \]
It is also easy to verify that $B_n^{(m)}(m - t) = (-1)^n B_n^{(m)}(t)$, and
\[
\frac{B_n^{(m)}(t)}{n!} = \sum_{k_0 + \cdots + k_{m-1} = n} \frac{B_{k_0}(t)}{k_0!} \prod_{0 < i < m} \frac{B_{k_i}}{k_i!} \quad \text{provided } m > 0.
\]

If $0 \leq n < p - 1$, then $B_0, \ldots, B_n$ are $p$-adic integers by the von Staudt-Clausen theorem (cf. [IR, p. 233]) or the recurrence $\sum_{k=0}^{l} \binom{l+1}{k} B_k = 0$ ($l = 1, 2, \ldots$), therefore $B_n^{(m)}(t) \in \mathbb{Z}_p[t]$ where $\mathbb{Z}_p$ is the ring of $p$-adic integers.

Our discovery of the next theorem was actually motivated by Theorem 1.1.

**Theorem 1.2.** Let $p$ be a prime, and let $n \in \mathbb{N}$ and $r \in \mathbb{Z}$. Set $n^* = \{−n\}^p_{p-1}$. For any integer $m \equiv n \, (\text{mod } p)$, if $m \geq 0$ then $(-1)^n F_p(n, r)$ is congruent to
\[
\sum_{k=0}^{n^*} \bar{S}(n^* - k + m, m) \frac{(-r)^k}{k!} = \sum_{k=0}^{n^*} \bar{S}(m + n^*, m + k) \binom{-r}{k}
\]
\[
= \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \binom{k-r}{m+n^*} \quad \text{(1.9)}
\]
modulo $p$; if $m \leq 0$ then we have
\[
F_p(n, r) \equiv \frac{(-1)^{n^*}}{n^*!} B_n^{(-m)}(-r) \equiv -(p-1-n^*)! B_n^{(-m)}(-r) \, (\text{mod } p). \quad \text{(1.10)}
\]

The following consequence determines $B_n^{(m)}(a)$ modulo a prime $p$ for $m \in \{1, \ldots, p\}$, $n \in \{0, \ldots, p-2\}$ and $a \in \mathbb{Z}$.

**Corollary 1.2.** Let $p$ be a prime and $r \in \mathbb{Z}$. Let $n_0 \in \{0, \ldots, p-1\}$ and $n_1 \in \{0, \ldots, p-2\}$. If $n_0 \leq n_1$, then
\[
B_{n_1-n_0}^{(p-n_0)}(-r) \equiv \frac{1}{(n_1)_{n_0}} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^{n_0-k} (k-r)^{n_1} \, (\text{mod } p). \quad \text{(1.11)}
\]

If $n_0 > n_1 = 0$, then
\[
B_{p-n_0+n_1-1}^{(p-n_0)}(-r) \equiv \frac{(-1)^{\{r\}_p-1}}{n_0!} \binom{n_0}{\{r\}_p} \, (\text{mod } p). \quad \text{(1.12)}
\]

If $n_0 > n_1 > 0$, then
\[
B_{p-n_0+n_1-1}^{(p-n_0)}(-r) \equiv \frac{(-1)^{n_1}}{(n_0-n_1)!(n_1-1)!} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{n_1} q_p(k-r) \, (\text{mod } p). \quad \text{(1.13)}
\]
Proof. Let \( n \) be a nonnegative integer with \( n \equiv n_0 - pn_1 \pmod{p(p - 1)} \). Applying (1.10) with \( m = n_0 - p \) we obtain

\[
F_p(n, r) \equiv \frac{(-1)^{n^*}}{n^*!} B_{n^*}^{(p-n_0)}(-r) \equiv -(p - 1 - n^*)!B_{n^*}^{(p-n_0)}(-r) \pmod{p},
\]

where \( n^* = \{-n\}_{p-1} \).

If \( n_0 \leq n_1 \), then \( n^* = n_1 - n_0 \) and hence

\[
B_{n^*_1-n^*_0}^{(p-n_0)}(-r) \equiv (-1)^{n_1-n_0}(n_1 - n_0)!F_p(n, r) \pmod{p},
\]

which implies (1.11) with the help of (1.2).

Now we consider the case \( n_0 > n_1 \). Clearly \( n^* = n_1 - n_0 + p - 1 \) and \( p - 1 - n^* = n_0 - n_1 \). Therefore

\[
F_p(n, r) \equiv -(n_0 - n_1)!B_{n^*_1-n^*_0+p-1}^{(p-n_0)}(-r) \pmod{p}.
\]

The case \( n_1 = 0 \) of this, together with (1.3), yields (1.12). When \( n_1 > 0 \), combining the last congruence with (1.4) we obtain (1.13). \( \square \)

Corollary 1.3. Let \( p \) be a prime and let \( n \in \mathbb{Z}^+ \). Then \( \text{ord}_p(C_p(n, r)) = \lfloor (n - 1)/(p - 1) \rfloor \) for at least \( p - n^* \geq 2 \) values of \( r \in \{0, \ldots, p - 1\} \), where \( n^* = \{-n\}_{p-1} \).

Proof. For any \( r \in \mathbb{Z} \), \( \text{ord}_p(C_p(n, r)) = \lfloor (n - 1)/(p - 1) \rfloor \) if and only if \( F_p(n, r) \not\equiv 0 \pmod{p} \). By Theorem 1.2,

\[
F_p(n, r) \equiv \frac{(-1)^{n^*}}{n^*!} B_{n^*_r}^{(p-n_0)}(-r) \pmod{p} \quad \forall r = 0, \ldots, p - 1.
\]

Recall that \( B_{n^*_r}^{(p-n_0)}(x) \in \mathbb{Z}_p[x] \) is monic and of degree \( n^* \). Also, a polynomial of degree \( n^* \) over the field \( \mathbb{Z}/p\mathbb{Z} \) cannot have more than \( n^* \) distinct zeroes in the field (cf. [IR, p. 39]). So the congruence equation \( F_p(n, r) \equiv 0 \pmod{p} \) has at most \( n^* \) solutions with \( r \in \{0, \ldots, p - 1\} \). This yields the desired result. \( \square \)

Corollary 1.4. Let \( p \) be a prime, and let \( n \in \mathbb{N} \) and \( n^* = \{-n\}_{p-1} \). Then

\[
(-1)^n F_p(n, 0) \equiv \tilde{S}(n^* + \{n\}_p, \{n\}_p) \equiv \frac{B_{n^*_m}^{(m)}}{n^*_m!} \pmod{p}, \quad \text{(1.14)}
\]

where \( m \) is any nonnegative integer with \( m + n \equiv 0 \pmod{m} \). Also,

\[
(-1)^n F_p(p n + p - 1, r) \equiv \frac{B_{n^*_r}(-r)}{n^*_r!} \equiv -(p - 1 - n^*)!B_{n^*_r}(r + 1) \pmod{p} \quad \text{(1.15)}
\]
for all \( r \in \mathbb{Z} \), and in particular
\[
\binom{2p-1}{p+r} + (-1)^p \binom{2p-1}{r} \equiv (-1)^r p^2 B_{p-2}(-r) \pmod{p^3} \tag{1.16}
\]
for every \( r = 0, \ldots, p-1 \).

**Proof.** Applying Theorem 1.2 with \( r = 0 \) we immediately get (1.14).

As \( pn + p - 1 \equiv -1 \pmod{p} \) and \( n^* = \{-(pn + p - 1)\}_{p-1} \), by the second part of Theorem 1.2 and the identity \((-1)^n B_{n^*}(x) = B_{n^*}(1-x)\), whenever \( r \in \mathbb{Z} \) we have
\[
(-1)^n F_p(pn + p - 1, r) \equiv \frac{B_{n^*}(-r)}{n^*!} \equiv (-1)^{n^*+1}(p - 1 - n^*)!B_{n^*}(-r) \equiv - (p - 1 - n^*)!B_{n^*}(r + 1) \pmod{p}
\]
and hence (1.15) holds.

Now let \( r \in \{0, \ldots, p - 1\} \). By (1.15) in the case \( n = 1 \),
\[
-F_p(2p - 1, r) \equiv -(p - 1 - (p-2))!B_{p-2}(r+1) \pmod{p}
\]
and hence
\[
F_p(2p - 1, r) \equiv B_{p-2}(1 - (-r)) = (-1)^{p-2} B_{p-2}(-r) \pmod{p}
\]
which is equivalent to (1.16). We are done. \( \square \)

Let \( p \) be an odd prime, and let \( h_p \) and \( h_p^+ \) denote the class numbers of the cyclotomic field \( \mathbb{Q}(\zeta_p) \) and its maximal real subfield \( \mathbb{Q}(\zeta_p + \zeta_p^{-1}) \) respectively, where \( \zeta_p \) is a primitive \( p \)-th root of unity in the complex field \( \mathbb{C} \). It is well known that \( h_p^- = h_p/h_p^+ \) is an integer. If \( p \) divides none of the numerators of the Bernoulli numbers \( B_0, B_2, \ldots, B_{p-3} \in \mathbb{Z}_p \), then \( p \) is said to be a *regular* prime. In 1850 E. Kummer proved that
\[
p \nmid h_p \iff p \nmid h_p^- \iff p \text{ is regular}
\]
\[
\implies x^p + y^p = z^p \text{ has no integer solution with } xyz \neq 0.
\]
Furthermore,
\[
h_p^- \equiv \prod_{0 < n \leq (p-3)/2} \left(-\frac{B_{2n}}{4n}\right) \pmod{p}
\]
by the proof of Theorem 5.16 in [Wa, p. 62].
Corollary 1.5. Let \( p \) be a prime.

(i) For every \( n = 2, \ldots, p \) we have

\[
\sum_{k=1}^{n} (-1)^{pk-1} \left( \frac{pn - 1}{pk - 1} \right) \equiv (n - 1)!B_{p-n} p^n \pmod{p^{n+1}}. \tag{1.17}
\]

(ii) Suppose that \( p > 3 \). Then \( p \) does not divide the class number \( h_p \) of the \( p \)-th cyclotomic field \( \mathbb{Q}(\zeta_p) \), if and only if

\[
\ord_p \left( \sum_{k=1}^{n} (-1)^k \left( \frac{pn - 1}{pk - 1} \right) \right) = n \quad \text{for all } n = 3, 5, \ldots, p - 2.
\]

Also,

\[
\sum_{k=1}^{(p-1)/2} (-1)^{k-1} \left( \frac{p(p - 1)/2 - 1}{pk - 1} \right) \equiv [4 \mid p + 1](-1)^{(h(-p)+1)/2} n(-p)p^{(p-1)/2} \pmod{p^{(p+1)/2}}, \tag{1.18}
\]

where \( h(-p) \) is the class number of the imaginary quadratic field \( \mathbb{Q}(\sqrt{-p}) \).

Proof. (i) Let \( n \in \{2, \ldots, p\} \). Then \([(pn - 1)/(p - 1)] = n \) and hence

\[
F_p(pn - 1, -1) = (-p)^{-n} C_p(pn - 1, -1) = (-p)^{-n} \sum_{k=1}^{n} \left( \frac{pn - 1}{pk - 1} \right) (-1)^{pk-1}.
\]

By Corollary 1.4, \((-1)^n F_p(pn - 1, -1)\) is congruent to

\[
(p - 1 - \{- (n - 1)\}_{p-1})!B_{\{- (n-1)\}_{p-1}} (-1 + 1) = (n - 1)!B_{p-n}
\]

modulo \( p \). Therefore (1.17) holds.

(ii) In view of part (i),

\[
\ord_p \left( \sum_{k=1}^{n} (-1)^k \left( \frac{pn - 1}{pk - 1} \right) \right) = n \quad \text{for } n = 3, 5, \ldots, p - 2
\]

\(\iff\) \( B_{p-n} \not\equiv 0 \pmod{p} \) for \( n = 3, 5, \ldots, p - 2 \)

\(\iff\) \( p \) is regular

\(\iff\) \( h_p \not\equiv 0 \pmod{p} \).

Taking \( n = (p - 1)/2 \) in (1.17) we get

\[
\sum_{k=1}^{(p-1)/2} (-1)^{k-1} \left( \frac{p(p - 1)/2 - 1}{pk - 1} \right) \equiv \frac{(p - 1)/2)! p^{(p-1)/2} B_{(p+1)/2} \pmod{p^{(p+1)/2}}.
\]
If $p \equiv 1 \pmod{4}$, then $B_{(p+1)/2} = 0$ since $(p + 1)/2 \in \{3, 5, \ldots\}$. If $p \equiv 3 \pmod{4}$, then $h(-p) \equiv -2B_{(p+1)/2} \pmod{p}$ (cf. [IR, p. 238]), and $((p - 1)/2)! \equiv (-1)^{(h(-p)+1)/2} \pmod{p}$ by Mordell [M]. So (1.18) follows from the above. This concludes the proof. □

Remark. Let $p$ be an odd prime. If $p \geq 5$, then (1.17) in the case $n = 2$ reduces to Wolstenholme’s congruence $(2^{p-1}) \equiv 1 \pmod{p^3}$ since $B_{p-2} = 0$. Taking $n = 3$ in (1.17) we get

$$\binom{3p-1}{p-1} - \binom{3p-1}{2p-1} + \binom{3p-1}{3p-1} \equiv 2B_{p-3}p^3 \pmod{p^4};$$

as $\binom{3p-1}{2p-1} = 2\binom{3p-1}{p-1}$ this yields the congruence

$$\binom{3p-1}{p-1} \equiv 1 - 2p^3B_{p-3} \pmod{p^4}.$$ 

This was first obtained by J.W.L. Glaisher (cf. [G1, p. 21] and [G2, p. 323]) who showed that

$$\binom{pn-1}{p-1} \equiv 1 - \frac{n(n-1)}{3}p^3B_{p-3} \pmod{p^4} \text{ for } n = 1, 2, 3, \ldots.$$ 

Corollary 1.6. Let $p$ be an odd prime, and let $n \in \{3, \ldots, p\}$ and $r \in \mathbb{Z}$. Then

$$F_p(pn-2, r) \equiv -n! \left(\frac{B_{p-n+1}(-r)}{n-1} + (r + 1)\frac{B_{p-n}(-r)}{n}\right) \pmod{p}. \quad (1.19)$$

Proof. Clearly $\{-(pn-2)\}_{p-1} = p - n + 1$. By Theorem 1.2, $F_p(pn-2, r)$ is congruent to

$$-(p - 1 - (p - n + 1))!B_{p-n+1}^{(2)}(-r) = -(n - 2)!B_{p-n+1}^{(2)}(-r)$$

modulo $p$.

Let $m = p - n + 1$. By [PS, (2.14)] or [SP, (1.12)],

$$\frac{(-1)^m}{m} \sum_{k=0}^{m} \binom{m}{k} B_k B_{m-k}(-x) - \frac{B_m(1-x)}{m} B_0$$

$$= - \sum_{k=0}^{1} \binom{1}{k} B_{1-k}(x)B_{m-1+k}(1-x) - B_1 B_{m-1}(1-x)$$

$$= - B_1(x)B_{m-1}(1-x) - B_0(x)B_m(1-x) - B_1 B_{m-1}(1-x)$$

$$= (-1)^m (B_1(x) + B_1)B_{m-1}(x) - B_m(x))$$

$$= (-1)^m ((x - 1)B_{m-1}(x) - B_m(x)).$$
It follows that
\[
B_m^{(2)}(-r) = \sum_{k=0}^{m} \binom{m}{k} B_k B_{m-k}(-r) \\
= (1 - m)B_m(-r) + m(-r - 1)B_{m-1}(-r) \\
\equiv (1 + n - 1)B_{p-n+1}(-r) - (r + 1)(-n + 1)B_{p-n}(-r) \\
\equiv n(n - 1) \left( \frac{B_{p-n+1}(-r)}{n - 1} + (r + 1)\frac{B_{p-n}(-r)}{n} \right) \pmod{p}.
\]

Combining the above we immediately obtain (1.19). □

By Theorem 1.1 or 1.2, for any prime \( p \) the Fleck quotient \( F_p(n, r) \) (with \( n \in \mathbb{N} \) and \( r \in \mathbb{Z} \)) modulo \( p \) only depends on \( p \) and \( r \) and the remainder of \( n \) modulo \( p(p - 1) \). This observation can be further extended as follows.

**Theorem 1.3.** Let \( p \) be a prime, and let \( a, l, n \in \mathbb{N} \) and \( r \in \mathbb{Z} \). Then
\[
\sum_{k=0}^{n} \binom{n}{k} (-1)^k F_p(kp^a(p - 1) + l, r) \\
\equiv 0 \pmod{p^{a + [(n - l^*)/(p - 1)]}},
\]
where \( l^* = \{-l\}_p \) and \( \lceil \cdot \rceil \) is the ceiling function.

The following consequence is somewhat similar to Kummer’s congruence for Bernoulli numbers (cf. [IR, pp. 238–241]).

**Corollary 1.7.** Let \( p \) be a prime, and let \( a, l \in \mathbb{N} \) and \( r \in \mathbb{Z} \). Then
\[
F_p(p^a(p - 1) + l, r) \equiv F_p(l, r) \pmod{p^a},
\]
\[
F_p(2p^a(p - 1) + l, r) \equiv 2F_p(p^a(p - 1) + l, r) - F_p(l, r) \pmod{p^{2a}},
\]
\[
F_p(3p^a(p - 1) + l, r) \equiv 3F_p(2p^a(p - 1) + l, r) - 3F_p(p^a(p - 1) + l, r) \\
+ F_p(l, r) \pmod{p^{3a}}.
\]

**Proof.** Simply apply (1.20) with \( n = 1, 2, 3 \). □

Let \( p \) be a prime, and let \( a \in \mathbb{Z}^+ \) and \( r \in \mathbb{Z} \). In 1977 C. S. Weisman [We] extended Fleck’s result by showing that if \( n \geq p^{a-1} \) then
\[
C_{p^a}(n, r) \equiv 0 \pmod{p^{[(n - p^{a-1})/\varphi(p^a)]}},
\]
where \( \varphi \) is Euler’s totient function. In view of this, we define the generalized Fleck quotient
\[
F_{p^a}(n, r) = (-p)^{-(n - p^{a-1})/\varphi(p^a)} C_{p^a}(n, r) + [n < p^{a-1}] \in \mathbb{Z}.
\]
Note that \( F_{p^a}(n, r) \equiv 1 \pmod{p} \) for \( n = 0, \ldots, p^{a-1} - 1 \).
Theorem 1.4. Let $p$ be a prime, and let $a, n \in \mathbb{Z}^+$ with $n \geq p^{a-1}$.

(i) For any $r \in \mathbb{Z}$ we have

$$F_{p^a}(n, r) \equiv \sum_{k=0}^{d} \binom{r + k - 1}{k} F_{p^a}(n + k, 0) \pmod{p},$$

where $d = \{p^{a-1} - 1 - n\} \varphi(p^a)$ is the least nonnegative integer with $n + d \equiv p^{a-1} - 1 \pmod{\varphi(p^a)}$.

(ii) We have

$$\text{ord}_p (C_{p^a}(n, r)) = \left\lfloor \frac{n - p^{a-1}}{\varphi(p^a)} \right\rfloor \quad (i.e., \, p \nmid F_{p^a}(n, r)) \quad \text{for some } r \in \mathbb{Z}.$$ (1.22)

If $n \geq 2p^{a-1}$, then

$$F_{p^a}(n + p^a(p - 1), r) \equiv F_{p^a}(n, r) \pmod{p} \quad \text{for all } r \in \mathbb{Z}. \quad (1.23)$$

In view of the first congruence in Corollary 1.7 and the last congruence in Theorem 1.4, we propose the following conjecture.

Conjecture 1.1. Let $p$ be a prime, and let $a, b, n \in \mathbb{Z}^+$ and $r \in \mathbb{Z}$. If $n \geq 2p^{a+b-2}$, then

$$F_{p^a}(n + \varphi(p^{a+b}), r) \equiv F_{p^a}(n, r) \pmod{p^b}.$$

Theorems 1.1, 1.2 and 1.3 will be proved in Sections 2, 3 and 4 respectively. In Section 5 we will first give a new proof of Weisman’s congruence via roots of unity, and then establish Theorem 1.4.

2. Proof of Theorem 1.1

Lemma 2.1. Let $p$ be a prime, and let $n \in \mathbb{N}$ and $n^* = \{-n\}_{p-1}$. Define $G(n) = \sum_{a=1}^{p-1} a^n \zeta_p^a$ and $\pi = 1 - \zeta_p$, where $\zeta_p$ is a primitive $p$-th root of unity in the complex field $\mathbb{C}$. Then

$$G(n) \equiv (-1)^{n^*-1} \sum_{m=n^*}^{p-2} s(m, n^*) \frac{\pi^m}{m!} \pmod{p}, \quad (2.1)$$

where $s(m, 0), \ldots, s(m, m)$ are Stirling numbers of the first kind defined by $(x)_m = \sum_{k=0}^{m}(-1)^{m-k}s(m, k)x^k$. 

Proof. Clearly,

\[ G(n) = \sum_{a=1}^{p-1} a^n (1 - \pi)^a = \sum_{a=1}^{p-1} a^n \sum_{m=0}^{a} \binom{a}{m} \left(-\pi\right)^m \]

\[ = \sum_{m=0}^{p-1} \frac{(-\pi)^m}{m!} \sum_{a=1}^{p-1} a^n (a)_m \]

\[ = \sum_{m=0}^{p-1} \frac{(-\pi)^m}{m!} \sum_{a=1}^{p-1} a^n \sum_{k=0}^{m} (-1)^{m-k} s(m, k) a^k \]

\[ = \sum_{m=0}^{p-1} \frac{(-\pi)^m}{m!} \sum_{k=0}^{m} (-1)^{m-k} s(m, k) \sum_{a=1}^{p-1} a^n+k. \]

Since

\[ 1 + x + \cdots + x^{p-1} = \frac{x^p - 1}{x-1} = \prod_{a=1}^{p-1} (x - \zeta_p^a), \]

we have

\[ \frac{p}{\pi^{p-1}} = \prod_{a=1}^{p-1} \frac{1 - \zeta_p^a}{\pi} = \prod_{a=1}^{p-1} \frac{1 - (1 - \pi)^a}{\pi} \equiv \prod_{a=1}^{p-1} a \equiv -1 \pmod{\pi} \]

with the help of Wilson’s theorem. Note also that

\[ \sum_{a=1}^{p-1} a^{n+k} \equiv -\left[ p - 1 \mid n + k \right] \pmod{p} \]

by elementary number theory (see, e.g., [IR, pp. 235–236]). Therefore

\[ G(n) \equiv \sum_{m=0}^{p-1} \frac{\pi^m}{m!} \sum_{k=0}^{m} (-1)^k s(m, k) (-[k = n^*]) \]

\[ \equiv (-1)^{n^* - 1} \sum_{m=n^*}^{p-2} s(m, n^*) \frac{\pi^m}{m!} \pmod{p}. \]

This concludes the proof. □

Remark. Let \( p \) be an odd prime. For each \( a \in \mathbb{Z} \) let \( \bar{a} = a + p\mathbb{Z} \in \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \). Let \( \omega \) be the Teichmüller character of the multiplicative group \( \mathbb{F}_p^* = \mathbb{F}_p \setminus \{0\} \). For \( \bar{a} \in \mathbb{F}_p^* \), \( \omega(\bar{a}) \) is just the \((p - 1)\)-th root of unity in the unique unramified extension of the \( p \)-adic field \( \mathbb{Q}_p \) with \( \omega(\bar{a}) \equiv a \pmod{p} \).

(See, e.g., [Wa, p. 51].) If \( \zeta_p \) is a primitive \( p \)-th root of unity in the algebraic closure of \( \mathbb{Q}_p \), then for \( n \in \mathbb{N} \) and \( \pi = 1 - \zeta_p \) we have

\[ \sum_{a=1}^{p-1} a^n \zeta_p^a \equiv \sum_{a=1}^{p-1} \omega^n(\bar{a}) \zeta_p^a \equiv -\frac{(-\pi)^n}{n^*!} \pmod{\pi^{n^* + 1}} \]

with \( n^* = \{-n\}_{p-1} \), by Stickelberger’s congruence for Gauss’ sums (cf. [BEW, pp. 344–345]).
Lemma 2.2. Let $p$ be a prime, and let $\zeta_p$ be a primitive $p$-th root of unity in $\mathbb{C}$. Let $n = p^a m + n_0 > 0$ with $a \in \mathbb{Z}^+$ and $m, n_0 \in \mathbb{N}$. Then, for any $r \in \mathbb{Z}$ we have

$$\pi^{-p^a m} C_p(n, r) - \left[ p - 1 \mid m \right] C_p(n_0, r) \equiv G(p^a m) \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k - r)^{p^a m - \left( \mod p^{a-1} \pi \min\{n_0+1, p-1\} \right)},$$

where $\pi = 1 - \zeta_p$ and $m^* = \{-m\}_{p-1}$.

Proof. Let $j \in \{1, \ldots, p-1\}$. Then

$$\left( \frac{1 - \zeta_p^j}{\pi} \right)^m = \left( \frac{1 - (1 - \pi)^j}{\pi} \right)^m = \left( \sum_{i=1}^{j} \binom{j}{i} (-\pi)^{i-1} \right)^m = j^m + \beta_j \pi,$$

where $\beta_j$ is a suitable element in the ring $\overline{\mathbb{Z}}$ of algebraic integers. For $i = 0, 1, \ldots$, if

$$\left( \frac{1 - \zeta_p^j}{\pi} \right)^{p^i m} = j^{p^i m} + p^i \pi \beta_j^{(i)}$$

for some $\beta_j^{(i)} \in \overline{\mathbb{Z}}$, then

$$\left( \frac{1 - \zeta_p^j}{\pi} \right)^{p^{i+1} m} = \left( j^{p^i m} + p^i \pi \beta_j^{(i)} \right)^p = j^{p^{i+1} m} + p^{i+1} \pi \beta_j^{(i+1)}$$

for some $\beta_j^{(i+1)} \in \overline{\mathbb{Z}}$. So

$$\left( \frac{1 - \zeta_p^j}{\pi} \right)^{p^a m} \equiv j^{p^a m} \mod p^a \pi.$$
modulo \( p^a \pi^{n_0+1} \), where

\[
S_{k-r} = \sum_{j=1}^{n-1} j^{p^a m} \zeta_p^{j(k-r)}.
\]

If \( k \not\equiv r \pmod{p} \), then

\[
S_{k-r} = (k-r)^{p^a m} \sum_{j=1}^{p-1} (j(k-r))^{p^a m} \zeta_p^{j(k-r)} \equiv (k-r)^{p^a m} G(p^a m) \pmod{p^{a+1}}.
\]

(Note that if \( j(k-r) \equiv t \pmod{p} \) then \( (j(k-r))^{p^a m} \equiv t^{p^a m} \pmod{p^{a+1}} \).)

Choose a primitive root \( g \) modulo \( p \). Since

\[
(g^{p^a m} - 1) \sum_{j=1}^{p-1} j^{p^a m} = \sum_{j=1}^{p-1} (gj)^{p^a m} - \sum_{t=1}^{p-1} t^{p^a m} \equiv 0 \pmod{p^{a+1}},
\]

if \( p-1 \nmid m \) then \( g^{p^a m} - 1 \not\equiv 0 \pmod{p} \) and so \( \sum_{j=1}^{p-1} j^{p^a m} \equiv 0 \pmod{p^{a+1}} \). Thus, when \( k \equiv r \pmod{p} \) we have

\[
S_{k-r} = \sum_{j=1}^{p-1} j^{p^a m} \equiv (p-1)[p-1 \mid m] \pmod{p^{a+1}}.
\]

Recall that \( p/\pi^{p-1} \equiv -1 \pmod{\pi} \). In view of the above,

\[
\pi^{-p^a m} p C_p(n, r) - \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k-r)^{p^a m} G(p^a m)
\]

\[
\equiv \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k \left( [p-1 \mid m](p-1) - (k-r)^{p^a m} G(p^a m) \right)
\]

\[
\equiv C_p(n_0, r) [p-1 \mid m] p \pmod{p^a \pi^{\min\{n_0+1, p-1\}}},
\]

where we have noted that if \( p-1 \mid m \) (i.e., \( m^* = 0 \)) then

\[
p - 1 - G(p^a m) \equiv p - \sum_{t=0}^{p-1} \zeta_p^t = p - \frac{1 - \zeta_p^p}{1 - \zeta_p} = p \pmod{p^{a+1}}.
\]

Therefore the desired congruence follows. \( \square \)
Proof of Theorem 1.1. In the case \( n = 0 \), (1.2) holds since \( n_1 = n_0 = 0 \) and \( F_p(n, r) = - p C_p(0, r) + 1 \). Below we assume \( n > 0 \).

Let \( \zeta_p \) be a primitive \( p \)-th root of unity in \( \mathbb{C} \), and set \( \pi = 1 - \zeta_p \). By Lemma 2.2 in the case \( a = 1 \),

\[
\pi^{ - p/\lfloor n/p \rfloor} C_p(n, r) - \lfloor n_1 = 0 \rfloor C_p(n_0, r) \equiv \left( -1 \right)^{n_1 - 1} \sum_{m = n_1}^{n - 2} s(m, n_1) \frac{\pi^m}{m!} \left( \mod \pi^{\min\{n_0 + 1, p - 1\}} \right). \]

In view of Lemma 2.1,

\[
G \left( p \left\lfloor \frac{n}{p} \right\rfloor \right) \equiv G \left( \left\lfloor \frac{n}{p} \right\rfloor \right) \equiv \left( -1 \right)^{n_1 - 1} \sum_{m = n_1}^{n - 2} s(m, n_1) \frac{\pi^m}{m!} \left( \mod p \right). \]

If \( n_0 > n_1 \), then

\[
\sum_{k = 0}^{n_0} \left( \begin{array}{c} n_0 \\ k \end{array} \right) (-1)^k (k - r)^{pn_1} \equiv \sum_{k = 0}^{n_0} \left( \begin{array}{c} n_0 \\ k \end{array} \right) (-1)^k (k - r)^{n_1} = 0 \left( \mod p \right),
\]

where we have applied Fermat’s little theorem and Euler’s identity (mentioned in Section 1). Therefore

\[
\pi^{ - p/\lfloor n/p \rfloor} C_p(n, r) - \lfloor n_1 = 0 \rfloor C_p(n_0, r) \equiv \left( -1 \right)^{n_1 - 1} \sum_{m = n_1}^{n - 2} s(m, n_1) \frac{\pi^m}{m!} \left( \mod \pi^{\min\{n_0 + 1, p - 1\}} \right).
\]

Recall that \( -p/\pi^{p-1} \equiv 1 \left( \mod \pi \right) \). Since \( s(n_1, n_1) = 1 \) and

\[
\frac{p^{\min\{n_0 + 1, p - 1\}}}{\pi^{n_1}} \equiv 0 \left( \mod \pi \right),
\]

by the above we have

\[
\frac{p^{\min\{n_0 + 1, p - 1\}}}{\pi^{p/\lfloor n/p \rfloor + n_1}} C_p(n, r) - p^{\min\{n_0 + 1, p - 1\}} \equiv \left( -1 \right)^{n_1 - 1} \sum_{k = 0}^{n_0} \left( \begin{array}{c} n_0 \\ k \end{array} \right) (-1)^k (k - r)^{pn_1} \left( \mod \pi \right).
\]

Note that

\[
\left\lfloor \frac{n - 1}{p - 1} \right\rfloor = \left\lfloor \frac{p/\lfloor n/p \rfloor + n_0 - 1}{p - 1} \right\rfloor = \frac{p/\lfloor n/p \rfloor + n_1}{p - 1} - \lfloor n_0 \leq n_1 \rfloor.
\]
and hence
\[
\frac{(-p)^{[n_0 < n_1]} C_p(n, r)}{\pi^{[n/p]+n_1}} = \frac{C_p(n, r)}{(-p)^{[(n-1)/(p-1)]}} \left( \frac{-p}{\pi^{p-1}} \right)^{(p[n/p]+n_1)/(p-1)} \equiv F_p(n, r) \pmod{\pi}.
\]

In view of the above,
\[
(-1)^{[n_0 < n_1]} F_p(n, r) - [n_0 > n_1 = 0] C_p(n_0, r)
\]
\[
\equiv \frac{(-1)^{n_1}}{p^{[n_0 > n_1]} . n_1!} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k - r)^{pn_1} \pmod{\pi}.
\]

As the rational p-adic integer
\[
D = F_p(n, r) - [n_0 > n_1 = 0] C_p(n_0, r)
\]
\[
= \frac{(-1)^{n_1}}{(-p)^{[n_0 > n_1]} . n_1!} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k - r)^{pn_1}.
\]
is divisible by \( \pi \), we have \( D^{p-1} \equiv 0 \pmod{p} \) and hence \( D \equiv 0 \pmod{p} \).

Thus
\[
F_p(n, r) - [n_0 > n_1 = 0] C_p(n_0, r)
\]
\[
\equiv \frac{(-1)^{n_1}}{(-p)^{[n_0 > n_1]} . n_1!} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k - r)^{pn_1} \pmod{p}.
\]

In the case \( n_0 \leq n_1 \), (2.2) reduces to (1.2). When \( n_0 > n_1 = 0 \), (2.2) yields (1.3) since \( C_p(n_0, r) = (-1)^{[r]} p^{n_0} \binom{n_0}{r} \) and \( \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k = (1 - 1)^{n_0} = 0 \).

Now assume that \( n_0 > n_1 > 0 \). As \( \sum_{k=0}^{n_0} \binom{n_0}{k} (k - r)^{n_1} = 0 \) by Euler’s identity, (2.2) implies that
\[
F_p(n, r) \equiv \frac{(-1)^{n_1-1}}{n_1!} \sum_{k=0}^{n_0} \binom{n_0}{k} (-1)^k (k - r)^{pn_1} - (k - r)^{n_1} p \pmod{p}.
\]

If \( n_1 = 1 \), then
\[
\frac{(k - r)^{pn_1} - (k - r)^{n_1}}{p} = (k - r)^{n_1} q_p(k - r);
\]
if \( n_1 \geq 2 \) and \( k \equiv r \pmod{p} \), then
\[
\frac{(k - r)^{pn_1} - (k - r)^{n_1}}{p} \equiv 0 \equiv (k - r)^{n_1} q_p(k - r) \pmod{p};
\]
if \( a = k - r \not\equiv 0 \pmod{p} \), then
\[
\frac{(k - r)^{pn_1} - (k - r)^{n_1}}{p} = a^{n_1} (1 + p \cdot q_p(a))^{n_1} - 1 \equiv a^{n_1} q_p(a) \pmod{p}.
\]

Therefore (1.4) follows.

The proof is now complete. \( \square \)
3. Proof of Theorem 1.2

The following lemma is a refinement of an induction technique used by Sun [S06].

**Lemma 3.1.** Let $p$ be a prime, and let $n \in \mathbb{N}$ with $n \geq p$. Then

$$F_p(n, r) \equiv - \sum_{j=1}^{p-1} \frac{1}{j} \sum_{i=0}^{j-1} F_p(n - p + 1, r - i) \pmod{p}. \quad (3.1)$$

**Proof.** Set $n' = n - (p - 1) > 0$. By the Chu-Vandermonde convolution identity (cf. [GKP, (5.27)]),

$$F_p(n, r) = (-p)^{-\lfloor (n-1)/(p-1) \rfloor} \sum_{0 \leq k \leq n \atop k \equiv r \pmod{p}} \sum_{j=0}^{k} \binom{p-1}{j} \left( \binom{n'}{k-j} \right) (-1)^k$$

$$= -\frac{1}{p} \sum_{j=0}^{p-1} \binom{p-1}{j} (-p)^{-\lfloor (n'-1)/(p-1) \rfloor} \sum_{j \leq k \leq n \atop p \nmid k-j} \binom{n'}{k-j} (-1)^k$$

$$= -\frac{1}{p} \sum_{j=0}^{p-1} \binom{p-1}{j} (-1)^j F_p(n', r - j).$$

For any $j = 0, \ldots, p-1$, clearly

$$\binom{p-1}{j} (-1)^j = \prod_{0 < i \leq j} \left( 1 - \frac{p}{i} \right)$$

$$\equiv 1 - \sum_{0 < i \leq j} \frac{p}{i} \equiv (-1)^{p-1} + p \sum_{j < k < p} \frac{1}{k} \pmod{p^2}.$$

(Note that $2 \sum_{k=1}^{p-1} 1/k = \sum_{k=1}^{p-1} (1/k + 1/(p-k)) \equiv 0 \pmod{p}$.) Also,

$$\sum_{j=0}^{p-1} F_p(n', r - j) = (-p)^{-\lfloor (n'-1)/(p-1) \rfloor} \sum_{k=0}^{n'} \binom{n'}{k} (-1)^k = 0.$$ 

Therefore

$$F_p(n, r) \equiv - \sum_{j=0}^{p-1} \sum_{j < k < p} \frac{F_p(n', r - j)}{k} = -\sum_{k=1}^{p-1} \frac{1}{k} \sum_{j=0}^{k-1} F_p(n', r - j) \pmod{p}.$$

This proves (3.1). \qed
Proof of Theorem 1.2. (i) Suppose \( m \geq 0 \). Then

\[
\sum_{k=0}^{n^*} \bar{S}(m + n^* - k, m) \frac{(-r)^k}{k!}
= [x^{m+n^*}] \sum_{l=m}^{\infty} \bar{S}(l, m)x^l \sum_{k=0}^{\infty} \frac{(-rx)^k}{k!}
= [x^{m+n^*}](e^x - 1)^m e^{-rx} = [x^{n^*}] \left( \frac{e^x - 1}{x} \right)^m e^{-rx}
= [x^{m+n^*}] \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} e^{(k-r)x} = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \frac{(k-r)^{m+n^*}}{(m+n^*)!}.
\]

By the identity (2.4) of Sun [S03], for any \( l = 0, 1, \ldots \) we have

\[
\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} (k+1)^{m+n^*} = \sum_{j=0}^{l} \binom{l}{j} (m+j)! S(m+n^*, m+j)
= \sum_{j=0}^{n^*} \binom{l}{j} (m+j)! S(m+n^*, m+j).
\]

Thus

\[
\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} (k+x)^{m+n^*} = \sum_{j=0}^{n^*} \binom{x}{j} (m+j)! S(m+n^*, m+j)
\]

and hence

\[
\sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \frac{(k-r)^{m+n^*}}{(m+n^*)!} = \sum_{j=0}^{n^*} \binom{-r}{j} \bar{S}(m+n^*, m+j).
\]

If \( m \leq 0 \), then

\[
\frac{B_{n^*}(-m)}{n^!} (-r) = [x^{n^*}] \left( \frac{x}{e^x - 1} \right)^{-m} e^{-rx} = [x^{n^*}] \left( \frac{e^x - 1}{x} \right)^m e^{-rx}.
\]

Note also that

\[
\frac{1}{n^*!} = \prod_{j=1}^{p-1-n^*} \frac{p-j}{(p-1)!} \equiv (-1)^{n^*+1}(p-1-n^*)! \pmod{p}
\]

by Wilson’s theorem.
In view of the above, whether \( m \geq 0 \) or \( m \leq 0 \), we only need to show that
\[
(-1)^n F_p(n, r) \equiv [x^{n^*}] \left( \frac{e^x - 1}{x} \right)^m e^{-rx} \pmod{p}.
\]

(ii) All those formal power series \( f(x) = \sum_{k=0}^{\infty} a_k x^k \) with \( a_k \in \mathbb{Q} \) and \( a_0, \ldots, a_{n^*} \in \mathbb{Z}_p \) form a ring \( R_{n^*} \) under the usual addition and multiplication. In particular, this ring contains

\[
e^{-rx} = \sum_{k=0}^{\infty} (-r)^k \frac{x^k}{k!}, \quad \frac{e^x - 1}{x} = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} \quad \text{and} \quad \frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.
\]

(Recall that \( n^* < p - 1 \) and \( B_0, \ldots, B_{n^*} \in \mathbb{Z}_p \).) If \( f(x) = \sum_{k=0}^{\infty} a_k x^k \) and \( g(x) = \sum_{k=0}^{\infty} b_k x^k \) belong to \( R_{n^*} \), then

\[
[x^{n^*}] f(x) g(x)^p = [x^{n^*}] \sum_{j=0}^{n^*} a_j x^j \left( \sum_{k=0}^{n^*} b_k x^k \right)^p
\]

\[
\equiv [x^{n^*}] \sum_{j=0}^{n^*} a_j x^j \sum_{k=0}^{n^*} b_k^p x^{pk} = a_{n^*} b_0^p \equiv [x^{n^*}] f(x)[x^0] g(x) \pmod{p}.
\]

Consequently, for any \( a \in \mathbb{Z} \) we have

\[
[x^{n^*}] \left( \frac{e^x - 1}{x} \right)^m e^{ax} \equiv [x^{n^*}] \left( \frac{e^x - 1}{x} \right)^n e^{ax} \pmod{p}
\]

since \( m \equiv n \pmod{p} \). By this and part (i), it suffices to use induction on \( n \) to show that

\[
(-1)^n F_p(n, r) \equiv [x^{n^*}] \left( \frac{e^x - 1}{x} \right)^n e^{-rx} \pmod{p}.
\]  

(3.2)

(iii) Obviously

\[
(-1)^0 F_p(0, r) = -pC_p(0, r) + 1 \equiv 1 = [x^0] \left( \frac{e^x - 1}{x} \right)^0 e^{-rx} \pmod{p}.
\]

So (3.2) holds for \( n = 0 \).

Suppose that \( 0 < n \leq p - 1 \). Then \( n^* = p - 1 - n \) and

\[
[x^{n^*}] \left( \frac{e^x - 1}{x} \right)^n e^{-rx} = [x^{p-1}](e^x - 1)^n e^{-rx}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} [x^{p-1}] e^{(k-r)x} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (k-r)^{p-1} (p-1)!
\]

\[
\equiv (-1)^{n-1} \sum_{k \not\equiv r \pmod{p}} \binom{n}{k} (-1)^k \pmod{p}.
\]

\[
\|\text{Suppose that } 0 < n \leq p - 1. \text{ Then } n^* = p - 1 - n \text{ and}
\]

\[
[x^{n^*}] \left( \frac{e^x - 1}{x} \right)^n e^{-rx} = [x^{p-1}](e^x - 1)^n e^{-rx}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} [x^{p-1}] e^{(k-r)x} = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (k-r)^{p-1} (p-1)!
\]

\[
\equiv (-1)^{n-1} \sum_{k \not\equiv r \pmod{p}} \binom{n}{k} (-1)^k \pmod{p}.
\]
(To get the last congruence we have applied Wilson’s theorem and Fermat’s little theorem.) Since

\[- \sum_{k \not\equiv r \pmod{p}} \binom{n}{k} (-1)^k = \sum_{k \equiv r \pmod{p}} \binom{n}{k} (-1)^k = F_p(n, r),\]

the desired (3.2) follows.

Now fix \(n \geq p\) and assume that (3.2) holds for smaller values of \(n\). Clearly \(n' = n - (p - 1) > 0\) and \(\{-n\}_{p-1} = n^*\). In light of Lemma 3.1,

\[F_p(n, r) \equiv - \sum_{j=1}^{p-1} \sum_{k=0}^{j-1} F_p(n', r - k) \pmod{p}.\]

By the induction hypothesis and part (ii),

\[(-1)^n F_p(n', r - k) \equiv [x^{n^*}] \left( \frac{e^x - 1}{x} \right)^{n'} e^{-(r-k)x} \equiv [x^{n^*}] \left( \frac{e^x - 1}{x} \right)^{n+1} e^{(k-r)x} \pmod{p}.\]

Thus \((-1)^{n-1} F_p(n, r)\) is congruent to

\[\sum_{j=1}^{p-1} \sum_{k=0}^{j-1} [x^{n^*}] \left( \frac{e^x - 1}{x} \right)^{n+1} e^{(k-r)x},\]

modulo \(p\). This yields

\[(-1)^n F_p(n, r) \equiv - [x^{n^*}] \left( \frac{e^x - 1}{x} \right)^n e^{-rx} \sum_{j=1}^{p-1} \sum_{k=1}^{j} \frac{e^{jx} - 1}{j} \equiv [x^{n^*}] \left( \frac{e^x - 1}{x} \right)^n e^{-rx} \pmod{p},\]

since \(n^* < p - 1\) and \(\sum_{j=1}^{p-1} j^{k-1} \equiv -[p-1 \mid k-1] \pmod{p}\).

In view of the above, we have completed the proof. □
4. Proof of Theorem 1.3

Proof of Theorem 1.3. Let $\zeta_p$ be a primitive $p$-th root of unity in $\mathbb{C}$, and set $\pi = 1 - \zeta_p$. For any $k = 0, \ldots, n$, we have

\[
pC_p(kp^a(p - 1) + l, r) = \sum_{j=0}^{p-1} \zeta_p^{-jr} (1 - \zeta_p^j)^{kp^a(p-1)+l}
\]

and thus

\[
F_p(kp^a(p - 1) + l, r) = (-p)^{-\lceil (kp^a(p-1)+l-1)/(p-1) \rceil} C_p(kp^a(p - 1) + l, r) + [k = l = 0]
\]

Therefore, for $S_n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k F_p(kp^a(p - 1) + l, r)$ we have

\[
S_n = - \sum_{j=1}^{p-1} \zeta_p^{-jr} (1 - \zeta_p^j)^{l} (-p)^{-\lceil (l-1)/(p-1) \rceil} \cdot c_{n,j}, \quad (4.1)
\]

where

\[
c_{n,j} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (-p)^{-kp^a} (1 - \zeta_p^j)^{kp^a(p-1)}
\]

\[
= \left( 1 - (-p)^{-p^a} (1 - \zeta_p^j)^{p^a(p-1)} \right)^n.
\]

Let $j \in \{1, \ldots, p - 1\}$. Clearly

\[
\left( \frac{1- \zeta_p^j}{\pi} \right)^{p-1} = \left( \frac{1 - (1 - \pi)^j}{\pi} \right)^{p-1} \equiv j^{p-1} \equiv 1 \pmod{\pi}
\]

and hence

\[
b_j := \frac{(1 - \zeta_p^j)^{p-1}}{-p} = \left( \frac{1 - \zeta_p^j}{\pi} \right)^{p-1} \frac{\pi^{p-1}}{-p} \equiv 1 \pmod{\pi}.
\]

(Recall the congruence $p/\pi^{p-1} \equiv -1 \pmod{\pi}$.) It follows that $b_j^{pa} \equiv 1 \pmod{p^a \pi}$ and

\[
c_{n,j} = \left( 1 - b_j^{pa} \right)^n \equiv 0 \pmod{p^a \pi^n}. \quad (4.2)
\]
Since \((1 - \zeta_p^l) \equiv 0 \pmod{\pi^l}\) and \(\text{ord}_p(\pi) = 1/(p - 1)\), in view of (4.1) and (4.2) we have

\[
\text{ord}_p(S_n) \geq \frac{l + n}{p - 1} + a_n - \left\lfloor \frac{l - 1}{p - 1} \right\rfloor - 1 = a_n + \frac{l + n}{p - 1} - \frac{l + l^*}{p - 1} = a_n + \frac{n - l^*}{p - 1}
\]

and hence \(\text{ord}_p(S_n) \geq a_n + \left\lceil (n - l^*)/(p - 1) \right\rceil\). This proves (1.20). □

5. ON GENERALIZED FLECK QUOTIENTS

**Lemma 5.1.** Let \(d, q \in \mathbb{Z}^+, n \in \mathbb{N}\) and \(r \in \mathbb{Z}\). Let \(\zeta_{dq}\) be a primitive \(dq\)-th root of unity in \(\mathbb{C}\). Then

\[
C_{dq}(n, r) = \frac{1}{d} \sum_{k=0}^{n} \binom{n}{k} C_q(k, r) \sum_{j=0}^{d-1} \zeta_{dq}^{j(k-r)} \left(1 - \zeta_{dq}^j\right)^{n-k}. \tag{5.1}
\]

**Proof.** Note that \(\zeta = \zeta_{dq}^d\) is a primitive \(q\)-th root of unity. Thus

\[
q \sum_{k=0}^{n} \binom{n}{k} C_q(k, r) \sum_{j=0}^{d-1} \zeta_{dq}^{j(k-r)} \left(1 - \zeta_{dq}^j\right)^{n-k}
\]

\[= \sum_{s=0}^{q-1} \sum_{j=0}^{d-1} \zeta_{dq}^{(ds+j)r} \sum_{k=0}^{n} \binom{n}{k} \left(\zeta_{dq}^s (1 - \zeta_{dq}^{ds})^k \left(1 - \zeta_{dq}^j\right)^{n-k}ight)
\]

\[= \sum_{s=0}^{q-1} \sum_{j=0}^{d-1} \zeta_{dq}^{-(ds+j)r} \left(1 - \zeta_{dq}^{ds+j}\right)^n
\]

\[= \sum_{t=0}^{dq-1} \zeta_{dq}^{-(tr)} \left(1 - \zeta_{dq}^t\right)^n = dqC_{dq}(n, r).
\]

So we have (5.1). □

With the help of Lemma 5.1 we can prove the following result via roots of unity.

**Theorem 5.1** (Weisman, 1977). Let \(p\) be a prime, and let \(a \in \mathbb{Z}^+, n \in \mathbb{N}\) and \(r \in \mathbb{Z}\). Then \(F_{p^a}(n, r) \in \mathbb{Z}\).

**Proof.** We use induction on \(a\).

The case \(a = 1\) reduces to Fleck's result. A proof of Fleck's result via roots of unity was given by A. Granville [Gr].
Now let \( a \geq 2 \) and assume that \( F_{p^a}(n', r') \in \mathbb{Z} \) for all \( n' \in \mathbb{N} \) and \( r' \in \mathbb{Z} \). If \( n < p^a \), then \( [(n - p^a - 1) / \phi(p^a)] \leq 0 \) and hence \( F_{p^a}(n, r) \in \mathbb{Z} \). Below we suppose \( n \geq p^a \) and let \( \zeta_{p^a} \) be a primitive \( p^a \)-th root of unity in \( \mathbb{C} \).

By Lemma 5.1,

\[
C_{p^a}(n, r) = \frac{1}{p} \sum_{k=0}^{n} \binom{n}{k} C_{p^a-1}(k, r) \sum_{j=0}^{p-1} \zeta_{p^a}^{j(k-r)} \left(1 - \zeta_{p^a}^j\right)^{n-k}. \tag{5.2}
\]

Observe that

\[
\prod_{j=1}^{p^a-1} \left(1 - \zeta_{p^a}^j\right) = \prod_{\gamma \neq 1}^{\gamma^{p^a-1} = 1} (1 - \gamma) = \lim_{x \to 1} \frac{x^{p^a-1} - 1}{x - 1} = \frac{p^a}{p^{a-1} - 1} = p.
\]

If \( p \nmid j \), then \( (1 - \zeta_{p^a}^j)/(1 - \zeta_{p^a}) \) is a unit in the ring \( \mathbb{Z}[\zeta_{p^a}] \) and thus

\[
\text{ord}_p(1 - \zeta_{p^a}^j) = \text{ord}_p(1 - \zeta_{p^a}) = \frac{1}{\phi(p^a)}.
\]

By this and the induction hypothesis, for any \( k = 0, \ldots, n \) we have

\[
\text{ord}_p \left( C_{p^a-1}(k, r) \sum_{j=0}^{p-1} \zeta_{p^a}^{j(k-r)} \left(1 - \zeta_{p^a}^j\right)^{n-k} \right) \\
\geq \max \left\{ 0, \left[ \frac{k - p^a - 2}{\phi(p^a-1)} \right] \right\} + \frac{n - k}{\phi(p^a)} \\
= \max \left\{ 0, \frac{pk - p^a - 1}{\phi(p^a)} - \left\{ \frac{k - p^a - 2}{\phi(p^a-1)} \right\} \right\} + \frac{n - k}{\phi(p^a)} \\
= \max \left\{ \frac{n - k}{\phi(p^a)}, \frac{n - p^a - 1}{\phi(p^a)} + \frac{k}{p^{a-1}} - \frac{k - p^a - 2}{\phi(p^{a-1})} \right\} > \frac{n - p^a - 1}{\phi(p^a)}.
\]

(Note that if \( k \geq p^a - 1 \) then \( k/p^{a-1} \geq 1 > \{(k - p^a - 2)/\phi(p^{a-1})\} \). Therefore, from (5.2) we get that

\[
\text{ord}_p(C_{p^a}(n, r)) > \frac{n - p^a - 1}{\phi(p^a)} - 1 \geq \left[ \frac{n - p^a - 1}{\phi(p^a)} \right] - 1.
\]

So \( F_{p^a}(n, r) = (-p)^{\{n - p^a - 1, \phi(p^a)\}} C_{p^a}(n, r) \in \mathbb{Z} \) as desired. \( \Box \)

**Proof of Theorem 1.4.** (i) Write \( n + d = p^a - 1 + m\phi(p^a) \) with \( m \in \mathbb{N} \).

Then, for any \( k = 0, \ldots, d \) we have

\[
\left[ \frac{n + k - p^a - 1}{\phi(p^a)} \right] = \left[ \frac{m - d - k + 1}{\phi(p^a)} \right] = m - 1.
\]
Below we use induction on $d$ to show the desired congruence (1.21).

In the case $d = 0$ (i.e., $n - p^a - 1 \equiv -1 \pmod{\varphi(p^a)}$), we have $F_{p^a}(n, r) \equiv F_{p^a}(n, 0) \pmod{p}$ because

$$F_{p^a}(n, i) - F_{p^a}(n, i - 1) = (-p)^{-m+1}C_{p^a}(n + 1, i) = -pF_{p^a}(n + 1, i)$$

for all $i \in \mathbb{Z}$. Furthermore, by a result of Weisman [We] (see also [SW, Theorem 1.5]), $F_{p^a}(n, r) \equiv 1 \pmod{p}$ if $d = 0$.

Now let $d > 0$ and assume that the desired result holds for smaller values of $d$. Clearly, $(n + 1) + (d - 1) = p^a - 1 - 1 + m\varphi(p^a)$ and

$$\left\lfloor \frac{n + 1 + k - p^a - 1}{\varphi(p^a)} \right\rfloor = m - 1 \quad \text{for} \quad k = 0, \ldots, d - 1.$$

If $r \geq 0$ then

$$C_{p^a}(n, r) - C_{p^a}(n, 0) = \sum_{0 < i \leq r} (C_{p^a}(n, i) - C_{p^a}(n, i - 1)) = \sum_{0 < i \leq r} C_{p^a}(n + 1, i);$$

if $r < 0$ then

$$C_{p^a}(n, r) - C_{p^a}(n, 0) = \sum_{r < i \leq 0} (C_{p^a}(n, i - 1) - C_{p^a}(n, i))$$

$$= - \sum_{r < i \leq 0} C_{p^a}(n + 1, i).$$

Therefore

$$F_{p^a}(n, r) - F_{p^a}(n, 0) = \begin{cases} \sum_{0 < i \leq r} F_{p^a}(n + 1, i) & \text{if} \quad r \geq 0, \\ - \sum_{r < i \leq 0} F_{p^a}(n + 1, i) & \text{if} \quad r < 0. \end{cases}$$

By the induction hypothesis, whenever $i \in \mathbb{Z}$ we have

$$F_{p^a}(n + 1, i) \equiv \sum_{k=0}^{d-1} \binom{i + k - 1}{k} F_{p^a}(n + 1 + k, 0) \pmod{p}.$$ 

For any $k = 0, \ldots, d - 1$, if $r \geq 0$

$$\sum_{0 < i \leq r} \binom{i + k - 1}{k} = \sum_{j=0}^{r+k-1} \binom{j}{k} = \binom{r+k}{k+1}$$

by an identity of S.-C. Chu (cf. [GKP, (5.10)]); if $r < 0$ then

$$- \sum_{r < i \leq 0} \binom{i + k - 1}{k} = (-1)^{k+1} \sum_{r < i \leq 0} \binom{-i}{k} = (-1)^{k+1} \sum_{j=0}^{-r-1} \binom{j}{k} = (-1)^{k+1} \binom{-r}{k+1} = \binom{r+k}{k+1}.$$
Thus, by the above, $F_{p^a}(n, r)$ is congruent to
\[
F_{p^a}(n, 0) + \sum_{k=0}^{d-1} \binom{r+k}{k+1} F_{p^a}(n+1+k, 0) = \sum_{k=0}^{d} \binom{r+k-1}{k} F_{p^a}(n+k, 0)
\]
modulo $p$. This concludes the induction proof of (1.21). □

(ii) In the case $a = 1$, the desired results in Theorem 1.4(ii) follow from Corollaries 1.3 and 1.7.

Now we let $a \geq 2$ and $r \in \mathbb{Z}$. Write $n = p^{a-2}(pn_1 + n_0) + s$ and $r = p^{a-2}(pr_1 + r_0) + t$, where $s, t \in \{0, \ldots, p^{a-2}-1\}$, $n_0, r_0 \in \{0, \ldots, p-1\}$ and $n_1 \in \mathbb{N}$ and $r_1 \in \mathbb{Z}$.

If $p^{a-1} \leq n < p^a$, then
\[
F_{p^a}(n, r) = C_{p^a}(n, r) = \binom{n}{\{r\}_{p^a}} (-1)^{\{r\}_{p^a}},
\]
and in particular $\text{ord}_p(C_{p^a}(n, 0)) = 0 = \lfloor (n - p^{a-1}) / \varphi(p^a) \rfloor$.

Below we assume that $n \geq 2p^{a-1}$ (i.e., $n_1 \geq 2$). By [SD, Theorem 1.7],
\[
F_{p^a}(n, r) \equiv (-1)^t \binom{s}{t} F_{p^a}(pn_1 + n_0, pr_1 + r_0) \pmod{p}.
\]

If $p | n_1$, or $p - 1 \nmid n_1 - 1$, or $n_0 = r_0 = p - 1$, then by [SW, Theorem 1.2] in the case $l = 0$, we have
\[
F_{p^2}(pn_1 + n_0, pr_1 + r_0) \equiv (-1)^{r_0} \binom{n_0}{r_0} F_p(n_1, r_1) \pmod{p}
\]
and hence $F_{p^a}(n, r) \equiv b_{n, r} F_p(n_1, r_1) \pmod{p}$, where
\[
b_{n, r} := (-1)^{\{r\}_{p^a-1}} \binom{n}{\{r\}_{p^a-1}} = (-1)^{p^{a-2}r_0 + t} \binom{p^{a-2}n_0 + s}{p^{a-2}r_0 + t} \\
\equiv (-1)^t \binom{s}{t} (-1)^{r_0} \binom{n_0}{r_0} \pmod{p} \text{ (by Lucas' theorem (cf. [HS])).}
\]

By Corollary 1.3, there is an $r'_1 \in \mathbb{Z}$ such that $F_p(n_1, r'_1) \not\equiv 0 \pmod{p}$.
Thus, if $p | n_1$ or $p - 1 \nmid n_1 - 1$, then
\[
F_{p^a}(n, p^{a-1}r'_1) \equiv F_p(n_1, r'_1) \not\equiv 0 \pmod{p}.
\]

If $n_0 = p - 1$, then
\[
F_{p^a}(n, p^{a-2}(pr'_1 + p - 1)) \equiv (-1)^{p-1} \binom{p-1}{p-1} F_p(n_1, r'_1) \not\equiv 0 \pmod{p}.
\]
Therefore

\[ F_{p^2}(pn_1 + n_0, pr_1 + r_0) \equiv [n_1 > 1] \frac{(-1)^{n_0} n_1}{r_0^{(r_0-1)/n_0}} = \frac{(-1)^{n_0} n_1}{r_0^{(r_0-1)/n_0}} \pmod{p} \]

and hence

\[ F_{p^2}(n, r) \equiv (-1)^{n_0 + r_0} \frac{n_1 \binom{s}{i}}{r_0^{(r_0-1)/n_0}} \pmod{p}. \]

In particular, if \( p \nmid n_1, p - 1 \mid n_1 - 1 \) and \( n_0 < p - 1 \), then

\[ F_{p^2}(n, p^{a-2}(n_0 + 1)) \equiv \frac{(-1)^{n_0} n_1}{n_0 + 1} \not\equiv 0 \pmod{p}. \]

In view of the above, we already have (1.22).

To prove the congruence in (1.23), we should also consider the case \( p \nmid n_1, p - 1 \mid n_1 - 1 \) and \( n_0 \geq r_0 \). By [SW, Lemmas 3.2 and 3.3],

\[
p^{-\lfloor(p_1+n_0-p)/\varphi(p^2)\rfloor} C_{p^2}(pn_1 + n_0, pr_1 + r_0) - (-1)^{r_0} \binom{n_0}{r_0} p^{-\lfloor(n_1-1)/(p-1)\rfloor} C_p(n_1, r_1)
\]

\[ \equiv (-1)^{n_1-1} p^{-\lfloor(n_1-1)/(p-1)\rfloor} C_p(n_1 - 1, r_1) (-1)^{n_1 + r_0} n_1 \binom{n_0}{r_0} \sigma_{n_0, r_0}(n_1) \pmod{p}, \]

where

\[ \sigma_{n_0, r_0}(n_1) = 1 + (-1)^p \prod_{1 \leq i \leq p, i \neq p - r_0} (p(n_1 - 1) + r_0 + i) \prod_{1 \leq i \leq p, i \neq p - (n_0 - r_0)} (n_0 - r_0 + i) \equiv 0 \pmod{p}. \]

Therefore

\[ F_{p^2}(pn_1 + n_0, pr_1 + r_0) - (-1)^{r_0} \binom{n_0}{r_0} F_p(n_1, r_1) \equiv (-1)^{r_0} \binom{n_0}{r_0} F_p(n_1 - 1, r_1) n_1 \sigma_{n_0, r_0}(n_1) \pmod{p}, \]

and hence

\[ F_{p^a}(n, r) \equiv b_{n, r} \left( F_p(n_1, r_1) + F_p(n_1 - 1, r_1) n_1 \sigma_{n_0, r_0}(n_1) \right) \pmod{p}, \]

Observe that \( n + p^a(p-1) = p^{a-2}(pn_1 + n_0) + s \) with \( n_1' = n_1 + p(p-1) \). Clearly \( F_p(n_1', r_1) \equiv F_p(n_1, r_1) \pmod{p} \) by Corollary 1.7, and \( \sigma_{n_0, r_0}(n_1') \equiv \sigma_{n_0, r_0}(n_1) \pmod{p^2} \) if \( n_0 \geq r_0 \). Thus, by the above, \( F_{p^a}(n + p^a(p-1), r) \equiv F_{p^a}(n, r) \pmod{p} \). This concludes the proof.
References


[S02] Z. W. Sun, On the sum \( \sum_{k \equiv r \pmod{m}} {n \choose k} \) and related congruences, Israel J. Math. 128 (2002), 135–156.


