Title
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Publication Date
1972-03-01
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March 1972

AEC Contract No. W-7405-eng-48

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CUBIC SPLINE FORMULATION FOR
MATRIX METHOD FOR SECOND ORDER
ORDINARY DIFFERENTIAL EIGENVALUES

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March 1972

ABSTRACT

This report describes a cubic spline formulation for the matrix method in solving for second order ordinary differential eigenvalues. The matrix method as described in Reference 1, formulates the matrices in terms of finite difference approximations. Rows of the matrices corresponding to boundary points relate to boundary conditions only and not to the differential equation.

The cubic spline formulation constructs the "boundary point" rows in such a way that both boundary condition and differential equation are satisfied.

For each eigenvalue so approximated, a corresponding eigenfunction is computed. An integral ratio (modified Rayleigh quotient) process is applied to this function to improve the eigenvalue approximation.

Numerical examples are given to illustrate the method and compare it with the finite difference formulation.
INTRODUCTION

Given the differential equation,
\[ p(x)u'' + q(x)u' + g(x)u + \lambda \{ \psi(x)u' + \phi(x)u \} = 0 \]  
(1)
on the interval, [0, 1], with boundary conditions,
\[ a_0 u(0) + b_0 u'(0) = 0 \]  
(2)
\[ a_1 u(1) + b_1 u'(1) = 0 \]  
(3)
we wish to find some \( \lambda \) (eigenvalues) for which non-trivial solutions, \( u \), (eigenfunctions) exist.

We employ the matrix method\(^1\), specifying on \([0, 1]\) a mesh:
\[ 0 = x_1, \ldots x_n = 1 \quad \text{with} \quad n \geq 4. \]
In the finite difference formulation, Equation (2) is approximated at \( x_1 \), Equation (1) is approximated at \( x_2, \ldots x_{n-1} \) and Equation (3) at \( x_n \) by finite difference ratios. Thus, we obtain a linear system of \( n \) equations in \( n \) unknowns which we can write in the form:
\[ A\bar{u} + B\bar{u} = 0 \]  
(4)
The involvement of \( \lambda \) in the first and last equations of the system is obtained by writing Equations (2) and (3) in the form
\[ a_0 u(0) + b_0 u'(0) + \lambda \cdot 0 \cdot u(0) = 0 \]  
(5)
\[ a_1 u(1) + b_1 u'(1) + \lambda \cdot 0 \cdot u(1) = 0 \]  
(6)
Hence, the first and last rows of \( B \) will have all zero elements. The matrix \( B \) is always singular, consequently the system should be "reduced."\(^1\) (That \( A \) may be non-singular and solution for \(-1/\lambda \) may be possible is not considered in Reference 1). Usually \( n-2 \) values for \( \lambda \) may be obtained from the reduced system.
In the cubic spline formulation, described fully in the next section, a similar mesh is used and the differential eigenvalues are again found as matrix eigenvalues. The matrices are consistent with both boundary conditions and the differential equation at the boundary points. Usually the system does not need to be reduced, although it may be necessary to solve for \(-1/\lambda\) instead of \(-\lambda\). With this formulation is usually possible to construct an eigenfunction corresponding to any eigenvalue obtained. The integral ratio process, described in detail later, may be applied to this eigenfunction to produce an improved approximation of the eigenvalue.
Cubic Spline Formulation

For a chosen (usually but not necessarily uniform mesh)

\[ 0 = x_1, x_2, \ldots, x_n = 1 \quad \text{with} \quad n \geq 4 \]

we seek a cubic spline, \( s \), with knots at the \( x_i \) such that \( s \) satisfies Equations (1), (2) and (3) at each of the \( x_i \) for \( i=1 \) to \( n \). The space of all cubic splines with knots at the \( x_i \) and satisfying Equations (2) and (3), (boundary conditions), is a linear space \( \mathbb{R}^n \) having dimension \( n \).

A convenient basis for the above space consists of the cubic splines, \( s_j \), for \( j=1 \) to \( n \) determined, as follows, by their values at the points, \( x_i \), and by their terminal derivatives:

\[
\begin{align*}
\text{for } j & = 1, \quad & s_1(x_1) & = 1, & s_1'(x_1) & = -a_0/b_0 \\
& & s_1(x_1) & = 0, & s_1'(x_1) & = 1 \\
& & s_1(x_1) & = 0 & & \\
& & s_1(x_n) & = 0 & & \\
\text{for } j & = 2, \ldots, n-1, & s_j(x_1) & = 0, & s_j'(x_1) & = 0, \\
& & s_j(x_i) & = \delta_{ij}, & s_j'(x_n) & = 0 \\
\text{for } j & = n, & s_n(x_1) & = 0, & s_n'(x_1) & = 0, \\
& & s_n(x_i) & = 0 & & \text{for } i=1, \ldots, n-1
\end{align*}
\]
if \( b_1 \neq 0 \)
\[
\begin{align*}
  s_n(x_n) &= 1 & s_n(x_n) &= 0 \\
  s_n'(x_n) &= -a_1/b_1 & s_n'(x_n) &= 1
\end{align*}
\]

For each of the "basic splines" we can compute
\[
 s_j'(x_i) \text{ for } i=2, n-1
\]
and
\[
 s_j''(x_i) \text{ for } i=1, n
\]

Any cubic spline in the space is a linear combination of the \( s_j \),
in particular the one we seek can be written:
\[
 s = \sum_{j=1}^{n} c_j s_j
\]

We have, also
\[
 s' = \sum_{j=1}^{n} c_j s'_j
\]
and
\[
 s'' = \sum_{j=1}^{n} c_j s''_j
\]

Substituting \( s, s' \) and \( s'' \) for \( u, u' \) and \( u'' \), respectively, in Equation (1)
at each of the \( x_i \) for \( i=1 \) to \( n \), we obtain a linear system of \( n \) equations
in the \( n \) unknowns, \( c_j \), for \( j=1 \) to \( n \) of the form:
\[
 \nabla \tilde{c} + \lambda W \tilde{c} = 0 \tag{5}
\]
where
\[
 \tilde{c} = (c_1, c_2, \ldots, c_n).
\]

Obviously, \( \tilde{c} = 0 \), satisfies (5). We must find values for \( \lambda \) for
which non-trivial solutions exist. To this we need to replace
Equation (5) by a "standard" 1 eigenvalue equation of the form:
\[
 P \tilde{c} - \mu \tilde{c} = 0 \tag{6}
\]

If \( W \) is non-singular, we can let
\[
\mathbf{p} = \mathbf{W}^{-1} \mathbf{v}
\]
\[
\mu = -\lambda
\]
Otherwise if \( V \) is non-singular, we let
\[
\mathbf{p} = \mathbf{V}^{-1} \mathbf{w}
\]
\[
\mu = -1/\lambda
\]
In these cases, we can find the eigenvalues, \( \mu \), for \( \mathbf{p} \) and then determine values, \( \lambda \), for \( \mu \neq 0 \). The case where both \( W \) and \( V \) are singular (which occurs rarely) can be handled by a reduction process.

**APPROXIMATE EIGENFUNCTIONS**

Now for any value, \( \lambda \), obtained by the foregoing we wish to find a corresponding cubic spline, \( s \), or equivalently, to find a non-trivial vector, \( \bar{\mathbf{c}} \), as a solution to Equation (5). For a particular, \( \lambda \), Equation (5) gives a homogeneous linear system of the form
\[
H \bar{\mathbf{c}} = \mathbf{0}
\]

where
\[
H = \mathbf{V} + \lambda \mathbf{W}
\]
The non-trivial vector, \( \bar{\mathbf{c}} \), can be determined (except for a multiplicative constant) from Equation (6). Usually this can be done by arbitrarily assigning \( c_1 = 1 \), and solving the last \( n-1 \) linear equations for the remaining \( c_j \) for \( j = 2 \) to \( n \).

Then we can use the cubic spline,
\[
\mathbf{s} = \sum_{j=1}^{n} c_j \mathbf{s}_j
\]
as an approximate eigenfunction corresponding to the approximate eigenvalue, $\lambda$.

Under the finite difference formulation the best we can obtain is approximate values for an eigenfunction at the $x_i$ for $i=1$ to $n$, whereas, the cubic spline, $s$, is well defined for every $x \in [0, 1]$.

**INTEGRAL RATIO PROCESS**

We now seek an improved approximation for any approximate eigenvalue, $\lambda$, obtained from the cubic spline formulation. First we define an integral ratio (modified Rayleigh quotient) for Equation (1) and compute its value for the approximate eigenfunction, $s$, obtained in the previous section:

$$ R = -\frac{\int_0^1 (\psi s' + \phi s)(ps'' + qs' + gs)dx}{\int_0^1 (\psi s' + \phi s)^2dx} $$

Of course if $s$ were actually an eigenfunction we should find $R = \lambda$, but it is unlikely that this will be true. If not, we can take $R$ as a new approximation for $\lambda$.

In all cases tried (where the value of $\lambda$ was known or where a presumable better value had been obtained by use of larger $n$) the value for $R$ was a better approximation for the eigenvalue.

**NUMERICAL EXAMPLES**

We consider some simple examples which significantly illustrate and compare the processes described above. In each case, we use $n=4$, 6, and 11. Uniform steps of $1/(n-1)$ are used. The construction is
in detail for \( n=4 \), but for \( n=6 \) and 11, only a summary of results is given. Where known, analytic solutions are given for comparison.

Example 1

Differential Equation

\[ u'' + u = 0 \]

(a) Boundary Condition

\[ u(0) = 0 \quad u(1) = 0 \]

Analytic Solution

\[ \lambda = k^2 \pi^2, \text{ for } k=1,2,... \]

Matrix method with

\[ n=4, \ h=1/3 \]

Finite Difference Formulation*

\[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 9 & -18 & 9 & 0 \\ 0 & 9 & -18 & 9 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \]

Already in reducible form

Reduced form

\[ \begin{pmatrix} -18 & 9 \\ 9 & -18 \end{pmatrix} + \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]

Standard form

\[ \begin{pmatrix} -18 & 9 \\ 9 & -18 \end{pmatrix} - \mu I \quad \mu = -\lambda \]

Matrix eigenvalues, \( \mu = -9, -27 \)

Differential eigenvalues, \( \lambda = 9, 27 \)

*Stepsize, \( h \), is incorporated in first matrix rather than the second.
Cubic Spline Formulation

\[ s'' + \lambda s = 0 \]

\[
\begin{pmatrix}
-10.4 & 50.4 & -14.4 & -0.4 \\
2.8 & -48.8 & 28.8 & 0.8 \\
-0.8 & 28.8 & -46.8 & -2.8 \\
0.4 & 14.4 & 50.4 & 10.4
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4
\end{pmatrix}
+
\lambda
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4
\end{pmatrix}
= 0

Second matrix is singular but first is non-singular

\(
\mu = -1/\lambda \\
-\mu \mathbf{I}
\)

Matrix eigenvalues, \( \mu = 0, -0.924, -0.4629, 0 \).

Differential eigenvalues \( \lambda = 10.8, 54.0 \)

Approximate Eigen Function

\[
s(10.8)
\]

\[
s = s_1 + 2.778s_2 + 2.778s_3 - s_4
\]

\[
s'' = s''_1 + 2.778s''_2 + 2.778s''_3 - s''_4
\]

\[
R(10.8) = -\int_0^1 s''dx/\int_0^1 s^2dx = 9.87
\]

\[
s(54)
\]

\[
R(54) = -\int_0^1 s''dx/\int_0^1 s^2dx = 39.95
\]
### Summary

<table>
<thead>
<tr>
<th>n=4</th>
<th>Finite Differences</th>
<th>Spline</th>
<th>Integral Ratio</th>
<th>Analytic</th>
</tr>
</thead>
<tbody>
<tr>
<td>λ₁</td>
<td>9.0</td>
<td>10.8</td>
<td>9.87</td>
<td>9.8696</td>
</tr>
<tr>
<td>λ₂</td>
<td>27.0</td>
<td>54.0</td>
<td>39.95</td>
<td>39.4784</td>
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<td>n=6</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>λ₁</td>
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<td>10.20</td>
<td>9.86963</td>
<td>9.869604</td>
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<td>λ₂</td>
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<td>44.89</td>
<td>39.4884</td>
<td>39.47842</td>
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<tr>
<td>λ₃</td>
<td>65.46</td>
<td>116.12</td>
<td>89.2912</td>
<td>88.8264</td>
</tr>
<tr>
<td>λ₄</td>
<td>90.40</td>
<td>227.84</td>
<td>165.4427</td>
<td>157.9136</td>
</tr>
<tr>
<td>n=11</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>λ₂</td>
<td>38.197</td>
<td>40.794</td>
<td>39.47851</td>
<td>39.478418</td>
</tr>
<tr>
<td>λ₃</td>
<td>82.443</td>
<td>95.575</td>
<td>88.8295</td>
<td>88.82644</td>
</tr>
<tr>
<td>λ₄</td>
<td>138.197</td>
<td>179.553</td>
<td>157.9535</td>
<td>157.91367</td>
</tr>
<tr>
<td>λ₅</td>
<td>200.000</td>
<td>300.000</td>
<td>247.0588</td>
<td>246.74011</td>
</tr>
</tbody>
</table>

b) Boundary conditions

- $u'(0) = 0$
- $u'(1) = 0$

Analytic Solution

- $\lambda = k^2 \pi^2$ for $k=1, 2, \ldots$

Matrix method $n=4$

#### Finite Difference Formulation

\[
\begin{pmatrix}
-4.5 & 6 & -1.5 & 0 \\
9 & -18 & 9 & 0 \\
0 & 9 & -18 & 9 \\
0 & 1.5 & -6 & 4.5 \\
\end{pmatrix}
\lambda
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
Reducible form

\[
\begin{pmatrix}
-4.5 & 0 & 0 & 0 \\
9 & -6 & 6 & 0 \\
0 & 6 & -6 & 9 \\
0 & 0 & 0 & 4.5
\end{pmatrix}
\] + \lambda
\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Reduced form

\[
\begin{pmatrix}
-6 & 6 \\
6 & -6
\end{pmatrix}
\]
\[
\lambda
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
\]

Standard form

\[
\begin{pmatrix}
-6 & 6 \\
6 & -6
\end{pmatrix}
\]
\[-\mu I, \mu = -\lambda
\]

Matrix eigenvalues, \(\mu = 0, -12\)

Differential eigenvalues, \(\lambda = 12.0\)

Cubic Spline Formulation \(s'' + \lambda s = 0\)

\[
\begin{pmatrix}
-39.6 & 50.4 & -14.4 & 3.6 \\
25.2 & -46.8 & 28.8 & -7.2 \\
-7.2 & 28.8 & -46.8 & 75.2 \\
3.6 & -14.4 & 50.4 & -39.6
\end{pmatrix}
\]
\[
\lambda
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Second matrix is identity (non-singular)

\[
\begin{pmatrix}
-39.6 & 50.4 & -14.4 & 3.6 \\
35.2 & -46.8 & 28.8 & -7.2 \\
-7.2 & 28.8 & -46.8 & 25.2 \\
3.6 & -14.4 & 50.4 & -39.6
\end{pmatrix}
\]
\[-\mu I, \mu = -\lambda
\]

Matrix eigenvalues, \(\mu = 0, -10.8, -54.0, -108.0\)

Differential eigenvalues, \(\lambda = 10.8, 54.0, 108.\)
Integral ratio process

After computing a non-trivial solution, \( \tilde{r}(\lambda) \), for each of the above \( \lambda \), and integrating numerator and denominator, we find

\[
\begin{align*}
R(10.8) &= 9.87 \\
R(54.0) &= 39.95 \\
R(108.0) &= 88.94
\end{align*}
\]

Summary

<table>
<thead>
<tr>
<th>( n=4 )</th>
<th>Finite Difference</th>
<th>Cubic Spline</th>
<th>Integral Ratio</th>
<th>Analytic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>12.</td>
<td>10.8</td>
<td>9.87</td>
<td>9.8696</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>Not found</td>
<td>54.0</td>
<td>39.95</td>
<td>39.4784</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>Not found</td>
<td>108.0</td>
<td>88.94</td>
<td>88.8264</td>
</tr>
</tbody>
</table>

<table>
<thead>
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<th>( n=6 )</th>
<th>Finite Difference</th>
<th>Cubic Spline</th>
<th>Integral Ratio</th>
<th>Analytic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>10.23</td>
<td>10.19</td>
<td>9.86963</td>
<td>9.86960</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>41.67</td>
<td>44.89</td>
<td>39.4884</td>
<td>39.4784</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>81.43</td>
<td>116.12</td>
<td>89.2912</td>
<td>88.8264</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>Not found</td>
<td>227.84</td>
<td>165.4427</td>
<td>157.9136</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( n=11 )</th>
<th>Finite Difference</th>
<th>Cubic Spline</th>
<th>Integral Ratio</th>
<th>Analytic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>9.8815</td>
<td>9.9510</td>
<td>9.869605</td>
<td>9.8696044</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>39.4879</td>
<td>40.7936</td>
<td>39.47851</td>
<td>39.47842</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>87.6838</td>
<td>95.5755</td>
<td>88.8295</td>
<td>88.8264</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>150.5689</td>
<td>179.5525</td>
<td>157.9535</td>
<td>157.9136</td>
</tr>
<tr>
<td>( \lambda_5 )</td>
<td>221.0940</td>
<td>300.0000</td>
<td>247.0588</td>
<td>246.7401</td>
</tr>
</tbody>
</table>
c) Boundary conditions

\[ u(0) + u'(0) = 0 \quad u(1) + u'(1) = 0 \]

Analytic Solution

\[ \lambda = -1, \quad \lambda = k^2 \pi^2 \text{ for } k=1,2,\ldots \]

Matrix method \( n=4 \)

Finite Difference Formulation

\[
\begin{pmatrix}
-3.5 & 6.0 & -1.5 & 0 \\
9.0 & -18.0 & 9.0 & 0 \\
0 & 9.0 & -18.0 & 9.0 \\
0 & 1.5 & -6.0 & 3.5
\end{pmatrix} + \lambda \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Reducible

\[
\begin{pmatrix}
-3.5 & 0 & 0 & 0 \\
9.0 & -2.5714 & 5.1429 & 0 \\
0 & 6.5455 & -8.1818 & 9.0 \\
0 & 0 & 5.5 & 0
\end{pmatrix} + \lambda \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Reduced

\[
\begin{pmatrix}
-2.5714 & 5.1429 \\
6.5455 & -8.1818
\end{pmatrix}
\]

\[-\mu \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} \quad \mu = -\lambda
\]

whence \( \mu = 1.068, \ -11.82 \)

and \( \lambda = -\mu = -1.068, \ 11.82 \)

Cubic Spline Formulation

\[
\begin{pmatrix}
-29.2 & 50.4 & -14.4 & 4.0 \\
22.4 & -46.8 & 28.8 & -8.0 \\
-6.4 & 28.8 & -46.8 & 28.0 \\
3.2 & -14.4 & 50.4 & -50.0
\end{pmatrix} + \lambda \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
Setting $\mu = -\lambda$

We obtain $\mu = 0.9909, -10.8, -54.0, 108.9909$

Whence $\lambda = -0.9909, 10.8, 54.0, 108.9909$

Integral Ratio Process

For each of the above $\lambda$, we solve for a set of non-trivial $c_j$ for $j=1$ to $n$. Integrating to form the "modified" Rayleigh quotient we find

$$ R(-0.9909) = -1.000000 $$
$$ R(10.8) = 9.8703 $$
$$ R(54.0) = 39.95 $$
$$ R(108.9909) = 88.95 $$

Summary

<table>
<thead>
<tr>
<th>$n=4$</th>
<th>etc.</th>
<th>Cubic Spline</th>
<th>Integral Ratio</th>
<th>Analytic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>-1.068</td>
<td>-.9909</td>
<td>-1.000000</td>
<td>-1.000000</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>11.82</td>
<td>10.8</td>
<td>9.8703</td>
<td>9.8696</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>Not found</td>
<td>54.0</td>
<td>39.95</td>
<td>39.4784</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>Not found</td>
<td>108.9909</td>
<td>88.63</td>
<td>88.8264</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$n=6$</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>-1.0261</td>
<td>-.9967</td>
<td>-1.000000</td>
<td>-1.000000</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>10.180</td>
<td>10.198</td>
<td>9.86963</td>
<td>9.869604</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>41.577</td>
<td>44.888</td>
<td>39.488</td>
<td>39.4784</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>81.396</td>
<td>116.117</td>
<td>89.291</td>
<td>88.826</td>
</tr>
</tbody>
</table>
\[ n = 11 \]

\[
\begin{array}{cccc}
\lambda_1 & -1.0069 & -0.99917 & -1.0000000 & -1.0000000 \\
\lambda_3 & 39.4619 & 40.7936 & 39.4785 & 39.47842 \\
\lambda_4 & 87.6451 & 95.5755 & 88.8295 & 88.8264 \\
\lambda_5 & 150.5284 & 179.5525 & 157.9535 & 157.9136 \\
\end{array}
\]

d) Boundary conditions

\[ u(0) = 0 \quad u'(1) = 0 \]

Analytic solution

\[ \lambda = (k\pi/2)^2 \quad \text{for} \quad k=1,3,\ldots \]

Matrix method \( n=4 \)

Finite Difference Formulation

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
9 & -18 & 9 & 0 \\
0 & 9 & -18 & 9 \\
0 & 3 & -12 & 9
\end{pmatrix}
\begin{pmatrix}
\lambda
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Reducible form

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
9 & -18 & 9 & 0 \\
0 & 6 & -6 & 9 \\
0 & 0 & 0 & 9
\end{pmatrix}
\begin{pmatrix}
\lambda
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Reduced form

\[
\begin{pmatrix}
-18 & 9 \\
6 & -6
\end{pmatrix}
\begin{pmatrix}
-\mu I \\
\mu = -\lambda
\end{pmatrix}
\]

\[ \mu = -2.513, -21.487 \]

\[ \lambda = 2.513, 21.487 \]
The second matrix is singular but the first is not, hence we insert it, multiply through by the inverse and let \( \mu = -1/\lambda \) obtaining
\[
\begin{bmatrix}
0 & -0.3333 & -0.3333 & -0.1667 \\
-\mu I & 0 & -0.0926 & -0.1111 & -0.0556 \\
& 0 & -0.1111 & -0.2037 & -0.1111 \\
& & 0 & -0.1111 & -0.2222 & -0.1481
\end{bmatrix}
\]
whence \( \mu = -0.3962, -0.0370, -0.0113, 0 \)
and \( \lambda = 2.524, 27.0, 88.860 \)

Integral Ratio
\[
R(2.524) = 2.4674 \\
R(27.0) = 22.2353 \\
R(88.860) = 65.4976
\]
Summary

<table>
<thead>
<tr>
<th>n=4</th>
<th>Finite Differences</th>
<th>Cubic Spline</th>
<th>Integral Ratio</th>
<th>Analytic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>2.51</td>
<td>2.52</td>
<td>2.4674</td>
<td>2.4674</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>21.49</td>
<td>27.00</td>
<td>22.235</td>
<td>22.207</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>Not found</td>
<td>88.86</td>
<td>65.498</td>
<td>61.685</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n=6</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>2.470</td>
<td>2.488</td>
<td>2.467401</td>
<td>2.467401</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>21.921</td>
<td>23.894</td>
<td>22.2074</td>
<td>22.2066</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>55.273</td>
<td>75.000</td>
<td>61.7647</td>
<td>61.6850</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>87.002</td>
<td>168.648</td>
<td>123.0271</td>
<td>120.9027</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n=11</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>2.4653</td>
<td>2.4724</td>
<td>2.467401</td>
<td>2.46701</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>22.0165</td>
<td>22.6205</td>
<td>22.620662</td>
<td>22.620661</td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>59.9504</td>
<td>64.9165</td>
<td>61.6857</td>
<td>61.6850</td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>113.2033</td>
<td>133.4992</td>
<td>120.9146</td>
<td>120.9027</td>
</tr>
<tr>
<td>$\lambda_5$</td>
<td>176.5348</td>
<td>234.7112</td>
<td>199.9776</td>
<td>199.8595</td>
</tr>
</tbody>
</table>

Example 2

$u'' - 2u' + \lambda u = 0$ on $[0, 1]$

Boundary conditions $u(0) = 0$, $u(1) = 0$

Analytic Solution $\lambda = 1 + k^2 \pi^2$ for $k=1,2,...$

Matrix method $n=4$
Finite Difference Formulation

\[ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 12 & -18 & 6 & 0 \\ 0 & 12 & -18 & 6 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

In Reducible form

Reduced form

\[ \begin{pmatrix} -18 & 6 \\ 12 & -18 \end{pmatrix} \]

\[ \mu = -1.31, -10.69 \]

\[ \lambda = 9.51, 26.48 \]

Cubic Spline Formulation

\[ \begin{pmatrix} -12.4 & 50.4 & -14.4 & -0.4 \\ 3.33 & -48.0 & 24.0 & 0.67 \\ -0.93 & 33.6 & 45.6 & -2.27 \\ 0.4 & -14.4 & 50.4 & 8.4 \end{pmatrix} \]

The second matrix is singular but the first is not, multiplying through by its inverse and letting \( \mu = -1/\lambda \) we obtain

\[ \begin{pmatrix} 0 & 0.1648 & -0.0577 & 0 \\ 0 & -0.0516 & -0.0278 & 0 \\ 0 & -0.0437 & -0.0556 & 0 \\ 0 & 0.1813 & 0.2885 & 0 \end{pmatrix} \]

whence \( \mu = -0.08845, -0.01869, 0, 0 \)

and \( \lambda = 11.30, 53.49 \)
Integral Ratio

Computing the integral ratios we obtain

\[ R(11.30) = 10.56 \]
\[ R(53.49) = 39.74 \]

Summary

<table>
<thead>
<tr>
<th>n=4</th>
<th>Finite Differences</th>
<th>Cubic Spline</th>
<th>Integral Ratio</th>
<th>Analytic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>9.51</td>
<td>11.30</td>
<td>10.56</td>
<td>10.8696</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>26.48</td>
<td>53.49</td>
<td>39.74</td>
<td>40.4784</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n=6</th>
<th>Finite Differences</th>
<th>Cubic Spline</th>
<th>Integral Ratio</th>
<th>Analytic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>10.366</td>
<td>11.010</td>
<td>10.755</td>
<td>10.8696</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>34.861</td>
<td>45.199</td>
<td>40.029</td>
<td>40.4784</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>65.139</td>
<td>115.807</td>
<td>89.298</td>
<td>89.8264</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>103.664</td>
<td>227.027</td>
<td>164.616</td>
<td>158.9136</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n=11</th>
<th>Finite Differences</th>
<th>Cubic Spline</th>
<th>Integral Ratio</th>
<th>Analytic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_1 )</td>
<td>10.742</td>
<td>10.903</td>
<td>10.8403</td>
<td>10.8696</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>39.008</td>
<td>41.603</td>
<td>40.3546</td>
<td>40.4784</td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>83.032</td>
<td>96.164</td>
<td>89.5572</td>
<td>89.8264</td>
</tr>
<tr>
<td>( \lambda_4 )</td>
<td>138.506</td>
<td>179.862</td>
<td>158.4864</td>
<td>158.9136</td>
</tr>
<tr>
<td>( \lambda_5 )</td>
<td>200.000</td>
<td>300.000</td>
<td>247.3517</td>
<td>246.7401</td>
</tr>
</tbody>
</table>

Example 3

\[(\cos x) u'' + 2xu' + (x+1)u + \lambda (u' + e^x u) = 0 \quad \text{on} \quad [0, 1]\]

Boundary condition \( u(0) = 0 \), \( u'(1) = 0 \)

Analytic solution unknown

Matrix method \( n=4 \)
Finite Difference Formulation

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
7.5046 & -15.6759 & 9.5046 & 0 & 0 \\
0 & 5.0730 & -12.4793 & 9.0730 & 0 \\
0 & 1.5 & -6.0 & 4.5 & 0
\end{pmatrix} + \lambda
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
-1.5 & 1.3956 & 1.5 & 0 & 0 \\
0 & -1.5 & 1.9477 & 1.5 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Reducible Form

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
7.5046 & -15.6759 & 9.5046 & 0 & 0 \\
0 & 2.0487 & -.3820 & 9.0730 & 0 \\
0 & 0 & 0 & 4.5 & 0
\end{pmatrix} + \lambda
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
-1.5 & 1.3956 & 1.5 & 0 & 0 \\
0 & -2.0 & 3.9478 & 1.5 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Reduced Form

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
7.5046 & -15.6759 & 9.5046 & 0 & 0 \\
0 & 2.0487 & -.3820 & 9.0730 & 0 \\
0 & 0 & 0 & 4.5 & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
7.6335 & 4.4767 & 0 & 0 & 0 \\
2.1712 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Inverting second matrix and multiplying thru by inverse and letting \( \mu = -\lambda \)

\[
\begin{pmatrix}
-7.6335 & 4.4767 \\
-3.3483 & 2.1712
\end{pmatrix}
\]

Values for \( \mu \) are 0.276, -5.7384

hence for \( \lambda \) we have -0.276, 5.7384
Cubic Spline Formulation

\[
\begin{pmatrix}
-10.4 & 50.4 & -14.4 & 3.6 \\
2.4681 & -42.4907 & 28.8148 & -7.2037 \\
-0.5398 & 19.4336 & -35.9129 & 23.0044 \\
0.2161 & -7.7804 & 27.2312 & -19.3960
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-0.2667 & 1.9956 & 2.4 & -0.6 \\
0.0667 & -2.4 & 1.3477 & 2.4 \\
0 & 0 & 0 & 2.7183
\end{pmatrix}
\]

The second \((\lambda)\) matrix is non-singular, inverting, multiplying thru by the inverse and letting \(\mu=\lambda\) we obtain

\[
\begin{pmatrix}
-10.4 & 50.4 & -14.4 & 3.6 \\
-0.0305 & -11.4243 & 21.6993 & -13.0089 \\
-0.0820 & -3.3207 & -5.1325 & 6.4315 \\
0.0795 & -2.8622 & 10.0178 & -7.1354
\end{pmatrix}
\]

whence \(\mu = 0.258, -9.11 + 5.061, -9.11 - 5.061, -16.13\)

and \(\lambda = -0.258, 9.11 - 5.061, 9.11 + 5.061, 16.13\)

Integral Ratio

Computing \(R\) for the real values of \(\lambda\) only we obtain

\[R(-0.258) = -0.261\]

\[R(16.13) = 31.02\]
Summary

(Real \( \lambda \) only)

<table>
<thead>
<tr>
<th>( n )</th>
<th>Finite Difference</th>
<th>Cubic Spline</th>
<th>Integral Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n=4 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \lambda_1 )</td>
<td>-0.276</td>
<td>-0.258</td>
<td>-0.281</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>5.738</td>
<td>16.130</td>
<td>3.102</td>
</tr>
<tr>
<td>( n=6 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \lambda_1 )</td>
<td>-0.303</td>
<td>-0.278</td>
<td>-0.286</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>15.031</td>
<td>18.851</td>
<td>13.569</td>
</tr>
<tr>
<td>( n=11 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \lambda_1 )</td>
<td>-0.297</td>
<td>-0.287</td>
<td>-0.289</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>34.712</td>
<td>2.840</td>
<td></td>
</tr>
<tr>
<td>( \lambda_3 )</td>
<td>335.083</td>
<td>0.308</td>
<td></td>
</tr>
<tr>
<td>( n=21 )</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>( \lambda_1 )</td>
<td>-0.2923</td>
<td>-0.2891</td>
<td>-0.2896</td>
</tr>
<tr>
<td>( \lambda_2 )</td>
<td>69.326</td>
<td>58.690</td>
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</tr>
<tr>
<td>( \lambda_3 )</td>
<td>1328.413</td>
<td>0.300</td>
<td></td>
</tr>
</tbody>
</table>

From the above, it would appear that the only real eigen value we have any reasonable approximation for is \( \lambda_1 \). The failure to find any other by finite difference formulation (except when constrained to do so by the evenness of the system) supports this view. The discrepancy in cubic spline values and integral ration values is a supporting argument. There does seem to be at least one (possibly only one) real eigen value in the neighborhood of -0.290.
Computer Code

A computer code, EYGMAT, has been written in FORTRAN for the CDC 6600, which will perform the computation described and illustrated in the foregoing. The user need only supply a subroutine to compute the coefficient functions of the differential equation and read in boundary conditions. A listing and description of the code is available from the author. The code limits the problem to the interval, [0, 1] and is written for uniform partitioning.

CONCLUSION

Our first intention was simply to implement the matrix method of by use of cubic splines instead of finite difference formulas. Since the cubic spline in general, has discontinuous third derivatives we limited our consideration to second order differential equations.

In developing the formulation we discovered that

1. both differential equation and boundary condition could be satisfied at the boundary points

2. in most cases, where the "\( \lambda \)" matrix was singular, the other matrix was not, and it was possible to solve for values of \( \lambda \) without reducing the system

3. in most cases, it was possible to compute an approximate eigen function corresponding to each approximate eigen value

4. that an integral ratio for the approximate eigen function gave a much better approximation for the eigenvalue

5. incorporation of the step-size in the first rather than the second ("\( \lambda \)") matrix was more convenient for mixed boundary conditions (involving both \( u \) and \( u' \))
(6) if non-uniform steps were used that cubic spline formulation would be less intricate than finite difference formulation

(7) a single general computational process could be devised which would apply to all suitable differential equations and boundary conditions

It is realized that some of the above could be accomplished in the finite difference formulation with small changes in the technique given in 1. The really significant items (and least achievable under finite differences) are (3) and (4). In examples where eigenvalues could be determined analytically, comparison of approximations attained by finite differences and cubic splines (per se) revealed no particular advantage for either. However, the integral ratio process applied to the latter, in examples given here and in all other examples tried, gave a startling improvement in the approximation. We believe this is the principle advantage for the cubic spline formulation.

ACKNOWLEDGEMENT

This work was done in part under the auspices of the Atomic Energy Commission.
REFERENCES


This report was prepared as an account of work sponsored by the United States Government. Neither the United States nor the United States Atomic Energy Commission, nor any of their employees, nor any of their contractors, subcontractors, or their employees, makes any warranty, express or implied, or assumes any legal liability or responsibility for the accuracy, completeness or usefulness of any information, apparatus, product or process disclosed, or represents that its use would not infringe privately owned rights.