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An Analytical Solution for the Electrical Properties of a Radio-Frequency Quadrupole (RFQ) with Simple Vanes

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Summary

Although the SUPERFISH program is used for calculating the design parameters of an RFQ structure with complex vanes, an analytical solution for electrical properties of an RFQ with simple vanes provides insight into the parametric behavior of these more complicated resonators.

The fields in an inclined plane wave guide with proper boundary conditions match those in one quadrant of an RFQ. The principle of duality is used to exploit the solutions to a radial transmission line in solving the field equations. Calculated are the frequency equation, frequency sensitivity factors (S), electric field (E), magnetic field (H), stored energy (U), power dissipation (P), and quality factor (Q).

Simple Vanes

For this derivation, simple vanes are vanes whose sides form one straight line from vane tip to vane base, Fig. 1.

Fig. 1a, b: Examples of Simple Vanes

Principle of Duality

A structure which describes the fields in one resonator of an RFQ is an inclined plane wave guide. The fields in this structure are the duals of those in a radial transmission line. Therefore, all of the well-known equations for radial lines can be used for the inclined plane wave guide with the substitutions shown in Fig. 2.

Inclined Plane

Fig. 2: Replacements by Symmetry

The field equations then are:

\[ E_\phi = n G_0(kr) [A e^{j\phi(kr)} - B e^{-j\phi(kr)}] \]
\[ H_z = G_0(kr) [A e^{j\omega(kr)} + B e^{-j\omega(kr)}] \]

where

\[ n = \sqrt{n_k}, \quad k = 2\pi/\lambda, \]
\[ G_0(kr) = \frac{\sqrt{2} G_0(kr) + n_0^2(kr)}{G_0(kr) + n_1^2(kr)}, \]
\[ G_1(kr) = \frac{\sqrt{2} G_1(kr) + n_1^2(kr)}{G_1(kr) + n_0^2(kr)}, \]
\[ \phi(kr) = \tan^{-1}[N_0(kr)/J_0(kr)], \]
\[ \psi(kr) = \tan^{-1}[J_1(kr)/-N_1(kr)]. \]

A and B are constants to be determined. The time varying part of the equations is not shown, and the end effects of the RFQ resonator are not taken into account.

Referring to Fig. 3, the boundary conditions are:

\[ E_\phi = E_i \text{ at } r = r_i, \]
\[ E_\phi = 0 \text{ at } r = r_L. \]

Fig. 3: One RFQ Resonator

Solving for the constants A and B gives for the field equations:

\[ E_\phi = E_i \frac{G_i}{G_i + \frac{j}{n_i} \sin(\phi - \psi_i)}, \]
\[ H_z = \frac{E_i}{J_i} \frac{G_0}{G_i + \frac{j}{n_i} \sin(\phi_i - \psi_i)}, \]

where \( G_i = G_i(kr_i), \quad \phi = \psi(kr_i), \quad G_i = G_i(kr), \)

etc.; \( j = \sqrt{-1}. \)

Because the vanes create a highly foreshortened structure, the small argument approximations for the Bessel functions apply. Using these gives for the field equations:

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The stored energy in a resonant structure can be expressed as:

\[ U = \frac{\varepsilon}{2} \int |E|^2 \, dv = \frac{\varepsilon}{2} \int_{r_i}^{r_f} |E|^2 L_0 \phi_0 \, r \, dr, \]
giving for one resonator:

\[ U = \frac{\varepsilon \varepsilon_0^2 L_0 \phi_0}{2 r_i^2} \frac{(r_f^4 - r_i^4)}{4} - r_f^2(r_f^2 - r_i^2) \]
\[ + r_f^4 \ln(r_f/r_i) \]
\[ U_{TOTAL} = 4U. \]

The power dissipated in an RFQ resonator is composed of two parts, that lost in the vane walls, and that lost in the cylindrical surface. The two parts are given approximately by:

\[ P = 2 R_S/2 \int |J_r|^2 \, dS_{vane} + R_S/2 \int |J_{r_L}|^2 L_0 \phi_0 r r_L \]
where

\[ R_S = 1/\sigma \alpha, \ \sigma = \text{conductivity}, \ \alpha = \text{skin depth}, \]
\[ J_r = H_z(kr_f) = \text{current per unit width}. \]

At resonance,

\[-[\gamma + \ln(kr_f/2)] = 2/(kr_f)^2\]
(see frequency equation below), and P can be written as:

\[ P = \frac{R_S E_f^2}{2n^2 (kr_f)^2 [1 - (r_f/r_i)^2]^2/4} \times \]
\[ \left\{ (\frac{r_f^4 - r_i^4)}{4} - r_f^2(r_f^2 - r_i^2) \right\} \phi_0 r_L \]
\[ + \frac{(kr_f)^4}{2} [r_L - 2r_i + r_L(\ln(r/L/r_i) - 1)^2]; \]

At resonance, \( H_z \) can also be written as:

\[ H_z = (E_i/n) \frac{1 + (kr_f)^2 [\gamma + \ln(kr_f/2)]/2}{kr_i [1 - (r_i/r_f)^2]^2/2}. \]

### Other Parameters

The frequency equation

The field shapes in an RFQ are greatly distorted from the TE210 mode in a conventional cylindrical resonator. This results in a resonant frequency on the order of four times lower than the classical TE210 mode.

The input wave admittance, Fig. 2, for an inclined wave guide is given by:

\[ Y_i = Y_{L0} \cos(\phi_i - \psi_i) + j Y_{OL} \sin(\phi_i - \psi_i). \]

Given that the structure, Fig. 3, is resonant, we have:

\[ Y_i = 0, \text{boundary condition}, \]
\[ Y_i > 0, \text{resonant condition}, \]

therefore

\[ \phi_i - \psi_L = \gamma/2 \]

and \( \tan(-\phi_i) = \cot \psi_L \), giving for the frequency equation:

\[ N_0(kr_f)/J_0(kr_f) = N_1(kr_f)/J_1(kr_f). \]

Using small argument approximations for the Bessel functions gives for the frequency equation:

\[ -\ln(kr_f^2) = \gamma + 2 \left(\frac{k}{kr_f}\right)^2 \]
where \( \gamma = 0.5772 \ldots \) = Euler's constant.

One immediate result is the fact that the resonant frequency does not depend on the angle of inclination of the two planes.

Two examples:

1. RFQ model for Bevatron linac preinjector.
   (For cross section, see Fig. 1b.)
   \[ r_f = 0.145 \text{ cm} \]
   \[ r_L = 8.5 \text{ cm} \]
   Calculation = 370.6 MHz

\[ P_{TOTAL} = 4P. \]

The quality factor (Q) for the resonator is defined as:

\[ Q = \omega_0 U/P. \]

Substituting for \( U \) and \( P \) gives:

\[ Q = \frac{\varepsilon \varepsilon_0 k_0^2 n^2}{4R_S} \left\{ (r_f^4 - r_i^4)/4 - r_f^2(r_f^2 - r_i^2) \right\} \]
\[ + r_f^4 \ln(r_f/r_i) \left\{ \left[ 1 - \frac{\gamma + \ln(kr_f/2)}{\gamma + \ln(kr_f/2)} \right] \phi_0 r_L \right\} \]
\[ + \frac{(kr_f)^4}{2} [r_L - 2r_i + r_L(\ln(r/L/r_i) - 1)^2]. \]
2. RFQ model I for the Numatron Project.\(^2\)

(For cross section, see Fig. 1a.)

\[ r_i = 1.0 \text{ cm} \]
\[ r_L = 9.5 \text{ cm} \]

Resonant Frequency

SUPERFISH \(= 453.9 \text{ MHz} \)
Calculation \(= 452.9 \text{ MHz} \)

A quick look-up graph for RFQ resonant frequency vs. dimension is given in Fig. 5 for ranges of common interest.

\[ r_i (\text{em}) \]
\[ r_L (\text{em}) \]

Resonant Frequency

SUPERFISH \(= 53.9 \text{ MHZ} \)
Calculation \(= 452.9 \text{ MHZ} \)

\[ \text{SUPERFISH} \]

1.0 9.5

A quick look-up graph for RFQ resonant frequency vs. dimension is given in Fig. 5 for ranges of common interest.

\[ r_L (\text{cm}) \]
\[ r_i (\text{cm}) \]

\( r_i (\text{cm}) \)
\( (0.1, 0.4, 1.0) \)

\( FREQ (\text{MHz}) \)

\[ r_i (\text{cm}) \]
\[ (0.1, 0.4, 1.0) \]

\( FREQ (\text{MHz}) \)

\[ r_i (\text{cm}) \]
\[ (0.1, 0.4, 1.0) \]

\( FREQ (\text{MHz}) \)

\[ r_i (\text{cm}) \]
\[ (0.1, 0.4, 1.0) \]

\( FREQ (\text{MHz}) \)

\[ r_i (\text{cm}) \]
\[ (0.1, 0.4, 1.0) \]

Sensitivity Factors

The variation in resonator frequency due to dimensional errors can be found by taking differentials of the frequency equation. The sensitivity factors for errors in input and output radius are shown below and plotted in Figs. 6 and 7 for typical radii.

\[ S_i = \left| \frac{df/f}{dr_i/r_i} \right| = 4/\left[(kr_i)^2 - 1\right]^{-1} \]

\[ r_i \text{ = const.} \]

\[ S_L = \left| \frac{df/f}{dr_L/r_L} \right| = \frac{4/\left[(kr_L)^2 - 1\right]}{4/\left[(kr_L)^2 - 1\right]} \]

\[ r_i \text{ = const.} \]

Example:

\( f = 200 \text{ MHz}, r_i = 0.31 \text{ cm}, r_L = 15.8 \text{ cm}, Q = 4000 \)

\( df = 50 \text{ KHz} \) one-half power bandwidth

from figure 6,

\[ dr_i = \frac{r_i}{S_i} \times \frac{df}{f} = 6 \times 10^{-4} \text{ cm} \]

from figure 7,

\[ dr_L = \frac{r_L}{S_L} \times \frac{df}{f} = 3.5 \times 10^{-3} \text{ cm} \]

This demonstrates the extreme frequency sensitivity due to dimensional changes.

The sensitivity factor, \( S_\phi \), for vane tilt errors can be written in terms of the other factors as:

\[ S_\phi = \frac{df/f}{dr_i/r_i} = -S_L \phi \cot \phi \]

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References


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