Title
UNITARIZATION OF THE DUAL RESONANCE AMPLITUDE. III. GENERAL RULES FOR THE ORIENTABLE AND NONORIENTABLE MULTI-LOOP AMPLITUDES

Permalink
https://escholarship.org/uc/item/1vt627t8

Author
Kaku, Michio.

Publication Date
1970-11-01
UNITARIZATION OF THE DUAL RESONANCE AMPLITUDE.
III. GENERAL RULES FOR THE ORIENTABLE AND
NONORIENTABLE MULTILOOP AMP T I T U D E S

Michio Kaku and Loh-ping Yu

November 5, 1970

AEC Contract No. W-7405-eng-48

TWO-WEEK LOAN COPY
This is a Library Circulating Copy which may be borrowed for two weeks.
For a personal retention copy, call Tech. Info. Division, Ext. 5545

LAWRENCE RADIATION LABORATORY
UNIVERSITY of CALIFORNIA BERKELEY
DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.
UNITARIZATION OF THE DUAL RESONANCE AMPLITUDE.

III. GENERAL RULES FOR THE ORIENTABLE AND NONORIENTABLE MULTILOOP AMPLITUDES

Michio Kaku§ and Loh-ping Yu

Lawrence Radiation Laboratory
University of California
Berkeley, California 94720

November 5, 1970

ABSTRACT

We complete the KSV unitarization scheme for the dual resonance amplitude by presenting the nonorientable and overlapping multiloop amplitudes. We also present rules for writing down arbitrarily mixed planar, nonplanar, nonorientable, and overlapping multiloop amplitudes by inspection.
I. INTRODUCTION

This paper is the last in a series of three papers on the unitarization of the Veneziano amplitude.\textsuperscript{1-5} Remarkably, the nonorientable and overlapping loop amplitudes retain most of the essential features of the integrand of the planar and nonplanar multiloop amplitudes. Each nonorientable loop is distinguished by the fact that its projective transformation is loxodromic (i.e., its multiplier is negative). The overlapping loop amplitudes have hyperbolic projective transformations (as in the case with the planar and nonplanar loops); the difference lies in the fact that the two pairs of invariant points are found to have overlapping orderings.

Fortunately, the momentum-dependent factors in the integrand are independent of the arrangement of the various loops in a multiloop amplitude. We give simple rules for writing down by inspection the factors raised to the $\alpha_0 - 1$ power, which are sensitive to the topology of the diagram.
II. GENERAL MULTILOOP FORMULAS

A. The Nonorientable Multiloop Amplitude

In our previous papers,\textsuperscript{1-3} we went into considerable detail in deriving the planar and nonplanar multiloop amplitudes by joining the factorized legs of the N-reggeon tree as well as Scoito three-reggeon vertices. Because all multiloop amplitudes share many of the same characteristics, we shall present only the nonorientable and overlapping loop formulas without the derivation.

In Ref. 3, we found that the particular factorized tree used to calculate the multiloop amplitudes was often not the most desirable one from the viewpoint of obtaining symmetry among all quark loops and in discussing the limits of integration. In Fig. 1, for example, we display the factorized tree which was used in the calculation of the nonplanar multiloop amplitude. To restore the symmetry among all the quark loops, however, we found it convenient to move all external lines lying on the outer quark loop away from the loops. In Ref. 3 we found that these external variables, when bunched together, are confined to lie between the invariant points of the product of all projective transformations taken in order (see Fig. 2). The symmetry among all quark loops is restored, because the Koba-Nielsen variables associated with external legs trapped within a loop are confined to lie between the invariant points of that loop, whereas the variables associated with the outer quark loop are confined to lie between the invariant points of the product of all projective transformations.
Essentially all features of the multiloop amplitude remain the same if we go from Fig. 1 to Fig. 2, except for the factors raised to the power $\alpha_0 - 1$ and the region of integration.

Using the notation of Refs. 1, 2, and 3, we let $y_i$ represent each external leg and $y_\beta$ and $y_\alpha$ the invariant points of the $(\alpha \beta)$ loop. As we move external legs lying on the outer quark loop away from the loops, the critical $\alpha_0 - 1$ factors, which are sensitive to the quark topology, change from

\[
\chi \left[ \frac{(y_{\beta+1} - y_\beta)(R_{\beta \alpha}(y_{\beta+1}) - y_{\alpha-1})}{(y_\beta - R_{\beta \alpha}(y_{\beta+1}))} \right]^{\alpha_0 - 1}
\]

(2.1)

to

\[
\chi \left[ \frac{(y_{\beta+1} - y_\beta)(R_{\beta \alpha}(y_{\beta+1}) - y_{\alpha-1})}{(y_\beta - R_{\beta \alpha}(y_{\beta+1}))} \right]^{\alpha_0 - 1}
\]

\[
\left[ \frac{(y_0 - x^{(2)})(R(y_0) - y_{\alpha-1})}{(x^{(2)} - \hat{R}(y_0))} \right]^{\alpha_0 - 1}
\]

(2.2)

where $\hat{R} = R_{\beta \alpha} R_{\gamma \delta} \cdots R_{\lambda \sigma}$ and $x^{(2)} = \hat{R}^\infty(z)$. [Notice that we have dropped all factors like $(y_1 - y_{1+1})^{\alpha_0 - 1}$ for simplicity.]

$(x_1$ is the multiplier of $R_1$. Also, $x^{(2)}_{\alpha \beta} = y_\alpha = R_{\beta \alpha}^\infty(z_1)$, $z_1 \neq x^{(1)}_{\alpha \beta}$ and $x^{(1)}_{\alpha \beta} = y_\beta = R_{\beta \alpha}^\infty(z_2)$, $z_2 \neq x^{(2)}_{\alpha \beta}$.)
The factor raised to the $\left(\frac{1}{2}(\alpha_0 - 1)\right)$ power is projectively invariant and symmetric in all quark loops. It arises only when all external lines are removed from the region between two or more loops.

Now assume that we add an extra twist operator to Fig. 1 for each loop, so that it now describes the nonorientable multiloop amplitude. Applying the techniques used in Refs. 1 and 2, we find that the projective transformation describing each loop is loxodromic (i.e., it has a negative multiplier lying between zero and minus one) and that the $\alpha_0 - 1$ factors change slightly from the analogous factors given in (2.1):

$$\prod_{(\alpha\beta)=(\alpha')} \left( \frac{(y_{\beta+1} - y_\beta)(y_\alpha - y_{\alpha+1})}{y_\beta - R_{\beta\alpha}(y_{\beta+1})} \right)^{\alpha_0 - 1} \left( \frac{(y_{\beta-1} - y_\beta)(y_\alpha - y_{\alpha-1})}{y_\beta - R_{\beta\alpha}(y_{\beta-1})} \right)^{\alpha_0 - 1} . \quad (2.3)$$

Notice that the only difference between the nonplanar and nonorientable factors is that we have exchanged the roles of some of the $y$'s.

In the nonplanar case, lines trapped between the invariant points of the same loop remain there, though lines occurring within nonorientable loops are free to move out, due to duality. Mathematically, this corresponds to:

**Nonplanar case:**

$$y_{\beta+1} < y_\beta < y_\alpha \rightarrow y_\beta < y_\alpha < R_{\beta\alpha}(y_{\beta+1})$$

and

$$y_\beta < y_{\beta-1} < y_{\beta-2} < y_\alpha \rightarrow y_\beta < y_{\beta-2} < R_{\beta\alpha}(y_{\beta-1}) < y_\alpha;$$
Nonorientatable case:

\[ y_{\beta+1} < y_{\beta} < y_{\beta-1} < y_{\alpha} \rightarrow y_{\beta} < y_{\beta-1} < R_{\beta\alpha}(y_{\beta+1}) < y_{\alpha} \]

and

\[ y_{\beta} < y_{\beta-1} < y_{\beta-2} < y_{\alpha} \rightarrow y_{\beta} < y_{\beta-2} < y_{\alpha} < R_{\beta\alpha}(y_{\beta-1}). \quad (2.4) \]

In the nonplanar case, the projective transformation is hyperbolic, and therefore can send external lines past the invariant points. In the nonorientatable case, the transformation is loxodromic, so \( R^2 \) is responsible for sending external lines past loops.

As before, Fig. 1 is an awkward configuration to discuss the limits of integration for the nonorientable case. Assume that we have pushed all external lines away from the loops, as in Fig. 2 (notice that the lines situated between the various invariant points no longer exist). Since \( R_{\beta1}^2 \) is responsible for sending external lines past the \( i \)th loop, we expect that the product of the squares of all projective transformations, taken in order, will send a line completely around the diagram.

We thus find that the analogous factors in Fig. 2 are

\[
\left[ \frac{(y_0 - x^{(2)}) (R(y_0) - y_{\alpha-1})}{(x^{(2)} - R(y_0))} \right]^{\alpha_0-1} \left[ \frac{1}{\prod_{(\alpha\beta)=(\beta\gamma)} x^{(2)} \alpha_\beta \alpha_\gamma \alpha_\lambda} \right]^{2} , \quad (2.5)
\]

where \( R = R_{\beta\alpha}^2 R_{\gamma\gamma}^2 \cdots R_{\lambda\sigma}^2 \) and \( x^{(2)} = R^{-\infty}(z) \).

As in Refs. 2 and 3, we expect the region of integration for Fig. 2 in the nonorientable case to be
\[ u_2 = \left\{ x^{(2)}_\alpha \leq x^{(1)}_\alpha \leq x^{(2)}_\sigma \leq \cdots \leq x^{(2)}_{\alpha\beta} \leq x^{(2)}_{\alpha\beta} \leq x^{(1)} \leq \right. \]
\[ \leq R_{\beta\alpha}^2 \cdots R_{\sigma\alpha}^2 (y_0) \leq y_{\alpha-1} < y_{\alpha-2} \leq \cdots \leq y_1 \leq y_0 \leq x^{(2)} \]

and \[ x^{(1)} \leq \bar{R}(\bar{y}) \leq y_0 \leq \bar{y} \leq x^{(2)} \right\} \]

(2.6)

\[ \bar{y}, \text{ which is arbitrary, is introduced to eliminate periodicities, exactly as in Refs. 2 and 3.} \]

All other factors in the integrand of the nonorientable loop amplitude are identical to the planar case (except for the 1-x's).

B. Overlapping Multiloop Amplitude

The double overlapping loop amplitude can be calculated by joining the (\(\alpha\beta\)) and (\(\gamma\delta\)) pairs as shown in Fig. 3. As in the case with the nonorientable loop, all the essential features of the integrand remain the same as in the nonplanar case, except for the \(\alpha_0 - 1\) factors. A direct calculation reveals that, for Fig. 3, we have the same form as the nonplanar case, except that the invariant points of different loops are allowed to overlap. Duality allows us to successively move external lines past the loops; mathematically, this corresponds to

\[ y_{\delta+1} < y_{\delta} < y_\beta < y_\gamma < y_\alpha \rightarrow y_{\delta} < y_\beta < y_\gamma < y_\alpha < R_{\delta\gamma}^0 (y_{\delta+1}) < y_\alpha \rightarrow \]
\[ y_{\delta} < y_\beta < R_{\alpha\beta}^{-1} R_{\delta\gamma}^0 (y_{\delta+1}) < y_\gamma < y_\alpha \rightarrow \]
\[ y_{\delta} < R_{\delta\gamma}^{-1} R_{\alpha\beta}^{-1} R_{\delta\gamma}^0 (y_{\delta+1}) < y_\beta < y_\gamma < y_\alpha \rightarrow \]
\[ y_{\delta} < y_\beta < y_\gamma < y_\alpha < R_{\alpha\beta}^{-1} R_{\delta\gamma}^{-1} R_{\alpha\beta}^{-1} R_{\delta\gamma}^0 (y_{\delta+1}). \]
We are then free to bunch all external lines, as in Fig. 4. In this case, we find that the operator $R_1 R_2^{-1} R_1^{-1} R_2^{-1}$ is responsible for sending external lines past loops 1 and 2. Again, from projective invariance, we find that the $\alpha_0 - 1$ factors become

$$\left[ \frac{\left( y_{02} - x^{(2)} \right) \left( R(y_0) \right)_{-1}^{\frac{1}{2}} y_{s-2}}{\left( x^{(2)} - R(y_0) \right)} \right]^{\alpha_0 - 1}$$

$$\left[ \frac{x_1^2 x_2^2 x_{R-1}}{2} \right]^{\alpha_0 - 1}, \quad (2.8)$$

where $R = R_1 R_2^{-1} R_1^{-1} R_2^{-1}$ and $x^{(2)} \equiv R^{-\infty}(z)$.

As in the previous cases, we find that the limits of integration for Fig. 4 can be expressed as

$$x^{(2)} \leq x^{(1)} \leq x^{(2)} \leq x^{(2)} \leq x^{(1)} \leq R(y_0) \leq y_{S-3} \leq y_{S-4} \leq \cdots \leq y_1 \leq y_0 \leq x^{(2)}$$

and

$$x^{(1)} \leq R(\bar{y}) \leq \bar{y} \leq \bar{y} \leq x^{(2)} \quad (2.9)$$

We end our discussion by stating that the linear dependence correction factor is $(1 - X)$ for each planar loop, $(1 - X)^2$ for each nonplanar and overlapping loop, and $(1 + |X|)^2$ for each nonorientable loop.
III. ILLUSTRATIVE EXAMPLE

We shall illustrate our principles, which apply to all multiloop diagrams, by writing down the amplitude for a specific example. We shall find it convenient to make certain dual manipulations on the multiloop diagram such that (a) no loop ever occurs within another loop, (b) external legs confined between nonorientable and overlapping loops are removed, (c) all external legs not confined within nonplanar loops are bunched together, (d) no more than two loops ever overlap. In this configuration, all multiloop diagrams can be reduced to planar, nonplanar, nonorientable, and double overlapping loops placed in series along a chain.

Now consider the diagram corresponding to Fig. 5, which contains two planar, two nonplanar, one double overlapping, and one nonorientable loops. Applying our techniques, we know that the operator

\[ R_7^{-1} R_6^{-1} R_5^{-1} R_6^{-1} R_5^{-1} R_4^{-1} R_2^{-1} R_1^{-1} \]

is responsible for moving external lines past the region occupied by the loops. Therefore, we find

\[
\chi \left( \frac{(y_a - y_b)(y_b - y_c)(y_c - y_a)}{dy_a dy_b dy_c} \right) \prod_{(1 - x_q)^{-1}} (Q) \]

\[
(1 - x_1)(1 - x_2)(1 - x_3)^2(1 - x_4)^2(1 - x_5)^2(1 - x_6)^2(1 + |x_7|)^2
\]

Equation continued on next page
Equation continued.

\[ \sum_{i \neq \ell, m, S}^{S} (y_{i} - y_{i+1})^{\alpha_{0}^{-1}} \left\{ \frac{[y_{S+1} - R_{S}(y_{m})][x_{S}^{(1)} - y_{m}]}{[x_{S}^{(1)} - R_{S}(y_{m})]} \right\}^{\alpha_{0}^{-1}} \]

\[ \times \left\{ \frac{[y_{m+1} - R_{m}(y_{S})][x_{m}^{(1)} - y_{S}]}{[x_{m}^{(1)} - R_{m}(y_{S})]} \right\}^{\alpha_{0}^{-1}} \left\{ \frac{[y_{1} - \hat{R}(y_{1})][x_{1}^{(2)} - y_{1}]}{[x_{1}^{(2)} - \hat{R}(y_{1})]} \right\}^{\alpha_{0}^{-1}} \]

\[ \times \sum_{i, j = 1}^{L, L', L''} \sum_{n=0}^{\infty} \left\{ y_{i}^{\frac{1}{2}k_{i}k_{j}^{-1}} \right\} \]

\[ \times \sum_{i = 1}^{S} \sum_{i \neq \ell, m, S}^{S} \sum_{n=0}^{\infty} \left\{ y_{i}^{\frac{1}{2}k_{i}k_{j}^{-1}} \left[ \frac{[x_{i}^{(1)} - [R_{i}^{+}]^{(n)}(x_{i}^{(1)})]}{[x_{i}^{(1)} - [R_{i}^{+}]^{(n)}(x_{i}^{(1)})]} \right]^{\frac{1}{2}k_{i}k_{j}^{-1}} \right\}^{\frac{1}{2}k_{i}k_{j}^{-1}} \]

\[ \times \sum_{i = 1}^{S} \sum_{i \neq \ell, m, S}^{S} \sum_{n=0}^{\infty} \left\{ y_{i}^{\frac{1}{2}k_{i}k_{j}^{-1}} \left[ \frac{[x_{i}^{(1)} - [R_{i}^{+}]^{(n)}(x_{i}^{(1)})]}{[x_{i}^{(1)} - [R_{i}^{+}]^{(n)}(x_{i}^{(1)})]} \right]^{\frac{1}{2}k_{i}k_{j}^{-1}} \right\}^{\frac{1}{2}k_{i}k_{j}^{-1}} \]

\[ \times \sum_{i = 1}^{S} \sum_{i \neq \ell, m, S}^{S} \sum_{n=0}^{\infty} \left\{ y_{i}^{\frac{1}{2}k_{i}k_{j}^{-1}} \left[ \frac{[x_{i}^{(1)} - [R_{i}^{+}]^{(n)}(x_{i}^{(1)})]}{[x_{i}^{(1)} - [R_{i}^{+}]^{(n)}(x_{i}^{(1)})]} \right]^{\frac{1}{2}k_{i}k_{j}^{-1}} \right\}^{\frac{1}{2}k_{i}k_{j}^{-1}} \]
\( u_2 = [x^{(2)} \leq x_7^{(1)} \leq x_7^{(2)} \leq x_6^{(1)} \leq x_3^{(1)} \leq x_6^{(2)} \leq x_3^{(2)} \leq x_4^{(1)} \]

\[ \leq y_S \leq y_{S-1} \leq \ldots \leq y_{m+2} \leq y_{m+1} \leq R(y_S) \]

\[ \leq x_4^{(2)} \leq x_3^{(1)} \leq y_m \leq y_{m-1} \leq \ldots \leq y_{\ell+2} \leq y_{\ell+1} \]

\[ \leq R(y_m) \leq x_3^{(2)} \leq x_2^{(1)} \leq x_2^{(2)} \leq x_1^{(1)} \leq x_1^{(2)} \]

\[ \leq x_1^{(1)} \leq y_{\ell} \leq y_{\ell-1} \leq \ldots \leq y_2 \leq y_1 \leq \hat{R}(y_2) \leq x^{(2)} ] \]

and \( x_4^{(1)} \leq \overline{y}_4 \leq y_S \leq R(y_4) \leq x_4^{(2)} \)

and \( x_3^{(1)} \leq \overline{y}_3 \leq y_m \leq R(y_3) \leq x_3^{(2)} \)

and \( x_1^{(1)} \leq \overline{y}_1 \leq y_{\ell} \leq \hat{R}^{-1}(y_1) \leq x^{(2)} \),

where \( x_1^{(1)} = R_1^{-\infty}(z_1), \quad z_1 \neq x_1^{(2)} \),

\( x_1^{(2)} = R_1^{\infty}(z_1), \quad z_2 \neq x_1^{(1)} \),

\( x_1^{(1)} = \hat{R}^{\infty}(z_1), \quad z_3 \neq x_1^{(2)} \),

\( x_1^{(2)} = \hat{R}^{-\infty}(z_1), \quad z_4 \neq x_1^{(1)} \).

\[ Q = \text{set of all closed paths} = \text{set of all group elements generated by products of } R_1^+ \text{ such that cyclic permutations and inverses of such permutations of previously included members are excluded.} \]
\[ [R^+ (n)] = \text{set of all open paths} = \text{set of all group elements} \]

generated by products of \( R^+_1 \), such that first (last) element is \( R^+_L \) (last is \( R^+_L \)), and \( n \) represents the number of \( R \)'s in the product.

\( U_1 \) is determined implicitly from the conditions on \( U_2 \).

In passing, we note that the renormalization program\(^7\) is complicated by the fact that the invariant points can overlap in a general multiloop amplitude. The results for the divergence of the \( N \)-loop planar amplitude\(^8,9\) do not apply in this case (or in the case where we have loxodromic transformations).

**ACKNOWLEDGMENTS**

We thank Professor S. Mandelstam for his encouragement, guidance, and advice.
FOOTNOTES AND REFERENCES

* This work was supported in part by the U.S. Atomic Energy Commission.

+ Special Award in Theoretical Physics Fellowship, University of California-Berkeley.


6. Going from (2.1) to (2.2) is complicated by the fact that the momentum-dependent factors contribute $\alpha_0$ factors if we let $y \rightarrow R(y)$. Equation (2.2) has been rigorously verified for those factors raised to the minus one power. The "proof" of (2.2) for the $\alpha_0$ terms is indirect: suppose we had sewn two different multiloop diagrams, one with $m$ external legs between two loops and one with only $m - 1$ such external legs ($m > 1$). Since we know the form of both amplitudes, the substitution $y \rightarrow R(y)$ (which should remove one of the $m$ legs) does not contribute any $\frac{\alpha_0 - 1}{2}$ factors to the $m - 1$ case. But the momentum-dependent factors are insensitive to whether we have $m$ legs or one. The $\frac{\alpha_0 - 1}{2}$ factors, which arise only when all legs are removed from between two loops, then do not receive any contribution from the momentum-dependent factors.


9. M. Kaku and J. Scherk, Divergence of the N-Loop Planar Graph in the Dual Resonance Model (to be published).
FIGURE CAPTIONS

Fig. 1.  The general nonplanar double loop amplitude.

Fig. 2.  The ordering of the variables in a general nonplanar multiloop amplitude (notice that variables located between different loops have been moved via duality).

Fig. 3.  The general overlapping double loop amplitude.

Fig. 4.  The ordering of the variables in the overlapping double loop amplitude (notice that variables located within the loops have pushed out via duality).

Fig. 5.  An illustrative example: a multiloop amplitude containing two planar, two nonplanar, one overlapping, and one nonorientable loops.
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:

A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or

B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.

As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.