Title
Surgery on piecewise linear manifolds and applications

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1. Introduction and statement of results. In this note we indicate a method of performing surgery on piecewise linear (PL) manifolds, and show how to prove piecewise linear analogs of theorems on the homotopy type and classification of smooth manifolds (Browder [1], Novikov [10], Wall [13]).

The basic principles are two: to use normal microbundles instead of normal vector bundles, and to put a differential structure $\sigma$ on a neighborhood $V$ of an embedded sphere $S \subset M$ that represents a homotopy class we wish to kill. Then smooth ambient surgery can be performed on $V$, and the resulting cobordism triangulated.

Let $M_1, M_2$ be closed PL $n$-manifolds embedded in $S^{n+k}$ with normal microbundles $v_1, v_2$. A normal equivalence $b : (M_1, v_1) \rightarrow (M_2, v_2)$ is a microbundle equivalence $b : v_1 \rightarrow v_2$ covering a homotopy equivalence $M_1 \rightarrow M_2$.

Let $T(v_i)$ be the Thom complex of $v_i$ (see [12]), and let $c_i \in \pi_{n+k}T(v_i)$ be the homotopy class of the collapsing map $S^{n+k} \rightarrow T(v_i)$. We call $c_i$ a normal invariant for $M_i$. If $\partial M \neq 0$, a similar construction defines a normal invariant for $M$ as an element in $\pi_{n+k}(T(v_M), T(v_M|\partial M))$.

**Theorem 1.** Let $X$ be a 1-connected polyhedron satisfying Poincaré duality in a dimension $n \geq 5$. Let $\xi$ be a PL $k$-microbundle over $X$, and let $\alpha \in \pi_{n+k}T(\xi)$ be such that $h(\alpha) = \Phi(g)$, where $h : \pi_{n+k}T(\xi) \rightarrow H_{n+k}(\xi)$ is the Hurewicz homomorphism, $\Phi : H_n(X) \rightarrow H_{n+k}(\xi)$ is the Thom isomorphism, and $g \in H_n(X)$ is a generator. Assume $k \geq n$. Then $X$ has the homotopy type of a closed PL $n$-manifold $M \subset S^{n+k}$ such that

(a) If $n$ is odd, or if $n = 4q$ and the signature of $X$ is $\langle L_0(\xi), \cdots, \xi_q(\xi), g \rangle$, then $M$ has a normal microbundle induced from $\xi$, and $\alpha$ is a normal invariant of $M$;

(b) If $n$ is even, $M - \{\text{point}\}$ has a normal microbundle induced from $\xi$.

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1 Work partially supported by the National Science Foundation (USA) and Department of Scientific and Industrial Research (UK) at the Cambridge Topology Symposium, 1964.

2 We are informed that some of our results have been obtained independently by R. Lashof and M. Rothenberg.
Theorem 1 is the PL analog of [1]; see also [10].

**Theorem 2.** Let $M_1, M_2$ be PL closed 1-connected $n$-manifolds $n \geq 5$. Then $M_1$ and $M_2$ are combinatorially equivalent if and only if there are normal microbundles $\nu_i$ ($i = 1, 2$) of embeddings $M_i \subset S^{n+k}$, with normal invariants $c_i \in \pi_{n+k}T(\nu_i)$, and a normal equivalence $b : (M_1, \nu_1) \rightarrow (M_2, \nu_2)$ such that $T(b)_* (c_1) = c_2$.

Theorem 2 is the PL analog of a theorem of Novikov [10].

**Corollary.** Let $M$ be a PL closed 1-connected $n$-manifold, $n \geq 5$. Suppose the natural map $k_{PL}(M) \rightarrow k_{Top}(M)$ is injective and that $k_{PL}(\Sigma M) \rightarrow k_{Top}(\Sigma M)$ is surjective (see [8] and [9]). Then the PL structure on the underlying topological manifold $M$ is unique up to isomorphism.

**Proof.** Let $\nu_1, \nu_2$ be normal microbundles of two PL structures $M_1, M_2$ on $M$. By the stable uniqueness of a topological normal microbundle of $M$ [8], and the injectivity of $k_{PL}(M) \rightarrow k_{Top}(M)$, it follows that $\nu_1$ and $\nu_2$ are stably equivalent as PL microbundles. Let $c_i \in \pi_{n+k}T(\nu_i)$ be the normal invariant of $M_i$. Since $M_1$ and $M_2$ are the same topological manifold, it follows that (for sufficiently large $k$) there is a topological microbundle equivalence $b : \nu_1 \rightarrow \nu_2$ such that $T(b)_* (c_1) = c_2$. (The stable tubular neighborhood theorem [4], [7] is needed.) Using the surjectivity of $k_{PL}(\Sigma M) \rightarrow k_{Top}(\Sigma M)$ we can choose $b$ to be a PL microbundle equivalence. The Corollary follows from Theorem 2.

**Theorem 3.** Let $(X, A)$ be a polyhedral pair with both $X$ and $A$ 1-connected, satisfying Poincaré duality in a dimension $n \geq 6$. Let $\xi$ be a PL $k$-microbundle over $X$ with $k > n$, let $e \in H_n(X, A)$ be a generator, and suppose there exists $\beta \in \pi_{n+k}(T(\xi), T(\xi[A]))$ such that $h(\beta) = \Phi(e)$. Then $(X, A)$ is homotopy equivalent to PL manifold with boundary $(M, \partial M)$ having a normal microbundle induced from $\xi$, and having $\beta$ for a normal invariant. Moreover, $M$ is unique up to PL homeomorphism.

This is the PL analog of a result of Wall [13].

2. **Proofs of theorems.** We indicate the modification in the proofs of the analogous smooth theorems that are required in the PL case. To prove Theorem 1, by using the transverse regularity theorem
of Williamson [12] we may assume that there is a PL closed $n$-manifold $N \subset S^{n+k}$ such that:

(i) if $\tilde{f}: S^{n+k} \rightarrow T(\xi)$ represents $\alpha$, then $\tilde{f}^{-1}(X) = N$;

(ii) if $\tilde{f}|N = f$, then $f^*\xi = \nu$, the normal microbundle of $N$ in $S^{n+k}$;

(iii) $f: N \rightarrow X$ has degree 1.

(See [1].)

**Main Lemma.** Let $S \subset N$ be a PL embedded $p$-sphere, $p < n/2$, such that $f|S: S \rightarrow X$ is null homotopic. Then there exists a PL surgery killing the homotopy class of $S$. If $N'$ is the resulting $n$-manifold the trace of the surgery (an elementary PL cobordism $K$ between $N$ and $N'$) can be embedded in $S^{n+k} \times I$ with $K \cap (S^{n+k} \times 0) = N = N \times 0$ and $K \cap (S^{n+k} \times 1) = N'$. Moreover, $K$ has a PL normal microbundle $\eta$ in $S^{n+k} \times I$ with $\eta = g^*\xi$, where $g: K \rightarrow X$ extends $f|N \rightarrow X$.

**Proof.** Let $U \subset N$ be an open regular neighborhood of $S$. Then $f^*\xi|U = \nu|U$ is trivial because $f|U$ is null homotopic. Therefore there is a PL embedding $\phi: U \times R^k \rightarrow S^{n+k}$ such that $\phi(x, 0) = x$ and $\phi^{-1}N = U \times 0$. By the product theorem of [5], the smoothing of $U \times R^k$ induced by $\phi$ is concordant to a product smoothing. In fact, there is an open neighborhood $V$ of $S$ in $N$ with $\overline{V} \subset U$, a smoothing $\sigma$ of $V$, and a piecewise differentiable isotopy $\phi_t: U \times R^k \rightarrow S^{n+k}$ such that

(i) $\phi_0 = \phi$,

(ii) $\phi_t = \phi$ outside $V \times R^k$,

(iii) $\phi_t|V \times D^k$ is a smooth embedding $V \times D^k \rightarrow S^{n+k}$.

Observe now that $\phi_1(V \times 0)$ is a smooth submanifold of $S^{n+k}$ and $\phi_t$ provides a trivialization of its normal vector bundle. Let $V' \subset V_e$ be a smooth closed neighborhood of $S$, and put $W_0 = \phi_1(V' \times 0)$. Let $W_1 \subset S^{n+k}$ be the smooth submanifold obtained from $W_0$ by a smooth surgery killing the homotopy class of $\phi(S \times 0)$. By Haefliger [2] the trace of the surgery is a cobordism $L$ between $W_0$ and $W_1$ smoothly embedded in $S^{n+k} \times I$ such that $\partial L = W_0 \times 0 \cup (\partial W_0) \times I \cup W_1 \times 1$, and such that the embedding is the product embedding in a neighborhood of $\partial W_0 \times I$. Furthermore, the map $f': W_0 \times 0 \cup (\partial W_0) \times I \rightarrow X$, defined to be the composition

$$(W_0 \times 0) \cup (\partial W_0) \times I \rightarrow W_0 \xrightarrow{\phi_1^{-1}} N \rightarrow X$$

extends to $f'': L \rightarrow X$ such that $f''^*\xi$ is the normal bundle of $L$ in $S^{n+k} \times I$. 

The cobordism $L$ and the product cobordism $(N - \text{int } V') \times I$ fit together to form a cobordism $K_1 \subset S^{n+k} \times I$ between $N \times 0$ and $((N - \text{int } V') \cup W_1) \times 1$. The composition

$$(N - \text{int } V') \times I \rightarrow N \xrightarrow{f} X$$

and $f'' : L \rightarrow X$ fit together to give a map $g : K_1 \rightarrow X$. The microbundle $\nu$ extends to a microbundle $\eta$ over $K_1$ that coincides with $\nu$ over $N \times 0$, with $\nu \times I$ over $(N - \text{int } V') \times I$, and such that $\phi_1$ is a trivialization of $\eta|_{W_1 \times 1}$. In fact, $\eta = g^* \xi$. The isotopy $\phi_t$ provides an embedding $G : E_\eta \rightarrow S^{n+k} \times I$ of the total space $\eta$ which is the identity on $E_\nu$. Consider $G$ as a smooth triangulation of an open subset of $S^{n+k} \times I$.

Whitehead's triangulation theorems show that there is a neighborhood $E_0$ of the zero section of $\eta$ and a homeomorphism $H$ of $S^{n+k} \times I$ such that $HG|_{E_0}$ is PL, and $H|_{S^{n+k} \times 0}$ is the identity. Thus $K = HG(K_1)$ is the desired cobordism. This completes the proof of the Main Lemma.

The proof of Theorem 1 proceeds as in the smooth case if $n$ is odd.

If $n$ is even, we proceed until we have an $N$ such that $f : N \rightarrow X$ is an isomorphism in homotopy below the middle dimension. Following the procedure of the proof of the main lemma, we find just as in the smooth case that the obstruction $c$ to surgery is a signature or Kervaire-Arf invariant of the intersection quadratic form on the kernel $K_r$ of $f^*$ in $H_r(N)$, $2r = n$. If the signature of $X$ is as in (a) of Theorem 1, then $c = 0$; otherwise $c \equiv 0 \mod 8$. (To see this, recall that a non-singular quadratic form taking only even values has signature divisible by 8. It suffices to prove $x \# x = 0$ for $x \in \ker(f^*|_{H_*(N; Z_2)}).$ If $P : H^*(N; Z_2) \rightarrow H_*(N; Z_2)$ is Poincaré duality, then $x \# y = \langle P^{-1}x \cup P^{-1}y, N \rangle$ for $x, y \in H_*(N; Z_2).$ Let $P^{-1}x = z$. Then $x \# x = \langle Sqz, N \rangle = \langle z \cup U_N, N \rangle$ where $U_N \subset H^*(N)$ is the total Wu class. Since $Sq^{-1}U_N = W_N$ (the total Stiefel-Whitney class of $N$), if we define $U_X = Sq^{-1}W(\xi)^{-1}$ it follows that $U_N = f^* U_X$, and $x \# x = \langle Z \cup f^* U_X, N \rangle = x \# Pf^* U_X$. By [1], $K_r$ is orthogonal to $Pf^*(H^*(X))$. Hence $x \# x = 0.$)

There exists an oriented PL closed $(r - 1)$-connected $2r$-manifold $P$ with signature $-8$ if $r = 2q$, and with Kervaire-Arf invariant 1 if $r = 2q + 1$. Moreover $P - \{ \text{point} \}$ is parallelizable smoothable. It follows [3] that there is a PL embedding $P \subset S^{2r+2}$ having a trivial normal bundle on $P_0$ (the complement of a highest dimensional cell). Therefore the connected sum $N \# P$ embeds in $S^{n+k}$ with a normal microbundle $\nu'$ on $(N \# P)_0$ which coincides with the normal microbundle $\nu$ of $N$ on $N_0$, and which is trivial on the rest of $(N \# P)_0$. 
Let \( N' = N \# P \) if \( r = 2g+1 \), and let \( N' \) be the connected sum of \( N \) with \( c/8 \) copies of \( P \) if \( r = 2g \). Define \( f': N' \to X \) by \( f'|_N_0 = f \), and \( f'|_N' - N_0 \) constant. Since \( N' - N_0 \) is trivial, \( f' \) is covered by a microbundle map \( \nu' \to \xi \). The obstruction to surgery on \( N' \) now vanishes. Hence by surgery we obtain a manifold \( M \subset S^{n+k} \) with a normal microbundle \( \nu \) on \( M_0 \) and a homotopy equivalence \( f: M \to X \) such that \( f|_{ M_0 } \) is covered by a microbundle map \( \nu \to \xi \).

Alternatively, in the middle dimension we could use the method of [14].

Theorem 2 is proved in a similar way, using the same trick to extend Novikov’s proof to the PL case. Since for \( n \geq 5 \) any PL homotopy sphere \( T \) is a combinatorial sphere (Smale [11]), the conclusion of the smooth case, that \( M_1 \# T = M_2 \) becomes \( M_1 = M_2 \) in the PL case.

For Theorem 3 we imitate the proof of Theorem 2 of Wall [13] with the following modification of the immersion argument of [13]. Given a PL map \( f: D^{k+1} \to M^{2k+1} \) (in the notation of [13]), assume that \( f \) has generic singularities. It follows that \( H_f(D^{k+1}) = 0 \) for \( i > 2 \). Since \( \Gamma_i = 0 \) for \( i \leq 2 \), it follows from [5] that a neighborhood \( V \) of \( f(D^{k+1}) \) in \( M^{2k+1} \) has a smoothing \( \sigma \). Then we approximate \( f \) by a smooth map into \( V_\sigma \) and proceed as in [13].

**Bibliography**


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