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HOMOLOGICAL INDICES OF COLLECTIONS OF 1-FORMS.

E. GORSKY AND S.M. GUSEIN-ZADE

Abstract. Homological index of a holomorphic 1-form on a complex analytic variety with an isolated singular point is an analogue of the usual index of a 1-form on a non-singular manifold. One can say that it corresponds to the top Chern number of a manifold. We offer a definition of homological indices for collections of 1-forms on a (purely dimensional) complex analytic variety with an isolated singular point corresponding to other Chern numbers. We also define new invariants of germs of complex analytic varieties with isolated singular points related to “vanishing Chern numbers” at them.

1. Introduction.

For an isolated singular point of a vector field or of a 1-form on a smooth manifold one has a well-known integer invariant – the index. It can be defined for vector fields or 1-forms on a complex-analytic manifold as well. The notions of the indices of isolated singular points of a vector field or of a 1-form have several generalizations to singular (real or complex) analytic varieties: see, e.g., [2]. In particular, there are defined (in somewhat different settings) the so called GSV-index, the radial index and the Euler obstruction of a 1-form. One of the generalizations for a 1-form on a complex analytic variety with an isolated singular point at the origin is the so called homological index: [5].

The sum of indices of (isolated) singular points of a vector field or of a 1-form on a compact smooth (differentiable) manifold without boundary is equal to the Euler characteristic of the manifold. The sum of indices of (isolated) singular points of a complex-valued 1-form on an \( n \)-dimensional compact complex analytic manifold \( X^n \) is equal to \((-1)^n\) times the Euler characteristic of the manifold which coincides with the top Chern number \( \langle c_n(T^*X^n),[X^n]\rangle \) of the cotangent bundle. Thus one can say that the indices of singular points of vector fields or of 1-forms on complex analytic varieties correspond to the top Chern number.

Other Chern numbers correspond to indices of singular points of collections of 1-forms on varieties. On an \( n \)-dimensional compact complex analytic manifold \( X^n \) the Chern number \( \langle \prod_{i=1}^s c_{k_i}(T^*X^n),[X^n]\rangle \) with \( \sum_{i=1}^s k_i = n \) is equal to the sum of the (properly defined) indices of isolated singular points of a collection \( \{\omega_j^{(i)}\} \) of 1-forms \( (i=1,\ldots,s,\ j=1,\ldots,n-k_i+1) \): see, e.g., [1]. Analogues of the GSV-index for collections of 1-forms on an isolated complete intersection singularity were defined in [1]. If all the forms in the collection are complex analytic, this index is expressed as the dimension of a certain algebra. In [3], there was defined an analogue of the Euler obstruction for a collection of 1-forms on a purely \( n \)-dimensional complex analytic variety called the Chern obstruction.

Here we offer a definition of homological indices for collections of 1-forms on a (purely dimensional) complex analytic variety with an isolated singular point corresponding to Chern numbers different from the top one. We also define new invariants of germs of complex analytic varieties with isolated singular points related to “vanishing Chern numbers” at them.

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2. HOMOLOGICAL INDEX OF A 1-FORM.

For a germ of a holomorphic 1-form \( \omega = \sum_{i=1}^{n} A_i(\overline{z})dz_i \) with an isolated singular point (zero) at the origin in \( \mathbb{C}^n \) its index \( \text{ind}(\omega; \mathbb{C}^n, 0) \) is equal to the multiplicity of the map \( A = (A_1, \ldots, A_n) : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^n, 0) \) which, in turn, is equal to

\[
\dim_{\mathbb{C}} \left( \mathcal{O}_{\mathbb{C}^n, 0}/\langle A_1, \ldots, A_n \rangle \right),
\]

where \( \mathcal{O}_{\mathbb{C}^n, 0} \) is the ring of germs of holomorphic functions on \( \mathbb{C}^n \) at the origin, \( \langle A_1, \ldots, A_n \rangle \) is the ideal generated by the corresponding elements. This relation can be reformulated in the following way. Let \( \Omega_{\mathbb{C}^n, 0} \) be the module of germs of (holomorphic) differential \( i \)-forms on \( (\mathbb{C}^n, 0) \). Then

\[
\text{ind}(\omega; \mathbb{C}^n, 0) = \dim_{\mathbb{C}} \left( \Omega_{\mathbb{C}^n, 0}^{n-1}/\omega \wedge \Omega_{\mathbb{C}^n, 0}^{n-1} \right).
\]

An analogue of this equation holds for a 1-form on an isolated complete intersection singularity (ICIS) as well. Let \( (X, 0) \subset (\mathbb{C}^N, 0) \) be an \( n \)-dimensional isolated complete intersection singularity defined by the equations \( f_1 = f_2 = \ldots = f_{N-n} = 0 \), \( f_i \in \mathcal{O}_{\mathbb{C}^N, 0} \), and let \( \Omega_{X,0}^{i} = \Omega_{\mathbb{C}^N,0}^{i}/(f_i \Omega_{\mathbb{C}^N,0}^{i}, df_i \wedge \Omega_{\mathbb{C}^N,0}^{i-1}) \) be the module of germs of differential \( i \)-forms on \( (X, 0) \). For a 1-form (not necessarily holomorphic) with an isolated singular point at the origin its GSV-index \( \text{ind}_{\text{GSV}}(\omega; X, 0) \) was defined in [5]. If the 1-form \( \omega \) is holomorphic, one has [5]:

\[
(1) \quad \text{ind}_{\text{GSV}}(\omega; X, 0) = \dim_{\mathbb{C}} \left( \Omega_{X,0}^{1}/\omega \wedge \Omega_{X,0}^{0} \right).
\]

The usual index and the GSV-index possess the following “law of conservation of number”. If \( \omega' \) is a small deformation of the 1-form \( \omega \), the sum of indices of the singular points of the 1-form \( \omega' \) split from the origin is equal to the index of the 1-form \( \omega \) at the origin. This means that

\[
(2) \quad \text{ind}_{\text{GSV}}(\omega; X, 0) = \text{ind}_{\text{GSV}}(\omega'; X, 0) + \sum_{x} \text{ind}_{\text{GSV}}(\omega'; X, x),
\]

where the sum on the right hand side is over all singular points \( x \) of the 1-form \( \omega' \) in a small punctured neighbourhood of the origin \( 0 \) in \( X \). (Let us recall that for all points \( x \) from a punctured neighbourhood of the origin \( 0 \) in \( X \) the GSV-index \( \text{ind}_{\text{GSV}}(\omega'; X, x) \) is actually the usual index of the 1-form \( \omega' \) on the complex analytic manifold \( X \setminus \{0\} \).

This property does not hold in general for a 1-form on a germ of a complex analytic variety with an isolated singular point at the origin if the index is defined by Equation (1). A way to bypass this problem is to consider the homological index of a 1-form [3].

Let \( (X, 0) \subset (\mathbb{C}^N, 0) \) be a germ of a complex analytic variety of pure dimension \( n \) with an isolated singular point at the origin and let \( \omega \) be a holomorphic 1-form on \( (X, 0) \) (that is the restriction to \( (X, 0) \) of a holomorphic 1-form on \( (\mathbb{C}^N, 0) \) without singular points (zeroes) outside of the origin. Let \( \Omega_{X,0}^{i} \) be the module of germs of differential \( i \)-forms on \( (X, 0) \). Let us consider the complex \( (\Omega_{X,0}^{*}; \wedge \omega) \):

\[
0 \rightarrow \Omega_{X,0}^{0} \rightarrow \Omega_{X,0}^{1} \rightarrow \ldots \rightarrow \Omega_{X,0}^{n} \rightarrow 0,
\]

where the arrows are the exterior products by the 1-form \( \omega \wedge \cdot \). This complex has finite-dimensional (co)homology groups \( H^i(\Omega_{X,0}^{*}; \wedge \omega) \). (This follows from the fact that the corresponding complex of sheaves consists of coherent sheaves and its cohomologies are concentrated at the origin.)
Definition 4. ([5]) The homological index of the 1-form $\omega$ on $(X,0)$ is defined by

$$\text{ind}_{\text{hom}}(\omega; X,0) = \sum_{i=0}^{n} (-1)^{n-i} \dim \mathcal{H}^i(\Omega^*_{X,0}, \wedge \omega).$$

The homological index satisfies the law of conservation of number [8]. If $X$ is an ICIS, the homological index of a holomorphic 1-form coincides with its GSV-index [5] Theorem 3.2.

In [10] there was considered an equivariant (with respect to a finite group action) version of the homological index of a 1-form. It takes values in the ring of representations of the group. It was shown that on a smooth manifold this index coincides with the reduction of the equivariant index with values in the Burnside ring of the group defined earlier.

3. Indices of collections of forms.

Let $k_i, i = 1, \ldots, s$, be positive integers such that $\sum_{i=1}^{s} k_i = n$. We shall consider collections of 1-forms $\{\omega_j^{(i)}\}, i = 1, \ldots, s, j = 1, \ldots, n - k_i + 1$, on (purely) $n$-dimensional varieties or on germs of $n$-dimensional varieties. One can say that collections of this sort correspond to the Chern number $(\prod_{i=1}^{s} c_{k_i}[\bullet])$ in the following sense. Let $X$ be a (non-singular) compact complex manifold of dimension $n$ and let $\{\omega_j^{(i)}\}, i = 1, \ldots, s, j = 1, \ldots, n - k_i + 1$, be a collection of 1-forms on it (continuous, but not necessarily holomorphic).

Definition 6. A point $x \in X$ is called a singular point of the collection $\{\omega_j^{(i)}\}$ if for each $i$ the 1-forms $\omega_1^{(i)}, \ldots, \omega_{n-k_i+1}^{(i)}$ at the point $x$ are linearly dependent.

For an isolated singular point $x$ of a collection $\{\omega_j^{(i)}\}$ one can define the notion of its index $\text{ind}(\{\omega_j^{(i)}\}; X, x)$ (see a more general definition for a collection of 1-forms on an ICIS below). If the collection $\{\omega_j^{(i)}\}$ has only isolated singular points on $X$, the sum of their indices is equal to the characteristic number $(\prod_{i=1}^{s} c_{k_i}(T^*V^n), [V^n])$.

For positive integers $N$ and $m$ with $N \geq m$, let $\mathcal{M}(N,m)$ be the space of $N \times m$ matrices with complex entries and let $D(N,m)$ be the subspace of $\mathcal{M}(N,m)$ consisting of degenerate matrices, that is of matrices of rank less than $m$. (The subset $D(N,m)$ is an irreducible subvariety of $\mathcal{M}(N,m)$ of codimension $N - m + 1$.) For a sequence $\mathbf{m} = (m_1, \ldots, m_s)$ of positive integers, let $\mathcal{M}_{N,\mathbf{m}} := \prod_{i=1}^{s} \mathcal{M}(N, N - m_i + 1)$ and let $D_{N,\mathbf{m}} := \prod_{i=1}^{s} D(N, N - m_i + 1)$. The variety $D_{N,\mathbf{m}}$ is irreducible of codimension $m = \sum_{i=1}^{s} m_i$, and therefore its complement $W_{N,\mathbf{m}} = \mathcal{M}_{N,\mathbf{m}} \setminus D_{N,\mathbf{m}}$ is $(2m - 2)$-connected, $H_{2m-1}(W_{N,\mathbf{m}}) \cong \mathbb{Z}$, and there is a natural choice of a generator of the latter group. This choice defines the degree (an integer) of a map from an oriented manifold of dimension $2m - 1$ to $W_{N,\mathbf{m}}$.

Let, as above, $(X,0) \subset (\mathbb{C}^N,0)$ be an $(n$-dimensional) isolated complete intersection singularity defined by the equations $f_1 = f_2 = \ldots = f_{N-n} = 0$, $f_i \in \mathcal{O}_{\mathbb{C}^N,0}$. Let $k_i, i = 1, \ldots, s$, be positive integers such that $\sum_{i=1}^{s} k_i = n$ and $\{\omega_j^{(i)}\}, i = 1, \ldots, s, j = 1, \ldots, n - k_i + 1$, be a collection of 1-forms on $(X,0)$ (that is restrictions to $(X,0)$ of 1-forms on $(\mathbb{C}^N,0)$) without singular points on $X$ outside the origin. Let $U$ be a neighbourhood of the origin in $\mathbb{C}^n$ where all the functions $f_r (r = 1, \ldots, N - n)$ and the 1-forms $\omega_j^{(i)}$ are defined and such that the restriction of the collection $\{\omega_j^{(i)}\}$ of 1-forms to $(X \cap U) \setminus \{0\}$ has no singular points. Let $S_\delta \subset U$ be a sufficiently small sphere around the origin.
which intersects $X$ transversally and denote by $K = X \cap S_\delta$ the link of the ICIS $(X, 0)$.

Let $\mathfrak{k} := (k_1, \ldots, k_s)$ and let $\Psi_X$ be the map from $X \cap U$ to $\mathcal{M}_{n, \mathfrak{k}}$ which sends a point $x \in X \cap U$ to the collection of $N \times (N - k_i + 1)$-matrices

$$\{(df_1(x), \ldots, df_{N-n}(x), \omega_{1}^{(i)}(x), \ldots, \omega_{n-k_i+1}^{(i)}(x))\}, \quad i = 1, \ldots, s.$$ 

Its restriction $\psi_X$ to the link $K$ maps $K$ to $W_{N, \mathfrak{k}}$. The following notion was introduced in $[1]$.

**Definition 7.** The GSV index $\text{ind}_{GSV}^{}(\{\omega_{j}^{(i)}\}; X, 0)$ of the collection of 1-forms $\{\omega_{j}^{(i)}\}$ on the ICIS $(X, 0)$ is the degree of the mapping $\psi_X : K \to W_{N, \mathfrak{k}}$ or, equivalently, the intersection number $(\text{Im} \Psi_X \cap D_{N, \mathfrak{k}})$.

Assume now that all the 1-forms $\omega_{j}^{(i)}$ in the collection are complex analytic. In this case one has the following (“algebraic”) formula for the index $\text{ind}_{GSV}^{}(\{\omega_{j}^{(i)}\}; X, 0)$. Let $I_{X,\{\omega_{j}^{(i)}\}}$ be the ideal in the ring $\mathcal{O}_{\mathbb{C}^n, 0}$ generated by the functions $f_1, \ldots, f_{N-n}$ and by the $(N - k_i + 1) \times (N - k_i + 1)$ minors of all the matrices

$$(df_1(x), \ldots, df_{N-n}(x), \omega_{1}^{(i)}(x), \ldots, \omega_{n-k_i+1}^{(i)}(x))$$

for all $i = 1, \ldots, s$. Then one has (II)

$$\text{ind}_{GSV}^{}(\{\omega_{j}^{(i)}\}; X, 0) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0}/I_{X,\{\omega_{j}^{(i)}\}}.$$ 

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an arbitrary germ of an analytic variety (not necessarily with an isolated singularity) and let $\{\omega_{j}^{(i)}\}, i = 1, \ldots, s, j = 1, \ldots, n - k_i + 1$, be a collection of 1-forms on $(X, 0)$ $(\omega_{j}^{(i)}$ is the restriction of a 1-form on $(\mathbb{C}^N, 0)$ which will be denoted by $\omega_{j}^{(i)}$ as well.)

**Definition 9.** A point $x \in X$ is called a special point of the collection $\{\omega_{j}^{(i)}\}$ if there exists a sequence $\{x_m\}$ of points from the non-singular part $X_{\text{reg}}$ of the variety $X$ converging to $x$ such that the sequence $T_{x_m}X_{\text{reg}}$ of the tangent spaces at the points $x_m$ has a limit $L$ as $m \to \infty$ (in the Grassmann manifold $G(n, N)$ of $n$-dimensional vector subspaces of $\mathbb{C}^N$) and the restrictions of the 1-forms $\omega_{1}^{(i)}, \ldots, \omega_{n-k_i+1}^{(i)}$ to the subspace $L \subset T_x\mathbb{C}^N$ are linearly dependent for each $i = 1, \ldots, s$.

The collection $\{\omega_{j}^{(i)}\}$ of 1-forms has an isolated special point on the germ $(X, 0)$ if it has no special points on $X$ in a punctured neighbourhood of the origin. (The condition for a special point of a collection of holomorphic 1-forms to be non-isolated is a condition of infinite codimension.) For a collection of 1-forms on $(X, 0)$ with an isolated special point at the origin there is defined the local Chern obstruction $\text{Ch}^{}(\{\omega_{j}^{(i)}\}; X, 0)$: $[3]$. It is defined in terms of the Nash transform $\hat{X}$ of the variety $X$: the closure in $\mathbb{C}^N \times G(n, N)$ of the set $\{(x, T_xX_{\text{reg}})\}$ for all point $x$ from the non-singular part $X_{\text{reg}}$ of $X$. Over the Nash transform $\hat{X}$ one has the Nash bundle which extends the tangent bundle over the non-singular part of $X$. The forms $\omega_{j}^{(i)}$ define sections of the dual bundle. The Chern obstruction $\text{Ch}^{}(\{\omega_{j}^{(i)}\}; X, 0)$ is the primary (and in fact the only) obstruction to extend these sections from the preimage of the intersection of a small sphere around the origin with $X$ to the Nash transform $\hat{X}$ so that there are no points where for each $i = 1, \ldots, s$ the extensions of the forms $\omega_{j}^{(i)}, j = 1, \ldots, n - k_i + 1$, are linear dependent. For a generic collection $\{\omega_{j}^{(i)}\}$ (in particular, for a collection consisting of the differentials of a generic collection of linear functions on $\mathbb{C}^N$) the Chern obstruction is equal to zero. If $(X, 0)$ is
non-singular, the Chern obstruction coincides with the (usual) index of the collection of 1-forms.

4. Homological Index for a Collection of Forms

Let \((X, 0)\) be a germ of an algebraic variety of dimension \(n\) with an isolated singular point at the origin. As above, let \(k_i, i = 1, \ldots, s\), be positive integers such that \(\sum_{i=1}^{s} k_i = n\) and let \(\{\omega_j^{(i)}\}, i = 1, \ldots, s, j = 1, \ldots, n - k_i + 1\), be a collection of 1-forms on \((X, 0)\).

Let \(W_i = \mathbb{C}^{n-k_i+1}\) be an auxiliary vector space with a basis \(u_1, \ldots, u_{n-k_i+1}\). We define a chain complex \(\mathcal{C}(i) = \mathcal{C}(\omega_1^{(i)}, \ldots, \omega_{n-k_i+1}^{(i)})\) of sheaves of \(\mathcal{O}_{X,0}\)-modules as following:

\[
\mathcal{C}_0^{(i)} = \Omega_{X,0}^n, \quad \mathcal{C}_i^{(i)} = \Omega_{X,0}^{k_i-t} \otimes S^{t-1}W_i, \quad 1 \leq t \leq k_i.
\]

The differential \(d_i : \mathcal{C}_i^{(i)} \rightarrow \mathcal{C}_{i-1}^{(i)}\) is defined by the equations:

\[
d_1(\beta) = \beta \wedge \omega_1^{(i)} \wedge \ldots \wedge \omega_{n-k_i+1}^{(i)},
\]

\[
d_i(\beta \otimes \varphi(u)) = \sum_{l=1}^{n-k_i+1} (\beta \wedge \omega_l^{(i)}) \otimes \frac{\partial \varphi}{\partial u_l}, \quad 2 \leq t \leq k_i.
\]

**Lemma 11.** One has \(d^2 = 0\), so \((\mathcal{C}(i), d)\) is a chain complex.

**Proof.** One has

\[
d_0d_1(\beta \otimes u_l) = (\beta \wedge \omega_l^{(i)}) \wedge \omega_1^{(i)} \wedge \ldots \wedge \omega_{n-k_i+1}^{(i)} = 0,
\]

and

\[
d_id_{i+1}(\beta \otimes \varphi(u)) = \sum_{l,l'} (\beta \wedge \omega_l^{(i)} \wedge \omega_{l'}^{(i)}) \otimes \frac{\partial^2 \varphi}{\partial u_l \partial u_{l'}} = 0, \quad t > 0.
\]

We can also define \(\mathcal{C}(i)\) (at least its part of positive degree) using the notion of the exterior power of a chain complex. Recall that if \(\mathcal{E}\) and \(\mathcal{F}\) are two chain complexes of modules over a commutative ring, then \(\mathcal{E} \otimes_R \mathcal{F} \simeq \mathcal{F} \otimes_R \mathcal{E}\), and the isomorphism is given by \(a \otimes b \mapsto (-1)^{\deg(a) \deg(b)} b \otimes a\). Using this isomorphism, one can define the action of the symmetric group \(S_k\) on \(\mathcal{E}^\otimes_k\), and define \(\wedge^k(\mathcal{E})\) as the sign component for this action. One can check that the exterior powers of a two-term complex have the form:

\[
\wedge^k [A \leftarrow B] \simeq [\wedge^k A \leftarrow \wedge^{k-1} A \otimes B \leftarrow \cdots \leftarrow A \otimes S^{k-1}B \leftarrow S^k B].
\]

Here we assume that \(A\) has homological degree 0 and \(B\) has homological degree 1.

**Proposition 13.** One has \(\mathcal{C}_{>i} \simeq \wedge^{k_i-1} [\Omega_{X,0}^1 \leftarrow \mathcal{O}_{X,0} \otimes W_i]\), where the differential in the two-term complex \(\Omega_{X,0}^1 \leftarrow \mathcal{O}_{X,0} \otimes W_i\) over \(\mathcal{O}_{X,0}\) sends \(u_i\) to \(\omega_i\).

**Proof.** Follows from \((12)\) and the fact that \(\wedge^i(\Omega_{X,0}^1) = \mathcal{O}_{X,0}\) (over \(\mathcal{O}_{X,0}\)).

**Lemma 14.** The cohomologies of \(\mathcal{C}(i)\) are supported on the subvariety \(Z(X; \omega_1^{(i)}, \ldots, \omega_{n-k_i+1}^{(i)})\) consisting of the point \(x \in X\) where the forms \(\{\omega_j^{(i)}\}\) are linearly dependent.

**Proof.** Indeed, suppose that at some point \(x \in X\) the forms \(\omega_j^{(i)}\) are linearly independent, in particular, neither of them vanishes. Since \(X\) has an isolated singularity at the origin, we can assume that \(x\) is a smooth point of \(X\). Then we can choose local coordinates at \(x\) such that at this point \(\omega_j^{(i)} = dx_j\), and one can easily check that \(\mathcal{C}(dx_1, \ldots, dx_{n-k_i+1})\) is acyclic.
Now we can define a complex
\[ C = C(\{\omega_j^{(i)}\}) = \bigotimes_{i=1}^{s} C^{(i)}, \]
where the tensor product is taken over \( O_{X,0} \). Note that by construction the complex \( C^{(i)} \) has length \( k_i \), so the complex \( C \) has length \( \sum_{i=1}^{s} k_i = n \).

**Definition 15.** The homological index of the collection of 1-forms \( \{\omega_j^{(i)}\} \) is defined as the Euler characteristic of the complex \( C \):

\[ \text{ind}_{\text{hom}} \left( \{\omega_j^{(i)}\} \right) = \sum_{t=0}^{n} (-1)^t \dim H^t(C). \]

By Lemma 14 if the collection \( \{\omega_j^{(i)}\} \) has an isolated singular point on \((X,0)\), at each point of \( X \) outside of the origin at least one of the complexes \( C^{(i)} \) is acyclic and therefore the complex \( C \) is acyclic as well. This means that the cohomologies of \( C \) are supported at the origin and therefore are finite-dimensional. This implies that the homological index is well-defined.

**Example 17.** Suppose that \( k_1 = n \), that is the collection consists of a single 1-form \( \omega = \omega_1^{(1)} \). Then the complex \( C = C^{(1)} \) agrees with \( (3) \), and the definitions of the homological index agree.

**Proposition 18.** The homological index for a collection of 1-forms with an isolated singular point satisfies the law of conservation of number (like \( (2) \)).

**Proof.** This is a direct consequence of [3].

Since any collection of holomorphic 1-forms on \((\mathbb{C}^n,0)\) can be deformed to a one with non-degenerate singular points (that is to a collection with singular points of index 1) and for a non-degenerate singular point the homological index is equal 1 as well, Proposition 18 implies that on a non-singular manifold the homological index coincides with the usual one. In the next section we shall show that on an isolated complete intersection singularity the homological index coincides with the GSV one.

5. **The case of complete intersections**

**Theorem 19.** Let \((X,0)\) be an isolated complete intersection singularity and let \( \{\omega_j^{(i)}\} \) be a collection of holomorphic 1-forms on \((X,0)\) with an isolated singular point. Then the homological index defined by \( (15) \) agrees with the GSV-index defined by \( (5) \).

The key role in the proof is played by the classical Eagon-Northcott complex \([6, 7]\), and in the next subsection we remind its definition and properties.

5.1. **Eagon-Northcott complex.** The complex \( (10) \) is very similar to the so-called Eagon-Northcott complex \([6, 7]\) which we now review. Let \( R \) be a commutative ring, and let \( M = (m_{ij}) \) be an \( s \times r \) matrix \((s \leq r)\) with entries in \( R \). For \( 1 \leq i_1 < \ldots < i_s \leq r \) we denote by \( \Delta_{i_1,\ldots,i_s}(M) \) the corresponding \( s \times s \) minor of \( M \). As above, let \( W \) be a \( s \)-dimensional space with basis \( u_1,\ldots,u_s \), and let \( V \) be an \( r \)-dimensional space with basis \( e_1,\ldots,e_r \). The complex \( E(M) \) has the chain groups

\[ E_0 = R, \quad E_j = R \otimes \wedge^{s+j-1}V \otimes S^{j-1}W, \quad 1 \leq j \leq r-s+1, \]
and the differentials $d_j : \mathcal{E}_j \to \mathcal{E}_{j-1}$ are given by the equations
\[
d_j(e_i \wedge \ldots \wedge e_{i_s}) = \Delta_{i_1, \ldots, i_s}(M),
\]
\[
d_j(e_i \wedge \ldots \wedge e_{i_s}) \otimes \varphi(u) = \sum_{i=1}^{s+j-1} \sum_{s=1}^{s+j-1} (-1)^{i-1} m_{it} e_i \wedge \ldots \wedge e_{i_{s-1}} \wedge e_{i_{s}} \wedge \ldots \wedge e_{i_{s+j-1}} \otimes \frac{\partial \varphi}{\partial u_t}
\]
for $j > 1$. One can check that $d^2 = 0$.

**Theorem 21.** Suppose $R$ is Noetherian and the depth of the ideal $(\Delta_{i_1, \ldots, i_s}(M))$ equals $r - s + 1$. Then $H^j(\mathcal{E}, d) = 0$ for $j > 0$ and
\[
H^0(\mathcal{E}, d) = \langle \Delta_{i_1, \ldots, i_s}(M) \rangle.
\]

**Corollary 22.** If $R$ and $R/\langle \Delta_{i_1, \ldots, i_s}(M) \rangle$ are Cohen-Macaulay of dimensions $N$ and $N - r + s - 1$, respectively, then $H^j(\mathcal{E}, d) = 0$ for $j > 0$.

Theorem 21 can be used to study the complex (10) on the affine space. Suppose that $X = \mathbb{C}^n$, then the coefficients of the forms $\omega_i = \sum_{i=1}^n m_i dx_i$ define an $(n - k + 1) \times n$ matrix $M = (m_{it})$. Let $R = \mathcal{O}_{\mathbb{C}^n, 0}$.

**Proposition 23.** The complexes $\mathcal{C}(\omega_1, \ldots, \omega_k)$ and $\mathcal{E}(M)$ are isomorphic.

**Proof.** Let us identify the chain groups first:
\[
\mathcal{C}_j = \Omega^{k-j}_{\mathbb{C}^n} \otimes S^{j-1}W = R \otimes \wedge^{k-j}V^* \otimes S^{j-1}W \simeq R \otimes \wedge^{n-k+j}V \otimes S^{j-1}W = \mathcal{E}_j.
\]
Under this identification, the matrix coefficients of $d_1$ are given by the coefficients of the $(n - k + 1)$-form $\omega_1 \wedge \ldots \wedge \omega_{n-k+1}$ which are just the $(n - k + 1) \times (n - k + 1)$ minors of the matrix $M$. Furthermore,
\[
d_j(dx_{i_1} \wedge \ldots \wedge dx_{i_{k-j}} \otimes \varphi(u)) = \sum_t dx_{i_1} \wedge \ldots \wedge dx_{i_{k-j}} \wedge \omega_t \otimes \frac{\partial \varphi}{\partial u_t} = \sum_{i,t} m_{it} dx_{i_1} \wedge \ldots \wedge dx_{i_{k-j}} \wedge dx_i \otimes \frac{\partial \varphi}{\partial u_t}.
\]
\]

**Corollary 24.** Suppose that $Z(\mathbb{C}^n; \omega_1, \ldots, \omega_{n-k+1})$ is Cohen-Macaulay of dimension $k$. Then $\mathcal{C}(\omega_1, \ldots, \omega_{n-k+1})$ is a free resolution of the local ring of $Z(\mathbb{C}^n; \omega_1, \ldots, \omega_{n-k+1})$.

5.2. **Forms on complete intersections.** Let $(X, 0) = \{f_1 = \ldots = f_{N-n} = 0\}$ be an isolated complete intersection singularity in $(\mathbb{C}^n, 0)$, and let $\omega_1, \ldots, \omega_{n-k+1}$ be a sequence of 1-forms on $X$. Let $R = \mathcal{O}_{X,0} = \mathcal{O}_{\mathbb{C}^n,0}/(f_1, \ldots, f_{N-n})$, and let $F \simeq \mathbb{C}^{N-n}$. The following statement is well known, but we present its proof for the reader’s convenience.

**Lemma 25.** The module $\Omega_{X,0}^j$ of $j$-forms on $X$, $j \leq n$), has the following free resolution over $R$:
\[
\Omega_{X,0}^j \simeq \left[ R \otimes \wedge^j \mathbb{C}^N \leftarrow R \otimes \wedge^{j-1} \mathbb{C}^N \otimes F \leftarrow \cdots \leftarrow R \otimes S^j F \right],
\]
where the differentials are induced by the map $(df_1, \ldots, df_{N-n}) : R \otimes F \to R \otimes \mathbb{C}^N$ which sends the $i$th basis element of $F$ to $df_i$.

**Proof.** The module of $j$-forms is defined as the $j$-th exterior power of the module of 1-forms. The latter has a natural two-term free resolution over $R$:
\[
\Omega_{X,0}^1 \simeq \left[ R \otimes \mathbb{C}^N \xleftarrow{d} R \otimes F \right],
\]
where the map $d = (df_1, \ldots, df_{N-n}) : R \otimes F \to R \otimes \mathbb{C}^N$ sends the $i$th basis element of $F$ to $df_i$. The maximal minors of $d$ vanish on the set of singular points of $(X, 0)$, which
has codimension $n$ in $R$. Therefore by \[12\] Theorem 1 for $j \leq n$ the free resolution of the module $\Omega_{X,0}^j \simeq \wedge^j (\Omega_{X,0}^1)$ coincides with the $j$-th exterior power of the complex (27), which by \[12\] agrees with (26).

Remark 34. For $k_1 = n$ this follows from the proof of \[9\] Lemma 5.3].
Proof. By definition,
\[
\text{ind}_{GSV} (\{\omega_j^{(i)}\}; X, 0) = \dim \frac{R \otimes \wedge^N C^N}{\phi \left( \sum_i \left( (\wedge_j \omega_j^{(i)}) \wedge \Omega^{k_i-1}_{X,0} \right) \right)} = \\
\text{dim}(\mathcal{T}) + \dim \frac{\text{Im}(\phi)}{\phi \left( \sum_i \left( (\wedge_j \omega_j^{(i)}) \wedge \Omega^{k_i-1}_{X,0} \right) \right)} = \\
\text{dim}(\mathcal{T}) - \dim(\mathcal{T}') + \dim \frac{\Omega^{k_i-1}_{X,0}}{\sum_i \left( (\wedge_j \omega_j^{(i)}) \wedge \Omega^{k_i-1}_{X,0} \right)}.
\]

Now the statement follows from Lemma \[\text{[31]}\]. \qed

5.3. Homology vanishing.

**Theorem 35.** Suppose that, as above, \((X, 0)\) is an isolated complete intersection singularity and \(Z(X; \omega_1, \ldots, \omega_{n-k+1})\) is Cohen-Macaulay of dimension \(k\). Then the complex \(\mathcal{C}(\omega_1, \ldots, \omega_{n-k+1})\) defined by \((10)\) has no higher cohomologies.

**Proof.** Let \(W \cong C^{n-k+1}\). For \(j \leq n\), let us replace \(\Omega^j_{X,0}\) appearing in the definition of \(\mathcal{C}(\omega_1, \ldots, \omega_{n-k+1})\) by its free \(R\)-resolution \([26]\). We obtain the following bicomplex:

\[
\begin{array}{cccccc}
\Omega^1_{X,0} & \overset{\omega_1 \cdots \omega_{n-k+1}}{\longrightarrow} R \otimes \wedge^1 C^N & \overset{d_F}{\longrightarrow} R \otimes \wedge^2 C^N \otimes W & \overset{d_W}{\longrightarrow} R \otimes \wedge^3 C^N & \overset{d_F}{\longrightarrow} R \otimes \wedge^4 C^N \otimes W & \overset{d_W}{\longrightarrow} \ldots \\
\downarrow d_F & \downarrow d_F & \downarrow d_F & \downarrow d_F & \downarrow d_F & \notag \\
R \otimes \wedge^2 C^N \otimes F & \overset{d_W}{\longrightarrow} R \otimes \wedge^3 C^N \otimes F \otimes W & \overset{d_W}{\longrightarrow} R \otimes \wedge^4 C^N \otimes S^2 F \otimes W & \overset{d_W}{\longrightarrow} \ldots \\
\downarrow d_F & \downarrow d_F & \downarrow d_F & \notag \\
\ldots & \notag 
\end{array}
\]

The horizontal differentials \(d_W\) are induced by the map
\[
(\omega_1, \ldots, \omega_{n-k+1}) : R \otimes W \rightarrow R \otimes C^N
\]
(which sends the \(i\)th basis element of \(W\) to \(\omega_i\)), and the vertical differentials \(d_F\) are induced by the map
\[
(d_1, \ldots, d_{N-n}) : R \otimes F \rightarrow R \otimes C^N
\]
(which sends the \(i\)th basis element of \(F\) to \(d_i\)).

If one replaces the differential \(d_1\) in the complex \(\mathcal{C}\) by its composition with \(\phi\):
\[
\wedge d_1 \cdots d_{N-n} \wedge \omega_1 \cdots \omega_{n-k+1} : R \otimes \wedge^{k-1} C^N \rightarrow R \otimes \wedge^n C^n,
\]
one obtains just the EN complex associated to the matrix
\[
M = (d_1, \ldots, d_{N-n}, \omega_1, \ldots, \omega_{n-k+1}) : R \otimes (F \oplus W) \rightarrow R \otimes C^N.
\]
Indeed, \(S^j(F \oplus W) = \bigoplus_{i=0}^j S^i F \otimes S^{j-i} W\), and the differentials agree. Let us denote the total complex of the resulting bicomplex by \(\widetilde{\mathcal{C}}(\omega_1 \cdots \omega_{n-k+1})\).

By assumption, \(\omega_1, \ldots, \omega_{n-k+1}\) are linearly dependent on a codimension \(k\) subvariety \(Z(X; \omega_1, \ldots, \omega_{n-k+1})\) in \(X\). This subvariety is cut out by \((N-k+1) \times (N-k+1)\)-minors
of the \((N-k+1) \times N\) matrix \(M\), so by Theorem 21 the complex \(\tilde{C}(\omega_1 \cdots \omega_{n-k+1})\) does not have higher homologies. Now
\[
\text{Ker}(d_1) = \text{Ker}(\omega_1 \wedge \cdots \wedge \omega_k) \subset \text{Ker}(df_1 \wedge \cdots \wedge df_{N-n} \wedge \omega_1 \wedge \cdots \wedge \omega_k) = \text{Ker}(\phi \circ d_1) \subset \text{Im}(d_2),
\]
so the original bicomplex has no higher cohomologies as well.

On the other hand, consider the associated spectral sequence. By Lemma 25 the homologies in columns agree with \(\Omega^k_{X,0}\) and are concentrated in the top row. Therefore the spectral sequence collapses and \(C(\omega_1 \cdots \omega_{n-k+1})\) has no higher cohomologies.

Consider now a collection of forms \(\{\omega_j^{(i)}\}\) on a complete intersection \((X,0)\). Let \(\tilde{C}^{(i)} = \tilde{C}(\omega_1^{(i)}, \ldots, \omega_{n-k+1}^{(i)})\) where the complex \(\tilde{C}\) was defined in the proof of Theorem 35. Let
\[
\tilde{C}(\{\omega_j^{(i)}\}) = \tilde{C}^{(1)} \otimes_R \tilde{C}^{(2)} \cdots \otimes_R \tilde{C}^{(s)}.
\]

**Theorem 37.** Suppose that the collection of forms \(\{\omega_j^{(i)}\}\) on a complete intersection \((X,0)\) has an isolated singular point at the origin. Then the homologies \(H^t(\tilde{C}(\{\omega_j^{(i)}\}))\) vanish for \(t > 0\).

**Proof.** Suppose that \((X,0)\) is defined in \((\mathbb{C}^N,0)\) by the equations \(f_1 = \ldots = f_{N-n} = 0\). As above, consider the locus \(Z(X;\omega_1^{(i)}, \ldots, \omega_{n-k+1}^{(i)})\) where the forms \(\omega_1^{(i)}, \ldots, \omega_{n-k+1}^{(i)}\) are linearly dependent on \(X\) or, equivalently, the forms \(\omega_1^{(i)}, \ldots, \omega_{n-k+1}^{(i)}, df_1, \ldots, df_{N-n}\) are linearly dependent in \((\mathbb{C}^N,0)\). By construction, \(Z(X;\omega_1^{(i)}, \ldots, \omega_{n-k+1}^{(i)})\) is the intersection of \((X,0)\) with the determinantal variety defined by the vanishing of maximal minors of an \(N \times (N-k+1)\) matrix. Therefore the codimension of \(Z(X;\omega_1^{(i)}, \ldots, \omega_{n-k+1}^{(i)})\) in \((X,0)\) is less than or equal to \(k\). On the other hand, by assumption the intersection of the loci \(Z(X;\omega_1^{(i)}, \ldots, \omega_{n-k+1}^{(i)})\) for all \(i\) is zero-dimensional, so it has codimension \(n = \sum_{i=1}^s k_i\). Therefore for all \(i\) the subvarieties \(Z(X;\omega_1^{(i)}, \ldots, \omega_{n-k+1}^{(i)})\) have pure dimensions \(n - k_i\), which are equal to their expected dimensions. Therefore all \(Z(X;\omega_1^{(i)}, \ldots, \omega_{n-k+1}^{(i)})\) are Cohen-Macaulay [7]. Furthermore, the intersections of \(Z(X;\omega_1^{(i)}, \ldots, \omega_{n-k+1}^{(i)})\) for various \(i\) are all Cohen-Macaulay of correct dimension.

Let us prove by induction in \(i\) that the product \(\tilde{C}^{(1)} \otimes_{O_{X,0}} \tilde{C}^{(2)} \otimes \cdots \otimes_{O_{X,0}} \tilde{C}^{(i)}\) has no higher homologies. For \(i = 1\) this follows from the proof of Theorem 35. Assume that this is true for some \(1 \leq i \leq s\). Then \(\tilde{C}^{(1)} \otimes_{O_{X,0}} \tilde{C}^{(2)} \otimes \cdots \otimes_{O_{X,0}} \tilde{C}^{(i)}\) is a (not free in general) resolution of the structure sheaf of the Cohen-Macaulay scheme
\[
Z = Z(X;\omega_1^{(1)}, \ldots, \omega_{n-k+1}^{(1)}) \cap \cdots \cap Z(X;\omega_1^{(i)}, \ldots, \omega_{n-k+1}^{(i)}).
\]

Now
\[
\tilde{C}^{(1)} \otimes_{O_{X,0}} \tilde{C}^{(2)} \otimes \cdots \otimes_{O_{X,0}} \tilde{C}^{(i)} \otimes_{O_{X,0}} \tilde{C}^{(i+1)} \simeq O_{Z,0} \otimes_{O_{X,0}} \tilde{C}^{(i+1)}.
\]

Similarly to the proof of Theorem 35, we can replace \((38)\) by the complex \((36)\) with \(R\) replaced by \(O_{Z,0}\). Since \(Z\) and \(Z \cap Z(X;\omega_1^{(i+1)}, \ldots, \omega_{n-k+1+1}^{(i+1)})\) are both Cohen-Macaulay and \(Z \cap Z(X;\omega_1^{(i+1)}, \ldots, \omega_{n-k+1+1}^{(i+1)})\) has codimension \(k_{i+1}\) in \(Z\), by Theorem 21 the bicomplex has no higher homologies. Therefore the complex \((38)\) has no higher homologies as well. □

**Corollary 39.** Suppose that \((X,0)\) is an isolated complete intersection singularity, and the collection \(\{\omega_j^{(i)}\}\) of holomorphic 1-forms has an isolated singular point at the origin. Then the homological and the GSV indices for this collection of 1-forms agree:
\[
\text{ind}_{\text{hom}}(\{\omega_j^{(i)}\};X,0) = \text{ind}_{\text{GSV}}(\{\omega_j^{(i)}\};X,0).
\]
Proof. By Theorem 37 the complex $\bar{C}^{(i)}$ does not have higher homologies. Its zeroth homology equals

$$\mathcal{O}_{X,0} \otimes \wedge^N \mathbb{C}^N / \left( df_1 \cdots df_{N-n} \wedge \omega_1^{(i)} \cdots \omega_{n-k_i+1}^{(i)} \wedge \mathcal{O}_{X,0} \otimes \wedge^{k_i-1} \mathbb{C}^N \right) \cong \mathcal{O}_{X,0} / I^{(i)},$$

where the ideal $I^{(i)}$ is generated by the $(N - k_i + 1) \times (N - k_i + 1)$-minors of the $N \times (N - k_i + 1)$ matrix of coefficients of the forms $df_1, \ldots, df_{N-n}, \omega_1^{(i)}, \ldots, \omega_{n-k_i+1}^{(i)}$. Therefore the zeroth homology of the complex $\tilde{C} \left( \{\omega_j^{(i)}\} \right)$ is equal to $\mathcal{O}_{X,0} / \sum_i I^{(i)}$. On the other hand, by Equation (35) the GSV-index of the collection $\{\omega_j^{(i)}\}$ equals the dimension of the same space, so

$$\text{ind}_{\text{GSV}} (\{\omega_j^{(i)}\}; X, 0) = \chi \left( \tilde{C} \left( \{\omega_j^{(i)}\} \right) \right).$$

Therefore, it is sufficient to prove that the Euler characteristics of the complexes $\tilde{C}$ and $C$ agree.

Indeed, let $C'$ denote the complex (36) where $R \otimes \wedge^N \mathbb{C}^N$ is replaced by $\text{Im}(\phi)$. Then by Lemma 31 we have:

$$C^{(i)} / T' \cong \tilde{C}^{(i)} / C^{(i)} \cong T,$$

so

$$\chi(C) = \sum_{A \in \{1, \ldots, s\}} \chi \left( T^{\otimes |A|} \otimes \bigotimes_{i \notin A} \tilde{C}^{(i)} \right) = \sum_{A \in \{1, \ldots, s\}} \chi \left( T^{\otimes |A|} \otimes \bigotimes_{i \notin A} \tilde{C}^{(i)} \right) = \chi(\tilde{C}).$$

Note that since $T$ and $T'$ are supported at the origin, the cohomology of all complexes in the sum are also supported at the origin, and hence are finite-dimensional. Therefore all the Euler characteristics are finite.

6. Invariants of isolated singularities.

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a germ of a complex analytic variety of pure dimension $n$ with an isolated singular point at the origin, let $k_i, i = 1, \ldots, s$, be positive integers such that $\sum_i k_i = n$, and let $\{\omega_j^{(i)}\}, i = 1, \ldots, s, j = 1, \ldots, n - k_i + 1$, be a collection of holomorphic 1-forms on $(X, 0)$ without singular points on a punctured neighbourhood of the origin in $X$. Since both the homological index and the Chern obstruction satisfy the law of conservation of number and coincide with each other (and with the usual index) on a smooth manifold, one has the following statement.

Proposition 40. The difference $\text{ind}_{\text{hom}} (\{\omega_j^{(i)}\}; X, 0) - \text{Ch} (\{\omega_j^{(i)}\}; X, 0)$ between the homological index and the Chern obstruction of a collection of 1-forms on an isolated $n$-dimensional singularity does not depend on the collection of 1-forms $\{\omega_j^{(i)}\}$.

Thus this difference is an invariant of the singularity $(X, 0)$. For a generic collection $\{\omega_j^{(i)}\}$ (say, for the collection of differentials of a generic collection of linear functions on $\mathbb{C}^N$), the Chern obstruction $\text{Ch} (\{\omega_j^{(i)}\}; X, 0)$ is equal to zero: see Proposition 1.1. Therefore this difference is equal to the homological index of a generic collection of 1-forms on $(X, 0)$. If $(X, 0)$ is an ICIS and $s = 1$ (and therefore $k_1 = k$), this index is equal to $\mu(X) + \mu'(X)$, where $\mu(X)$ is the Milnor number of the ICIS $(X, 0)$ (the rank of the middle homology group of its smoothing) and $\mu'(X)$ is the Milnor number of its generic hyperplane section. If $(X, 0)$ is an isolated hypersurface singularity, up to a constant factor
(half of the volume of the $2n$-dimensional sphere) it is equal to the limit of the integral over the Milnor fibre of the Gauss curvature of the fibre: \[11\]. Thus one can say that (up to a constant factor) it is equal to “the vanishing curvature of the singularity”. One can conjecture that the introduced invariants in a similar way are related with vanishing integrals of the forms defined in terms of the curvature tensor and giving the corresponding Chern characteristic numbers.

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