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Resolvent Estimates and Semigroup Expansions for Non-self-adjoint Schrödinger Operators

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Resolvent Estimates and Semigroup Expansions
for Non-self-adjoint Schrödinger Operators

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Ben Bellis

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ABSTRACT OF THE DISSERTATION

Resolvent Estimates and Semigroup Expansions for Non-self-adjoint Schrödinger Operators

by

Ben Bellis

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2018

Professor Mikhail Khitrik, Chair

In this work we examine aspects of spectral theory for semiclassical non-self-adjoint Schrödinger operators. First, we show a subelliptic resolvent estimate for spectral parameters in an unbounded cubic neighborhood of the imaginary axis for a broad class of semiclassical Schrödinger operators with complex potentials of at most quadratic growth. We then generalize this result by showing that the same type of resolvent estimate also holds for non-self-adjoint magnetic Schrödinger operators under suitable growth conditions on the magnetic potential. Lastly, we show how this resolvent estimate can be applied to yield a large time expansion for the semigroup generated by such an operator in terms of spectral data near the origin.
The dissertation of Ben Bellis is approved.

Terence Chi-Shen Tao

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Mikhail Khitrik, Committee Chair

University of California, Los Angeles

2018
To my family.
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VITA

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PUBLICATIONS


Resolvent estimates for non-self-adjoint magnetic Schrödinger operators.
CHAPTER 1

Introduction

1.1 Overview

In this dissertation we study non-self-adjoint Schrödinger operators on Euclidean space in the semiclassical limit. The new results established here concern resolvent estimates for broad classes of such operators, including those in the presence of magnetic fields, in the interior of the pseudospectrum. We also apply these estimates to obtain an expansion for the evolution semigroup generated by such operators, in the limit of large times.

Semiclassical non-self-adjoint Schrödinger operators are those of the form

\[ P = -\hbar^2 \Delta + V(x), \]

where \( 0 < \hbar \leq 1 \) is the semiclassical parameter and \( V \) is a complex-valued potential function. In this work we will be taking \( V \in C^\infty(\mathbb{R}^n, \mathbb{C}) \) with \( \text{Re} V \geq 0 \). Such operators arise in a variety of settings. These include the Orr-Sommerfeld equation in fluid dynamics [19], the study of scattering resonances [21], and the Ginzburg-Landau theory in superconductivity [1]. Throughout the thesis, we shall work in the semiclassical limit; i.e. we will be examining the behavior of these operators as the parameter \( \hbar \to 0^+ \). Depending on the context, this might represent the low viscosity limit, the high frequency limit, or considering data that is large relative to Planck’s constant. We also work with magnetic Schrödinger operators, which are of particular importance in Ginzburg-Landau applications. These generalize regular Schrödinger operators, taking the form

\[ P = (hD_x - A(x))^2 + V(x), \quad D = i^{-1} \partial, \]

where \( 0 < h \leq 1 \) is the semiclassical parameter and \( V \) is a complex-valued potential function. In this work we will be taking \( V \in C^\infty(\mathbb{R}^n, \mathbb{C}) \) with \( \text{Re} V \geq 0 \). Such operators arise in a variety of settings. These include the Orr-Sommerfeld equation in fluid dynamics [19], the study of scattering resonances [21], and the Ginzburg-Landau theory in superconductivity [1]. Throughout the thesis, we shall work in the semiclassical limit; i.e. we will be examining the behavior of these operators as the parameter \( h \to 0^+ \). Depending on the context, this might represent the low viscosity limit, the high frequency limit, or considering data that is large relative to Planck’s constant. We also work with magnetic Schrödinger operators, which are of particular importance in Ginzburg-Landau applications. These generalize regular Schrödinger operators, taking the form

\[ P = (hD_x - A(x))^2 + V(x), \quad D = i^{-1} \partial, \]
where $A \in C^\infty(\mathbb{R}^n, \mathbb{C}^n)$ is a (magnetic) vector potential.

One of the primary technical tools we will use in studying such operators is the calculus of pseudodifferential operators, introduced via the Weyl quantization. Here the Weyl quantization of a symbol $a(x, \xi)$, where $(x, \xi) \in \mathbb{R}^{2n}$, is defined by

$$a_w(x, D_x) u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y) \cdot \xi} a\left(\frac{x + y}{2}, \xi\right) u(y) \, dy \, d\xi,$$

and the semiclassical Weyl quantization is defined by

$$a_w(x, hD_x) u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y) \cdot \xi} a\left(\frac{x + y}{2}, h\xi\right) u(y) \, dy \, d\xi.$$

Thus for a Schrödinger operator of the form (1.1.1) we have that $P = p_w(x, hD_x)$ where $p(x, \xi) = \xi^2 + V(x)$, or $p(x, \xi) = (\xi - A(x))^2 + V(x)$ for the magnetic case (1.1.2).

We shall now briefly discuss some of the challenges in the spectral analysis of non-self-adjoint operators, which are not present in the self-adjoint case. For a self-adjoint operator $A$, acting on some complex Hilbert space $\mathcal{H}$, the size of the resolvent, $(A - z)^{-1}$, with $z \in \mathbb{C}$, as an operator on $\mathcal{H}$, is given by the inverse of the distance between $z$ and the spectrum of $A$. However, if one no longer demands that $A$ be self-adjoint, the resolvent can grow large even when the spectral parameter is far from the spectrum. The spectrum consequently becomes highly sensitive to even small perturbations of the operator and one is naturally led to the fruitful notion of pseudospectrum, which is defined, roughly speaking, as a region in the complex spectral plane where the norm of the resolvent is large.

In numerous problems involving non-self-adjoint operators, including evolution problems, pseudospectrum plays an important role and should be recognized as being interesting in its own right. For semiclassical non-self-adjoint Schrödinger operators studied here, it is well known [7], that the semiclassical pseudospectrum, defined as the set of $z \in \mathbb{C}$ for which the resolvent grows larger than any power of $h^{-1}$ as $h \to 0^+$, encompasses large subsets of the interior of the range of the symbol $p(x, \xi)$. Thus, if we want to have some control over the size of the resolvent, we should look at suitable $h$-dependent neighborhoods of the boundary of the range of the symbol. As we will be taking $\text{Re} V \geq 0$
in (1.1.1), (1.1.2), the range will be confined to the right half-plane, and we will be focusing on \( z \) near the imaginary axis.

A significant application of the boundary estimates for the resolvent concerns evolution equations. Under mild assumptions on the electromagnetic potentials, guaranteeing the \( m \)-accretivity of our Schrödinger operators (1.1.1), (1.1.2), we can define the evolution semigroup \( e^{-tP/h}, \ t \geq 0 \), and represent it as the Laplace transform of the resolvent, via a Cauchy integral formula,

\[
e^{-tP/h} = \frac{1}{2\pi i} \int_{\gamma} e^{-tz/h} (z - P)^{-1} \, dz.
\]

Here \( \gamma = a + i \mathbb{R} \), first for some \( a < 0 \). We would like to obtain an expansion for the semigroup in terms of the true eigenfunctions of \( P \) and an exponentially decaying remainder, and to this end, we wish to perform a contour deformation to the right in (1.1.3), reaching positive values of \( a \). Clearly, the key to our ability to do that is some precise semiclassical resolvent estimates for \( P \) in a suitable region near the imaginary axis.

### 1.2 Statement of Results

This dissertation is comprised of three main results, each occupying its own chapter. The first one establishes a subelliptic resolvent estimate for a non-self-adjoint Schrödinger operator \( P = -h^2 \Delta + V(x) \), for spectral parameters \( z \) in an unbounded cubic neighborhood of the imaginary axis, bounded away from the origin. The second generalizes that result to apply to a magnetic Schrödinger operators \( P = (hD_x - A(x))^2 + V(x) \), where \( A \in C^\infty(\mathbb{R}^n, \mathbb{R}^n) \) is a magnetic potential. Lastly, the third one establishes a large time expansion for the evolution semigroup generated by \( P \), relying on the resolvent estimates of the previous chapters. While the third result has been fully established, some writing still remains to be done and a detailed paper based on this chapter will be prepared during the summer 2018.

Let us now proceed to describe the assumptions and state the results in the precise form.
When doing so, we use the notation ”$f \lesssim g$” to denote that there exists a constant $C > 0$ independent of $h, z$ and any other relevant parameters, such that $f \leq Cg$. In this work, the following conditions are placed on the potential $V$ in (1.1.1),

\begin{align*}
\text{Re} V & \geq 0, \quad (1.2.1) \\
|\partial^\alpha V| & \lesssim 1, \quad |\alpha| \geq 2, \quad (1.2.2)
\end{align*}

and for some $T \geq 0$,

\begin{equation}
|\text{Im} V| - T \lesssim \text{Re} V + |\text{Im} V'|^2. \quad (1.2.3)
\end{equation}

Here the first condition guarantees that the spectrum and semiclassical pseudospectrum of $P$ stays within the right half-plane. The second serves two purposes. It limits the growth of the potential to be at most quadratic, as in the work [11], where similar resolvent estimates are established for the Kramers-Fokker-Planck operator. It also enables us to use systematically the Calderón-Vaillancourt theorem to obtain some $L^2$ control for various error terms that arise in the proof, as well as the sharp Gårding inequality, which remains valid for non-negative smooth symbols enjoying growth conditions as in (1.2.2). The third condition is more technical. It is used in conjunction with the crucial subellipticity property

\begin{equation}
\text{Re} p + H_{\text{Im} p}^2 \text{Re} p \geq 0, \quad p(x, \xi) = \xi^2 + V(x), \quad (1.2.4)
\end{equation}

permitting us to control contributions from the region of the phase space where the expression in (1.2.4) is not large enough. Here $H_f$ stands for the Hamilton vector field of $f \in C^\infty(\mathbb{R}^n, \mathbb{R})$.

The main result established in Chapter 2 is the following theorem.

**Theorem 1.2.1.** Let $P = -h^2\Delta + V$ with $V \in C^\infty(\mathbb{R}^n, \mathbb{C})$ satisfying (1.2.1), (1.2.2), and (1.2.3). Then there exist constants $C_0, M, h_0 > 0$ such that for all $0 < h \leq h_0$ and $z \in \mathbb{C}$ such that $|z| \geq 2T + Mh$ and $\text{Re} z \leq C_0 h^{2/3} (|z| - T)^{1/3}$, it holds that

$$
\| (P - z)^{-1} \|_{L^2 \to L^2} \lesssim h^{-2/3} (|z| - T)^{-1/3}.
$$
The condition $|z| \geq 2T + Mh$ in Theorem 1 can be understood as keeping $z$ away from the spectrum of $P$. Indeed, the general results of [7] indicate that away from the purely imaginary values of the symbol $p$ corresponding to the critical values of $\text{Im} V$, the eigenvalues of $P$ do not come close to the imaginary axis. The condition (1.2.3) implies that such values are confined to the interval $i[-T, T]$.

There has been significant previous work on establishing resolvent estimates for non-self-adjoint Schrödinger operators. What distinguishes Theorem 1 from prior results is that contrary to the previous such estimates, we have fairly loose conditions on the potential, with no requirement of ellipticity at infinity, and the resolvent estimate obtained is valid in an unbounded spectral region. This result most closely resembles one proven by Herau, Sjöstrand, and Stolk for the Kramers-Focker-Planck operator [11], which established a similar resolvent estimate for that operator by a more involved, although related, approach.

Chapter 3 generalizes the results of Chapter 2 to show that the same estimate holds for magnetic Schrödinger operators, $P = (hD_x - A(x))^2 + V(x)$, with the same conditions on $V$, and the following conditions on the magnetic potential $A \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$:

$$|\partial^\alpha A| \lesssim 1, \quad |\alpha| \geq 1,$$

$$|\partial^\alpha A(x)| \lesssim (1 + |x|^2)^{-1/2}, \quad |\alpha| \geq 2, \quad x \in \mathbb{R}^n.$$  \hfill (1.2.5) \hfill (1.2.6)

Similarly to the condition (1.2.2) on $V$, these conditions imply that the symbol $p(x, \xi) = (\xi - A(x))^2 + V(x)$ grows at most quadratically, and that derivatives of order at least two of $|A|^2$ are bounded. However an obstacle arises in the magnetic case that was not previously present in the $A = 0$ case. Any derivative of the symbol with respect to $x$ is typically unbounded as it will contain a term of the form $-2\xi \cdot \partial_x^\alpha A$, which grows in $\xi$ as long as the corresponding derivative of $A$ is nonzero. This makes it more difficult to establish a priori $L^2$ bounds using the previous approach, when working with such operators. The way this problem is overcome is by working with a modified symbol $q$
which is equal to $p$ in the region where $|x| \gtrsim |\xi|$ and cuts off the magnetic potential when $|\xi|$ is much larger than $(1 + |x|^2)^{1/2}$. The modified symbol behaves sufficiently similarly to $p$ because in the region in which $q \neq p$ both symbols are elliptic of the same order. However the decay of higher derivatives of $A$ in (1.2.6) guarantees that the symbol $q$ has derivatives of order at least two bounded, just as $p$ did when $A = 0$. This allows us to show that the same estimate holds for $P$ in the presence of a magnetic potential. The following is the main result established in Chapter 3.

**Theorem 1.2.2.** Let $P = (hD_x - A(x))^2 + V$ with $V \in C^\infty (\mathbb{R}^n, \mathbb{C}), \ A \in C^\infty (\mathbb{R}^n, \mathbb{R}^n)$ satisfying (1.2.1), (1.2.2), (1.2.3), (1.2.5), and (1.2.6). Then there exists positive constants $C_0, M,$ and $h_0$ such that for all $0 < h \leq h_0$ and $z \in C$ such that $|z| \geq 2T + Mh$ and $Re z \leq C_0 h^{2/3} (|z| - T)^{1/3}$ it holds that

$$\| (P - z)^{-1} \|_{L^2 \to L^2} \lesssim h^{-2/3} (|z| - T)^{-1/3}.$$ 

We finally turn to a brief description of the results of the final Chapter 4 of the thesis. Here we combine the resolvent estimates of Chapters 2 and 3 with some results of [12], giving a precise description of the (discrete) spectrum of $P$ in an $O(h)$-neighborhood of the origin, to establish a large time expansion for the evolution semigroup $e^{-tP/h}$, along the lines indicated in the overview, see (1.1.3).

This allows us to show a theorem of the following form, here stated somewhat informally. As stated above, a detailed paper based on this chapter will be prepared during the summer 2018.

**Theorem 1.2.3.** Assume that $P$ satisfies the hypotheses of Theorem 1.2.2 and some additional assumptions so that Theorem 1.1 of [12] applies. Assume also that some eigenvalues of certain quadratic approximations of $P$ are simple and distinct and let $b > 0$ be such that the vertical line $Re z = b$ avoids these eigenvalues. Then we have,

$$e^{-tP/h} = \sum_{Re \lambda < bh} e^{-\lambda t/h} \Pi_\lambda + O_{L^2 \to L^2} (e^{-tb}), \quad t \geq 0,$$

6
where each $\lambda$ is an eigenvalue of $P$ and $\Pi_\lambda$ is the corresponding spectral projection. We have $\Pi_\lambda = O(1) : L^2 \to L^2$ uniformly as $h \to 0$. 

CHAPTER 2

Subelliptic Resolvent Estimates for Non-Self-Adjunct Schrödinger Operators

2.1 Introduction

Non-self-adjoint Schrödinger operators can appear in a variety of settings. These settings can range physical problems to purely mathematical ones. Such examples include the study of the Ginzburg-Landau equation in superconductivity [1], [9], the Orr-Sommerfeld operator in fluid dynamics [19], [20], the theory of scattering resonances [21], [24], or non-self-adjoint perturbations of self-adjoint operators [14]. In the self-adjoint case, the spectral theorem provides a powerful tool to control the resolvent of Schrödinger operators. However, there is no suitable analog to this for non-self-adjoint operators.

In this paper, we study semiclassical non-self-adjoint differential operators, and are thus concerned with the behavior of the resolvent as the semiclassical parameter $h$ tends towards $0$. The general difficulty is that for non-self-adjoint semiclassical operators the spectrum does not control the resolvent, which may become very large far away from the spectrum as $h \to 0$. By a theorem of Davies [6] and Dencker, Sjöstrand, and Zworski [7], for a non-self-adjoint semiclassical Schrödinger operator of the form $P = -h^2 \Delta + V(x)$, for $V \in C^\infty(\mathbb{R}^n)$, and any $z$ of the form $z = \xi_0^2 + V(x_0)$ where $(x_0, \xi_0) \in \mathbb{R}^{2n}$ and $\text{Im} \xi_0 \cdot V'(x_0) \neq 0$, $z$ is an “almost eigenvalue” of $P$, in the sense that there exists a family of functions $u(h) \in L^2$ for which $\| (P - z) u(h) \|_{L^2} = \mathcal{O}(h^\infty) \| u(h) \|_{L^2}$. Thus, when $\text{Re} V \geq 0$ we should not generally expect to have much control over the resolvent of such an operator in the interior of the right half-plane. So instead we will study resolvent...
estimates of such operators when the spectral parameter $z$ is near the boundary of this region.

In this paper we show that for a broad class of non-self-adjoint semiclassical Schrödinger operators there is an unbounded parabolic region near the imaginary axis where the resolvent is well controlled. Let us now introduce the precise assumptions on our operators.

Let $p \in C^\infty (\mathbb{R}^{2n})$ be such that

$$p(X) = |\xi|^2 + V(x),$$  \hspace{1cm} (2.1.1)

where $V = V_1 + iV_2$ with $V_1, V_2$ real valued and $X = (x, \xi)$, with $x, \xi \in \mathbb{R}^n$.

We place the following conditions on the potential $V$:

$$V_1(x) \geq 0, \quad x \in \mathbb{R}^n$$  \hspace{1cm} (2.1.2)

$$|V_2(x)| \leq 1 + |V_2'(x)|^2, \quad x \in \mathbb{R}^n,$$  \hspace{1cm} (2.1.3)

$$\partial^\alpha V \in L^\infty (\mathbb{R}^n), \quad |\alpha| \geq 2.$$  \hspace{1cm} (2.1.4)

Here, and throughout the paper, we use the notation “$f \lesssim g$” to denote that there exists a constant $c > 0$ such that $f \leq cg$. We define the Weyl quantization of a symbol $a(x,\xi)$ by

$$a^w(x,D_x)u(x) = \int_{\mathbb{R}^{2n}} e^{2\pi i (x-y) \cdot \xi} a\left(\frac{x+y}{2}, \xi\right) u(y) \, dy \, d\xi,$$

and the semiclassical Weyl quantization by

$$a^w(x,hD_x)u(x) = \int_{\mathbb{R}^{2n}} e^{2\pi i (x-y) \cdot \xi} a\left(\frac{x+y}{2}, h\xi\right) u(y) \, dy \, d\xi,$$

where $0 < h \leq 1$. Note that

$$p^w = -\frac{h^2}{4\pi^2} \Delta + V(x).$$

We first prove the following a priori estimate for this operator.

**Theorem 2.1.1.** For such $p$, let $T \geq 0$ be such that

$$|V_2(x)| - T \lesssim |V_2'(x)|^2, \quad x \in \mathbb{R}^n,$$  \hspace{1cm} (2.1.5)
and choose any $K \in \mathbb{R}$, $K > 1$. Then there exist positive constants $h_0$, $A$, and $M$ such that for all $0 < h < h_0$, $z \in \mathbb{C}$ with $|z| \geq KT + Mh$ and $Re z \leq Ah^{2/3}(|z| - T)^{1/3}$, and $u \in \mathcal{S}$,

$$\|(p^w(x, hD_x) - z)u\|_{L^2} \gtrsim h^{2/3}(|z| - T)^{1/3}\|u\|_{L^2}.$$  

We then use this to get a resolvent estimate on $L^2$.

**Theorem 2.1.2.** For $p$ as above, $P$, the $L^2$-graph closure of $p^w(x, hD_x)$ on $\mathcal{S}$ is the maximal realization of $p^w(x, hD_x)$ equipped with the domain $D_{max} = \{u \in L^2 : p^w u \in L^2\}$.

For $T$, $h$ and $z$ as above we have the resolvent estimate

$$\| (P - z)^{-1}\|_{L^2 \to L^2} \lesssim h^{-2/3}(|z| - T)^{-1/3}.$$  

Remark. For such $P$, we have that $P$ is accretive because

$$Re (Pu, u)_{L^2} = \left( \left( -\frac{h^2}{4\pi^2}\Delta + V_1 \right) u, u \right)_{L^2} \geq 0, \quad u \in D_{max}.$$  

Thus Theorem 2.1.2 implies that $P$ is maximally accretive.

![Figure 2.1: The shaded region indicates the values of z for which the Theorems 2.1.1 and 2.1.2 apply.](image)

Similar resolvent estimates have been attained for different classes of semiclassical non-self-adjoint operators. Herau, Sjöstrand, and Stolk proved a similar resolvent estimate for
the Kramers-Fokker-Planck operator under certain conditions [11]. We use a multiplier method inspired by one used in [11], but our proof proceeds quite differently. Theirs uses the FBI transform in a compact region of phase space and Weyl-Hörmander calculus with a suitable metric near infinity, while ours works globally using the Wick quantization and some standard Weyl calculus. Hitrik and Sjöstrand attained a similar estimate for certain one-dimensional non-self-adjoint Schrödinger operators [14], with ellipticity assumptions on the potential. Also, Dencker, Sjöstrand, and Zworski showed that for non-self-adjoint semiclassical operators, under suitable assumptions including ellipticity at infinity, the resolvent can be similarly estimated in a small region near a boundary point of the range of the symbol, away from critical values of the symbol [7]. What distinguishes our result, in addition to the relatively direct proof, is that we have fairly loose conditions on the potential, with no requirement of ellipticity, and we attain a resolvent estimate for $z$ in an unbounded region.

To demonstrate the applicability of this result, here are some examples of cases where it can be used.

**Example.** Let $V(x) = q(x)$ for $q$ any quadratic form with $\text{Re } q(x) \geq 0$. By diagonalization we can see that $|\text{Im } q(x)| \lesssim |\text{Im } q'(x)|^2$, and $q''$ is constant so we can apply the above theorems to $p = |\xi|^2 + q(x)$ with $T = 0$. Thus for some $h_0$, $A$, and $M$,

$$\left\| \left( -\frac{h^2}{4\pi^2} \Delta + q(x) - z \right)^{-1} \right\|_{L^2 \to L^2} \lesssim h^{-2/3} |z|^{-1/3},$$

for all $z \in \mathbb{C}$ with $|z| > Mh$ and $\text{Re } z \leq Ah^{2/3} |z|^{1/3}$ and $0 < h \leq h_0$.

We can apply these theorems to many other classes of potentials. Note that the condition $|V_2(x)| - T \lesssim |V'_2(x)|^2$ implies that $T$ will be at least as large as the maximum absolute value of a critical value of $V_2$.

**Example.** Let $V \in C^\infty(\mathbb{R}^2)$ be given by $V(x_1, x_2) = ix_1^2 + isin(x_2)$. Then $|V(x_1, x_2)| - 1 \lesssim |V'(x_1, x_2)|^2$ so applying the above to $p = |\xi|^2 + V$ with $T = 1$ and any $K > 1$ yields, for
some \( h_0, A, \) and \( M, \)
\[
\left\| \left( -\frac{h^2}{4\pi^2} \Delta + i(x_1^2 + \sin(x_2)) - z \right)^{-1} u \right\|_{L^2} \lesssim h^{-2/3} (|z| - 1)^{-1/3},
\]
for all \( z \in \mathbb{C} \) with \( |z| > K + Mh \) and \( \text{Re} \ z \leq Ah^{2/3} (|z| - 1)^{1/3} \) and \( 0 < h \leq h_0. \)

For a broader example we also have the following:

**Example.** Let \( V_2 \in C^\infty (\mathbb{R}^n; \mathbb{R}) \) be a Morse function with finitely many critical points that satisfies (2.1.4). Furthermore suppose that \( |V_2'(x)| \gtrsim |x| \) for all \( x \in \mathbb{R}^n \) with \( |x| > R \) for some \( R > 0. \) Let \( x_1, ..., x_N \in \mathbb{R}^n \) be the critical points of \( V \), and let \( T = \max_{1 \leq j \leq N} |V_2(x_j)|. \)

Since \( V_2 \) is Morse, in a neighborhood of each \( x_j, \)
\[
V_2(x) = V_2(x_j) + q_j(x - x_j) + \mathcal{O}(|x - x_j|^3)
\]
for some nondegenerate quadratic form \( q_j. \) So \( V_2'(x) = q_j'(x - x_j) + \mathcal{O}(|x - x_j|^2) \) and \( |q_j'(x - x_j)| \sim |x - x_j|. \) Then, locally near \( x_j \) we have
\[
|V_2(x)| - T \lesssim |q_j(x - x_j)| + \mathcal{O}(|x - x_j|^3)
\]
\[
\lesssim |x - x_j|^2 \lesssim |V_2'(x)|^2.
\]
Thus \( |V_2(x)| - T \lesssim |V_2'(x)|^2 \) in a neighborhood of each critical point. For \( x \) away from critical points and \( |x| \leq R, \) \( |V_2'(x)| \) is bounded below away from 0 and \( |V_2(x)| \) is bounded above, so \( |V_2(x)| - T \lesssim |V_2'(x)|^2 \) here as well. Lastly (2.1.4) implies that \( |V_2(x)| \lesssim 1 + |x|^2 \)
so \( |V_2(x)| \lesssim |V_2'(x)|^2 \) for \( |x| > R, \) and we see that the preceding theorems can be applied to \( p = |\xi|^2 + V_1(x) + iV_2(x) \) for any such \( V_2 \) and any \( V_1 \) satisfying (2.1.2) and (2.1.4).

The plan of the paper is as follows. In Section 2 we will construct a bounded weight function \( g \) to be used in proving Theorem 2.1.1. Then in Section 3 we will provide a brief overview of the Wick quantization. In Section 4 we prove Theorem 2.1.1 by using the weight function as a bounded multiplier to prove an estimate for the Wick quantization of \( p \) and use the relationship between the Wick and Weyl quantizations as well as some Weyl symbol calculus to get the desired estimate. In Section 5 we prove Theorem 2.1.2 by showing the estimate from Theorem 2.1.1 can be extended to the maximal domain of \( P \). In Section 6, we show how the preceding proofs can be modified to prove a similar result.
for a larger class of potential functions if we additionally require that $|z|$ be bounded above.

### 2.2 The Weight Function

Let

$$\lambda (X) := (|\xi|^2 + V_1 (x) + |V_2' (x)|^2)^{1/2}.$$  

It is worth noting that for this $p$ we have that

$$\lambda (X)^2 \lesssim \Re p + H_{\text{Im} p} \Re p \lesssim \lambda (X)^2,$$

because this motivates our choice of weight function. Here, for $f \in C^1 (\mathbb{R}^{2n})$, we use the notation $H_f$ to denote the Hamiltonian vector field of $f$, i.e. given $f (x, \xi), g (x, \xi) \in C^1 (\mathbb{R}^{2n})$ we define

$$H_f g = \{f, g\} = \partial_\xi f \cdot \partial_x g - \partial_x f \cdot \partial_\xi g.$$  

**Lemma 2.2.1.** Let $p \in C^\infty (\mathbb{R}^{2n})$ be given by $p (x, \xi) = |\xi|^2 + V (x)$ with $V = V_1 + iV_2$, $V_1, V_2$ real valued, $V'' \in L^\infty$, and $V_1 \geq 0$. Let $\psi \in C^\infty (\mathbb{R}; [0, 1])$ be a cutoff with $\psi (t) = 1$ for $|t| \leq 1$ and $\psi (t) = 0$ for $|t| \geq 2$.

There exist $0 < \epsilon < 1$ and $0 < h_0 \leq 1$ depending on $p$ such that for all $0 < h \leq h_0$ and $X$ with $\lambda (X) \geq h^{1/2}$, the smooth weight function $G$ given by

$$G (X) = \epsilon h^{-1/3} \frac{H_{\text{Im} p} \Re p}{\lambda (X)^{4/3}} \psi \left( \frac{4 \Re p}{(h \lambda (X))^{2/3}} \right)$$

satisfies

$$|G (X)| = O (\epsilon), \quad (2.2.1)$$

$$|G' (X)| = O (\epsilon h^{-1/2}), \quad (2.2.2)$$

and

$$\Re p (X) + h H_{\text{Im} p} G (X) \gtrsim h^{2/3} \lambda (X)^{2/3}. \quad (2.2.3)$$
Proof. The support of $G$ is contained in the region where $|\xi|^2 \leq \frac{1}{2} (h\lambda(X))^{2/3}$, so we see that since $\psi \leq 1$ we have

$$
|G(X)| \leq \epsilon h^{-1/3} |V_1'(x)| |\xi| \left(\frac{\text{Re} p(X) \psi}{(h\lambda(X))^{2/3}}\right)
\lesssim \epsilon h^{-1/3} \frac{\lambda(X)(h\lambda(X))^{1/3}}{\lambda(X)^{4/3}} \lesssim \epsilon,
$$

which verifies that $G$ satisfies (2.2.1). Note that as $V_1'' \in L^{\infty}$ and $V_1 \geq 0$ we have, using a standard inequality (Lemma 4.31 of [23]), that

$$
|V_1'(x)| \lesssim V_1(x)^{1/2}. \quad (2.2.5)
$$

This and (2.1.4) then imply that

$$
\partial^\alpha \lambda^2 = O(\lambda), \quad |\alpha| = 1.
$$

(2.2.6)

Now, to check (2.2.2), one can use (2.1.4), (2.2.5), (2.2.6), and the fact that $|\xi| \lesssim (h\lambda(X))^{1/3}$ on the support of $G$ to get the following estimates on the support of $G$:

$$
\frac{|H V_2| |\xi|^2}{\lambda(X)^{4/3}} = O(h^{1/3}),
$$

(2.2.7)

$$
\left| \partial^\alpha \frac{H V_2 |\xi|^2}{\lambda(X)^{4/3}} \right| = O(\lambda^{-1/3}) = O(h^{-1/6}), \quad |\alpha| = 1,
$$

(2.2.8)

$$
\left| \partial^\alpha \left( \psi \left( \frac{4(|\xi|^2 + V_1(x))}{(h\lambda(X))^{2/3}} \right) \right) \right| = O\left( \frac{|\xi| + |V_1'(x)|}{(h\lambda(X))^{2/3} + \lambda(X)^{-1}} \right),
$$

(2.2.9)

Thus by (2.2.7), (2.2.8), and (2.2.9),

$$
|G'(X)| \lesssim \epsilon h^{-1/3} \left( O(h^{-1/6}) + O(h^{1/3} h^{-1/2}) \right) = O(\epsilon h^{-1/2}),
$$

which verifies (2.2.2).
Now we shall attain (2.2.3) in the case where \(|\xi|^2 + V_1(x) \leq \frac{1}{4} (h\lambda(X))^{2/3} \leq \frac{1}{4} \lambda(X)^2\), and so \(|V_2'(x)| \geq \frac{3}{4} \lambda(X)^2\). In this region \(\psi \left( \frac{4\text{Re} p}{(h\lambda(X))^{2/3}} \right) \equiv 1\), and so \(G(X) = \epsilon h^{-1/3} |V_2'|/\lambda(X)^{4/3}\).

Now we get
\[
H_{V_2} G = \epsilon h^{-1/3} \left( \frac{2|V_2'(x)|^2}{\lambda(X)^{4/3}} - \frac{8 (V_2'(x) \cdot \xi)^2}{3\lambda(X)^{10/3}} \right).
\]  
(2.2.10)

Thus
\[
\text{Re} p(X) + h H_{lim} p G(X) = \text{Re} p(X) + \epsilon h^{2/3} \left( \frac{2|V_2'(x)|^2}{\lambda(X)^{4/3}} - \frac{8 (V_2'(x) \cdot \xi)^2}{3\lambda(X)^{10/3}} \right) 
\geq \text{Re} p(X) + \epsilon h^{2/3} \left( \frac{2|V_2'(x)|^2}{\lambda(X)^{4/3}} - \frac{2|V_2'(x)|^2}{3\lambda(X)^{10/3}} \right) \geq \epsilon h^{2/3} \left( \frac{4|V_2'(x)|^2}{3\lambda(X)^{4/3}} \right) 
\geq \epsilon h^{2/3} |V_2'(x)|^2 \lambda(X)^{-4/3} \geq \epsilon h^{2/3} \lambda(X)^2.
\]

It remains to show the bound in the region where \(|\xi|^2 + V_1(x) \geq \frac{1}{4} (h\lambda(X))^{2/3}\). Using (2.2.7), (2.2.8), and (2.2.9) we get that
\[
|h H_{V_2} G| \leq \epsilon h^{2/3} \lambda(X) O \left( \lambda(X)^{-1/3} \right) 
+ \epsilon h^{2/3} \lambda(X) O \left( h^{1/3} \left( (h\lambda(X))^{-1/3} + \lambda(X)^{-1} \right) \right) 
= O \left( \epsilon (h\lambda(X))^{2/3} \right).
\]

Here, fixing \(\epsilon\) sufficiently small yields
\[
|\xi|^2 + V_1(x) + h H_{V_2} G \gtrsim h^{2/3} \lambda(X)^{2/3} - O \left( \epsilon h^{2/3} \lambda(X)^{2/3} \right) \gtrsim h^{2/3} \lambda(X)^{2/3}.
\]

This completes the proof of the lemma.

**Corollary 2.2.2.** For such \(p\) as above, there exists a bounded real weight function \(g \in C^\infty(\mathbb{R}^{2n})\) and constants \(C_0, h_0 > 0\) such that for all \(0 < h \leq h_0\) and all \(X \in \mathbb{R}^{2n}\)
\(|g(X)| \leq 1, |g'(X)| = O \left( h^{-1/2} \right)\) and
\[
\text{Re} p(X) + h H_{lim} p g(X) + C_0 h \gtrsim h^{2/3} \lambda(X)^{2/3}.
\]

(2.2.11)

**Proof.** Let \(G\) be a weight function for \(p\) as constructed in Lemma 2.2.1, and set \(\epsilon\) small enough that \(|G| \leq 1\). Now we extend \(G\) to all of \(\mathbb{R}^{2n}\) by defining
\[
g(X) = \left( 1 - \psi \left( \frac{\lambda(X)^2}{h} \right) \right) G(X),
\]
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where $\psi \in C_c^\infty (\mathbb{R}; [0, 1])$, $\psi (t) = 1$ for $|t| \leq 1$, $\psi (t) = 0$ for $|t| \geq 2$, as before. By (2.2.2) and (2.2.6),

$$|g'| \lesssim \frac{\lambda (X)}{h} \left| \frac{\lambda (X)^2}{h} \right| |G| + \left( 1 - \frac{\lambda (X)^2}{h} \right) |G'| \lesssim h^{-1/2}.$$  

By Lemma 2.2.1, (2.2.11) holds in the region where $\lambda (X) > \sqrt{2} h^{1/2}$ for $h$ sufficiently small since $g = G$ there. When $\lambda (X) < h^{1/2}$ we have $H_{V_2} g (X) = 0$ and $h^{2/3} \lambda (X)^{2/3} < h$ so the inequality holds in this region as well. When $h^{1/2} \leq \lambda (X) \leq \sqrt{2} h^{1/2}$, using (2.2.12) we get

$$|\xi|^2 + V_1 (x) + hH_{V_2} g (X) + C_0 h \geq C_0 h - \mathcal{O} (h^{1/2} \lambda (X)) \gtrsim h^{2/3} \lambda (X)^{2/3},$$

for $C_0$ sufficiently large. \hfill \square

### 2.3 Wick quantization overview

Before proving Theorem 2.1.1 we first will note some facts about the Wick quantization. For $Y = (y, \eta) \in \mathbb{R}^{2n}$ and $x \in \mathbb{R}^n$ define

$$\phi_Y (x) = 2^{n/4} e^{-\pi |x-y|^2} e^{2\pi i \eta \cdot (x-y)}.$$

Then for $u \in L^2 (\mathbb{R}^n)$ define the wave packet transform of $u$ by

$$W u (Y) = (u, \phi_Y),$$

where $(\cdot, \cdot)$ denotes the $L^2$ scalar product. As proven in [16], $W$ is an isometry from $L^2 (\mathbb{R}^n)$ to $L^2 (\mathbb{R}^{2n})$ and continuous from $\mathcal{S} (\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{2n})$. The function $\phi_Y$ is $L^2$ normalized, so the rank-one orthogonal projection of $u$ onto $\phi_Y$ is given by

$$\Pi_Y u = (u, \phi_Y) \phi_Y.$$

For a symbol $a (x, \xi) \in L^\infty (\mathbb{R}^{2n})$ the Wick quantization of $a$ is given by

$$a^{Wick} = W^* a \mu W,$$  

(2.3.1)
where $a^\mu$ denotes multiplication by $a$ and $W^* : L^2 (\mathbb{R}^{2n}) \rightarrow L^2 (\mathbb{R}^n)$ is the adjoint of $W$, or equivalently

$$a^{\text{Wick}} = \int_{\mathbb{R}^{2n}} a (Y) \Pi_Y dY.$$  

We can see from (2.3.1) that for $a \in L^\infty (\mathbb{R}^{2n})$ then $a^{\text{Wick}}$ is a bounded operator on $L^2 (\mathbb{R}^n)$ with

$$\|a^{\text{Wick}}\|_{L^2 \rightarrow L^2} \leq \|a\|_{L^\infty} \tag{2.3.2}$$

and that

$$(a^{\text{Wick}})^* = (\overline{a})^{\text{Wick}}. \tag{2.3.3}$$

We can extend this definition of the Wick quantization to symbols in the space of tempered distributions, $S' (\mathbb{R}^{2n})$ [4]. For a tempered distribution $a \in S' (\mathbb{R}^{2n})$, $a^{\text{Wick}}$ is a map from $S (\mathbb{R}^n)$ to $S (\mathbb{R}^n)$ defined by

$$a^{\text{Wick}} u (x) = (a, \Pi_Y u), \quad u \in S (\mathbb{R}^n). \tag{2.3.4}$$

To see that $a^{\text{Wick}}$ maps $S (\mathbb{R}^n) \rightarrow S (\mathbb{R}^n)$ continuously we can follow the proof from Proposition 5 of [4]. First note that for $u \in S (\mathbb{R}^n)$ we have

$$(u, \phi_Y) \in S (\mathbb{R}^{2n}).$$

Also

$$\Pi_Y u (x) = (u, \phi_Y) \phi_Y (x) \in S (\mathbb{R}^{3n}),$$

and we can see that $u \rightarrow \Pi_Y u$ is continuous $S (\mathbb{R}^n) \rightarrow S (\mathbb{R}^{3n})$. Finally, $\Pi_Y u \rightarrow (a, \Pi_Y u)$ is continuous $S (\mathbb{R}^{3n}) \rightarrow S (\mathbb{R}^n)$, so we can conclude that the claimed property holds.

Furthermore, it follows shortly from the definition (2.3.4) that for a symbol $a \in S' (\mathbb{R}^{2n})$ and all $u \in S (\mathbb{R}^n)$

$$a \geq 0 \Rightarrow (a^{\text{Wick}} u, u) \geq 0. \tag{2.3.5}$$

Let $S (m)$ denote the symbol space

$$S (m) = \{ f \in C^\infty (\mathbb{R}^{2n}) : |\partial f (X)| \leq C_m (X), \forall \alpha \in \mathbb{N}^{2n} \},$$

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where $m$ is an order function on $\mathbb{R}^{2n}$ (cf. section 4.4 of [23]). Another fact we will need from [16] is that for $a \in S(m)$,

$$a^{\text{Wick}} = a^w + r(a)^w, \quad (2.3.6)$$

where

$$r(a)(X) = \int_0^1 \int_{\mathbb{R}^{2n}} (1 - t) a''(X + tY) Y^2 e^{-2\pi|Y|^2} 2^n dY dt. \quad (2.3.7)$$

For smooth symbols $a$ and $b$ with $a \in L^\infty(\mathbb{R}^{2n})$ and $\partial^\alpha b \in L^\infty(\mathbb{R}^{2n})$ for $|\alpha| = 2$ we have the following composition formula proven in [17],

$$a^{\text{Wick}} b^{\text{Wick}} = \left(ab - \frac{1}{4\pi} a' \cdot b' + \frac{1}{4\pi i} \{a, b\}\right)^{\text{Wick}} + R, \quad (2.3.8)$$

where $\|R\|_{L^2 \to L^2} \lesssim \|a\|_{L^\infty} \sup_{|\alpha| = 2} \|\partial^\alpha b\|_{L^\infty}$. We can see that the right-hand side is well defined as an operator $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ because for $|\alpha_1| = |\alpha_2| = 1$,

$$(\partial^{\alpha_1}a)(\partial^{\alpha_2}b) = \partial^{\alpha_1} (a\partial^{\alpha_2}b) - a (\partial^{\alpha_1 + \alpha_2}b).$$

As $|a(X) \partial^{\alpha_2}b(X)| \lesssim 1 + |X|$ we can see that the symbol on the right-hand side of (2.3.8) is indeed a tempered distribution.

### 2.4 Proving the a priori estimate

Now we will use the Wick quantization and the weight function from Lemma 2.2.1 to prove Theorem 2.1.1.

**Proof of Theorem 2.1.1.** We will now follow a multiplier method based on section 4 of [12]. Let $g$ be a bounded real weight function for $p$ as constructed in Corollary 2.2.2. We first note that for $u \in \mathcal{S}$, by (2.3.3),

$$\text{Re} \left( \left[p \left(\sqrt{h}X\right) - z\right]^{\text{Wick}} u, \left[2 - g \left(\sqrt{h}X\right)\right]^{\text{Wick}} u \right) = \text{Re} \left( \left[2 - g \left(\sqrt{h}X\right)\right]^{\text{Wick}} \left[p \left(\sqrt{h}X\right) - z\right]^{\text{Wick}} u, u \right) = \left(\text{Re} \left( \left[2 - g \left(\sqrt{h}X\right)\right]^{\text{Wick}} \left[p \left(\sqrt{h}X\right) - z\right]^{\text{Wick}} u, u \right)\right). \quad (2.4.1)$$

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From (2.3.3) it follows that
\[
\text{Re} a^\text{Wick} = \frac{1}{2} (a^\text{Wick} + (a^\text{Wick})^*) = \frac{1}{2} (a^\text{Wick} + (a^\text{Wick})^*) = (\text{Re} a)^\text{Wick}.
\]

Using this fact and the composition formula for the Wick quantization (2.3.8),
\[
\text{Re} \left[ \left(2 - g \left(\sqrt{\hbar X}\right)\right) \left[p \left(\sqrt{\hbar X}\right) - z\right] \right]^\text{Wick} = \quad (2.4.2)
\]

\[
\text{Re} \left[ \left(2 - g \left(\sqrt{\hbar X}\right)\right) \left(p \left(\sqrt{\hbar X}\right) - z\right) + \frac{1}{4\pi} \nabla \left(g \left(\sqrt{\hbar X}\right)\right) \cdot \nabla \left(p \left(\sqrt{\hbar X}\right)\right) \right] + S_h
\]

\[
= \left[ \left(2 - g \left(\sqrt{\hbar X}\right)\right) \left(\text{Re} p \left(\sqrt{\hbar X}\right) - \text{Re} z\right) \right]
\]

\[
+ \frac{h}{4\pi} g' \left(\sqrt{\hbar X}\right) \cdot \text{Re} p' \left(\sqrt{\hbar X}\right) + \frac{H Vor}{4\pi} H V^2 g \left(\sqrt{\hbar X}\right) + S_h,
\]

where \(\|S_h\|_{L^2 \rightarrow L^2} = O(h)\). Using (2.2.5) and (2.2.12) we have
\[
\left|h g' \left(\sqrt{\hbar X}\right) \cdot \text{Re} p' \left(\sqrt{\hbar X}\right)\right| \lesssim h^{1/2} \left(\text{Re} p \left(\sqrt{\hbar X}\right)\right)^{1/2}
\]

\[
\lesssim rh + \frac{1}{r} \text{Re} p \left(\sqrt{\hbar X}\right),
\]

for arbitrary \(r > 0\). By taking \(r\) large enough the \(\frac{1}{r} \text{Re} p \left(\sqrt{\hbar X}\right)\) term can be absorbed by \(\left(2 - g \left(\sqrt{\hbar X}\right)\right) \text{Re} p \left(\sqrt{\hbar X}\right)\).

Let
\[
y = |z| - T \geq (K - 1) T + M h.
\]

Because \(K > 1\) we have that \(y \gtrsim T\) and \(y \geq M h\). By using (2.2.11) we get that for some \(C_1, C_2 > 0\) and arbitrary \(A > 0\),
\[
\left(2 - g \left(\sqrt{\hbar X}\right)\right) \left(\text{Re} p \left(\sqrt{\hbar X}\right) - \text{Re} z\right)
\]

\[
+ \frac{h}{4\pi} g' \left(\sqrt{\hbar X}\right) \cdot \text{Re} p' \left(\sqrt{\hbar X}\right) + \frac{H Vor}{4\pi} H V^2 g \left(\sqrt{\hbar X}\right)
\]

\[
\gtrsim \text{Re} p \left(\sqrt{\hbar X}\right) - 3\max (0, \text{Re} z) + \frac{h}{4\pi} H V^2 g \left(\sqrt{\hbar X}\right) + O(h) \quad (2.4.3)
\]
\[ \gtrsim h^{2/3} \lambda \left( \sqrt{\hbar X} \right)^{2/3} - C_1 \text{max} \left( 0, \text{Re} \ z \right) - C_2 h \]
\[ \gtrsim h^{2/3} \left( \lambda \left( \sqrt{\hbar X} \right)^{2/3} - 2AC_1 y^{1/3} \right) + AC_1 h^{2/3} y^{1/3} \]
\[ + C_1 \left( A h^{2/3} y^{1/3} - \text{max} \left( 0, \text{Re} \ z \right) \right) - C_2 h. \]

As we required that \( \text{Re} \ z \leq A h^{2/3} y^{1/3} \) we have that

\[ h^{2/3} \left( \lambda \left( \sqrt{\hbar X} \right)^{2/3} - 2AC_1 y^{1/3} \right) + C_1 \left( A h^{2/3} y^{1/3} - \text{max} \left( 0, \text{Re} \ z \right) \right) \]
\[ \gtrsim -2AC_1 h^{2/3} y^{1/3} \psi \left( \frac{B \lambda \left( \sqrt{\hbar X} \right)^2}{y} \right), \]  

where

\[ B = \frac{1}{(2AC_1)^3}, \]

and \( \psi \) is the same cutoff as before. Fix the value of \( A \) by choosing it small enough such that we can use that

\[ |V_2(x)| - T \lesssim V_1(x) + |V_2'(x)|^2 \]  

to get

\[ |p(X)| - T \leq \frac{B \lambda (X)^2}{4}, \quad X \in \mathbb{R}^{2n}. \]

Substituting (2.4.4) into (2.4.3) gives

\[ \left( 2 - g \left( \sqrt{\hbar X} \right) \right) \left( \text{Re} \ p \left( \sqrt{\hbar X} \right) - \text{Re} \ z \right) + \frac{h}{4\pi} g' \left( \sqrt{\hbar X} \right) \cdot \text{Re} \ p' \left( \sqrt{\hbar X} \right) \]
\[ + \frac{h}{4\pi} H V_2 g \left( \sqrt{\hbar X} \right) \]
\[ \gtrsim -2AC_1 h^{2/3} y^{1/3} \psi \left( \frac{B \lambda \left( \sqrt{\hbar X} \right)^2}{y} \right) - C_2 h + AC_1 h^{2/3} y^{1/3}. \]

Now (2.3.5), (2.4.1), (2.4.2), and (2.4.7) imply that, for \( h \) sufficiently small, \( \text{Re} \ z \leq A h^{2/3} y^{1/3} \), and some \( C_4, C_5 > 0 \)

\[ \text{Re} \left( \left[ p \left( \sqrt{\hbar X} \right) - z \right]^{\text{Wick}} u, [2 - g \left( \sqrt{\hbar X} \right)]^{\text{Wick}} u \right) + C_4 h \| u \|^2_{L^2} + \]
\[ C_5 h^{2/3} y^{1/3} \left( \psi \left( \frac{B \lambda \left( \sqrt{\hbar X} \right)^2}{y} \right)^{\text{Wick}} u, u \right) \gtrsim h^{2/3} y^{1/3} \| u \|^2_{L^2}. \]
By the Cauchy-Schwarz inequality and (2.3.2) we get that
\[
\| [p(\sqrt{\hbar}X) - z]^{\text{Wick}} u \|_{L^2} + \hbar \| u \|_{L^2} + \hbar^{2/3} y^{1/3} \left\| \psi \left( \frac{B\lambda (\sqrt{\hbar}X)^2}{y} \right)^{\text{Wick}} u \right\|_{L^2} \\
\gtrsim \hbar^{2/3} y^{1/3} \| u \|_{L^2}.
\]

Now we pick \( M \) sufficiently large so that the \( \hbar \| u \|_{L^2} \) term can be absorbed by the right-hand side to get
\[
\| [p(\sqrt{\hbar}X) - z]^{\text{Wick}} u \|_{L^2} + \hbar^{2/3} y^{1/3} \left\| \psi \left( \frac{B\lambda (\sqrt{\hbar}X)^2}{y} \right)^{\text{Wick}} u \right\|_{L^2} \\
\gtrsim \hbar^{2/3} y^{1/3} \| u \|_{L^2}.
\]

This resembles the desired inequality, but we still need to switch from the Wick quantization to the Weyl quantization, and we need to deal with the term involving \( \psi \). First we will switch to the Weyl quantization. The Calderón-Vaillancourt Theorem (Theorem 4.23 in [23]) states that for \( a \in S(1) \) there exists a universal constant \( \lambda \) such that
\[
\| a^w (x, D_x) \|_{L^2 \rightarrow L^2} \lesssim \sup_{|\alpha| \leq \lambda n} \| \partial^\alpha a \|_{L^\infty}.
\]

From (2.1.4) we have that
\[
\partial^\alpha \left( p \left( \sqrt{\hbar}X \right) \right) = \mathcal{O} \left( \hbar^{|\alpha|/2} \right), \quad |\alpha| \geq 2,
\]
so we can apply the Calderón-Vaillancourt theorem to the remainder term in (2.3.6) with \( a (X) = p \left( \sqrt{\hbar}X \right) - z \) to get
\[
\left\| p \left( \sqrt{\hbar}X \right)^{\text{Wick}} u - zu \right\|_{L^2} = \left\| p \left( \sqrt{\hbar}X \right)^{w} u - zu \right\|_{L^2} + \mathcal{O} \left( \hbar \right) \| u \|_{L^2}.
\]

To do the same thing to the other term on the left side of (2.4.8) we need to estimate the derivatives of \( \psi \left( \frac{B\lambda (\sqrt{\hbar}X)^2}{y} \right)^{\text{Wick}} u \).

**Lemma 2.4.1.**
\[
\left\| \partial^\alpha \left( \psi \left( \frac{B\lambda (\sqrt{\hbar}X)^2}{y} \right) \right) \right\|_{L^2} \lesssim \frac{\hbar^{1/2}}{y^{1/2}}, \quad |\alpha| \geq 1.
\]
Proof. First, note that because $V'' \in S(1)$ we have

$$\partial^\alpha \lambda (X)^2 = \partial^\alpha (|\xi|^2 + V_1(x) + |V'_2(x)|^2) \lesssim 1 + |V'_2| \lesssim 1 + \lambda, \quad |\alpha| \geq 2. \quad (2.4.12)$$

Also, for $X$ in the support of $\psi \left( B\lambda(\sqrt{h}x)^2 \right) \frac{y}{y}$ we have

$$\lambda(\sqrt{h}X) \lesssim y^{1/2},$$

and so, by (2.2.6)

$$\left| \partial^\alpha \left( \frac{\lambda(\sqrt{h}X)^2}{y} \right) \right| \lesssim \frac{h^{1/2} \lambda(\sqrt{h}X)}{y} \lesssim \frac{h^{1/2}}{y^{1/2}}, \quad |\alpha| = 1,$$

and by (2.4.12)

$$\left| \partial^\alpha \left( \frac{\lambda(\sqrt{h}X)^2}{y} \right) \right| \lesssim \frac{h^{1/2}(1 + \lambda(\sqrt{h}X))}{y} \lesssim \frac{h}{y} + \frac{h}{y^{1/2}} \lesssim \frac{h^{1/2}}{y^{1/2}}, \quad |\alpha| \geq 2.$$

We can express $\partial^\alpha \left( \psi \left( B\lambda(\sqrt{h}x)^2 \right) \frac{y}{y} \right)$ as a linear combination of terms of the form

$$\psi^{(k)} \left( \frac{B\lambda(\sqrt{h}X)^2}{y} \right) \partial^{\gamma_1} \left( \frac{\lambda(\sqrt{h}X)^2}{y} \right) \ldots \partial^{\gamma_k} \left( \frac{\lambda(\sqrt{h}X)^2}{y} \right),$$

where $\alpha = \gamma_1 + \ldots + \gamma_k$, $|\gamma_i| \geq 1$ for all $i$, $1 \leq k \leq |\alpha|$. Each such term is of size $O \left( \left( \frac{h}{y} \right)^{k/2} \right)$, proving the lemma.

Using Lemma 2.4.1 and (2.3.6) we get

$$\left\| \psi \left( \frac{B\lambda(\sqrt{h}X)^2}{y} \right)^{wick} u \right\|_{L^2} = \left\| \psi \left( \frac{B\lambda(\sqrt{h}X)^2}{y} \right)^w u \right\|_{L^2} + O \left( M^{-1/2} \right) \|u\|_{L^2}. \quad (2.4.13)$$
By substituting (2.4.10) and (2.4.13) into (2.4.8) and taking $M$ sufficiently large we get

$$\left\| \left[ p \left( \sqrt{h}X \right) - z \right] u \right\|_{L^2} + h^{2/3} y^{1/3} \left\| \psi \left( B\lambda \left( \sqrt{h}X \right)^2 \right) y \right\|_{L^2} \geq h^{2/3} y^{1/3} \left\| u \right\|_{L^2}. \quad (2.4.14)$$

Now all that remains is to deal with the $\psi$ term, which we will accomplish by showing, with some basic Weyl calculus, that it can be absorbed by the other two terms.

Since $\psi$ is real valued $\psi^w$ is self-adjoint. Therefore

$$\left\| \psi \left( \frac{B\lambda \left( \sqrt{h}X \right)^2}{y} \right)^w y \right\|_{L^2}^2 = \left( \left( \psi \left( \frac{B\lambda \left( \sqrt{h}X \right)^2}{y} \right)^w y \right)^2 u, u \right).$$

For the sake of brevity we will henceforth use the notation

$$\Psi (X) := \psi \left( \frac{B\lambda \left( \sqrt{h}X \right)^2}{y} \right).$$

Lemma 2.4.1 can then be rephrased as:

$$\Psi' (X) \in S \left( \frac{h^{1/2}}{y^{1/2}} \right).$$

Let us now recall some basic Weyl calculus. For symbols $a$ and $b$ in $S (1)$, we have the following composition formula for their Weyl quantizations [16],

$$a^w b^w = (a \# b)^w = \left( ab + \frac{1}{4\pi i} \{a, b\} + R \right)^w,$$

where

$$R = -\frac{1}{16\pi^2} \int_0^1 (1 - t) e^{\frac{it}{4\pi i}} (D_x \cdot D_y - D_x \cdot D_n) \left( \left( D_x \cdot D_y - D_x \cdot D_n \right)^2 a (x, \xi) b (y, \eta) dt \right) \bigg|_{(y, \eta) = (x, \xi)}.$$

Thus, using that $\{ \Psi, \Psi \} = 0$,

$$\Psi (X) \# \Psi (X) = \Psi^2 (X) - \frac{1}{16\pi^2} \int_0^1 (1 - t) e^{\frac{it}{4\pi i}} (D_x \cdot D_y - D_x \cdot D_n) \left( \left( D_x \cdot D_y - D_x \cdot D_n \right)^2 \Psi (x, \xi) \Psi (y, \eta) dt \right) \bigg|_{(y, \eta) = (x, \xi)}.$$
By Lemma 2.4.1

\[ (D_x \cdot D_y - D_t \cdot D_t)^2 \Psi (x, \xi) \Psi (y, \eta) = O_{S(1)} \left( \frac{h}{y} \right), \]

where “\( F_1 = O_{S(1)} (F_2) \)” means \( \partial^\alpha F_1 = O (F_2) \), for all \( \alpha \). By Theorem 4.17 in [23] the operator \( e^{\frac{u}{2} (D_x \cdot D_y - D_t \cdot D_t)} \) maps \( S (m) \) to \( S (m) \) continuously for any order function \( m \), so by the above we get that

\[ (\Psi (X)^w)^2 = \Psi^2 (X)^w + \frac{h}{y} R_1^w, \]

for some \( R_1 = O_{S(1)} (1) \). Thus by applying (2.4.9) we get

\[ \| \Psi (X)^w u \|_{L^2}^2 = (\Psi^2 (X)^w u, u) + O \left( \frac{h}{y} \right) \| u \|_{L^2}^2. \]  
(2.4.16)

To control the first term on the right-hand side we follow a method similar to Lemma 8.2 from [11].

**Lemma 2.4.2.** \( (\Psi^2 (X)^w u, u) \leq \left( \left( 4 \frac{|p (\sqrt{h}X) - z|^2}{y^2} \Psi^2 (X)^w \right) u, u \right) + O \left( \frac{h^{1/2}}{y^{1/2}} \right) \| u \|_{L^2}^2. \)

**Proof.** Recalling (2.4.6), we see that on the support of \( \Psi (X) \) we have that

\[ \left| p \left( \sqrt{h}X \right) \right| - T \leq \frac{B \lambda \left( \sqrt{h}X \right)^2}{4} \leq \frac{y}{2}. \]  
(2.4.17)

Thus

\[ \frac{1}{y} \left| p \left( \sqrt{h}X \right) - z \right| \geq \frac{1}{y} \left( |z| - \left| p \left( \sqrt{h}X \right) \right| \right) \]

\[ = \frac{1}{y} \left( y + T - \left| p \left( \sqrt{h}X \right) \right| \right) \geq \frac{1}{2}, \]

and so

\[ \Psi^2 (X) \leq 4 \frac{|p \left( \sqrt{h}X \right) - z|^2}{y^2} \Psi^2 (X). \]  
(2.4.18)

Let

\[ Q (X) = 4 \frac{|p \left( \sqrt{h}X \right) - z|^2}{y^2} \Psi^2 (X) - \Psi^2 (X) \geq 0. \]  
(2.4.19)
By (2.3.5), (2.3.6), and (2.3.7) we get that
\begin{equation}
(Q^w(x, D_x)u, u)_{L^2} + \left\| \left( \int_0^1 \int_{\mathbb{R}^{2n}} (1-t) Q''(X + tY) Y^2 e^{-2\pi |Y|^2} 2^n dY dt \right)^w u \right\|_{L^2} \|u\|_{L^2} \geq 0. \tag{2.4.20}
\end{equation}

To estimate the second term, (2.4.9) implies that we need to estimate the derivatives of order two and higher of $Q$.

As $|z| > KT + Mh$ and $K > 1$,
\begin{align*}
y &= |z| - T > (K - 1) T \geq T.
\end{align*}

So, for $X$ in the support of $\Psi$, using (2.4.17), $y \gtrsim T$, and $y \gtrsim |z|$, we get the following
\begin{align*}
\left| p\left(\sqrt{h}X\right) - z \right| &\lesssim \frac{1}{y} (y + T + |z|) \lesssim 1.
\end{align*}

For such $X$, using (2.2.5) we also have
\begin{align*}
\left| \partial^\alpha p\left(\sqrt{h}X\right) - z \right| &\lesssim \frac{h^{1/2}}{y} \lambda \left(\sqrt{h}X\right) \lesssim \frac{h^{1/2}}{y^{1/2}}, \quad |\alpha| = 1 \tag{2.4.21}
\end{align*}

and
\begin{align*}
\left| \partial^\alpha p\left(\sqrt{h}X\right) - z \right| &\lesssim \frac{h^{1/2}}{y}, \quad |\alpha| \geq 2. \tag{2.4.22}
\end{align*}

By the above and (2.4.11), for $|\alpha| \geq 1$,
\begin{align*}
|\partial^\alpha Q(X)| &\lesssim \frac{h^{1/2}}{y^{1/2}}.
\end{align*}

Thus by applying the Calderón-Vaillancourt theorem (2.4.9) we can bound the latter term of (2.4.20) as follows.
\begin{align*}
\left\| \left( \int_0^1 \int_{\mathbb{R}^{2n}} (1-t) Q''(X + tY) Y^2 e^{-2\pi |Y|^2} 2^n dY dt \right)^w u \right\|_{L^2} \lesssim \frac{h^{1/2}}{y^{1/2}} \|u\|_{L^2}.
\end{align*}

Therefore (2.4.20) implies a variant of the sharp Gårding inequality (cf. Theorem 4.32 of [23]) for $Q$,
\begin{align*}
(Q^w(x, D_x)u, u)_{L^2} + \mathcal{O}\left(\frac{h^{1/2}}{y^{1/2}}\right) \|u\|_{L^2}^2 \geq 0.
\end{align*}
And so by (2.4.19) we attain the desired inequality,

\[
(\Psi^2 (X)^w u, u) \leq \left( \frac{4 |p(\sqrt{\hbar}X) - z|^2}{y^2} \Psi^2 (X) \right)^w u, u \right) + O \left( \frac{h^{1/2}}{y^{1/2}} \|u\|_{L^2}^2 \right). \tag{2.4.23}
\]

Finally, we have to understand the first term on the right side of (2.4.23). The estimates (2.4.11), (2.4.21), and (2.4.22) imply that

\[
\partial^\alpha \left( \frac{p(\sqrt{\hbar}X) - z}{y} \Psi (X) \right) = O \left( \left( \frac{h}{y} \right)^{1/2} \right), \quad |\alpha| \geq 1.
\]

Thus, using this and (2.4.15) and repeating the same Weyl calculus argument used to attain (2.4.16) we get

\[
4 \left| \frac{p(\sqrt{\hbar}X) - z}{y^2} \right|^2 \Psi^2 (X)
\]

\[
= 4 \left( \frac{p(\sqrt{\hbar}X) - z}{y} \Psi (X) \# \frac{p(\sqrt{\hbar}X) - z}{y} \Psi (X) \right)
\]

\[
- \frac{1}{\pi i} \left\{ \left( \frac{p(\sqrt{\hbar}X) - z}{y} \Psi (X) \right), \left( \frac{p(\sqrt{\hbar}X) - z}{y} \Psi (X) \right) \right\} + \frac{h}{y} R_2
\]

\[
= 4 \left( \frac{p(\sqrt{\hbar}X) - z}{y} \Psi (X) \# \frac{p(\sqrt{\hbar}X) - z}{y} \Psi (X) \right) + \frac{h}{y} R_3,
\]

where \( R_2, R_3 = O_{S(1)}(1) \). We also similarly get from (2.4.11), (2.4.21), (2.4.22) and (2.4.15) that

\[
\Psi (X) \# \frac{p(\sqrt{\hbar}X) - z}{y} = \frac{p(\sqrt{\hbar}X) - z}{y} \Psi (X) + \frac{h}{y} R_4,
\]

for \( R_4 = O_{S(1)}(1) \).
Now, using (2.4.16), Lemma 2.4.2, the fact that $\frac{h}{y} \leq \frac{1}{M}$, and that $\|\Psi^w\|_{L^2 \to L^2} = O(1)$, we can conclude that

$$\|\Psi (X)^w u\|_{L^2}^2 \lesssim \left\| (\Psi (X))^w \left( \frac{p (\sqrt{h}X) - z}{y} \right)^w \right\|_{L^2}^2 + O \left( \frac{h^{1/2}}{y^{1/2}} \right) \|u\|_{L^2}^2,$$

and

$$\lesssim \frac{1}{y^2} \left\| p \left( \sqrt{h}X \right) - z \right\|_{L^2}^2 + O \left( \frac{1}{M^{1/2}} \right) \|u\|_{L^2}^2.$$

Plugging this in to (2.4.14) we get

$$\left\| \left[ p \left( \sqrt{h}X \right) - z \right]^w \right\|_{L^2} + \frac{h^{2/3}}{y^{2/3}} \left\| \left[ p \left( \sqrt{h}X \right) - z \right]^w \right\|_{L^2}$$

$$+ O \left( \frac{1}{M^{1/4}} \right) h^{2/3} y^{1/3} \|u\|_{L^2}^2 \gtrsim h^{2/3} y^{1/3} \|u\|_{L^2}^2.$$

Then taking $M$ sufficiently large yields

$$\left\| \left[ p \left( \sqrt{h}X \right) - z \right]^w \right\|_{L^2} \gtrsim h^{2/3} y^{1/3} \|u\|_{L^2}.$$

Finally, by making the symplectic change of coordinates $x \to \frac{x}{\sqrt{h}}$, $\xi \to \frac{\sqrt{h} \xi}{y}$, we obtain the desired estimate,

$$\left\| (p^w (x, hD_x) - z) u \right\|_{L^2} \gtrsim h^{2/3} y^{1/3} \|u\|_{L^2} \label{eq:desiredestimate} \tag{2.5} \,.$$

\[\square\]

### 2.5 From a priori to a resolvent estimate

Now we will use Theorem 2.1.1 to prove Theorem 2.1.2. To do so it will be convenient to work in the standard, or Kohn-Nirenberg, quantization rather than the Weyl quantization. In the semiclassical case, this quantization is defined by

$$a^{KN} (x, hD_x) u(x) = \int_{\mathbb{R}^{2n}} e^{i\pi (x-y) \cdot \xi} a(x, h\xi) u(y) \, dyd\xi$$

$$= \mathcal{F}_{\xi \to x}^{-1} a(x, h\xi) \mathcal{F}_{y \to \xi} u(y),$$

where $\mathcal{F}$ denotes the Fourier transform. Note that just like in the Weyl quantization we have that

$$p^{KN} (x, hD_x) = -\frac{h^2}{4\pi^2} \Delta + V(x).$$
In this quantization we have the composition formula

\[
a^{KN}(x, hD_x) b^{KN}(x, hD_x) = \left(ab + \frac{ih}{2\pi} D_x a \cdot D_x b + R\right)^{KN}(x, hD_x),
\]

where

\[
R = -\frac{h^2}{4\pi^2} \int_0^1 (1-t) e^{i\frac{h}{2\pi} D_x(D_y) (D_x \cdot D_y)^2} a(x, \xi) b(y, \eta) dt\bigg|_{(y, \eta) = (x, \xi)}.
\]

The standard quantization of a symbol is equivalent to the Weyl quantization of a related symbol [24], specifically if \(a \in S(m)\) for some order function \(m\), we have

\[
a^{KN}(x, hD_x) = \left(e^{\frac{i}{4\pi} (D_x \cdot D_y)} a\right)^{w}(x, hD_x)
\]

and

\[
e^{\frac{i}{4\pi} (D_x \cdot D_y)} a \in S(m).
\]

This tells us that some properties of the Weyl quantization can be applied to the standard quantization as well, the Calderón-Vaillancourt theorem (2.4.9) among them.

**Proof of Theorem 2.** To show that \(P\), the graph closure of \(p^w(x, hD_x)\) on \(S(\mathbb{R}^n)\) has domain \(D_{max} = \{u \in L^2 : p^w u \in L^2\}\) we follow a method from Hörmander found in [15]. Let \(\chi_{\delta} : L^2 \to S\) be a family of operators parametrized by \(\delta > 0\) such that \(\chi_{\delta} u \to u \) in \(L^2\) as \(\delta \to 0\) for all \(u \in L^2\). If

\[
(P\chi_{\delta} - \chi_{\delta} P) u \to 0
\]

in \(L^2\) as \(\delta \to 0\) for all \(u \in D_{max}\) then we have that \(u_{\delta} := \chi_{\delta} u\) is a sequence of functions in \(S\) converging to \(u\) and that \(Pu_{\delta} \to Pu\), thus the domain of \(P\) is \(D_{max}\).

To accomplish this, let \(\phi \in C^\infty_c(\mathbb{R}^n, [0, 1])\) be a cutoff function with \(\phi(x) = 1\) for \(x\) in a neighborhood of 0. It suffices to consider the \(h = 1\) case as \(h\) is fixed independent of \(\delta\) and thus does not affect issues of convergence. Then define

\[
\chi_{\delta} u = (\phi(\delta x) \phi(\delta \xi))^{KN} u, \quad u \in L^2.
\]

We then have that \(\chi_{\delta} : L^2 \to S\) and \(\chi_{\delta} u \to u\) in \(L^2\) as \(\delta \to 0\) for all \(u \in L^2\) as desired. We then need to check (2.5.2). This can be accomplished using some standard quantization...
symbol calculus for the commutator \([P, \chi_\delta]\). By (2.5.1) we have

\[
[P, \chi_\delta] = \left( \frac{i}{2\pi} \{ p(x, \xi), \phi(\delta x) \phi(\delta \xi) \} + \mathcal{O}_{S(1)}(\delta^2) \right)^{KN}
\]

\[
= \frac{\delta}{\pi i} (\xi \cdot \phi'(\delta x) \phi(\delta \xi))^{KN} - \frac{\delta}{2\pi i} (V'(x) \cdot \phi'(\delta \xi) \phi(\delta x))^{KN} u + \left( \mathcal{O}_{S(1)}(\delta^2) \right)^{KN} \tag{2.5.3}
\]

\[
= I + II + III.
\]

On the support of \(\phi(\delta x) \phi(\delta \xi)\) we have that \(|x| \lesssim \delta^{-1}\) and \(|\xi| \lesssim \delta^{-1}\) so, as \(\delta \to 0\),

\[
|\delta \partial^\alpha (\xi \cdot \phi'(\delta x) \phi(\delta \xi))| = \mathcal{O}(1), \quad \forall \alpha
\]

and, recalling (2.1.4),

\[
|\delta \partial^\alpha (V'(x) \cdot \phi'(\delta \xi) \phi(\delta x))| = \mathcal{O}(1), \quad \forall \alpha.
\]

Thus by (2.4.9)

\[
\|[P, \chi_\delta]\|_{L^2 \to L^2} = \mathcal{O}(1).
\]

It thus suffices to show that \([P, \chi_\delta]u \to 0\) for all \(u\) in a dense subset of \(L^2\). Term \(III\) is easily dealt with because as \(\delta \to 0\),

\[
\|IIIu\|_{L^2} = \mathcal{O}(\delta^2) \|u\|_{L^2} \to 0.
\]

To deal with terms \(I\) and \(II\), let \(u \in L^2\) be such that \(\mathcal{F}u \in C_c^\infty(\mathbb{R}^n)\). Then

\[
IIu = -\frac{\delta}{2\pi i} \phi(\delta x) V'(x) \cdot \mathcal{F}^{-1} (\phi'(\delta \xi) \mathcal{F}u)(\xi).
\]

Note that \(\phi'(\delta \xi)\) is supported where \(|\xi| \sim \delta^{-1}\) so for \(\delta\) sufficiently small

\[
\phi'(\delta \xi) (\mathcal{F}u)(\xi) = 0
\]

and so

\[
\|IIu\|_{L^2} \to 0.
\]

Also,

\[
Iu = \frac{\delta}{\pi i} \phi'(\delta x) \cdot \mathcal{F}^{-1} (\xi \phi(\delta \xi) \mathcal{F}u)(\xi).
\]

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Because $\mathcal{F} u (\xi)$ is compactly supported and $\phi = 1$ in a neighborhood of 0, for $\delta$ sufficiently small we have
\[
\phi (\delta \xi) (\mathcal{F} u) (\xi) = (\mathcal{F} u) (\xi).
\]
And then
\[
I u = \frac{\delta}{\pi i} \phi' (\delta x) \cdot \mathcal{F}^{-1} (\xi (\mathcal{F} u) (\xi)) = - \frac{\delta}{2\pi^2} \phi' (\delta x) \cdot u' (x).
\]
Since $\mathcal{F} u \in C^\infty_c$ we have $u' \in L^2$ so
\[
\| I u \|_{L^2} \to 0.
\]
Therefore (2.5.2) holds, which tells us that the graph closure of $p^w (x, hD_x)$ on $S$, has the domain $D_{\text{max}}$. Thus, for $z$ and $h$ satisfying the conditions in Theorem 2.1.1 we have
\[
\| (P - z) u \|_{L^2} \gtrsim h^{2/3} (|z| - T)^{1/3} \| u \|_{L^2} \quad \forall u \in D_{\text{max}}.
\]
We thus have that $P - z$ is injective on $D_{\text{max}}$ and has closed range. We can apply the same argument to the formal adjoint of $p^w$ on $S$, $\overline{p^w} - \overline{z} = (|\xi|^2 + \overline{V} (x))^w - \overline{z}$, and we similarly get its graph closure is $\overline{P} - \overline{z} = - \frac{h^2}{8\pi^2} \Delta + \overline{V} (x) - \overline{z}$ with domain $\{ u \in L^2 : \overline{p^w} u \in L^2 \}$, which is also injective with closed range. As $\overline{P} - \overline{z}$ has maximal domain we have that $\overline{P} - \overline{z} = (P - z)^*$. Thus $P - z$ is invertible, and we get the desired resolvent estimate,
\[
\| (P - z)^{-1} u \|_{L^2} \lesssim h^{-2/3} (|z| - T)^{-1/3} \| u \|_{L^2}.
\]

2.6 The bounded $z$ case

In the preceding sections, the condition that was placed on $V$ in (2.1.5), that
\[
|V_2 (x)| - T \lesssim |V'_2 (x)|^2, \quad \forall x \in \mathbb{R}^n,
\]
is only used once. It is used so that we can get the inequality (2.4.6)

\[ |p(X)| - T \leq \frac{B\lambda(X)^2}{4}, \quad \forall X \in \mathbb{R}^{2n}, \]

which is then implies (2.4.18)

\[ \psi^2 \left( \frac{B\lambda(\sqrt{h}X)^2}{y} \right) \leq 4 \frac{|p(\sqrt{h}X) - z|^2}{y^2} \psi^2 \left( \frac{B\lambda(\sqrt{h}X)^2}{y} \right) \lesssim 1. \]

We see that the condition on \( V_2 \) in (2.1.5) is only needed in the region where \( \lambda^2 \lesssim y \).

Thus if we instead only consider values of \( z \) such that \( |z| - T \leq R \) for some \( R > 0 \) we do not need this condition on \( V_2 \) to apply globally. In this case we can reach the same inequality if we instead require there to exist some constant \( L > 0 \) such that

\[ |V_2(x)| - T \lesssim |V_2'(x)|^2, \quad \forall x \in \{ x \in \mathbb{R}^n : |V_2'(x)| \leq L \}. \quad (2.6.1) \]

The \( \psi \) term is supported inside of the region where \( \lambda^2 \lesssim \frac{R}{B} \). As \( |V_2'| \leq \lambda \) we can choose \( B \) large enough that \( \psi \) is supported within the region where \( |V_2'| \leq L \). Then (2.6.1) applies and we can proceed exactly as before to get (2.4.18)

\[ \psi^2 \left( \frac{B\lambda(\sqrt{h}X)^2}{y} \right) \leq 4 \frac{|p(\sqrt{h}X) - z|^2}{y^2} \psi^2 \left( \frac{B\lambda(\sqrt{h}X)^2}{y} \right) \lesssim 1. \]

The rest of the proof can remain unchanged. This results in the following.

**Theorem 2.6.1.** Let \( p \) be in \( C^\infty(\mathbb{R}^{2n}) \) be given by \( p = \xi^2 + V(x) \) with \( V = V_1 + iV_2 \), \( V_1, V_2 \) real valued, \( V_1 \geq 0 \), \( V'' \in S(1) \), and

\[ V_2(x) - T \lesssim |V_2'(x)|^2, \quad \forall x \in \{ x \in \mathbb{R}^n : |V_2'(x)| \leq L \}, \]

for some \( L > 0, T \geq 0 \). Then for any \( R > 0, K > 1 \), there exist positive constants \( A, M, \) and \( h_0 \) such that for all \( 0 < h \leq h_0 \) and \( z \in \mathbb{C} \) with \( Mh \leq |z| - KT \leq R \) and \( \text{Re} z \leq Ah^{2/3}(|z| - T)^{1/3} \) we have

\[ \| [p^w(x, hD_x) - z] u \|_{L^2} \gtrsim h^{2/3}(|z| - T)^{1/3} \| u \|_{L^2}, \quad \forall u \in S(\mathbb{R}^n), \]

and taking \( P \), the \( L^2 \)-graph closure of \( p^w \) on \( S \), we have

\[ \| (P - z)^{-1} u \|_{L^2} \lesssim h^{-2/3}(|z| - T)^{-1/3} \| u \|_{L^2}, \quad \forall u \in L^2. \]
The set of potentials $V$ to which this can apply is very broad. Provided $V_1 \geq 0$ and $V'' \in S(1)$, then (2.6.1) will be satisfied for some $T$ and $L$ as long as there is no sequence of points $x_j$ along which $|V'_2(x_j)| \to 0$ and $V_2(x_j) \to \infty$. 
CHAPTER 3

Resolvent Estimates for Non-Self-Adjoint Magnetic Schrödinger Operators

3.1 Introduction

In this paper we study non-self-adjoint Schrödinger operators with magnetic potentials. Non-self-adjoint Schrödinger operators appear in a variety of contexts such as the study of resonances [21], Hamiltonians of open systems [8], and the damped wave equation [5]. Those with magnetic potential are of particular importance in Ginzburg-Landau theory in the study of superconductivity [1]. One difficulty of working with non-self-adjoint operators as opposed to the self-adjoint case is the lack of the spectral theorem. Whereas the size of the resolvent of a self-adjoint operator is a function of the distance between the spectral parameter and the operator’s spectrum, there is no analog for non-self-adjoint operators, for which the resolvent can grow large even far away from the spectrum [7], [22]. As such, it becomes of interest to study under what circumstances and in which regions of the complex spectral plane we can establish useful estimates of the size of the resolvent for such operators.

Non-self-adjoint magnetic Schrödinger operators have the form

\[ P = (hD_x - A(x))^2 + V(x), \quad x \in \mathbb{R}^n, \quad D_x := -i\partial_x \]

where we will take \( V = V_1 + iV_2 \), with \( V_1, V_2 \in C^\infty(\mathbb{R}^n; \mathbb{R}) \), \( A \in C^\infty(\mathbb{R}^n; \mathbb{R}^n) \), and \( h > 0 \). We refer to \( A \) as the magnetic potential and \( V \) as the electric potential. We shall study such operators in the semiclassical limit, i.e. we shall be concerned with the behavior of
the operator in the limit as $h \to 0$. In the context of quantum mechanics $h$ represents Planck’s constant and taking $h$ to be small models the situation where the data is large relative to the quantum scale. Our primary goal is to show a particular resolvent estimate for a broad class of such operators which generalizes one proven in [2] for the $A = 0$ case.

To study such an operator we will use methods involving pseudodifferential operators. For this we will primarily use the Weyl quantization. For a symbol $a : \mathbb{R}^{2n} \to \mathbb{C}$, the Weyl quantization of $a$ is given by

$$a^w(x, D_x) u := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y) \cdot \xi} a \left( \frac{x + y}{2}, \xi \right) u(y) \, dy \, d\xi,$$

and the semiclassical Weyl quantization is given by

$$a^w_h u = a^w(x, hD_x) u := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y) \cdot \xi} a \left( \frac{x + y}{2}, h\xi \right) u(y) \, dy \, d\xi.$$

Using this we can write $P = p^w_h$ for $p(x, \xi) = |\xi - A(x)|^2 + V(x)$.

Now let us introduce some notation that we will use throughout this paper. For $X \in \mathbb{R}^{2n}, X = (x, \xi)$ with $x, \xi \in \mathbb{R}^n$. The notation “$f \lesssim g$” means there exists a $C > 0$, independent of $X, h$ and other parameters, such that $f \leq Cg$. Additionally, the symbol class $S(m)$ is defined by

$$S(m) := \{ a \in C^\infty (\mathbb{R}^N) : |\partial^\alpha a| \lesssim m, \ \forall \alpha \},$$

for $m$ some positive function on $\mathbb{R}^N$. We shall place the following condition on the symbol $p$:

$$V_1 \geq 0$$

$$V'' \in S(1),$$

$$|A'(x)| \lesssim 1, \quad x \in \mathbb{R}^n,$$

$$A'' \in S(\langle x \rangle^{-1}),$$

$$|V_2(x)| \lesssim 1 + V_1(x) + |V'_2(x)|^2, \quad x \in \mathbb{R}^n.$$

One implication of these conditions that will be useful is that by (3.1.1) and (3.1.2)

$$|V'_1| \lesssim V_1^{1/2},$$

(3.1.6)
as this holds for any nonnegative $C^2$ function with bounded second derivatives [23]. It then follows that if we define

$$m_p (X) := 1 + \text{Re} p (X) + |V_2 (x)'|^2,$$

these conditions collectively imply that

$$p \in S (m_p). \quad (3.1.7)$$

Let $P := p^w (x, hD_x)$. When regarding $P$ as a closed, unbounded operator on $L^2$, we equip $P$ with the maximal domain $D (P) := \{ u \in L^2 (\mathbb{R}^n) : p^w_h u \in L^2 \}$. The following is the main result of this paper.

**Theorem 3.1.1.** Let $T \geq 0$ be such that $|V_2| - T \lesssim V_1 + |V_2|^2$. For such $p$ and any $K > 1$ there exist constants $C_0, M, h_0$ with $0 < h_0, C_0 \leq 1$ and $M \geq 2$ such that for all $0 < h \leq h_0$ and $z \in \mathbb{C}$ with $|z| \geq KT + Mh$ and $\text{Re} \ z \leq C_0 h^{2/3} (|z| - T)^{1/3} (P - z)^{-1}$ exists and we get

$$\| (P - z)^{-1} \|_{L^2 \to L^2} \lesssim h^{-2/3} (|z| - T)^{-1/3} \quad (3.1.8)$$

For convenience, we will write $y := |z| - T$, and so the conclusion of the theorem can be rewritten as

$$\| (P - z)^{-1} \|_{L^2 \to L^2} \lesssim h^{-2/3} y^{-1/3}$$

for $z \in \mathbb{C}$, $|z| \geq KT + Mh$, $\text{Re} \ z \leq C_0 h^{2/3} y^{1/3}$. Also, note that when $T \neq 0$ the constant $M$ can be made irrelevant by taking $h_0$ small enough, and when $T = 0$ the constant $K$ is irrelevant. Theorem 3.1.1 can thus be considered as applying to two distinct cases: when $T \neq 0$ and thus $|z| \geq KT$ and $y \geq (K - 1) T \gtrsim 1$, and a sharper estimate when $T = 0$ and so $|z| \geq Mh$ and $y = |z|$.

To prove this we start by proving the $L^2$ lower bound

$$\| (P - z) u \| \gtrsim h^{2/3} y^{1/3} \| u \|, \quad \forall u \in S. \quad (3.1.9)$$

We do this by showing that it suffices to prove the same estimate for the Weyl quantization of a modified symbol $q$, obtained by cutting off the magnetic potential in the region where
\[ |x| \ll |\xi|, \text{ taking advantage of the ellipticity of } p \text{ in this region.} \]

We then prove such an estimate for \( q_h^w \) by constructing a weight function to use as a bounded multiplier and then using some symbol calculus of pseudodifferential operators, following a method very similar to that used in [2]. Then we show that this lower bound extends to the maximal domain of \( P \) by using a graph closure argument, which then implies that the desired resolvent estimate (3.1.8) holds.

The plan of the paper is as follows. In section 2 we construct the modified symbol \( q \) and show that working with \( q_h^w - z \) suffices. In section 3 we construct a weight function \( g \) and show that it satisfies a list of properties which will be needed to use it as a bounded multiplier later. In section 4 we review some of the important properties of the Wick quantization. In section 5 we prove the desired \( L^2 \) lower bound for \( q_h^w - z \) using the weight function from section 3 and pseudodifferential symbol calculus in both the Wick and Weyl quantizations. This then implies that (3.1.9) holds due to section 2. In section 6 we prove that for a class of symbols which includes \( p \), the graph closure of the corresponding Weyl quantization on \( S \) has maximal domain. We then apply this to conclude that \( P - z \) is invertible and attain the desired resolvent estimate.

The reason why we work with a modified symbol is because when deriving \( L^2 \) estimates using pseudodifferential symbol calculus it will be necessary to bound certain derivatives of the symbol of our operator. Specifically, we will be using the Calderón-Vaillancourt theorem which states that for \( a \in S(1) \),

\[
\| a^w \|_{L^2 \to L^2} \lesssim \sup_{|\alpha| \leq M_n} \| \partial^\alpha a \|_{L^\infty}, \tag{3.1.10}
\]

where \( M > 0 \) is some global constant, see [23]. An obstacle for using this when working with a Schrödinger operator with magnetic potential is that any derivative of the symbol \( p \) with respect to \( x \) will have a term of the form \( \xi \cdot \partial^\alpha A(x) \), which is unbounded in \( \xi \), even when \( A \) is compactly supported. However, in the region where \( \langle x \rangle \gtrsim |\xi| \), the condition (3.1.4) implies that such a term is bounded for \( |\alpha| \geq 2 \). Thus it is preferable to work with a modified symbol, \( q \), which has the magnetic potential cut off in the region where \( |\xi| \gtrsim \langle x \rangle \).
3.2 Truncating the Magnetic Potential

We are able to modify \( p \) in the region where \( |\xi| \) is much larger than \( |x| \) while keeping the same the \( L^2 \) lower bound because the derivative bounds on \( A \) and \( V \) imply that \( |p| \sim |\xi|^2 \sim m_p \), and thus \( p \) is elliptic in this region. Thus, if we change the symbol there while keeping this ellipticity we can expect the new symbol to behave similarly to \( p \).

First we recall a couple standard facts of Weyl symbol calculus. We say that a function \( m > 0 \) on \( \mathbb{R}^N \) is an order function if \( m(X) \lesssim \langle X - Y \rangle^k m(Y) \) for all \( X,Y \in \mathbb{R}^N \) and some \( k \) (see [23]). Let \( m_1, m_2 \) be order functions on \( \mathbb{R}^{2n} \), and let \( f \in S(m_1), g \in S(m_2) \). We then have that \( fg, f \# g \in S(m_1 m_2) \) with \( f \# g \) defined by

\[
(f \# g)_h^w = (fg + \frac{h}{2i} \{f, g\} + h^2 r)_h^w ,
\]

where

\[
f \# g = e^{\frac{ih}{2} (D_\xi \cdot D_y - D_x \cdot D_\eta)} f(x, \xi) g(y, \eta) \bigg|_{(y, \eta) = (x, \xi)}
\]

and

\[
r = -\frac{1}{4} \int_0^1 (1 - t) e^{\frac{ih}{2} (D_\xi \cdot D_y - D_x \cdot D_\eta)} (D_\xi \cdot D_y - D_x \cdot D_\eta)^2 f(x, \xi) g(y, \eta) \ dt \bigg|_{(y, \eta) = (x, \xi)} .
\]

In particular, expanding to first order, given \( f \) and \( g \) with \( f' \in S(m_1) \) and \( g' \in S(m_2) \) it holds that

\[
f_h^w g_h^w = (fg)_h^w + h (r_1)_h^w ,
\]

for some \( r_1 \in S(m_1 m_2) \). Both (3.2.1) and (3.2.2) can also apply to the non-semiclassical Weyl quantization by taking \( \hbar = 1 \).

Let \( \chi \in C^\infty_c(\mathbb{R}^n; \mathbb{R}) \) be a smooth function with \( \chi(x) = 1 \) for all \( |x| \leq 1 \) and \( \chi(x) = 0 \) for all \( |x| \geq 2 \). We use \( \chi \) to cut off the magnetic potential in the region where \( |\xi| \) is large relative to \( \langle x \rangle \). Define \( \chi_1(X) := \chi \left( \frac{\xi}{t(x)} \right) \) and

\[
q(X) := |\xi - \chi_2 R(X) A(x)|^2 + V(x) .
\]
where $R > 0$ is chosen sufficiently large such that $|A(x)|^2, |V(x)| \leq \frac{1}{16} R^2 \langle x \rangle^2$.

Such an $R$ exists because conditions (3.1.2) and (3.1.3) imply that $|A(x)| \lesssim \langle x \rangle$ and $|V(x)| \lesssim \langle x \rangle^2$. Thus for $X \in \text{supp} (1 - \chi_R)$ we have $|\xi| \geq R \langle x \rangle$, and so $|\xi - A(x)| \geq \frac{3|\xi|}{4} \geq \frac{3R \langle x \rangle}{4}$ and

$$|V(x)| \leq |\xi - A(x)|^2 \sim |\xi|^2.$$

We then see that $|\xi|^2$ dominates the other terms of $\text{Re} p$ on the support of $1 - \chi_R$, and so

$$\text{Re} p \sim \text{Re} q \sim m_p (X) \sim |\xi|^2 \sim \langle X \rangle^2, \quad X \in \text{supp} (1 - \chi_R). \quad (3.2.4)$$

We can see from (3.2.3) that $\text{supp} (p - q) \subseteq \text{supp} (1 - \chi_{2R})$. Thus $p$ and $q$ are both elliptic of order $m_p$ in the region in which $p \neq q$.

Furthermore in this region, $\text{Re} p, \text{Re} q \geq \frac{R^2}{4} \langle x \rangle^2$ while $|\text{Im} p|, |\text{Im} q| \leq \frac{R^2}{16} \langle x \rangle^2$, so $\text{Re} \geq 8 |\text{Im} p|$ and the same holds for $q$. The conditions of Theorem 3.1.1 also require that $\text{Re} z \leq C_0 h^{2/3} y^{1/3}$ and $|z| \geq M h$ with $M \geq 2, C_0 \leq 1$. Thus $\text{Re} z \leq 2^{-2/3} |z|$ which implies that $\text{Re} z \leq |\text{Im} z|$. Thus for such $z$ we have that for $X \in \text{supp} (1 - \chi_R)$, $p(X)$ and $q(X)$ lie in a closed cone disjoint from one which contains $z$, as shown in Figure 3.1. It follows that $|p(X) - z| \sim |p(X)| + |z|$, and the same for $q$. From this and (3.2.4), we now have that:

$$|p(X) - z| \sim |q(X) - z| \sim \langle X \rangle^2 + |z|, \quad X \in \text{supp} (1 - \chi_R). \quad (3.2.5)$$

What is convenient about working with this $q$, given in (3.2.3) is that, unlike $p$, derivatives or order two and higher are bounded.

**Lemma 3.2.1.** For $q$ as defined in (3.2.3), it holds that

$$q'' \in S(1). \quad (3.2.6)$$

*Proof.* Let us expand $q$ and consider each term.

$$q = |\xi|^2 - 2 \xi \cdot A \chi_{2R} + \chi_{2R}^2 |A|^2 + V.$$
Figure 3.1: The lighter shaded region indicates the values of $z$ for which Theorem 3.1.1 applies. The ranges of $p(X)$ and $q(X)$ for $X \in \text{supp} \ (1 - \chi_R)$ lie within the darker shaded cone around the positive real axis.

It is trivial that derivatives of order two and higher are bounded for $|\xi|^2$ and $V$, due to (3.1.2). To see that the same holds for the other two terms, we first observe that

$$|\partial^\alpha \chi_t(X)| \lesssim \langle X \rangle^{-|\alpha|}, \quad |\alpha| \geq 0.$$  \hfill (3.2.7)

Then by (3.1.3) and (3.2.7),

$$|\partial^\alpha (A(x) \chi_{2R}(X))| \lesssim 1, \quad |\alpha| \geq 1,$$

and by (3.1.4) and (3.2.7)

$$|\partial^\alpha (A(x) \chi_{2R}(X))| \lesssim \langle X \rangle^{-1}, \quad |\alpha| \geq 2.$$

With these two estimates we can see that all derivatives of order at least two of $\xi \cdot A\chi_{2R}$ and $\chi_{2R}^2|A|^2$ are bounded.

So that we may work with $q$ instead of $p$ we will establish that

$$\| (q_h^w - z) u \| \lesssim \| (p_h^w - z) u \| + \mathcal{O}(h) \| u \|, \quad u \in \mathcal{S}, \quad \text{Re} \ z \leq |\text{Im} \ z|.$$  \hfill (3.1.5)

To see this we will use the following lemma.
Lemma 3.2.2. Define $F(X)$ by

$$F(X) = \frac{q(X) - z}{p(X) - z} (1 - \chi_R(X)) + \chi_R(X),$$

where $z \in \{ z \in \mathbb{C} : \text{Re } z \leq |\text{Im } z| \}$. The symbol $F$ satisfies

$$F \in S(1)$$

and

$$F' \in S(\langle X \rangle^{-1}).$$

Proof. Recall that when we say $a \in S(m)$ for a symbol $a$ and symbol class $S(m)$, the implicit constants in the derivative bounds are independent of $z$ even when $a$ or $m$ depends on $z$.

To understand $F$, note that, by (3.2.5), both $p - z$ and $q - z$ are elliptic on $\text{supp} (p - q) \subset \text{supp} (1 - \chi_R)$, so whenever $p - z = 0$ we also have that $q - z = 0$ and $F = 1$. Thus $F$ is equal to $(q - z) / (p - z)$ everywhere that the latter is defined, and $F(X) = 1$ when $|\xi| \leq 2R\langle x \rangle$. In particular

$$(p - z) F = q - z. \quad (3.2.8)$$

We can then see that the bounds in the lemma are only nontrivial when $p \neq q$, i.e. for $X \in \text{supp} (1 - \chi_{2R})$, and that $\text{supp} (F') \subset \text{supp} (1 - \chi_{2R})$. In that region, (3.2.5) implies that $|F| \lesssim 1$ uniformly in $z$ so it remains to check the size of the derivatives there.

By (3.2.6), $q' \in S(\langle X \rangle)$ and $q \in S(\langle X \rangle^2)$. Similarly, observe that the derivative bounds on $V$ and $A$, (3.1.2), (3.1.3), and (3.1.4), imply the same for $p$:

$$|\partial^\alpha p| \lesssim \langle X \rangle, \quad |\alpha| \geq 1. \quad (3.2.9)$$

So we can say $p' \in S(\langle X \rangle)$, and $p \in S(\langle X \rangle^2)$ by (3.1.7).

Thus $p - z, q - z \in S(\langle X \rangle^2 + |z|)$, and $|p - z| \gtrsim \langle X \rangle^2 + |z|$ on $\text{supp} (1 - \chi_R)$ by (3.2.5). It then follows from (3.2.5) and (3.2.7) that

$$\frac{1 - \chi_R}{p - z} \in S\left( \frac{1}{\langle X \rangle^2 + |z|} \right).$$
Using that $\chi_R = 0$ on $\text{supp} (F')$ and $p = q$ on $\text{supp} (\chi_R')$, we have

$$F' = (1 - \chi_R) \frac{(p - z) q' - (q - z) p'}{(p - z)^2} - \chi_R + \chi_R'$$

$$= \left( \frac{1 - \chi_R}{p - z} \right)^2 \left((p - z) q' - (q - z) p'\right).$$

Multiplying the symbol classes of these factors together yields

$$F' \in S \left( \frac{1}{\langle X \rangle^2 + |z|} \right) \left( \langle X \rangle^2 + |z| \langle X \rangle \right) \subseteq S \left( \langle X \rangle^{-1} \right).$$

As we have already shown that $|F| \lesssim 1$, this implies that $F \in S(1)$. 

This leads to the following.

**Corollary 3.2.3.** Suppose it is true that

$$\| (q^w - z) u \| \gtrsim h^{2/3} y^{1/3} \| u \|, \quad u \in S,$$

(3.2.10)

for $z \in \mathbb{C}$ satisfying the hypotheses of Theorem 3.1.1. Then

$$\| (p^w - z) u \| \gtrsim h^{2/3} y^{1/3} \| u \|.$$

**Proof.** As $F \in S(1)$ by Lemma 3.2.2, it follows from the Calderón-Vaillancourt Theorem (3.1.10) that

$$\| F^w_h (p^w_h - z) u \| \lesssim \| (p^w_h - z) u \|.$$

Then because $F' \in S (\langle X \rangle^{-1})$ and $p' \in S (\langle X \rangle)$, as noted in Lemma 3.2.2, using (3.2.2), (3.2.8) and (3.1.10) yields

$$\| F^w_h (p^w - z) u \| \gtrsim \| (p^w - z) u \| - \mathcal{O} (h) \| u \| = \| (q^w - z) u \| - \mathcal{O} (h) \| u \|.$$

Thus

$$\| (p^w_h - z) u \| \gtrsim \| (q^w - z) u \| - \mathcal{O} (h) \| u \| \gtrsim \left( h^{2/3} y^{1/3} - \mathcal{O} (h) \right) \| u \|.$$

We required that $y \geq Mh$ for some large $M > 0$, so taking $M$ large enough we get the desired estimate. 

Thus when proving Theorem 3.1.1 we can get the desired lower bound for $P$, (3.1.9), by showing that (3.2.10) holds. So our goal now is to show (3.2.10).
3.3 The Weight Function

Note that for the original symbol $p$ we have the subellipticity property

$$\Re p + H_{\Im p}^2 \Re p = |\xi - A|^2 + V_1 + 2|V_2'|^2 \geq 0.$$ 

We use this as a basis for constructing a weight function $g$, to be used to form a bounded multiplier.

Let

$$\lambda_p := |\xi - A|^2 + V_1 + 2|V_2'|^2 = \Re p + 2|V_2'|^2,$$

and

$$\lambda_q := |\xi - \chi_{2R} A|^2 + V_1 + 2|V_2'|^2 = \Re q + 2|V_2'|^2.$$ 

Let $\psi \in C_c^\infty (\mathbb{R}; [0, 1])$ be a cutoff function with $\psi(t) = 1$ for $|t| \leq \frac{1}{2}$ and $\psi(t) = 0$ for $|t| \geq 1$.

**Lemma 3.3.1.** There exists a function $g \in C^\infty (\mathbb{R}^{2n}; \mathbb{R})$ such that:

$$|g| \leq 1,$$  \hfill (3.3.1)

$$|g'| \lesssim h^{-1/2},$$  \hfill (3.3.2)

and

$$\Re(X) + h H_{\Im q} g(X) + C_2 h \gtrsim h^{2/3} \lambda_q(X)^{1/3}, \quad X \in \mathbb{R}^{2n}$$  \hfill (3.3.3)

for some $C_2 > 0$ and all $h > 0$ sufficiently small.

**Proof.** Define $G(X)$ in the region where $\lambda_p(X) \geq h$ by

$$G = \epsilon h^{-1/3} H_{\Im p}^2 \psi^\frac{2/3}{\lambda_p^{1/3}} \left( \frac{\Re p}{h^{2/3} \lambda_p^{1/3}} \right),$$

with $\epsilon > 0$, to be chosen later, independent of $h$. Then we define $g$ by

$$g = \left( 1 - \psi \left( \frac{\lambda_p}{2h} \right) \right) G.$$
We will first show that the following hold where $G$ is defined:

$$|G| \lesssim \epsilon,$$

$$|G'| \lesssim \epsilon h^{-1/2},$$

and

$$\text{Re } p + h H_{\text{Im } p} G \gtrsim h^{2/3} \lambda_p^{1/3}. \quad (3.3.4)$$

Then we will use these to show the desired properties for $g$. Proving this works almost identically to the $A = 0$ case from [2]. The support of $G$ is contained in the region where $\text{Re } p \leq h^{2/3} \lambda_p^{1/3}$, so we see that since $\psi \leq 1$

$$|G(X)| \leq \epsilon h^{-1/3} \frac{2|V_2'(x)||\xi - A|}{\lambda_p(X)^{2/3}} \psi \left( \frac{\text{Re } p}{h^{2/3} \lambda_p^{1/3}} \right)$$

$$\lesssim \epsilon h^{-1/3} \frac{\lambda_p^{1/2} \left( h^{1/3} \lambda_p^{1/6} \right)}{\lambda_p^{2/3}} \lesssim \epsilon.$$

Using (3.1.2), (3.1.3), and (3.1.6) we get that

$$|\text{Re } p'| \lesssim (\text{Re } p)^{1/2}, \quad (3.3.5)$$

and

$$|\lambda_p'| \lesssim |\text{Re } p'| + |V_2'| \lesssim \lambda_p^{1/2}. \quad (3.3.6)$$

Then to estimate $|G'|$ we have the following estimates on the support of $G$, using the above and that $|\xi - A| \leq h^{1/3} \lambda_p^{1/6}$ in this region:

$$\left| \frac{H_{\text{Im } p} \text{Re } p}{\lambda_p^{2/3}} \right| = \mathcal{O} \left( h^{1/3} \right), \quad (3.3.7)$$

$$\left| \partial^\alpha \frac{H_{\text{Im } p} \text{Re } p}{\lambda_p^{2/3}} \right| = \mathcal{O} \left( \lambda_p^{-1/6} \right) = \mathcal{O} \left( h^{-1/6} \right), \quad |\alpha| = 1, \quad (3.3.8)$$

$$\left| \partial^\alpha \left( \psi \left( \frac{\text{Re } p}{h^{2/3} \lambda_p^{1/3}} \right) \right) \right| = \mathcal{O} \left( h^{-1/3} \lambda_p^{-1/6} + \lambda_p^{-1/2} \right) = \mathcal{O} \left( h^{-1/2} \right), \quad |\alpha| = 1. \quad (3.3.9)$$
Thus by (3.3.7), (3.3.8), and (3.3.9),

$$|G'| = \epsilon h^{-1/3} \left( O(h^{-1/6}) + O(h^{1/3}h^{-1/2}) \right) = O(\epsilon h^{-1/2}),$$

which verifies that $|G'| \lesssim \epsilon h^{-1/2}$.

Now we shall attain (3.3.4) in the region where

$$\Re p \leq \frac{1}{4} h^{2/3} \lambda_p^{1/3},$$

and so $2|V'_{2}(x)|^2 \geq \frac{3}{4} \lambda_p(X)$. In this region $\psi \left( \frac{\Re p}{h^{2/3} \lambda_p^{1/3}} \right) \equiv 1$, and so $G = \epsilon h^{-1/3} \frac{H_{\text{imp}} \Re p}{\lambda_p^{2/3}}$.

Now we get

$$H_{\text{imp}} G = \epsilon h^{-1/3} \left( \frac{2|V'_{2}(x)|^2}{\lambda_p^{2/3}} - \frac{8(V'_2(x) \cdot (\xi - A))^2}{3 \lambda_p^{5/3}} \right). \quad (3.3.10)$$

Thus

$$\Re p (X) + h H_{\text{imp}} G (X) = \Re p (X) + \epsilon h^{2/3} \left( \frac{2|V'_{2}(x)|^2}{\lambda_p^{2/3}} - \frac{8(V'_2(x) \cdot (\xi - A))^2}{3 \lambda_p^{5/3}} \right)$$

$$\geq \Re p (X) + \epsilon h^{2/3} \left( \frac{2|V'_{2}(x)|^2}{\lambda_p^{2/3}} - \frac{2|V'_{2}(x)|^2}{3 \lambda_p^{2/3}} \right) \geq \epsilon h^{2/3} \frac{4|V'_{2}(x)|^2}{3 \lambda_p^{2/3}}$$

$$\geq \frac{1}{2} \epsilon h^{2/3} \lambda_p^{1/3}.$$ 

It remains to show the bound in the region where $\Re p \geq \frac{1}{4} h^{2/3} \lambda_p^{1/3}$. Using (3.3.7), (3.3.8), and (3.3.9) we get that

$$|h H_{\text{imp}} G| \leq \epsilon h^{2/3} \lambda_p^{1/2} O\left( \lambda_p^{-1/6} \right)$$

$$+ \epsilon h^{2/3} \lambda_p^{1/2} O\left( h^{1/3} \left( \lambda_p^{-1/3} + \lambda_p^{-1/2} \right) \right)$$

$$= O(\epsilon h^{2/3} \lambda_p^{1/3}).$$

Now for $\epsilon$ sufficiently small we get

$$\Re p + h H_{\text{imp}} G \gtrsim h^{2/3} \lambda_p^{1/3} - O(\epsilon h^{2/3} \lambda_p^{1/3}) \gtrsim h^{2/3} \lambda_p^{1/3}.$$ 

Thus $G$ has all of the claimed properties, and we will now show the corresponding properties for $g$. 

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As $|G| \lesssim \epsilon$, the same holds for $g$, and we now fix the value of $\epsilon$ by choosing it small enough such that $|g| \leq 1$. To check that $|g'| \lesssim h^{-1/2}$ we note that in the region where $\lambda_p \geq 2h$, we have already shown that it holds as $g = G$ there. When $\lambda_p < h$ it holds trivially as $g = 0$ there. In the intermediate region where $h \leq \lambda_p \leq 2h$, we see, by using (3.3.6),

$$|g'| \lesssim |G'| + \left| \frac{\lambda'_p}{2h} G \right| \lesssim h^{-1/2} + \frac{\lambda_p^{1/2}}{h} |G| \lesssim h^{-1/2},$$

thus verifying (3.3.2). To attain (3.3.3) we will show the corresponding bound with $q$ replaced by $p$,

$$\text{Re} p(X) + hH_{\text{im} p} g(X) + C_2 h \gtrsim h^{2/3} \lambda_p (X)^{1/3}, \quad \forall X \in \mathbb{R}^{2n},$$  \hspace{1cm} (3.3.11)

and then show that this implies desired inequality with $q$. To check (3.3.11) we first note that we have already shown that it holds where $\lambda_p \geq 2h$ in (3.3.4). When $\lambda_p < 2h$ we use (3.3.2) to see that

$$|h H_{\text{im} p} g| \lesssim h^{1/2} |V_2'| \lesssim h.$$  \hspace{1cm} (3.3.12)

Thus, choosing $C_2$ sufficiently large,

$$\text{Re} p(X) + hH_{\text{im} p} g(X) + C_2 h \gtrsim h \gtrsim h^{2/3} \lambda_p^{1/3}.$$  

Now it remains to show that this implies the related fact for $q$, (3.3.3). In the region where $p = q$ this implication is trivial. In the region where $p$ and $q$ can differ, i.e. where $|\xi| \geq 2R(x)$, we recall from (3.2.4) that

$$\text{Re} q \sim \lambda_q \sim |\xi|^2 \gtrsim 1.$$  

Then, as

$$|h H_{\text{im} q} g| \lesssim h^{1/2} |V_2'| \lesssim h^{1/2} \lambda_q^{1/2},$$

we can see that $\text{Re} q$ is the dominant term on the left-hand side of (3.3.3) in this region, and so

$$\text{Re} q - \mathcal{O} \left( h^{1/2} (\text{Re} q)^{1/2} \right) + C_2 h \gtrsim h^{2/3} (\text{Re} q)^{1/3} \gtrsim h^{2/3} \lambda_q^{1/3}$$

for all $h$ sufficiently small. \hfill \Box
Lemmas 3.2.1 and 3.3.1, and the fact that \( \text{Re} q \geq 0 \) are the main ingredients needed to adapt the proof from Section 4 of [2] to prove (3.2.10).

### 3.4 Wick Quantization

In addition to the Weyl quantization, we will also work with pseudodifferential operators in the Wick quantization. Here we provide a brief summary of the relevant properties. More detail can be found in [4] and [16]. Let \( Y = (y, \eta) \in \mathbb{R}^{2n} \). Define \( \phi_Y \) by

\[
\phi_Y (x) = \pi^{-n/4} e^{-\frac{1}{2} |x-y|^2} e^{i(x-y) \cdot \eta}, \quad \| \phi_Y \|_{L^2} = 1.
\]

Then define

\[
\Pi_Y u (x) = (u, \phi_Y) \phi_Y (x),
\]

where \((\cdot, \cdot)\) denotes the \( L^2 \) scalar product. Then for \( a \in S' (\mathbb{R}^{2n}) \) and \( u \in S (\mathbb{R}^n) \) we can define \( a^{\text{Wick}} : S (\mathbb{R}^n) \to S (\mathbb{R}^n) \) by

\[
a^{\text{Wick}} u := a_Y (\Pi_Y u).
\]

To see that this indeed maps to \( S (\mathbb{R}^n) \) we observe that for \( u \in S (\mathbb{R}^n) \)

\[
(u, \phi_Y) \in S (\mathbb{R}^{2n}_{(y,\eta)}),
\]

which is verified in Proposition 3.1.6 of [18]. So \( \Pi_Y u \in S (\mathbb{R}^{3n}_{(y,\eta,x)}) \), and then applying \( a \) in the first two variables leaves \( a_Y (\Pi_Y u) \in S (\mathbb{R}^n) \). It follows shortly from the definition that for a symbol \( a \in S' (\mathbb{R}^{2n}) \) and all \( u \in S (\mathbb{R}^n) \)

\[
a \geq 0 \Rightarrow (a^{\text{Wick}} u, u) \geq 0, \quad (3.4.1)
\]

and, just as for the Weyl quantization, formal \( L^2 \) adjoints are attained by taking the complex conjugate of the symbol.

\[
(a^{\text{Wick}})^* = (\overline{a})^{\text{Wick}}. \quad (3.4.2)
\]

Furthermore, if \( a \in L^\infty \), then \( a^{\text{Wick}} : L^2 \to L^2 \) and

\[
\| a^{\text{Wick}} \|_{L^2 \to L^2} \leq \| a \|_{L^\infty}. \quad (3.4.3)
\]
We also can relate the Wick quantization of a symbol \(a \in S(m)\), for some order function \(m\), to its Weyl quantization by

\[
a^{\text{Wick}} = a^w + r(a)^w, \tag{3.4.4}
\]

where

\[
r(a)(X) = (\pi)^{-n/2} \int_0^1 \int_{\mathbb{R}^{2n}} (1 - t) a''(X + tY) Y \cdot Y e^{-|Y|^2} dY dt. \tag{3.4.5}
\]

For smooth symbols \(a\) and \(b\) with \(a \in L^\infty(\mathbb{R}^{2n})\) and \(\partial^\alpha b \in L^\infty(\mathbb{R}^{2n})\) for \(|\alpha| = 2\) we have the following composition formula proven in [17],

\[
a^{\text{Wick}} b^{\text{Wick}} = \left( ab - \frac{1}{2} a' \cdot b' + \frac{1}{2i} \{a, b\} \right)^{\text{Wick}} + R, \tag{3.4.6}
\]

where \(\|R\|_{L^2 \to L^2} \lesssim \|a\|_{L^\infty} \sup_{|\alpha| = 2} \|\partial^\alpha b\|_{L^\infty}\). We will now use this to show that (3.2.10), the desired \(L^2\) lower bound for \(q_w^h - z\) on \(S(\mathbb{R}^n)\), holds.

### 3.5 Proving the Lower Bound for \(q_h^w - z\)

Let \(u \in S(\mathbb{R}^n)\), and let \(z \in \mathbb{C}\) satisfy the hypotheses of Theorem 3.1.1. We will start by using Wick symbol calculus to use \(g^{\text{Wick}}\) as a bounded multiplier for \(q^{\text{Wick}}\), which will be related back to \(q^w\). By (3.4.2), Wick operators with real symbols are formally self adjoint. Thus,

\[
\text{Re} \left( \left[ q \left( \sqrt{\hbar}X \right) - z \right]^{\text{Wick}} u, \left[ 2 - g \left( \sqrt{\hbar}X \right) \right]^{\text{Wick}} u \right) = \\
\text{Re} \left( \left[ 2 - g \left( \sqrt{\hbar}X \right) \right]^{\text{Wick}} \left[ q \left( \sqrt{\hbar}X \right) - z \right]^{\text{Wick}} u, u \right) = \\
\left( \text{Re} \left( \left[ 2 - g \left( \sqrt{\hbar}X \right) \right]^{\text{Wick}} \left[ q \left( \sqrt{\hbar}X \right) - z \right]^{\text{Wick}} \right) u, u \right). \tag{3.5.1}
\]

For any Wick symbol \(a\) it is true that

\[
\text{Re} a^{\text{Wick}} = \frac{1}{2} \left( a^{\text{Wick}} + (a^{\text{Wick}})^* \right) = \frac{1}{2} \left( a^{\text{Wick}} + (\overline{a})^{\text{Wick}} \right) = (\text{Re} a)^{\text{Wick}}.
\]

Using this fact, and the composition formula for the Wick quantization, (3.2.1),

\[
\text{Re} \left( \left[ 2 - g \left( \sqrt{\hbar}X \right) \right]^{\text{Wick}} \left[ q \left( \sqrt{\hbar}X \right) - z \right]^{\text{Wick}} \right) = \tag{3.5.2}
\]

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\[
\text{Re} \left[ \left( 2 - g \left( \sqrt{h}X \right) \right) \left( q \left( \sqrt{h}X \right) - z \right) + \frac{1}{2} \nabla \left( g \left( \sqrt{h}X \right) \right) \cdot \nabla \left( q \left( \sqrt{h}X \right) \right) \right]^{\text{Wick}} + S_h
\]
\[
= \left[ \left( 2 - g \left( \sqrt{h}X \right) \right) \left( \text{Re} \ q \left( \sqrt{h}X \right) - \text{Re} \ z \right) \right]
\]
\[
+ \frac{h}{2} g' \left( \sqrt{h}X \right) \cdot \text{Re} \ q' \left( \sqrt{h}X \right) + \frac{h}{2} H_{V_2} g \left( \sqrt{h}X \right) \]^{\text{Wick}} + S_h,
\]
where \( \|S_h\|_{L^2 \to L^2} = O(h) \), because \( |g| \leq 1 \) and \( q'' \in S(1) \) by Lemma 3.2.1. Then, because \( |\text{Re} \ q'| \lesssim (\text{Re} \ q')^{1/2} \) and, by (3.3.2), \( |g'| \lesssim h^{-1/2} \) we have
\[
\left| h g' \left( \sqrt{h}X \right) \cdot \text{Re} \ q' \left( \sqrt{h}X \right) \right| \lesssim h^{1/2} \left( \text{Re} \ q \left( \sqrt{h}X \right) \right)^{1/2}
\]
\[
\lesssim rh + \frac{1}{r} \text{Re} \ q \left( \sqrt{h}X \right),
\]
for arbitrary \( r > 0 \). By taking \( r \) large enough the \( \frac{1}{r} \text{Re} \ q \left( \sqrt{h}X \right) \) term can be absorbed by \( \left( 2 - g \left( \sqrt{h}X \right) \right) \text{Re} \ q \left( \sqrt{h}X \right) \).

By using (3.3.3) we get that for some \( C_1, C_2 > 0 \) and arbitrary \( C_0 > 0 \),
\[
\left( 2 - g \left( \sqrt{h}X \right) \right) \left( \text{Re} \ q \left( \sqrt{h}X \right) - \text{Re} z \right)
\]
\[
+ \frac{h}{2} g' \left( \sqrt{h}X \right) \cdot \text{Re} \ q' \left( \sqrt{h}X \right) + \frac{h}{2} H_{V_2} g \left( \sqrt{h}X \right)
\]
\[
\gtrsim \text{Re} \ q \left( \sqrt{h}X \right) - 3 \max (0, \text{Re} \ z) + \frac{h}{2} H_{V_2} g \left( \sqrt{h}X \right) + O(h)
\]
\[
\gtrsim h^{2/3} \lambda_q \left( \sqrt{h}X \right)^{1/3} - C_1 \max (0, \text{Re} \ z) - C_2 h
\]
\[
\gtrsim h^{2/3} \left( \lambda_q \left( \sqrt{h}X \right)^{1/3} - 2 C_0 C_1 y^{1/3} \right) + C_0 C_1 h^{2/3} y^{1/3}
\]
\[
+ C_1 \left( C_0 h^{2/3} y^{1/3} - \max (0, \text{Re} \ z) \right) - C_2 h.
\]
As we required that \( \text{Re} \ z \leq C_0 h^{2/3} y^{1/3} \) it follows that
\[
h^{2/3} \left( \lambda_q \left( \sqrt{h}X \right)^{1/3} - 2 C_0 C_1 y^{1/3} \right) + C_1 \left( C_0 h^{2/3} y^{1/3} - \max (0, \text{Re} \ z) \right)
\]
\[
\geq -2 C_0 C_1 h^{2/3} y^{1/3} \psi \left( \frac{B \lambda_q \left( \sqrt{h}X \right)}{y} \right),
\]
48.
where
\[ B = \frac{1}{(4C_0C_1)^3}, \] (3.5.5)
and \( \psi \) is the same cutoff as before. Fix the value of \( 0 < C_0 \leq 1 \) by choosing it small enough such that we can use that \( |V_2(x)| - T \lesssim V_1(x) + |V_2'(x)|^2 \) to get
\[ |q(X)| - T \leq \frac{B\lambda_q(X)}{2}, \quad X \in \mathbb{R}^{2n}, \] (3.5.6)
which we will need later. Substituting (3.5.4) into (3.5.3) gives
\[
\left( 2 - g\left( \sqrt{h}X \right) \right) \left( \text{Re} q\left( \sqrt{h}X \right) - \text{Re} z \right) + \frac{h}{2} g' \left( \sqrt{h}X \right) \cdot \text{Re} q' \left( \sqrt{h}X \right)
\]
\[ + \frac{h}{2} H_{V_2} g \left( \sqrt{h}X \right)
\]
\[ \gtrsim -2C_0C_1 h^{2/3} y^{1/3} \psi \left( \frac{B\lambda_q \left( \sqrt{h}X \right)}{y} \right) - C_2 h + C_0C_1 h^{2/3} y^{1/3}. \] (3.5.7)

Now (3.4.1), (3.5.1), (3.5.2), and (3.5.7) imply that for \( h \) sufficiently small, \( \text{Re} z \leq C_0 h^{2/3} y^{1/3} \), and some \( C_4, C_5 > 0 \)
\[
\text{Re} \left( [q \left( \sqrt{h}X \right) - z]^\text{Wick} u, [2 - g \left( \sqrt{h}X \right)]^\text{Wick} u \right) + C_4 h \|u\|^2 +
\]
\[ C_5 h^{2/3} y^{1/3} \left( \psi \left( \frac{B\lambda_q \left( \sqrt{h}X \right)}{y} \right)^\text{Wick} u, u \right) \gtrsim h^{2/3} y^{1/3} \|u\|^2_{L^2}. \]

By the Cauchy-Schwarz inequality and (3.4.3) we get that
\[
\left\| [q \left( \sqrt{h}X \right) - z]^\text{Wick} u \right\| + h \|u\| + h^{2/3} y^{1/3} \left\| \psi \left( \frac{B\lambda_q \left( \sqrt{h}X \right)}{y} \right)^\text{Wick} u \right\|
\]
\[ \gtrsim h^{2/3} y^{1/3} \|u\|
\]

Now, as \( y \geq Mh \), we pick \( M \) sufficiently large so that the \( h \|u\| \) term can be absorbed by the right-hand side to get
\[
\left\| [q \left( \sqrt{h}X \right) - z]^\text{Wick} u \right\| + h^{2/3} y^{1/3} \left\| \psi \left( \frac{B\lambda_q \left( \sqrt{h}X \right)}{y} \right)^\text{Wick} u \right\|
\] (3.5.8)
\[ \gtrsim h^{2/3} y^{1/3} \| u \|. \]

We now want to get the same bound for the Weyl quantizations of these symbols. Because of (3.2.6), (3.4.4), and the Calderón-Vaillancourt Theorem (3.1.10),
\[ q \left( \sqrt{h} X \right)^{\text{Wick}} = q \left( \sqrt{h} X \right)^{w} + O_{L^2 \to L^2}(h). \] (3.5.9)

In order to do the same for the \( \psi \) term, we need bounds on its derivatives. To do this we will first note the following bounds on the derivatives of \( \lambda_q \), which follow from (3.1.2), (3.2.6), and that \( \Re q \geq 0 \).
\[ |\partial^\alpha \lambda_q| \lesssim |\Re q'| + |V_q'| \lesssim \lambda_q^{1/2}, \quad |\alpha| = 1. \] (3.5.10)
\[ |\partial^\alpha \lambda_q| \lesssim 1 + |V_q'| \lesssim 1 + \lambda_q^{1/2}, \quad |\alpha| \geq 2. \] (3.5.11)

We use these bounds to prove the following lemma.

**Lemma 3.5.1.** The derivatives of the \( \psi \) term obey the following estimate.
\[ \left| \partial^\alpha \left( \psi \left( \frac{B \lambda_q \left( \sqrt{h} X \right)}{y} \right) \right) \right| \lesssim \frac{h^{1/2} \lambda_q \left( \sqrt{h} X \right)^{1/2}}{y^{1/2}}, \quad |\alpha| \geq 1. \] (3.5.12)

**Proof.** For \( X \in \text{supp} \left( \psi \left( \frac{B \lambda_q \left( \sqrt{h} X \right)}{y} \right) \right) \) we have
\[ \lambda_q \left( \sqrt{h} X \right) \lesssim y, \]
and so, by (3.5.10)
\[ \left| \partial^\alpha \left( \frac{\lambda_q \left( \sqrt{h} X \right)}{y} \right) \right| \lesssim \frac{h^{1/2} \lambda_q \left( \sqrt{h} X \right)^{1/2}}{y} \lesssim \frac{h^{1/2}}{y^{1/2}}, \quad |\alpha| = 1, \]
and by (3.5.11)
\[ \left| \partial^\alpha \left( \frac{\lambda_q \left( \sqrt{h} X \right)}{y} \right) \right| \lesssim h^{\alpha^{1/2}} \frac{1 + \lambda_q \left( \sqrt{h} X \right)^{1/2}}{y} \lesssim \frac{h}{y} + \frac{h}{y^{1/2}} \lesssim \frac{h^{1/2}}{y^{1/2}}, \quad |\alpha| \geq 2. \]
We can express \( \partial^\alpha \left( \psi \left( \frac{B\lambda_y(\sqrt{h}X)}{y} \right) \right) \) as a linear combination of terms of the form

\[
\psi^{(k)} \left( \frac{B\lambda_y(\sqrt{h}X)}{y} \right) \partial^{\gamma_1} \left( \frac{\lambda_y(\sqrt{h}X)}{y} \right) \ldots \partial^{\gamma_k} \left( \frac{\lambda_y(\sqrt{h}X)}{y} \right),
\]

where \( \alpha = \gamma_1 + \ldots + \gamma_k, \ |\gamma_i| \geq 1 \) for all \( i, 1 \leq k \leq |\alpha| \). Each such term is of size \( O \left( \left( \frac{h}{y} \right)^{k/2} \right) \), proving the lemma.

Thus

\[
\psi \left( \frac{B\lambda_y(\sqrt{h}X)}{y} \right)_{\text{Wick}} = \psi \left( \frac{B\lambda_y(\sqrt{h}X)}{y} \right) + O_{L^2 \rightarrow L^2} \left( \frac{h^{1/2}}{y^{1/2}} \right). \tag{3.5.13}
\]

It then follows from (3.5.8), (3.5.9), and (3.5.13), taking \( M \) sufficiently large, that

\[
\left\| q \left( \sqrt{h}X - z \right) u \right\| + h^{2/3}y^{1/3} \left\| \psi \left( \frac{B\lambda_y(\sqrt{h}X)}{y} \right) u \right\| \gtrsim h^{2/3}y^{1/3} \| u \|. \tag{3.5.14}
\]

Now if we can show that the \( \psi \) term can be absorbed into the other two we will get the desired inequality, (3.2.10). For the sake of brevity we will henceforth use the notation

\[
\Psi (X) := \psi \left( \frac{B\lambda_y(\sqrt{h}X)}{y} \right).
\]

Lemma 3.5.1 can then be rephrased as:

\[
\Psi' (X) \in S \left( \frac{h^{1/2}}{y^{1/2}} \right).
\]

Thus, by applying (3.2.2) for the \( h = 1 \) quantization we get that

\[
(\Psi (X)^w)^2 = \Psi^2 (X)^w + \frac{h}{y} R_1^w,
\]

for some \( R_1 \in S \left( 1 \right) \). Then by applying (3.1.10) and using that \( \Psi^w \) is self adjoint we get

\[
\| \Psi (X)^w u \|_{L^2}^2 = (\Psi^2 (X)^w u, u) + O \left( \frac{h}{y} \right) \| u \|_{L^2}^2. \tag{3.5.15}
\]

To control the first term on the right-hand side we follow a method similar to Lemma 8.2 from [11] and Lemma 3 from [2].
Lemma 3.5.2. For all $u \in S$, $h > 0$ sufficiently small, and $z \in C$ with $|z| > KT + Mh$

$$(\Psi^2 (X)^w u, u) \leq \left( \left( \frac{4}{y^2} \frac{q (\sqrt{h}X) - z}{y^2} \Psi^2 (X) \right)^w u, u \right) + O \left( \frac{h^{1/2}}{y^{1/2}} \right) \|u\|_{L^2}^2.$$ 

Proof. Recalling (3.5.6), we see that for $X \in \text{supp} (\Psi)$

$$\left| q \left( \sqrt{h}X \right) \right| - T \leq \frac{B\lambda_q \left( \sqrt{h}X \right)}{2} \leq \frac{y}{2}.$$ \hspace{1cm} (3.5.16)

This property is precisely what condition (3.1.5) is needed for. We then have

$$\frac{1}{y} \left| q \left( \sqrt{h}X \right) - z \right| \geq \frac{1}{y} \left( |z| - \left| q \left( \sqrt{h}X \right) \right| \right)$$

$$= \frac{1}{y} \left( y + T - \left| q \left( \sqrt{h}X \right) \right| \right) \geq \frac{1}{2},$$

and so

$$\Psi^2 (X) \leq 4 \frac{\left| q \left( \sqrt{h}X \right) - z \right|^2}{y^2} \Psi^2 (X), \quad X \in \mathbb{R}^{2n}. \hspace{1cm} (3.5.17)$$

Let

$$Q (X) = 4 \frac{\left| q \left( \sqrt{h}X \right) - z \right|^2}{y^2} \Psi^2 (X) - \Psi^2 (X) \geq 0. \hspace{1cm} (3.5.18)$$

By (3.4.1), (3.4.4), and (3.4.5) we get that

$$(Q^w (x, D_x) u, u)_{L^2} +$$

$$\left\| \pi^{-n/2} \left( \int_0^1 \int_{\mathbb{R}^{2n}} (1 - t) Q'' (X + tY) Y \cdot Y e^{-|Y|^2} dY dt \right)^w u \right\| \|u\| \geq 0. \hspace{1cm} (3.5.19)$$

To estimate the second term, (3.1.10) implies that we need to estimate the derivatives of order two and higher of $Q$.

As $|z| > KT + Mh$ and $K > 1$,

$$y = |z| - T > (K - 1) T \geq T.$$ 

So, for $X \in \text{supp} (\Psi)$, using (3.5.16), $y \geq T$, and $y \geq |z|$, we get the following

$$\frac{\left| q \left( \sqrt{h}X \right) - z \right|}{y} \approx \frac{1}{y} (y + T + |z|) \approx 1.$$ \hspace{1cm} (3.5.20)
For such $X$, using that $|\text{Re} q| \lesssim (\text{Re} q)^{1/2}$, we also have

$$
\left| \frac{q}{y} \left( \sqrt{h} X \right) - z \right| \lesssim \frac{h^{1/2}}{y} \lambda_q \left( \sqrt{h} X \right)^{1/2} \lesssim \frac{h^{1/2}}{y^{1/2}}, \quad |\alpha| = 1
$$

(3.5.21)

and

$$
\left| \frac{\partial^\alpha q}{y} \left( \sqrt{h} X \right) - z \right| \lesssim \frac{h^{1/2}}{y^{1/2}|\alpha|}, \quad |\alpha| \geq 2.
$$

(3.5.22)

By the above and (3.5.12), for $|\alpha| \geq 1$,

$$
|\partial^\alpha Q(X)| \lesssim \frac{h^{1/2}}{y^{1/2}}, \quad X \in \mathbb{R}^{2n}.
$$

Thus by applying the Calderón-Vaillancourt theorem (3.1.10) we can bound the latter term of (3.5.19) as follows,

$$
\left\| \left( \int_0^1 \int_{\mathbb{R}^{2n}} (1 - t) Q''(X + tY) Y \cdot Y e^{-|Y|^2/2} dY dt \right) u \right\| \lesssim \frac{h^{1/2}}{y^{1/2}} \|u\|.
$$

Therefore (3.5.19) implies a variant of the sharp Gårding inequality (cf. Theorem 4.32 of [23]) for $Q$,

$$
(Q^w(x, D_x) u, u)L^2 + O \left( \frac{h^{1/2}}{y^{1/2}} \right) \|u\|_L^2 \geq 0.
$$

By (3.5.18) we attain the statement in the lemma.

Combining (3.5.15) and Lemma 3.5.2 we get that

$$
\|\Psi u\|^2 \leq \left( \left( \frac{1}{4} \left| \frac{q}{y^2} \left( \sqrt{h} X \right) - z \right|^2 \Psi^2(X) \right)^w u, u \right) + O \left( \frac{h^{1/2}}{y^{1/2}} \right) \|u\|_L^2.
$$

(3.5.23)

Finally, we have to understand the first term on the right side of (3.5.23). The estimates (3.5.12), (3.5.20), (3.5.21), and (3.5.22) imply that

$$
\partial^\alpha \left( \frac{q}{y} \left( \sqrt{h} X \right) - z \right) \Psi(X) = O \left( \left( \frac{h}{y} \right)^{1/2} \right), \quad |\alpha| \geq 1.
$$
Then by applying (3.2.2) we get
\[
\frac{1}{4y^2} \left| q \left( \sqrt{h}X \right) - z \right|^2 \Psi^2 (X)
\]
\[
= 4 \left( \frac{q \left( \sqrt{h}X \right) - z}{y} \Psi (X) \right) \# \left( \frac{q \left( \sqrt{h}X \right) - z}{y} \Psi (X) \right) + \frac{h}{y} R_2,
\]
where \( R_2 \in S (1) \). We also similarly get from (3.5.12), (3.5.21), (3.5.22) and (3.2.2) that
\[
\Psi (X) \# \left( \frac{q \left( \sqrt{h}X \right) - z}{y} \right) = \left( \frac{q \left( \sqrt{h}X \right) - z}{y} \right) \Psi (X) + \frac{h}{y} R_3,
\]
for \( R_3 \in S (1) \).

Now, using this, (3.5.23), the fact that \( \frac{h}{y} \leq \frac{1}{M} \), and (3.1.10), we can conclude that
\[
\| \Psi (X)^w u \|^2_{L^2} \lesssim \left( \Psi (X) \right)^w \left( \frac{q \left( \sqrt{h}X \right) - z}{y} \right)^w u \|^2_{L^2} + O \left( \frac{h^{1/2}}{y^{1/2}} \right) \| u \|^2_{L^2}
\]
\[
\lesssim \frac{1}{y^2} \left\| q \left( \sqrt{h}X \right) - z \right\|^w u \|^2_{L^2} + O \left( \frac{1}{M^{1/2}} \right) \| u \|^2_{L^2}.
\]

Plugging this into (3.5.14) we get
\[
\left\| q \left( \sqrt{h}X \right) - z \right\| u \| + \frac{h^{2/3}}{y^{2/3}} \left\| q \left( \sqrt{h}X \right) - z \right\| u \|
\]
\[
+ O \left( \frac{1}{M^{1/3}} \right) h^{2/3} y^{1/3} \| u \| \gtrsim h^{2/3} y^{1/3} \| u \|.
\]

Then taking \( M \) sufficiently large yields
\[
\left\| q \left( \sqrt{h}X \right) - z \right\| u \| \gtrsim h^{2/3} y^{1/3} \| u \|.
\]

Finally, by making the symplectic change of coordinates \( x \rightarrow \frac{x}{\sqrt{h}}, \xi \rightarrow \sqrt{h} \xi \) we obtain the desired estimate,
\[
\| (q^w (x, hD_x) - z) u \| \gtrsim h^{2/3} y^{1/3} \| u \|.
\]

The results of this section can summarized by the following.
Proposition 3.5.3. For any $K > 1$ there exists constants $0 < C_0 \leq 1$, $M \geq 2$, and $h_0 > 0$ such that for all $z \in \mathbb{C}$ with $|z| \geq KT + Mh$ and $\text{Re} z \leq C_0 h^{2/3} y^{1/3}$ and all $0 < h \leq h_0$,  
$$
\| (q_h^w - z) u \| \gtrsim h^{2/3} y^{1/3} \| u \|, \quad u \in \mathcal{S}.
$$

Thus, by Corollary 3.2.3 the same holds for $p_h^w$. All that remains to complete the proof of Theorem 3.1.1 is to extend this lower bound to the maximal domain of $p_h^w$ so that we may conclude the corresponding upper bound for its resolvent.

3.6 Attaining the Resolvent Estimate

We will use the following to finish the proof of Theorem 3.1.1.

Proposition 3.6.1. Let $a \in C^\infty (\mathbb{R}^{2n})$ with 
$$
a' \in S (\langle X \rangle) . \tag{3.6.1}
$$
Then the $L^2$-graph closure of $a^w$ on $\mathcal{S}$ has the maximal domain 
$$
D_{\text{max}} := \{ u \in L^2 : a^w u \in L^2 \} .
$$

Proof. To show that the graph closure of $a^w (x, D_x)$ on $\mathcal{S} (\mathbb{R}^n)$ has domain $D_{\text{max}}$ we follow a method from Hörmander found in [15]. Let $\chi_\delta \in S (\mathbb{R}^{2n})$ be a family of symbols parametrized by $\delta > 0$ such that $\chi_\delta^w : L^2 \to \mathcal{S}$ is a bounded family of operators with $\chi_\delta^w u \to u$ in $L^2$ as $\delta \to 0$ for all $u \in L^2$. If 
$$
(a^w \chi_\delta^w - \chi_\delta^w a^w) u \to 0 \tag{3.6.2}
$$
in $L^2$ as $\delta \to 0$ for all $u \in D_{\text{max}}$ then $u_\delta := \chi_\delta^w u$ is a sequence of functions in $\mathcal{S}$ converging to $u$ and $a^w u_\delta \to a^w u$ in $L^2$, thus the domain of the graph closure of $a^w$ is $D_{\text{max}}$.

To accomplish this, let $\phi \in C^\infty_0 (\mathbb{R}^n, [0, 1])$ be a cutoff function with $\phi (x) = 1$ for $x$ in a neighborhood of 0. Then define 
$$
\chi_\delta = (\phi (\delta x) \phi (\delta \xi)) .
$$
We then have that \( \chi^w_\delta : L^2 \to S \) and \( \chi^w_\delta u \to u \) in \( L^2 \) as \( \delta \to 0 \) for all \( u \in L^2 \) as desired, which is quick to verify using Weyl calculus and Parseval’s theorem. We then need to check (3.6.2). This can be accomplished using some Weyl symbol calculus for the commutator \([a^w, \chi^w_\delta]\). For \( X \in \text{supp} (\chi_\delta) \) it holds that \(|X| \lesssim \delta^{-1}\). We also have that \(|\partial^\alpha \chi_\delta| \lesssim \delta^2\) for \(|\alpha| \geq 2\), and so \( \chi''_\delta \in S (\delta \langle X \rangle^{-1}) \) uniformly in \( \delta \). Using this and (3.6.1) it follows that for some \( R_4 \in S(1) \) we have

\[
[a^w, \chi^w_\delta] = (-i \{a (x, \xi), \phi (\delta x) \phi (\delta \xi)\} + \delta R_4)^w
= -i\delta (\partial_x a \cdot \phi' (\delta x) \phi (\delta \xi))^w + i\delta (\partial_x a \cdot \phi' (\delta \xi) \phi (\delta \xi))^w u + \delta R_4^w \tag{3.6.3}
\]

Using (3.6.1) and the fact that \(|X| \lesssim \delta^{-1}\) on \( \text{supp} (\phi (\delta x) \phi (\delta \xi)) \) we get

\[
|\delta \partial^\alpha (\partial_x a \cdot \phi' (\delta x) \phi (\delta \xi))| = O (1), \quad \forall \alpha \tag{3.6.4}
\]

and

\[
|\delta \partial^\alpha (\partial_x a \cdot \phi' (\delta \xi) \phi (\delta \xi))| = O (1), \quad \forall \alpha. \tag{3.6.5}
\]

Thus by (3.1.10)

\[
\|[a^w, \chi^w_\delta]\|_{L^2 \to L^2} = O (1).
\]

It thus suffices to show that \([a^w, \chi^w_\delta] u \to 0\) for all \( u \) in a dense subset of \( L^2 \). Term \( III \) is easily dealt with because as \( \delta \to 0 \),

\[
\|[IIIu]\| = O (\delta) \|u\| \to 0, \quad u \in L^2.
\]

To deal with terms \( I \) and \( II \), let \( u \in C^\infty_c (\mathbb{R}^n) \). Note that the Weyl symbol from \( II \) is supported where \(|\xi| \sim \delta^{-1}\) and is in \( S(1) \) by (3.6.5). Then, using the definition of the Weyl quantization and integration by parts,

\[
IIu = \frac{i\delta}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y) \cdot \xi} (\partial_x a) \left( \frac{x+y}{2}, \xi \right) \cdot \phi' (\delta \xi) \phi \left( \frac{\delta (x+y)}{2} \right) u(y) \, dyd\xi
= -\frac{i\delta}{(2\pi)^n} \int_{\mathbb{R}^{2n}} \frac{e^{i(x-y) \cdot \xi}}{|\xi|^2} \Delta_y \left( (\partial_x a) \left( \frac{x+y}{2}, \xi \right) \cdot \phi' (\delta \xi) \phi \left( \frac{\delta (x+y)}{2} \right) u(y) \right) \, dyd\xi.
\]

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where each $b_{\alpha} \in S(\delta^2)$. Thus
\[\|Iu\| \lesssim \delta^2 \to 0.\]

Similarly, the Weyl symbol in $I$ is in $S(1)$ by (3.6.4), and
\[Iu = \frac{-i\delta}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot \xi} (\partial_\xi a) \left( \frac{x+y}{2}, \xi \right) \cdot \phi' \left( \frac{\delta(x+y)}{2} \right) \phi(\delta \xi) u(y) \, dy \, d\xi \]
\[= \frac{i\delta}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot \xi} \left( \frac{x+y}{2}, \xi \right) \cdot \phi' \left( \frac{\delta(x+y)}{2} \right) \phi(\delta \xi) u(y) \, dy \, d\xi.\]
The integrand is supported where $|x+y| \sim \delta^{-1}$ and $|y| \lesssim 1$. So for $\delta$ sufficiently small $|x+y| \sim \delta^{-1}$. Thus
\[Iu = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y)\cdot \xi} p(x, y, \xi) u(y) \, dy \, d\xi,\]
where $\partial^\alpha p = O(\delta^2)$ for all $\alpha$. Thus by Theorem 4.20 of [23]
\[\|Iu\| \lesssim \delta^2 \to 0.\]

Therefore (3.6.2) holds, which tells us that the graph closure of $a^u$ on $\mathcal{S}$ has the domain $D_{\text{max}}$. \hfill \Box

The above lemma applies to $p^w(x, hD_x) - z$ as we have $p' \in S(\mathcal{A}(X))$ from (3.2.9). Let $P$ denote $p^w(x, hD_x)$ extended by graph closure to its maximal domain, $D(P) := \{u \in L^2 : Pu \in L^2\}$. We then have that
\[\| (P - z) u \| \gtrsim h^{2/3} y^{1/3} \|u\|, \quad \forall u \in D(P),\]
and so $P - z$ is injective with closed range. We can apply the same argument to the formal adjoint of $p^w - z$ on $\mathcal{S}$, $\overline{p}^w - \overline{z} = (hD_x - A)^2 + V(x) - \overline{z}$, and we similarly get its graph closure, $\overline{P} - \overline{z}$, is also injective with maximal domain and closed range. Thus $\overline{P} = P^*$, and $P - z$ in invertible. We then get the desired resolvent estimate,
\[\| (P - z)^{-1} u \| \lesssim h^{-2/3} (|z| - T)^{-1/3} \|u\|.\]
CHAPTER 4

Semigroup Expansions for Non-Selfadjoint Magnetic Schrödinger Operators

4.1 Introduction

In this paper, we study the heat semigroup generated by a class of non-self-adjoint semiclassical Schrödinger operators with magnetic potential, i.e. $e^{-tP/h}$ where $P = (hD_x - A(x))^2 + V(x)$ with $V$ a complex-valued potential, $A$ a real valued magnetic vector potential, and $h > 0$ a semiclassical parameter.

Let $p = |\xi - A(x)|^2 + V(x)$ where $V = V_1 + iV_2$ with $V_1, V_2 \in C^\infty(\mathbb{R}^n; \mathbb{R})$ and $A \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$.

We place the following conditions on $A$ and $V$:

$$|A'| \lesssim 1,$$  \hspace{1cm} (4.1.1)

$$A'' \in S(\langle x \rangle^{-1}),$$  \hspace{1cm} (4.1.2)

$$V'' \in S(1),$$  \hspace{1cm} (4.1.3)

$$V_1 \geq 0,$$  \hspace{1cm} (4.1.4)

and

$$|V_2(x)| \lesssim 1 + V_1(x) + |V_2'(x)|^2, \quad x \in \mathbb{R}^n.$$  \hspace{1cm} (4.1.5)

Here we use the notation

$$S(f) := \{ \phi \in C^\infty : |\partial^\alpha \phi| \leq C_\alpha f, \quad \forall \alpha \}$$
as well as
\[ f \lesssim g \iff \exists C > 0 \text{ s.t. } f \leq Cg. \]

Define the \( \lambda(X), m(X) \in C^\infty(\mathbb{R}^{2n}; \mathbb{R}) \), \( X = (x, \xi) \), by
\[ \lambda(X) = |\xi - A|^2 + V_1(x) + |V'_2(x)|^2, \quad (4.1.6) \]
and
\[ m(X) = 1 + \lambda(X). \quad (4.1.7) \]

We will later show that \( m \) is an order function (as defined in section 4.4 of [23]). We also require that \( p \) obeys the ellipticity condition
\[ \Re p(X) \gtrsim m(X), \quad |X| \geq C; \quad (4.1.8) \]
for some \( C > 0 \). This condition is equivalent to
\[ V_1(x) \gtrsim 1 + |V'_2(x)|^2, \quad |x| \geq C \quad (4.1.9) \]
for a possibly different value of \( C \).

Suppose further that \( \Re p \) has a finite number of zeros, which we will write as
\[ (\Re p)^{-1}(0) = \{X_1, \ldots, X_N\}. \quad (4.1.10) \]

Note that each \( X_j \) is of the form \( X_j = (x_j, A(x_j)) \). We require that these zeros be doubly characteristic points of \( p \) in the sense that
\[ p(X_j) = p'(X_j) = 0, \quad \forall j = 1, \ldots, N; \quad (4.1.11) \]
or equivalently
\[ V(x_j) = V'(x_j) = 0. \quad (4.1.12) \]

For each \( X_j \in \Re p^{-1}(0) \) define \( q_j(X) \) to be the quadratic approximation of \( p \) in a neighborhood of \( X_j \) such that
\[ p(X_j + Y) = q_j(Y) + \mathcal{O}(Y^2). \]
Letting \( Y = (y, \eta) \), we can express
\[
q_j(Y) = |\eta - A'(x_j)y|^2 + \frac{1}{2} y \cdot V''(x_j)y.
\] (4.1.13)

We shall require that each \( q_j \) be nondegenerate in the sense that
\[
\ker V''(x_j) = \{0\}.
\] (4.1.14)

Let \( P = p^w(x, hD_x) \) denote the semiclassical Weyl quantization of \( p \), i.e.
\[
P = (hD_x - A(x))^2 + V(x).
\]

Such \( P \) is maximal accretive on \( L^2 \) with domain \( D(P) = \{u \in L^2 : Pu \in L^2\} \). To see this we note that for \( u \in \mathcal{S}(\mathbb{R}^n) \),
\[
\text{Re} \ (Pu, u) = ((hD_x - A)u, (hD_x - A)u) + (V_1u, u) \geq 0.
\]

Then, by Proposition 6.1 of [3], the \( L^2 \) graph closure of \( P \) on \( \mathcal{S} \) has domain \( D(P) \) and so \( P \) is maximally accretive with that domain. Thus by the Hille-Yosida theorem, the semigroup \( e^{-tP/h} \) is a strongly continuous contraction semigroup on \( L^2 \) for \( t \geq 0 \). Our goal is to show the following.

**Theorem 4.1.1.** Let \( p \) satisfy (4.1.3), (4.1.5), (4.1.8), (4.1.10), (4.1.11), and (4.1.14). Define \( E_b := \{z \in \sigma(P) : \text{Re} \ z \leq bh\}, b > 0 \). Then for all \( h \) sufficiently small \( E_b \) is finite, and the semigroup generated by \( P \) can be expanded as
\[
e^{-tP/h} = \sum_{z_{j,k} \in E_b} e^{-z_{j,k}t/h} \Pi_{j,k} + \mathcal{O}(e^{-tb})_{L^2 \to L^2},
\] (4.1.15)

where \( \Pi_{j,k} \) denotes projection onto the corresponding generalized eigenspace of \( P \).

We will obtain a more precise description of the eigenvalues \( z_{j,k} \) using a theorem of Hitrik and Pravda-Starov from [12], Theorem 4.2.1.

Our primary means for establishing the semigroup expansion is the following theorem of Helffer and Sjöstrand [10].
Theorem 4.1.2. Let $B$ be the generator of the semigroup $e^{tB}$. Suppose that for some $\omega \in \mathbb{R}$ the resolvent $(B - z)^{-1}$ obeys the uniform bound
\[ \| (B - z)^{-1} \|_{L^2 \to L^2} \lesssim 1, \quad \forall z \in \mathbb{C}, \ Re z \geq \omega. \]

Let $f(t)$ be a continuous positive function such that $\| e^{tB} \|_{L^2 \to L^2} \leq f(t)$. Let $\omega_0 < \omega$ and assume that $B$ has no spectrum on the line $Re z = \omega_0$ and that the spectrum of $B$ in the strip $\omega_0 < Re z < \omega$ is compact. Assume that $\| (B - z)^{-1} \|_{L^2 \to L^2}$ is uniformly bounded for $z \in \{ \omega_0 < Re z < \omega \} \setminus U$ where $U$ is any neighborhood of $\sigma_+ := \{ z \in \sigma(B) : \omega_0 < Re z < \omega \}$.

Define $r(\omega_0)$ by
\[ \frac{1}{r(\omega_0)} = \sup_{Re z = \omega_0} \| (B - z)^{-1} \|. \]

Then for every $t > 0$ we have
\[ e^{tB} = e^{tB} \Pi_+ + R(t) = e^{tB} \Pi_+ + e^{tB} (1 - \Pi_+), \]
and for any $a_1, a_2 > 0$ with $a_1 + a_2 = t$
\[ \| R(t) \| \leq \frac{e^{\omega_0 t}}{r(\omega_0)} \frac{e^{\omega_0 a_1}}{|| e^{\omega_0} ||_{L^2([0,a_1])}} \frac{e^{\omega_0 a_2}}{|| e^{\omega_0} ||_{L^2([0,a_2])}}} || 1 - \Pi_+ ||, \quad (4.1.16) \]
where $\Pi_+$ is the spectral projection onto $\sigma_+$.

As the spectrum of our operator $P$ is contained in the right half-plane we will be applying this theorem with $-P$ in the place of $B$. We will thus need to understand the spectrum and resolvent of $P$. We do this by applying two theorems- one which describes the resolvent and spectrum of $P$ in a small neighborhood of the origin, and another which provides a resolvent estimate for $P$ in an unbounded parabolic region bounded away from the origin. These combined will allow us to apply this theorem and attain the desired result.

### 4.2 The Resolvent and Spectrum of $P$

First, to control the resolvent for $z$ near the origin we will use Theorem 1.1 and equation (1.21) of [12]. In this case, due to $p$ satisfying (4.1.4), (4.1.8), (4.1.10), (4.1.11), and
Theorem 4.2.1. If \( m \) is an order function and \( m, p \in S(m) \), then for any \( C > 0 \) there exists \( h_0 > 0 \) such that for all \( 0 < h \leq h_0 \), the spectrum of the operator \( P \) in the open disc in the complex plane \( D(0, Ch) \) is given by the eigenvalues of the form,

\[
z_{j,k} = h \left( \lambda_{j,k} + O \left( h^{1/r_{j,k}} \right) \right), \quad 1 \leq j \leq N, \quad k \in \mathbb{N}
\]

where the \( \lambda_{j,k} \) are the eigenvalues of each \( q^w_j(x, D_x) \), repeated by algebraic multiplicity, and \( r_{j,k} \) are the dimensions of the corresponding generalized eigenspaces. Furthermore, for all \( z \in \mathbb{C} \), \( |z| \leq C \), outside a fixed neighborhood of the set \( \{ \lambda_{j,k} : 1 \leq N, k \in \mathbb{N} \} \) we have that

\[
\| (P - hz)^{-1} \|_{L^2 \rightarrow L^2} \lesssim h^{-1}.
\]

Remark. The statement of Theorem 1.1 in [12] requires that each \( q_j \) be elliptic along its singular space, which we have replaced with the condition (4.1.14). Here we will show that these two conditions are equivalent for \( p \). The singular space, \( S \), of a quadratic form \( q \) is defined by

\[
S = \{ Y \in \mathbb{R}^{2n} : H_{\text{Im} q_j} \text{Re} q_j(Y) = 0, \quad k \in \mathbb{N} \}
\]

For \( q_j \) as defined in (4.1.13) we can compute the singular space, \( S_j \) as follows.

\[
\text{Re} q_j(Y) = |\eta - A'(x_j)y|^2 + \frac{1}{2} y \cdot V''_1(x_j)y.
\]

\[
H_{\text{Im} q_j} \text{Re} q_j(Y) = V''_2(x_j)y \cdot (\eta - A'(x_j)y).
\]

\[
H^2_{\text{Im} q_j} \text{Re} q_j(Y) = 2|V''_2(x_j)y|^2.
\]

\[
H^k_{\text{Im} q_j} \text{Re} q_j = 0, \quad k \geq 3.
\]

Starting with the \( k = 0 \) condition,

\[
\text{Re} q_j(Y) = 0 \implies \eta = A'(x_j)y, \quad y \cdot V''_1(x_j)y = 0.
\]

As \( V''_i \geq 0 \), this implies that \( y \in \ker (V''_1(x_j)) \). Looking at the \( k = 1 \) condition, \( H_{\text{Im} q_j} \text{Re} q_j = 0 \) whenever \( \text{Re} q_j = 0 \) so that condition is redundant. For \( k = 2 \)

\[
H^2_{\text{Im} q_j} \text{Re} q_j = 0 \implies y \in \ker (V''_2(x_j)).
\]
Thus
\[ S_j = \{ (y, \eta) \in \mathbb{R}^{2n} : y \in \ker (V''_1 (x_j)) \cap \ker (V''_2 (x_j)) , \ \eta = A' (x_j) y \} . \]

Then for \( Y \in S_j , q_j (Y) = 0 \). Thus \( q_j \) is elliptic along its singular space \( S_j \) if and only if \( S_j = \{ 0 \} \). To see that this is equivalent to (4.1.14), consider \( u = u_1 + i u_2 \in \mathbb{C}^n \) with \( u \in \ker (V'' (x_j)) \). Then
\[ 0 = \text{Re} \, u \cdot V'' (x_j) u = u_1 \cdot V''_1 (x_j) u_1 + u_2 \cdot V''_1 (x_j) u_2 . \]

As \( V''_1 (x_j) \) is positive definite we have \( u_1 , u_2 \in \ker (V''_1 (x_j)) \). Therefore \( V'' (x_j) u = i V''_2 (x_j) u \) so \( u_1 , u_2 \in \ker (V''_2 (x_j)) \) as well and thus \( S_j = \{ 0 \} \) if and only if \( \ker (V'' (x_j)) = \{ 0 \} \). So each \( q_j \) is elliptic along its singular space if and only if (4.1.14) holds.

It is worth noting that we can directly compute the values of \( \lambda_{j,k} \). We define the Hamilton map \( F_j \) associated to \( q_j \) by
\[ q_j (X,Y) = \sigma (X,F_j Y) , \quad X,Y \in \mathbb{C}^{2n} \]
where \( \sigma \) is the symplectic form on \( \mathbb{C}^{2n} \). Let \( \sigma (F_j) \) denote the spectrum of the \( F_j \). Then, as shown in [13] we have
\[ \sigma (q''_j (x,D_x)) = \left\{ \sum_{\lambda \in \sigma (F)} (r_\lambda + 2k_\lambda) (-i \lambda) : k_\lambda \in \mathbb{N} \right\} , \quad (4.2.2) \]
where \( r_\lambda \) denotes the dimension of the generalized eigenspace corresponding to \( \lambda \).

**Example.** Consider the case where \( A = 0 \). Then we can express
\[ F_j = \begin{bmatrix} 0 & I \\ -\frac{1}{2} V'' (x_j) & 0 \end{bmatrix} . \]

We can compute the eigenvalues as follows.
\[
\det (F_j - \lambda I) = \det \begin{bmatrix} -\lambda I & I \\ -\frac{1}{2} V'' (x_j) & -\lambda I \end{bmatrix} = \det \begin{bmatrix} -\frac{1}{2} V'' (x_j) - \lambda I & 0 \\ -\frac{1}{2} V'' (x_j) & -\lambda I \end{bmatrix}
\]
Thus for each eigenvalue \( \mu \in \sigma (V''(x_j)) \) there is a pair of corresponding \( \lambda \in \sigma (F_j) \) with
\[
\lambda = \pm \frac{i}{\sqrt{2}} \mu^{1/2},
\]
and the dimension of the generalized eigenspace for each \( \lambda \) is the same as the for corresponding \( \mu \). As \( \Re V'' \geq 0 \), each \( \mu \) lies in the right half-plane, so if we let \( \mu^{1/2} \) denote the branch of the square root that has positive real part, then (4.2.2) becomes
\[
\sigma (q_j^w (x, D_x)) = \left\{ \sum_{\mu \in \sigma (V''(x_j))} (r_{\mu} + 2k_{\mu}) \left( \frac{\mu}{2} \right)^{1/2} : k_{\mu} \in \mathbb{N} \right\}, \tag{4.2.3}
\]
where \( r_{\mu} \) denotes the dimension of the generalized eigenspace corresponding to \( \mu \).

In order to apply Theorem 4.2.1 we need to verify that \( m \) as defined in (4.1.7) is an order function, i.e. that \( m (X) \lesssim \langle X - Y \rangle^N m (Y) \) for some \( N > 0 \) and all \( X, Y \in \mathbb{R}^{2n} \), and that \( m \in S (m) \).

**Lemma 4.2.2.** For any \( f \in C^1 (\mathbb{R}^{2n}) \), if \( f \gtrsim 1 \) and \( |f'| \lesssim f^k \) for some \( k \neq 1 \) then \( f \) is an order function.

**Proof.** As \( |f^{-k} f'| \lesssim 1 \),
\[
f (X)^{1-k} - f (Y)^{1-k} = (1 - k) \int_0^1 f^{-k} \left( tX + (1 - t) Y \right) f' \left( tX + (1 - t) Y \right) \cdot (X - Y) \, dt \lesssim |X - Y|.
\]

Thus
\[
f (X)^{1-k} \lesssim |X - Y| + f (Y)^{1-k} \lesssim \langle X - Y \rangle f (Y)^{1-k},
\]
and so we get,
\[
f (X) \lesssim \langle X - Y \rangle^{1/k} f (Y).
\]
To apply this to \( m \) we note that, using (4.1.1) and (4.1.3),
\[
|m'| \lesssim |\xi - A| + |V'_1| + |V'_2|.
\]
By a standard inequality (see Lemma 4.31 in [23]), because \( V_1 \geq 0 \) and \( V''_1 = O(1) \), it follows that
\[
|V'_1| \lesssim |V_1|^{1/2}. \tag{4.2.4}
\]
So we get that \(|m'| \lesssim m_1^{1/2}\), and the lemma implies that \( m \) is an order function. To see that \( m \in S(m) \) note that from (4.1.3) we have
\[
\partial^{\alpha} |V'_2(x)|^2 \lesssim 1 + |V'_2(x)|, \quad |\alpha| \geq 1.
\]
By (4.1.1)
\[
\partial^{\alpha} |\xi - A|^2 \lesssim |\xi - A| \lesssim 1 + |\xi - A|^2, \quad |\alpha| = 1
\]
and by (4.1.1) and (4.1.2)
\[
\partial^{\alpha} |\xi - A|^2 \lesssim 1 + ||\alpha|||\partial^{\alpha} A| \lesssim 1 + |\xi - A|^2, \quad |\alpha| \geq 2.
\]
Thus, combining these with (4.1.3) and (4.2.4) we get
\[
\partial^{\alpha} m(X) \lesssim m(x), \quad \forall \alpha.
\]
Furthermore, it follows immediately from (4.1.5) that \(|p| \lesssim m\), so combining that with the above estimates and (4.1.3) we get that \( p \in S(m) \). Thus we can indeed apply Theorem 4.2.1 to understand the resolvent of \( P \) in an \( O(h) \) region near the origin.

*Remark.* Theorem 4.2.1 also helps us understand the spectral projectors, \( \Pi_{j,k} \). Suppose we additionally have that the eigenvalues \( \lambda_{j,k} \) of the quadratic forms \( q_j \) are simple and distinct. It follows that the eigenvalues \( z_{j,k} \) of \( P \) are as well and \(|z_{j_1,k_1} - z_{j_2,k_2}| \sim h \) for \((j_1,k_1) \neq (j_2,k_2)\). Then we can define \( \Pi_{j,k} \) by
\[
\Pi_{j,k} = \frac{1}{2\pi i} \int_{\gamma_{j,k}} (z - P)^{-1} dz,
\]
where \( \gamma_{j,k} = \{ z : |z - z_{j,k}| = C h \} \) with \( C > 0 \) chosen such that \( \gamma_{j,k} \) encloses \( z_{j,k} \) and no other part of the spectrum of \( P \). Since \( \| (P - z)^{-1} \|_{L^2 \rightarrow L^2} = O(h^{-1}) \) along \( \gamma_{j,k} \) we see that these projections are bounded uniformly in \( h \).
Next, to estimate the resolvent for $|z| \gtrsim h$ we use Theorem 1.1 from [3], which states the following:

**Theorem 4.2.3.** Let $p = |\xi - A(x)|^2 + V_1(x) + iV_2(x)$ satisfy (4.1.1), (4.1.2), (4.1.3), (4.1.4), and (4.1.5). Let $T \geq 0$ be such that

$$|V_2(x)| - T \lesssim V_1(x) + |V_2'(x)|^2, \quad x \in \mathbb{R}^n,$$

(4.2.5)

and choose any $K \in \mathbb{R}$, $K > 1$. There exist positive constants $C_0$, $h_0$, and $M$ such that for all $0 < h \leq h_0$ and $z \in \mathbb{C}$ with $|z| \geq KT + Mh$ and $\Re z \leq C_0^{2/3} (|z| - T)^{1/3}$ we have

$$\| (P - z)^{-1} \|_{L^2 \to L^2} \lesssim h^{-2/3} (|z| - T)^{1/3}.$$

In the case where $T$ in (4.2.5) can be taken to be 0, Theorem 4.2.3 provides a resolvent estimate outside of an $O(h)$ region of the origin. In that $O(h)$ region we can apply Theorem 4.2.1 which provides sufficient control of the resolvent there. This is then all we need to apply Theorem 4.1.2 and get the semigroup estimate. The condition that each $q_j$ be nondegenerate in the sense of (4.1.14) together with (4.1.5) implies that $p$ does indeed satisfy (4.2.5) with $T = 0$. To see this note that outside any neighborhood of the set of points $x_j$, the ellipticity condition (4.1.8) implies that $V_1 \gtrsim 1$, so

$$|V_2| \lesssim 1 + V_1 + |V_2'|^2 \implies |V_2| \lesssim V_1 + |V_2'|^2$$

in that region. For all $x_j + y$ in a small neighborhood of any $x_j$, the condition (4.1.14) implies that either $|y \cdot V_1''(x_j) y| \sim |y|^2$ or $|V_2''(x_j) y| \sim |y|^2$. Thus we have that

$$V_1(x_j + y) + |V_2'(x_j + y)|^2 \sim |y|^2 + O(|y|^3)$$

in this neighborhood. As $V_2(x_j) = 0$ and $|V_2''| \lesssim 1$ it follows that

$$|V_2(x_j + y)| \lesssim |y|^2 \sim V_1(x_j + y) + |V_2'(x_j + y)|^2.$$

Thus we globally have $|V_2| \lesssim V_1 + |V_2'|^2$, and we can take $T = 0$
4.3 Finishing the Proof

The results of the preceding section allow us sufficient control of the resolvent of $P$ to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. For our purposes, we will apply Theorem 4.1.2 with $-P$ in place of $B$ and $t/h$ in place of $t$. As the spectrum of $P$ is contained in the right half-plane, we can choose $\omega = -1$ and $f(t) = 1$. By 4.2.3, there exist $C_0, M, h_0 > 0$ such that,

$$\| (P - z)^{-1} \| \lesssim h^{-2/3}|z|^{-1/3},$$

for all $z \in \mathbb{C}$ with $|z| \geq Mh$ and $\text{Re} z \leq C_0 h^{2/3}|z|^{1/3}$ and all $h$ with $0 < h < h_0$. Inside the region $|z| \leq Mh$, Theorem 4.2.1 implies that the spectrum of $P$ is discrete and of the form

$$z_{j,k} = h\lambda_{j,k} + o(h).$$

We may thus choose some $0 < b < C_0 M^{1/3}$ so that $b \neq \text{Re} \lambda_{j,k}$ for any $\lambda_{j,k}$. Then for all $h$ sufficiently small, the line $\text{Re} z = bh$ is at least an $O(h)$ distance away from the spectrum of $P$. So we may use $\omega_0 = -bh$ and $\frac{1}{r(\omega_0)} \lesssim h^{-1}$. Letting $a_1 = a_2 = \frac{t}{2h}$ we get that

$$\left\| \frac{e^{\omega_0}}{m} \right\|_{L^2([0,a_1])} = \left\| e^{-bh} \right\|_{L^2([0,t/2h])} = O(h^{-1/2}).$$

Thus in this case (4.1.16) becomes

$$\| R(t) \| \lesssim \frac{h^{-1}e^{-bt}}{h^{-1/2}h^{-1/2}} \lesssim e^{-bt}.$$ 

We can therefore conclude that the semigroup can be expanded as

$$e^{-tP/h} = \sum_{z_{j,k} \in E_b} e^{-z_{j,k}t/h} \Pi_{j,k} + O(e^{-tb})_{L^2 \to L^2}. \tag{4.3.1}$$ 

$\square$
REFERENCES


