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MULTIPLE SCATTERING OF ELECTROMAGNETIC WAVES IN AN UNDERDENSE PLASMA AND THE RADIATION TRANSPORT EQUATION

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(Ph. D. Thesis)
April 29, 1971

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MULTIPLE SCATTERING OF ELECTROMAGNETIC WAVES IN AN UNDERDENSE PLASMA AND THE RADIATION TRANSPORT EQUATION

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April 29, 1971

ABSTRACT

The multiple scattering of electromagnetic wave in an underdense turbulent plasma is studied. Using familiar diagrammatic techniques, we derived integral equations for the coherent wave and radiation flux. On the basis of these integral equations, we examine the conditions under which coherent propagation can be described by a refractive index, and incoherent scatterings of the radiation flux can be described by the conventional radiation transport equation. Expressions are given relating the "absorption coefficient" and scattering function in the transport equation to the electron density fluctuations.

I. INTRODUCTION

Two approaches are available to describe scattering of electromagnetic waves from a turbulent plasma—one via the fundamental Maxwell wave equations and the other via the radiation transport equation. Due to mathematical complexity, the wave-equation approach is useful only for studying scatterings in the first Born approximation. On the other hand, the radiation transport equation, which is necessarily an approximation to the wave equations, is more amenable to analysis (often numerical). It is thus desirable to examine the approximation involved in using the transport equation by attempting to derive it from the wave equation.

Starting from scattering from individual electrons, Watson obtained the transport equation from the wave equations. It was found that the effect of multiple scatterings on the averaged (or coherent) wave could be described by a refractive index, which was related to the electron density fluctuations. The averaged radiation flux had two contributions, one due to the coherent wave (which traveled in the forward direction), and the other due to incoherent or independent scatterings of the coherent wave by the density fluctuations. Watson showed that the latter scatterings were describable by the conventional transport equation.

In his derivation, Watson kept only the lowest-order correlations between the electrons in the turbulent plasma. It is the purpose here to extend the study to higher-order correlations. The same physical picture as described in the last paragraph and similar results emerge, with modified expressions for the quantities that appear in the transport equation.
After specifying the notation in the next section, we derive the
expression for the refractive index in Secs. III and IV. We will
use familiar diagrammatic techniques. In Sec. V it is shown that the
two-point correlation function of the field satisfies an integral
equation, to which the conventional radiation transport equation is an
approximation. The scattering function that enters into the transport
equation is related to the kernel of the exact integral equation.
Finally, in the last section, the imaginary part of the refractive
index, which describes the "absorption" of the coherent wave, is
related to the total incoherent scattering through energy conservation.

For the sake of completeness, some of the developments in
Ref. 1 will be repeated.

II. STATISTICAL DESCRIPTION OF THE MEDIUM

We assume a time-stationary turbulent medium, characterized by
a probability distribution $P_N(z_1, \ldots, z_N)$ ($N = \text{total number of}$
electrons, $N \gg 1$), such that $P_N(z_1, \ldots, z_N) d^3 z_1 \ldots d^3 z_N$ is the
probability of finding $N$ electrons at $z_1, \ldots, z_N$ within the volume elements $d^3 z_1, \ldots, d^3 z_N$, respectively. It is assumed that the average
of a physical observable with respect to this probability distribution
is equal to the time average of that observable in the actual medium.
It is the time average of any observable, instead of its instantaneous
value, that is of physical interest. This is the usual statistical
approach.

The reduced probability distribution functions are defined as

$$P_k(z_1, \ldots, z_k) = \int d^3 z_{k+1} \ldots d^3 z_N P_N(z_1, \ldots, z_N), \ k \leq N.$$ 

We will work with the correlation functions $g_k$ defined as
usual:

$$P_2(z_1, z_2) = P_1(z_1) P_1(z_2)[1 + g_2(z_1, z_2)],$$

$$P_3(z_1, z_2, z_3) = P_1(z_1) P_1(z_2) P_1(z_3)[1 + g_2(z_1, z_2) + g_2(z_2, z_3) + g_2(z_3, z_1) + g_3(z_1, z_2, z_3)],$$

and so on.

It shall be assumed that the statistical correlation of the
electrons is characterized by a parameter $R_c$, the "correlation range,"
such that the correlation function $g_k(z_1, \ldots, z_k)$ is nonvanishing only
when all of its arguments lie within a range $R_c$ of one another. For
scatterings from turbulent flow, only the eddies with $R_c \sim k^{-1}$.
(k is the wave vector of the incident wave) contribute significantly to scattering into nonforward directions.\(^2\)

If one also takes into account the motion of the electrons, time-dependent probability distribution functions have to be introduced, which will lead to a Doppler shift in the frequency of the scattered radiation.\(^3\)

III. MULTIPLE SCATTERING

We now suppose a plane electromagnetic wave to be incident on the plasma (assumed to be finite in extent). The electric field is

\[ E_I(x) e^{-i\omega t} = E_0 e^{i(k \cdot x - \omega t)}. \]

The effects of the magnetic field and of the ions are neglected.

If the incident wave is scattered by an electron at position \( z \), then the scattered wave is given by

\[ E_{sc}(x) = f \cdot G_0(x, z) \cdot E_I(z), \tag{3.1} \]

where \( f = -\frac{e^2}{mc} \) is the negative of the classical electron radius, and \( G_0 \) is the tensor Green function of Maxwell's equation,

\[ (\nabla_x \times \nabla_z \times -k^2) G_0(x, z) = 4\pi \delta(x - z). \tag{3.2} \]

The explicit expression for \( G_0 \) is

\[ G_0(x, z) = (1 + \frac{1}{k^2} \nabla_x \nabla_z) G_0(R) \]

\[ = G_0(R) \left[ \frac{1}{k^2} \left( 1 + \frac{1}{kR} - \frac{1}{k^2 R^2} \right) - \frac{2}{k^2} \left( 1 + \frac{1}{kR} - \frac{2}{k^2 R^2} \right) \right] \]

\[ - \frac{1}{3k^2} \delta(R), \tag{3.3} \]

where \( R = z - x \),

and

\[ G_0(R) = \frac{e^{ikR}}{R}. \]

The effects of collision in the plasma can be included by replacing \( \omega \rightarrow \omega + i\nu_c \), where \( \nu_c \) is a constant collision frequency. This approximation is valid if \( \nu_c \ll \omega \). We shall not do this here.
Following Watson, we write the solution for the electric field in the plasma as a set of multiple scattering equations (unless otherwise stated, \( \mathbf{z}_\alpha, \mathbf{z}_\beta \) etc. denote electron coordinates):

\[
E(z_\alpha) = E_I(z_\alpha) + \sum_{\beta(\neq \alpha)}^N E(z_\alpha, z_\beta)
\]

(3.4)

and

\[
E(z_\alpha, z_\beta) = \int G_0(z_\alpha, z_\beta) \cdot E_I(z_\beta) + \sum_{\sigma(\neq \beta)}^N \int G_0(z_\alpha, z_\beta) \cdot E(z_\beta, z_\sigma) \cdot \nabla \cdot \mathbf{H}.
\]

(3.5)

The field \( E(z_\alpha, z_\beta) \) here is interpreted as the wave at the point \( z_\alpha \) scattered from the electron at \( z_\beta \). Although it is not explicitly stated, \( E(z_\alpha, z_\beta) \) is a function of all the electron coordinates.

Alternatively, we can eliminate \( E(z_\alpha, z_\beta) \) from the above equations and express \( E(z_\alpha) \) as an infinite scattering series:

\[
E(z_\alpha) = E_I(z_\alpha) + \sum_{\beta(\neq \alpha)}^N \int G_0(z_\alpha, z_\beta) \cdot E_I(z_\beta) + \sum_{\sigma(\neq \beta)} \int G_0(z_\alpha, z_\beta) \int G_0(z_\beta, z_\sigma) \cdot E_I(z_\sigma) \cdot \nabla \cdot \mathbf{H}.
\]

(3.6)

This equation gives the wave at the point \( z_\alpha \) as the sum of the incident wave, once-scattered wave, twice-scattered wave, etc.

The solution for the Green function \( G(z_\alpha, z_\beta) \) has the same form and interpretation:

\[
G(z_\alpha, z_\beta) = G_0(z_\alpha, z_\beta) + \sum_{\sigma(\neq \alpha)}^N \int G_0(z_\alpha, z_\beta) \cdot G_0(z_\beta, z_\sigma) \cdot \nabla \cdot \mathbf{H} + \sum_{\sigma(\neq \beta)} \sum_{\lambda(\neq \alpha)}^N \int G_0(z_\alpha, z_\beta) \int G_0(z_\beta, z_\lambda) \cdot G_0(z_\lambda, z_\sigma) \cdot \nabla \cdot \mathbf{H} + \cdots.
\]

(3.7)

In finding \( \langle E \rangle \), the averaged wave (also called the coherent wave, since it can interfere with the incident wave), it is convenient to separate out the effects owing to the statistical correlation of the electrons. Accordingly we take the statistical average of (3.6), assuming statistical independence of the electrons, i.e., all of the \( g_{\alpha \beta} \)'s vanish. Denoting the coherent wave in this case by \( \langle E \rangle^{(1)} = E_c^{(1)} \), we have

\[
E_c^{(1)}(z_\alpha) = E_I(z_\alpha) + \sum_{\beta(\neq \alpha)} \int d^3z_\beta P_1(z_\beta) \int G_0(z_\alpha, z_\beta) \cdot E_I(z_\beta) + \sum_{\sigma(\neq \beta)} \sum_{\lambda(\neq \alpha)}^N \int d^3z_\beta \int d^3z_\sigma P_1(z_\beta) P_1(z_\sigma) \int G_0(z_\alpha, z_\beta) \int G_0(z_\beta, z_\sigma) \cdot E_I(z_\sigma) \cdot \nabla \cdot \mathbf{H} + \cdots.
\]

or

\[
E_c^{(1)}(z_\alpha) = E_I(z_\alpha) + \int d^3z_\beta \rho(z_\beta) \int G_0(z_\alpha, z_\beta) \cdot E_I(z_\beta) + \int d^3z_\beta \int d^3z_\sigma \rho(z_\beta) \rho(z_\sigma) \int G_0(z_\alpha, z_\beta) \int G_0(z_\beta, z_\sigma) \cdot E_I(z_\sigma) + \cdots,
\]

(3.8)

where we have introduced the average electron density \( \rho(z) = N P_1(z) \). Furthermore, since \( N \gg 1 \), the restriction on the sums over electron coordinates is neglected.
Equation (3.8) is equivalent to the integral equation
\[ E^{(1)}_c(z_\alpha) = E^{(1)}_c(z_\alpha') + \int d^3z_c \rho(z_c) f G_0(z_\alpha - z_c) E^{(1)}_c(z_c) \] (3.9)

Hence \( E^{(1)}_c \) satisfies Maxwell's wave equation with a modified wave vector
\[ (\nabla_\alpha \times \nabla_\alpha - k^2) E^{(1)}_c(z_\alpha) = \int d^3z_c \rho(z_c) f 4\pi \delta(z_\alpha - z_c) E^{(1)}_c(z_c) \]
\[ = 4\pi f \rho(z_\alpha) E^{(1)}_c(z_\alpha), \] (3.10)

where (3.2) has been used.

We introduce the modified wave vector
\[ p_1^2(z_\alpha) = k^2 + 4\pi f \rho(z_\alpha). \] (3.11)

Equation (3.10) becomes
\[ [\nabla_\alpha \times \nabla_\alpha - p_1^2(z_\alpha)] E^{(1)}_c(z_\alpha) = 0. \] (3.12)

The corresponding Green function \( G^{(1)}_c(z_\alpha, z_\beta) \) satisfies the same equation, with a different boundary condition
\[ G^{(1)}_c(z_\alpha, z_\beta) = G_0(z_\alpha, z_\beta) + \int d^3z_c \rho(z_c) G_0(z_\alpha, z_c) G^{(1)}_c(z_c, z_\beta) \] (3.13)
or
\[ [\nabla_\alpha \times \nabla_\alpha - p_1^2(z_\alpha)] G^{(1)}_c(z_\alpha, z_\beta) = 4\pi f \delta(z_\alpha - z_\beta). \] (3.14)

In order to solve (3.12) or (3.14), assumptions about the spatial variation of the average electron density have to be made.

Introducing the refractive index \( n_1 = p_1/k \), we assume that
\[ |\nabla \ln n_1| << k \] (3.15)

and
\[ |n_1 - 1| \ll 1. \] (3.16)

For convenience in later derivations, we also assume that
\[ |\nabla \ln n_1| \ll k^{-1}. \] (3.17)

Relations (3.15) and (3.17) imply that the average electron density can be treated as constant within a wavelength or correlation length.

As is well known, (3.15) allows (3.12) or (3.14) to be solved in eikonal form,
\[ E^{(1)}_c(z_\alpha) = \exp \left[ i \int x ds_{p_1} E^{(1)}_c(z), \right] \] (3.18)

where \( x \) is a point on the ray path outside the plasma medium. For the Green function we have
\[ G^{(1)}_c(z_\alpha, z_\beta) = \exp \left[ i \int_\beta^\alpha ds_{p_1} \right] \frac{1}{|z_\alpha - z_\beta|} \left[ 1 - \hat{\mathbf{R}}_{\mathbf{Q}_\alpha} \hat{\mathbf{F}}_{\mathbf{Q}_\beta} \right], \] (3.19)

where \( |z_\alpha - z_\beta| \gg k^{-1} \). Assumption (3.16) allows the line integrals in Eqs. (3.18) and (3.19) to be taken along straight ray paths. The assumption is relaxed in Ref. 4, where curved ray paths are used, resulting in a rotation of polarization vectors along the ray.
IV. MASS OPERATOR OR EFFECTIVE PROPAGATION FUNCTION

In generalizing (3.9) or (3.13) to include electron correlations, it is more transparent to use diagrammatic representations, as shown in Fig. 1. The expression associated with a diagram is obtained by forming dot products of successive tensor quantities, and integrating over all internal coordinates. As an example, the expression for diagram (h) of Fig. 2 is

\[
\int d^3 z_1 d^3 z_2 d^3 z_3 d^3 z_4 \rho(z_1) \rho(z_2) \rho(z_3) \rho(z_4)
\]

\[
\times \delta(z_2 - z_1) \delta(z_3 - z_4) \delta(z_3 - z_2) \delta(z_4 - z_1)
\]

The generalization of (3.13) then is the infinite series shown in Fig. 2. It should be noted that the Green function \( G_0 \) has replaced \( G_c \), and that all the indicated scatterings are "correlated scatterings," i.e., every scattering electron is correlated with some other electrons. For homogeneous media, the rules for the diagrams can also be given in Fourier-transformed wave-vector space. However, we do not need them here.

A connected diagram is one which cannot be split into two separate diagrams by cutting a solid line. We define the effective propagation function (also called mass operator) \( S \) as the sum of all connected diagrams shown in Fig. 3. The function \( S \) is a second-rank tensor, given by

\[
S(z_\alpha, z_\lambda) = e^2 \rho(z_\alpha) \rho(z_\lambda) \delta(z_\alpha - z_\lambda) G^{(1)}(z_\alpha, z_\lambda)
\]

\[+
\int d^3 z_\kappa d^3 z_\kappa \rho(z_\kappa) \rho(z_\lambda) \epsilon_j(z_\kappa, z_\lambda, z_\alpha, z_\beta)
\times G^{(1)}(z_\alpha, z_\kappa) G^{(1)}(z_\lambda, z_\beta)
\]

The infinite series in Fig. 2 can be re-expressed as an integral equation (see Fig. 4),

\[
G_c(z_\alpha, z_\beta) = G^{(1)}(z_\alpha, z_\beta) + \int d^3 z_\alpha d^3 z_\lambda G^{(1)}(z_\alpha, z_\lambda) G_c(z_\lambda, z_\beta)
\times S(z_\alpha, z_\lambda) G_c(z_\lambda, z_\beta)
\]

The equation for the coherent wave is evidently

\[
\xi_c(z_\alpha) = \xi^{(1)}(z_\alpha) + \int d^3 z_\alpha d^3 z_\lambda \epsilon_j(z_\alpha, z_\lambda) \xi_c(z_\lambda, z_\alpha) G_c(z_\alpha, z_\lambda)
\]

Equation (4.2) is (3.13) generalized to include electron correlations. It is formally analogous to Dyson's equation in field theory, with \( G^{(1)}_c \) corresponding to the unrenormalized propagator, \( G_c \) to the renormalized propagator, and \( S \) to the proper self-energy operator.

Following the procedures of Sec. III, we find that (4.2) is equivalent to the nonlocal differential equation

\[
[\nabla_\alpha \nabla_\alpha - p^2(\alpha)] G_c(z_\alpha, z_\beta) = 4\pi \int d^3 z_\lambda S(z_\alpha, z_\lambda) G_c(z_\lambda, z_\beta)
\times S(z_\alpha, z_\lambda)
\]

This equation cannot be solved in general, and one has to resort to approximations.
Let us first consider the simpler case of a homogeneous and isotropic medium, for which (4.4) can be exactly solved using Fourier transformation. Define the Fourier transforms,

\[
\mathcal{G}(z^\alpha, z^\beta) = \int \frac{d^3 p}{(2\pi)^3} e^{i \mathbf{R} \cdot \mathbf{p}} \left[ (1 - \hat{\mathbf{p}} \hat{n}) \mathcal{G}_1(p^2) + \hat{\mathbf{p}} \hat{n} \mathcal{G}_2(p^2) \right]
\]

and

\[
\mathcal{G}_c(z^\alpha, z^\beta) = \int \frac{d^3 p}{(2\pi)^3} e^{i \mathbf{R} \cdot \mathbf{p}} \left[ (1 - \hat{\mathbf{p}} \hat{n}) \mathcal{G}_1(p^2) + \hat{\mathbf{p}} \hat{n} \mathcal{G}_2(p^2) \right],
\]

where \( \mathcal{G}_1,2 \) and \( \mathcal{G}_1,2 \) are scalar functions. The specific forms of (4.5) and (4.6) are consequences of the assumptions of homogeneity and isotropy.

Transforming (4.4) to \( p \)-space, we obtain the solution

\[
\mathcal{G}_1(p^2) = \frac{\hbar c}{p^2 - p_e^2 - 4\pi \mathcal{G}_1(p^2)}
\]

and

\[
\mathcal{G}_2(p^2) = \frac{\hbar c}{p^2 - p_e^2 - 4\pi \mathcal{G}_2(p^2)}.
\]

To find the explicit form of \( \mathcal{G}_c \) we need to know the functions \( \mathcal{G}_1(p^2) \) and \( \mathcal{G}_2(p^2) \), which involves the practically impossible task of calculating and summing an infinite number of integrals. However, we can draw some general conclusions without knowing the functions.

Consider the singularities of \( \mathcal{G}_1(p^2) \) and \( \mathcal{G}_2(p^2) \) in the complex \( p \)-plane. Let us assume that the electron density fluctuations are "small," in the sense that

\[
P_e^2 \gg |\mathcal{G}_1(p_e^2)|,
\]

and

\[
1 \gg \left| \frac{d \mathcal{G}_1(p^2)}{dp^2} \right|_{p = p_e}.
\]

Then \( \mathcal{G}_1(p^2) \) will have a simple pole at

\[
p^2 = p_e^2 = p_1^2 + 4\pi \mathcal{G}_1(p_1^2),
\]

or

\[
p_e \approx p_1 + \frac{2\pi}{p_1} \mathcal{G}_1(p_1^2).
\]

It is reasonable to assume that \( |\mathcal{G}_2(p^2)| \sim |\mathcal{G}_1(p^2)| \). Therefore, \( \mathcal{G}_2(p^2) \) has no pole at the point \( p = p_e \).

Calculating the Fourier integral in (4.6), we find that the contribution of this pole to \( \mathcal{G}_c \) is

\[
\mathcal{G}_c(z^\alpha, z^\beta) = \left[ \frac{\hbar c}{p_1} \frac{d \mathcal{G}_1(p_1^2)}{dp_1^2} \right]^{-1} \left[ 1 + \frac{1}{p_e^2} \nabla \alpha \nabla \alpha \right] e^{i \mathbf{p}_e \mathbf{R}_0} \frac{e^{i \mathbf{p}_e \mathbf{R}_0}}{R_0}.
\]

\[
\approx \left[ 1 + \frac{1}{p_e^2} \nabla \alpha \nabla \alpha \right] e^{i \mathbf{p}_e \mathbf{R}_0} \frac{e^{i \mathbf{p}_e \mathbf{R}_0}}{R_0}.
\]

\[
\rightarrow \frac{1}{R_0} \left[ 1 - \frac{e^{i \mathbf{p}_e \mathbf{R}_0}}{R_0} \right] e^{i \mathbf{p}_e \mathbf{R}_0}.
\]
In general, \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) have other singularities which contribute to \( G_c \). However, using simple model calculations, we find that the amplitudes of these other modes are negligible if the electron density fluctuations satisfy equation (4.9). Hence the coherent Green function can be taken to be the expression (4.14). This result is physically expected, and in principle it can be checked by calculating the function \( \mathcal{S} \).

The above result can be easily generalized to the case of a "weakly homogeneous" and "locally isotropic" medium--one in which the average electron density satisfies Eqs. (3.15), (3.16), and (3.17), and the correlation functions \( g_k \) satisfy

\[
ge_k(z, \omega; R) \approx g_k(z; R_R),
\]

(4.15)

\[
g_1(z, \omega; R) \approx g_1(z; R_R),
\]

(4.16)

and so on.

Examination of the terms of \( \mathcal{S}(z_\alpha, z_\beta) \) shows that this function is negligible for \( R_R > R_c \). Hence the above assumptions imply that \( \mathcal{S} \) can be Fourier-transformed in the form

\[
\mathcal{S}(z_\alpha, z_\beta) \approx \int \frac{d^3 p}{(2\pi)^3} e^{i p_R R_R} \left[ (1 - \hat{k} \cdot \hat{k}) \mathcal{S}_1(z_\alpha; p^2) + \hat{k} \cdot \hat{k} \mathcal{S}_2(z_\alpha; p^2) \right].
\]

(4.16)

Furthermore, the assumptions (4.9) and (4.10) of small density fluctuations become

\[
p_1^2(z_\alpha) \gg |\mathcal{S}_1(z_\alpha; p_1^2)|.
\]

(4.17)

The validity of solution (4.19) can be seen by substituting it and expression (4.16) into the original equation (4.4) and checking the consistency.

The coherent wave is evidently given by

\[
E_c(z) = E_0(z) \left[ i \int_{\infty}^{z} \left. \frac{\partial \mathcal{S}_1(z_\alpha; p^2)}{\partial p^2} \right|_{p=p_1} \exp \left( i \int_{z}^{z_\alpha} p_e ds \right) \right],
\]

(4.18)

for all \( z_\alpha \). In this case the asymptotic solution for \( G_c \) is just the eikonal form of (4.14),

\[
G_c(z_\alpha, z_\beta) \approx (1 - R_{\alpha\beta} \hat{R}_{\alpha\beta}) \exp \left( i \int_{\infty}^{z_\alpha} p_e ds \right),
\]

(4.19)

where now

\[
p_e(z_\alpha) \approx p_1^2(z_\alpha) + \frac{2\pi}{p_1} \mathcal{S}_1(z_\alpha; p_1^2(z_\alpha))
\]

(4.20)

or

\[
p_e(z_\alpha) \approx p_1(z_\alpha) + \frac{2\pi}{p_1} \mathcal{S}_1(z_\alpha; p_1^2).
\]

The validity of solution (4.19) can be seen by substituting it and expression (4.16) into the original equation (4.4) and checking the consistency.

The coherent wave is evidently given by

\[
E_c(z) = E_0(z) \left[ i \int_{\infty}^{z} \left. \frac{\partial \mathcal{S}_1(z_\alpha; p^2)}{\partial p^2} \right|_{p=p_1} \exp \left( i \int_{z}^{z_\alpha} p_e ds \right) \right],
\]

(4.21)

where \( \chi \) is a point on the ray path outside the medium.

In general, \( \mathcal{S}_1 \) and hence \( p_e \) has a positive imaginary part. We define
as the "mean free path," or attenuation distance for coherent propagation in the turbulent medium. This attenuation phenomenon is a phase-interference effect resulting from multiple scatterings, and it is not due to any physical dissipation. (There would be physical dissipation if we had included the effect of electron collisions.)

The condition (4.17) implies that

\[ \text{Re } P_e \approx P_1 \]  

(4.24)

and

\[ P_1 \gg \text{Im } P_e. \]

The latter is equivalent to

\[ \lambda \gg P_1^{-1}. \]  

(4.25)

In the next section we shall find that the averaged radiation flux (in contrast to the coherent wave) is nonvanishing in directions other than the forward direction. These energy fluxes can be considered as due to incoherent scatterings of the coherent wave by the electron density fluctuations. From (4.25), it is seen that these incoherent scatterings take place mainly in the wave zone. As a result the coherent Green function can be approximated by (4.19).

The implication of condition (4.18) depends on the ratio of the wavelength to the correlation length. For the case of interest here \( k^{-1} \sim R_c \), and if a smooth dependence of the function \( \mathcal{G}_1 \) on \( \frac{1}{P_1} \) is assumed, it follows that

\[ \left| \frac{3 \mathcal{G}_1}{\partial P_1^2} \right| \approx \frac{\mathcal{G}_1}{P_1^2}. \]

Thus nothing new is added to (4.25), which in this case can also be written as

\[ \lambda \gg R_c. \]

For general values of \( k^{-1} \) and \( R_c \), it is expected from dimensional considerations that the conditions (4.25) and (4.26) are equivalent to (4.17) and (4.18). Hence they are sufficient for the validity of the solution (4.19).

Finally, to end this section, we make two remarks: (i) if "local isotropy" as in (4.15) is not assumed, we would have obtained a tensor for \( P_e \); (ii) if \( S \) is approximated by its first term [diagram (a) in Fig. 5], as was done by Watson, the coherent Green function is equivalent to the infinite series in Fig. 5. Analytically, \( S \) is then given by

\[ S(z_\alpha, z_\lambda) = \rho^2 \rho(z_\alpha) \rho(z_\lambda) e^{i P_1 R} \left( \frac{\sin(p_1 R)}{p_1 R} \right) \left( 1 + \frac{1}{p_1 R} - \frac{1}{p_1 R^2} \right), \]

and the refractive index \( n \) (\( \equiv \frac{P_e}{k} \)) has the form

\[ n^2(z_\alpha) \equiv 1 + \frac{4\pi \rho^2(z_\alpha)}{k^2} \left( \frac{4\pi}{k} \right)^2 \int_0^\infty dR R^2 \left( \frac{\sin(p_1 R)}{p_1 R} \right) \left( 1 + \frac{1}{p_1 R} - \frac{1}{p_1 R^2} \right) \]

\[ \times \rho^2(z_\alpha) \rho(z_\alpha; R) e^{i P_1 R} \left[ \frac{\sin(p_1 R)}{p_1 R} \left( 1 + \frac{1}{p_1 R} - \frac{1}{p_1 R^2} \right) \right] - \frac{1}{(p_1 R)^2} \left( \frac{\sin(p_1 R)}{p_1 R} - \cos(p_1 R) \right) \left( 1 + \frac{3}{p_1 R} - \frac{3}{p_1 R^2} \right) \]

(4.26)
For short waves \( p R_c \gg 1 \), this complicated formula is reduced to that given by Watson:

\[
\eta^2(z_o) \approx 1 + \frac{4\pi f \rho(z_o)}{k^2} + \frac{1}{k^2} \int d^2 R \rho^2(z_o) \exp[i p_R \cdot (R - k \cdot \hat{R})] \tag{4.29}
\]

where \( \eta_T(z_o) = \frac{1}{2} f^2 [1 + (k \cdot \hat{R})^2] \) is the Thomson differential cross section.

The condition for the validity of the first-order approximation of \( \eta \) is readily estimated to be (for \( k^{-1} \sim R_c \)):

\[
| \xi f R_c^2 | \ll 1 \tag{4.30}
\]

where \( \xi \) is a parameter characterizing the magnitude of the mean square density fluctuation \( \left( \rho^2 \right)^{\frac{1}{2}} \). For general values of \( k^{-1} \) and \( R_c \), the condition depends on the functional forms of the electron density correlation functions. However, from dimensional considerations, it is sufficient that the following condition be satisfied:

\[
| \xi f L^2 | \ll 1,
\]

where \( L \) is the larger of \( k^{-1} \) and \( R_c \).

\section{V. The Radiation Transport Equation}

In this section we derive the transport equation for radiation flux from the multiple scattering equations (3.6) or (3.7). It is assumed that the conditions under which the coherent Green function is given by (4.19) are satisfied.

We first introduce the generalized energy density,

\[
\eta(a', a) = \langle \hat{u}^*(a') \otimes \hat{u}(a) \rangle \tag{5.1}
\]

and the averaged double Green function,

\[
\hat{W}(a', a; \beta', \beta) = \langle \hat{G}^*(a', \beta') \otimes \hat{G}(a, \beta) \rangle \tag{5.2}
\]

where, for simplicity of notation, \( \alpha \) stands for \( a, a' \), and \( \otimes \) denotes the direct product.

We define \( \eta(a) = \eta(a, a) \) and \( \hat{W}(a) = \hat{W}(a, a) \). Then the power flux (in unrationalized Gaussian units) for waves with linear polarization at the point \( \hat{r} \) is given by \( \hat{c} \cdot \hat{W}(\hat{r}) \cdot \hat{e} \).

Finally the coherent energy density is defined by

\[
\eta_c(a', a) = \langle \hat{E}^*(a') \otimes \hat{E}(a) \rangle \tag{5.3}
\]

and

\[
\eta_c(a) = \eta_c(a, a),
\]

where \( \hat{E}_c \) is given by (4.21).

To find an equation for \( \hat{W} \), we substitute for \( \eta \) the multiple scattering series in (3.7) and take the average. Again, a diagrammatic representation of the terms can be given. The rules are the same as those given earlier except for the additional ones: (i) functions
associated with the bottom part of a diagram are complex conjugated (thus a wavy line at the bottom part of a diagram represents $g^*_{-c}$); (ii) a direct product is taken between the top and bottom parts of a diagram. As an example, Fig. 6 represents the following analytic expression:

$$
\int d^3 z_\lambda d^3 z_\alpha d^3 z_\beta d^3 z_\delta \ell^3 \rho(\lambda) \rho(\sigma) \rho(\nu) \rho(\kappa) g_-(\sigma, \nu')
$$

The result of averaging (5.2) is then given diagrammatically in Fig. 7. Introducing the "intensity function" $K$, which is the sum of all connected diagrams (Fig. 8), we can rewrite the infinite series as an integral equation (Fig. 9). It is analogous to the Bethe-Salpeter equation in field theory. The analytic expressions for Figs. 8 and 9 are, respectively,

$$
K(\sigma', \sigma; \lambda', \lambda) = R^2 \rho(\sigma') \rho(\sigma) g_2(\sigma', \sigma) \delta(\xi_\sigma - \xi_\lambda) \delta(\xi_{\sigma'} - \xi_{\lambda'})
$$

The equation for $K(\sigma', \sigma)$ is analogous to (5.5):

$$
\bar{V}(\sigma', \sigma) = \int d^3 z_\sigma d^3 z_\lambda d^3 z_\alpha d^3 z_\beta \delta(\xi_\sigma - \xi_\lambda) \delta(\xi_{\sigma'} - \xi_{\lambda'}) [g^*_c(\sigma', \sigma') \otimes g_c(\sigma', \sigma)]
$$

It should be noted that $K$, like $\bar{V}$, is in the form of a series of direct products of second-rank tensors.
\[
\mathcal{E}_c'(\alpha') \otimes \mathcal{E}_c(\alpha) \equiv \mathcal{E}_c(\alpha) \otimes \mathcal{E}_c'(\alpha) \exp[-i p_1 \hat{\mathbf{k}} \cdot \mathbf{R}_{\alpha'\alpha}]
\]  
\[
= \int d\hat{\omega} \delta(\hat{\omega} - \hat{\mathbf{k}}) \mathcal{U}_c(\alpha) \exp[-i p_1 \hat{\omega} \cdot \mathbf{R}_{\alpha'\alpha}], \tag{5.10}
\]

where \( \hat{\mathbf{k}} \) is the direction of propagation of the coherent wave, \( \delta(\hat{\omega} - \hat{\mathbf{k}}) \) is the angular delta function, and the integration is over the unit sphere. To the same order of approximation, \( p_c \) in the exponential factor is replaced by \( p_1 \).

Similarly, since \( R_{\alpha\gamma} R_{\sigma'\gamma} \sim R_c \), the coherent Green functions can be approximated by

\[
\mathcal{G}_c'(\alpha', \sigma') \equiv \mathcal{G}_c'(\alpha', \gamma) \exp(-i p_1 \hat{\mathbf{k}} \cdot \mathbf{R}_{\alpha'\gamma}) \tag{5.11}
\]

and

\[
\mathcal{G}_c(\alpha, \sigma) \equiv \mathcal{G}_c(\gamma, \gamma) \exp(ip_1 \hat{\mathbf{k}} \cdot \mathbf{R}_{\alpha\gamma}). \tag{5.12}
\]

When \( R_{\alpha'\alpha} \sim R_c \), \( R_{\alpha\gamma} \sim e \), we have

\[
\mathcal{G}_c'(\alpha', \gamma) \equiv \mathcal{G}_c'(\gamma, \gamma) \exp(-i p_1 \hat{\mathbf{k}} \cdot \mathbf{R}_{\alpha'\gamma}),
\]

so that

\[
\mathcal{G}_c'(\alpha', \sigma') \otimes \mathcal{G}_c(\alpha, \sigma) \equiv \mathcal{G}_c(\gamma, \gamma) \otimes \mathcal{G}_c(\gamma, \gamma) \exp[ip_1 \hat{\mathbf{k}} \cdot \mathbf{R}_{\alpha'\gamma} (R_{\alpha'\gamma} - \mathbf{R}_{\alpha'\alpha})], \tag{5.13}
\]

where we have set \( \hat{\mathbf{k}} \cdot \mathbf{R}_{\alpha'\gamma} = \hat{\mathbf{k}} \cdot \mathbf{R}_{\alpha'\gamma}' \). We write the integral \( \int d^3 \mathbf{z}_\gamma \) as

\[
\int d\hat{\omega} \int_{(-\hat{\omega})} dR_{\alpha\gamma} R_{\alpha\gamma}^2 \tag{5.14}
\]

where the line integral \( \int_{(-\hat{\omega})} dR_{\alpha\gamma} \) is taken along the straight-line path in the direction \( \hat{\omega} \) and starting from the point \( \mathbf{z}_\alpha \). Equation (5.9) can then be cast into the form

\[
\mathcal{U}(\alpha'; \alpha) \equiv \int d\hat{\omega} \delta(\hat{\omega} - \hat{\mathbf{k}}) \mathcal{U}_c(\alpha) \exp(-i p_1 \hat{\omega} \cdot \mathbf{R}_{\alpha'\alpha})
\]

\[
+ \int d\hat{\omega} \exp(-i p_1 \hat{\omega} \cdot \mathbf{R}_{\alpha'\alpha}) \int_{(-\hat{\omega})} dR_{\alpha\gamma} \mathcal{U}_c(\alpha), \tag{5.15}
\]

where \( \{\ldots\} \) represents an expression independent of \( \alpha' \). Thus the quantity \( \mathcal{U}(\alpha'; \alpha) \) for \( R_{\alpha'\alpha} \sim R_c \) has the form

\[
\mathcal{U}(\alpha'; \alpha) \equiv \int d\hat{\omega} \mathcal{U}(\alpha; \hat{\omega}) \exp(-i p_1 \hat{\omega} \cdot \mathbf{R}_{\alpha'\alpha}). \tag{5.16}
\]

Physically the quantity \( \mathcal{U}(\alpha; \hat{\omega}) \) represents the energy density arriving at \( \mathbf{z}_\alpha \) in the direction \( \hat{\omega} \). Using the approximation (5.14) for both \( \mathcal{U}(\alpha'; \alpha) \) and \( \mathcal{U}(\alpha'; \alpha) \) in (5.9), and removing an angular integration common to all terms, we finally write (5.9) in the form

\[
\mathcal{U}(\alpha; \hat{\omega}) = \delta(\hat{\omega} - \hat{\mathbf{k}}) \mathcal{U}_c(\alpha) + \int_{(-\hat{\omega})} dR_{\alpha\gamma} R_{\alpha\gamma}^2 \mathcal{U}(\alpha; \hat{\omega}) \tag{5.17}
\]

\[
\chi [\mathcal{G}_c(\alpha, \gamma) \otimes \mathcal{G}_c(\gamma, \gamma)] \int d^3 \mathbf{z}_{\gamma'} \mathcal{G}_c(\gamma', \gamma') \mathcal{U}(\gamma; \mathbf{z}_{\gamma'}), \tag{5.18}
\]

where

\[
\mathcal{E}_c(\alpha, \gamma) \otimes \mathcal{E}_c(\gamma, \gamma) \exp[ip_1 \hat{\omega} \cdot \mathbf{R}_{\alpha'\gamma} (R_{\alpha'\gamma} - \mathbf{R}_{\alpha'\alpha})]. \tag{5.19}
\]

Equation (5.15) is essentially the transport equation derived by Watson, and it has a simple physical interpretation. The iterative solution of this equation can be written

\[
\mathcal{U}(\alpha; \hat{\omega}) = \sum_{n=0}^{\infty} \mathcal{U}_n(\alpha; \hat{\omega}),
\]
where

\[ U_n(\alpha; \hat{\omega}) = \int \frac{d\Omega}{4\pi} \frac{R_{\alpha'y}}{R_{\alpha'y}} \mathcal{G}_{c}(\alpha, r) \mathcal{G}_{c}(\alpha, \hat{r}) \]

and

\[ U_0(\alpha; \hat{\omega}) = \delta(\hat{\omega} - \hat{k}) U_c(a). \]

Hence the energy density at \( z_\alpha \) equals the coherent energy density \( (U_0) \) plus contributions due to waves multiply scattered by the electron density fluctuations (the \( U_n \)'s, \( n \geq 1 \)). Note that these latter contributions add together incoherently and that the wave nature of the process is manifested only through the structure of the scattering function \( m_1 \).

It can be seen from the above derivation that two factors make it possible to transform Eq. (5.7) to the approximate transport equation (5.15): first, the mean free path \( \ell \) is very much larger than \( R_c \) and \( k^{-1} \) (as we have assumed); second, the first-order intensity function \( K_1 \) [Eq. (5.8)] is localized within a range \( R_c \).

Consequently the Green functions can be approximated as in (5.13) and \( U(\alpha', \alpha) \) as in (5.14), from which the transport equation follows.

In extending the above approximation procedure to include higher-order terms of the intensity function \( K \), one encounters the difficulty that some of the terms are not manifestly localized [e.g., diagrams 8(g), (h), (i) ...]. As an example we consider diagram 8(g). Its contribution to \( U(\alpha) \), in first order, corresponds to the physical scattering process depicted in Fig. 10, and is given by the analytical expression

\[ \int d^2z_\beta \int d^2z_\lambda \int d^2z_\kappa \rho(\beta) \rho(\lambda) \rho(\kappa) g_2(\lambda, \kappa) g_2(\beta, \kappa) \]

\[ \mathcal{G}_{c}(\alpha, \beta) \mathcal{G}_{c}(\alpha, \lambda) \mathcal{G}_{c}(\lambda, \kappa) \mathcal{G}_{c}(\kappa, \beta) \] \[ \cdots \]

Thus the radiation intensity at the observation point \( z_\alpha \) is due to waves scattered from two different correlation clusters. As a consequence of the phase difference between the two waves, the major contribution to the integral (5.17) comes from the integration domain where the two clusters are within a distance of the order of a wavelength from each other. In this domain the Green function \( \mathcal{G}_{c}(\alpha, \beta) \) in (5.17) can be approximated as

\[ \mathcal{G}_{c}(\alpha, \beta) \approx \mathcal{G}_{c}(\alpha, \beta) \exp[-ip_{1\alpha} - p_{2\beta}], \]

so that the oscillatory phase factors associated with propagation from \( z_\beta \) to \( z_\alpha \) cancel out. The relative error committed in the above approximation is of the order of \( 1/\ell k \). Thus, to this order of accuracy the term in \( K \) corresponding to diagram 8(g) can be considered to be effectively localized in the sense that only the localized part contributes to the radiation energy.

Similar considerations can be extended to other higher-order terms of \( K \). Hence we arrive at the conclusion that, neglecting contributions with relative order \( 1/\ell k \), only the localized part of \( K \) contributes to the radiation energy. Physically this amounts to neglecting wave interference effects associated with coherent propagation between scatterings. To carry out the transformation of Eq. (5.7), we first rewrite it as (we set \( z_{\alpha'} = z_\alpha \)
Since only the localized part of $\mathbf{K}$ contributes to the integral, the four arguments of $\mathbf{K}$, $z_{\sigma'}$, $z_{\sigma}$, $z_{\lambda'}$, and $z_{\lambda}$, and their "center of mass" $z_{\gamma'}$ are separated from each other by a distance very much less than the mean free path $\ell$. Consequently, we can make the approximations as done previously.

\[ \mathbf{g}^s(\alpha, \sigma') \otimes \mathbf{G}_c(\alpha, \sigma) \approx \mathbf{g}^s(\alpha, \gamma) \otimes \mathbf{G}_c(\alpha, \gamma) \exp[\text{i} \mathbf{p}_c \cdot \mathbf{R}_{\sigma'}] \]

and

\[ \mathcal{U}(\lambda', \lambda) \approx \int \text{d}z' \mathcal{U}(\gamma; \hat{\omega}, \hat{\omega}') \exp[-\text{i} \mathbf{p}_c \cdot \mathbf{R}_{\lambda'}]. \tag{5.7'} \]

Using these approximations in Eq. (5.7'), we obtain the radiation transport equation in the form

\[ \mathcal{U}(\alpha; \hat{\omega}) = \mathcal{S}(\hat{\omega} - \hat{k}) \mathcal{U}_c(\alpha) + \int (-\hat{\omega}) \mathcal{E} \mathbf{G}_c(\alpha; \gamma) \mathcal{M}(\gamma; \hat{\omega}, \hat{\omega}') \mathcal{U}(\gamma; \hat{\omega}'), \tag{5.18} \]

where the scattering function $\mathcal{M}$ is defined as

\[ \mathcal{M}(\gamma; \hat{\omega}, \hat{\omega}') = \int \text{d}^3z_\sigma \text{d}^3z_{\sigma'} \text{d}^3z_\lambda \text{d}^3z_{\lambda'} \delta(z_{\gamma'} - \frac{1}{\mathbf{p}_c} (z_{\sigma'} + z_{\sigma} + z_{\lambda'} + z_{\lambda})) \times \exp[\text{i} \mathbf{p}_c \cdot \mathbf{R}_{\sigma'}] \mathbf{K}(\sigma'; \sigma, \lambda') \exp[-\text{i} \mathbf{p}_c \cdot \mathbf{R}_{\lambda'}]. \tag{5.19} \]

The function $\mathcal{M}$, of course, has the same tensorial character as $\mathbf{K}$ [Eq. (5.6)].

To put Eq. (5.18) in a more transparent form, we introduce the following unit polarization vectors for a transverse wave travelling along the direction $\hat{\omega}$:

\[ \hat{e}_0 (\hat{\omega}) = \frac{\hat{\omega} \times \hat{k}}{|\hat{\omega} \times \hat{k}|}. \tag{5.20} \]

and

\[ \hat{e}_1 (\hat{\omega}) = \hat{e}_0 (\hat{\omega}) \times \hat{\omega}. \tag{5.21} \]

In terms of these unit vectors, the coherent Green function (4.19) can be written as

\[ \mathcal{G}_c(\alpha, \beta) \approx \frac{\exp \left( \int \frac{\alpha_p e^{-\text{d}s}}{R_{\text{OB}}} \left[ \hat{e}_0 (\hat{R}_{\text{OB}}) \hat{e}_1 (\hat{R}_{\text{OB}}) + \hat{e}_2 (\hat{R}_{\text{OB}}) \hat{e}_2 (\hat{R}_{\text{OB}}) \right] \right)}{\int \text{d}\mathbf{R}_{\text{OB}}}. \tag{5.22} \]

The components of the energy densities are given by

\[ \mathcal{E}_i (\alpha; \hat{\omega}) = \hat{e}_i (\hat{\omega}) \cdot \mathcal{U}(\alpha; \hat{\omega}) \cdot \hat{e}_j (\hat{\omega}) \tag{5.23} \]

and

\[ \mathcal{E}_i (\alpha; \hat{k}) = \hat{e}_i (\hat{k}) \cdot \mathcal{U}_c (\alpha) \cdot \hat{e}_j (\hat{k}), \tag{5.24} \]

where $\hat{e}_i (\hat{\omega})$ are the polarization vectors of the incident plane wave.

In terms of these components, Eq. (5.18) can be rewritten as

\[ \mathcal{U}_{ij} (\alpha; \hat{\omega}) = \mathcal{S}(\hat{\omega} - \hat{k}) \mathcal{E}_i (\hat{k}) \mathcal{U}_c (\alpha) \mathcal{E}_j (\hat{k}) + \int (-\hat{\omega}) \mathcal{E} \mathbf{G}_c(\alpha; \gamma) \mathcal{M}(\gamma; \hat{\omega}, \hat{\omega}') \mathcal{U}(\gamma; \hat{\omega}'), \tag{5.25} \]

where $\mathcal{E}_i (\hat{\omega})$ are the polarization vectors of the incident plane wave.
where the components of the scattering function are [recall (5.6)]

\[ \langle ij | \mathcal{M}(r; \hat{\omega}_f, \hat{\omega}_s) | kl \rangle = \int d^3 z_\sigma \, d^3 z_\lambda \, \delta[z_\gamma - \frac{1}{k}(\hat{z}_\sigma + \hat{z}_\lambda)] \sum_{a,b=1} \hat{\xi}_a(\hat{\omega}) \cdot K_a(\sigma'; \sigma, \lambda', \lambda) \cdot \hat{\xi}_b(\hat{\omega}') \exp[ip_1(\hat{\omega} \cdot R_{\sigma \sigma'} - \hat{\omega}' \cdot R_{\lambda \lambda'})]. \] (5.26)

Keeping only the first term in the series for \( K \) [as in Eq. (5.8)] gives the same scattering function as in Ref. 1, that is,

\[ \langle ij | \mathcal{M}_1(r; \hat{\omega}_f, \hat{\omega}_s) | kl \rangle = \int d^3 z_\sigma \, d^3 z_\lambda \, \delta[z_\gamma - \frac{1}{k}(\hat{z}_\sigma + \hat{z}_\lambda)]. \]

\[ \chi \hat{\xi}_a(\hat{\omega}) \cdot K_a(\sigma', \sigma; \lambda', \lambda) \cdot \hat{\xi}_b(\hat{\omega}') \cdot F(\hat{\omega} \cdot R_{\sigma \sigma'} - \hat{\omega}' \cdot R_{\lambda \lambda'}). \] (5.27)

Consequently, \( \mathcal{M}_1 \) and \( U \) satisfy

\[ \langle ij | \mathcal{M}_1 | kl \rangle^* = \langle ji | \mathcal{M}_1 | kl \rangle \] (5.28)

and

\[ U_{ij}^* = U_{ji}. \] (5.29)

Relation (5.29) is consistent with the definition of \( U \) in (5.1).

The radiation flux \( I_{ij}(\alpha; \hat{\omega}) = \frac{c}{k} U_{ij}(\alpha; \hat{\omega}) \) satisfies the same Eq. (5.25). Differentiation along a path element parallel to \( \hat{\omega} \) gives the final form

\[ \hat{\omega} \cdot \nabla_{\alpha} I_{ij}(\alpha; \hat{\omega}) = -\frac{1}{k(\alpha)} I_{ij}(\alpha; \hat{\omega}) + \int d^3 \hat{\omega}' \sum_{k, \ell=1} \langle ij | \mathcal{M}(\alpha; \hat{\omega}, \hat{\omega}') | kl \rangle I_{kl}(\alpha; \hat{\omega}'). \] (5.30)

Equation (5.30) is the conventional radiation transport equation. The quantities in the equation have the usual meanings: \( \hat{\omega} \) is the "absorption coefficient" for the radiation flux, and \( \mathcal{M}(\alpha; \hat{\omega}, \hat{\omega}') \) describes the scattering of a pencil of radiation in the direction \( \hat{\omega}' \) to the direction \( \hat{\omega} \). Both effects are due to multiple scattering from the electron density fluctuations, and they are connected through energy conservation (see next section).

The radiation transport equation (5.30) can be written down intuitively by use of energy-conservation arguments. The derivation given above shows that it is an approximation to the exact wave equation. Here we briefly summarize the fundamental assumptions under which the transport equation is valid:

1. \( \Im n \ll \Re n \), where \( n \) is the refractive index. This assumption is equivalent to \( \Delta k \ll 1 \).
2. \(|\nabla \rho_n | < k\), where \(\rho_n\) is the mean electron density; and
3. \(\ell \gg R_o\).

Assumption (1) is necessary in order that we can use a refractive index to describe the coherent wave. Assumptions (2) and (3) enable us to use the eikonal approximation to describe coherent wave propagation between incoherent scatterings. The eikonal approximation is necessary if we are to obtain a transport equation of the conventional form. Finally, assumptions (1) and (3) allow us to transform Eq. (5.7) to the transport Eq. (5.30).

It should also be mentioned that if curved ray paths are used to describe coherent propagation, there will be an additional term of the form \(\sum_{k, \ell=1}^2 (i\mathcal{R}_{k,\ell} I_{k,\ell})\), describing the rotation of polarization vectors along the ray.

VI. THE OPTICAL THEOREM

As mentioned earlier, the depletion of the coherent wave is due to incoherent scatterings by the electron density fluctuations as described by the function \(E\). (If collisions have been included, there would be additional depletion.) Consideration of energy conservation then leads us to expect the following relation between the "absorption" and total scattering of radiation flux:

\[
\frac{1}{\delta(\alpha)} = \sum_{s=1}^2 \int d\hat{q} (s s|N(\alpha; \hat{q}, \hat{q}成立以来)|11). \tag{6.1}
\]

The final polarization index \(s\) is summed over because the total radiation flux is given by \(I_{11} + I_{22}\).

In this final section we prove the relation (6.1). Recalling the definitions of \(\delta\) and \(M\), we first rewrite (6.1) in the form to be proved:

\[
\frac{\hbar}{\mathcal{F}_1(\alpha)} \text{Im} \mathcal{F}_1(\alpha; p_1 \mathcal{P}) = \int d^3z_o d^3z_o d^3z_o d^3z_o \int d\hat{q} \sum_{s=1}^2 \sum_{a, b=1}^\infty \hat{e}_s(\hat{q}) \hat{e}_a(\hat{q}) \hat{e}_b(\hat{q}) \left[ \delta(\alpha) - \frac{1}{4} (\hat{e}_{\sigma, \sigma} + \hat{e}_{\sigma, \lambda} + \hat{e}_{\lambda, \lambda} + \hat{e}_{\lambda, \sigma}) \right] \hat{K}_s(\sigma,\sigma; \lambda,\lambda) \hat{K}_a(\sigma,\sigma; \lambda,\lambda) \hat{K}_b(\sigma,\sigma; \lambda,\lambda)
\]

\[
\times \exp[i p_1 (\hat{q} \cdot R_o \sigma - \hat{q} \cdot R_o \lambda)]. \tag{6.2}
\]

The basis for proving (6.2) is the optical theorem.
The relation between the effective propagation function $\mathcal{S}$ and the intensity function $\mathcal{I}$ is solely a consequence of a real scattering amplitude $f$ [Eq. (3.1)]. It can be derived from the basic multiple-scattering equation (3.7), Dyson's equation (5.4), and the Bethe-Salpeter equation (5.5). The derivation being somewhat lengthy, will not be given here. An alternative way to see the validity of (6.3) is to diagrammatically expand both sides and compare the results term by term.

The energy-conservation relation (6.2) can now be derived. Multiplying both sides in (6.3) by $\exp[-i\mathbf{p}_1 \cdot \mathbf{R}_\alpha', \alpha']$ and integrating, we have

$$
\int d^3z_y \left[ \delta(z_y - \frac{1}{2}(z_{\alpha'} + z_{\alpha})) \delta(z_y - \frac{1}{2}(z_{\alpha'} + z_{\alpha})) \right]
\chi \left[ \mathcal{S}(\alpha', \alpha) \otimes \frac{1}{2} \otimes \mathcal{S}^*(\alpha', \alpha) \right]
$$

or

$$
\int d^3z_y \left[ (1 - \frac{1}{2}) \mathcal{I}_1(p_1^2) + \frac{1}{2} \mathcal{I}_2(p_1^2) \right] \otimes \frac{1}{2}
$$

$$
- \int d^3z_y \left[ \frac{1}{2} \mathcal{I}_1(p_1^2) + \frac{1}{2} \mathcal{I}_2(p_1^2) \right]
$$

where on the left side we have integrated over $\frac{1}{2}(z_{\alpha'} + z_{\alpha})$, getting rid of the delta function, while on the right side we have changed variables: $d^3z_{\alpha'} \rightarrow d^2z_{\alpha'} d^2z_{\beta'}$. We now take away the integration over $z_y$, and take the dot products of both sides with $\hat{\mathcal{E}}_i(\mathbf{p}) \otimes \hat{\mathcal{E}}_3(\mathbf{p})$ [the $\hat{\mathcal{E}}$'s are defined in (5.21) and (5.22)]. This gives

$$
\begin{align*}
\int d^3z_y \chi \left[ \delta(z_y - \frac{1}{2}(z_{\alpha'} + z_{\alpha})) \delta(z_y - \frac{1}{2}(z_{\alpha'} + z_{\alpha})) \right]
\delta(z_y - \frac{1}{2}(z_{\alpha'} + z_{\alpha}))(g_c(\beta', \beta) \otimes \frac{1}{2} \otimes g_c^*(\beta', \beta))
\end{align*}
$$

$$
\left[ \mathcal{S}(\alpha', \alpha) \otimes \frac{1}{2} \otimes \mathcal{S}^*(\alpha', \alpha) \right]
$$

or

$$
\left[ \mathcal{S}(\alpha', \alpha) \otimes \frac{1}{2} \otimes \mathcal{S}^*(\alpha', \alpha) \right]
$$

Equation (6.5) is in the form

$$
A \otimes B = C \otimes D + C' \otimes D',
$$

from which it follows that

$$
A \cdot B = C \cdot D + C' \cdot D'.
$$
Hence it follows from (6.5) that

\[ \varepsilon_{ij} \left( \frac{2}{p_1} \text{Im } s'_{1}(p_1^2) \right) = \int d^2z_{\alpha'} d^2z_{\beta'} d^2z_{\alpha} d^2z_{\beta} \]

\[ \sum_{a,b} \left( g^{ab} [K_{a} \cdot \hat{a}_j] [K_{b} \cdot \hat{a}_j] - [K_{a} \cdot \hat{a}_j]^* G_{c} [K_{b} \cdot \hat{a}_j] \right) \]

(6.6)

Since \( G_{c} \) is a symmetric tensor, the expression in braces can be written as

\[ \left( [K_{a} \cdot \hat{a}_j] G_{c} [K_{b} \cdot \hat{a}_j] - [K_{a} \cdot \hat{a}_j]^* G_{c} [K_{b} \cdot \hat{a}_j] \right) \]

(6.7)

When \( R_{B'} = R_{c} \) or \( k^{-1} \), it is straightforward to derive the relation

\[ Q_{c}(\beta', \beta) - Q_{c}^{*}(\beta', \beta') \equiv 2ip_{1} \int \frac{d\hat{q}'}{4\pi} \exp(ip_{1} \cdot \hat{q} \cdot \hat{B} \cdot \beta) \]

\[ \sum_{s=1}^{2} \sum_{a,b} \hat{e}_{s}(\hat{q}') \cdot \hat{r}_{s}(\hat{q}') \]

(6.8)

where we have approximated \( p_{e} \) by \( p_{1} \) with relative error \( k^{-1/2} \) or \( R_{c}/d \). Now to the same order of approximation only the localized part of \( \hat{K} \) (i.e., for \( z_{\alpha'} \), \( z_{\beta'} \), \( z_{B} \) all within a range of the order of \( R_{c} \) or \( k^{-1} \) from each other) contributes to the integral on the right side of (6.6). Hence we may use (6.8) to rewrite (6.7) as

\[ \left\{ \frac{2ip_{1}}{4\pi} \int d\hat{q}' \exp(ip_{1} \cdot \hat{q} \cdot \hat{B}_{c} \cdot \beta) \sum_{s=1}^{2} \hat{e}_{s}(\hat{q}') \cdot \hat{r}_{s}(\hat{q}') \right\} \]

(6.9)

Replacing the expression in curly brackets on the right side of (6.6) by the expression (6.9), we finally have the relation

\[ \varepsilon_{ij} \left( \frac{2}{p_1} \text{Im } s'_{1}(p_1^2) \right) = \int d^2z_{\alpha'} d^2z_{\beta'} d^2z_{\alpha} d^2z_{\beta} \]

\[ \sum_{a,b} \sum_{s=1}^{2} \hat{e}_{s}(\hat{q}') \cdot \hat{r}_{s}(\hat{q}') \]

(6.10)

For the special case \( i = j \), this coincides with the energy-conservation relation (6.2).
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FOOTNOTES AND REFERENCES

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6. Note that we neglect the so-called "strictly incoherent" scatterings, i.e., rescatterings by the same electrons. This is permissible if, as we shall assume, there are a large number of electrons in a correlation cluster, i.e., \( R_c^3 \gg 1 \).
7. The transport-equation approximation to the scalar wave equation was studied by L. L. Foldy, Phys. Rev. 65, 107 (1945). Foldy considered only statistically uncorrelated scatterers. The generalization to correlated scatterers was made by Y. N. Barabanenko and V. M. Finkel'berg, Sov. Phys.-JETP 26, 587 (1968), but only for the limiting cases \( kR_c \gg 1 \) and \( kR_c \ll 1 \).
8. See Ref. 1, Sec. 5. Watson derived the transport equation by a seemingly different procedure. He rewrote the basic multiple scattering equations (3.4) and (3.5) in terms of coherent waves and coherent Green functions. In calculating the averaged power flux, only "cross-correlated" parts were kept. In actuality this is equivalent to the procedure used here.


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FIGURE CAPTIONS

Fig. 1. Diagrammatic representations.

Fig. 2. Infinite series expression for $G_{\infty}$.

Fig. 3. Definition of the effective propagation function $S$.

Fig. 4. Integral equation for $G_{\infty}$.

Fig. 5. First order approximation for $G_{\infty}$.

Fig. 6. Example of a double-lined diagram.

Fig. 7. Infinite series expression for $W(\alpha',\alpha; \beta',\beta)$.

Fig. 8. Definition of the intensity function $K$.

Fig. 9. Integral equation for $\tilde{W}$.

Fig. 10. Physical scattering process corresponding to expression (5.17).
\[ G^{(1)}_{\alpha \beta}(Z_{\alpha}, Z_{\beta}) \]

\[ G^{(2)}_{\alpha \beta}(Z_{\alpha}, Z_{\beta}) \]

\[ f^2 \rho(Z_{\sigma}) \rho(Z_{\lambda}) g_1(Z_{\sigma}, Z_{\lambda}) \]

\[ f^3 \rho(Z_{\sigma}) \rho(Z_{\lambda}) \rho(Z_{\beta}) g_3(Z_{\sigma}, Z_{\lambda}, Z_{\beta}) \]

\[ \ldots \]

Fig. 1.
\[
\begin{align*}
\tilde{z} \sigma & \leftrightarrow S(\tilde{z} \sigma, \tilde{z} \lambda) \\
\tilde{z} \lambda & = (a) + (b) + (c) + (d) + (e) + (f) + \cdots
\end{align*}
\]

Fig. 3.

\[
\begin{align*}
\tilde{z} \sigma & \leftrightarrow S(\tilde{z} \sigma, \tilde{z} \lambda) \\
\tilde{z} \lambda & = (a) + (b) + (c) + (d) + (e) + (f) + \cdots
\end{align*}
\]

Fig. 4.
Fig. 5.

Fig. 6.
Fig. 7.

\[ \omega (a', \alpha ; \beta', \beta ) \sim \]

\[ \omega (a, \alpha ; \beta, \beta') \]

\[ = \]

\[ (a) + (b) + (c)
\]

\[ + (d) + (e) + (f)
\]

\[ + (g) + (h) + (i)
\]

\[ + \ldots \]

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Fig. 8.

\[ K (\sigma', \sigma : \lambda', \lambda) \sim \]

\[ = \]

\[ (a) + (b) + (c) + (d) + (f)
\]

\[ + (f) + (g) + (h) + (i)
\]

\[ + \ldots \]

\[ XBL714-3353 \]
Fig. 9.

Fig. 10.
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