Title
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Journal
Journal of Hydrology, 389(1-2)

ISSN
0022-1694

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Publication Date
2010-07-01

DOI
10.1016/j.jhydrol.2010.05.044

Journal of Hydrology 389 (2010) 177–185

Contents lists available at ScienceDirect

Journal of Hydrology

journal homepage: www.elsevier.com/locate/jhydrol

Manning's equation and two-dimensional flow analogs

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ARTICLE INFO

Article history:
Received 10 November 2009
Received in revised form 26 March 2010
Accepted 29 May 2010

This manuscript was handled by Konstantine P. Georgakakos, Editor in Chief, with the assistance of Kieran M. O'Conner, Associate Editor

Keywords:
Two-dimensional flow
Manning's equation
Mathematical modeling

SUMMARY

Two-dimensional (2D) flow models based on the well-known governing 2D flow equations are applied to floodplain analysis purposes. These 2D models numerically solve the governing flow equations simultaneously or explicitly on a discretization of the floodplain using grid tiles or similar tile cell geometry, called "elements". By use of automated information systems such as digital terrain modeling, digital elevation models, and GIS, large-scale topographic floodplain maps can be readily discretized into thousands of elements that densely cover the floodplain in an edge-to-edge form. However, the assumed principal flow directions of the flow model analog, as applied across an array of elements, typically do not align with the floodplain flow streamlines. This paper examines the mathematical underpinnings of a four-direction flow analog using an array of square elements with respect to floodplain flow streamlines that are not in alignment with the analog's principal flow directions. It is determined that application of Manning's equation to estimate the friction slope terms of the governing flow equations, in directions that are not coincident with the flow streamlines, may introduce a bias in modeling results, in the form of slight underestimation of flow depths. It is also determined that the maximum theoretical bias, occurs when a single square element is rotated by about 13°, and not 45° as would be intuitively thought. The bias as a function of rotation angle for an array of square elements follows approximately the bias for a single square element. For both the theoretical single square element and an array of square elements, the bias as a function of alignment angle follows a relatively constant value from about 5° to about 85°, centered at about 45°. This bias was first noted about a decade prior to the present paper, and the magnitude of this bias was estimated then to be about 20% at about 10° misalignment. An adjustment of Manning's *n* is investigated based on a considered steady state uniform flow problem, but the magnitude of the adjustment (about 20%) is on the order of the magnitude of the accepted ranges of friction factors. For usual cases where random streamline trajectory variability within the floodplain flow is greater than a few degrees from perfect alignment, the apparent bias appears to be implicitly included in the Manning's *n* values. It can be concluded that the array of square elements may be applied over the digital terrain model without respect to topographic flow directions.

1. Introduction

Two-dimensional grid type mathematical models are increasingly used in civil engineering and planning for the analysis of two-dimensional unsteady flow effects. The diffusion formulation of the governing flow equations is readily applied to such models. The earliest analysis and use of the diffusion formulation of the governing flow equations is discussed by a number of researchers including Xanthopoulos and Koutitas (1976), Ponce et al. (1978), Akan and Yen (1981), Hromadka and Lai (1985), and Hromadka et al. (1987). Perhaps the earliest such general use two-dimensional flow model is the public domain Diffusion Hydrodynamic Model developed for the US Geological Survey (USGS DHM, Hromadka and Yen (1987) among other publications by those authors) which has been used for a variety of two-dimensional unsteady flow studies including coupled two-dimensional overland flow with one-dimensional channel flow problems where channel flow interfaces as both a source or sink to the overland flow grid system depending on current hydraulic conditions being modeled. Subsequently, proprietary models have been developed that "implement[s] the Diffusion Hydrodynamic Model (DHM) created by Hromadka and Yen" (see Bertolo and Wieczorek (2005)

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doi:10.1016/j.jhydrol.2010.05.044

among others). Hromadka and Yen (1987) showed that the diffusion formulation of the flow equations adequately portrays flows with Froude numbers up to 4. Another two-dimensional diffusion model developed by G.L. Guymon for applications in alluvial fan flow modeling in Maricopa County, Arizona, USA applies a probabilistic extension to USGS DHM. Lal (2005), for example, stated, "These studies showed that diffusion flow models can be used successfully to simulate a variety of natural flow conditions". The diffusive wave approximation has been applied to overland and channel flows for a looped channel system (Luo, 2007). The diffusive wave approximation has also been used to model extreme flood events, where channel and overbank flows are routed, and the principal variable is Manning's n (Moussa and Bocquillon, 2008). A thorough investigation of "reduced complexity codes", including the diffusion formulation, and comprehensive literature review has been done by Hunter et al. (2007). Because of increasing use of the diffusion formulation of the flow equations and its application to grid type models of the problem overland flow domains, for example, US Army Corps of Engineers gridded surface/subsurface hydrologic analysis model GSSHA (Ogden et al., 2003), further research to improve computational efficiency and accuracy will continue to be needed.

GIS programs can be used to develop large databases of topographic mapping discretized into the elements used in such coupled 1D–2D models. The ease of computer graphics and GIS enable such 2D flow analogs to be readily applied to large 2D flow regions. For example, Fig. 1 from Jordan (2003) illustrates a USGS DHM model containing more than 2000 square grid elements ("elements"). Some flow models use regular polygon elements such a triangles, squares, hexagons, or octagons to cover the 2D problem domain, and other models use irregularly shaped polygonal elements. Wilson et al. (2007) report a model with 1.7 million

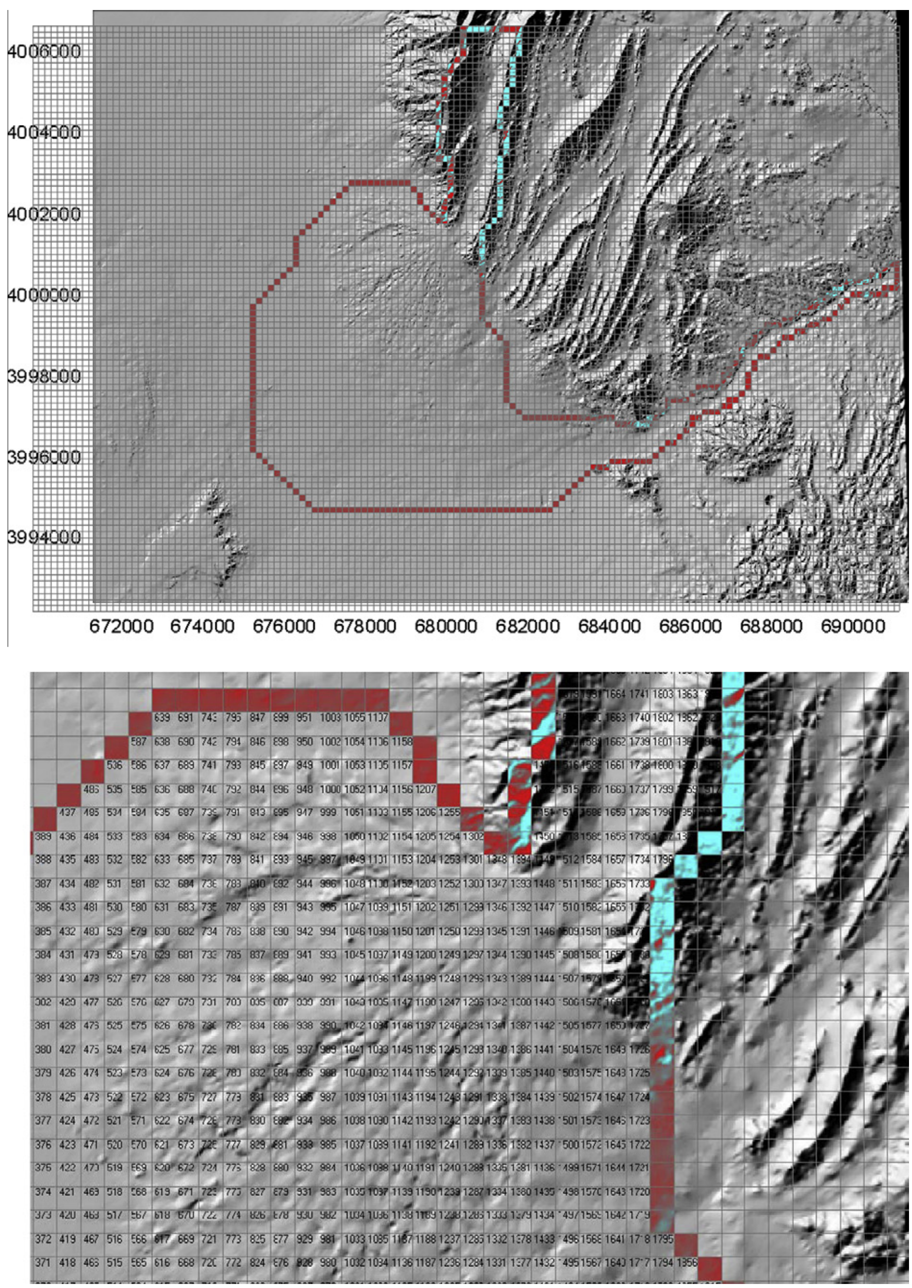

Fig. 1. USGS DHM surface model developed from USGS DEM data (152 m (500 ft.) grid element sides), with detail over alluvial fan.

square elements to investigate large-scale seasonal inundation of Amazon wetlands. It has been previously shown that an array of square elements (e.g. four-direction flow in the Cartesian coordinate system as used in USGS DHM) which are aligned with flow streamlines provides an unbiased estimate of steady state uniform flow (SSUF) depth, whereas use of three or greater than four flow directions per element does not. The bias in computations is seen as a loss in accuracy of estimates of flow depth associated with arrays of elements of other shapes (e.g. triangles, octagons). The mathematical conclusions were developed for an arbitrary number n of flow path directions, all equally spaced with angle $2\pi/n$, and included the theoretical case as n approaches infinity (Hromadka et al., 2007). In the current paper, only four-direction flow is investigated.

In the current paper, some issues are considered regarding the arbitrary placement and subsequent alignment of an array of square elements with respect to the underlying two-dimensional flow streamlines in the flow regime. For example, the computer program USGS DHM documentation (Hromadka and Yen, 1987) shows several application problems where elements are laid out by hand on topographic maps conforming to the anticipated streamline directions, such that axis orientations of individual elements are in alignment with anticipated flow streamlines. Use of GIS, however, for larger investigations containing thousands of elements, typically results in problem domain grid developments that either do not consider streamline directions, or are only approximately oriented with respect to topographic flow directions. Therefore, the flow analog used in USGS DHM, for example, is not necessarily being applied in perfect alignment with the streamlines, and therefore the application of Manning's equation to determine friction slope in the x- and y-directions (S_{fx} and S_{fy}) is not necessarily exact. It can be demonstrated that arbitrary alignment of elements with respect to flow streamlines may result in slightly different computational results unless attention is paid to such effects by modifying the Manning's friction factor as used in the diffusion formulation. The magnitude of this difference is small. This principle was first noted by Horritt and Bates (2001) a decade prior to the present paper. It was recognized that flow vectors differed by about 20% from theory, and more importantly, this effect reached a maximum at about 10° between alignment of free surface slope and alignment of one of the grid axes. The present paper provides a theoretical explanation of what was first recognized in practice.

By equating the diffusion flow equations to the standard energy equation as applied to steady state uniform flow (SSUF) of the flow regime set at various trajectory angles with respect to element alignment axis, the ratio of Manning's n at any angle to Manning's n for SSUF can be calculated and the magnitude of the difference from unity can be estimated. This friction factor ratio is a function of element alignment with the flow regime angle. This friction factor adjustment compensates for the effect of the modeling grid axis not being aligned with the flow regime. From the developed equation, it is seen that the greatest change of ratio with respect to angle occurs within very small angles of rotation from 0° to about 5°, and from about 85° to 90°. For greater angles of rotation (between about 5° and about 85° symmetrical about 45°), the ratio remains close to a constant value. This latter result may be significant when contemplating how the Manning's friction factor is estimated in the field. That is, field measurements of flow regimes typically involve flows where streamlines are not in parallel alignment and, therefore, would already be in the range of angles from 5° to 85° under the above computational model. When streamlines are parallel, the ratio has a value of 1.0. Otherwise, when streamlines are not parallel, the computational model predicts a ratio of about 1.2. However, should the friction factor be based upon field measurements where streamlines are very unlikely to be parallel, then such effects may already be included in the measure of the friction factor itself. In other words, field calibration makes the theoretical ratios developed in the computational model redundant. The implication for automated gridding of square elements with four-direction flow is that the array of square elements may be applied over the digital terrain model without respect to topographic flow directions.

In the following, the magnitude of bias for the conditions of SSUF where the flow analog principal flow directions are at an angle θ with respect to the flow streamlines is investigated, the ratio of Manning's n at any angle to Manning's n for SSUF is developed.

2. Mathematical development

To develop a theoretical analysis that can be verified by traditional calculation methods, the special flow condition of steady state, uniform turbulent flow (SSUF) is assumed throughout the 2D flow regime, R. Let Ω be a smaller region in R such that flow streamlines are all parallel in Ω such that the flow in Ω can be analyzed as one-dimensional flow in Ω even though application of a 2D flow analog on R would necessitate the application of the 2D analog in Ω.

The problem for analysis is the application of the four-direction flow analog, with square elements used in USGS DHM, to this steady-state, uniform 1D flow in Ω, with constant topographic slope, S_o, where the streamlines are at an angle θ with respect to the principal flow directions used in the four-direction flow analog. USGS DHM is used in this paper as a case study for analysis purposes because the model is not proprietary, boundary conditions may be easily established, and continuity may be easily verified.

The well-known partial differential equations (PDEs) that describe incompressible fluid flow in two dimensions, with all vertical components assumed invariant at a point (x, y), are given by one equation of mass continuity:

$$\frac{\delta q_x}{\delta x} + \frac{\delta q_y}{\delta y} + \frac{\delta H}{\delta t} = 0 \tag{1}$$

And two equations of motion:

$$\frac{\partial q_x}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q_x^2}{h}\right) + \frac{\partial}{\partial y}\left(\frac{q_x q_y}{h}\right) + gh\left(s_{fx} + \frac{\partial H}{\partial x}\right) = 0 \tag{2}$$

$$\frac{\partial q_x}{\partial t} + \frac{\partial}{\partial y}\left(\frac{q_y^2}{h}\right) + \frac{\partial}{\partial x}\left(\frac{q_x q_y}{h}\right) + gh\left(s_{fy} + \frac{\partial H}{\partial y}\right) = 0 \tag{3}$$

where (x, y) are the Cartesian coordinates; t is time; g is the gravitational acceleration; q_x and q_y are unit flows in the x and y Cartesian coordinate directions; S_{fx} and S_{fy} are friction slopes in the x, y directions; h is flow depth; and H is the water surface elevation. These three PDEs form the underpinning for computer models of two-dimensional (2D) flow and also computer models of one-dimensional (1D) channel flow networks coupled with 2D topographic flow models. For example, see the US Geological Survey computer program "Diffusion Hydrodynamic Model" (USGS DHM) by Hromadka and Yen (1987); also see Brater et al. (1996), Chapter 14, p. 33; and Maidment (1993), Chapter 21, pp. 26–27.

At issue is the 2D flow analog used and the application of Manning's equation in computing information that is subsequently used in the 2D flow analog when flow streamlines are not aligned with analysis principal flow directions. The governing flow Eqs. (1)–(3) involve the friction slope terms S_{fx} and S_{fy} which are typically computed by application of Manning's equation for an element aligned with principal flow directions. However, as will be shown below, additional mathematical considerations may be needed when arbitrarily using Manning's equation in a 2D flow analog for an element not so aligned.

For the SSUF problem considered, q_x, q_y, and h are all constant in Ω, and the 2D flow equations simplify to reduce the to the system of PDEs:

$$\left(S_{fx}+\frac{\partial H}{\partial x}\right)=0 \tag{4}$$

$$\left(S_{fy}+\frac{\partial H}{\partial y}\right)=0 \tag{5}$$

where $\frac{\partial H}{\partial x}$ and $\frac{\partial H}{\partial y}$ are constants in Ω, and where

$$\theta=\tan^{-1}\left(\frac{q_y}{q_x}\right) \tag{6}$$

Therefore, for the subject SSUF problem, the relevant friction slope terms are given by the partial derivatives,

$$S_{fx}=-\frac{\partial H}{\partial x} \tag{7}$$

$$S_{fy}=-\frac{\partial H}{\partial y} \tag{8}$$

which indicates that the friction slopes in the x, y directions are equal to the slope of the water surface in the same directions. A modeling approach typically used in 2D models is to extend the above results into a generalization,

$$S_{fz}=-\frac{\partial H}{\partial z};\quad z=x,y \tag{9}$$

for arbitrary direction z, and then substitute into Manning's equation (wherein shallow flow in a wide rectangular channel is assumed and all of the resistance is due to bottom friction, neglecting the side boundary layer effects) to obtain a unit flow rate, q_z,

$$q_z=\frac{1}{n}y^{5/3}s_z^{1/2};\quad z=x,y \tag{10}$$

where n is the Manning's friction factor; and y is the flow depth. However, as will be shown below, direct use of Eq. (10) may introduce a bias in computational results. It is noted that for the considered SSUF problem, the USGS DHM formulation solves the governing system of PDE of Eqs. (4)–(8). It has been noted that the governing system of equations is solved exactly only if time steps are sufficiently short to avoid computational instability (Hunter et al., 2005). USGS DHM employs a time-stepping algorithm that reduces or expands the time step size depending on hydraulic conditions anywhere in the model. To avoid computational instability, the time step may be reduced at any locality while the time step at other locations in the model may remain unchanged or expand.

A typical 2D modeling flow analog is to develop networks of connections between geometric elements, and then use q_s to compute flow rates that apply during a small model time step, Δt.

For the considered four-direction flow analog, flow directions are in the x, y directions only, whereas in an unaligned flow, streamlines are at an angle θ with the positive x-axis. For 2D grid size W, flow velocities in the projected x- and y-directions are obtained from the streamline flow velocity, v_s, by

$$\left.\begin{aligned}v_y&=v_s\sin\theta\\ v_x&=v_s\cos\theta\\ v_s^2&=v_x^2+v_y^2\end{aligned}\right\} \tag{11}$$

With flow depth a constant in Ω, under the considered SSUF problem assumptions,

$$h^2v_s^2=h_2v_x^2+h^2v_y^2 \tag{12}$$

or

$$q_s^2=q_x^2+q_y^2 \tag{13}$$

where q_s is the unit flow along the streamlines that are parallel in the considered SSUF problem.

From the flow assumptions,

$$\left.\begin{aligned}hv_x&=q_x=q_s\cos\theta\\ hv_y&=q_y=q_s\sin\theta\end{aligned}\right\} \tag{14}$$

typically, for the considered SSUF problem, modeled unit flows in the x- and y-directions are approximated by a similar application of Manning's equation, where the gradient of the water surface along same trajectory matches the gradient of the topography along the trajectory,

$$\left.\begin{aligned}q_x&=\frac{1}{n}h_4^{5/3}s_{ox}^{1/2}\\ q_y&=\frac{1}{n}h_4^{5/3}s_{oy}^{1/2}\end{aligned}\right\} \tag{15}$$

where h_4 is the resulting four-direction flow analog flow depth by use of the above application of Manning's equation, and where h is constant in Ω given the considered SSUF problem assumptions; and the topographic slopes in the x, y directions are S_{ox}, S_{oy} where

$$\left.\begin{aligned}S_{ox}&=s_o\cos\theta\\ S_{oy}&=s_o\sin\theta\end{aligned}\right\} \tag{16}$$

Therefore, combining Eqs. (15) and (16), we have the four-direction flow analog approximations for the subject problem assumptions,

$$\left.\begin{aligned}q_x&=\alpha h_4^{5/3}\cos^{1/2}\theta\\ q_y&=\alpha h_4^{5/3}\sin^{1/2}\theta\end{aligned}\right\} \tag{17}$$

where

$$\alpha=1\sqrt{s_o}/n \tag{18}$$

The flow width projection of the grid, W^*, is given by

$$W^*=W(\sin\theta+\cos\theta) \tag{19}$$

And unit flow across W^* with the streamlines is q_s, where

$$q_s=\alpha y_n^{5/3} \tag{20}$$

where y_n is the normal depth from Manning's equation.

Setting inflow to the grid equal to its flow analog outflow gives

$$q_sW^*=W(q_x+q_y) \tag{21}$$

or,

$$\alpha y_n^{5/3}W(\sin\theta+\cos\theta)=\alpha h_4^{5/3}W(\cos^{1/2}\theta+\sin^{1/2}\theta) \tag{22}$$

which reduces to

$$h_4^{5/3}=\left(\frac{\sin\theta+\cos\theta}{\cos^{1/2}\theta+\sin^{1/2}\theta}\right)y_n^{5/3} \tag{23}$$

or

$$h_4=\left(\frac{\sin\theta+\cos\theta}{\cos^{1/2}\theta+\sin^{1/2}\theta}\right)^{3/5}y_n \tag{24}$$

In Eq. (24), θ = 0° or $\theta=\pi/2$ radians places the streamlines in alignment with the principal flow directions of the four-direction flow analog, and also in alignment with the x and y axes, and Eq. (24) gives the solution,

$$h_4=y_n;\quad \theta=0,\quad \pi/2 \tag{25}$$

For values of θ = 0° and 90°, the aligned case, $h_4=y_n$, and the computed depth equals SSUF normal depth.

For other values of θ, the grid principal flow paths are not in alignment, and $h_4 < y_n$. Use of Manning's equation in Eq. (15) requires a factor, β, to make the computed depth h_4 equal to normal depth, y_n.

From the above equations, the factor, β, is given by,

$$\beta = \beta(\theta) = \left(\frac{\sin\theta + \cos\theta}{\cos^{1/2}\theta + \sin^{1/2}\theta}\right)^{-3/5} \tag{26}$$

where again,

$$\theta = \tan^{-1}\left(\frac{q_y}{q_x}\right)$$

To develop the factor, β, for any angle, the following trigonometric relationships apply:

$$\left.\begin{aligned} \sin\theta &= \frac{q_y}{\eta} \\ \cos\theta &= \frac{q_x}{\eta} \\ \eta &= (q_x^2 + q_y^2)^{1/2} \end{aligned}\right\} \tag{27}$$

Let r be defined by,

$$r = q_y/q_x, \quad \text{for} \quad q_x \pm 0 \tag{28}$$

Substituting Eq. (27) into Eq. (26) gives,

$$\beta(\theta) = \left(\frac{(q_y + q_x)\eta}{(\sqrt{q_x} + \sqrt{q_y})/\sqrt{\eta}}\right)^{-3/5} \tag{29}$$

or, after reducing,

$$\beta(\theta) = \left[\frac{(1 + r)}{(1 + \sqrt{r})(1 + r^2)^{1/4}}\right]^{-3/5} \tag{30}$$

Note that as $\theta \to \pi/2$, $r \to \infty$, and $\beta \to 1$. Also, at $\theta = 0$, $r = 0$, and $\beta = 1$. At $\theta = \pi/4$, which is the maximum angle out of alignment for the four-direction flow analog, $q_x = q_y$ and $r = 1$, giving $\beta = 2^{3/20}$ or approximately, $\beta = 1.11$.

Therefore, the factor, β, for any angle, can be expressed as a ratio of normal depth to computed depth

$$\beta(\theta) = y_n/h_4 \tag{31}$$

for θ values between θ and $\pi/2$. Because q_x and q_y are known by the flow analog application, Eq. (31) is readily applied.

3. Extension of Manning's equation

From the previous section, use of a similar application of Manning's equation to flow vectors that are not in alignment with the considered SSUF problem streamlines may introduce a bias in the estimation of hydraulic properties. In this section, the identified possible bias is addressed by redefining the application of the flow vector friction factor. For the considered SSUF problem, equating inflow into the grid to grid outflow by the four-direction flow analog gives,

$$\frac{1}{n} y_n^{5/3} s_o^{1/2} W(cos\theta + \sin\theta) = \frac{1}{\gamma n} y_n^{5/3} W(s_{oy}^{1/2} + s_{ox}^{1/2}) \tag{32}$$

where γ is a factor applied to Manning's n value as applied in the four-direction flow analog such that $h_4 = y_n$.

From Eq. (16) and combining with Eq. (32) gives γ as a function of angle θ and,

$$\gamma(\theta) = \frac{\sqrt{\cos\theta} + \sqrt{\sin\theta}}{\cos\theta + \sin\theta} \tag{33}$$

A plot of $\gamma(\theta)$ is shown in Fig. 2. From Fig. 2, the average value of $\gamma(\theta)$ taken at 1° increments from 0° to 90° is slightly greater than 1.19. The average value of $\gamma(\theta)$ taken at 1° increments from 5° to 85° is slightly greater than 1.20. That is, there is little variation in $\gamma(\theta)$ for almost all θ, and $\gamma(\theta) = 1.0$ only for $\theta = 0°$ and $\theta = 90°$. The value of $\gamma(\theta)$ at 45° is exactly $2^{1/4}$, or 1.189.

Combining Eqs. (32) and (33), the combination of $\gamma(\theta)$ and Manning's n (for the streamline direction) gives $N(\theta)$ where

$$N(\theta) = n\gamma(\theta) \tag{34}$$

where, approximately,

$$N(\theta) = \begin{cases} 1.2;\ 85 > \theta > 5^\circ \\ 1.1;\ 0 < \theta < 5^\circ \text{ or } 85 < \theta < 90^\circ \\ 1.0;\ \theta = 0^\circ \end{cases} \tag{35}$$

4. Application problem

For the considered SSUF problem, the mathematical (diffusion) formulation used in USGS DHM simplifies to Eqs. (7) and (8) as does a fully dynamic formulation. Therefore, both the USGS DHM flow analog that is based on the diffusion formulation (Hromadka and Yen, 1987), rather than the fully dynamic equation set, is equally relevant in solving the considered SSUF application problem herein. For other applications where there is a departure from SSUF, it has been shown that the diffusion formulation used in DHM produces very nearly the same results as a fully dynamic formulation (Hromadka and Yen, 1987) for Froude numbers less than about 4. This is consistent with Ponce et al. (1978), who developed applicability criteria for kinematic and diffusion models. Using the SSUF flow conditions described below with Ponce Eq. (17), the initial flow ramp of 2 h, followed by steady flow of 10 h meets the applicability criterion.

In constructing multi-element four-direction flow analog arrays to model SSUF with USGS DHM, it was found that a base SSUF flow field with 400 elements, each 30.5 m (100 ft.) wide, was sufficient to demonstrate the theory. The objective was to achieve a shallow uniform subcritical flow about 1 ft. deep. Theoretical model normal depth was 30.24 cm (0.992 ft.). A model in perfect alignment with the flow field had a constant topographic slope of 0.0016; discharge, q, of 0.093 m^3/s/m (unit discharge, q, of 1 cfs/ft.); and Manning's n of 0.050. The modeled flow was bounded at the upstream end by 20 inflow boundary elements with q sufficient to sustain normal depth of about 0.3 m (1 ft.) extending some distance downstream. The modeled flow was bounded at the downstream end by critical outflow boundary elements. The flow profile is described as the subcritical drawdown curve, M2 in Chow §9–3 and 9–4 (1959). As modeled, the Froude number at the upstream end of the model was about 0.31. Fig. 3 illustrates the aligned model.

Models not in alignment with SSUF consisted of the same 400 element array rotated about the lower right corner so that the slope measured along the angle of alignment remained at 0.00116. Flow paths were bounded at the left and right sides by elements with base elevations raised above flow depth. The rows of inflow elements upstream and outflow elements downstream of the modeled flow were truncated at the left and right boundary elements. Rotation angles were chosen at integral ratios of boundary elements, e.g. $1h{:}1v$ was $\tan^{-1}(1/1)$ or 45°; $1h{:}2v$ was $\tan^{-1}(1/2) = 26.6$ or ~27°; $1h{:}3v$ was $\tan^{-1}(1/3) = 18.4$ or ~18°; $1h{:}4v$ was $\tan^{-1}(1/4) = 14.0$ or ~14°; $1h{:}5v$ was $\tan^{-1}(1/5) = 11.3$ or ~11°; and $1h{:}10v$ was $\tan^{-1}(1/10) = 5.7$ or ~6°.

Upstream boundary elements received a unit flow discharge of about 1 cfs/ft. based on the width of the flow path measured between the innermost dimensions of the flow boundaries. Figs. 4 and 5 illustrate typical models for 14° and 27° rotation respectively.

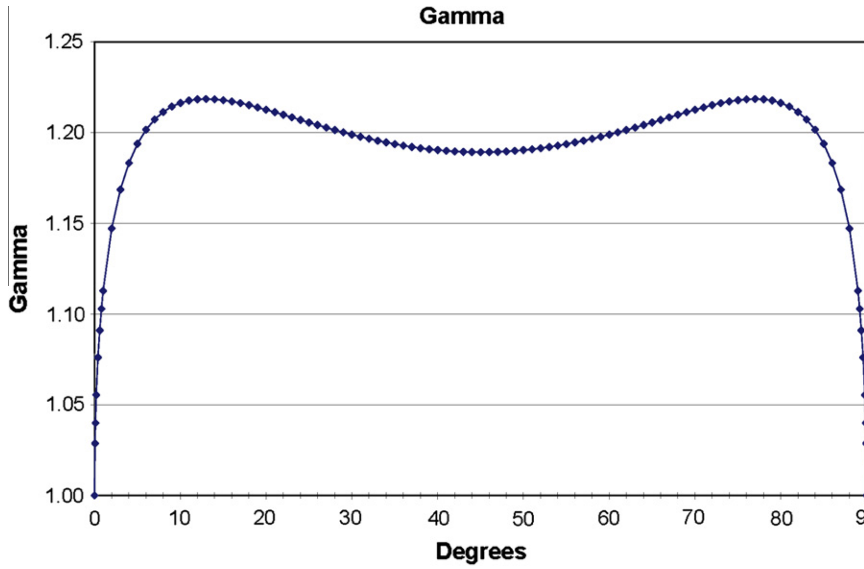

Fig. 2. Plot of $\gamma(\theta) = [(\sqrt{\cos\theta}) + (\sqrt{\sin\theta})]/(\cos\theta + \sin\theta)$.

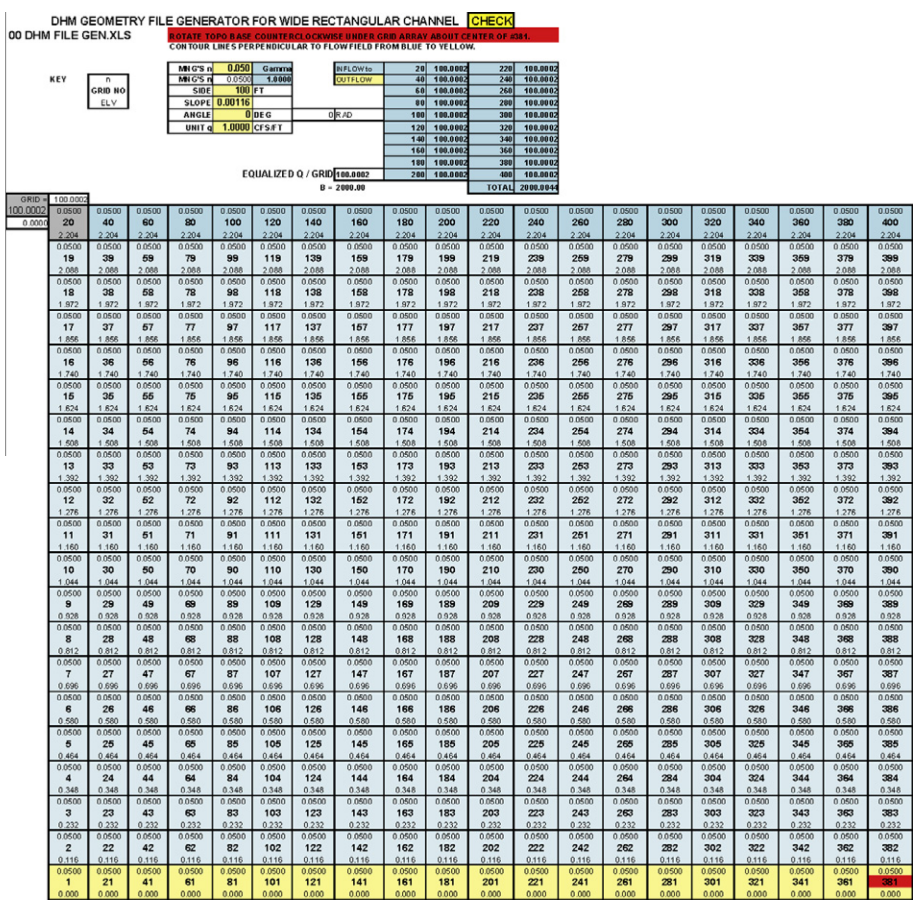

Fig. 3. Aligned model.

Each of the rotated models had base topography contour lines perpendicular to the northwest to southwest flow directions. The base contour lines were at perfect right angles with respect to flow streamlines, but in varying degrees of rotation with respect to element orientation.

Continuity for all models was verified by comparing total outflow over all the outflow elements with total inflow, at hour 12 of the SSUF modeling period. Outflow discharges matched inflow discharges within 0.01%.

Flow uniformity was tested and achieved by analyzing USGS DHM output data for velocities at each element, focusing on the central elements used for flow depth analysis. USGS DHM output includes flow velocities in the four Cartesian coordinate directions, N, E, S, and W. For steady flow, averages of N and S velocities

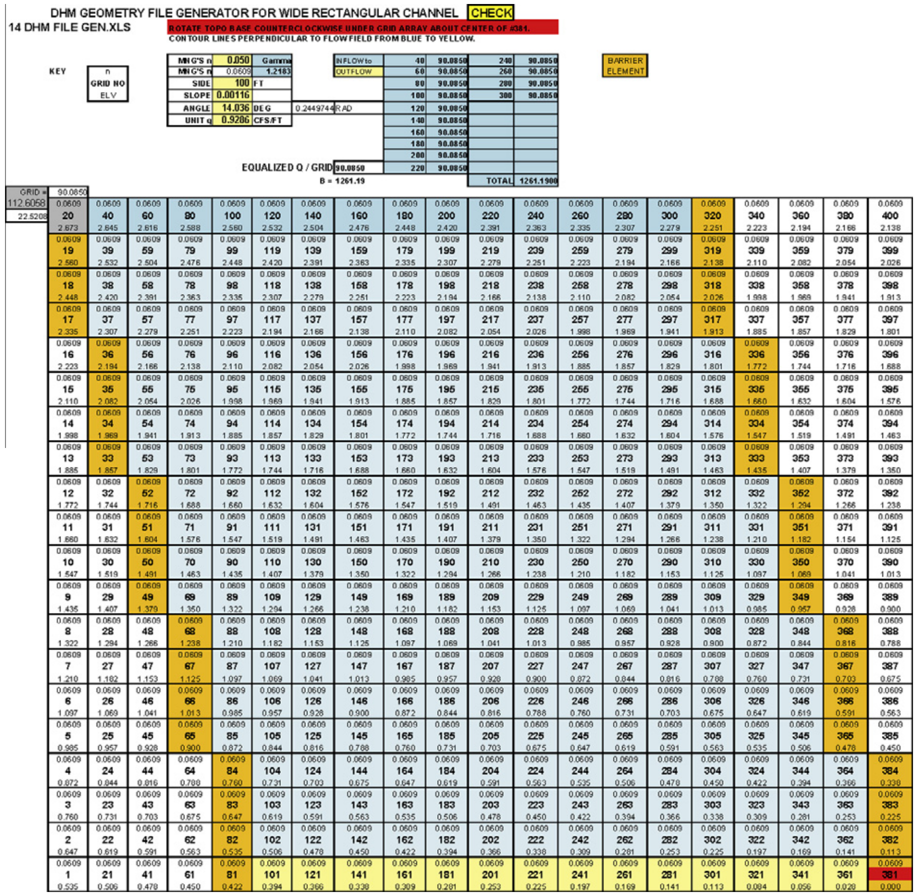

Fig. 4. Model rotated 14°.

provide the velocity in the N–S direction, and similarly for the E–W direction. Resolving these velocities into angular and velocity components yields flow direction through each element, which compared well with theoretical flow directions. Table 1 summarizes the results.

For both the aligned and rotated models, $\gamma(\theta)$ was estimated by first analyzing each rotation model with Manning's $n = 0.050$. Consistent with theory, the flow depths in all rotated models were slightly less than the computed normal depth. Manning's n was increased according to Eq. (33) and a second analysis was made. In most cases, the computed depth was not quite equal to normal depth, so a third value of Manning's n was interpolated or extrapolated based on the results of the first two analyses, and a third analysis was made. If the computed flow depth was equal to normal depth, the actual value of $\gamma(\theta)$ was computed as model Manning's n /0.050. If the computed flow depth was not equal to normal depth, a three-point interpolation or extrapolation of previously-computed data was used to estimate a value of Manning's n that would result in computed depth equal to normal depth. The actual value of $\gamma(\theta)$ was computed as model Manning's n/0.050.

Table 1 and Fig. 6 summarize the results for the aligned and rotated cases. For rotation angles other than 0° (and 90° by symmetry), flow depths were lower than normal depth. Manning's n values needed to develop a computed depth equal to normal depth were within the range reported in the literature, with the highest being 0.061. For example, Chow (1959) reports floodplain n-values ranging from 0.035 to 0.070 for a normal n-value of 0.050. Several general conclusions are readily apparent:

Computed values of $\gamma(\theta)$ generally follow the trend of the theoretical values.

The aligned model computed $\gamma(\theta)$ at zero (and 90° by symmetry) matches theoretical $\gamma(\theta)$ exactly.

The rotated model computed $\gamma(\theta)$ at 45° matches theoretical $\gamma(\theta)$ exactly.

The rotated models computed $\gamma(\theta)$ at 6°, 11°, 14°, 18°, and 27° (and 63°, 72°, 76°, 79°, and 84° by symmetry) closely approximate theoretical $\gamma(\theta)$.

5. Discussion

In the field estimation of Manning's friction factor values, watercourses and floodplains are typically identified that approximately satisfy steady-state flow conditions, and that also satisfy approximately one-dimensional uniform flow conditions. Application of a grid tiling of elements to such areas, using very small elements (i.e., with side dimension approaching the limit established by the Courant criterion), could result in a mathematical situation analogous to the considered SSUF problem setting examined in this paper. From the results summarized in Fig. 6, it is logical to hypothesize that the flow streamlines are typically randomly varying in trajectory with respect to the grid flow analog's principal flow directions, and at angles oftentimes greater than a few degrees. In such a case, the field-estimated friction factor value, n, used to calibrate the model should already include the $\gamma(\theta)$ factor, which is essentially a constant value except for trajectories in nearly perfect alignments with the principal flow directions. Therefore, the issue may be viewed that $\gamma(\theta)$ is already included in the Manning's n values, except in those rare conditions where random streamline trajectory variability within the channel flow does not vary more than a few degrees from perfect alignment. Adjustments, if applied, would be on the order of magnitude of the

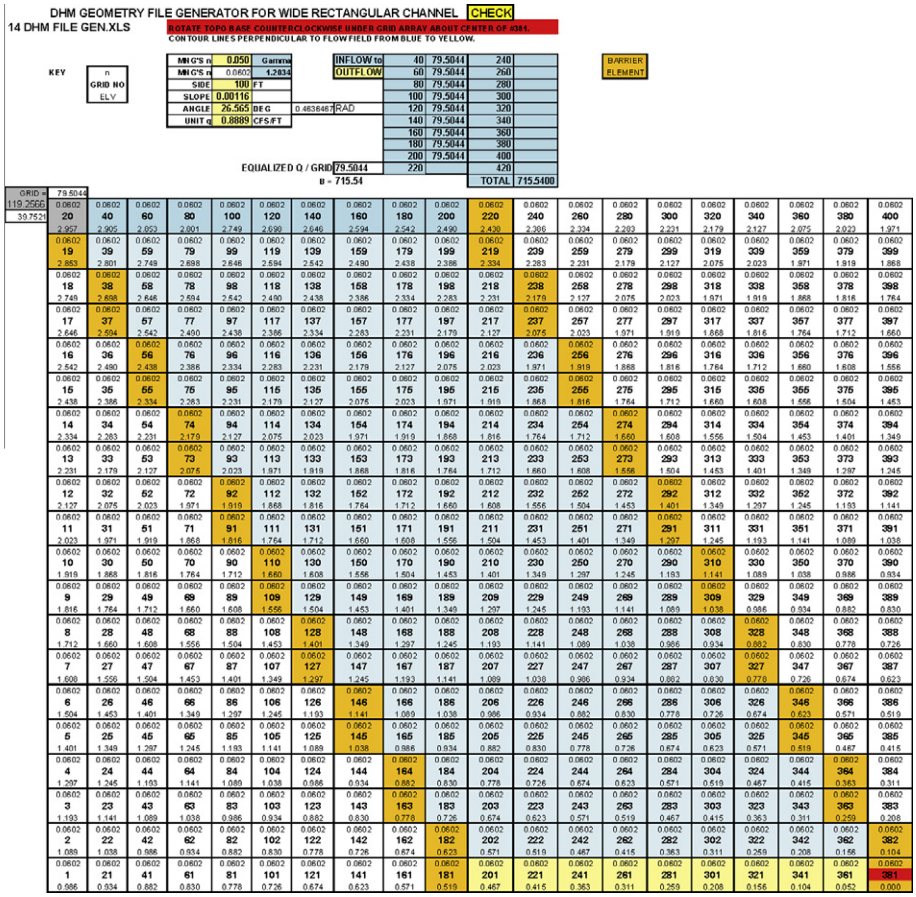

Fig. 5. Model rotated 27°.

Table 1
Summary of results – angular analysis and gamma computations.

Nominal angle[a] (deg)	Angle (deg)	Computed angle (low, deg)	Computed angle (average, deg)	Computed angle (high, deg)	Theoretical depth (cm)	Depth at n = 0.050 (cm)	n To achieve D = 30.24	Computed gamma	Theoretical gamma
0, 90	0.0	0.0	0.0	0.0	30.24	30.24	0.0500	1.000	1.000
6, 84	5.7	4.6	6.7	8.6	30.24	27.34	0.0596	1.192	1.120
11, 79	11.3	9.8	11.8	12.8	30.24	26.97	0.0606	1.212	1.218
14, 76	14.0	12.9	14.5	17.2	30.24	26.58	0.0612	1.224	1.218
18, 72	18.4	17.9	19.5	21.2	30.24	26.88	0.0604	1.208	1.215
27, 63	26.6	25.4	25.7	26.2	30.24	27.13	0.0597	1.195	1.203
45	45.0	45.0	45.0	45.0	30.24	27.28	0.0594	1.188	1.189

[a] angle by symmetry.

accepted ranges of friction factors. It follows that an array of square elements applied over the digital terrain model without respect to topographic flow directions would not require any adjustment to account for variability of streamline trajectory.

Results from an early application of USGS DHM support the hypothesis that $\gamma(\theta)$ is already included in the Manning's n values, and support the application of an array of square elements over the terrain model without respect to topographic flow directions. Synthetic unit hydrographs (s-graphs) developed from USGS DHM correlated well with the NRCS unit hydrographs, for an array of square elements laid over a gaged mountain watershed with complex topography (Hromadka and Nestlinger, 1985).

Nonetheless, use of the $\gamma(\theta)$ term brings into consistency the numerical solution of the governing flow equations, for the considered SSUF problem, for the considered flow analog and tiling of elements.

6. Conclusions

Application of Manning's equation to compute x and y axis projected flow direction friction slopes for use in the governing 2D flow equations may produce a biased result in hydraulic computations in situations where flow streamlines exceed a few degrees from perfect alignment. To investigate the nature and magnitude of this possible bias, a steady state uniform flow problem is examined and ratios of computed Manning's n to SSUF Manning's n with respect to angle are derived. Investigation of a ratio with respect to Manning's n, as opposed to introducing a new factor into Manning's equation, is justified for the typical application of USGS DHM to analyze shallow overland flow in floodplains. Engman (1989) has shown that the governing flow equations can be solved with proper boundary conditions and the selection of only one parameter, Manning's n. For elements aligned with principal flow

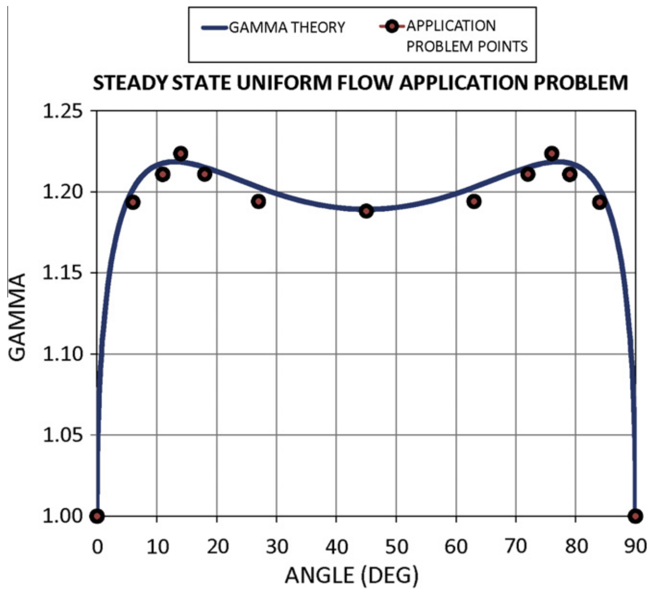

Fig. 6. Application problem – compare Gamma theory with USGS DHM model results.

streamlines, the ratio has a value of 1.0. Otherwise, when elements are not aligned with streamlines, the computational model predicts a ratio of about 1.2.

It might be concluded that Manning's n could be adjusted for each element so that computed depths match actual depths. However, the small variation in Manning's n across the wide range of streamline flow angles with respect to the element alignments makes this an ineffective process that might indeed be superfluous. For usual cases where random streamline trajectory variability within the floodplain flow is greater than a few degrees from perfect alignment, the ratio $\gamma(\theta)$ appears to be implicitly included in the Manning's n values. It can be concluded that the array of square elements may be applied over the digital terrain model without respect to topographic flow directions.

Acknowledgement

C.C. Yen's review of this manuscript is gratefully acknowledged.

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