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Conservation Laws and Inertial Parameters in the Quasiparticle Random Phase Approximation at Finite Temperature and High Spin.¹

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Abstract:

The conservation laws in the finite temperature RPA with emphasis on the angular momentum and particle number are discussed. Realistic calculations of the inertial tensor for the nucleus $^{164}$Er are performed for several temperatures and angular momenta.

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1. Introduction

The high spin region of deformed nuclei has been the object of intensive study in the last decade (see ref. 1 for a recent review). New experimental techniques have pushed to higher and higher spin ($\sim 40\hbar$) the resolution of discrete lines in the yrast limit and at low excitation energy above it. For high spin and high excitation energy correlation methods have been developed\(^{(2)}\) to disentangle the seemingly structureless spectra. In this regard, very recently, Deleplanque et al.,\(^{(3)}\) using the feeding correction method, successfully measured very detailed features, such as the dynamic moment of inertia, in the quasicontinuum of some transitional Rare-Earth nuclei. Now several experimental groups are studying the quasicontinuum trying to understand, among other things, how the gamma decay proceeds after the neutron-evaporation in a compound nucleus reaction.

On the theoretical side most of the effort has been concentrated on the yrast line and the side bands, the basic ingredient in all theories being the mean field approach (Hartree-Fock-Bogoliubov, HFB) combined with the cranking method.\(^{(4)}\) The variety of cranking theories rank from the simplest one\(^{(5)}\) which only includes Coriolis effects passing by the selfconsistent theories\(^{(6,7)}\) to the modern symmetry-conserving mean field theories.\(^{(8)}\) To take into account further correlations along the yrast band and to describe properly the low-lying vibrations, some authors\(^{(9-13)}\) proposed to go beyond the mean field by taking into account the RPA corrections to the self-consistent cranking model, the cranked RPA or CRPA. A further property which makes the RPA appealing is the restoration of the broken asymmetries in the RPA order, separating the Goldstone modes from the normal modes. This property in the context of high-spin states has been discussed by Marshalek.\(^{(14)}\)

To study the nuclear properties above the yrast line till the continuum the finite temperature HFB theory (FTCHFB) has been developed.\(^{(15-17)}\) The experimental evidence\(^{(18)}\) of giant resonances based on highly excited rotating compound states has motivated the formulation\(^{(19-20,21)}\) of RPA theories at finite temperature in the rotating system (FTCRPA). So far, the for-
mulations of the FTCRPA have concentrated on solutions corresponding to real vibrations, i.e. the normal modes, but no attention has been paid to the solutions of the FTCRPA in the presence of Goldstone modes. On the other hand it is well known that the mass parameter $\mu_2$ associated with the Goldstone modes are equivalent to the dynamic moment of inertia for rotations, i.e. their masses, for the angular momentum operator, correspond to the moments of inertia recently measured by Deleplanque et al.\textsuperscript{(3)}.

In this paper we concentrate on this more general problem. We first introduce some notation in section 2.1 and review the FTCRPA equations in section 2.2. In section 2.3 we explicitly show that the Goldstone modes satisfy the FTCRPA. The inertial tensor associated with the angular momentum and the particle number operators is discussed in section 2.4 and in part 3.

We do a numerical application of the theory to the realistic case of the nucleus $^{164}\text{Er}$ in part 4. Several properties of a hot nucleus as a function of the cranking velocity and the temperature are investigated. The results are summarized in part 5.

2. Theory

The FTCRPA equations have been studied mostly in the context of the Linear Response Theory. However in the time independent picture the FTCRPA has received little attention. Therefore we give here a short derivation for the sake of introducing some notation adequate for our purposes and the convenience of the reader. We follow to a large extent the lines of ref. 21.

2.1. General considerations of the FTHFB and FTRPA theory

In ref. 21 a notation was introduced which allows a very compact formalism for the temperature dependent problem in the presence of pairing correlation.

Let $\{\alpha_k, \alpha_k^\dagger; k = 1, \ldots, M\}$ be a set of quasiparticle operators which define the vacuum $\phi >$, i.e.
\[ \alpha_k \phi = 0, \quad k = 1, \ldots, M \]  

We define the operators

\[ a_\mu = \begin{cases} 
\alpha_m & \text{for } \mu = m \\
\alpha_m^\dagger & \text{for } \mu = -m 
\end{cases}, \quad m = 1, \ldots, M, \]

their anticommutation relations are

\[ \{a_\mu, a_\nu\} = \delta_{\mu, \nu}, \]

where \( \nu \) stands for \(-\nu\). In terms of these operators any single-particle operator

\[ F = F^0 + \sum_{m,m'} F_{mm'}^{11} \alpha_m^\dagger \alpha_{m'} + \sum_{m < m'} \left( F_{mm'}^{20} \alpha_m^\dagger \alpha_{m'}^\dagger + F_{mm'}^{02} \alpha_m \alpha_{m'} \right) \]

can be written as

\[ F = \mathcal{F}^0 + \frac{1}{2} \sum_{\mu\mu'} \mathcal{F}_{\mu\mu'} a_\mu^\dagger a_{\mu'}, \]

with

\[ \mathcal{F} = \begin{pmatrix} F^{11} & F^{20} \\ -F^{02} & -F^{11}^* \end{pmatrix} \]

\[ \mathcal{F}^0 = F^0 + \frac{1}{2} Tr_M(F^{11}) \]

The more general Bogoliubov transformation can be represented by

\[ a_\mu^\dagger = \sum_{\nu} \mathcal{F}_{\nu\mu} c_\nu^\dagger \]

with
or by
\[ a_\mu^\dagger = e^{iZ_c^\mu} e^{-iZ_c^\mu} \]
with
\[ Z = \frac{1}{2} \sum_{\mu} \sum_{\nu} \mathcal{X}_{\mu\nu} a_\mu^\dagger a_\nu \]
and
\[ \mathcal{W} = \exp(iZ) \]

We shall use script notation when working with the generalized quasiparticle operators of eq. (2).

Since we are interested in the description of hot nuclei, we have to replace the individual compound systems with definite excitation energy, particle number and angular momentum by the grand ensemble of nuclei. The ensemble is suppose to be in thermal equilibrium with a heat bath. The nuclear temperature, chemical potential and cranking velocity determine the average quantities for the energy, particle number and angular momentum respectively. The expectation value of any operator \( F \) is obtained by averaging over all multiquasiparticle states, i.e.
\[ \langle F \rangle_T = \sum_n p_n \langle n | F | n \rangle \]

with
\[ | n \rangle = a_{\mu_1} \cdots a_{\mu_r} | \phi \rangle \]

The probabilities \( p_n \) are given by
\[ p_n \sim \exp\left[ -\beta(E_{\mu_1} + E_{\mu_2} + \cdots + E_{\mu_s}) \right] \]  
(14)

and are normalized to unity; the quantities \( E_{\mu} \) are the quasiparticle energies in the intrinsic frame.

\[ E_{\mu} = \langle \phi \left| \left[ a_{\mu} A H^\dagger, a_{\mu}^\dagger \right] \right| \phi \rangle \]

\[ H' = H - \omega J_x - \lambda N \]  
(15)

The parameters \( \omega \) and \( \lambda \) are the cranking velocity and the chemical potential above mentioned.

Furthermore, since we have taken the states \( |n> \) to be generalized Slater determinants, we can use the generalized Wick theorem to calculate the thermal averages (12) in terms of the single particle density matrix

\[ \mathcal{R}_{\mu} = \langle a_{\mu}^\dagger a_{\mu} \rangle_T \]  
(16)

In particular for a single particle operator we get from eq. (5) and (6)

\[ <F>_T = \mathcal{F}^0 + \frac{1}{2} Tr_{2M}(\mathcal{F}, \mathcal{F}) \]  
(17)

2.2. The HFB and RPA equations at finite temperature

The FTHFB equation determines the equilibrium distribution of nuclei in the grand canonical ensemble and their properties in the independent particle approximation. The corresponding density matrix \( \mathcal{R}^0 \) is given by \( \mathcal{F}^0 \)

\[ \mathcal{R}(\mathcal{R}^0), \mathcal{R}^0 | = 0 \]  
(18)

with

\[ \mathcal{R}_{\mu} = \langle \left| a_{\mu} A H^\dagger, a_{\mu}^\dagger \right| \rangle_T \]  
(19)
In the self-consistent basis determined by equation (18) \( \mathcal{H} \) and \( \mathcal{A} \) are both diagonal

\[
\mathcal{H} = \begin{pmatrix}
E_m & 0 \\
0 & -E_m
\end{pmatrix}
\quad \text{and} \quad
\mathcal{A}^0 = \begin{pmatrix}
f_m & 0 \\
0 & 1-f_m
\end{pmatrix}.
\]

(20)

The finite temperature RPA determines the vibration of a nucleus close to thermal equilibrium in the limit of small amplitudes. If we assume

\[
\mathcal{A}(t) = \mathcal{A}^0 + (\delta \mathcal{A} \cdot e^{-i\Omega t} + h.c.),
\]

(21)

the amplitudes \( \delta \mathcal{A} \) can be obtained\(^{21} \) from the equations of motion; they are given by

\[
(\Omega - E_\mu + E_{\mu'})\delta \mathcal{A}_{\mu\mu'} = \frac{1}{2} (f_\mu' - f_\mu) \sum_{\nu\nu'} \gamma_{\mu\nu'\nu'} \delta \mathcal{A}_{\nu\nu'}
\]

(22)

with

\[
\Omega = \begin{pmatrix}
\Omega_K \\
-\Omega_K
\end{pmatrix}
\]

(23)

and the effective interaction

\[
\gamma_{\mu\nu'\nu}(f_\nu' - f_\nu) = \langle [a_\nu^\dagger a_\nu, [a_\mu H] a_\mu^\dagger] \rangle_T.
\]

(24)

The eq. (22) looks rather asymmetric, and the norms of the vectors \( \delta \mathcal{A} \) are dependent of the temperature factors \( f_\mu \). To avoid these problems we define

\[
\delta \mathcal{A}_{\mu\mu'} = (f_\mu' - f_\mu) \mathcal{Z}_{\mu\mu'}
\]

(25)

and take advantage of the hermiticity of the density matrix \( \mathcal{A} \) to order the states \( \mu, \mu' \) so that \( E_\mu > E_{\mu'} \). Introducing these changes in eq. (22) we obtain

\[
\mathcal{F} \mathcal{A} = \mathcal{N} \mathcal{A} \Omega
\]

(26)

in exactly the same way as the usual RPA\(^{23} \) with the following identifications:
\[ \mathcal{X}_{\mu\nu} = (E_\mu - E_\nu)\delta_{\mu\nu} + \frac{1}{2} (f_\mu - f_\nu)^{1/2} \mathcal{X}_{\mu\nu} (f_\nu - f_\mu)^{1/2} \] (27)

\[ \mathcal{X}_{\mu\nu} = (f_\mu - f_\nu)^{1/2} \mathcal{X}_{\mu\nu} \] (28)

and the norm matrix

\[ \mathcal{N} = \begin{pmatrix} 1 & -1 \end{pmatrix} \] (29)

Afterwards we shall refer to eq. (26) as the Finite Temperature RPA equations.

As with the zero-temperature RPA equation, the FTRPA equation can be derived in a time independent picture, from the ansatz

\[ \langle [a_\mu^T a_\nu] | H', B^\dagger \rangle >_T = \Omega \langle [a_\mu^T a_\nu] B^\dagger \rangle >_T \] (30)

with

\[ B^T = \sum_{\mu < \mu'} \mathcal{X}_{\mu\mu'} a_{\mu'}^T a_{\mu} \] (31)

The components of the operator \( B^T \) are related to the eigenvectors of (26) by (28), as one would expect from the normalization condition

\[ \delta_{KL} = \langle [B_K, B_L^T] >_T = \sum_{\mu < \mu'} \mathcal{X}_{KK', \mu \mu'} L(f_\mu - f_\mu') = \sum_{\mu < \mu'} \mathcal{X}_{KK', \mu \mu'} \] (32)

Some other properties of the FTRPA can be found in ref. 20.

2.3. Conservation laws in the Finite Temperature RPA

We shall now concentrate on problems arising from zero frequencies. The nuclear many-body problem is very complex and the reason why simple-minded theories, like the mean-field approximation, work is that by breaking symmetries one can take into account many correla-
tions. The most commonly broken symmetries in mean-field theories are the translational invariance, particle number (if pairing correlations are included) and rotational invariance (in deformed nuclei).

We shall first examine the zero-temperature case: let us assume that the two body Hamiltonian

\[ H = \sum_{k,l} T_{kl} c_k^\dagger c_l + \frac{1}{4} \sum_{k,l,n,m} V_{klmn} c_k^\dagger c_l c_n c_m \] (33)

is invariant under a continuous symmetry operation generated by a hermitian one-body operator \( P \), i.e.

\[ [H, P] = 0 \] (34)

Let us furthermore assume that this symmetry is broken in the HF approximation; that is, the single-particle density matrix does not commute with \( P \). With these two assumptions it is easy to show\(^{(24)}\) that the RPA approximation, calculated with single particle energies and wave functions obtained in the selfconsistent HF calculations, has one solution at zero energy. Thus,

\[ [H_{RPA}, P_{RPA}] = 0 \] (35)

where \( H_{RPA} \) and \( P_{RPA} \) are the RPA approximations to the operators \( H \) and \( P \). Since the RPA solutions appear pairwise at \( +\Omega \) and \( -\Omega \), and the \( P_{RPA} \) operator is hermitian, it is obvious that at zero energy one solution is missing from a complete set. The missing equation can be found working with a set of canonical conjugated operators as Marshalek and Weneser\(^{(24)}\) did. The equation they found is

\[ [H_{RPA}, Q_{RPA}] = -i \frac{P_{RPA}}{M} \] (36)

where \( Q \) is the canonical conjugated variable to \( P \), and the mass parameter \( M \) is determined by the commutation relation
\[ [P_{RPA}, Q_{RPA}] = -i \]  

(37)

This property of the RPA of separating the Goldstone mode from the other solutions is a great advantage of the RPA over other theories where the spurious solutions are mixed up with the physical modes. It is therefore interesting to see if this important property of the RPA is still present in the finite-temperature limit, where thermal averages have been taken. We will show below that this property of the RPA is still maintained at finite temperatures.

Let us first show that if the symmetry associated with the operator \( P \) is broken in the mean field approximation, then the transformed density matrix under this symmetry \( \mathcal{A} \) is also a solution of the FTHF equations (18), provided the Hamiltonian is symmetry-invariant. The transformed density matrix under \( P \) is

\[ \mathcal{A}_{\mu'} = e^{i\alpha P} \mathcal{A}_{\mu} e^{-i\alpha P} = e^{i\alpha P} \langle a^\dagger_{\mu} a_{\mu} \rangle_T e^{-i\alpha P} = \langle \overline{a}^\dagger_{\mu} \overline{a}_{\mu} \rangle_T \neq \overline{\mathcal{A}}_{\mu'} \]  

(38)

and the transformed Hamiltonian (19) is

\[ \mathcal{H}_{\mu\nu}(\mathcal{A}) = e^{i\alpha P} \mathcal{H}_{\mu\nu}(\mathcal{A}) e^{-i\alpha P} = e^{i\alpha P} \langle [a_{\mu}, H] a^\dagger_{\nu} \rangle_T e^{-i\alpha P} \]

\[ = \langle [\overline{a}_{\mu}, H] \overline{a}^\dagger_{\nu} \rangle_T = \langle [\overline{a}_{\mu}, H] \overline{a}^\dagger_{\nu} \rangle = \mathcal{H}_{\mu\nu}(\mathcal{A}) \]  

(39)

In the last line we have made use of the invariance of the Hamiltonian under the symmetry \( P \) and of eq. (9) to obtain the transformed operators \( \overline{a}^\dagger_{\mu} \). Then from eqs. (18), (38) and (39) we finally obtain

\[ [\mathcal{H}(\mathcal{A}), \mathcal{A}] = 0 \]  

(40)

If we now expand \( \mathcal{A} \) and \( \mathcal{H}(\mathcal{A}) \) around \( \mathcal{A} \) and \( \mathcal{H}(\mathcal{A}) \), respectively, we get to first order in \( \alpha \)

\[ \mathcal{A}_{\mu'} = \langle e^{i\alpha P} a^\dagger_{\mu} a_{\mu} e^{-i\alpha P} \rangle_T \approx \mathcal{A}_{\mu'} - i\alpha \langle [P, a^\dagger_{\mu} a_{\mu}] \rangle_T \]
We have introduced the notation $\delta \mathcal{R}$ and $\delta \mathcal{R}'$ in the right hand side of both equations. By substitution in eq. (40) we get to first order in $\alpha$

$$|\delta \mathcal{R}, \mathcal{R}| + |\mathcal{R}, \delta \mathcal{R}| = 0$$

Substitution of eq. (24) in the expression (42) for $\delta \mathcal{R}'$ and the use of the basis in which $\mathcal{R}$ and $\mathcal{R}'$ are diagonal allows us to write eq. (43) as

$$(E_\mu - E_\mu') \delta \mathcal{R}_{\mu'\mu} = \frac{1}{2} (f_{\mu'} - f_\mu) \sum_{\nu} V_{\mu'\nu} \delta \mathcal{R}_{\nu\mu}$$

Comparison with eq. (22) clearly shows the existence of a solution at zero energy with amplitudes

$$\delta \mathcal{R}_{\mu'\mu} = |\mathcal{R}, \mathcal{R}' \rangle = (f_{\mu'} - f_\mu) \mathcal{R}_{\mu'\mu}$$

defined in eq. (41).

In terms of (30) the statement above means

$$\langle a_{\mu'}^\dagger a_\mu | H', P | \rangle >_T = 0$$

For the operator $P$ we have employed the notation of eq. (5). The meaning of eq. (46) is clear: The broken symmetry in the FTHFB approximation is recovered in the FTRPA.

The equation for the canonical conjugated operator can be obtained, as in the zero-temperature limit, by the Marshalek-Weneser method, and is given by
The mass parameter $M$ is given by the normalization condition

$$< [a^*_{\mu} a_{\nu}] | H', Q | >_T = \frac{-i}{M} < [a^*_{\mu} a_{\nu}] P | >_T \quad (47)$$

The equations (46-48) determine completely the operator $P, Q$ which appears in the presence of zero frequencies, and these together with the operators $B_K, B_K^+$ form a complete set in the Finite Temperature RPA.

It is interesting to notice that in the FTRPA the whole operator $P$ appears and not just the $20$ and $02$ parts of its quasiparticle representation. This does not mean that the broken symmetry is now exactly conserved, since the equations (43) were obtained in the limit of small amplitude, i.e. small $\alpha$'s.

2.4. The inertial tensor.

In the former paragraph we have obtained the equations for the Goldstone mode in the case of only one broken symmetry. In general there are several, depending on the particular case. In what follows we shall concentrate on the conserved symmetries of the high spin region, namely, the angular momentum $\vec{J}$ and the number operators $N_P$ and $N_N$, for protons and neutrons, respectively.

The conservations laws are

$$[H, \vec{J}] = 0 \quad , \quad [H, N_P] = 0 \quad , \quad [H, N_N] = 0 \quad (49)$$

In terms of $H'$, eq.(15), we obtain

$$[H', J_x] = 0 \quad , \quad [H', N_P] = 0 \quad , \quad [H', N_N] = 0 \quad (50)$$

$$[H', J_z] = \pm \omega J_z \quad (51)$$
where $J_\pm$ are defined by

$$J_\pm = J_z \pm iJ_y \quad (52)$$

These equations imply, as we have seen, the existence of zero-frequency solutions in the FTRPA, which satisfy the following relations:

$$\langle[a_\mu^* a_\nu | H', J_x] \rangle_T = 0, \quad \langle[a_\mu^* a_\nu | H', N_P] \rangle_T = 0, \quad \langle[a_\mu^* a_\nu | H', N_N] \rangle_T = 0 \quad (53)$$

and

$$\langle[a_\mu^* a_\nu | H', J_\pm] \rangle_T = \pm \omega \langle[a_\mu^* a_\nu J_\pm] \rangle \quad (54)$$

In calculations involving high spin states it is very convenient to work in the Goodman basis\(^{16}\) to get operators with good signature because then the HFB- as well as the RPA matrix factorize in this representation. The operators $J_x, N_P$, and $N_N$ have a positive signature, and $J_\pm$ a negative one.

We first analyze the Goldstone modes in the positive signature subspace. Let us call $\phi_x, \phi_p$ and $\phi_N$ the canonical-conjugate operators of $J_x, N_P$, and $N_N$. They satisfy

$$\langle[J_x, \phi_x] \rangle_T = -i, \quad \langle[N_P, \phi_p] \rangle_T = -i, \quad \langle[N_N, \phi_N] \rangle_T = -i \quad (55)$$

Aside from the normalization relations for the normal modes, (see eq. (32)), all other relations of the type $\langle[C, D] \rangle_T$ are zero, $C$ and $D$ belonging to the complete set $\{B_K, B_K^*: J_x, \phi_x; N_P, \phi_p; N_N, \phi_N\}$.

The generalization of relation (47) to define the canonical conjugate variables to the case of several zero frequencies is

$$\langle[a_\mu^* a_\nu | H', i\phi_x] \rangle_T = g_{JJ} \langle[a_\mu^* a_\nu J_x] \rangle_T + g_{JP} \langle[a_\mu^* a_\nu N_P] \rangle_T + g_{JN} \langle[a_\mu^* a_\nu N_N] \rangle_T$$

$$\langle[a_\mu^* a_\nu | H', i\phi_p] \rangle_T = g_{JP} \langle[a_\mu^* a_\nu J_x] \rangle_T + g_{PP} \langle[a_\mu^* a_\nu N_P] \rangle_T + g_{PN} \langle[a_\mu^* a_\nu N_N] \rangle_T$$
\[ < [a_a^t a_r | H^{'}_r i \phi_N ] > T = g_{2N} < [a_a^t a_r J_x ] > T + g_{PN} < [a_a^t a_r N_P ] > T + g_{NN} < [a_a^t a_r N_N ] > T \]  \hspace{1cm} (56)

The quantities \( g_{ij}^{-1} \) are the elements of the inertial tensor associated with the Goldstone modes. In particular \( g_{JJ}^{-1} \) is the generalization of the Thouless-Valatin moment of inertia, \( \mathcal{\mathcal{T}}_{TV} \) for finite temperatures, \( g_{PP}^{-1}(g_{NN}^{-1}) \) are the equivalent quantities for rotations in the gauge space associated with the operator \( N_p \) \((N_N)\). The equations (56), together with the relation (55), define \( g_{ij} \) completely. They are given by

\[ g_{ij} = \frac{C_{ij}}{D} \]  \hspace{1cm} (57)

where \( C_{ij} \) is a cofactor of the determinant

\[ D = \begin{vmatrix}
< [J_x, i \theta_1 ] > T & < [J_x, i \theta_2 ] > T & < [J_x, i \theta_3 ] > T \\
< [N_N, i \theta_1 ] > T & < [N_N, i \theta_2 ] > T & < [N_N, i \theta_3 ] > T
\end{vmatrix} \]  \hspace{1cm} (58)

\( \Theta_1, \Theta_2 \) and \( \Theta_3 \) are one-body operators of the form

\[ \Theta_i = \frac{1}{2} \sum_{\mu\nu} \Theta_{i, \mu\nu} a_{\mu}^t a_{\nu} \]

defined by the \( g_{ij} \) independent equations

\[ < [a_a^t a_r | H^{'}_r i \theta_1 ] > T = < [a_a^t a_r J_x ] > T \]

\[ < [a_a^t a_r | H^{'}_r i \theta_2 ] > T = < [a_a^t a_r N_P ] > T \]

\[ < [a_a^t a_r | H^{'}_r i \theta_3 ] > T = < [a_a^t a_r N_N ] > T \]  \hspace{1cm} (59)

We turn now to the solutions of eq. (54) in the negative signature subspace. First we observe that because of the cranking term, there appears a solution at the cranking frequency \( \omega \). The normalization of this solution is given, as for the normal modes, by eq. (32). Since
the normalized operator

$$\Gamma' = \frac{1}{(2\langle \mathbf{J}_x \rangle_T)^{1/2}} \mathbf{J}_+.$$  \hspace{1cm} (61)

satisfies the FTRPA equation

$$\langle [a^*_\mu a^*_\nu, [H, \Gamma'] ] \rangle_T = \omega \langle [a^*_\mu a^*_\nu, \Gamma'] \rangle_T.$$  \hspace{1cm} (62)

Since $\Gamma'$ and $\Gamma$ are non-hermitian operators, it is not necessary to introduce canonical conjugate operators, and the complete set is given by $\{ B_K, B_\ell^*, \Gamma, \Gamma' \}$. $B_K$ are the normal modes in the negative-signature subspace.

3. Application for separable forces

We shall now obtain the expression for the inertial tensor $g_{ij}$ in the simple case of separable forces. In this case the hamiltonian of eq. (23) takes the form

$$H = \varepsilon + \frac{1}{2} \sum_\rho \chi_\rho D_\rho^{*}D_\rho$$  \hspace{1cm} (63)

where $\varepsilon$ are the spherical single-particle energies, $D_\rho$ are one-body operators either hermitian or antihermitian, and $\chi_\rho$ the force constants.

With this hamiltonian the effective interaction $\mathcal{V}_{\mu\nu\sigma}$ of eq. (24) takes the form

$$\mathcal{V}_{\mu\nu\sigma} = \sum_\rho \chi_\rho D_\rho^{*}D_\rho.$$  \hspace{1cm}

The FTRPA equations for the $\theta_j$ operators of eq. (59), is given by

$$(E_\mu - E_\nu) \theta_{\mu\nu} + \frac{1}{2} \sum_\rho \chi_\rho \mathcal{D}_{\mu\nu}^{*} \sum_\nu' \mathcal{D}_{\nu\nu'}^{*} (f_{\nu'} - f_{\nu'}) \theta_{\mu\nu'} = \mathcal{P}_{\mu\nu}.$$  \hspace{1cm} (64)
for \( j = 1, 2, 3 \) and \( P_1 = J_x, P_2 = N_p \) and \( P_3 = N_N \). Equivalently

\[
\theta_{j, \mu' \nu'} = \frac{\mathcal{P}_{j, \mu' \nu'}}{E_\mu - E_{\mu'}} - \frac{1}{2} \sum_{\rho} \Phi_{j, \rho} \chi_{\rho} \frac{\mathcal{P}_{\rho, \mu' \nu'}}{E_\mu - E_{\mu'}}
\]

(65)

with

\[
\Phi_{j, \rho} = \sum_{\nu < \nu'} \Phi_{j, \nu} (f_{\nu'} - f_{\nu}) \mathcal{P}_{\mu' \nu'}^* = \langle [\theta_j, D_\rho] \rangle_T
\]

(66)

One can extract an equation for the \( \Phi_{j, \rho} \) simply by multiplying eq. (65) by \((f_{\mu'} - f_{\mu}) \mathcal{P}_{\mu' \nu'}^*\), and summing up over \( \mu, \mu' \).

The last step in the calculation is to evaluate the scalar products \( \langle [P_k, i \theta_j] \rangle \) of the determinant (58).

They are given by

\[
\langle [P_k, i \theta_j] \rangle = \sum_{\mu < \mu'} \mathcal{P}_{k, \mu \nu} (f_{\mu'} - f_{\mu}) \mathcal{P}_{j, \mu' \nu'} - \frac{1}{2} \sum_{\rho} \Phi_{j, \rho} \chi_{\rho} \sum_{\mu < \mu'} \mathcal{P}_{k, \mu \nu} (f_{\mu'} - f_{\mu}) \mathcal{P}_{\rho, \mu' \nu'},
\]

(67)

and the components of the inertial tensor \( g_{ij} \) are given by expression (57).

To find some limiting cases we shall now assume that there is only one broken symmetry.

In this case

\[
\mathcal{I}_{T'} = g_{11}^{-1} = \langle [P, i \phi] \rangle
\]

(68)

where \( \mathcal{I}_{T'}(P) \) represents the inertial parameter for rotations in the space associated with the operator \( P \). If we take in eq. (67), the zero temperature limit

\[
f_{\mu} \to 0 \quad, \quad f_{\mu'} \to 1
\]

(69)

we get from (68) the well-known expression\(^{22}\), from the linear-response theory, for the
Thouless-Valatin moment of inertia.

If the limit $x_n \to 0$, we get from eq. (68) the generalization to finite temperature of the Inglis moment of inertia

$$\mathcal{J}_{\text{ng}}(P) = \sum_{\mu \mu'} \frac{\mathcal{P}_{\mu \mu}^* (f_{\mu} - f_{\mu'}) \mathcal{P}_{\mu' \mu}}{E_\mu - E_{\mu'}}$$  \hspace{1cm} (70)

The zero temperature limit on eq. (70) gives the well-known expression

$$\mathcal{J}_{\text{ng}}(P) = \sum_{m, m'} \frac{|P_{mm'}^{20}|^2}{E_m + E_{m'}}$$  \hspace{1cm} (71)

4. Numerical results

For a realistic calculation of the inertial tensor, eq. (57), in the FTRPA with separable forces, we use the configuration space and the effective interaction of Kumar-Baranger.\(^{(25)}\) This hamiltonian contains the pairing-plus-quadrupole force

$$H = \varepsilon - \frac{1}{2} x \mathbf{Q}^\dagger \cdot \mathbf{Q} - G_p P_p^\dagger P_p - G_N P_N^\dagger P_N$$  \hspace{1cm} (72)

where $\varepsilon$ are the spherical single-particle energies, $\mathbf{Q}$ is the quadrupole operator symmetrized with respect to the Goodman symmetry\(^{(16)}\) and the operators $P_p^\dagger (P_N)$ creates proton (neutron) Cooper pairs. In the notation of eq. (63) the operators $D_p$ will run over $Q$ and $P_r^\dagger \pm P_r$. The configuration space contains the spherical oscillator shells with the principal quantum numbers $N = 4$ and $5$ for protons and $N = 5$ and $6$ for neutrons. The force constants $G_p$ and $G_N$ were adjusted to the ground-state properties of the rare-earth nuclei. Further details can be found in ref. 25.

To solve selfconsistently the FTRPA we first have to solve the FTHFB equations (18) to determine the basis in which the one-body hamiltonian $\mathcal{H}$ and the density matrix $\mathcal{A}$ are diagonal (see eq. 20). This was done for the nucleus $^{164}$Er, as reported in ref. 17, for temperatures
$0 \leq T \leq 1$ MeV in intervals $\Delta T = .05$ MeV and for angular velocities $0 \leq \omega \leq .6$ MeV in intervals $\Delta \omega = .025$ MeV. In terms of excitation energy and angular momentum we cover the range $0 \leq E \leq 15$ MeV and $0 \leq I \leq 60$ h, respectively. On each point of this grid the FTRPA equations are solved, in the way indicated in the preceding paragraph, to get the quantities $g_{ij}^{-1}$.

We shall concentrate on the diagonal terms of the inertial tensor $g_{ii}^{-1}$, the inverse of which are the Thouless-Valatin moments of inertia in the space associated with the corresponding operators. The Thouless-Valatin moment of inertia for rotations corresponds therefore to $g_{ii}^{-1}$. This moment of inertia is also known in the literature as the dynamic moment of inertia $\mathcal{J}^{(2)}$ to differentiate it from the usual (kinematic) moment of inertia $\mathcal{J}^{(1)}$. In the framework of our theory the last one would correspond to the selfconsistent moment of inertia. The proper definition of this quantity is given by

$$<J_x>_T = I = \mathcal{J}^{(1)}, \quad \omega = \mathcal{J}^{(2)} \cdot \omega,$$

whereas

$$\Delta I = \mathcal{J}^{(2)} \cdot \Delta \omega = \mathcal{T} \cdot \Delta \omega,$$

i.e. $\mathcal{J}^{(1)}$ corresponds to the finite quotient of $I$ and $\omega$, and $\mathcal{J}^{(2)}$ is the slope of the curve $I$ as a function of $\omega$.

In Fig. 1a we show the angular momentum $I$ as a function of the angular velocity for constant temperatures for the nucleus $^{164}$Er. It is important to notice that we have solved the FTHFB equations with constant $\omega$, in steps of $\Delta \omega = .025$, instead of constant $I$ (i.e. for simplicity we did not use the gradient method). Because of this we do not see the backward parts of the backbendings, if any, but only alignments. For $T = 0.0$ we see in Fig. 1a a strong neutron alignment at $\omega \approx .225$, a smaller proton alignment at $\omega \approx .325$, and a much smaller neutron alignment at $\omega \approx .525$. At $T = 0.2$ and 0.4 MeV we still observe some structure, although smoothed out.
In Fig. 1b we draw $\mathcal{F}_{TV}$ ($\mathcal{F}^{(2)}_{V}$) for different temperatures as a function of the angular velocity. From eq.(74) we know that $\mathcal{F}_{TV}$ goes to $\pm \infty$ at each up-backbending. Since we are solving the equations for finite step size $\Delta \omega$, the height of the peak in each upbending does not necessarily mean larger alignment but only how near our actual $\omega$ is to $\omega_{\text{crit}}$. At $T = 0$, as expected, we get three peaks. At $T = 0.2$ and $0.4$ MeV only the peaks corresponding to large alignments remain; the smaller one at $\omega \approx 0.525$ is already washed out at these low temperatures. At higher temperatures little structure remains, and the moment of inertia converges to the rigid-body value. The comparison* with the experimental values(1) is not straightforward: for low angular velocities their values correspond mostly to the yrast line, and the agreement with ours is rather good; for higher angular velocities their window in energy is much larger, so that it is difficult to compare. Nevertheless the general trends to smooth out do agree in both experiment and theory.

In Figs. 2 and 3 the mass parameter $g_{NN}^{-1}$, $g_{pp}^{-1}$ for neutrons and protons for different temperatures is shown as function of the cranking velocity. The general behaviour that one expects for this parameter is related, in the RPA approximation, to the energy gap $\Delta$: If the nucleus is in the superconducting phase, i.e. finite $\Delta$, it is possible to have collective pairing-rotations, and as long as $\Delta$ is more or less constant we expect the collective parameter $g^{-1}$ to behave smoothly. At the normal phase (large $\omega$), $\Delta \rightarrow 0$, it is not possible to have such collective motion, and one expects $g^{-1} \rightarrow 0$ as $\Delta \rightarrow 0$. At $T = 0.0$ we observe for $g_{NN}^{-1}$ in Fig. 2 a deep dip at $\omega \approx 0.225$ MeV caused by the coupling of $g_{NN}^{-1}$ to $g_{JJ}^{-1}$ that as mentioned goes to infinity at this frequency. The peak at $\omega \approx 0.3$ MeV can be interpreted as a single-particle effect: In an alignment process a pair of particles decouples from the Cooper-pair condensate and in the same way as it produces an increase to $g_{JJ}^{-1}$, due to single particle contribution, does it for $g_{NN}^{-1}$.

This can also be understood considering the expression

*We are not directly comparing our results with theirs because they measured $^{156,158,160}$Er but not $^{164}$Er. In the results presented here we have used the FTHFB minima of ref. 17, which were calculated before the experimental values were available. The CPU time needed to calculate these minima for another nucleus is high.
\[ \Delta N = g_{NN}^{-1} \cdot \Delta \lambda_N \] (75)

analog to (74), \( N \) is the number of neutrons. The quantity \( g_{NN}^{-1} \) measures the change in the particle number when we change the chemical potential \( \lambda_N \) in \( \Delta \lambda_N \). In an alignment process the two particles sit at the Fermi surface, and the change in the particle number has to increase for fixed \( \Delta \lambda_N \), i.e. \( g_{NN}^{-1} \) has to get larger. The abrupt dropping from \( g_{NN}^{-1} \approx 17 \) at \( \omega \approx 0.30 \text{ MeV} \) to zero at \( \omega \approx 0.35 \text{ MeV} \) is due to the mean field approximation where one observes a sharp phase transition. In a number-projected theory this feature would not appear.\(^7\)

For higher temperatures the expected general features appear: the domain for which \( g_{NN}^{-1} \) is different from zero decreases as \( T \) increases (the gap parameter \( \Delta \to 0 \)) and the single-particle effects are washed out. It may seem surprising that the single-particle effects change so fast from \( T = 0 \) to \( T = 0.1 \text{ MeV} \), but this is only a consequence of our not calculating with \( \omega \) as a continuous parameter.

In Fig. 3 the quantity \( g_{pp}^{-1} \), i.e. the proton mass parameter, is shown. The same general properties are expected, only the discontinuities are now expected where the proton alignment occurs, namely at \( \omega \approx 0.325 \text{ MeV} \). Furthermore, since the proton gap lasts longer, the whole picture extends to higher angular velocities. The wiggles at temperatures 0.2, 0.3 and 0.4 MeV and high angular velocities (\( \omega > 0.3 \text{ MeV} \)) are probably due to band crossings. Then, although \( \Delta_p \) is very small,\(^{(17)}\) the matrix elements of the number operator are not exactly zero, and because of eqs. (68)-(67) this causes a maximum at each band crossing.

In ref. 14 it was shown that the RPA correlated energy \( E(I) \) is given by the sum of the contribution of the normal modes \( E_{NM} \) and the Goldstone modes \( E_{GM} \). In ref. 13 it was shown that the contribution of the last one is crucial to smooth out the strange behavior of \( E_{NM} \). The contribution of the Goldstone modes in the positive signature subspace is given by\(^{(13)}\)

\[ E^{(+)}_{GM} = E_{JJ} + E_{NN} + E_{ZZ} + E_{JN} + E_{JZ} \] (76)

with
\[ E_{JJ} = - \frac{1}{2} g_{JJ} \langle \Delta J_x^2 \rangle_T, \quad E_{NN} = - \frac{1}{2} g_{NN} \langle \Delta N_N^2 \rangle_T, \quad E_{ZZ} = - \frac{1}{2} g_{ZZ} \langle \Delta N_p^2 \rangle_T \]

\[ E_{JN} = - g_{JN} \langle \Delta J_x N_N \rangle_T, \quad E_{JZ} = - g_{JZ} \langle \Delta J_x N_p \rangle \quad (77) \]

and in the negative signature by

\[ E_{GM}^{(-)} = - \frac{\langle J_x^2 + J_y^2 \rangle_T}{2 J_c} \quad (78) \]

In Fig. 4 the fluctuations \( \langle \Delta J_x^2 \rangle_T, \langle \Delta J_y^2 \rangle_T \) and \( \langle \Delta J_z^2 \rangle_T \) are depicted for different temperatures as a function of the angular velocity. At \( T = 0 \) we observe in all three quantities the irregularities related to the alignments mentioned above. At \( T = .2 \) there is a smoothing associated with admixture of other configurations. At \( T = .4, .6 \) there is an increase in \( \langle \Delta J_x^2 \rangle_T \) and \( \langle \Delta J_y^2 \rangle_T \) at small angular velocities caused by the vanishing of pairing correlations (the critical temperature for \( \Delta \rightarrow 0 \) is \( T \approx .5 \text{ MeV} \)); in \( \langle \Delta J_z^2 \rangle_T \) the loss of pairing does not show up because of the axial symmetry at \( I = 0 \). Up to \( T = .6 \text{ MeV} \) the three quantities behave very smoothly. The overall reduction in \( \langle \Delta J_x^2 \rangle_T \) and \( \langle \Delta J_y^2 \rangle_T \) at \( T = .8, 1.0 \text{ MeV} \) is due\(^{(17)}\) to the decrease of the deformation \( \beta \) at high temperatures.

In Fig. 5 the fluctuations in the number operator for neutron \( \langle \Delta N_N^2 \rangle_T \) and proton \( \langle \Delta N_p^2 \rangle_T \) is depicted. The general behavior of these quantities is determined by the pairing gap. For \( \Delta \rightarrow 0 \) the number operator becomes a conserved symmetry, and the fluctuations are zero. What one sees in Fig. 5 is the dramatic effect of the cranking velocity and the temperature on the pairing correlations.

In Fig. 6 we show the quantities \( E_{JJ}, E_{NN} \) and \( E_{ZZ} \) of eq. (77). Again, only at low temperatures do the irregularities associated with the alignments show up. For \( T \geq .6 \) only the contribution stemming from the angular momentum, which is itself rather smooth, is different from zero. The nondiagonal contribution \( E_{JN} \) and \( E_{JZ} \) are very small everywhere. Finally, in Fig. 7, the total contribution of the Goldstone modes to the correlation energy is drawn: on the
upper part of Fig. 7 the quantity $E_{GM}^{(+)}$ of eq. (76); at $T = 0$ and $\omega = 0$ gives a lowering of the energy by $\sim 2$ MeV, rising to $\sim 1.1$ MeV at $\omega \sim .225$, to $\sim .6$ at $\omega \sim .325$ MeV and to $\sim .4$ MeV at $\omega \sim .5$. At temperatures $T = .2$ and .4 MeV we still observe angular-velocity dependence, but again for $T \geq .6$ MeV the contribution is rather smooth. On the lower part of Fig. 7 is shown the total contribution of the negative-signature Goldstone mode, eq. (78), to the total energy. At $T = 0$ and $\omega = 0$ the contribution is $\sim .53$ MeV; at $\omega \sim .225$ it increases to $-.32$ MeV and then smoothly decreases to $-.35$ at $\omega = .6$ MeV. At $T = .2$, .4 their sharp edges are slowly smoothed and up to $T > .6$ MeV the behavior is flattened at around $-.3$ MeV.

5. Conclusion

We have investigated the role of the Goldstone modes at high excitation energy and high angular momentum. We have shown that in the Finite Temperature Cranked Random Phase Approximation the modes associated with the broken symmetries, i.e., angular momentum and particle number, separate from the normal modes as in the zero-temperature theory. We have done a numerical application to a realistic calculation, namely the nucleus $^{164}$Er. For this nucleus, using the Baranger-Kumar Hamiltonian and configuration space, we have solved the FT-CRPA equations for several temperatures and angular velocities. We have discussed in detail the mass tensor, the fluctuations of the angular momentum and particle number operators as well as the contribution of the Goldstone modes to the correlation energy.

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Figure Captions

Fig. 1. (a) The angular momentum $I(h)$ as a function of the angular velocity for different temperatures for $^{164}$Er.

(b) The dynamic moment of inertia $\mathcal{J}^{(2)}$ versus the angular velocity for different temperatures for $^{164}$Er.

Fig. 2. The mass parameter $g_{NN}^{-1}$ for pairing rotations in the neutron gauge space as a function of the cranking velocity for different temperatures.

Fig. 3. Same as Fig. 2 for protons.

Fig. 4. The fluctuations of the angular momentum operators as a function of the angular velocity for different temperatures.

Fig. 5. The fluctuations of the particle number operator (upper part for neutrons, lower part for protons) as a function of the angular velocity for different temperatures.

Fig. 6. The contributions of the Goldstone modes to the correlation energy, eq. (77), as a function of the angular velocity for different temperatures.

Fig. 7. The total energies $E_{GM}^{(+)}$, eq. (76) $E_{GM}^{(-)}$, eq. (78) as a function of the angular velocity for different temperatures.
Fig. 1
Fig. 2
Fig. 3
Fig. 4
Fig. 5
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