**Title**
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**Journal**
Journal of Mathematical Physics, 55(10)

**ISSN**
0022-2488



**Author**
Schechter, Martin



**Publication Date**
2014-10-01

**DOI**
10.1063/1.4897443

Peer reviewed

# GROUND STATE SOLUTIONS FOR NON-AUTONOMOUS DYNAMICAL SYSTEMS

**Martin Schechter** *

*Department of Mathematics, University of California, Irvine, CA 92697-3875, U.S.A.*

March 27, 2014

**Abstract**

We study the existence of periodic solutions for a second order non-autonomous dynamical system. We allows both sublinear and superlinear problems. We obtain ground state solutions.

## 1 Introduction

We consider the following problem. One wishes to solve

$$-\ddot{x}(t) = B(t)x(t) + \nabla_x V(t, x(t)), \tag{1}$$

where

$$x(t) = (x_1(t), \cdots, x_n(t)) \tag{2}$$

is a map from $I = [0, T]$ to $\mathbb{R}^n$ such that each component $x_j(t)$ is a periodic function in $H^1$ with period $T$, and the function $V(t, x) = V(t, x_1, \cdots, x_n)$ is continuous from $\mathbb{R}^{n+1}$ to $\mathbb{R}$ with

$$\nabla_x V(t, x) = (\partial V/\partial x_1, \cdots, \partial V/\partial x_n) \in C(\mathbb{R}^{n+1}, \mathbb{R}^n). \tag{3}$$

We shall study this problem under several sets of assumptions. Our assumption on $B(t)$ is

*Mathematics Subject Classification (2000): Primary 35J20, 35J25, 35J60, 35Q55, 35J65, 47J30, 49B27, 49J40, 58E05. Key words and phrases: Critical points, linking, dynamical systems, periodic solutions.

$(B1)$ The elements of the symmetric matrix $B(t)$ are real-valued functions $b_{jk}(t) = b_{kj}(t) \in L^1(I)$.

Although this assumption is very weak, it is sufficient to allow us to find an extension $\mathcal{D}$ of the operator

$$\mathcal{D}_0 x = -\ddot{x}(t) - B(t)x(t)$$

having a discrete, countable spectrum consisting of isolated eigenvalues of finite multiplicity with a finite lower bound -L

$$-\infty \ < \ -L \ \leq \ \lambda_0 \ < \ \lambda_1 \ < \ \lambda_2 \ < \ \ldots \ < \ \lambda_l \ < \ \ldots \ . \tag{4}$$

.

Let $\lambda_l$ be the first positive eigenvalue of $\mathcal{D}$. We allow $\lambda_{l-1} = 0$. We have

**Theorem 1.1.** *Assume*

1. $$2V(t,x) \geq \lambda_{l-1}|x|^2, \quad t \in I, \ x \in \mathbb{R}^n.$$
2. *There are positive constants $\mu < \lambda_l$ and $m$ such that*
$$2V(t,x) \leq \mu|x|^2, \quad |x| \leq m, \ x \in \mathbb{R}^n.$$
3. *There is a constant $C$ such that*
$$V(t,x) \leq C(|x|^2 + 1), \quad t \in I, \ x \in \mathbb{R}^n.$$
4. *The function given by*
$$H(t,x) = 2V(t,x) - \nabla_x V(t,x) \cdot x \tag{5}$$
*satisfies*
$$H(t,x) \leq W(t) \in L^1(I), \quad t \in I, \ x \in \mathbb{R}^n, \tag{6}$$
*and*
$$H(t,x) \to -\infty, \quad |x| \to \infty, \ t \in I, \ x \in \mathbb{R}^n. \tag{7}$$

*Then the system*

$$\mathcal{D}x(t) = \nabla_x V(t, x(t)) \tag{8}$$

*has a solution.*

**Theorem 1.2.** *If, in addition, we assume*

*There are a constant $\gamma > \lambda_l$ and a function $W(t) \in L^1(I)$ such that*

$$2V(t,x) \geq \gamma|x|^2 - W(t), \quad t \in I,\ x \in \mathbb{R}^n.$$

*then the system (8) has a nontrivial solution.*

**Theorem 1.3.** *The system (8) has a solution if we assume*

1.
$$2V(t,x) \geq \lambda_{l-1}|x|^2 - W(t), \quad t \in I,\ x \in \mathbb{R}^n.$$

2.
$$2V(t,x) \leq \lambda_l|x|^2 + W(t), \quad t \in I,\ x \in \mathbb{R}^n.$$

3. *There is a constant $C$ such that*
$$V(t,x) \leq C(|x|^2 + 1), \quad t \in I,\ x \in \mathbb{R}^n.$$

4. *The function given by*
$$H(t,x) = 2V(t,x) - \nabla_x V(t,x) \cdot x \tag{9}$$
*satisfies*
$$H(t,x) \leq W(t) \in L^1(I), \quad t \in I,\ x \in \mathbb{R}^n, \tag{10}$$
*and*
$$H(t,x) \to -\infty, \quad |x| \to \infty,\ t \in I,\ x \in \mathbb{R}^n, \tag{11}$$
*where $W(t) \in L^1(I)$.*

**Theorem 1.4.** *The conclusions of Theorems 1.1 - 1.3 hold if we replace (10) and (7) with*

$$H(t,x) \geq -W_1(t) \in L^1(I), \quad x \in \mathbb{R}^n,\ t \in I, \tag{12}$$

*and*

$$H(t,x) \to \infty, \quad |x| \to \infty,\ t \in I,\ x \in \mathbb{R}^n. \tag{13}$$

**Theorem 1.5.** *The conclusions of Theorems 1.1 - 1.3 hold if in place of (10), (7) we assume*

$$H(t,x) \geq -W_1(t)(|x|^\alpha + 1), \quad x \in \mathbb{R}^n,\ t \in I, \tag{14}$$

*and*

$$\nu(t) := \liminf_{|x|\to\infty} H(t,x)/|x|^\alpha > 0 \quad a.e. \tag{15}$$

*for some $\alpha$ satisfying $0 < \alpha \leq 2$, where $W_1(t) \in L^1(I)$.*

**Theorem 1.6.** *The conclusions of Theorems 1.1 - 1.3 hold if in place of (10), (7) we assume that there is an* $\alpha$ *satisfying* $0 < \alpha \leq 2$ *such that*

$$H(t,x) \leq W_1(t)(|x|^\alpha + 1), \quad t \in I,\ x \in \mathbb{R}^n, \tag{16}$$

*and*

$$\nu(t) := \limsup_{|x|\to\infty} H(t,x)/|x|^\alpha < 0 \quad a.e., \tag{17}$$

*where* $W_1(t) \in L^1(I)$.

Let $\mathcal{M}$ be the set of all solutions of

$$\mathcal{D}x(t) = \nabla_x V(t, x(t)). \tag{18}$$

A solution $\tilde{x}$ is called a "ground state solution" if it minimizes the functional

$$G(x) = d(x) - 2\int_I V(t,x)\,dt \tag{19}$$

over the set $\mathcal{M}$.

We have

**Theorem 1.7.** *Under the hypotheses of any of the Theorems 1.1 - 1.6, system (18) has a ground state solution.*

The periodic non-autonomous problem

$$\ddot{x}(t) = \nabla_x V(t, x(t)), \tag{20}$$

has an extensive history in the case of singular systems (cf., e.g., Ambrosetti-Coti Zelati [1]). The first to consider it for potentials satisfying (3) were Berger and the author [5] in 1977. We proved the existence of solutions to (8) under the condition that

$$V(t,x) \to \infty \quad \text{as} \quad |x| \to \infty$$

uniformly for a.e. $t \in I$. Subsequently, Willem [54], Mawhin [25], Mawhin-Willem [27], Tang [47, 48], Tang-Wu [51, 52], Wu-Tang [55] and others proved existence under various conditions (cf. the references given in these publications).

Most previous work considered the case when $B(t) = 0$. Ding and Girardi [11] considered the case of (1) when the potential oscillates in magnitude and sign,

$$-\ddot{x}(t) \;=\; B(t)x(t) \;+\; b(t)\nabla W(x(t))\;, \tag{21}$$

and found conditions for solutions when the matrix $B(t)$ is symmetric and negative definite and the function $W(x)$ grows superquadratically and satisfies a homogeneity condition. Antonacci [3, 4] gave conditions for existence of solutions with stronger constraints on the potential but without the homogeneity condition, and without the negative definite condition on the matrix. Generalizations of the above results are given by Antonacci and Magrone [2], Barletta and Livrea [6], Guo and Xu [16], Li and Zou [24], Faraci and Livrea [15], Bonanno and Livrea [7, 8], Jiang [21, 22], Shilgba [39, 40], Faraci and Iannizzotto [14] and Tang and Xiao [53].

Some authors considered the second order system (1) where the potential function $V(t,x)$ is quadratically bounded as $|x| \to \infty$. Berger and Schechter [5] considered the case of (1) where $B(t)$ is a constant symmetric matrix that is positive definite, and showed existence of solutions when the magnitude of $\nabla_x V(t,x)$ is uniformly bounded, the potential is strictly convex, and if $y(t)$ is a $T$-periodic solutions of the linear system $-\ddot{y} = Ay$, then there exists a function $x(t)$ which is weakly differentiable with $\dot{x} \in L^2(I, \mathbb{R}^n)$ and satisfies

$$\int_0^T \langle \nabla_x V(t, x(t)),\, y(t) \rangle_{\mathbb{R}^n}\, dt \;=\; 0\;.$$

Han [17] gave conditions for existence of solutions when $B(t)$ was a multiple of the identity matrix, the system satisfies the resonance condition, and the potential has upper and lower subquadratic bounds. Li and Zou [24] considered the case where $B(t)$ is continuous and nonconstant and the system satisfies the resonance condition, and showed existence of solutions when the potential is even and grows no faster than linearly. Tang and Wu [49] required the function that satisfies the resonance condition to pass through the zero vector, and gave upper and lower conditions for subquadratic growth of the magnitude of $V(t,x)$ without the requirement that the potential be even. Faraci [13] considered the case where for each $t \in I$, $B(t)$ is negative definite with elements that are bounded but not necessarily continuous and the potential has an upper quadratic bound as $|x| \to \infty$, showing existence of a solution when the gradient of the potential is bounded near the origin and exceeds the matrix product in at least one direction.

We shall prove Theorems 1.1 – 1.7 in Section 3 after we introduce the operator $\mathcal{D}$ in the next section. We use linking and sandwich methods of critical point theory and then apply the monotonicity trick introduced by Struwe in [42, 43] for minimization problems. (This trick was also used by others to solve Landesman-Lazer type problems, for bifurcation problems, for Hamiltonian systems and Schrödinger equations.)

The theory of sandwich pairs began in [41] and [34, 35] and was developed in subsequent publications such as [36, 37].

# 2 The operator $\mathcal{D}$

In proving our theorems we shall make use of the following considerations.

We define a bilinear form $a(\cdot,\,\cdot)$ on the set $L^2(I,\mathbb{R}^n)\times L^2(I,\mathbb{R}^n)$,

$$a(\,u,\,v\,)\;=\;(\,\dot{u},\,\dot{v}\,)\;+\;(\,u,v\,)\;. \tag{22}$$

The domain of the bilinear form is the set $D(a)=H$, cosisting of those periodic $x(t)=(x_1(t),\cdots,x_n(t))\in L^2(I,\mathbb{R}^n)$ having weak derivatives in $L^2(I,\mathbb{R}^n)$. $H$ is a dense subset of $L^2(I,\mathbb{R}^n)$. Note that $H$ is a Hilbert space. Thus we can define an operator $\mathcal{A}$ such that $u\in D(\mathcal{A})$ if and only if $u\in D(a)$ and there exists $g\in L^2(I,\mathbb{R}^n)$ such that

$$a(\,u,\,v\,)\;=\;(\,g,\,v\,),\quad v\in D(a). \tag{23}$$

If $u$ and $g$ satisfy this condition we say $\mathcal{A}u=g$.

**Lemma 2.1.** *The operator $\mathcal{A}$ is a self-adjoint Fredholm operator from $L^2(I,\mathbb{R}^n)$ to $L^2(I,\mathbb{R}^n)$. It is one-to-one and onto.*

*Proof.* Let $f\in L^2(I,\mathbb{R}^n)$. Then

$$(v,f)\le\|v\|\cdot\|f\|\le\|v\|_H\|f\|,\quad v\in H.$$

Thus $(v,f)$ is a bounded linear functional on $H$. Since $H$ is complete, there is a $u\in H$ such that

$$(u,v)_H=(f,v),\quad v\in H.$$

Consequently, $u\in D(\mathcal{A})$ and $\mathcal{A}u=f$. Moreover, if $\mathcal{A}u=0$, then

$$(u,v)_H=0,\quad v\in H.$$

Thus, $u=0$. Hence, $\mathcal{A}$ is one-to-one and onto.

For any two functions $x,y\in D(\mathcal{A})$,

$$(\,\mathcal{A}x,y\,)\;=\;(\,\dot{x},\dot{y}\,)\;+\;(\,x,y\,)\;=\;(\,x,\mathcal{A}y\,)\;. \tag{24}$$

Thus, $\mathcal{A}$ is symmetric. It is now easy to show that $D(\mathcal{A})\subset D(a)$ is also a dense subset of $L^2(I,\mathbb{R}^n)$. In fact, if $f\in L^2(I,\mathbb{R}^n)$ satisfies $(f,v)=0\ \forall v\in D(\mathcal{A})$, then $w=\mathcal{A}^{-1}f$ satisfies $(w,\mathcal{A}v)=(\mathcal{A}w,v)=0\ \forall\ v\in D(\mathcal{A})$. Since $\mathcal{A}$ is onto, $w=0$. Hence , $f=\mathcal{A}w=0$.

Next, we show that $\mathcal{A}$ is self-adjoint. Consider any $u,f\in L^2(I,\mathbb{R}^n)$, and suppose for any $v\in D(\mathcal{A})$,

$$(\,u,\mathcal{A}v\,)\;=\;(\,f,v\,)\;. \tag{25}$$

Since $\mathcal{A}$ is onto and $f \in L^2(I, \mathbb{R}^n)$, there exists $w \in D(\mathcal{A})$ such that $\mathcal{A}w = f$. Then using (24),

$$( u - w, \mathcal{A}v ) \ = \ ( f, v ) \ - \ ( \mathcal{A}w, v ) \ = \ 0 \ .$$

Since $u - w \in L^2(I, \mathbb{R}^n)$, we can find a $v \in D(\mathcal{A})$ such that $\mathcal{A}v = u - w$, and

$$\|u \ - \ w\|^2 \ = \ 0.$$

This implies $u = w$ in the space $L^2(I, \mathbb{R}^n)$, and therefore $u \in D(\mathcal{A})$. Hence, $\mathcal{A}u = \mathcal{A}w = f$. □

**Lemma 2.2.** *The essential spectrum of $\mathcal{A}$ is the null set.*

*Proof.* By (2.1), $\mathcal{A}$ is linear, self-adjoint, and onto $L^2(I, \mathbb{R}^n)$.

Next, we note that

$$\| \mathcal{A}^{-1} u \| \ \leq \ \|u\| \ .$$

To see this, let $f = \mathcal{A}u$. Then $u = \mathcal{A}^{-1}f$, and

$$(u, v)_H = (f, v), \quad v \in H.$$

Thus,

$$\|u\|_H^2 \leq \|f\| \cdot \|u\| \leq \|f\| \cdot \|u\|_H.$$

Hence, $\|u\| \leq \|f\|$.

Now we show that the inverse operator $\mathcal{A}^{-1}$ is compact on $L^2(I, \mathbb{R}^n)$. Let $(u_k)$ be a bounded sequence in $L^2(I, \mathbb{R}^n)$, and let $C > 0$ satisfy for each $k$, $\|u_k\| \leq C$. By applying the inverse operator, let $(x_k)$ be the sequence such that for each $k$, $\mathcal{A}x_k = u_k$. From the above statements, for each $k$, $\|x_k\| \leq C$. From the definition of the operator $\mathcal{A}$, for any $x \in D(\mathcal{A})$,

$$( \mathcal{A}x, x ) \ = \ ( \dot{x}, \dot{x} ) \ + \ ( x, x ) \ = \ \|x\|_H^2 \ \geq \ 0 \ .$$

Hence, $\mathcal{K} = \mathcal{A}^{-1}$ is a positive compact operator, and the eigenvalues $\mu_k$ of $\mathcal{K}$ are denumerable and have 0 as their only possible limit point. The eigenfunctions $\phi_k$ of $\mathcal{K}$ are also eigenfunctions of $\mathcal{K}^{-1} = \mathcal{A}$ and satisfy

$$\mathcal{A}\, \phi_k \ = \ \frac{1}{\mu_k}\, \phi_k \ .$$

Since the values $\mu_k$ are bounded and have no limit point except 0, there are no limit points of the set $(1/\mu_k)$ and the essential spectrum of $\mathcal{A}$ is the null set. □

We will use two theorems of Schechter [30] on bilinear forms to prove Lemma 2.5.

**Theorem 2.3.** *Let $a(\cdot,\,\cdot)$ be a closed Hermitian bilinear form with dense domain in $L^2(I,\mathbb{R}^n)$. If for some real number $N$,*

$$a(\,u,\,u\,)\;+\;N\|u\|^2\;\geq\;0\;, \tag{26}$$

*then the operator $\mathcal{A}$ associated with $a(\cdot,\,\cdot)$ is self-adjoint and $\sigma(A)\subset[-N,\infty)$.*

**Theorem 2.4.** *Suppose $a(\cdot,\,\cdot)$ is a bilinear form satisfying the hypotheses of Theorem 2.3. Let $b(\cdot,\,\cdot)$ be a Hermitian bilinear form such that $D(a)\subset D(b)$ and for some positive real number $K$, for any $u\in D(a)$,*

$$|b(\,u,\,u\,)|\;\leq\;Ka(\,u,\,u\,)\;. \tag{27}$$

*Assume that every sequence $(u_k)\subset D(a)$ which satisfies*

$$\|u_k\|^2\;+\;a(\,u_k,\,u_k\,)\;\leq\;C \tag{28}$$

*has a subsequence $(v_j)$ such that*

$$b(\,v_j-v_k,\,v_j-v_k\,)\;\to\;0\;. \tag{29}$$

*Assume also that if (28),*(29) *hold and $v_j\to 0$ in the $L^2(I,\mathbb{R}^n)$ norm, then $b(\,v_j,\,v_j\,)\to 0$. Set*

$$c(\,u,\,v\,)\;=\;a(\,u,\,v\,)\;+\;b(\,u,\,v\,)\;. \tag{30}$$

*and let $\mathcal{A},\mathcal{C}$ be the operators associated with $a,c$, respectively. Then*

$$\sigma_e(\mathcal{A})\;=\;\sigma_e(\mathcal{C})\;.$$

Let

$$b(\,u,\,v\,)\;=\;-\sum_{j=1}^{n}\sum_{k=1}^{n}\int_0^T(b_{jk}(t)\;+\;\delta_{jk})\,u_k(t)v_j(t)\,dt \tag{31}$$

and

$$d(u,v)=a(u,v)+b(u,v). \tag{32}$$

We shall prove

**Lemma 2.5.** *The operator $\mathcal{D}$ associated with the bilinear form $d(\cdot,\,\cdot)$ under assumption (B1) is self-adjoint. Its essential spectrum is the null set and there exists a finite real value L such that $\sigma(\mathcal{D})\subset[-L,\infty)$. $\mathcal{D}$ has a discrete, countable spectrum consisting of isolated eigenvalues of finite multiplicity with a finite lower bound -L*

$$-\infty\;<\;-L\;\leq\;\lambda_0\;<\;\lambda_1\;<\;\lambda_2\;<\;\ldots\;<\;\lambda_l\;<\;\ldots\;. \tag{33}$$

To show the bilinear form $b(\cdot,\,\cdot)$ is Hermitian, we can use the symmetry of the matrix $B(t)+I$ to rearrange the order of the finite summation,

$$\begin{aligned} b(\,u,\,v\,) \;&=\; -\sum_{j=1}^{n}\sum_{k=1}^{n}\int_0^T (b_{jk}(t)\;+\;\delta_{jk})\,u_k(t)v_j(t)\,dt \\ &=\; -\sum_{k=1}^{n}\sum_{j=1}^{n}\int_0^T (b_{jk}(t)\;+\;\delta_{jk})\,v_j(t)u_k(t)\,dt \\ &=\; -\sum_{k=1}^{n}\sum_{j=1}^{n}\int_0^T (b_{kj}(t)\;+\;\delta_{kj})\,v_j(t)u_k(t)\,dt \\ &=\; b(\,v,\,u\,)\;. \end{aligned}$$

Also the magnitude of $b(u)=b(u,u)$ is bounded by a multiple of the bilinear form $a(\cdot,\,\cdot)$ and satisfies (27),

$$\begin{aligned} |b(\,u\,)| \;&\le\; K_B\|u\|^2_{L^\infty(I,\mathbb{R}^n)} \\ &\le\; K_B\,(M\|u\|_H)^2 \\ &\le\; K_B\cdot M^2\|u\|^2_H \;=\; Ka(\,u\,)\;. \end{aligned} \tag{34}$$

Consider a sequence $(x_k)\subset D(\mathcal{A})$ which is bounded by a constant $C$ in the $H$ norm. Then each term of the sequence satisfies

$$\|x_k\|^2\;+\;a(\,x_k\,)\;=\;2(\,x_k,\,x_k\,)\;+\;(\,\dot{x}_k,\,\dot{x}_k\,)\;\le\;2\|x_k\|^2_H\;\le\;4C^2\;.$$

Since,

$$\|u\|_{L^\infty(I,\mathbb{R}^n)}\le C\|u\|_H,\quad u\in H, \tag{35}$$

we can find a subsequence $(x_{\bar{k}})$ which converges weakly in $H$ and strongly in $L^\infty(I,\mathbb{R}^n)$ and $L^2(I,\mathbb{R}^n)$ to some function $x\in H$. Because the subsequence is convergent in $L^\infty(I,\mathbb{R}^n)$ it is also Cauchy under this norm. As $\bar{\jmath},\bar{k}\to\infty$ we can apply (34) to show this subsequence satisfies (29),

$$|b(\,x_{\bar{\jmath}}-x_{\bar{k}}\,)|\;\le\;K_B\|x_{\bar{\jmath}}-x_{\bar{k}}\|^2_{L^\infty(I,\mathbb{R}^n)}\;\to\;0\;. \tag{36}$$

If in addition the subsequence $(x_{\bar{k}})$ converges to zero in $L^2(I,\mathbb{R}^n)$, the subsequence must also converge in $L^\infty(I,\mathbb{R}^n)$ to the zero function, and

$$b(\,x_{\bar{k}}\,)\to 0\;.$$

Then the bilinear forms $a(\cdot,\,\cdot)$ and $b(\cdot,\,\cdot)$ satisfy the conditions of Theorem 2.4. The bilinear form $d(\cdot,\,\cdot)$ is the sum of these two bilinear forms as in (30). By this theorem, the operator $\mathcal{D}$ associated with this bilinear form has the same essential spectrum as the operator $\mathcal{A}$ associated with the bilinear form $a(\cdot,\,\cdot)$.Now we show that for any constant $\epsilon>0$ there exists a positive constant $K_\epsilon$ such that

$$|\,b(\,x\,)\,|\;\le\;\epsilon\|\dot{x}\|^2\;+\;K_\epsilon\|x\|^2\quad x\in D(\mathcal{A}). \tag{37}$$

We can use (34) to find a constant $K_B$, and for any $\epsilon > 0$, let $\xi = \epsilon / K_B$. Then there is a constant $C_\xi$ which satisfies

$$
\begin{aligned}
|\, b(\,x\,)\,| \;&\le\; K_B \|x\|^2_{L^\infty(I,\mathbb{R}^n)} \\
&\le\; K_B \left( \frac{\epsilon}{K_B} \|\dot{x}\|^2 \;+\; C_\xi \|x\|^2 \right) \\
&\le\; \epsilon \|\dot{x}\|^2 \;+\; \left( K_B \cdot C_\xi \right) \|x\|^2 \,\big) \;.
\end{aligned}
$$

Setting $K_\epsilon = K_B \cdot C_\xi$ gives the stated inequality.To show $d(\cdot,\,\cdot)$ is closed, first apply (37) with $\epsilon = 1/2$. Thus there is a constant $C_0$ such that

$$
|\, b(\,u\,)\,| \;\le\; \frac{1}{2}\, a(\,u\,) \;+\; C_0 \|u\|^2 \;. \tag{38}
$$

Now suppose a sequence $(u_k) \subset D(d)$ satisfies

$$
d(\,u_j \;-\; u_k\,) \;\to\; 0 \;, \tag{39}
$$

and $(u_k) \to u$ in $L^2(I,\mathbb{R}^n)$. The sequence is Cauchy in $L^2(I,\mathbb{R}^n)$ and as $j,k$ increase

$$
\|\, u_j \;-\; u_k \,\|^2 \;\to\; 0 \;.
$$

Suppose that $u \notin D(d)$. Because the domains of $d(\cdot,\,\cdot)$ and $a(\cdot,\,\cdot)$ are the same, $u \notin D(a)$. We have shown above that $a(\cdot,\,\cdot)$ is closed, so the sequence cannot be Cauchy and as $j,k$ increase $a(\,u_j - u_k\,)$ does not approach zero. But by (39),

$$
a(\,u_j - u_k\,) \;-\; b(\,u_j - u_k\,) \;\to\; 0 \;.
$$

Applying the inequality in (38) bounds the magnitude of each $b(\cdot,\,\cdot)$ term, and since $a(u,u) \ge 0$, the following inequality is satisfied,

$$
a(\,u_j - u_k\,) \;-\; b(\,u_j - u_k\,) \;\ge\; \tfrac{1}{2} a(\,u_j - u_k\,) \;-\; C_0 \|u_j - u_k\|^2 \;.
$$

Adding the last term to both sides leaves only the positive bilinear form on the right side,

$$
\begin{aligned}
a(\,u_j - u_k\,) - b(\,u_j - u_k\,) \;+\; C_0 \|u_j - u_k\|^2 & \\
\ge\; \tfrac{1}{2} a(\,u_j - u_k\,) & \\
\ge\; 0 \;. &
\end{aligned}
$$

As $j,k$ increase the left side of this equation approaches zero so the center term must also approach zero, a contradiction to the statement above. Therefore, $u \in D(a) = D(d)$, and $d(\cdot,\,\cdot)$ is also a closed bilinear form.

Next we show that there exists a positive constant $N$ such that for any $x \in D(a)$,

$$
d(\,x\,) \;+\; N \|x\|^2 \;\ge\; 0 \;. \tag{40}
$$

For any positive constant $\epsilon > 0$ we can find $K_\epsilon$ which satisfies (37) and thereby find a lower bound for $b(\,x\,,x\,)$,

$$a(\,x\,)\;+\;b(\,x\,)\;+\;N\|x\|^2\;\geq\;a(\,x\,)\;-\;\epsilon\|\dot{x}\|^2\;-\;K_\epsilon\|x\|.$$

We have shown that $d(\cdot,\,\cdot)$ is closed, and as the sum of two Hermitian bilinear forms, $d(\cdot,\,\cdot)$ is clearly Hermitian. Its domain is dense in $L^2(I,\mathbb{R}^n)$ and the $N$ in (40) satisfies the conditions of Theorem 2.3, so the operator $\mathcal{D}$ associated with this bilinear form is self-adjoint and has its spectrum bounded below by $-N$. We have shown that the essential spectrum of this operator is the null set, so the spectrum is discrete and we can number the eigenvalues in increasing order, and each eigenvalue is of finite multiplicity.

# 3 Proofs

Proof of Theorem 1.1.

Define the subspaces $M$ and $N$ of $H$ as

$$N\;=\;\bigoplus_{k\,<\,l} E(\lambda_k)\;,\quad M\;=\;N^{\perp}\;,\quad H\;=\;M\oplus N\;.$$

Let

$$G(x) = d(x) - 2\int_I V(t,x)\,dt. \tag{41}$$

We note that Hypothesis 1 implies

$$G(v) \leq 0, \quad v \in N. \tag{42}$$

In fact, we have

$$G(x) = d(x) - 2\int_I V(t,x)\,dt \leq \int_I [\lambda_{l-1}|x|^2 - 2V(t,x)]\,dt \leq 0, \quad x \in N.$$

Note that there is a positive $\rho > 0$ such that

$$|x(t)| < m$$

when $\|x\|_H = \rho$. In fact, we have $|x(t)| \leq c_0\|x\|_H$. If $x \in M$, then

$$G(x) = d(x) - 2\int_I V(t,x)\,dt \geq d(x)[1 - \frac{2\mu\|x\|^2}{d(x)}] > \varepsilon > 0.$$

Take

$$\begin{aligned} A &= \partial\mathbf{B}_\rho \cap M, \\ B &= N, \end{aligned}$$

where

$$\mathbf{B}_\rho = \{x \in H : \|x\|_H < \rho\}.$$

By Example 8, p. 22 of [37], $A$ links $B$. Moreover,

$$\sup_A[-G] \leq 0 \leq \inf_B[-G]. \tag{43}$$

Hence, we may apply Corollary 2.8.2 of [31] to conclude that that there is a sequence $\{x^{(k)}\} \subset H$ such that

$$G(x^{(k)}) = d(x^{(k)}) - 2\int_I V(t, x^{(k)}(t))\, dt \to c \leq 0, \tag{44}$$

$$(G'(x^{(k)}), z)/2 = d(x^{(k)}, z) - \int_I \nabla_x V(t, x^{(k)}) \cdot z(t)\, dt \to 0, \quad z \in H \tag{45}$$

and

$$(G'(x^{(k)}), x^{(k)})/2 = d(x^{(k)}) - \int_I \nabla_x V(t, x^{(k)}) \cdot x^{(k)}\, dt \to 0. \tag{46}$$

Moreover, (44) and (46) imply that

$$\int_I H(t, x^{(k)}(t))\, dt \to -c. \tag{47}$$

If

$$\rho_k = \|x^{(k)}\|_H \leq C,$$

there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in H$ weakly in $H$ and uniformly on $I$. From (45) we see that

$$(G'(x), z)/2 = d(x, z) - \int_I \nabla_x V(t, x(t)) \cdot z(t)\, dt = 0, \quad z \in H,$$

from which we conclude easily that $x$ is a solution of (1). On the other hand, if

$$\rho_k = \|x^{(k)}\|_H \to \infty,$$

let $\tilde{x}^{(k)} = x^{(k)}/\rho_k$. Then there is a renamed subsequence such that $\tilde{x}^{(k)}$ converges to a limit $\tilde{x} \in H$ weakly in $H$ and uniformly on $I$. Let

$$x^{(k)} = w^{(k)} + v^{(k)},\ w^{(k)} \in M,\ v^{(k)} \in N.$$

Since

$$d(\tilde{x}^{(k)}) - 2\int_I V(t, x^{(k)}(t))\, dt/\rho_k^2 \to 0$$

and there is a constant $c_0 > 0$ such that

$$c_0\|x^{(k)}\|_H^2 \leq d(w^{(k)}) - d(v^{(k)}) + \|g^{(k)}\|^2,$$

where $g^{(k)}$ is the component of $x^{(k)}$ in $N(\mathcal{D})$, we have

$$\begin{aligned} c_0 = c_0\|\tilde{x}^{(k)}\|_H^2 &\leq -2d(\tilde{v}^{(k)}) + \|\tilde{g}^{(k)}\|^2 + 2\int_I V(t, x^{(k)}(t))\,dt/\rho_k^2 + o(1) \\ &\leq -2d(\tilde{v}^{(k)}) + \|\tilde{g}^{(k)}\|^2 + 2C\int_I (|\tilde{x}^{(k)}(t)|^2 + \rho_k^{-2})\,dt + o(1) \\ &\to -2d(\tilde{v}) + \|\tilde{g}\|^2 + 2C\int_I |\tilde{x}(t)|^2\,dt. \end{aligned}$$

Hence, $\tilde{x}(t) \not\equiv 0$. Let $\Omega_0 \subset I$ be the set on which $\tilde{x}(t) \neq 0$. The measure of $\Omega_0$ is positive. Moreover, $|x^{(k)}(t)| \to \infty$ as $k \to \infty$ for $t \in \Omega_0$. Thus,

$$\int_I H(t, x^{(k)}(t))\,dt \leq \int_{\Omega_0} H(t, x^{(k)}(t))\,dt + \int_{I\setminus\Omega_0} W(t)\,dt \to -\infty$$

by hypothesis. But this contradicts (47). Hence, the $\rho_k$ are bounded, and the proof is complete.

Proof of Theorem 1.3. We note that Hypothesis 1 implies

$$G(v) \leq Q, \quad v \in N \tag{48}$$

where

$$Q = \int_I W(t)\,dt.$$

In fact, we have

$$G(x) = d(x) - 2\int_I V(t,x)\,dt \leq \int_I [\lambda_{l-1}|x|^2 - 2V(t,x)]\,dt \leq Q, \quad x \in N.$$

If $x \in M$, we have by Hypothesis 2 that

$$\begin{aligned} G(x) &\geq d(x) - \int \lambda_l |x(t)|^2\,dt - Q \\ &\geq (\lambda_l - \lambda_l)\|x\|^2 - Q \geq -Q. \end{aligned} \tag{49}$$

By Theorem 3.17, p.26 of [37], $M$ and $N$ form a sandwich pair. We can now follow the proof of Theorem 1.1 to come to the same conclusion.

Proof of Theorem 1.2 Let

$$y(t) = v + sw_0,$$

where $v \in N$, $s \geq 0$, and $w_0 \in M$ is an eigenfunction of $\mathcal{D}$ corresponding to $\lambda_l$. Consequently,

$$\begin{aligned} G(y) &= s^2 d(w_0) + d(v) - 2\int_I V(t, y(t))\, dt \\ &\leq \lambda_l s^2 \|w_0\|^2 + \lambda_{l-1}\|v\|^2 - 2\gamma \int_I |y(t)|^2\, dt + Q \\ &\leq (\lambda_{l-1} - \gamma)\|v\|^2 + (\lambda_l - \gamma)s^2\|w_0\|^2 + Q \\ &\rightarrow -\infty \ \text{ as } \ s^2 + |v|^2 \rightarrow \infty, \end{aligned}$$

where

$$Q = \int_I W(t)\, dt.$$

Take

$$\begin{aligned} A &= \{v \in N : \|v\|_H \leq R\} \cup \{sw_0 + v : v \in N, s \geq 0, \|sw_0 + v\|_H = R\}, \\ B &= \partial \mathbf{B}_\rho \cap M,\ 0 < \rho < R. \end{aligned}$$

By Example 3, p.38, of [31], $A$ links $B$. Moreover, if $R$ is sufficiently large,

$$\sup_A G \leq 0 < \varepsilon \leq \inf_B G. \tag{50}$$

We may now apply Corollary 2.8.2 of [31] to conclude that that there is a sequence $\{x^{(k)}\} \subset H$ such that

$$G(x^{(k)}) = d(x^{(k)}) - 2\int_I V(t, x^{(k)}(t))\, dt \rightarrow c \geq \varepsilon > 0, \tag{51}$$

$$(G'(x^{(k)}), z)/2 = d(x^{(k)}, z) - \int_I \nabla_x V(t, x^{(k)}) \cdot z(t)\, dt \rightarrow 0, \quad z \in H \tag{52}$$

and

$$(G'(x^{(k)}), x^{(k)})/2 = d(x^{(k)}) - \int_I \nabla_x V(t, x^{(k)}) \cdot x^{(k)}\, dt \rightarrow 0. \tag{53}$$

If

$$\rho_k = \|x^{(k)}\|_H \leq C,$$

there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in H$ weakly in $H$ and uniformly on $I$. From (52) we see that

$$(G'(x), z)/2 = d(x, z) - \int_I \nabla_x V(t, x(t)) \cdot z(t)\, dt = 0, \quad z \in H,$$

from which we conclude easily that $x$ is a solution of (1). Moreover, (53) implies

$$d(x^{(k)}) \rightarrow \int_I \nabla_x V(t, x) \cdot x\, dt = d(x). \tag{54}$$

Consequently,

$$x^{(k)} \to x \tag{55}$$

strongly in $H$. This means that

$$G(x) = d(x) - 2\int_I V(t,x)\,dt = c \geq \varepsilon > 0. \tag{56}$$

But

$$G(0) = -2\int_I V(t,0)\,dt \leq 0.$$

Hence, $x(t) \neq 0$. On the other hand, if

$$\rho_k = \|x^{(k)}\|_H \to \infty,$$

we can follow the proof of Theorem 1.1 to obtain a contradiction. Thus $\rho_k$ is bounded, and the theorem is proved.

In proving Theorem 1.4 we merely replace $H(t,x)$ by $-H(t,x)$ in Theorems 1.1 and 1.2.

Proof of Theorem 1.5.
In this case we have

$$\underline{\lim} \int_{\Omega_0} \frac{H(t,x^{(k)})}{\rho_k^\alpha} \geq \int_{\Omega_0} \underline{\lim}\, \frac{H(t,x^{(k)})}{|x^{(k)}|^\alpha}\,|\tilde{x}^{(k)}|^\alpha \geq \int_{\Omega_0} \nu(t)\,|\tilde{x}|^\alpha \geq 0,$$

where $\Omega_0$ is the set where $|x(t)^{(k)}| \to \infty$. Moreover, $\tilde{x} = 0$ on $\Omega_1 = I\backslash\Omega_0$. Since $\nu(t) > 0$ a.e., it follows that $\tilde{x} = 0$ a.e. The rest of the proof proceeds as before.

Proof of Theorem 1.6.
In this case we have

$$\overline{\lim} \int_{\Omega_0} \frac{H(t,x^{(k)})}{\rho_k^\alpha} \leq \int_{\Omega_0} \overline{\lim}\, \frac{H(t,x^{(k)})}{|x^{(k)}|^\alpha}\,|\tilde{x}^{(k)}|^\alpha \leq \int_{\Omega_0} \nu(t)\,|\tilde{x}|^\alpha \leq 0,$$

where $\Omega_0$ is the set where $|x(t)^{(k)}| \to \infty$. Moreover, $\tilde{x} = 0$ on $\Omega_1 = I\backslash\Omega_0$. Since $\nu(t) < 0$ a.e., it follows that $\tilde{x} = 0$ a.e. The rest of the proof proceeds as before.

Proof of Theorem 1.7.

Let

$$\alpha = \inf_{\mathcal{M}} G(x).$$

There is a sequence $\{x^{(k)}\} \in \mathcal{M}$ such that

$$G(x^{(k)}) = d(x^{(k)}) - 2\int_I V(t, x^{(k)}(t))\,dt \to \alpha, \tag{57}$$

$$(G'(x^{(k)}), z)/2 = d(x^{(k)}, z) - \int_I \nabla_x V(t, x^{(k)}) \cdot z(t)\,dt = 0, \quad z \in H \tag{58}$$

and

$$(G'(x^{(k)}), x^{(k)})/2 = d(x^{(k)}) - \int_I \nabla_x V(t, x^{(k)}) \cdot x^{(k)}\,dt = 0. \tag{59}$$

Thus,

$$\int_I H(t, x^{(k)}(t))\,dt = -G(x^{(k)}) \to -\alpha.$$

This implies that

$$\rho_k = \|x^{(k)}\|_H \le C.$$

Hence, there is a renamed subsequence such that $x^{(k)}$ converges to a limit $x \in H$ weakly in $H$ and uniformly on $I$. From (57) and (58) we see that

$$G(x) = d(x) - 2\int_I V(t, x(t))\,dt = \alpha, \tag{60}$$

and

$$(G'(x), z)/2 = d(x, z) - \int_I \nabla_x V(t, x(t)) \cdot z(t)\,dt = 0, \quad z \in H,$$

from which we conclude easily that $x$ is a solution of (1). Hence, $x \in \mathcal{M}$ and $G(x) = \alpha$. This completes the proof.

# References


[1] A. Ambrosetti and V. Coti Zelati, Periodic Solutions of Singular Lagrangian Systems, *Birkhäuser, Boston,* 1993.

[2] F. Antonacci and P. Magrone, Second order nonautonomous systems with symmetric potential changing sign, Rend. Mat. Appl., 1998, 18, 367–379.

[3] F. Antonacci, Periodic and homoclinic solutions to a class of Hamiltonian systems with indefinite potential in sign, Boll. Un. Mat. Ital., 1996, 10, 303–324

[4] F. Antonacci, Existence of periodic solutoins of Hamiltonian systems with potential indefinite in sign, Nonlinear Anal., 1997, 29, 1353–1364

[5] M.S. Berger and M. Schechter, On the solvability of semilinear gradient operator equations, *Adv. Math.* **25** (1977) 97-132.

[6] G. Barletta and R. Livrea, Existence of three periodic solutions for a non-autonomous second order system, Matematiche (Catania), 2002, 57, 205–215

[7] G. Bonanno and R. Livrea, Periodic solutions for a class of second-order Hamiltonian systems, Electron. J. Diff. Eq., 2005, 115, 13

[8] G. Bonanno and R. Livrea, Multiple periodic solutions for Hamiltonian systems with not coercive potential, J. Math. Anal. Appl., 2010, 363, 627–638

[9] H. Brézis and L. Nirenberg, Remarks on finding critical points. *Comm. Pure Appl. Math.* **44** (1991) 939–964.

[10] A. K. Ben Naoum, C. Troestler and M. Willem, Existence and multiplicity results for homogeneous second order differential equations, *J. Diff. Eq.* **112** (1994) 239–249.

[11] Y. Ding and M. Girardi, Periodic and homoclinic solutions to a class of Hamiltonian systems with the potentials changing sign, Dynam. Systems Appl., 1993, 2, 131–145.

[12] I. Ekeland and N. Ghoussoub, Certain new aspects of the calculus of variations in the large, *Bull. Amer. Math. Soc.* **39** (2002) 207-265.

[13] F. Faraci, Multiple periodic solutions for second order systems with changing sign potential, J. Math. Anal. Appl., 2006, 319, 567–578

[14] F. Faraci and A. Iannizzotto, A multiplicity theorem for a perturbed second-order non-autonomous system, Proc. Edinb. Math. Soc. (2), 2006, 49, 267–275

[15] F. Faraci and G. Livrea, Infinitely many periodic solutions for a second-order nonautonomous system, Nonlinear Anal., 2003, 54, 417–429

[16] Z. Guo and Y. Xu, Existence of periodic solutions to second-order Hamiltonian systems with potential indefinite in sign, Nonlinear Anal., 2002, 51, 1273–1283

[17] Z. Han, $2\pi$-periodic solutions to ordinary differential systems at resonance, Acta Math. Sinica (Chinese), 2000, 43, 639–644

[18] Z. Q. Han, Periodic solutions to Second Differential Systems

[19] L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on $\mathbf{R}^N$, *Proc. Roy. Soc. Edinburgh* **A 129** (1999), 787-809.

[20] L. Jeanjean, Local conditions insuring bifurcation from the continuous spectrum. Math. Z. **232**, 651-664(1999)

[21] M. Jiang, Periodic solutions of second order superquadratic Hamiltonian systems with potential changing sign I, J. Diff. Eq., 2005, 219, 323–341

[22] M. Jiang, Periodic solutions of second order superquadratic Hamiltonian systems with potential changing sign II, J. Diff. Eq., 2005, 219, 342–362

[23] Y. M. Long, Nonlinear oscillations for classical Hamiltonian systems wit bi-even subquadritic potentials, *Nonlinear Anal.* **24** (1995) 1665–1671.

[24] S. Li and W. Zou, Infinitely many solutions for Hamiltonian systems, J. Diff. Eq., 2002, 186, 141–164

[25] J. Mawhin, Semi coercive monotone variational problems, *Acad. Roy. Belg.Bull. Cl. Sci.* **73** (1987) 118–130.

[26] J. Mawhin and M. Willem, Critical points of convex perturbations of some indefinite quadratic forms ans semilinear boundary value problems at resonance, *Ann. Inst. H. Poincaré Anal. Non Linearé* **3** (1986) 431–453.

[27] J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems, *Springer-Verlag, New York,* 1989.

[28] J. Ma and C. L. Tang, Periodic solutions for some nonautonomous second-order systems. *J. Math. Anal. Appl.* **275** (2002), no. 2, 482–494.

[29] P. H. Rabinowitz, On subharmonic solutions of Hamiltonian systems, *Comm. Pure. Appl. Math.* **33** (1980) 609–633.

[30] M. Schechter, Operator Methods in Quantum Mechanics, Elsevier, New York, 1981

[31] M. Schechter, Linking methods in critical point theory, *Birkhäuser, Boston,* 1999.

[32] M. Schechter, Periodic non-autonomous second-order dynamical systems. J. Differential Equations 223 (2006), no. 2, 290–302.

[33] M. Schechter, Periodic solutions of second-order nonautonomous dynamical systems. Bound. Value Probl. 2006, Art. ID 25104, 9 pp.

[34] M. Schechter, New saddle point theorems. Generalized functions and their applications (Varanasi, 1991), 213–219, Plenum, New York, 1993.

[35] M. Schechter, A generalization of the saddle point method with applications, Ann. Polon. Math. 57 (1992), no. 3, 269–281.

[36] M. Schechter, Sandwich pairs in critical point theory. Trans. Amer. Math. Soc. 360 (2008), no. 6, 2811–2823.

[37] M. Schechter, Minimax Systems and Critical Point Theory, Birkhauser Boston, 2009.

[38] M. Schechter, Custom monotonicity methods, J. Differential Equations. 246 (2009), no. 8, 3398–3415.

[39] L. K. Shilgba, Periodic solutions of non-autonomous second order systems with quasisubadditive potential, J. Math. Anal. Appl., 1995, 189, 671–675

[40] L. K. Shilgba, Existence result for periodic solutions of a class of Hamiltonian systems with super quadratic potential, Nonlinear Anal., 2005, 63, 565–574

[41] E.A. de B.e. Silva, Linking theorems and applications to semilinear elliptic problems at resonance, Nonlinear Analysis TMA 16(1991) 455–477.

[42] M. Struwe, The existence of surfaces of constant mean curvature with free boundaries. *Acta Math.* **160**, 19-64(1988)

[43] M. Struwe, Variational Methods, *Springer-Verlag, Second Edition,* 1996

[44] M. Schechter and W. Zou, Double linking theorem and multiple solutions. *J. Funct. Anal.* 205 (2003), no. 1, 37–61.

[45] M. Schechter and W. Zou, Superlinear problems. *Pacific J. Math.* 214 (2004), no. 1, 145–160.

[46] M. Schechter and K. Tintarev, Pairs of critical points produced by linking subsets with applications to semilinear elliptic problems, *Bull. Soc. Math. Belg.* **44** (1994), 249-261.

[47] C. L. Tang, Periodic solutions of nonautonomous second order systems with $\gamma$ -quasisubadditive potential, *J. Math. Anal. Appl.* **189** (1995) 671–675.

[48] C. L. Tang, Periodic solutions of nonautonomous second order systems with sublinear nonlinearity, *Proc. Amer. Math. Soc.* **126** (1998) 3263–3270.

[49] C. Tang and X. Wu, Periodic solutions for a class of nonautonomous subquadratic second Hamiltonian systems, J. Math. Anal. Appl., 2002, 275, 870–882

[50] C. L. Tang and X. P. Wu, Notes on periodic solutions of subquadratic second order systems. *J. Math. Anal. Appl.* **285** (2003), no. 1, 8–16.

[51] C. L. Tang and X. P. Wu, Periodic solutions for a class of nonautonomous subquadratic second order Hamiltonian systems. *J. Math. Anal. Appl.* **275** (2002), no. 2, 870–882.

[52] C. L. Tang and X. P. Wu, Periodic solutions for second order systems with not uniformly coercive potential. *J. Math. Anal. Appl.* **259** (2001), no. 2, 386–397.

[53] X. H. Tang and L. Xiao, Existence of periodic solutions to second-order Hamiltonian systems with potential indefinite in sign, Nonlinear Anal., 2008, 69, 3999–4011

[54] M. Willem, Oscillations forcées systèmes hamiltoniens, *Public. Sémin. Analyse Non Linéarie, Univ. Beseancon,* 1981.

[55] X. P. Wu, C. L. Tang, Periodic solutions of a class of nonautonomous second order systems, *J. Math. Anal. Appl.* **236** (1999) 227–235.

[56] M. Willem and W. Zou, On a semilinear Dirichlet problem and a nonlinear Schrödinger equation with periodic potential. (Preprint)

[57] Z. Wang and J. Zhang, Periodic solutions of second order non-autonomous Hamiltonian systems with local superquadric potential