Role of Convective Cells in Nonlinear Interaction of Kinetic Alfven Waves

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Publication Date
2015

Peer reviewed|Thesis/dissertation
Role of Convective Cells in Nonlinear Interaction of Kinetic Alfvén Waves

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Physics

by

Onnie Luk

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2015
DEDICATION

To my family...
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ACKNOWLEDGMENTS

First and foremost, I would like to thank my thesis advisor Professor Zhihong Lin for all the support he has provided for me throughout graduate school. He always keeps his office door open to make sure his help is accessible. Whenever I knock on his office door to ask for guidance on my research project, he would drop his current work when possible and give me his full attention. Also, his patience for me is tremendous. He provided much-needed help for me during the revision process of my thesis. I cannot thank him enough.

I thank the plasma physics professors in the department for sharing their knowledge and insights. Professor Liu Chen provides me inspiring feedbacks on my research project whenever we have a discussion. I thank Professors Bill Heidbrink, Roger McWilliams and Toshi Tajima for sharing their extensive knowledge in plasma physics in the courses they taught. I would also like to thank Professor Norman Rostoker. Norman was always very kind to me. I enjoyed listening to the life stories he shared with me.

I thank my current and former colleagues and friends in the Plasma Theory Group, including Animesh Kuley, Calvin Lau, Dan Fulton, Dongjian Liu, Huasen Zhang, Ihor Holod, Jian Bao, Joey McClanaghan, Kurt Tummel, Peng Jiang, Peter Porazik, Sam Taimourzadeh, Wenjun Deng, Wenlu Zhang, Xi Cheng, Xin Wang, Yong Xiao, Zehua Guo, Zhixuan Wang, Zhiyong Qiu. I thank them for their kindness, meaningful discussions on our research projects, and the help they offered whenever I need help on programming or understanding some physics concepts.

I thank the good friends I’ve made in graduate school, including Arthur Yu, Basilio Yniguez, Chilun Jiang, Chin-Chang (Hoopi) Kuo, Chris Trinh, Gaël Nguyen, Jie Yu, Jinrui Huang, Lucas Wagner, Shiu Liu, Shu Zhou, Suzanne Genc and Zheng Duan. I had great times studying and/or hanging out with them while I was living away from home.

I thank my current vanpool partners Araceli Rossi, Daisey Urena, Delsa Langford, Esther Chung, Nicole Michele and Preston Reed for being amazing. Carpooling with them make the horrible daily commute on the 605/405/73 freeways tolerable. They help take my mind off of work with lively discussions on topics such as food, family and the daily news.

I thank my parents and sisters for their support throughout the years. They are always there for me and help me get through all the ups and downs in my life. I also want to thank my high school career counselor Miss Carolyn Dunn. Not only was she my counselor, she is also a mother figure to me. She is always one phone call away when I need advice in life.

Last but not least, I thank my best friend and life partner Josh Moody for his unconditional love and support throughout my journey in graduate school. He believes in me when I have doubt in myself. I would never have the mental strength to complete my doctorate degree if it wasn’t for his continuous encouragement.

I would like to express my sincere gratitude to the Gates Millennium Scholars Program, Graduate Assistance in areas of National Need Fellowship, Graduate Student Federal Work
Study Program and grants from Dr. Lin for their financial support throughout my study in graduate school.
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Role of Convective Cells in Nonlinear Interaction of Kinetic Alfvén Waves

By

Onnie Luk

Doctor of Philosophy in Physics

University of California, Irvine, 2015

Professor Zhihong Lin, Chair

The convective cells are observed in the auroral ionosphere and they could play an important role in the nonlinear interaction of Alfvén waves and disrupt the kinetic Alfvén wave (KAW) turbulence. Zonal fields, which are analogous to convective cells, are generated by microturbulence and regulate microturbulence inside toroidally confined plasmas. It is important to understand the role of convective cells in the nonlinear interaction of KAW leading to perpendicular cascade of spectral energy. A nonlinear gyrokinetic particle simulation has been developed to study the perpendicular spectral cascade of kinetic Alfvén wave. However, convective cells were excluded in the study. In this thesis project, we have modified the formulation to implement the convective cells to study their role in the nonlinear interactions of KAW. This thesis contains detail description of the code formulation and convergence tests performed, and the simulation results on the role of convective cells in the nonlinear interactions of KAW. In the single KAW pump wave simulations, we observed the pump wave energy cascades to waves with shorter wavelengths, with three of them as dominant daughter waves. Convective cells are among those dominant daughter waves and they enhance the rate of energy transfer from pump to daughter waves. When zonal fields are present, the growth rates of the dominant daughter waves are doubled. The convective cell (zonal flow) of the zonal fields is shown to play a major role in the nonlinear wave interaction, while the linear zonal vector potential has little effects. The growth rates of the daughter waves
linearly depends on the pump wave amplitude and the square of perpendicular wavenumber. On the other hand, the growth rates do not depend on the parallel wavenumber in the limit where the parallel wavenumber is much smaller than the perpendicular wavenumber. The nonlinear wave interactions with various perpendicular wavenumbers are also studied in this work. When convective cells are excluded, the nonlinear wave interactions show exponential growth on the daughter waves, but at a rate about half of that of the wave interactions with convective cells. In the two pump wave simulations, six daughter waves dominate in the energy cascade process, and three of them are convective cells. The growth rates of the daughter waves are doubled compared with the growth rates of the daughter waves generated in single KAW pump wave simulation. The relationship between the growth rates of the daughter waves and pump wave parameters are studied. The growth rates of the daughter waves have a linear relationship with both pump wave amplitudes and the square of perpendicular wavenumber of the pump waves. On the other hand, the growth rates do not depend on the parallel wavenumber of the pump waves in the limit where the parallel wavenumber is much smaller than the perpendicular wavenumber. The growth rate dependence on one of the two pump waves shows that the time variation on the pump wave amplitudes must be considered.
Chapter 1

Introduction

1.1 Background

1.1.1 Alfvén turbulence

Alfvén waves are commonly found in space and astrophysical plasmas. These waves often interact with one another. The nonlinear interactions cause wave energy to cascade, which is anisotropic by observation, [Bieber et al., 1996, Podesta, 2009] through the intermediate spatial scale, then dissipate to the background plasma through smaller spatial scale. The intermediate spatial scale is called the inertial range and the smaller spatial scale is called the dissipation range. The point that separates the two ranges is about the size of an ion Larmor radius (gyroradius) or ion inertial length. This phenomenon of Alfvénic turbulence is one of the active areas of research in the space plasma community, [Goldstein, 2001] for there is no concrete explanations to why energy cascade is anisotropic and what causes the waves’ energy to dissipate into the plasmas.
To understand how energy cascades in Alfvénic turbulence, one must know the different types of plasma waves exist at different spatial and temporal scales. Ideal magnetohydrodynamic (MHD) waves are waves of large spatial and long temporal scales. These waves include the shear Alfvén wave (SAW), slow and fast magnetosonic waves. However, the latter two are compressional waves that could experience transit time damping and therefore dissipate quickly. The SAW could interact nonlinearly with other waves, especially at small/shorter scales. For example, at perpendicular spatial scale approximately the size of the ion gyroradius, SAW could couple to the ion acoustic wave (IAW) and become a kinetic Alfvén wave (KAW).[Hasegawa and Chen, 1976] KAW has parallel electric field and therefore Landau damping could occur. On the other hand, at parallel spatial scale approximately the size of the ion gyroradius and temporal scale similar to the ion gyrofrequency, the SAW could couple to the ion cyclotron wave and lower hybrid waves. The resulting waves could get suppressed by ion cyclotron resonance or ion/electron Landau damping. All these wave couplings at various spatial and temporal scales could cause Alfvénic turbulence.

The energy cascade processes result in some forms of power spectra of Alfvén turbulence. Power spectra (of wavevector) were plotted using the observational data from satellites in space. At the inertial range, full scale MHD turbulence occurs and the power spectrum satisfies the Kolmogorov power law of $P \propto k^{-5/3}$ in globally isotropic system with a strong background magnetic field. However, in an anisotropic system with magnetic perturbation perpendicular to the background magnetic field, the observed power law becomes $P \propto k_{\perp}^{-3/2}$, [Müller and Grappin, 2005, Vasquez et al., 2007, Shivamoggi, 2008] as predicted by Kraichnan.[Kraichnan, 1965] At high frequency and spatial scale of approximately the size of ion gyroradius, the spectrum at the dissipation range steepens.[Leamon et al., 1998, Goldstein and Roberts, 1999] Many hypotheses to explain the steepened power spectrum at dissipation range were proposed. These include ion cyclotron damping [Marsch and Tu, 2001], Landau damping of KAW [Chen and Hasegawa, 1991], stochastic heating by dispersive Alfvén
wavesx [Chaston et al., 2004], weakly damped magnetosonic and whistler waves.[Beinroth and Neubauer, 1981, Stawicki et al., 2001, Gary and Li, 2000]

The anisotropic energy cascades of Alfvénic turbulence were observed in interplanetary plasmas and the direction of the ambient magnetic field seems to play a role. The energy cascades from waves of larger scales to waves of smaller scales, and they are different based on the direction of cascade, e.g. parallel or perpendicular to the ambient magnetic field. Observations indicate that majority of the wavevectors point in the direction perpendicular to the magnetic field. Theoretical and numerical attempts were made to understand this anisotropic cascade, [Shebalin et al., 1983, Matthaeus et al., 1998] which may be caused by the magnetic field. However, Dasso et. al. [Dasso et al., 2005] showed that the type of solar wind being analyzed could also determine the nature of the anisotropic cascade. Turbulence in the fast solar wind is dominated by the parallel modes, while the slow solar wind is dominated by perpendicular modes.

The basic nonlinear process underlying the Alfvén cascade is three-wave coupling that leads to parametric decay instability. Even though parametric instability of Alfvén waves have not yet been demonstrated in laboratory, a recent experimental work on nonlinear excitation of ion acoustic wave by two oppositely propagating Alfvén waves was demonstrated by Dorfman and Carter.[Dorfman and Carter, 2013] In the experiment, two counterpropagating Alfvén waves of similar amplitudes were launched by antennas placed at the two ends inside the Large Plasma Device (LAPD). They performed a frequency scan on the beat wave of these nonlinearly interacting Alfvén waves. To do so, one of the Alfvén wave has a different frequency at every discharge, while the other one is fixed. The result of the frequency scan shows the beat wave resonates at a frequency that matches with the ion acoustic wave. They know this is a resonant interaction because the mode did not immediately disappear after
the Alfvén waves discharge is off, and the time it took to dissipate is close to the ion-neutral collision time. [Janev et al., 1987] They were able to plot the beat wave spatial structure and its amplitude is 3.5% of the mean ion saturation current. This is consistent with the prediction by theory of three-wave coupling caused by pondermotive force.

Before one could understand how the Alfvén waves could decay into other modes, knowledge on nonlinear physics of SAW is important. Chen and Zonca [Chen and Zonca, 2013] have written an elaborate discussion on that and what causes destruction of the pure Alfvénic state. Those factors include the finite ion compressibility and non-ideal kinetic effects. Both finite ion compressibility and non-ideal kinetic effects were analyzed using the gyrokinetic theory. By including the Finite ion Larmor radius (FLR) effects in the nonlinear studies, the pure Alfvénic states were destroyed and the study of parametric decay process is more complete compared to the one explained using the ideal MHD theory. For example, SAW could go through the parametric decays into an IAW and a back-scattered SAW.

1.1.2 Convective cell

A possible mechanism preferring perpendicular cascade is the generation of convective cell (CC). CC is a two-dimensional plasma wave traveling in the direction perpendicular to the magnetic field ($k_\parallel = 0$). Not only is convective cell quasi-stationary, but also spatially periodic. [Shukla et al., 1984, Diamond et al., 2005] CC has been found to play an important role in the nonlinear interaction of Alfvén waves. It is possible that the parametric instability of KAWs generates CCs. The coupling of CCs with KAWs could then lead to turbulence in the upper ionosphere. [Sagdeev et al., 1978, Onishchenko et al., 2004] Furthermore, Zhao et. al. [Zhao et al., 2012] shows that the large-scale CCs could disrupt KAW turbulence and
therefore contributes to the evolution of ionospheric plasma turbulence.

There are two types of convective cells: electrostatic (ESCC) and magnetostatic (MSCC). As the name suggests, the ESCCs can be described by the zonal component of scalar potential; they could drive macroscopic plasma convection, and therefore affect plasma transport and space weather.[Shukla et al., 1984, Diamond et al., 2005, Lin et al., 1978, Popel et al., 2011]. On the other hand, MSCCs can be described by the zonal component of vector potential; they are tied to magnetic-field bending.[Shukla et al., 1984, Chu et al., 1978, Pavlenko and Weiland, 1981]. A theoretical study found that KAW in the inertial range can excite electrostatic convective cell with certain spatial scale. However, KAW in the kinetic range can excite another type of convective cell called magnetostatic convective cell.[Zhao et al., 2011]

There is an issue brought up by Zonca and Chen. [Zonca and Chen, 2014] As they suggested, many theoretical analyses in the past made assumptions that finite Larmor radius effects are negligible in Reynold stress calculations and that the ESCCs could be decoupled from MSCCs. Those assumptions were proven incorrect. They showed that KAWs can excite ESCCs and MSCCs at the same time, but only at short-wavelength limit. This process could lead to diffusive isotropization of the perpendicular KAW spectrum. These studies suggest that kinetic simulation is needed for CC.

CC has been observed in simulation work, which offered some explanation to the energy transport in large range of spatial scales. Lin et. al. [Lin et al., 2012] performed a 3-dimensional ion particle simulation to study the mode conversation and parametric decay of KAW. In the simulation, a fast magnetosonic mode with wavenumber \( k = (k_x,0,k_z) \) is allowed to propagate into a region of increasing Alfvén velocity, with the background magnetic field pointing in the z-direction and the gradient of background magnetic field (\( \nabla B \))
and density \((\nabla n)\) pointing in the x-direction. KAW of \(k_y\rho_i \sim 1\) grows when amplitude of KAWs generated by linear mode conversion becomes large to drive parametric decay instability. A mode with \(k_z = 0\) also gets excited during parametric decay instability. This mode has characteristics of a zonal flow mode or convective cell, with a large \(k_y\) value similar to the \(k_y\) value of the daughter KAW excited during the decay. Wave energy transfers to large \(k_y\) modes to drive large transport across plasma boundaries. Understanding transport involves mode conversion, which may explain massive transport at magnetopause into plasma sheet during linear mode conversion.

CCs are commonly detected in space, however, another mode of similar form could also be detected in fusion experiments and numerical simulations. A residual flow is spontaneously excited in ion temperature gradient (ITG) turbulence inside a magnetically confined toroidal plasma. This \(E \times B\) residual flow is known as zonal flow, which is an electrostatic mode propagates in the direction of nonuniformity. The direction of nonuniformity is defined in the x-direction, and hence \(k_y = k_z = 0\). A massively-parallel global particle-in-cell (PIC) code composed by Lin et. al. [Lin et al., 1998] demonstrates that zonal flow could reduce transport and regulate the ITG turbulence. Therefore, understanding zonal flow becomes a popular research topic in confinement physics. Zonal flow is slightly different from CC, for it could not propagate in the y-direction. However, both zonal flow and CC have wavevector perpendicular to the magnetic field. While the zonal flow is restricted to propagate in one direction (in the x-direction), the CC can propagate in the plane perpendicular to the background magnetic field and therefore has more degree of freedom.

The zonal dynamic is described not only by the zonal flow, but also by the zonal current. As mentioned earlier, zonal flow is an \(E \times B\) residual flow excited in microturbulence represented by scalar potential \(\delta \phi\). On the other hand, the zonal current is represented by parallel vector potential \(\delta A_\parallel\). A side-by-side analytical and numerical study on the zonal dynamics in
tokamak plasmas was carried out by Chen et. al.[Chen et al., 2001] They showed that zonal flow was spontaneously excited by ITG and drift Alfvén turbulence. The zonal flow is important in regulating the turbulence. Zonal current was also excited in microturbulence, but unlike zonal flow, its presence does not affect the turbulence significantly.

1.2 Objective of this Research Project

The objective of this thesis project is to understand the role of convective cell (CC) in the nonlinear interaction of kinetic Alfvén wave (KAW) that leads to the perpendicular cascade of spectral energy. There were previous studies on spectral cascade in Alfvénic turbulence; namely, a nonlinear gyrokinetic particle-in-cell code was developed by Cheng [Cheng, 2011] to study the perpendicular spectral cascade and collisionless dissipation of KAW. However, CCs were excluded in Cheng’s studies. We implemented CCs into this code to allow their presence during the KAW interactions. Convergence tests were first performed on the new code in order to find the time step size, number of grids on each side of the simulation box and number of particles per wavelength that provide accurate simulation results and yet save overall computing time. Those convergence tests were performed on linear single-wave, nonlinear single-wave and multiple-wave interactions simulations, which verify the simulation code.

The role of CCs in the KAW interactions of a single KAW pump wave and two KAW pump waves were then studied in this project. In the single pump wave simulations, we observed that the pump wave energy cascades to waves of shorter wavelengths during KAW interactions. There are a total of three daughter waves, in which one of them is a CC, that are essential in producing the energy cascade of a single standing pump wave. Simulations that include the pump wave and those three daughter waves show that while the real frequencies of the waves are unaffected in the presence of CC (zonal flow), the growth rates of the daughter
waves increased by almost a factor of two. However, linear zonal vector potential of the zonal fields plays a minor role during the KAW interactions. Studies on the daughter wave growth rate dependence on pump wave parameters were also included. The growth rates are linearly dependent on the pump wave amplitude and the square of perpendicular wavenumber. On the other hand, the growth rates are independent of the parallel wavenumber in the low-parallel-wavenumber limit. These findings are consistent with the theory of modulational instability [Chen et al., 2001] and the understanding of the dominant \( E \times B \) drift in the nonlinear KAW interaction. The study on KAW interactions of the pump wave and dominant daughter waves with different perpendicular wavenumbers is also mentioned in this thesis. We found that no coupling occurred when the daughter wave with the highest wavenumber (non-CC) is excluded. On the other hand, when CC is excluded from the KAW interactions, KAW interactions via parametric decay instability occurs and the growth rates correlate to the perpendicular wave number. Comparisons between simulations of KAW interactions with and without CCs show the growth rates of the daughter waves with CC increased by almost a factor of two in the higher-perpendicular wavenumber tested.

In the simulations of two pump waves, we observed a total of six daughter waves that are dominant in the energy cascade process, and three of them are CCs. The extra pump wave provides an additional source of energy, therefore the growth rates of the daughter waves in two-pump simulations are doubled from the growth rates of daughter waves in single-pump simulations. Studies on the daughter wave growth rate dependence on pump wave parameters are also included. The growth rates are linearly dependent on the pump wave amplitude and the square of perpendicular wavenumber. On the other hand, the growth rates are independent of the parallel wavenumber in the low-parallel wavenumber limit. The growth rate dependence on one pump wave amplitude, with the other wave amplitude fixed, show that the change in pump wave energies need to be taken into account.
1.3 Thesis Outline

The rest of this thesis will be displayed in the following order: Chapter 2 introduces the gyrokinetic theory and outlines the numerical code that we use for this project. Chapter 3 shows the convergence tests and benchmark performed on linear and nonlinear single-mode simulations and nonlinear multiple-mode simulation. Chapter 4 shows simulation results on the role of convective cell in nonlinear interaction of KAWs generated by a single pump wave. Those simulations help demonstrate the energy cascade of the pump wave into the dominant daughter waves; determine the importance of convective cell during KAW interactions; determine the growth rate dependence on pump wave parameters; and show the variations of KAW interactions under certain perpendicular wavevector values. Chapter 5 studies the role of convective cell during nonlinear interaction of KAWs generated by two pump waves. This includes identifying the dominant daughter waves involved in the KAW interactions and finding the growth rate dependence on the pump wave parameters. Chapter 6 summarizes all the findings in this thesis project and outlines the future plan for this project.
Chapter 2

Gyrokinetic Theory and Code

Formulation

We are interested in studying kinetic Alfvén waves (KAWs) that could be found in space and laboratory plasmas. In general, Alfvén waves are low-frequency electromagnetic fluctuations. These fluctuations have a frequency $\omega$ much lower than the ion cyclotron frequency $\left(\Omega_i \equiv \frac{eB_0}{m_i c}\right)$ due to strong background magnetic field. Here, $e$ is the ion charge, $B_0$ is the background magnetic field strength, $m_i$ is the ion mass, and $c$ is the speed of light. With the same reasoning, the ion gyro orbit has a small ion gyroradius $\rho_i = v_i / \Omega_i$ compared to plasma characteristic length, with $v_i$ as the ion thermal speed. In other words, the spatial scale of background plasmas is much larger than the ion gyroradius. However, the finite Larmor radius (FLR) effects could greatly alter my nonlinear physics studies, since $k_\perp \rho_i \sim 1$, where $k_\perp \rho_i$ is the magnitude of wavevector pointing in the direction perpendicular to the background magnetic field. In Alfvénic turbulence, the energy of the compressional modes are suppressed due to transit-time magnetic pumping. It is very difficult to excite large $k_\parallel$ modes due to field-line bending effects, where $k_\parallel$ is the magnitude of wavevector pointing in the direction parallel to the background magnetic field. That reduces the power of modes.
with high \( k_\parallel \) values. Therefore, the perpendicular wave number \( k_\perp \) is much bigger than the parallel wave number \( k_\parallel \). I want constant background magnetic field in my study for simplicity; therefore, the equilibrium scale length of the magnetic field \( L \) is much larger than \( \rho_i \). The perturbed potential energy and perturbed magnetic field \( \delta B \) is much smaller than kinetic energy and \( B_0 \), respectively.

### 2.1 Gyrokinetic Theory

The KAW described in the Introduction can be most efficiently studied using nonlinear gyrokinetic theory.[Brizard and Hahm, 2007] By utilizing the gyrokinetic theory, I eliminate the uninteresting gyro-motion of ions by averaging over cyclotron orbit, while all low-frequency interesting physics are kept in my work.

In gyrokinetic theory, the orderings are

\[
\frac{\omega}{\Omega_i} \sim \frac{\rho}{L} \sim k_\parallel \rho_i \sim \frac{e\delta \phi}{T_e} \sim \frac{\delta B}{B_0} \sim \epsilon
\]

\[k_\perp \rho_i \sim 1,\]  

(2.1)

where \( \delta \phi \) is the perturbed electrostatic potential energy, \( T_e \) is the electron temperature, and \( \epsilon \) is the smallness number.

In gyrokinetic theory, the phase space coordinates are transformed from particle coordinates \((x, v)\) to gyrocenter coordinates \((X, \mu, v_\parallel, \theta)\), where \( x \) is particle position, \( v \) is the particle velocity, \( X \) is the gyrocenter position, \( \mu \) is the magnetic moment, \( v_\parallel \) is the gyrocenter velocity along the background magnetic field and \( \theta \) is the gyroangle. With the transformed coordinate system, the gyroangle-averaged Vlasov equation becomes the nonlinear gyrokinetic equation.
for species $\alpha$:[Brizard and Hahm, 2007]

$$\left( \frac{\partial}{\partial t} + \dot{X} \cdot \nabla + v_\parallel \frac{\partial}{\partial v_\parallel} \right) f_\alpha (X, \mu, v_\parallel, \theta) = 0,$$

(2.2)

where $f_\alpha$ is the gyrocenter distribution function for the species $\alpha$. More specifically,

$$\dot{X} = v_\parallel \frac{B}{B_0} + \mathbf{v}_E + \mathbf{v}_c + \mathbf{v}_g = v_\parallel \frac{B}{B_0} + \frac{c \mathbf{b}_0 \times \nabla \delta \phi}{\Omega_\alpha} + \frac{v_\parallel^2}{\Omega_\alpha} \nabla \times \mathbf{b} + \frac{\mu}{m_\alpha \Omega_\alpha} \mathbf{b}_0 \times \nabla B,$$

(2.3)

$$v_\parallel = -\frac{1}{m_\alpha B_0} \cdot (\mu \nabla B + q_\alpha \nabla \delta \phi) - \frac{q_\alpha}{m_\alpha c} \frac{\partial}{\partial t} \delta A_\parallel,$$

(2.4)

$$B^* = B_0 + \frac{B_0 v_\parallel}{\Omega_\alpha} \nabla \times \mathbf{b} + \delta B_\perp,$$

(2.5)

with $B_0 \equiv B_0 \mathbf{b}_0$ as background magnetic field, $B \equiv B_0 + \delta B$ as total magnetic field, $\mathbf{v}_E$ as $E \times B$ drift, $\mathbf{v}_c$ as the curvature drift, $\mathbf{v}_g$ as the gradient-B drift, $\delta \phi$ as the gyroaveraged scalar potential, $\Omega_\alpha$ as the gyroradius of species $\alpha$, $\mathbf{b}$ as the unit vector of total magnetic field, $\mu$ as the magnetic moment, $m_\alpha$ as the particle mass of species $\alpha$, $q_\alpha$ as the charge of species $\alpha$, and $\delta A_\parallel$ represents the gyroaveraged vector potential along $B_0$.

I also made the assumption that $\delta B$ has no compressional component for $\beta \ll 1$. Therefore, the perturbed magnetic field can be expressed as

$$\delta B \approx \delta B_\perp = \nabla_\perp \times \delta A_\parallel.$$

(2.6)

The gyrocenter distribution function for species $\alpha$ could be expanded to distinguish background particle distribution and perturbed distribution: $f_\alpha = f_{0\alpha} + \delta f_\alpha$. Here, $f_{0\alpha}$ satisfies
the equilibrium equation: \( L_{0\alpha}f_{0\alpha} = 0 \), and \( L_{0\alpha} \) represents the equilibrium part of propagator for species \( \alpha \). The distributions of background particles of species \( \alpha \), or \( f_{0\alpha} \), are uniform in real space and Maxwellian in velocity space. Mathematically, \( f_{0\alpha} \) is expressed as:

\[
f_{0\alpha} = \frac{n_{0\alpha}}{(2\pi T_{0\alpha}/m_{\alpha})^{3/2}} \exp \left( \frac{-2\mu B_0 + m_{\alpha}v_{\parallel}^2}{2T_{0\alpha}} \right).
\]

(2.7)

The perturbed distribution function for species \( \alpha \) (\( \delta f_{\alpha} \)) can be expressed using Equation 2.2:

\[
L_{\alpha} \delta f_{\alpha} = -\delta L_{\alpha} f_{0\alpha}
\]

(2.8)

From the above equation, \( L_{\alpha} \) represents the complete propagator for species \( \alpha \) and \( \delta L_{\alpha} \) represents the perturbation of the propagator away from the equilibrium for species \( \alpha \). The complete propagator for \( \alpha \) satisfies \( L_{\alpha} \equiv L_{0\alpha} + \delta L_{\alpha} \).

The particle density and current density of species \( \alpha \) (\( \delta n_{\alpha} \) and \( \delta u_{\parallel\alpha} \), respectively) can be calculated using \( \delta f_{\alpha} \) with gyro-averaging when transforming from \( x \) to \( X \).

For perturbed particle density of species \( \alpha \):

\[
\delta n_{\alpha}(x) = \frac{1}{2\pi} \oint d\theta \int \delta f_{\alpha}(X) \delta(x - X - \rho_{\alpha}) \frac{B_0}{m_{\alpha}} dX d\nu_{\parallel} d\mu
\]

(2.9)

For perturbed particle current density of species \( \alpha \):

\[
n_0 \delta u_{\parallel\alpha}(x) = \frac{1}{2\pi} \oint d\theta \int \nu_{\parallel} \delta f_{\alpha}(X) \delta(x - X - \rho_{\alpha}) \frac{B_0}{m_{\alpha}} dX d\nu_{\parallel} d\mu
\]

(2.10)

When we focus on the electron species, equations 2.9 and 2.10 can be simplified by taking the drift-kinetic limit: \( k_{\perp} \rho_e \to 0 \).

The field equations are the gyrokinetic Poisson’s equation[Lee, 1983] and gyrokinetic Ampère’s
law, which can be expressed as

\[
\frac{\tau}{\lambda_D} \left( \delta \phi - \delta \hat{\phi} \right) = 4\pi \sum_{\alpha} q_{\alpha} \delta n_{\alpha},
\]

(2.11)

\[
-\nabla_{\perp}^{2} \delta A_{\parallel} = \frac{4\pi e n_0}{c} \left( \delta u_{i\parallel} - \delta u_{e\parallel} \right),
\]

(2.12)

respectively. Equations 2.2, 2.11, and 2.12 form a complete set of nonlinear equations that describe the plasma wave.

### 2.2 Linear Dispersion Relation

To test the set of gyro-averaged nonlinear equations (Equations 2.2, 2.11, and 2.12), we take the linear limit of those equations in attempt to recover the linear drift kinetic dispersion relation. In the linear limit, waves are decomposed to normal modes, i.e., to have the form \( e^{i(k \cdot x - \omega t)} \), where \( k \) is the wavevector of the wave and \( \omega \) is the frequency of the wave. By transforming the field equations 2.11 and 2.12 from real space to k-space, gyroaveraging the field terms would turn into the form \( \langle e^{i(k \cdot \rho)} \rangle_{\theta} = J_0(k_{\perp} \rho_i) \). The function \( J_0(k_{\perp} \rho_i) \) is the Bessel function of the first kind. If we take the nonlinear gyrokinetic equation, expand the terms by order of \( \delta B/B_0 \) and keep up to the 1st order terms, the equation becomes the linear gyrokinetic equation, or

\[
\left( \frac{\partial}{\partial t} + v_{\parallel} \cdot \nabla \right) \delta f_{\alpha} = -\frac{q_{\alpha}}{m_{\alpha}} \frac{\partial E_{\parallel}}{\partial v_{\parallel}} \frac{\partial}{\partial v_{\parallel}} f_{0\alpha}.
\]

(2.13)
δE∥ is the perturbed electric field that is defined as:

\[
\delta E_\parallel \equiv -\nabla_\parallel \delta \psi = -\nabla_\parallel \delta \phi - \frac{1}{c} \frac{\partial}{\partial t} \delta A_\parallel,
\]

(2.14)

with \( \delta \psi \) is called the effective scalar potential.

Integrating the linear gyrokinetic equation (Equation 2.13) and plug in the gyroaveraged field quantities found from equations 2.11 and 2.12, then the linear dispersion relation is obtained:

\[
\left[ \frac{\omega^2}{k_\parallel^2 v_A^2} - \left( 1 + k_\perp^2 \rho_i^2 \right) \right] \left[ 1 + \zeta_e Z(\zeta_e) + J_0^2(k_\perp \rho_i)(\tau + \tau \zeta_i Z(\zeta_i)) \right] = k_\perp^2 \rho_s^2
\]

(2.15)

where Alfvén speed is \( v_A = \sqrt{\frac{B_0^2}{4\pi n_0 m_i}} \), ion gyroradius is \( \rho_i = \frac{v_{th,i}}{\Omega_i} \), \( \zeta_\alpha = \frac{\omega}{\sqrt{2k_\parallel v_{th,\alpha}}} \), background density is \( n_0 \), thermal velocity of species \( \alpha \) of temperature \( T_\alpha \) is \( v_{th,\alpha} = \sqrt{\frac{T_\alpha}{m_\alpha}} \), and the plasma dispersion function \( Z(\zeta_\alpha) = \frac{1}{\sqrt{\pi}} \int \frac{e^{-t^2}}{i-\zeta_\alpha} dt \).

Equation 2.15 could be separated into two branches or normal modes: kinetic Alfvén wave (KAW) and ion acoustic wave (IAW).

If the plasma meets the limit \( \zeta_e \to 0, \zeta_i \to 0 \) and \( k_\perp^2 \rho_s^2 \to 0 \), where \( rho_s \) is the ion acoustic gyroradius, then the real part of equation 2.15 turns into two separate modes

\[
\omega_1^2 = k_\parallel^2 v_A^2,
\]

which is known as KAW, and

\[
\omega_2^2 = k_\parallel^2 c_s^2,
\]

which is known as IAW. Since \( \tau = 1 \) throughout the project, \( \rho_i = \rho_s \). We need to note that \( \rho_s \) is ion acoustic gyroradius. When we set the limit \( k_\perp^2 \rho_s^2 \to 0 \), we also set \( k_\perp^2 \rho_i^2 \to 0 \).
For finite $\zeta_e$ and $\zeta_i$, Z-function contains singularity, which gives rise to linear Landau damping due to wave-particle resonance.

When the real part of equation 2.15 is set to 0, an expression for the real part of wave frequency $\omega_r$ appears. Assuming $\omega = \omega_r + i\gamma$ and $\frac{\gamma}{\omega_r} \ll 1$, the linear dispersion relation could be expanded and an expression for $\gamma$ could be found. $\gamma$ is known as the linear Landau damping rate.

### 2.3 Nonlinear $\delta f$ Method

In the nonlinear gyrokinetic simulation, the perturbed particle distribution is governed by the gyrokinetic equation:

$$
\left( \frac{\partial}{\partial t} + \mathbf{\dot{X}} \cdot \nabla + v_\parallel \frac{\partial}{\partial v_\parallel} \right) \delta f_\alpha = -v_\parallel \frac{\partial}{\partial v_\parallel} f_{0\alpha}
$$

(2.16)

The terms in red contain nonlinear terms of the 2nd order, as expressed in Equations 2.3 and 2.4. When the wave amplitudes become significant, the nonlinear terms must be included in the simulation. For a single wave, particle trapping occurs and particles bounce back and forth in a potential well. This will be demonstrated in section 3.2 for nonlinear verification of the simulation code. Notice that linear dispersion relation could be recovered if those nonlinear terms were removed.

### 2.4 Electron Model

The code we use for our research is a gyrokinetic particle-in-cell (PIC) code developed to study the kinetic Alfvén wave turbulence.[Cheng, 2011] The ion dynamics in the code is gyrokinetic, while the electron dynamics are hybrid, with the 0th order motion treated as
a massless fluid and higher orders as kinetic. [Lin and Chen, 2001] We treat electrons as a
massless fluid because electron beta in our research is 0.16, where \( v_e \gg v_A \sim v_i \) and thus ion
Landau damping effect is dominant compared to electron Landau damping. Electron beta
\( \beta_e \) is defined as the electron-plasma pressure to magnetic pressure ratio. With that in mind,
some of the terms from equations that form closed system for electrons could be dropped
because they have much weaker effects than other terms.

In the lowest order of the hybrid electron model, if electron is adiabatic, \( k_i v_e \gg \omega \), then

\[
e^{-e\delta\psi/T_e} = 1 + \frac{\delta n_e}{n_0}
\]  \hspace{1cm} (2.17)

The nonlinear gyrokinetic equation (see equation 2.16) could provide the dynamics of elec-
trons. If we integrate equation 2.16 in the velocity space and keep terms of up to the 2\textsuperscript{nd}
order, then the time evolution of the electron charge density \( \delta n_e \) can be described by the
electron continuity equation:

\[
\frac{\partial}{\partial t} \delta n_e + n_0 b_0 \cdot \nabla \delta u_{e\parallel} + n_0 \delta b \cdot \nabla \delta u_{e\parallel} + \frac{c b_0 \times \nabla \delta \phi}{B_0} \cdot \nabla \delta n_e = 0,
\]  \hspace{1cm} (2.18)

The dynamic equation that solves for the time evolution of vector potential \( \delta A_{\parallel} \) can be
obtained by rearranging Equation 2.14:

\[
\frac{1}{c} \frac{\partial}{\partial t} \delta A_{\parallel} = b_0 \cdot \nabla (\delta \psi - \delta \phi).
\]  \hspace{1cm} (2.19)

Equation 2.19 is the definition of parallel electric field.
### 2.5 Ion Equations

The dynamics of the ions rely on the field quantities, which are calculated in Equations 2.11, 2.12 and 2.14. The dynamic equations that describe the ion particles’ motions are:

\[
\frac{dX_i}{dt} = v_\parallel b_0 + v_\parallel \delta b + \frac{e b_0 \times \nabla \phi}{B_0},
\]

(2.20)

\[
\frac{dv_\parallel}{dt} = \frac{e}{m_e} \delta E_\parallel,
\]

(2.21)

\[
\frac{dw_i}{dt} = -(1 - w_i) \frac{e}{m_i v_{th,i}^2} \nabla_\parallel \delta \psi.
\]

(2.22)

Equation 2.20 contains the dominant terms from equation 2.3, equation 2.21 contains the dominant terms from equation 2.4, and equation 2.22 is the time evolution of the ion weight function. Ion weight is the perturbed particle distribution function of an ion marker particle defined as

\[
w_i \equiv \delta f_i / f_i,
\]

(2.23)

where \(w\) is the weight and \(f_i\) is the complete particle distribution function \((f_i = f_{i0} + \delta f_i)\). Since the gyrokinetic PIC code employs the \(\delta f\) method [Hu and Krommes, 1994] to sample phase space, knowing the dynamic evolution of \(\delta f\), or equivalently \(w\), is important. The dynamic expression for ion weight (equation 2.22) was obtained by replacing \(f_i = f_{i0} + \delta f_i\) in equation 2.1 with \(f_{i0} + wf_i\). The terms, in equations 2.18 - 2.22, highlighted in red are nonlinear terms of 2nd order.
When calculating the field equations, the ion density and ion current density are in particle positions. However, the gyrokinetic equation (equation 2.16) is defined in gyro-center position. The gyro-averaged ion density and ion current density could be converted from gyro-center position to particle position, as follows:

\[
\delta n_i(x) = \sum_k \int J_0(k_{\perp} \rho) \delta f_{i,k} B d\mu d\nu\parallel
\]  

(2.24)

\[
n_0 \delta u_i(x) = \sum_k \int J_0(k_{\perp} \rho) v_{\parallel} \delta f_{i,k} B d\mu d\nu\parallel
\]  

(2.25)

### 2.6 Zonal Fields

The zonal fields, or convective cells, are modes with \(k_{\parallel} = 0\). The zonal fields are composed of zonal potential \(\delta \phi\) and zonal vector potential \(\delta A_{\parallel}\). Zonal fields are known for regulating turbulence, and implementing zonal fields in our existing code could enable us to study the role of convective cells in the nonlinear KAW interaction.

Derivations of zonal scalar and vector potentials were shown by Holod et. al. [Holod et al., 2009] in their electromagnetic formulation of the electron hybrid model in toroidal geometry. To carry Holod’s work further, Wang [Wang, 2014] has shown that a nonlinear term in the gyrokinetic theory that was left out before is actually important to the zonal magnetic fields in a toroidal plasma. However, this new term does not play a major role to the overall simulation outcome.
2.6.1 Zonal potential

The zonal potential is very straight-forward to obtain from the expression on scalar potential (see Equation 2.11). Therefore, the zonal potential could be expressed as:

$$\tau \frac{\lambda_D}{D} (\delta \phi - \delta \tilde{\phi}) = 4\pi \sum_{\alpha} q_\alpha \delta n_\alpha,$$

(2.26)

2.6.2 Zonal vector potential

In Wang’s formulation on zonal fields [Wang, 2014], for zonal current with scale length $k^{-1}$ larger than electron collisionless skin depth $\delta_e$, the Ampère’s law for zonal field is expressed as

$$\delta A_{\|} = 4\pi e \delta_e^2 \left( \int_{GC} \delta f_i v_{\|} d\mathbf{v} - \int_{GC} \delta h v_{\|} d\mathbf{v} \right) - c \int \frac{\delta B}{B_0} \cdot \nabla \delta \phi dt,$$

(2.27)

where $\delta h$ is the part of electron distribution function that describes higher order non-adiabatic response. The first term on the right-hand-side of equation 2.27 is parallel current screened by the electron collisionless skin depth $\delta_e$. The second term is a nonlinear ponderomotive force arising from derivation of ideal MHD. The overbar represents flux surface average or the zonal component. Since electrons are adiabatic in my simulation, $\delta h = 0$.

The perturbed magnetic field can be expressed in terms of vector potential. For the limit $k_{\|} \ll k_{\perp}$, in particular, $\delta B = \delta B_{\perp} \approx \nabla_{\perp} \times \delta A_{\|}$.

After normalizations, equation 2.27 becomes

$$\bar{\delta A}_{\|} = \frac{m_e}{m_i} \delta u_{\| i} - \int \left( \nabla_{\perp} \times \delta A_{\|} \right) \cdot \nabla \delta \phi dt,$$

(2.28)

which is unitless.
The fluid electron response to the zonal vector potential is screened by the electron collisionless skin depth $\delta_e$:

\[
\frac{4\pi en_0}{c} \delta u^{(0)}_{\parallel e} = \frac{1}{\delta_e^2} \delta A_{\parallel}.
\]  

(2.29)

$\delta u^{(0)}_{\parallel e}$ is needed in the electron continuity equation.

### 2.6.3 Incorporation of zonal fields with non-zonal fields

Any field quantity can be found by adding both zonal and nonzonal components together. For example, the perturbed parallel vector potential $\delta A_{\parallel}$ is equivalent to

\[
\delta A_{\parallel} = \delta A_{\parallel}^{NZ} + \delta A_{\parallel}^{L},
\]  

(2.30)

in which $\delta A_{\parallel}^{NZ}$ is the nonzonal component of $\delta A_{\parallel}$ and could be obtained by solving equation 2.19 and $\delta A_{\parallel}^{L}$ by solving equation 2.28.

The perturbed parallel electron current density in k-space $\delta u_{\parallel e,k}$ is

\[
\delta u_{\parallel e,k} = \delta u_{\parallel e,k}^{NZ} + \overline{\delta u_{\parallel e,k}}.
\]  

(2.31)

Substitute the nonzonal component of parallel electron current density $\delta u_{\parallel e,k}^{NZ}$ with equation 2.12 Fourier-transformed into k-space and $\overline{\delta u_{\parallel e,k}}$ with equation 2.29 Fourier transformed into k-space, then equation 2.31 becomes:

\[
\delta u_{\parallel e,k} = \left( \delta u_{\parallel e,k}^{NZ} - \frac{c}{4\pi en_0 k^2} \delta A_{\parallel}^{NZ} \right) + \frac{e}{m_e c} \overline{\delta A_{\parallel,k}}
\]  

(2.32)

where $\overline{\delta A_{\parallel,k}} \approx \delta A_{\parallel,k}^{L}$. Note that the flux surface average can be calculated by taking the average value along the field line $b_0$. 

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2.7 Units and Normalization

The code employs the following basic units:

- inverse ion gyrofrequency for time \( \Omega_i^{-1} = \frac{e m_i}{e B_0} \),
- ion gyroradius for length \( \rho_s = c_s / \Omega_i \),
- proton charge for charge \( e \),
- proton mass for mass \( m_i \),

where \( c_s = \sqrt{T_e/m_i} \) is the speed of sound and \( T_e \) is the electron temperature.

Using the basic units above, the dynamic and field quantities can be normalized in the following manner:

- position \( \dot{X} = X / \rho_s \),
- velocity \( \dot{v} = v / c_s \),
- density \( \dot{n} = n / n_0 \),
- magnetic field \( \dot{B} = B / B_0 \),
- vector potential \( \dot{A} = A e c_s / c T_e \),
- scalar potential \( \dot{\phi} = \phi e T_e \),
- electric field \( \dot{E} = E e \rho_s / T_e \).

The code also employs the following dimensionless numbers in the dynamic and field equations:

- electron plasma beta \( \beta_e = \frac{8 \pi n_e T_e}{B_0^2} \),
- electron-to-ion temperature ratio \( \tau = T_e / T_i \).

After applying normalization to the field and dynamic quantities, the dimensionless field and dynamic equations (see equations 2.11, 2.12, 2.17 - 2.22, 2.28) become (exclude hats above the dimensionless quantities):
Field equations

\[ \tau (\delta \phi - \delta \tilde{\phi}) = \delta n_i - \delta n_e \]  
(2.11')

\[ \delta u_{e\|} = \delta u_{i\|} - \frac{2}{\beta_e} \nabla^2 \delta A_{\|} \]  
(2.12')

\[ \delta \psi = \delta n_e \]  
(2.17')

Note: Equation 2.12’ is the Ampères Law for nonzonal component. Equation 2.17’ is obtained by taking the limit \( \delta \psi \ll \frac{T_e}{e} \) from equation 2.17.

Dynamic equations

\[ \frac{\partial}{\partial t} \delta n_e = -b_0 \cdot \nabla \delta u_{e\|} - \delta b \cdot \nabla \delta u_{e\|} - b_0 \times \nabla \delta \phi \cdot \nabla \delta n_e \]  
(2.18')

\[ \frac{\partial}{\partial t} \delta A_{\|} = b_0 \cdot \nabla (\delta \psi - \delta \phi) \]  
(2.19')

\[ \frac{dX_\alpha}{dt} = v_{\parallel} b_0 + v_{\parallel} \delta b + e b_0 \times \nabla \delta \phi \]  
(2.20’)

\[ \frac{dv_{\alpha\|}}{dt} = -\nabla_{\|} \delta \psi \]  
(2.21’)

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\[
\frac{dw_i}{dt} = -\tau (1 - w_i) v_{i\parallel} \nabla_{\parallel} \delta \psi
\] (2.22')

Zonal fields

\[
\delta A_{\parallel} = \frac{m_e}{m_i} \delta u^{(0)}_{i\parallel} \varepsilon - \int \delta b \cdot \nabla \delta \phi \, dt,
\] (2.28')

where, \( \delta b = \nabla_{\perp} \times \delta A_{\parallel} \).

### 2.8 Code Formulation

The geometry in the simulation is a slab with periodic boundary conditions. The coordinate system used here is the Cartesian coordinate system, with the z-axis being parallel to the background magnetic field \( B_0 \equiv B_0z \).

The particle-in-cell method divides the real space into grids and following the four steps in every time loop to advance each particle to its next phase space position:

1. Accumulate ion particles’ charge and current densities on grids;

2. Solve for electromagnetic field quantities (effective scalar potential, scalar potential, and electron current density) using equations 2.17, 2.11, and 2.12, respectively;

3. Gather particle force from grids to particle positions;

4. Integrate particle orbit and place each particle to its new phase space coordinates using equations 2.18 - 2.22.

When accumulating ion charge and velocity, we use linear interpolation to determine the
weight of each particle on the grid. The further the particle is away from the grid of interest, the less weight/contribution it has towards the charge and current densities on that grid. This process is called particle-to-grid scattering. While the electron is a massless fluid, the ions are gyrokinetic. As a consequence, the ions’ coordinates have to be gyro-phase averaged to correspond to equation 2.2. An effective method to carry out gyro-phase averaging is called the N-points averaging method.[Lee, 1987] This method essentially calculates the Bessel function of the first kind, as mentioned in section 2.1.

The field quantities are solved in k-space. Therefore, all the terms in the Maxwell’s equations involved to calculate those field quantities are Fourier-transformed from real to k-space. To ensure there is complete control to which wave of a specific $k$-mode to evolve in time, a mode filter is set up. The original version of the code has mode filter set up to filter out $k_\perp$ modes, but no filter to $k_\parallel$. Therefore, I implemented a filter that controls which mode of a particular $k_\parallel$ to study. After the field quantities are transformed back to real space via inverse Fourier transform, the central finite difference method was used to calculate spatial derivatives from the dynamic equations 2.18 - 2.22, on every grid.

Next, the perturbed parallel electric field, magnetic field, and perpendicular differential of scalar potential on the grids are linearly extrapolated to the nearby particles. This process is called the grid-to-particle gathering.

Finally, the time integration was performed using the 2nd order Runge-Kutta method.[Dormand and Prince, 1980] This method allows the dynamic quantities to advance by half a time step. Then the process is iterated to find the dynamic quantities in one complete time step, by using the fields found in half a time step ago. This is called the time-centered method, which has a better numerical stability. The dynamic quantities are calculated using the dynamic equations 2.18 - 2.22.

Utilization of parallel computing helps shorten computer run time by dividing computing
tasks among processors. Two types of parallelizations are used in the code: OpenMP [Chandra, 2001] and MPI [Gropp et al., 1996]. OpenMP is an Application Program Interface that allows one master thread to perform on-node tasks until it reaches massive do-loops that explicitly ask parallel threads to split the calculations within the loops. Each thread would rejoin the master thread once the code reaches the command to end OpenMP. The other parallelization method is the Massage Passing Interface (MPI). In this code, MPI is used to perform particle decomposition. The particles are divided amongst computer nodes. Each node independently performs grid-to-particle gathering and particle-to-grid scattering. Then, the MPI nodes communicate and gather all the particle charge and current densities to obtain the total of both on each grid. Both methods acquire splitting computational tasks and therefore help the code run more efficiently.

For further details on how the code is structured, see Appendix A.
Chapter 3

Linear and Nonlinear Benchmark

The simulation results from the gyrokinetic particle-in-cell (PIC) code are not deemed reliable until the code is benchmarked with theory. Therefore, linear and nonlinear simulations of a kinetic Alfvén wave (KAW) were compared with theoretical results before physics study of 3-wave coupling can be carried out.

Ideally, having the smallest time step size, as many spatial grids and marker particles as possible would provide the most accurate simulation results. However, this is computationally costly, if not impossible. Therefore, performing convergence tests on those parameters helps in finding the time step small enough to give accurate results and yet large enough that it would not become a burden for the computer to run. Similarly, convergence tests can help in finding the most suitable numbers of spatial grids and marker particles.

As expressed in the theory, KAW experiences linear and nonlinear Landau damping. Therefore, I decide to simulate a KAW and use it to find the convergence parameters and make sure the results agree with theory. The gyrokinetic PIC code simulates plasma in slab geometry with periodic boundary conditions. The plasma is composed of singly charged-ions and electrons with an electron beta $\beta_e$ of 0.4. The electron beta is defined as the ratio...
of electron’s thermal pressure to magnetic field pressure, or \( \beta_e \equiv \frac{8\pi n_e T_e}{B_0^2} \). Here, \( n_e \) is the electron density, \( T_e \) is the electron temperature, and \( B_0 \) is the magnetic of the background magnetic field. The ions motions are calculated using gyrokinetic equation while electrons are treated as massless fluid. The initial electron and ion temperatures are the same \( (\tau \equiv T_e/T_i = 1.0) \). Both species of particles initially have uniform distribution in space and Maxwellian distribution in velocity space. The background magnetic field is uniform and points in the z-direction \( (B_0 \equiv B_0 z) \). The size of the simulation box is \( L_x \times L_y \times L_z \), with \( L_a/\rho_s = 2\pi/ (k_a\rho_s) \).

## 3.1 Linear Convergence and Benchmark

As mentioned earlier, the purpose of performing convergence tests is to find:

- minimum number of marker particles per wavelength of each dimension \( (N_p) \),
- minimum number of grids per wavelength of each dimension \( (N_g) \), and
- small enough time step \( (\Delta t) \)

to accurately simulate the wave. The KAW I chose to study in the linear test is the fundamental mode of the simulation domain, which has \( k_\perp \rho_s = 0.4, \ k_x \rho_s = k_y \rho_s, \ k_z \rho_s = 0.004 \) (or mode 111) and initial perturbation of \( \delta B_\perp/B_0 = 0.0012 \).

The following are the equations that govern the linear simulations:

\[
\tau \frac{\lambda_D^2}{\chi} \left( \delta \phi - \delta \tilde{\phi} \right) = 4\pi \sum_{\alpha} q_{\alpha} \delta n_{\alpha},
\]

\[
\frac{4\pi n_0 e}{c} \delta u_e || = \nabla_\perp^2 \delta A || + \frac{4\pi n_0 e}{c} \delta u_i ||,
\]

\[
e^{\delta \psi/T_e} = 1 + \frac{\delta n_e}{n_0},
\]
\[
\frac{\partial}{\partial t} \delta n_e + n_0 b_0 \cdot \nabla \delta u_e = 0, \\
\frac{1}{c} \frac{\partial}{\partial t} \delta A_\parallel = b_0 \cdot \nabla (\delta \psi - \delta \phi), \\
\frac{dX_\alpha}{dt} = v_{\alpha \parallel} z, \\
\frac{dv_{\alpha \parallel}}{dt} = 0, \\
\frac{dw_i}{dt} = -\frac{e}{m_i v_{th,i}^2} v_\parallel \nabla \parallel \delta \psi. 
\]

3.1.1 Time step

The size of the time step sufficient to simulate a KAW must be small enough such that \( \omega_{KAW} \Delta t \ll 1 \), in which \( \omega_{KAW} \) is the real part of the KAW frequency and \( \Delta t \) is the size of the time step. The complete expression for \( \omega_{KAW} \) can be obtained by solving equation 2.15. As for all the simulation results presented in this thesis, \( \Delta t \) is normalized by the shear Alfvén wave (SAW) frequency: \( \omega_A \Delta t \).

The \( \omega_A \Delta t \) tested as a possible convergence parameter are 0.186, 0.093, 0.046, 0.023, and 0.012. In Figure 3.1, the damping rate versus \( \omega_A \Delta t \) is shown. A systematic way was used to find the damping rate: nonlinear regression was used to obtain the real frequency and damping rate. As demonstrated, the damping rate begins to converge at \( \omega_A \Delta t = 0.093 \), and the discrepancy between this and the damping rate from \( \omega_A \Delta t = 0.046 \) is 1.5%. Even though \( \omega_A \Delta t = 0.093 \) is a relatively converged value, but \( \omega_A \Delta t = 0.046 \) is even closer to convergence. Therefore, \( \omega_A \Delta t = 0.046 \) is the best convergence parameter, while setting \( N_g = 32 \) and \( N_p = 2e4 \). The real frequencies \( \omega_r \) were also measured from the convergence tests. However, the discrepancy between any of those \( \omega_r \) is always less than 1%. Therefore,
they were not shown nor used to find the converged value of $\Delta t$.

### 3.1.2 Perpendicular grid numbers

The wave is symmetric along x- and y-axes because $k_x = k_y$. Consequently, the number of grids per wavelength along x and y; $N_x$ and $N_y$, respectively; should also be the same. The four test cases on number of grids per wavelength along both x- and y-axes are $N_x = N_y = 8$, 16, 32, and 64, while $\omega_A \Delta t = 0.093$; $N_p = 5000$ for $N_x = 8$, $N_p = 1e4$ for $N_x = 16$, $N_p = 2e4$ for $N_x = 32$, $N_p = 4e4$ for $N_x = 64$; and number of grids along z ($N_z$) is 32. Figure 3.2 shows the damping rate of mode 111 versus number of grids along x. As shown, the damping rate becomes relatively constant starting at $N_x = 16$. However, $N_x = 32$ is the chosen converged parameter. The discrepancy between the damping rates using $N_x = 32$ and $N_x = 64$ is 3.9%.
Similar convergence tests were conducted before [Cheng, 2011], and those tests show that using \( N_x \) over 64 lowers the discrepancy between the growth rates. Therefore it indicates that further tests on any \( N_x > 64 \) would show a plateau in Figure 3.2.

### 3.1.3 Parallel grid number

A similar convergence test on parallel grid number \( N_z \) was carried out. The \( N_z \) I tried are 8, 16, 32, and 64. The rest of the parameters are fixed, and they are \( \omega_A \Delta t = 0.093; N_p = 5000 \) for \( N_z = 8 \), \( N_p = 1e4 \) for \( N_z = 16 \), \( N_p = 2e4 \) for \( N_z = 32 \), \( N_p = 4e4 \) for \( N_z = 64 \); and \( N_x = N_y = 32 \). Figure 3.3 shows the damping rate of mode 111 versus \( N_z \). The damping rate at \( N_z = 32 \) is where convergence occurs. The discrepancy between the damping rates
Figure 3.3: Linear damping rate $\gamma/\omega_A$ of KAW of mode 111 versus $N_z$.

using $N_z = 32$ and $N_z = 64$ is 2.9%, which is small. Therefore, $N_z = 32$ is chosen as the converged parameter.

3.1.4 Number of particle per wavelength

The various $N_p$ tested for possible convergence parameter are 1250, 2500, 5000, 10000, and 20000. The fixed parameters of the wave includes $\omega_A \Delta t = 0.093$ and $N_g = 32$. The damping rate versus $N_p$ is plotted in Figure 3.4. The damping rates in all five test cases do not vary much from one another, by 2% at most from its neighbor’s damping rate. Therefore, $N_p = 1250$ is good enough to use as a converged value, but I picked $N_p = 20000$ instead.
Figure 3.4: Linear damping rate $\gamma/\omega_A$ of KAW of mode 111 versus $N_p$. 
Table 3.1: Comparisons between theoretical and computational values of KAW real frequency and linear damping rate.

<table>
<thead>
<tr>
<th></th>
<th>$\omega_r/\omega_A$</th>
<th>$\gamma/\omega_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>theory</td>
<td>1.17e0</td>
<td>-1.02e-2</td>
</tr>
<tr>
<td>simulation</td>
<td>1.16e0</td>
<td>-9.46e-3</td>
</tr>
<tr>
<td>variation</td>
<td>0.94%</td>
<td>7.69%</td>
</tr>
</tbody>
</table>

The theoretical expressions for real frequency and linear damping rate of a KAW were shown in Chapter 2. I use those expressions and take the appropriate limit on the Z-function in order to find the theoretical values. I then compare them to the linear simulation results of a KAW using the convergence parameters determined in earlier in this Section ($\omega_A\Delta t = 0.046$, $N_g = 32$, $N_p = 2e4$). The comparisons were displayed in Table 3.1. The variation was defined as the division of the difference between theoretical and simulation values by the theoretical value. While the gyrokinetic PIC code could closely recover $\omega_r/\omega_A$ to the theoretical calculations (with less than 1% variation), the $\gamma/\omega_A$ has variation of less than 8%. The reason why the variation between the theoretical and simulation value on $\gamma/\omega_A$ is this high may have to do with two reasons. First, as mentioned earlier, the analytic value of $\gamma/\omega_A$ was obtained by making approximation on the Z-function. This approximation could deviate the analytic value from the simulated value. Second, measurement error on the simulated growth rate could also be a possibility. The regression fit analyzed the entire time history of the wave to determine the value for $\omega_r/\omega_A$ and $\gamma/\omega_A$. While the wave period is established from the beginning, linear damping rate is not clear until about a wave period, therefore, the regression fit on the damping rate has more room for error.

With the same convergence parameters, I could demonstrate the gyrokinetic PIC code can equally simulate modes with the same $k_{\perp}\rho_s = 0.4$ and $k_{\parallel}\rho_s = 0.004$ but with different $k_x$ and $k_y$. Earlier I found the convergence parameters of mode 111 ($k_x = k_y$, $k_{\perp}\rho_s = 0.4$, and $k_{\parallel}\rho_s = 0.004$). I simulated 2 more KAW in separate occurrences: mode with $k_x\rho_s = k_{\perp}\rho_s = 0.4$, $k_y\rho_s = 0$, $k_{\parallel}\rho_s = 0.004$ (mode 101) and mode with $k_x\rho_s = 0$, $k_y\rho_s = k_{\perp}\rho_s = 0.4$, $k_{\parallel}\rho_s = 0.004$.  

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Figure 3.5: The time evolution of KAWs of mode 111, 101, and 011. All three waves have the same wavenumber: $k_{\parallel} \rho_s = 0.4$ and $k_{\parallel} \rho_s = 0.004$

$k_{\parallel} \rho_s = 0.004$ (mode 011). The time evolutions of the vector potential in k-space of all three modes are plotted in Figure 3.5, with the time normalized by KAW frequency $\omega_{KAW}$. The figure shows that no matter which direction the $k_{\perp} \rho_s$ vector points, the time evolution of the mode is unaffected. Therefore, I choose to simulate mode 101 instead of mode 111 in when searching for nonlinear convergence parameters.

### 3.2 Nonlinear Convergence and Benchmark

In nonlinear simulation, not only does the KAW linearly Landau damped, but also oscillates nonlinearly due to particles trapped in potential well created by the finite wave amplitude. As a consequence, the nonlinear simulation is more difficult compared to the linear ones,
for more physics has to be considered in the calculations. From the GK theory and code formulation section, all the terms in red are the terms that are ignored in linear calculations, but become important when the wave amplitude becomes significant. In this section, I include those terms in all my convergence tests and benchmark my results with theory.

The following are the equations that govern the nonlinear simulations:

\[
\frac{\tau}{\lambda_D} (\delta \phi - \delta \tilde{\phi}) = 4\pi \sum_\alpha q_\alpha \delta n_\alpha,
\]

\[
\frac{4\pi n_0 e}{c} \delta u_{e||} = \nabla_\perp^2 \delta A_{||} + \frac{4\pi n_0 e}{c} \delta u_{i||},
\]

\[
e^{\delta \psi/T_e} = 1 + \frac{\delta n_e}{n_0},
\]

\[
\frac{\partial}{\partial t} \delta n_e + n_0 b_0 \cdot \nabla \delta u_{e||} + n_0 \delta b \cdot \nabla \delta u_{e||} + \frac{c b_0 \times \nabla \delta \phi}{B_0} \cdot \nabla \delta n_e = 0, \tag{3.2}
\]

\[
\frac{1}{c} \frac{\partial}{\partial t} \delta A_{||} = b_0 \cdot \nabla (\delta \psi - \delta \phi),
\]

\[
\frac{dX_i}{dt} = v_{i||} b_0 + v_{i||} \delta b + \frac{c b_0 \times \nabla \delta \phi}{B_0},
\]

\[
\frac{dv_{i||}}{dt} = \frac{e}{m_e} \delta E_{||},
\]

\[
\frac{dw_i}{dt} = -(1 - w_i) \frac{e}{m_i v_{th,i}^2} \nabla_{||} \delta \psi.
\]

The same procedures from the linear convergence tests were followed in the nonlinear regime. Before searching for the parameters that show convergence, I want to find the suitable initial wave amplitude that allows time for linear regime to dominate before nonlinear effects become significant. In the gyrokinetic PIC code, a parallel vector potential \(\delta A_{||,k}\) with wavenumber \(k\) is generated initially. I varied the amplitude of the initial wave and allow each case to involve
Figure 3.6: The nonlinear time evolution of mode 101, with four cases of adjusted magnitude of $\delta B/B_0$ plotted against time.

in time. The cases I tested are $\delta B/B_0 = 0.012, 0.12, 0.24$ and 0.40. The best way to compare these three cases was to overlap them and have them begin at the same initial amplitude. Therefore, I adjusted the initial amplitudes of all three scenarios by multiplying a constant factor in front of $\delta B/B_0$. The nonlinear time evolutions of the KAW mode with various initial amplitudes are plotted in Figure 3.6. These magnitudes have various constants multiplied in front such that all cases could overlap with the $\delta B/B_0 = 0.24$ case at the starting point. In the figure, the wave with initial amplitude of $\delta B/B_0 = 0.24$, is the best one that allows linear physics to dominate for a few oscillations before nonlinear physics become significant. All of these cases were simulated using $\omega_A \Delta t = 0.046$, $N_g = 32$ and $N_p = 2e4$. In the rest of the convergence tests, we will use $\delta B/B_0 = 0.24$.

The figure also shows the wave exhibits nonlinear oscillations. These nonlinear oscillations
are due to particle trapping inside the potential. The bounce frequency $\omega_b$ of the deeply trapped particles can be expressed as $\omega_b^2 = \left| \frac{q k_\parallel \delta E}{m} \right|$. As the initial amplitude increases, so does $\omega_b$, as predicted by the expression. As for the case when $\delta B/B_0 = 0.012$, no nonlinear oscillation was observed. That is because the initial amplitude of the wave is too low and particle trapping does not occur during the time span of the simulation.

### 3.2.1 Time step

The time evolution of mode 101 using various $\omega_A \Delta t$ are being compared. The four $\omega_A \Delta t$ tested are 0.093, 0.046, 0.023, and 0.012, and are shown in Figure 3.7. The only one that does not converge is $\omega_A \Delta t = 0.093$, while the rest showed convergence. Therefore, choosing $\omega_A \Delta t = 0.046$ is a good enough convergence parameter. The other parameters used are $N_g = 32$ and $N_p = 2e4$.

### 3.2.2 Parallel grid number

The next parameters I test for convergence is $N_z$. I let $N_z = 16, 32, 64, \text{ and } 128$ in the simulations, and the nonlinear evolution of mode 101 is shown in Figure 3.8. The simulation with $N_z$ set to 16 is the only one that stands out while the rest shows converged results. Therefore, I chose $N_z = 32$ as the convergence parameter, with $N_x = N_y = 32$, $\omega_A \Delta t = 0.046$; and $N_p = 1e4$ for $N_z = 16$, $2e4$ for $N_z = 32$, $4e4$ for $N_z = 64$ and $8e4$ for $N_z = 128$. By the same logic, $N_x$ and $N_y$ should also converge at 32.
Figure 3.7: Nonlinear time evolution of mode 101 with various $\omega_A \Delta t$. 
Figure 3.8: Nonlinear time evolution of mode 101 with various $N_z$. 
3.2.3 Number of particles per wavelength

The last parameter to test for convergence is $N_p$. The $N_p$ I tested are 625, 1250, 2500, 5000, 10000, and 20000. The nonlinear time evolutions of the KAW with various $N_p$ are plotted in Figure 3.9. While all the $N_p$ look very similar, the result converged at $N_p = 5000$ because its KAW time evolution look almost indistinguishable from higher $N_p$. The other parameters used in this convergence test are $\omega_A \Delta t = 0.046$ and $N_g = 32$.

The convergence tests performed on the nonlinear simulation of a KAW shows that setting $\omega_A \Delta t = 0.046$, $N_g = 32$ and $N_p = 5000$ can give us minimal numerical instability using the least amount of computer time to achieve.
3.3 Multiple Modes

The addition of multiple modes into the code could alter the convergence parameters. Therefore I perform growth rate convergence tests with respect to time step size ($\omega_A \Delta t$), number of grids in each direction ($N_g$), and number of particles per (shortest) wavelength ($N_p$) in this section.

The KAW I chose to study in the convergence test is the fundamental mode of the simulation domain, which has $k_x \rho_s = k_x \rho_s = 0.2$, $k_y \rho_s = 0$, $k_z \rho_s = 0.002$ (or mode 101) and initial perturbation of $\delta B/B_0 = 0.012$. The waves generated by mode 101 via parametric decay instability are $121(k_x \rho_s = 0.2, k_y \rho_s = 0.4, k_z \rho_s = 0.002)$ and $222 (k_x \rho_s = 0.4, k_y \rho_s = 0.4, k_z \rho_s = 0.004)$. The plasma has an electron beta of $\beta_e = 0.16$, $\tau = 1.0$. The composition of the plasma is singly-charged ions and electrons, in which the ion motion is treated as gyrokinetic and electron motion as massless fluid.

3.3.1 Time step

The $\omega_A \Delta t$ tested as a possible convergence parameter are 0.196, 0.098, 0.049, 0.025, 0.012 and 0.006. In Figure 3.10, the growth rates of all three daughter waves versus $\omega_A \Delta t$ are shown. As demonstrated, the growth rate begins to converge at $\omega_A \Delta t = 0.098$, and the discrepancy between this and the damping rate from $\omega_A \Delta t = 0.049$ is about 2% for both daughter modes. Even though $\omega_A \Delta t = 0.098$ is a relatively converged value, $\omega_A \Delta t = 0.049$ is even closer to convergence. Therefore, $\omega_A \Delta t = 0.049$ is the best convergence parameter, while setting $N_g = 64$ and $N_p = 25600$. 

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Figure 3.10: Convergence of the growth rate ($\gamma/\omega_A$) with respect to the size of each time step ($k||v_A\Delta t$).
Figure 3.11: Convergence of the growth rate ($\gamma/\omega_A$) with respect to the number of grids in each direction ($N_g$).

### 3.3.2 Grid number

The test cases on number of grids per grid along any Cartesian coordinate axes are $N_g = 16, 32, \text{ and } 64$, while $\omega_A \Delta t = 0.006$, $N_p = 25600$. Figure 3.11 shows the growth rate of modes 121 and 222 versus number of grids along an axis. As shown, the growth rate converges at $N_g = 64$.

### 3.3.3 Number of particles per wavelength

The various $N_p$, or the number of particles per (shortest) wavelength, tested for possible convergence parameter are 800, 1600, 3200, 6400, 12800, and 25600. Fixed parameters
Figure 3.12: Convergence of the growth rate ($\gamma/\omega_A$) with respect to the number of particles per wavelength ($N_p$).

include the number of grids ($N_g = 64$) and size of each time step ($\omega_A \Delta t = 0.006$). The growth rate versus $N_p$ is plotted in Figure 3.12. The growth rates in all 6 test cases do not vary much from one another, by 2% at most from its neighbor’s damping rate. Therefore, $N_p = 3200$ is good enough to use as a converged value, but I picked $N_p = 6400$ instead.

The 3 convergence tests performed on the multiple-mode simulation helped us determined that by setting $\omega_A \Delta t = 0.049$, $N_g = 64$ and $N_p = 6400$, we could minimize numerical deviation and run the code in the shortest amount of computer time.
Now that the convergence parameters are determined for the KAW simulations with sideband generation, we could use the PIC code to understand the effects on convective cell (CC) being driven by a KAW. As mentioned in Chapter 1, CC is a plasma flow that propagates in the direction perpendicular to the ambient field $B_0z$. In fusion plasmas, zonal flow does not propagate along the y and z directions ($k_y = k_z = 0$); it has a non-zero wave vector pointing in the x-direction, in which the plasma is nonuniform. 3-D simulation shows that the zonal flow is responsible for the suppression of microturbulence in toroidal plasma, which could enhance plasma confinement.[Lin et al., 1998] Similarly in space plasmas, a flow plays an important role in breaking apart KAW turbulence, and it is known as a convective cell.[Zhao et al., 2012] Convective cell is a wave that propagates only along the direction perpendicular to the background magnetic field $B_0$ ($k_\parallel = 0$). There is no distinction between the x- and y-directions when plasma is uniform in the current work.
The plasma in this simulation is composed of gyrokinetic ions, fluid electrons, with electron plasma beta ($\beta_e$) of 0.16 and electron-to-ion temperature ratio ($\tau$) of 1.0. The convergence parameters mentioned in Section 3.3 are employed throughout the simulations in this chapter, which include a number of particles per (shortest) wavelength ($N_p$) of 6400, a number of grids ($N_g$) of 64 for each direction and a size of each time step ($\omega_A \Delta t$) of 0.049.

4.1 Direction of KAW Energy Cascade

4.1.1 Daughter waves with both longer and shorter wavelengths

Before we focus on CC effects in the simulation, we want to understand the wave energy transfer of a KAW into other possible waves.

To test the energy channel of a KAW into other waves, a pump wave with an initial magnetic field perturbation of $\delta B/B_0 = 0.012$ is introduced into the plasma. This pump wave is a standing wave and has a wavevector of $k_x \rho_s = k_{\perp} \rho_s = 0.2$, $k_y \rho_s = 0$ and $k_{\parallel} \rho_s = 0.002$. All other waves have no amplitudes at $t = 0$. The energy of the pump wave is allowed to transfer into waves with $k_x \rho_s$, $k_y \rho_s$, $k_{\parallel} \rho_s$ equal to 0, 0.5, 1.5, and 2.0 times of the pump wave. The time history of the scalar potential of all dominant waves is plotted in Figure 4.1. In this figure, the size of the simulation box is $2\pi (L_x^{-1} \times L_y^{-1} \times L_z^{-1}) \rho_s = 0.1 \times 0.1 \times 0.001$. A wave along any axis in the Cartesian coordinate with wavelength equivalent to the size of the simulation box axis is labeled as mode with index 1, wavelength equivalent to half the size of the simulation box is labeled as mode with index 2, etc. The simulation allows waves with wavelengths as short as a quarter of the size of the simulation box and filtered out the rest. Hence, the pump wave is labeled as wave of mode 202. The mode filter in the code was set up such that the box could fit four wavelengths of the shortest wave in each direction in the Cartesian coordinate system. That way the pump wave energy could transfer to both
longer and shorter wavelengths.

As Figure 4.1 demonstrates, energy gets transfer mostly to waves with shorter wavelengths. The daughter waves that get the highest growth rate $\gamma$ are modes 040 ($k_x \rho_s = 0, k_y \rho_s = 0.4, k_\parallel \rho_s = 0$), 242 ($k_x \rho_s = 0.2, k_y \rho_s = 0.4, k_\parallel \rho_s = 0.002$) and 444 ($k_x \rho_s = 0.4, k_y \rho_s = 0.4, k_\parallel \rho_s = 0.004$). The growth rates of those 3 dominant modes are displayed in Table 4.1, which shows that their $\gamma$ are approximately the same. Theoretically all three daughter waves should have identical growth rates due to wave-wave interaction. However, the measured values are slightly different (less than 1%). The discrepancy among growth rates may have to do with the numerical noise or the measurement method. The method employed to obtain the growth rate is by taking the peaks of the oscillations (after the transient period) from each mode and then connect the peaks with a straight line in a semi-log plot. The slope of the line is the measured value of the growth rate. Even though the measured growth rates are not the same, they are off by at most 0.8% from each other.

One interesting daughter wave from Figure 4.1 is mode 102 ($k_x \rho_s = 0.1, k_y \rho_s = 0, k_\parallel \rho_s = 0.002$). This mode has the strongest amplitude at the earliest part of the energy cascade. Modes 102 and 040 together drives the growth of mode 142 ($k_x \rho_s = 0.1, k_y \rho_s = 0.4, k_\parallel \rho_s = 0.002$). This wave-wave interaction seems to hinder the growth of mode 142 and therefore saturates at later time. This could also be a transient behavior of mode 102. These two hypotheses require further investigation.

### 4.1.2 Daughter waves with shorter wavelengths

Now that we know energy of the pump wave gets mainly transferred to waves with higher $k$. The next step to take is to set up pump wave with the same wavevector as the one from subsection 4.1.1 ($k_x \rho_s = k_\perp \rho_s = 0.2, k_y \rho_s = 0, k_\parallel \rho_s = 0.002$) and reduce the box size by half. That way, the pump wave energy is limited to transfer to waves with higher $k$, up to
Figure 4.1: The time history of scalar potential $\delta \phi$ of the pump wave 202 along with the daughter waves generated with the highest amplitudes.
Figure 4.2: The time history of scalar potential $\delta \phi$ of the pump wave 101 along with the daughter waves generated with the highest amplitudes.

2 wavelengths fitted in each direction in the Cartesian coordinate system. This is easier to simulate. The time history of the $\delta \phi$ is displayed in Figure 4.2. Since the simulation box size is halved, the pump wave is labeled as mode 101 because its wavelength along the x- and z- axis is equivalent to the two sides of the box. Using the same logic, modes 040, 242 and 444 in subsection 4.1.1 become modes 020, 121 and 222, respectively.

The growth rates of the three dominant modes are displayed in Table 4.1, which shows that the growth rates are higher by at most 3% compared to the ones from Figure 4.1. This may have to do with the fact that less channels were available for the pump wave energy to get transferred to in the current case (Figure 4.2). This shows that even the simulation box size is cut down by half and energy of the pump wave was only allowed to transfer to waves with shorter wavelengths, we could recover the wave-wave coupling if energy were allowed
to transfer to waves with either longer or shorter wavelengths. Therefore, the pump wave energy cascades to waves with shorter wavelengths during KAW turbulence.

Modes 102 and 142 observed in Figure 4.1 are not present in Figure 4.2 because the size of the simulation box simply does not hold waves with equivalent wavelengths as those two modes.

4.1.3 Identification of dominant daughter waves

The energy of the pump wave being transferred to every possible mode up to the 2\textsuperscript{nd} harmonic is displayed in Figure 4.2. However, not every one of these daughter waves plays a major role in the wave-wave coupling. In this section, we find those dominant modes.

To figure out which mode plays a dominant role, a mode filter is set up in the code. This allows us to control which k-mode is filtered out and which one is kept.

The first test we perform is to filter all the mode except for the pump wave of mode 101, and daughter waves of mode 020, 121 and 222. The time history of the scalar potential of those four modes is displayed in Figure 4.3. The growth rates of the daughter waves are measured and listed in Table 4.1.

If we compare the growth rates of the daughter waves in Figure 4.3 with the same waves in Figure 4.2, they are almost identical. The discrepancy between the growth rates from those two simulations is at most 1.2%. Therefore, the three daughter waves were proven to be the ones that received the majority of the energy from the pump wave.

The next step is to determine importance of each daughter wave in different sets of wave-wave coupling. One of the simulation tests we performed is to filter out all the modes except the pump wave of mode 101 and daughter waves of mode 020 and 121. The time history of the
Figure 4.3: The time history of scalar potential $\delta \phi$ of the pump wave 101 along with the three daughter waves permitted to grow in the simulation.
Figure 4.4: The time history of the scalar potential $\delta \phi$ of the pump wave (mode 101) and daughter waves. Solid lines represent the modes from the simulation in which mode 101, 020 and 121 are kept in one simulation; while dash lines represents modes from simulation in which mode 101, 020, 121 and 222 are kept in another simulation.

The scalar potential of those three modes is displayed in Figure 4.4. This shows that daughter waves would not grow exponentially without the existence of mode 222. The minimum model requires three daughter waves (modes 020, 121 and 222).

The other simulation tests we performed is to filter out all the modes except the pump wave (mode 101) and daughter modes 121 and 222. The time history of the scalar potential of those three modes is displayed in Figure 4.5. This shows that although daughter waves 121 and 222 grow exponentially without mode 020, the growth rate of mode 121 is 46\% lower compared to the case when 020 is included in the simulation and growth rate of mode 222 is 45\% lower.
Figure 4.5: The time history of the scalar potential $\delta \phi$ of the pump wave (mode 101) and daughter waves. Solid lines represent the modes from the simulation in which mode 101, 121 and 222 are kept; while dash lines represents modes from simulation in which mode 101, 020, 121 and 222 are kept.

Table 4.1: The growth rates of the dominant daughter waves generated by pump wave 101 (top 3 rows) and pump wave 202 (bottom 2 rows).
In summary, a standing KAW is set up as the pump wave and is allowed to interact with other KAWs in the simulations. The pump wave energy being transferred to daughter modes with higher $k$ is much more preferable than being transferred to daughter modes with lower $k$. In particular, the daughter waves that gained the most energy out of this wave-wave coupling are modes 020, 121 and 222. There would not be any wave-wave coupling if mode 222 were suppressed. The convective cell (mode 020) enhances energy transfer from the pump wave into the three dominant daughter modes. Four waves (101, 020, 121, 222) are required minimal waves to model the energy cascade of a single pump wave. In addition, these four modes exhibit behavior of standing waves: the time history of all modes are sinusoidal. The real frequency of mode 020 and 121 are about the same as the real frequency of the pump wave ($\omega_{r,101} = \omega_{KAW}$). On the other hand, the real frequency of mode 222 is approximately twice the frequency of the pump wave.

4.2 Importance of Convective Cell

In this section we will study the time evolution of scalar potential, vector potential, electron and ion densities of pump wave and daughter waves under three different simulation scenarios: 1) no zonal fields present, 2) zonal flow only, and 3) zonal flow and zonal current.

4.2.1 No zonal fields

The simulation result in Figure 4.3 shows the result of a pump wave of mode 101 generating 3 daughter waves of modes 020, 121 and 222 when zonal fields are present. If zonal fields were taken out, then there are no zonal components present to influence the wave-wave interaction. The time histories of scalar potential, vector potential, electron density and ion density when zonal components are removed are displayed in Figure 4.6. The time history of
Figure 4.6: The time histories of $\delta \phi$, $\delta A_\parallel$, $\delta n_e$, $\delta n_i$ displayed in a, b, c, d, respectively, when all zonal fields were suppressed.

$\delta \phi$ here in Figure 4.6a is identical to the solid traces in Figure 4.5, but over a longer period of time. The growth rates of mode 121 and 222 are measured; the growth rate of mode 121 is $\gamma/\omega_{SAW} = 0.136$ and the growth rate of mode 222 is $\gamma/\omega_{SAW} = 0.137$. The numbers are basically identical, when we consider room for measurement error. It is not surprising to see the same growth rates for both modes because theoretically daughter waves produced from parametric decay instability (PDI) gain energy from the pump wave at the same rate.
4.2.2 Zonal flow only

Addition of zonal mode 020 in scalar potential (but not zonal vector potential) to the simulation gives result of wave-wave coupling when only zonal flow present. The time histories of scalar potential, vector potential of nonzonal modes, electron density and ion density when zonal flow is present are displayed in Figure 4.7. This shows the presence of zonal flow increases the growth rates; the growth rates of mode 020, 121 and 222 are $\gamma/\omega_{SAW} = 0.262, 0.262$ and 0.265, respectively. When zonal flow is present during wave-wave interaction, the growth rates of the daughter waves are doubled compared to the ones without zonal fields.

4.2.3 Zonal flow and zonal current

The linear zonal current and zonal vector potential are now added into the simulation setup from the previous section. The time histories of scalar potential, vector potential of nonzonal modes, electron density and ion density when zonal fields are present are displayed in Figure 4.8. The growth rates obtained from the time histories of scalar potential from all three scenarios are summarized in Table 4.2. Growth rates almost double when zonal flow is included in the simulation. However, when linear zonal current is added to the zonal flow, growth rates decreased slightly (by about 5-6%). This shows that, zonal flow can strengthen the exponential growth of daughter waves, while the addition of linear zonal current could barely affect KAW wave-wave interaction. In all three scenarios, all the modes remain as standing waves. Whether zonal fields are present or not, the real frequency of any of the four modes are unaffected. All simulations in the remaining thesis include section 2.7. Note that the nonlinear part of the zonal current, $\delta A^NL$, is not included in this simulation because Wang has shown in his work that the nonlinear term is not important in the wave-wave interaction.[Wang, 2014]
Figure 4.7: The time histories of $\delta \phi$, non-zonal $\delta A_\parallel$, $\delta n_e$, $\delta n_i$ displayed in a, b, c, d, respectively, when all zonal components were suppressed except for zonal flow.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>$\gamma_{020}/\omega_A$</th>
<th>$\gamma_{121}/\omega_A$</th>
<th>$\gamma_{222}/\omega_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>No zonal components</td>
<td>N/A</td>
<td>0.136</td>
<td>0.137</td>
</tr>
<tr>
<td>Zonal flow only</td>
<td>0.262</td>
<td>0.262</td>
<td>0.265</td>
</tr>
<tr>
<td>Zonal flow and zonal current</td>
<td>0.248</td>
<td>0.250</td>
<td>0.250</td>
</tr>
</tbody>
</table>

Table 4.2: Growth rates (normalized by SAW wave frequency) of modes 020, 121 and 222 generated by KAW of mode 101 under three different scenarios: no zonal components exist, zonal flow only, and zonal flow along with linear zonal current in the simulations.
Figure 4.8: The time histories of $\delta\phi$, non-zonal $\delta A_\parallel$, $\delta n_e$, $\delta n_i$ displayed in a, b, c, d, respectively, when zonal flow and linear zonal current calculations are included.
4.3 Growth Rates’ Dependence on Pump Wave Parameters

The growth rates of daughter waves determined so far in this chapter is using fixed parameters for the pump wave \( \left( \delta B/B_0 = 0.012, k_{\perp} \rho_s = 0.2, k_{\parallel} \rho_s = 0.002 \right) \). However, different initial amplitudes and wavenumbers of the pump wave affect growth rates of the daughter waves differently. In this section, we determine the growth rates’ dependencies on the pump wave parameters to provide further insights on the PDI process.

4.3.1 Pump wave amplitudes

The pump wave is generated by an initial perturbation of the electron current density \( \delta j_{\parallel e} \). The amplitude of this pump wave perturbation affects the later growth of other daughter waves. A scan of the pump wave amplitudes and the growth rates in every case is conducted and the result is displayed in Figure 4.9. The parallel wavevector \( k_{\parallel} \rho_s \) is fixed at 0.2 and \( k_{\perp} \rho_s \) at 0.002. A linear fit is plotted on top of the simulation results, which indicates that the threshold to exponential growth is at \( \delta B_{\perp}/B_0 = 0.005 \). The linear fit is consistent with the theory of modulational instability.[Chen et al., 2001]

4.3.2 Parallel wavenumber \( k_{\parallel} \rho_s \)

The parallel component of the wavevector (parallel wavenumber) of the pump wave plays a role in the daughter wave growth rates. A scan of the parallel wavenumbers is displayed in Figure 4.10. The perpendicular wavevector \( k_{\perp} \rho_s \) is fixed at 0.2 and the perturbed magnetic field \( \delta B/B_0 \) at 0.012. In the figure, the product of the parallel wavenumber and the growth rates of the daughter waves \( \left( k_{\parallel} \rho_s \gamma/\omega_A = \gamma \rho_s / v_A \right) \) are plotted against the wavenumber ratio
Figure 4.9: The growth rates (normalized by the SAW frequency $\omega_A$) of daughter waves with respect to the pump wave amplitude $\delta B_\perp / B_0$. 

$$\frac{\gamma_{020}}{\omega_A} = 35.08 \left| \frac{\delta B_\perp}{B_0} \right| - 0.17$$
Figure 4.10: The $\kappa_\parallel \rho_s \gamma / \omega_A$ of daughter waves with respect to $k_\parallel / k_\perp$, with various $k_\parallel \rho_s$ and fixed $k_\perp \rho_s$ at 0.2

($k_\parallel / k_\perp$). $v_A$ is the Alfvén speed. At lower-$k_\parallel \rho_s$ values, $\gamma$ is independent of $k_\parallel \rho_s$ as expected, since the dominant nonlinear term $v_{E \times B}$ only involves $k_\perp$. The growth rate can be fitted as the normalized equation $\gamma = \gamma_0 + a k_\parallel$, with $\gamma_0$ and $a$ as constants.

4.3.3 Perpendicular wavenumber $k_\perp \rho_s$

A scan on the perpendicular wavenumber is also conducted while pump wave initial amplitude with parallel wavenumber fixed. The dependence of growth rates on $(k_\perp \rho_s)^2$ is displayed in Figure 4.11. The parallel wavevector $k_\parallel \rho_s$ is fixed at 0.002 and $\delta B / B_0$ at 0.012. The growth rate's dependency on $(k_\perp \rho_s)^2$ can be fitted into a linear function, which is consistent with the dominant nonlinear $v_{E \times B}$ convection that leads to the parametric decay instability. Based
on extrapolation, the growth rate of the daughter waves are zero when $k_{\|\rho_s} = 0$. When $(k_{\perp\rho_s})^2$ reaches 0.2, $\gamma/\omega_A$ reaches 1.2.

### 4.4 Growth Rate Comparison for the High and Low $k_{\perp}$ Modes

We have seen the convective cell (mode 020) and other daughter waves (mode 121 and 222) generated using a pump wave (mode 101) with $k_{\perp\rho_s} = 0.2$ and $k_{\|\rho_s} = 0.002$. Now let us study how the generation and evolution of those daughter waves vary in the lower and higher end of the pump wave $k_{\perp\rho_s}$ value ($k_{\perp\rho_s} = 0.1$ and $k_{\perp\rho_s} = 0.4$, respectively).
More importantly, we want to know how the importance of the daughter waves varies under different $k_\perp$ values.

### 4.4.1 $k_\perp \rho_s = 0.1$

First test we choose to run is to allow a lower $k_\perp$-pump wave ($k_\perp \rho_s = 0.1$) to excite daughter waves of mode 020 and 121. The initial amplitude of the pump wave is $\delta B_\perp/B_0 = 0.012$ and its parallel wavenumber is $k_\parallel \rho_s = 0.002$. The time history of these three modes is displayed in Figure 4.12. This shows that daughter waves barely show signs of exponentially growth.

Next test we choose to run is to allow the pump wave (with the same initial condition as the previous test) to excite daughter waves of mode 121 and 222. The time history of these
three modes are displayed in Figure 4.13. This shows that daughter wave 121 is the only one with exponential growth, with a rate of $\gamma_{121}/\omega_{SAW} = 0.05$.

Finally, we allow the pump wave to excite all three daughter waves (mode 020, 121 and 222). The time history of these four modes is displayed in Figure 4.14. This shows that daughter wave 121 is the only one with exponential growth, with a rate of $\gamma_{121}/\omega_{SAW} = 0.047$. The growth rate from simulation with the presence of mode 101, 121 and 222 (this is known as the parametric decay instability) is off from the growth rate with the presence of mode 101, 020, 121 and 222 by 6%. In the low-$k_\perp$ limit, convective cell (mode 020) is not necessary in order to simulate the optimal 3-wave interaction. All the modes exhibit standing waves even when $k_\perp$ is lowered. The daughter modes 020 and 121 have real frequency approximately the same as the frequency of the pump wave, while mode 222 have real frequency approximately
Figure 4.14: The time history of the scalar potential of the pump wave (mode 101) and daughter waves, in which mode 101, 020, 121 and 222 are kept.

double of the frequency of the pump wave.

4.4.2 \( k_{\perp} \rho_s = 0.4 \)

We are also interested in knowing the importance of each daughter wave when the perpendicular wavevector of the pump wave is increased to \( k_{\perp} \rho_s = 0.4 \). The initial amplitude of the pump wave is set to \( \delta B_{\perp}/B_0 = 0.012 \) and the pump wave parallel wavenumber is fixed at \( k_{\parallel} \rho_s = 0.002 \).

The first test we performed is to allow the pump wave (mode 101) to excite daughter waves of mode 020 and 121. The time history of these three modes is displayed in Figure 4.15. This shows that daughter waves of mode 020 and 121 grow exponentially, with \( \gamma_{020}/\omega_{SAW} = 0.076 \).
and $\gamma_{121}/\omega_{SAW} = 0.079$.

Next test is to allow mode 101 to excite modes 121 and 222. The time history of these three modes is displayed in Figure 4.16. This shows that daughter waves 121 and 222 grow exponentially, with a rate of $\gamma_{121}/\omega_{SAW} = 0.511$ and $\gamma_{222}/\omega_{SAW} = 0.509$.

Finally, we allow mode 101 to excite modes 020, 121 and 222. The time history of these four modes is displayed in Figure 4.17. This shows that daughter waves 020, 121 and 222 grow exponentially, with a rate of $\gamma_{020}/\omega_{SAW} = 0.99$, $\gamma_{121}/\omega_{SAW} = 0.999$ and $\gamma_{222}/\omega_{SAW} = 1.004$. The growth rate from simulation with the presence of mode 101, 121 and 222 is off from the growth rate here by 50%. In the high-$k_\perp$ limit, convective cell (mode 020) is necessary in order to simulate the optimal 3-wave interaction. While the pump wave and mode 020 remain
Figure 4.16: The time history of the scalar potential of the pump wave (mode 101) and daughter waves, in which mode 101, 121 and 222 are kept.
Figure 4.17: The time history of the scalar potential of the pump wave (mode 101) and daughter waves, in which mode 101, 020, 121 and 222 are kept.

as standing waves at this higher-$k_\perp$ limit, modes 121 and 222 are turning into traveling wave, although the transition is not complete. Mode 020 at this $k_\perp \rho_s = 0.4$ has real frequency approximately the same as frequency of the pump wave, which is consistent with the trend at lower-$k_\perp$. However, modes 121 and 222 break away from the trend observed at lower-$k_\perp$: mode 121’s real frequency is approximately twice the frequency of the pump wave and mode 222’s real frequency is undetermined. Mode 222 appears to be superposition of two waves of different frequencies.

In this section, it is shown that pump wave with $k_\perp \rho_s$ ranging from 0.1 to 0.4 cannot couple to mode 020 and 121 efficiently. When convective cell (mode 020) is excluded and pump wave is allowed to couple to modes 121 and 222, then growth rates of those daughter waves increase as $k_\perp \rho_s$ of the pump wave increases. If pump wave is allowed to couple to modes
Table 4.3: Growth rates (normalized by SAW wave frequency) of daughter waves generated by KAW of mode 101 under three different $k_\perp \rho_s$ values: 0.1, 0.2, and 0.4 in the simulations with and without convective cell.

<table>
<thead>
<tr>
<th>$k_\perp \rho_s$</th>
<th>$\gamma_{121}/\omega_A$</th>
<th>$\gamma_{222}/\omega_A$</th>
<th>$\gamma_{020}/\omega_A$</th>
<th>$\gamma_{121}/\omega_A$</th>
<th>$\gamma_{222}/\omega_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.05</td>
<td>N/A</td>
<td>N/A</td>
<td>0.047</td>
<td>N/A</td>
</tr>
<tr>
<td>0.2</td>
<td>0.136</td>
<td>0.137</td>
<td>0.248</td>
<td>0.250</td>
<td>0.250</td>
</tr>
<tr>
<td>0.4</td>
<td>0.511</td>
<td>0.509</td>
<td>0.99</td>
<td>0.999</td>
<td>1.004</td>
</tr>
</tbody>
</table>

020, 121 and 222, then the percentage of increase from the case without convective cell goes up as $k_\perp \rho_s$ of the pump wave goes up. Also, the presence of CC (mode 020) can increase the growth rates by a factor of approximately two. This is true for the two higher-$k_\perp$ cases we tested; presence of CC did not affect the growth rate of mode 121 when $k_\perp \rho_s = 0.1$. The comparison could be found in table 4.3. Finally, the daughter waves, especially mode 121 and 222, are turning from standing to traveling waves as $k_\perp$ increases.
Chapter 5

Role of Convective Cell in Nonlinear Interaction of Two KAW Pump Waves

We have successfully demonstrated the excitation of convective cell (CC) by a single KAW. We have showed that energy of a single pump wave cascades to waves of higher wavenumber. The three daughter waves mode 101 produced are 020, 121 and 222. Without mode 222, there would not be any wave-wave interaction. On the other hand, without convective cell (mode 020), the three waves undergoes parametric instability. The daughter waves do gain energy, but growth rates are only half of the growth rates from the case including CC (mode 020). In that case with CC, the growth rates of the daughter waves have a linear dependence on the pump wave amplitude; no dependence on $k_\parallel \rho_s$ in the low-parallel wavenumber limit; and linear dependence on $(k_\perp \rho_s)^2$ of the pump waves .

In this chapter, we perform simulations with two kinetic Alfvén waves serving as pump waves to study direct generation of CC by two KAW pump waves. The pump waves have
wavevectors perpendicular to each other, since the dominant nonlinear interaction is $E \times B$ drift. We perform the same set of analyses as we did for a single pump wave. A comparison between the results from a single and two pump waves is displayed at the end of the chapter.

5.1 Direction of KAW Energy Cascade of KAWs of Same Phase

The energy cascade from the two pump waves to daughter waves are simulated. The plasma is composed of GK ions and fluid electrons. The temperature of the two species are the same ($\tau = 1$) and $\beta_e = 0.16$. The parameters tested in the multiple-mode convergence tests (see Chapter 3) are employed in this chapter: they are $\omega_A \Delta t = 0.049$, $N_g = 64$ and $N_p = 6400$.

5.1.1 Daughter waves with shorter wavelengths

The two pump waves are initial perturbations introduced into the plasma. They have wavevectors perpendicular to each other; one has $\mathbf{k}$ in the xz-direction (mode 101) and the other in the yz-plane (mode 011). The wave amplitudes are the same for both pump waves and the wavelengths are the same as the size of the simulation box, which gives $k_x \rho_s = 0.2$, $k_y \rho_s = 0$ and $k_z \rho_s = 0.002$ for mode 101 and $k_x \rho_s = 0$, $k_y \rho_s = 0.2$ and $k_z \rho_s = 0.002$ for mode 011. They are allowed to interact with waves of shorter wavelengths, of up to half the size of the simulation box (mode index $\leq 2$). The time history of the pump waves and the daughter waves with the strongest interaction are displayed in Figure 5.1. As shown in the figure, pump wave 101 is able to couple to mode 020, 121, and 222, as demonstrated in single pump wave simulation. On the other hand, pump wave 011 excites 200, 211 and 222. Mode 110 has the highest amplitude among the daughter waves and is a product of 3 wave coupling with the 2 pump waves.
Figure 5.1: The time history of $\delta \phi$ of the pump waves (mode 101 and 011 with the same phase) and daughter waves generated with the highest amplitudes.
Figure 5.2: The time history of $\delta \phi$ of the pump waves (mode 101 and 011 with a phase difference of $\pi/2$) and daughter waves generated with the highest amplitudes.

Besides launching two completely in-phase pump waves, we also did a simulation with two pump waves that are $\pi/2$ out-of-phase from each other. The time history of such run is displayed in Figure 5.2. The dominant daughter waves are the same as before. However, the growth rates are slightly lower from the earlier case. The growth rates here are about 3% - 10% lower compared to the growth rates excited by the two in-phase pump waves. The difference is not significant, especially when measurement error has not been taken into account. There seems to be no significant or meaningful relationship between growth rates of daughter waves and the phase difference between the two pump waves.

A list of the daughter waves generated by the two pump waves (mode 101 and 011) and the corresponding growth rates is displayed in Table 5.1. The growth rates of the daughter waves generated by a single pump wave (mode 101) are also listed for comparison. The growth
Table 5.1: The growth rates of the dominant daughter waves generated by a single pump wave 101 (middle column) and two pump waves 101 and 011 with the same phase (right column).

<table>
<thead>
<tr>
<th></th>
<th>two pump wave 101 and 011</th>
<th>single pump wave 101</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_{110}/\omega_A$</td>
<td>0.451</td>
<td>N/A</td>
</tr>
<tr>
<td>$\gamma_{121}/\omega_A$</td>
<td>0.453</td>
<td>0.25</td>
</tr>
<tr>
<td>$\gamma_{211}/\omega_A$</td>
<td>0.45</td>
<td>N/A</td>
</tr>
<tr>
<td>$\gamma_{020}/\omega_A$</td>
<td>0.455</td>
<td>0.248</td>
</tr>
<tr>
<td>$\gamma_{200}/\omega_A$</td>
<td>0.442</td>
<td>N/A</td>
</tr>
<tr>
<td>$\gamma_{222}/\omega_A$</td>
<td>0.451</td>
<td>0.25</td>
</tr>
</tbody>
</table>

rates of the daughter waves produced by two pump waves of the same initial amplitude are 80% - 83% higher compared to the ones produced by a single pump wave.

5.1.2 Identification of dominant daughter waves

Now that we identify the six daughter waves with the strongest growth from the interaction with two pump waves, the next step is to understand the importance of these daughter waves to drive the exponential growth observed from Figure 5.1.

The first test is the simplest, or most straight-forward, three-wave interaction: two pump waves (101 and 011) coupled to the daughter wave of mode 110. The time history of these three modes are being compared with the ones from the simulation in which all waves with shorter wavelengths (up to half the size of the simulation box) are kept. The comparison is displayed in Figure 5.3, which shows that daughter wave (mode 110) would not grow exponentially without the existence of the rest of the daughter waves.

The second test is to simulate the wave-wave interaction of the pump waves (mode 101 and 011) with daughter waves 110 and 112 only. Even though mode 112 is not one of the modes with the strongest growth (see Figure 5.1), it is one of the possible daughter waves when the two pump waves directly couple together. The time history of these four modes
Figure 5.3: The time history of the scalar potential $\delta \phi$ of the pump waves (mode 101 and 011) and daughter wave (mode 110). Solid lines represent the modes from the simulation in which mode 101, 011 and 110 are kept; while dash lines represent modes from simulation in which all modes with shorter wavelengths of up to half of the size of the simulation box are kept.
is displayed in Figure 5.4, which shows that daughter waves (mode 110 and 112) would not grow exponentially without the existence of the rest of the daughter waves.

The third test is to simulate the wave-wave interactions of the pump waves (mode 101 and 011) with daughter waves 110, 121 and 211 only. These three daughter waves displayed the strongest growth among the six in Figure 5.1. The time history of these five modes is displayed in Figure 5.5, which shows that daughter waves (mode 110, 121 and 211) would not grow exponentially without the existence of the rest of the daughter waves.

The fourth test is to simulate the wave-wave interactions of the pump waves (mode 101 and 011) with daughter waves 110, 121, 211 and 222 only. Mode 110 is chosen in this
Figure 5.5: The time history of the scalar potential of the pump waves (mode 101 and 011) and daughter waves (mode 110, 121 and 211). Solid lines represent the modes from the simulation in which mode 101, 011, 110, 121 and 211 are kept; while dash lines represent modes from simulation in which all modes with shorter wavelengths of up to half of the size of the simulation box are kept.
Figure 5.6: The time history of the scalar potential of the pump waves (mode 101 and 011) and daughter waves (mode 110, 121, 211 and 222). Solid lines represent the modes from the simulation in which mode 101, 011, 110, 121, 211 and 222 are kept; while dash lines represent modes from simulation in which all modes with shorter wavelengths of up to half of the size of the simulation box are kept.

test because it is suppose to be the result of direct interaction between the two pump waves. Mode 121, 211 and 222 are chosen in this test because they are essential for PDI of the KAW pump waves. The time history of these six modes is displayed in Figure 5.6. This shows that daughter waves (mode 110, 121, 211 and 222) could grow exponentially without the existence of the rest of the daughter waves. However, the growth rate is lower. The growth rate of mode 110 is $\gamma_{110}/\omega_A = 0.334$, growth rate of mode 121 and 211 are $\gamma_{121}/\omega_A = \gamma_{211}/\omega_A = 0.337$, and growth rate of mode 222 is $\gamma_{222}/\omega_A = 0.333$.

The fifth test is to simulate the wave-wave interactions of the pump waves (mode 101 and 011) with daughter waves 020, 200, 121, 211 and 222 only. The goal of this test is to see how
the energy cascade without the direct interaction between the two pump waves, since mode 110 is filtered out. The time history of these seven modes is displayed in Figure 5.7. This shows the daughter waves (mode 110, 020, 200, 121, 211 and 222) could have exponential growth in time even without mode 110. However, the wave-wave interaction is less efficient compared to the case in which we kept all modes with shorter wavelengths (up to half the size of the simulation box). The growth rates are: \( \gamma_{020}/\omega_A = 0.422 \), \( \gamma_{200}/\omega_A = 0.416 \), \( \gamma_{121}/\omega_A = 0.427 \), \( \gamma_{211}/\omega_A = 0.418 \), and \( \gamma_{222}/\omega_A = 0.421 \).

The sixth and final test is to simulate the wave-wave interactions of the pump waves (mode 101 and 011) with daughter waves 110, 020, 200, 121, 211 and 222 only. We add two
convective cells (020, 200) to see if convective cells could recover the growth rates observed in Figure 5.1. The time history of these eight modes is displayed in Figure 5.8. This shows that the six daughter waves (mode 110, 020, 200, 121, 211 and 222) are sufficient to reproduce the wave-wave coupling process amongst all modes with shorter wavelengths of up to half of the size of the simulation box are kept. The comparison between growth rates in simulations with and without various CCs are shown in Table 5.2. The growth rates of the daughter waves in simulation without CCs of mode 020 and 200 are 25% - 26% lower than the ones with the three CCs (mode 110, 020 and 200). On the other hand, the growth rates of the daughter waves in simulation without CC of mode 110 are 6% - 7% lower than the ones with the three CCs (mode 110, 020 and 200).
Table 5.2: The growth rates of the dominant daughter waves generated by pump waves 101 and 011, when convective cells of mode 020 and 200 are absent (2nd column), convective cell of mode 110 is absent (3rd column) or all three convective cells are present (right column).

<table>
<thead>
<tr>
<th>Mode</th>
<th>Growth Rate</th>
<th>Mode</th>
<th>Growth Rate</th>
<th>Mode</th>
<th>Growth Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma_{110}/\omega_A$</td>
<td>0.334</td>
<td>$\gamma_{020}/\omega_A$</td>
<td>N/A</td>
<td>$\gamma_{110}, 020, 200$</td>
<td>0.451</td>
</tr>
<tr>
<td>$\gamma_{200}/\omega_A$</td>
<td>N/A</td>
<td>$\gamma_{020}/\omega_A$</td>
<td>0.422</td>
<td>$\gamma_{020}$</td>
<td>0.455</td>
</tr>
<tr>
<td>$\gamma_{121}/\omega_A$</td>
<td>0.337</td>
<td>$\gamma_{121}/\omega_A$</td>
<td>0.427</td>
<td>$\gamma_{200}$</td>
<td>0.442</td>
</tr>
<tr>
<td>$\gamma_{211}/\omega_A$</td>
<td>0.337</td>
<td>$\gamma_{211}/\omega_A$</td>
<td>0.418</td>
<td>$\gamma_{211}$</td>
<td>0.450</td>
</tr>
<tr>
<td>$\gamma_{222}/\omega_A$</td>
<td>0.333</td>
<td>$\gamma_{222}/\omega_A$</td>
<td>0.421</td>
<td>$\gamma_{222}$</td>
<td>0.451</td>
</tr>
</tbody>
</table>

5.2 Growth Rates’ Dependence on Pump Wave Parameters

The growth rates of daughter waves determined in Section 5.1 have fixed parameters for the pump waves ($\delta B_{101}/B_0 = \delta B_{011}/B_0 = 0.012, k_\perp \rho_s = 0.2, k_\parallel \rho_s = 0.002$). However, different initial amplitudes and wavenumbers of the two pump waves affect growth rates of the daughter waves differently. In this section, we determine the six daughter wave growth rates dependencies on the pump wave parameters to provide further insights on the PDI process.

5.2.1 Both pump wave amplitudes

The pump wave is generated by an initial perturbation of the electron current density $\delta j_{\parallel e}$. The amplitude of this pump wave perturbation affects the growth of other daughter waves. A scan of the pump wave amplitudes and the growth rates is conducted and the result is displayed in Figure 5.9. The perpendicular wavenumber $k_\perp \rho_s$ is fixed at 0.2 and parallel wavenumber $k_\parallel \rho_s$ at 0.002. A linear fit is plotted on top of the simulation results, which indicates that the threshold to exponential growth is at $\delta B_\perp/B_0 = 0.004$. The linear fit is consistent with the theory of modulational instability.
Figure 5.9: The growth rates $\gamma/\omega_A$ of daughter waves with respect to the pump wave amplitudes $\delta B/B_0$ of both pump waves.

$\frac{\gamma_{020}}{\omega_A} = 56.63 \left| \frac{\delta B_\perp}{B_0} \right| - 0.23$
5.2.2 Parallel wavenumber $k_{\parallel} \rho_s$

The parallel component of the wavevector (parallel wavenumber) of the pump waves also plays a role in the daughter wave growth rates. A scan of the parallel wavenumbers is displayed in Figure 5.10. Note that the perpendicular wavenumber $k_{\parallel} \rho_s$ is fixed at 0.2 and $\delta B/B_0$ at 0.012 for both pump waves. As shown in the figure, $k_{\parallel} \rho_s \gamma/\omega_A = \gamma \rho_s/v_A$ is independent of $k_{\parallel}/k_\perp$ at low-$k_{\parallel} \rho_s$ and is linearly dependent of $k_{\parallel}/k_\perp$ at higher-$k_{\parallel} \rho_s$. This relationship is the same as the one observed in single pump wave simulations.
5.2.3 Perpendicular wavenumber $k_{\perp} \rho_s$

A scan on the perpendicular wavenumber is also conducted while both pump waves initial amplitude and parallel wavenumber are fixed. The dependence of growth rates on $(k_{\perp} \rho_s)^2$ is displayed in Figure 5.11. The parallel wavenumber $k_{\parallel} \rho_s$ is fixed at 0.002 and $\delta B/B_0$ at 0.012. The growth rates dependency on $(k_{\perp} \rho_s)^2$ can be fitted into a linear function. The fitted function shows that the $\gamma/\omega_A$ could reach a value of 1.05 at $(k_{\perp} \rho_s)^2 = 0.1$.

5.2.4 Pump wave 011 amplitudes

The last parameter scan we performed is $\gamma$ dependence on one of the two pump wave amplitudes. The dependence is displayed in Figure 5.12. Note that the amplitude $\delta B_{101}/B_0$ of
Figure 5.12: The growth rates $\gamma/\omega_A$ of daughter waves with respect to mode 011 amplitude $\delta B_{011}/B_0$.

mode 101 is fixed at 0.012, perpendicular wavenumber $k_\perp \rho_s$ at 0.2 and parallel wavenumber $k_\parallel \rho_s$ at 0.002. The growth rates of all the daughter waves do not overlap, except when $\delta B_{011}/B_0 = \delta B_{101}/B_0$. Intuitively, $\gamma/\omega_A$ of the daughter wave increases as $\delta B/B_0$ of the pump wave increases. This overall trend is indeed observed in the figure. However, when $\delta B_{011} < \delta B_{101}$, $\gamma_{200}$ and $\gamma_{211}$ (generated by mode 011) are higher than $\gamma_{020}$ and $\gamma_{121}$. When $\delta B_{011} > \delta B_{101}$, $\gamma_{200}$ and $\gamma_{211}$ (generated by mode 011) are lower than $\gamma_{020}$ and $\gamma_{121}$. The time history of the case when $\delta B_{011} < \delta B_{101}$, as displayed in Figure 5.13, shows that even though $\delta B_{011}$ was lower initially, it overtakes $\delta B_{101}$ as time progresses. This may explain the non-intuitive simulation results displayed in Figure 5.12. The time history of the case when $\delta B_{011} > \delta B_{101}$ is displayed in Figure 5.14, and shows that even though $\delta B_{011}$ was higher initially, it becomes less than $\delta B_{101}$ as time progresses.
Figure 5.13: The time history of $\delta \phi$ of the pump waves (with initial amplitude $\delta B_{101}/B_0 = 0.012$ and $\delta B_{011}/B_0 = 0.006$) and daughter waves generated with the highest amplitudes.
Figure 5.14: The time history of $\delta \phi$ of the pump waves (with initial amplitude $\delta B_{101}/B_0 = 0.012$ and $\delta B_{011}/B_0 = 0.018$) and daughter waves generated with the highest amplitudes.
In summary, two pump waves provide an extra energy source for the daughter waves. The growth rates of the daughter waves with strongest energy gain are almost doubled compared to the case of a single pump wave. These dominant daughter waves are produced by wave-wave interactions between the two pump waves and energy cascade process from a single pump wave. Adjusting the pump waves parameters can affect the growth rates of the daughter waves. The linear dependence between growth rates and both pump wave amplitudes is consistent with the finding from a single pump wave simulations. When one of the pump wave amplitude is held fixed while the other one varies, the growth rate dependence on the pump wave amplitude is tied to the time variation of pump wave energy. The growth rates have no dependence on the pump wave parallel wavevector in the limit where parallel wavenumber is much lower than the perpendicular wavenumber. Lastly, the growth rates of the daughter waves show linear dependence on \((k_{\perp} \rho_s)^2\) of the two pump waves. These results are consistent with the findings from single pump wave simulations.
Chapter 6

Summary

6.1 Conclusion

The nonlinear interaction of kinetic Alfvén waves (KAW) and the role of convective cells (CC) are studied. A parallelized gyrokinetic particle-in-cell code for studying KAW turbulence is modified to include convective cells. With the implementation of convective cells, the study of KAW and CC interactions is more complete.

The code we employed is a gyrokinetic particle-in-cell (PIC) code to study plasma in a slab. The plasma is consisted of gyrokinetic ions and massless electrons. An initial disturbance inside the plasma with features consistent with a KAW is generated. The pump wave is allowed to interact with other KAWs (including CCs) throughout the simulation. The 2nd order Runge Kutta method was used to obtain the dynamic quantities. The \( \delta f \) method was used to reduce simulation noise. Parallelization of the PIC code were implemented in the code, with the help of OpenMP and MPI, to reduce the total computer run time. Convergence tests were performed on the code to make sure the size of each time step \( \omega_A \Delta t \), the number of grids on each side of the simulation box \( N_g \) and the number of particles per wave-
length $N_p$ are sufficient to correctly simulate the physical phenomena in the plasma. Those convergence tests were conducted in three separately setups: linear single-wave, nonlinear single-wave, and multiple-wave interactions.

The simulation setup allows us to study the time evolution of KAW interactions with one or two pump wave(s) and its(their) interaction(s) with other plasma waves generated in the process. The energy cascade of a single KAW pump wave is simulated and we learned that energy of the pump wave overwhelmingly prefers to cascade into waves of shorter wavelengths in every direction. In particular, the pump wave (mode 101) and three of its daughter waves (mode 020, 121 and 222) are the required minimal waves to model the energy cascade process of a single KAW pump wave, with mode 020 being a convective cell ($k_\parallel = 0$). Among the three daughter waves, the pump wave cannot effectively couples to mode 020 nor mode 121 without the presence of mode 222, and therefore no three-wave interaction occurred. This is true for pump wave with perpendicular wavenumber $k_{\perp} \rho_s$ ranging from 0.1 to 0.4. On the other hand, if CC were absent and only mode 101, 121 and 222 were present, parametric decay instability of the pump wave occurs. However, growth rates of these two daughter waves are only about half of the growth rates from the case when CC were present, except when $k_{\perp} \rho_s = 0.1$. When $k_{\perp} \rho_s = 0.1$, the growth rates were about the same, whether the zonal fields were present or not. This shows that the presence of convective cell helps enhance the rate of energy transfer to daughter waves at higher $k_{\perp} \rho_s$. Zonal field is composed of zonal flow (CC) and zonal vector potential. When the zonal vector potential is missing and only CC is kept, the growth rates of the daughter waves are actually slightly higher than the growth rates from the case when both the CC and zonal vector potential are present. This indicates that the CC by itself is the most important part of zonal field when interacting with KAW. The real frequencies were also compared, and their values are unaffected whether CCs were present or not. We also look into the growth rate dependence on the pump wave parameters. The growth rates have a linear relationship with the pump wave amplitude, and the linear fit shows the pump wave amplitude has to be $|\delta B/B_0| \geq 0.005$ in order to see
exponential growth of the daughter waves. The growth rates are independent of the parallel wavenumber of the pump wave. However, this is true for low-\(k_{\parallel}\rho_s\) values. The growth rates dependence on the perpendicular wavenumber was also studied, and the growth rates are linearly proportional to \((k_{\perp}\rho_s)^2\).

When there are two pump waves, more daughter waves play dominant role in the energy cascade process. There are a total of six dominant daughter waves: mode 110, 020, 202, 121, 211 and 222. These daughter waves were produced by wave-wave interaction from individual pump wave and also the coupling between the two pump waves. Pump wave of mode 101 excites mode 020, 121, and 222, while pump wave of mode 011 excites mode 200, 211 and 222. Pump waves (mode 101 and 011) couple together to generate mode 110. The presence of these six daughter waves are sufficient to reproduce the wave-wave coupling process amongst all modes with shorter wavelengths. Since there are two pump waves, they provide an extra source of energy (compared to single pump wave) for the daughter waves. Therefore, the dominant daughter waves produced by two pump waves have twice the growth rates of the daughter waves produced by a single KAW pump wave. There is no major difference between launching two in-phase pump waves and two \(\pi/2\) out-of-phase pump waves. The growth rates of the daughter waves from these two cases are \(\leq 10\%\), but one interesting difference is that mode 222 is a standing wave in two in-phase pump wave simulation, and yet a traveling wave in two out-of-phase pump wave simulation. Also, the growth rates dependence on the pump wave parameters were studied. The growth rates have a linear relation with identical pump wave amplitudes, and the linear relation shows a threshold to exponential growth at \(|\delta B/B_0| = 0.004\). The growth rates of the daughter waves do not depend on the pump wave parallel wavenumber when \(k_{\parallel}\rho_s\) is much lower than \(k_{\perp}\rho_s\). On the other hand, the growth rates do depend on the pump wave perpendicular wavenumber, with \(\gamma/\omega_A\) increases linearly as \((k_{\perp}\rho_s)^2\) increases. These are the same type of relationships observed from single pump wave simulation results. However, when we study the growth rates dependence on one of the pump wave amplitudes \(\delta B_{011}/B_0\), the relationship is not straightforward. The daughter
waves excited by the pump wave with the lower amplitude have higher growth rate than the
daughter waves excited by the pump wave with higher amplitude. This is not consistent with
our finding that growth rates increases linearly with pump wave amplitude. However, the
higher-amplitude pump is losing energy while the lower-amplitude pump is gaining energy,
and this should be taken into account.

6.2 Future Work

So far we have only been focused on pump wave amplitude $\delta B/B_0$ ranges from 0.006 to
0.02, perpendicular wavenumber $k_\perp \rho_s$ ranges from 0.1 to 0.4 and plasmas with electron
thermal-to-magnetic ratio ($\beta_e$) of 16%. However, it is very common to observe $\beta_e$ of higher
values and different values for $\delta B/B_0$ and $k_\perp \rho_s$. The next task we could do is to study how
the role convective cells play is altered when interacting with KAW of higher $\delta B/B_0$ and
various values of $k_\perp \rho_s$, in higher-$\beta$ plasmas. Theoretically, a high-wavenumber is essential in
generating convective cells. We could try to do a scan on $k_\perp$ to study the energy cascade in
those situations. In the near future, we would study the role convective cells play when the
pump wave has higher wave amplitude. In addition, the simulation results will be compared
to analytic theory on the generation of CC.
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Appendix A

Code Structure

A.1 Initial loading (at t = 0)

1. Load dynamic quantities
   \( X \)  Gyrocenter position of every marker particle (uniform distribution)
   \( v_\parallel \) Parallel velocity of every marker particle (Maxwellian distribution)
   \( v_x, v_y \) Velocity of every marker ion particle along x- and y-axis, respectively
   \( w \) Weight of every marker particle

   Only if using initial perturbation to inject pump wave into plasma:
   \( \delta n_e \) (NZ) Perturbed electron charge density
   \( \delta A_{\parallel}^{NZ} \) (NZ) Perturbed vector potential

A.2 Field calculations (at t > 0)

Within every time step, there is a 2-step Runge Kutta (R-K) loop doing the following field calculations:
1. Particle-to-cell scattering

\[ \delta n_i \quad \text{(NZ+Zonal) Perturbed ion density} \]
\[ \delta u_{\parallel i} \quad \text{(NZ+Zonal) Perturbed parallel ion current density} \]

2. Fast Fourier Transform (FFT) along x-, y-, and z-axis

\[ \delta n_i (x,y,z) \rightarrow \delta n_{i,k} (k_x, k_y, k_z) \]
\[ \delta u_{\parallel i}(x,y,z) \rightarrow \delta u_{\parallel i,k} (k_x, k_y, k_z) \]
\[ \delta n_e (x,y,z) \rightarrow \delta n_{e,k} (k_x, k_y, k_z) \]
\[ \delta A_{\parallel}^{NZ} (x,y,z) \rightarrow \delta A_{\parallel,k}^{NZ} (k_x, k_y, k_z) \]
\[ \delta A_{\parallel} (x,y,z) \rightarrow \delta A_{\parallel,k} (k_x, k_y, k_z) \]

3. Set up mode filter

The filter is a 2\textsuperscript{nd} rank tensor that serves the purpose of eliminating unwanted waves with specific wavevector \( k \).

4. Solve field equations (see equations 2.11', 2.12', 2.17'), with mode filter applied

Define zonal components \( \bar{\delta n}_{e,k}, \bar{\delta n}_{i,k} \) and \( \bar{\delta u}_{\parallel i,k} \) by pulling \( k_S = 0 \) mode from \( \delta n_{e,k}, \delta n_{i,k} \) and \( \delta u_{\parallel i,k} \), respectively

\[ \delta \psi_k (k_x, k_y, k_z) = \delta n_{e,k} (k_x, k_y, k_z) \]
\[ \delta \phi_k (k_x, k_y, k_z) = \frac{1}{k_{\perp}^2} \left( 1 + \frac{k_{\perp}^2}{\tau} \right) \left( \delta n_{i,k} (k_x, k_y, k_z) - \delta n_{e,k} (k_x, k_y, k_z) \right) \]
\[ \delta u_{\parallel e,k} (k_x, k_y, k_z) = \left( \delta u_{\parallel i,k} - \overline{\delta u_{\parallel i,k}} \right) - \frac{2}{\beta_e} k_{\perp}^2 \delta A_{\parallel,k}^{NZ} + \frac{m_i}{m_e} \overline{\delta A_{\parallel,k}} \]
\[ \delta \psi_k^{NZ} (k_x, k_y, k_z) = \delta n_{e,k} - \overline{\delta n_{e,k}} \]
\[ \delta \phi_k^{NZ} (k_x, k_y, k_z) = \frac{1}{k_{\perp}^2} \left( 1 + \frac{k_{\perp}^2}{\tau} \right) \left( \left( \delta n_{i,k} - \overline{\delta n_{i,k}} \right) - \left( \delta n_{e,k} - \overline{\delta n_{e,k}} \right) \right) \]

\( \delta B_{x,k} (k_x, k_y, k_z) \) and \( \delta B_{y,k} (k_x, k_y, k_z) \) only calculated at every 1st step of R-K loop for diagnostic purpose

Note: filter is applied to \( \delta n_{e,k}, \delta A_{\parallel,k}^{NZ} \) and \( \overline{\delta A_{\parallel,k}} \) in this step
5. Inverse FFT along x-, y-, and z-axis
\[ \delta \psi_k (k_x, k_y, k_z) \rightarrow \delta \psi (x, y, z) \]
\[ \delta \phi_k (k_x, k_y, k_z) \rightarrow \delta \phi (x, y, z) \]
\[ \delta u_{||e,k} (k_x, k_y, k_z) \rightarrow \delta u_{||e} (x, y, z) \]
\[ \delta \psi^N_Z (k_x, k_y, k_z) \rightarrow \delta \psi^N_Z (x, y, z) \]
\[ \delta \phi^N_Z (k_x, k_y, k_z) \rightarrow \delta \phi^N_Z (x, y, z) \]
\[ \delta u_{||i,k} (k_x, k_y, k_z) \rightarrow \delta u_{||i} (x, y, z) \]

\( \delta u_{||i} (x, y, z) \) will be used to determine the static term in equation 2.28

6. Finite difference equations (calculate terms from dynamic equations)

From equation 2.18' (electron continuity equation):
\[ (b_0 + \delta b) \cdot \nabla \delta u_{||e} \]
\[ (b_0 \times \nabla \delta \phi) \cdot \nabla n_e \]

From equation 2.19' (Faraday’s law):
\[ b_0 \cdot \nabla (\delta \psi^N_Z - \delta \phi^N_Z) \]

From equation 2.20' (particle position):
\[ b_0 \times \nabla \delta \phi \]

From equations 2.21' (particle parallel velocity) and 2.22' (ion weight):
\[ -b_0 \cdot \nabla \delta \psi \text{ or } \delta E_{||} \]

7. Diagnostics (only calculated at every 1st step of R-K loop)
\[ \delta E_{||} (x, y, z) \rightarrow \delta E_{||,k} (1 : k_{x,max}, 1 : k_{y,max}, k_z) \]

Print k-space field and dynamic quantities, volume average quantities, root-mean-square quantities, and field energy
A.3 Particle push (at $t > 0$)

Within every time step, there is a 2-step R-K loop doing the following particle calculations:

If calculation is within the 1st step of R-K loop, then $\Delta t = \frac{\Delta t}{2}$.
If calculation is within the 2nd step of R-K loop, then $\Delta t = \Delta t \Omega_i$.

1. Grid-to-particle gathering

   $b_0 \cdot \nabla \delta u_{||}$ Gradient of parallel electron current density along background magnetic field line

   $\delta E_{||}$ Perturbed parallel electric field

   $b_0 \times \nabla \delta \phi$ E x B drift velocity

   $\delta B$ Perturbed magnetic field

2. Calculate dynamic quantities

   Solve dynamic equations 2.18' - 2.22' to obtain $\delta A_{||}^{NZ}$, $\delta n_e$, $X$, $w_i$, and $v_{||}$.

   Solve linear term from the dynamic equation 2.28 to obtain $\overline{\delta A_{||}}$

After step 2, the time loop would return back to field calculates (see section A.2) until the last time step is complete.
Appendix B

Collisional Effects on Nonlinear Wave-Particle Trapping in Mirror Instability and Landau Damping

B.1 Abstract

Mirror instability is a low-frequency compressible electromagnetic mode driven by the pressure anisotropy, and this can only occur in the high-beta plasmas. It has been widely studied due to its possible connection to the low-frequency compressible magnetic turbulence in magnetized plasma. A gyrokinetic particle simulation was formulated to study this instability, and it shows that at a relatively weak drive, phase-space trapping of the particles plays a dominant role in the nonlinear saturation [Qu et al., 2008]. Here, we extend this gyrokinetic particle simulation model of the mirror mode to include the Coulomb collisional effects by implementing the Lorentz pitch-angle scattering operator. We study the role of collision in the nonlinear saturation and evolution of mirror instability, and then compare the results
with the corresponding Landau damping cases. We find that collisions can destroy the non-linear oscillations of the electric and magnetic fields and allow the magnetic field to grow in mirror instability and damp in Landau damping. Collisions tend to destroy nonlinear oscillations more effectively in mirror instability.

B.2 Introduction

Mirror instability is often observed in space because it requires the plasma to have a high thermal to magnetic pressure ratio ($\beta$). This means the thermal pressure has to be at least approximately the same as the magnetic pressure. Plasma with a high $\beta$ is easy to find in space. Besides that, space is also a place where we can easily find temperature anisotropy, namely $T_\perp > T_\parallel$. This is also a requirement for mirror instability to occur. Once the magnetic pressure can no longer balance with the bulk plasma pressure, magnetic field will begin to grow in time. The mirror instability was treated as a magnetohydrodynamic instability [Rudakov and Sagdeev, 1961] until Southwood and Kivelson [Southwood and Kivelson, 1993] published their treatment of the instability as a kinetic problem. In their paper, mirror instability can be thought of as a magnetic mirror. Magnetic moment $\mu$ and $E_{total}$ on average are conserved. However, this only applies to the bulk plasma but not the resonant particles. The resonant particles have $v_\parallel \sim \gamma/k_\parallel \ll v_i$. In 2008, Qu et. al [Qu et al., 2008] utilized gyrokinetic theory to simulate the nonlinear mirror instability. Much effort had been put in to study this instability not only for the sole understanding of this phenomenon, but also to determine whether this instability plays a role to the low-frequency compressible magnetic turbulence in the magnetized plasmas. However, all the work above are only limited to a collisionless environment. In our work, we want to study how collisions affect the instability, especially the phase space trapping happened in the collisionless case [Qu et al., 2008]. Then we compare the outcome to a relatively well-known case, namely Landau damping. We see
that collisions can get rid of the nonlinear oscillations appeared in the collisionless simulation and let the mirror instability continue to grow in time and let the Landau damping continue to damp in time. In addition, collisions can diffuse phase space islands more effectively in mirror instability than in Landau damping.

We will first talk about our simulation result of how the electric and magnetic field change in time in both collisionless and collisional simulations. Next, we will observe what collisions can do to the phase space islands. And finally, we will study how numerical entropy changes in time.

**B.3 Time History of Field**

The 2-dimensional gyrokinetic particle-in-cell (PIC) code that simulates the nonlinear evolution of single-mode mirror instability was developed by Qu et. al. [Qu et al., 2008]. In this code they set $\beta = 2$, $A_i = 1$, $k_\perp \rho_i = 0.2$, and $k_\parallel/k_\perp = 0.2$. Ion mass $m_i$, ion charge $e$, and background magnetic field strength $|B_0|$ are all normalized to 1. Positions were normalized by ion’s Larmor radius $\rho_i$, time by inverse of ion gyrofrequency $\Omega_i^{-1}$, velocity by ion thermal velocity perpendicular to the background magnetic field vector $v_{Ti\perp}$, and magnetic moment $\mu$ by ion thermal temperature $T_{i\perp}$. This collisionless simulation shows the perturbed magnetic field component pointing along the ambient field $\delta B_\parallel/|B_0|$ grows linearly at a rate of $\gamma$ in the semi-log scale until it reaches the onset to nonlinear saturation at $t \approx \tau_b$. This is when nonlinear effects become important. The field then undergoes nonlinear oscillations with a frequency approximately equal to the analytical bounce frequency of a particle inside the magnetic well.
Collisions can play an interesting role to the time evolution of mirror instability. We modify Qu’s code [Qu et al., 2008] to include the Coulomb collisions into the simulation. The Lorentz collision operator

\[ C = \nu \frac{1}{2} \frac{\partial}{\partial \xi} (1 - \xi^2) \frac{\partial}{\partial \xi} \]

(B.1)
is the principle behind our operation. We can use the Monte Carlo pitch angle scattering model to find the new pitch angle due to the Coulomb collision in each time step [Lin et al., 1995]

\[ \xi = \xi_0 (1 - \nu \Delta t) + (R - 0.5) \left[ 12 (1 - \xi_0^2) \nu \Delta t \right]^{1/2} \]

(B.2)

Figure B.1 shows the time histories of the magnetic field when collisions of frequency \( \nu_{ji}/\gamma = 0, 0.1, 0.3 \) and 1% are included. As Figure B.1 indicates, collisions only slightly affect the linear growth rate of the instability. Once the instability reaches its onset to nonlinear saturation, the collisions can slightly delay the onset; with a higher collision frequency further delays the onset. After the onset, the collisions can damp out the nonlinear oscillations appear in the collisionless simulation. Besides damping out the nonlinear oscillations, the collisions allow the instability to grow as signs of random noise appear in the nonlinear evolution of magnetic field. Therefore, collisions do not play a significant role until the instability reaches its nonlinear state.

Collisional effects on mirror instability are different compared to the ones on Landau damp-
Figure B.1: Time history of mirror instability. Normalized perturbed particle distribution in phase space at time steps $a$ and $b$ will be studied.
ing. We used the one-dimensional PIC code developed by Lin to simulate the collisionless case. The parameter in this source code is $k_{\parallel} \rho_{i} = 0.4$. Electron mass $m_e$ and the absolute value of electron charge $e$ are normalized to 1. Position is normalized by Debye length $\lambda_D$, time by inverse of electron plasma oscillation frequency $\omega_{pe}^{-1}$, and velocity by electron thermal speed $v_{Te}$. To see what collisions can do to Landau damping, we implemented a subroutine into the PIC code to serve the purpose. We can simulate the nonlinear evolution of the electric field and it is displayed in Figure B.2. In this figure, only the top envelop of the electric field amplitude is plotted. Unlike mirror instability, Coulomb collisions do not have any apparent effects on Landau damping until after the onset to nonlinear saturation. After the onset, collisions clearly smooth out the random noise appeared in collisionless case and decrease the field strength, which are different from the collisional effects on mirror instability. We can see from the time evolution of fields that collisions remove nonlinear oscillations more effectively in mirror instability than in Landau damping.

\section*{B.4 Perturbed Particle Distributions in Phase Space (Mirror Instability)}

Next, we want to study the phase space structures after the onset to nonlinear saturation of mirror instability. We decide to focus on the times where we label $a$ and $b$ in Figure B.1. The perturbed particle distributions with $\mu/T_{i\perp} = 1$ at marker $a$, $\mu/T_{i\perp} = 1$ at $b$, $\mu/T_{i\perp} = 2$ at marker $a$, $\mu/T_{i\perp} = 2$ at $b$ are displayed in Figures B.3 - B.6, respectively. Collisions affect the phase space island more or less the same way regardless of time e.g. markers $a$ and $b$, and magnetic moment: phase-space island diffuses as collision frequency goes up. The effect of collisions look similar between the phase space at markers $a$ and $b$, therefore we focus on phase space at a particular time, e.g. marker $a$ (Figures B.3 and B.5). If we compare the phase space islands between $\mu/T_{i\perp} = 1$ and $\mu/T_{i\perp} = 2$, the phase space island
Figure B.2: Time history of the top envelope to the nonlinear Landau damping. Normalized perturbed particle distribution in phase space at time steps $c$ and $d$ will be studied.
for $\mu/T_{i\perp} = 2$ has a less-apparent island structure. The reason they are different is because phase space particle trapping in mirror instability depends on particle’s velocity component pointing along the background field $v_{\parallel}$ and $\mu$, more specifically the magnetic force can be expressed as

$$-i k_{\parallel} \mu \delta B_{\parallel}. \quad (B.3)$$

Collisions change the particle’s trajectory, e.g. change $v_{\parallel}$, and therefore affect particle trapping in phase space.
Figure B.4: Marker $b$ ($t\Omega_i = 6477$), $\mu/T_{i\perp} = 1$

Figure B.5: Marker $a$ ($t\Omega_i = 5908$), $\mu/T_{i\perp} = 2$
We do the same thing to study the phase space structures of Landau damping as we did to the mirror instability; we obtain the phase space structures at times labeled \( c \) and \( d \) in Figure B.2. Figure B.7 displays the perturbed particle distribution in phase space at marker \( c \). Likewise, Figure B.8 for marker \( d \). The collisionless case in these figures show phase space vortices/islands formed at \( v \approx v_\phi \) in phase space, which are consistent with Manfredi’s simulation [Manfredi, 1997]. The band structures appear in phase space represent the ballistic modes in Landau damping. The shape of the phase space island is long and narrow when electric field reaches its minimum. The phase space island is round in an oval-like geometry when field reaches its maximum. Once we move into the collisional cases, the band structures are destroyed and the severity depends on the collisional frequency: frequent
collisions enhance the destruction of these band structures while the phase space island remains. The island does show sign of diffusion as collision frequency increases, which also happens in mirror instability. However, unlike the mirror instability, particle trapping only depends on velocity component pointing along the electric field, $v_\parallel$, because the force exerted by the electric field onto the particles only pushes the particle along the field direction:

$$-ie k_\parallel \phi.$$  \hfill (B.4)
B.6 Entropy

We can also observe how collisions affect the entropy as time progresses. Basically, entropy is the marker particles’ average perturbed weight in numerical simulations. Calculating the entropy at each time step allows us to see the time evolution of entropy. There are some subtle differences between the time evolution of entropy for the mirror mode instability and the one for Landau damping.

Let us begin the entropy of mirror instability. Since the time evolution of entropy

\[ \sum_{i=1}^{N} \delta f_i^2 / N \]  

(B.5)
is strongly affected by the perturbed magnetic field, we divide the entropy by $|\delta B_\parallel / B_0|^2$ to minimize the influence due to the magnetic field, as shown in Figure B.9. Initially, this normalized entropy of the mirror instability has a value of almost 200. It then decreases and reaches its absolute minimum before the instability reaches its onset to nonlinear saturation. At that point on, the entropy fluctuates more or less about a constant value throughout the rest of the simulations; with fluctuations representing the random noise. This is true for the collisional simulation, except the case with collision frequency equivalent to 1% of the bounce frequency. For this case, the entropy increases linearly after the absolute minimum, as particle noise appears along the rest of the simulation.

The entropy of Landau damping was treated in a different manner. Since the time evolution of electric field doesn’t seem to have any apparent influence on entropy itself, we plot out the entropy in time without dividing it by electric field nor getting rid of any fast oscillations.
There is no entropy initially, as shown in Figure B.10. It then increases as time progresses, until the electric field reaches its onset to nonlinear saturation: the entropy reaches its local minimum, which sets it apart from mirror instability. From that point on, collisions change the time evolution of entropy, but more or less fluctuate between 0.07 and 0.09. Unlike the mirror instability, the entropy with collision frequency of 1% of bounce frequency has the lowest entropy throughout most of the simulation. That means frequent collisions can lower the entropy in Landau damping while increase the entropy in mirror instability.

**B.7 Conclusion**

In conclusion, collisions affect mirror instability and Landau damping in different ways. Collisions destroy nonlinear oscillations and allow instability continue to grow/damp for both mirror instability and Landau damping. When we study the perturbed particle distributions
in phase space, we see that phase-space trapping depends on \( \mu \) in mirror instability but not so in Landau damping. This can be explained by the equations of motion for both mirror instability and Landau damping. Last but not least, collisions destroy phase-space island more effectively in mirror instability than in Landau damping, destroying the nonlinear oscillations in the fields’ time histories. The next step we will take is to simulate multiple mirror modes. In particular, we will study their interactions and how they affect the nonlinear evolution to mirror instability.