Two Models of Coagulation With Instantaneous Gelation

by

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Abstract

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Two models of coagulation are presented: one, a system of coupled partial differential equations and the other microscopic Brownian particles. Both models feature a parameter that represents the tendency of two particles to coagulate at sufficiently close distances. Both models have a phase transition, viewed as the mass clumping together as a gel. Previous work has shown the models are connected, and in the same sense the phase transitions are too.
To my parents.
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Chapter 1

Introduction

1.1 Deterministic Smoluchowski Coagulation Equations

We consider a basic deterministic model of clusters with discrete integer masses which coagulate when sufficiently close. The rate at which a cluster of size $n$ and one of size $m$ collide is given by a coagulation kernel $K(n,m)$. Smoluchowski (1916) [Smo16] introduced a model to describe the clumping together of particles of radius $r_1$ and $r_2$ experiencing a Brownian motion in three dimensions at diffusion rate proportional to $\frac{1}{r_1}$ and $\frac{1}{r_2}$ respectively. He surmised that if the clusters were of uniform density, for a particle of mass $m$, $r = Cm^{1/3}$, two particles would meet at a rate given by the product of their radii and relative speed: $(r_1 + r_2)(\frac{1}{r_1} + \frac{1}{r_2})$. Therefore the rate of coagulation would be given by the following kernel:

$$K(m,n) = C(m^{1/3} + n^{1/3})(m^{-1/3} + n^{-1/3}).$$

Different kernels are introduced to represent varying models of motion and coalescence, and have been used to study the coagulation of polymerization in chemistry, the coagulation of aerosols, and coalescence in population genetics. For a given coagulation kernel $K$, we will define the homogeneous Smoluchowski coagulation equation by a differential equation

$$\frac{\partial}{\partial t} f(n,t) = \frac{1}{2} \sum_{k=0}^{n} K(k,n-k)f(k,t)f(n-k,t) - f(n,t) \sum_{k=0}^{\infty} K(n,k)f(k,t)$$

(1.1)

Here $f(n,t)$ means the proportion of clusters at time $t$ that are of size $n$. The interpretation of the equation is intuitive: the rate of change in the number of clusters of size $n$ is given by the addition of all collisions of clusters sized $m,k$, where $m + k = n$, minus the formation of clusters where one of size $n$ collides with any other cluster. We might define the mass of this function at time $t$ by

$$m_1(t) = \sum_{k=0}^{\infty} kf(k,t)$$
Theorem 1. (Ball and Costa 1990 [BC90]) For every coagulation kernel $K(n,m)$ satisfying $K(n,m) \leq C(n+m)$ for some constant $C$ not dependent on $n,m$, there exists a mass-conserving solution $f : \mathbb{N} \times [0, \infty) \to \mathbb{R}$ to 1.1. By mass-conserving, we mean that for all $t > 0$

$$m_1(t) = m_1(0) = \sum_{k=1}^{\infty} k f(k,0).$$

It is not hard to show, as we shall later, that for any solution to 1.1, in general it is true that for all $t > 0$,

$$m_1(t) \leq m_1(0).$$

It is possible for the mass function $m_1$ to drop, representing a phase change: the formation of a gel (as opposed to the sol), where the mass is viewed formally as lying in cluster of size infinity. Under slightly stricter conditions on the kernel, we also have a unique mass-conserving solution:

Theorem 2. [BC90] Let $0 \leq \alpha \leq 1/2$ and $C$ be some constant. If $K(n,m) \leq C(nm)^\alpha$, then the solution to 1.1 is unique.

That mass-conservation is not a general feature of all solutions of 1.1 is the subject of our investigation of increasingly complex models of Smoluchowski coagulation. Mathematically, we define a gelling kernel $K(n,m)$ as one where there exists a $T_{gel} < \infty$ for which

$$T_{gel} = \inf \{ t > 0 \mid m_1(t) < m_1(0) \}.$$  \hfill (1.2)

This terminology comes from the scientific modeling literature, which examined it in phase transitions. Formal proof of finite gelation time for the kernel $\alpha(m,n) = mn$ was given in [JBM62]. Rigorous proofs of certain classes of gelling kernels were given in [al03]. One interesting feature of the Smoluchowski coagulation model is that we can obtain the distribution of certain statistics for a number of stochastic processes that involve the formation of clusters, which are surveyed in [Ald99].

Example 1. [Ald99] Consider nodes on the integer line where the edge between $i$ and $i+1$ appear at a random time $T_i$, with $P(T_i > t) = \frac{2^t}{t+2}$, and assume the $T_i$ are independent. Then the density of mass $n$ clusters of edges at time $t$ is given by the solution to the Smoluchowski equation for $K(m,n) = 1$.

Below we see a representation of the number line as it progresses under an instance of this process.

Example 2. [Ald99] Place nodes at positions 0, 1, 2, ..., with edges between them. From this base, begin a Galton-Watson branching process with offspring distribution of a poisson random variable, $\lambda = 1$. 
Figure 1.1: From [Ald99], an example of the discrete coalescent with constant coagulation kernel.

If we view the edges of the completed branching tree as having appeared at a random time, say $T_i$ with a mean 1 exponential distribution, then the density of size $n$ clusters at time $t$, $f(n, t)$ is given by the unique solution to the Smoluchowski equation with coagulation kernel $K(m, n) = m + n$, with $f(n, 0) = 1$ for all $n$.

**Example 3.** Silk and White [SW78] consider the coagulation of clumps of matter (varying continuously, rather than discretely) in early galaxy formation in the expanding universe. They conclude with theoretical and empirical arguments that the number density $f(n, t)$ satisfies a continuous version of the Smoluchowski coagulation ODE with $K(kx, ky) = \frac{1}{3}K(x, y)$ with $\lambda = 4/3$.

### 1.2 Coupled System of Partial Differential Equations

For a more complex model, we might consider formulations of 1.1 but with some spatial inhomogeneity for cluster density. By this we mean that the clusters move like as colloidal particles suspended in a solution, so that their motion is given by Brownian motion. This diffusion may depend on their mass, designated by a function $d : \mathbb{N} \to \mathbb{R}$ for the rate of diffusion. Let $\alpha(m, n)$ be the coagulation kernel (denoted by $K$ earlier). Given the initial conditions $f_n(x, 0) \in C_1(\mathbb{R}^d)$, a system of functions $f_n : \mathbb{R}^d \times [0, \infty) \to \mathbb{R}$ solve Smoluchowski PDE with the kernel $\alpha$ and diffusion function $d(n)$ if they satisfy

\[
\frac{\partial}{\partial t} f_n(x, t) = d(n) \Delta f_n(x, t) + Q_n(f)(x, t) \tag{1.3}
\]
Figure 1.2: From [Ald99], an example of the discrete coalescent with additive coagulation kernel.

where \( Q_n(f)(x,t) = Q_n^+(f)(x,t) - Q_n^-(f)(x,t) \),

\[
Q_n^+(f)(x,t) = \frac{1}{2} \sum_{m=1}^{n-1} \alpha(m, n-m) f_m(x,t) f_{n-m}(x,t)
\]

and

\[
Q_n^-(f)(x,t) = f_n(x,t) \sum_{m=1}^{\infty} \alpha(m, n) f_m(x,t)
\]

Clearly, the first term in the right hand side of 1.3 represents the diffusion of the Brownian particles in the mixture (or whatever the ambient space represents). The operators \( Q_n^+ \) and \( Q_n^- \) represent, respectively, the formation of size \( n \) clusters from smaller particles and the loss of such particles as they form bigger ones. We define a weak solution to this coupled system of PDEs to satisfy

\[
f_n(x,t) = S_t^d(n) f_n(x,0) + \int_0^t S_{t-s}^d(n) Q_n(f)(x,s)
\]
where $S_t^{d(n)}$ is the semigroup operator associated with $u_t = d(n)\Delta u$ and $Q_n(f) \in L^1(\mathbb{R}^d \times [0,T])$ for every $T$. As before, we can define the mass at time $t$ by

$$M_1(t) = \sum_{m=1}^{\infty} m \int_{\mathbb{R}^d} f_m(x,t) dx$$

If for every $t$, $M_1(t) = M_1(0)$, the solution to 1.3 is called mass conserving. For any weak solution to 1.3, we have that $M_1(t) \leq M_1(0)$. To see this, consider the partial sums of the non-linear operators $Q_n$; it suffices to show that the weighted sum is non-positive:

$$P_k(t) = \sum_{m=1}^{k} mQ_m(x,t) \leq 0$$

To see this, note that by grouping the like terms from summing $nQ_n$, we get

$$P_k(x,t) = \sum_{m,n} ((m+n)1(m+n \leq k) - m1(m \leq k) - n1(n \leq k))f_m(x,t)f_n(x,t) \leq 0.$$ 

A rigorous proof of the existence of a solution in a finite domain $\Omega \in \mathbb{R}^d$ was given in [LM02] assuming that $\lim_{m \to \infty} \frac{\alpha(m,n)}{m} \to 0$. Further work by Wrzosek [Wrz04] showed that there exist unique solutions under mild conditions on the initial data and a subadditive kernel: if for some $K$,

$$\alpha(m,n) \leq K \min\{m+n, mn\},$$

a solution to 1.3 exists. The article by Wrzosek [Wrz04] also expanded the notion of a solution to all of $\mathbb{R}^d$, the setting in which we will work. He also showed in [Wrz02] that in the finite domain setting, under the fairly restrictive assumption that $\alpha(m,n) \leq K$, the solution is mass conserving. More recent work [HR06] has focused on precise estimates for the $p$-moments of solutions:

$$M_p(t) = \int_{\mathbb{R}^d} \sum_{k=1}^{\infty} k^pf_k(x,t) dx$$

finding that under mild restrictions on the function $d(n)$ and initial data, unique solution to 1.3 exist. If in addition the kernel is sub-additive, mass is conserved. Under a slightly more restrictive condition on the kernels, where

$$\alpha(m,n) \leq K(m^\gamma + n^\gamma)$$

for some $0 \leq \gamma < 1$, [HR06][Breden2016] find the $k$-moments lie in a certain $L^p$ space.
1.3 Microscopic Models of Coagulating Brownian Particles

The models above are sometimes [Fel10] referred to as mesoscopic models of coagulating models of coagulation: the equations are based on the interactions of clusters modeled by their local density at $x$, given by $f_n(x,t)$. A much more difficult problem is analyzing the behavior of a system of discrete colliding Brownian particles. By this we mean that given a collection of $N$ particles initially placed in a bounded set $\Gamma \in \mathbb{R}^d$, we would like to characterize the system’s behavior under the rule that they stick together if they approach each other at some "microscopic" distance – call this $\epsilon$. The exact mathematical nature of this process of collision will be explained in detail, but the first thing to consider is: for a given Brownian particle, what relationship between $N$ and $\epsilon$ will produce a positive, bounded number of collisions in one unit of time?

To derive this scaling, we must look at the asymptotics of the Wiener sausage. Given a Brownian path $\omega(t)$ and a radius $\delta$, this is defined as the image set

$$W_\delta(t) = \bigcup_{0 \leq s \leq t} B_\delta(\omega(s))$$

where $B_\delta(y)$ is defined to be the ball in $\mathbb{R}^d$ centered at $y$ with radius $\delta$.

Figure 1.3: Long, thin Wiener sausage in three dimensions. Reproduced from Wikipedia.

If $N$ Brownian particles begin their paths in $\Gamma \in \mathbb{R}^d$, it stands to reason that they will experience in one unit of time roughly

$$N \frac{\mathbb{E}[W_\epsilon(1)]}{\text{Vol}(\Gamma)}$$
collisions as defined above. To define the relationship between \( N \) and \( \epsilon \) we need an asymptotic formula for \( \mathbb{E} [W_\epsilon(1)] \) as \( \epsilon \to 0 \). From Brownian scaling we actually know that

\[
\mathbb{E} [W_\epsilon(1)] = \epsilon^d \mathbb{E} \left[ W_1(1/\epsilon^2) \right]
\]

It therefore suffices to analyze the long term asymptotics of the Wiener sausage of radius 1:

**Theorem 3.** (Donsker and Varadhan 1975) [DV75] For the dimension \( d \geq 3 \), as \( t \to \infty \),

\[
\mathbb{E} [W_1(t)] \sim C(d) t
\]

where \( C(d) \) is a constant proportional to the (Newtonian) capacity of the unit ball.

Putting this together, for our model we would like to scale the arms of our Brownian particles, \( \epsilon \), so that

\[
Z = N\epsilon^{d-2}
\]

where \( Z \) is a constant of proportionality that can be thought of as the mean-free time. A magnificent property these models might have is that they might give us, under the right initial conditions, the Smoluchowski coupled PDE system as a scaling limit. More formally, if the measure \( dx \) gives the position of particles \( x_i, i = 1, \cdots, N \), we can define the scaled empirical measure

\[
g_n^\epsilon(t, dx) = \epsilon^{d-2} \sum_{i=1}^N \delta_{x_i}(dx_i) 1(m_i(t) = n).
\]

A goal of the literature is to show that under some formalizations of this microscopic model, these measures \( g_n^\epsilon \) converge weakly to a coupled system of measures that solve the Smoluchowski PDE in the weak sense of 1.3. That is, there exists a system \( \{f_n\} \) solving 1.3 such that for each smooth test function \( J(x) \in \mathcal{C}_c^\infty(\mathbb{R}^d) \) and \( t > 0 \),

\[
\int_{\mathbb{R}^d} J(x) g_n^\epsilon(t, dx) \to \int_{\mathbb{R}^d} J(x) f_n(x, t) dx.
\]

Lang and Nguyen [Lan80] were the first to carry out such a project. In their formulation, particles were indistinguishable and moved about with a constant diffusion rate \( D \). They use the scaled density of particles measure \( \rho^\epsilon(t, dx) = \epsilon^{d-2} \sum_i \delta_{x_i}(dx) \) and derive the following PDE as a limit

\[
\frac{\partial}{\partial t} \rho(x, t) = D \Delta \rho(x, t) - \beta \rho(x, t)^2
\]

where \( \beta = ZC(d) \), where \( Z \) and \( C(d) \) are defined by 1.3 and 1.7, and \( \rho \) is the weak limit of \( \rho_\epsilon \) as \( \epsilon \to 0 \). To contrast their formal analysis of this Brownian particle system with later works, we will describe it in more detail here. They begin by defining the sets, \( j \in \mathbb{N} \),

\[
D_j = \{(x_1, \cdots, x_j) \mid |x_i - x_k| > \epsilon; i \neq k, i, k \leq j\}
\]

\[
B_j^\epsilon = \mathbb{R}^{3j} \setminus D_j^\epsilon.
\]
In words, $D^\epsilon_j$ represents $j$-tuples of particle positions where all pairs $i$ and $k$ are more than $\epsilon$ distance away from each other, while every $j$-tuple in $B^\epsilon_j$ includes a pair of particles within $\epsilon$ of each other. In order to derive 1.9, they needed a formula for the joint $k$-marginals at time $t > 0$, which they did by generating a series expansion for the BBGKY hierarchy: writing the $k$-marginals in terms of the $k + 1$-marginals. They were also able to show that in a model where particles have multiplicity, i.e. mass, the empirical measures satisfied the Smoluchowski partial differential equations with $\beta(m, n)$ a constant. Norris [Nor04] analyzes a model where Brownian particles have radii modeled as if their (continuous valued) mass is uniformly distributed, i.e. $m = r^{1/d}$. Therefore two particles $X_i$ and $X_j$ with masses $m_i, m_j$ collide when the distance satisfies

$$|X_i(t) - X_j(t)| < \epsilon(m_i^{1/d} + m_j^{1/d}).$$

By comparing this process to one where the particles do not disappear after collision (creating "ghost particles"), they are able to show that the empirical measures

$$\mu^N_t = \epsilon^{d-2} \sum_i \delta_{(x_i, m_i)}$$

converge weakly to a measure $\mu_t$ that satisfies the following evolution equation:

$$\frac{\partial}{\partial t} \mu_t = \frac{1}{2} d(m_i) \Delta \mu_t + K(\mu_t)$$

where $f : \mathbb{N} \rightarrow \mathbb{R}$ is a function on mass and $K$ is the following operator on measures:

$$\langle f, K(\mu_t) \rangle = \frac{1}{2} \int_{(0, \infty)^2} \{ f(y + y') - f(y) - f(y') \} \alpha(y, y') \mu(dy) \mu(dy').$$

To complete the proof, one must give a concentration bound on the number of particles in the ghost particle process described above, along with a careful analysis of the collision history using a set of trees. We shall not use such a model here.

### 1.4 The formulation used in this thesis

We are now ready to present the precise formulation of this problem to be used in this work. It is based on the formulation in Hammond and Rezakhanlou [HR07], with a technical modification that makes the analysis of large particle coagulation easier but does not change anything qualitatively. For each $N$ we will create a measure $\mathbb{P}_N$ that represents the trajectory of these particles.
CHAPTER 1. INTRODUCTION

Notation for particles and initial density.

We first specify For each \( t \geq 0 \) we denote the set of \( \mathcal{N} \) particles \((x_i(t), m_i(t)) \in \mathbb{R}^d \times [1, \mathcal{N}]\). A trajectory for particles is given by a function

\[
(x_i, m_i) : [0, \infty) \rightarrow (\mathbb{R}^d \times \mathbb{N}) \cup \{c\},
\]

where \( c \) is a cemetery state, i.e. the designation for a particle whose mass has been placed in another after coagulation. Given a set of initial density functions \( \{h_n\} \), each of which designate the initial spatial distribution of size \( n \) clusters. For consistency in terminology, we’ll say that

\[
Z = \sum_n n \int_{\mathbb{R}^d} h_n(x) dx,
\]

where \( Z \) represents the mean-free time discussed above. Of course, for our purposes we assume that the support of \( \{h_n\} \) is uniformly contained in some compact set in \( \mathbb{R}^d \).

Markov generator for particle dynamics

We can then define the trajectory of particles via a smooth functionals on particle configurations \( q \):

\[
F(q) : (\mathbb{R}^d \times \mathbb{N}) \cup \{c\}^N \rightarrow \mathbb{R}.
\]

Let us denote the surviving particles (those not in the cemetery state) by \( I(q) \). We would like the Markov generator to capture the three critical features of this model: the free motion of \( n \)-clusters diffusing at rate \( d(n) \) as independent Brownian trajectories until they collide, as well as the part that formalizes the precise notion of collision in this model. Thus we define the action of the generator on functionals \( F \) with these two components:

\[
\mathbb{L}F(q) = \mathbb{A}_f F(q) + \mathbb{A}_c F(q) \tag{1.10}
\]

The free motion operator \( \mathbb{A}_f F(q) \) is given by

\[
\mathbb{A}_f F(q) = \sum_{(x_i, m_i) \in I(q)} d(m_i) \Delta_{x_i} F(q)
\]

The collision term is substantially more complicated. First let us denote by \( V(x) \) a standard bump function with compact support, \( \text{supp}(V) \subset B_1(0) \) and \( \int_{\mathbb{R}^d} V(x) dx = 1 \). Define further the scaled functions

\[
V_{n,m}^\epsilon(x) = (\epsilon r(n) + r(m))^{-2} V \left( \frac{x}{\epsilon r(n) + r(m)} \right) \tag{1.11}
\]

By the definition of \( V \), the support of \( V_{n,m}^\epsilon \) is contained in \( B_{\epsilon(r(n)+r(m))}(0) \). What we are trying to capture here is a function that will be positive for the pairwise distance vector of
two particles $x_i - x_j$ if and only if it is within the collision radius. The collision operator $A_c$ is then:

$$A_c F(q) = \frac{1}{2} \sum_{i \neq j} \alpha(m_i, m_j) V_{n,m}^\epsilon(x_i - x_j) \left[ \frac{m_i}{m_i + m_j} F(q^{ij}) + \frac{m_j}{m_i + m_j} F(q^{ji}) - F(q) \right]$$  \hspace{1cm} (1.12)

where

$$q^{ij}(k) = \begin{cases} 
q(k) & k \neq i, j \\
(x_i, m_i + m_j) & k = i \\
c & k = j.
\end{cases}$$  \hspace{1cm} (1.13)

Let’s unpack this. The transformation $q^{ij}$ represents a new particle configuration where $(x_j, m_j)$ is sent to the cemetery state and sends its mass to $(x_i, m_i)$. The transformation $q^{ji}$ is the same but the particles are reversed. The function $\alpha(m, n)$ represents a microscopic collision rate kernel, though it will not play as much of a role in our analysis as it does in the mesoscopic model. For each pair $i, j$ within an appropriate collision radius $- \epsilon(r(m_j) + r(m_i))$ – the bracketed term represents the process of coagulation where one of the two particles is sent to the cemetery state, and its mass sent to the. The probability of $(x_i, m_i)$ or $(x_j, m_j)$ being sent to a cemetery state is proportional to their relative weights. What, then, explains the rate of this replacement, given by the constant factor in $V_{n,m}^\epsilon$? To understand this, we turn to a well known result from the theory of Brownian paths

**Theorem 4.** ([PM10]) Given a standard Brownian motion $B(t)$, with $B(0) = r$, let $\tau = \inf \{ t > 0 \mid B(t) \geq 2r \}$. Then $\mathbb{E}[\tau] = r^{-2}$

So the amount of time these two particles spend within the collision radius, where the distance between their centers is viewed a Brownian motions in $\mathbb{R}^d$, is proportional to the inverse of the square of the collision radius. This explains the scaling factor given by $V_{n,m}^\epsilon$.

**Mass-conserving solutions, collision radius, and correspondence of microscopic models to mesoscopic ones**

In their work, [HR07] sought to find a correspondence between microscopic models of discrete Brownian particles and the mesoscopic Smoluchowski PDEs discussed above. It turns out that the principal parameter by which one can model formation of fractal patterns of coagulation, such as those generated by diffusion-limited aggregation is to modify the relationship between the mass of a particle and the effective collision radius.
CHAPTER 1. INTRODUCTION

Recall that in the Norris paper [Nor99], the three-dimensional model assumed that for all particles the collision radius was related to the cube root of the radius, since the particles were modeled as uniform. As the above fractal picture shows, particles under different models may form in a more diffuse manner because new particles might stick to some outer edge rather than travel to the interior. We can therefore imagine a family of models

$$r(m) = m^\chi$$

where $$\chi = 1/d$$ corresponds to the uniform sphere model. Given a particular $$\chi$$, is there a scaling limit to a Smoluchowski PDE? If so, what is the relationship between and the coagulation kernel $$K(m,n)$$? These are the questions undertaken in [HR07][Ham12].

**Assumptions sufficient for scaling limit when $$\chi < 1/(d-2)$$:**

- The initial particle distributions $$h_n$$ are compactly supported and their densities bounded.
- The function $$d(n)$$ is non-increasing and $$\sup_n n^{-1}d(n)^{-d/2} < \infty$$.
- The microscopic coagulation kernel has a uniform bound $$\alpha(n,m) < C < \infty$$.

**Theorem 5.** [HR07] Given the assumptions above, $$r(m) = m^\chi$$ for $$\chi < 1/(d-2)$$, $$d \geq 3$$, if $$\{g_n\}$$ are a sequence of measures defined by

$$g_n^\epsilon(x,t)dx = \epsilon^{d-2} \sum_i \delta(x_{i}, m_{i})$$

and $$\{f_n\}$$ satisfy the Smoluchowski PDE 1.3 with kernel $$\beta(n,m)$$, then

$$\limsup_{n \to \infty} \mathbb{E}_n\left| \int_{[0,T]} \int_{\mathbb{R}^d} J(x,t) (g_n^\epsilon(x,t) - f_n(x,t)) dx \right| = 0. \quad (1.14)$$

In particular, the $$f_n$$ are a mass conserving solution with kernel $$\beta(m,n)$$, given by

$$\beta(n,m) = \alpha(n,m) \int_{\mathbb{R}^d} V_{n,m} (1 - u_{n,m}) (x) dx. \quad (1.15)$$

If the solutions are mass-conserving in the scaling limit, we must be able to formulate and prove some precise notion for this in the microscopic setting. By capacity theory [LL01], since the support of $$V_{m,n}$$ is contained in $$B(0, r(m) + r(n))$$, we know that

$$\beta(m,n) \leq C(d) \alpha(m,n) (r(m) + r(n))^{d-2}$$

where $$C(d)$$ is the volume of the unit ball in $$d$$ dimensions. This already suggests that if $$\chi > 1/(d-2)$$, we may end up in a regime corresponding to instantaneous gelation, since such behavior occurs for the homogenous Smoluchowski ODE. We will also prove it occurs...
for the PDE in Chapter 3.

Because the Markov generator 1.10 gives us a strong Feller process, we may use Dynkin’s formula [Gal16] to note that for any functional $F(q)$,

$$M_T = F(q(t)) - F(q(0)) - \int_0^t \mathbb{L}F(q)$$  

(1.16)

is a martingale.

Using Dynkin’s formula, we are able to derive the following theorem.

**Theorem 6.** [Rez06] Let assumptions be as above, and $\chi < 1/(d - 2)$. Fix a smooth $J(x)$, and define the coagulation functionals $T_{n,t}$ by

$$T_{n,t}(J) = 1/2 \int_0^t \sum_{i \neq j} \alpha(m_i, m_j)V_{m_i,m_j}^\epsilon(x_i - x_j)J(x_i)1(m_i + m_j = n)ds$$  

(1.17)

In other words, the number of coagulations forming $n$-sized clusters, weighted by $J(x)$. Then for each $t > 0$:

$$\lim_{l \to \infty} \sup_N \mathbb{E} \left[ \sum_{n=l}^t T_{n,t} \right] = 0$$  

(1.18)

**Proof.** We begin by setting $H = \Delta^{-1}(-J)$ (which certainly exists since $J \in C_c^\infty$), that is,

$$H(x) = c(d) \int_{\mathbb{R}^d} \frac{J(y)}{(x - y)^{d-2}} dy$$

Choose an integer $k$ so big that

- for $|x| \geq k$, $H(x)|x|^{d-2} \geq 1/2c(d)\eta$, with $\eta = \int J$. Such a $k$ exists because for $|x|$ large enough, $|x - y| \leq |x|$.

- For $x \in suppV_{m,n}^\epsilon$, $|x| \leq k\epsilon(\epsilon(m) + \epsilon(n))$.

Define the correlation functional $X_N(t)$, which is simply a measure of how close particles are to each other, weighted by their masses:

$$X_N(t) = \frac{1}{N^2} \sum_{i \neq j} H\left(\frac{x_i - x_j}{\epsilon}\right) m_i m_j$$

By definition, we know that $H \geq 0$ and $-\Delta H = J$, so by eqn: dynkin,

$$\mathbb{E}[X_N(q(0))] \geq \mathbb{E}[X_N(q(0))] - \mathbb{E}[X_N(q(t))] = \mathbb{E}\left[-\int_0^t \mathbb{A}_e X_N(q(s))ds \right. \left. - \int_0^t \mathbb{A}_f X_N(q(s))ds\right]$$
Since we have assumed the initial particle distribution is confined to a compact set, we clearly have
\[ \mathbb{E}[X_N(q(0))] < \infty. \]
Applying the infinitesimal operator to the functional, we have:
\[ A_f X_N(q) = \frac{1}{N^2} \sum_{i \neq j} \Delta H \left( \frac{x_i - x_j}{\epsilon} \right) m_i m_j (d(m_i) + d(m_j)) \]
and
\[ A_c X_N(q) = -\frac{1}{N^2} \sum_{i \neq j} \alpha(m_i, m_j) V_{m_i, m_j}^\epsilon H \left( \frac{x_i - x_j}{\epsilon} \right) m_i m_j. \]
The first is clear from the definition; the latter comes from the fact that \( X_N \) is itself a double sum over \( \{i, j\} \), so the double sum inside \( A_c \) cancels out unless the particle pair coincides with \( i \) and \( j \). Since \( -A_f X_N(q) \geq 0 \) because \( J \geq 0 \), we will show 1.18 by analyzing the quantity \( H \left( \frac{x_i - x_j}{\epsilon} \right) \) when \( \frac{x_i - x_j}{\epsilon} \) is less or more than \( k(r(m_i) + r(m_j)) \). In the former case,
\[ H \left( \frac{x_i - x_j}{\epsilon} \right) \leq C \epsilon^{2-d} \]
while for the latter,
\[ H \left( \frac{x_i - x_j}{\epsilon} \right) \leq C (r(m_i) + r(m_j)). \]
This shows that
\[ C \geq \frac{1}{N} \int_0^t \sum_{i \neq j} \alpha(m_i, m_j) V_{m_i, m_j}^\epsilon H \left( \frac{x_i - x_j}{\epsilon} \right) m_i m_j ds \]
\[ \geq \frac{1}{N} \int_0^t \sum_{i \neq j} \alpha(m_i, m_j) V_{m_i, m_j}^\epsilon (r(m_i) + r(m_j))^{2-d} m_i m_j ds \]
Since \( \chi < 1/(d-2) \), we are done.

**Killed Brownian motion and the Feynman-Kac equation**

Given a *killing rate* \( V(x) \in C_\gamma(\mathbb{R}^d) \) and an exponential random variable \( E \) with mean one, and a Brownian path \( B(t) \in \mathbb{R}^d \), define the interaction time by the following Wiener functional
\[ I(t) = \int_0^t V(B(t)) dt \]
and the *killing time*
\[ K_x = \inf \{ t > 0 \mid I(t) > E \}. \]
We define killed Brownian motion with killing rate $V$ to be the process:

$$X(s) = \begin{cases} B(s) & t < K_x \\ c & t \geq K_x \end{cases}$$

where $c$ is the cemetery state as defined before. The relevance for us is that this is an excellent model for the trajectory of a tagged particle that is colliding with another; if $(x_i, m_i)$ is within the collision radius of another, the difference $x_i - x_j$ of the particle centers may be understood as killed Brownian motion with killing rate $V_{m_i,m_j}^\epsilon$. We now turn to a relevant partial differential equation:

$$\partial_t u = -\Delta u - W u, u(x, 0) = f(x) \quad (1.19)$$

**Theorem 7.** (Feynman-Kac representation [Gal16]) Suppose $u$ is a solution to 1.19, where $W$ is a continuous non-negative function. Then given an initial datum $u_0 \in C_0(\mathbb{R}^d)$,

$$u(x, t) = \mathbb{E} \left[ \exp \left( -\int_0^t W(x + B(s)) ds \right) \right].$$

Moreover, for each $t$, $u(x, t) \in C^2(\mathbb{R}^d)$ and $t \to u_t(x) = u(x, t)$ is differentiable in $t$ as a function in $C^2$.

To see the relevance of this to our killed Brownian motion process, note that if we define $v(x)$ by

$$v(x) = \lim_{t \to \infty} u(x, t) \quad (1.20)$$

then $u_W = 1 - v$ solves

$$-\Delta u_W(x) = W(x)(1 - u_W(x)) \quad (1.21)$$

For killed motion with rate $W$, this is equal to the probability of finite killing time:

$$\mathbb{P}(K_x < \infty) = u_W(x).$$

To show that the limit $v$ exists, consider that for $t > s,$

$$|v(x, s) - v(x, t)| \leq (t - s) ||W||_\infty \sup_{r \in [s, t]} \mathbb{P}(x + B(r) \in supp(W))$$

As $t \to \infty$, we see by the transient nature of Brownian motion in $d \geq 3$, we may find a bound uniform in $x$ for sufficiently large $t$. The existence of a limit as a distribution of $v$, and it is not hard to see that the solution $v$ is smooth since $W$ is assumed to be [Eva98]. The connection to our model is this: in analyzing the behavior of this process we are aiming to compute – or at least approximate – the time integral of the microscopic coagulation kernel:

$$V_{m,n}^\epsilon(x) = \epsilon^{2-d}(r(m) + r(n))^{-2}V \left( \frac{x}{\epsilon(r(m) + r(n))} \right) \quad (1.22)$$

This is a very singular kernel indeed, and it may be much easier to analyze collisions as approximated by a much smoother one. The solution to 1.19 where 1.22 (multiplied by an appropriate constant) gives us a much needed tool, as we shall see in the next section.
CHAPTER 1. INTRODUCTION

Approximation by assumption of molecular chaos: Stosszahlensatz

In his analysis of the density of velocities of rarified gases with the equation

$$\frac{\partial}{\partial t} f(v, t) + v \cdot \nabla_x f(x, t) = Q(f, f)$$

(1.23)

with \(Q\) a binary collision operator, Boltzmann derived his famous \(H\)-theorem: entropy, denoted by

$$S(f) = -H(f) = -\int f \log(f) dv dx$$

is non-decreasing. Soon after, it was widely recognized that the critical mathematical assumption in the derivation was what Boltzmann called the Stosszahlensatz [Ham12][Vil02], which can roughly translated as the assumption of molecular chaos. That is, he assumed that the pre-collision velocities would be uncorrelated, while the post-collision velocities would be necessarily correlated. In a similar way, the rate of coagulation for the microscopic Smoluchowski model might be approximated – at a mesoscopic scale – by the local density of particles. Mathematically, we’d like to show that we can approximate the formation of \(n\) sized clusters via, e.g.

$$Y_{m,k,n}(t) = \int_0^t \frac{1}{2} \sum_{i \neq j} \alpha(m_i, m_j)V_{m_i, m_j}(x_i - x_j) J(x_i) ds$$

(1.24)

for some smooth compact \(J\) and any pair \(m, k\) such that \(m + k = n\). In search of this, suppose a function \(u_{m,n}\) solves

$$-\Delta u_{m,n} = \frac{\alpha(m, n)}{d(m) + d(n)} V_{m,n}(1 - u_{m,n})$$

(1.25)

This is the limit as \(t \to \infty\) of 1.19, with killing rate \(\frac{\alpha(m,n)}{d(m)+d(n)} V_{m,n}\). As with \(V_{m,n}\), we introduce a scaled version of this function:

$$u_{m,n}^\epsilon(x) = \epsilon^{2-d} u_{m,n}\left(\frac{x}{\epsilon}\right).$$

This scaled function solves a PDE very similar to 1.25:

$$-\Delta u_{m,n}^\epsilon = \frac{\alpha(m, n)}{d(m) + d(n)} V_{m,n}^\epsilon(1 - u_{m,n}^\epsilon)$$

(1.26)

Suppose \(J\) is any smooth, compactly supported function in \(\mathbb{R}^d\), and define

$$X_z(t) = \frac{1}{N^2} \sum_{i \neq j} u_{m_i, m_j}^\epsilon(x_i - x_j + z) J(x_i)$$

(1.27)
CHAPTER 1. INTRODUCTION

(The parameter \( z \) is a spatial perturbation which we will soon use to average out on a mesoscopic scale.) By Dynkin’s theorem, we know that

\[
\mathbb{E}[X_z(t) - X_z(0)] = \mathbb{E}\left[\int_0^t A_c(X_z - X_0)(s)\,ds\right] + \mathbb{E}\left[\int_0^t A_f(X_z - X_0)(s)\,ds\right].
\]  (1.28)

Now,

\[
A_cX_z(s) = E_z + \frac{1}{2} \sum_{i \neq j} \alpha(m_i, m_j)V_{m_i, m_j}^\epsilon(x_i - x_j)u_{m_i, m_j}(x_i - x_j + z) = E_z + H_2(z)
\]

\[
A_fX_z(s) = -\frac{1}{N^2} \sum_{i \neq j} \alpha(m_i, m_j)\Delta(d(m_i) + d(m_j))u_{m_i, m_j}^\epsilon(x_i - x_j + z)J(x_i, x_j, m_i, m_j)
\]

\[
= \frac{1}{N^2} \sum_{i \neq j} \alpha(m_i, m_j)V_{m_i, m_j}^\epsilon(x_i - x_j + z) \left(1 - u_{m_i, m_j}^\epsilon(x_i - x_j + z)\right) J(x_i, x_j, m_i, m_j)
\]

\[= A_1(z) - A_2(z)\]

where

\[A_1(z) = \frac{1}{N^2} \sum_{i \neq j} \alpha(m_i, m_j)V_{m_i, m_j}^\epsilon(x_i - x_j + z)J(x_i, x_j, m_i, m_j)\]

and

\[A_2(z) = \frac{1}{N^2} \sum_{i \neq j} \alpha(m_i, m_j)V_{m_i, m_j}^\epsilon(x_i - x_j + z)u_{m_i, m_j}^\epsilon(x_i - x_j + z)J(x_i, x_j, m_i, m_j).\]

To clarify, \( E_z \) is a monstrously complicated error term that represents the effect of removing or adding a term to the double sum in \( X_z \) when a third particle is colliding with only one in the pair. This importance of this term will be analyzed in more detail in the proceeding chapters, but importantly [HR07] found that in the setting where \( \chi < 1/(d - 2) \), its size was small in comparison to the major terms in the analysis. The truly useful part of this computation is that

\[H_2(0) = A_2(0)\]

so that, even as \( z \) and \( \epsilon \) go to zero, the only terms of unit order (as we shall see) cancel out.

\[
\mathbb{E}[X_z(t) - X_0(t)] = \mathbb{E}[X_z(0) - X_0(0)] + \mathbb{E}[Q_0(t) - Q_z(t)] + \mathbb{E}[E_z + H_2(z) - E_0]
\]  (1.29)

where \( Q_0 \) is the true rate of coagulation 1.24, and

\[Q_z(t) = \int_0^t \frac{1}{N^2} \sum_{i \neq j} \alpha(m_i, m_j)\Delta(d(m_i) + d(m_j))u_{m_i, m_j}^\epsilon(x_i - x_j + z)J(x_i, x_j, m_i, m_j)\,ds.\]  (1.30)
The clever choice of $u_{m,n}$, then, allows us to approximate something hopeless ($Q_0 = A_1(0)$) with something more tractable, $Q_z$. Specifically, they were able to show

$$Q_0(t) = Q_z(t) + Err(\epsilon, z)$$  \hspace{1cm} (1.31)

for a function $Err(\epsilon, z) \rightarrow 0$ as $\epsilon \rightarrow 0$ and then subsequently $|z| \rightarrow 0$, with $\epsilon << |z|$. This all assumes that the error terms $X_z$ and $E_z$ are themselves small and tractable, which is of course the bulk of the difficulty in [HR07]. But assuming this, we can average out $Q_z$ over a mesoscopic region to get a version of molecular chaos. To see this, pick your favorite bump function $\eta$ such that the support is contained in the unit ball of $\mathbb{R}^d$ and $\int \eta = 1$. Pick some mesoscopic $\delta$ and define the mesoscopic candidate densities:

$$f_{n,\delta}(x,t) = \epsilon^{d-2} \sum_i \delta^{-d} \eta(\frac{x_i - x}{\delta}) 1_{m_i = n}$$  \hspace{1cm} (1.32)

In other words, this smooth function roughly counts the number of size $n$ clusters in a $\delta$-sized ball around $x \in \mathbb{R}^d$. A simple calculation shows that, by averaging $Q_{z_2-z_1}$ over $\Gamma = B(0, \delta) \times B(0, \delta) \in \mathbb{R}^d \times \mathbb{R}^d$

$$Q_0(t) = Avg_{\Gamma}(Q_{z_2-z_1}(t)) + Err(\epsilon, \delta)$$  \hspace{1cm} (1.33)

where now

$$Err(\epsilon, \delta) = \sup_{z \in B(0, \delta)} |Err(\epsilon, z)|$$  \hspace{1cm} (1.34)

and

$$Avg_{z \in B(x, \delta)} f = \frac{1}{Vol(B(0, \delta))} \int_{B(0, \delta)} f(x) dx.$$  

Making a change of variables, where $\omega_1 = x_i - z_1$ and $\omega_2 = x_j - z_2$,

$$Avg_{\Gamma}(Q_{\omega_2-\omega_1}(t)) = -\int_0^t \int_{\Gamma} \sum_{i \neq j} \alpha(m_i, m_j) \Delta(d(m_i) + d(m_j)) u_{m_i, m_j}^\epsilon(\omega_2 - \omega_1)$$

$$\times J(x_i, x_j, m_i, m_j) f_n(\omega_1, t) f_m(\omega_2, t) d\omega_1 d\omega_2 ds + Err(\epsilon, \delta)$$

To simplify this, we can move to the diagonal, because for $|\omega_2 - \omega_1| \leq \epsilon$, $|x_i - x_j| \leq \epsilon$,

$$|f_{n,\delta}(\omega_1, t) - f_{n,\delta}(\omega_2, t)| \leq \epsilon^{d-2} \delta^{-d} (\epsilon \delta^{-1} ||\nabla \eta||_\infty)$$

Since we assumed $J$ was of the form $J = 1_{m_i=k, m_j=m}(x)$, with $m + k = n$,

$$Avg_{\Gamma}(Q_z(t)) = \beta(m, n) \int_0^t \int_{\mathbb{R}^d} f_{m,\delta}(x,t) f_{n,\delta}(x,t) + Err(\epsilon, \delta)$$  \hspace{1cm} (1.35)

where, again

$$\beta(m, n) = \alpha(m, n) \int_{\mathbb{R}^d} V_{m,n}(x)(1 - u_{m,n}(x)) dx$$  \hspace{1cm} (1.36)

This gives us a precise recipe to translate between models of microscopic coagulation and mass-conserving solutions to the Smoluchowski PDE.
Chapter 2

Main Result

2.1 Model parameters and meaning of gelation

Let us briefly review the parameters of our model of microscopic coagulation. For $N \in \mathbb{N}$ we construct a model of coagulating Brownian particles, where for each time $t$ the particles are represented by a position vector $q(t) : (\mathbb{R}^d \times \mathbb{N} \cup \{c\})^N$ and a Markov transition kernel $\mathbb{L}$ described in 1.10. For a position vector, the set $I(q)$ represents the set of particles not in the cemetery state.

- The diffusion function $d : \mathbb{N} \rightarrow \mathbb{R}$ is uniformly bounded: $d(n) \leq \bar{d}$.
- Importantly for our results, $r(n) = n^{\chi}$ with $\chi > 1/(d-2)$.
- As $n, m \rightarrow \infty$, the coagulation kernels
  \[
  \alpha(n, m)V_{n,m}(x) = V \left( \frac{x}{r(n) + r(m)} \right) \left[ \frac{\alpha(n, m)}{(r(n) + r(m))^2} \right]
  \]
  stay bounded away from 0, where $V$ is the reader’s favorite bump function that integrates to 1 in $\mathbb{R}^d$. By bounded away from 0, we mean that for $|x| \leq \frac{1}{2}(r(n) + r(m))$, the quantity is greater than some constant $K$ for $n, m$ sufficiently large.
- The initial location of each particle $i \in 1, 2, \ldots, N$ is selected independently from an initial distribution $h(x) \in C_c^\infty(\mathbb{R}^d)$, where for convenience the support of $h$ is contained in the unit ball and $\int_{\mathbb{R}^d} h(x)dx = 1$. The initial masses are sampled independently from the function $m(n) : \mathbb{N} \rightarrow \mathbb{R}$ with $\sum_{n=1}^\infty m(n) = 1$.
- For the coagulation part of the infinitesimal generator, we will replace
  \[
  A_c F(q) = \frac{1}{2} \sum_{i \neq j} \alpha(m_i, m_j) V_{n,m}^\epsilon(x_i - x_j) \left[ \frac{m_i}{m_i + m_j} F(q^{i,j}) + \frac{m_j}{m_i + m_j} F(q^{j,i}) - F(q) \right]
  \]
with
\[ A_c F(q) = \frac{1}{2} \sum_{i \neq j} \alpha(m_i, m_j) V^c_{n,m}(x_i - x_j) \left[ H(m_i, m_j) F(q^{i,j}) + H(m_j, m_i) F(q^{j,i}) - F(q) \right] \]

where
\[ H(m, n) = \begin{cases} \frac{m+n}{m+n} & 2n \geq m \geq n/2 \\ 1 & m \geq 2n \\ 0 & \text{otherwise} \end{cases} \]

This is a mostly cosmetic change from the model in [HR07], with the advantage of allowing us to control what happens when a large gel is formed. The mass of a given tagged particle is no longer a martingale, but this does not concern us in this thesis.

Recalling the definition for gelation introduced in the previous chapter for the deterministic Smolochowski PDE, let us now explain what it would mean in the microscopic setting. Given some smooth functional \( J(x) \) and a cutoff \( R \), define the proportion mass above \( R \):
\[ M_R(t) = \frac{1}{N} \sum_{x_i \in \Omega(t)} m_i J(x_i) 1_{m_i \geq R} \]

Fixing a positive constant \( c > 0 \), we may then define a stopping time representing gelation:
\[ \tau_g(c, R) = \tau_g = \inf \left\{ t > 0 \mid M_R(t) > c \right\} \]

This is the natural microscopic analogue to the “loss” of mass in 1.2: in that model, the mass of the tails \( \sum_{n=k} \infty n f(n, t) \) did not go to 0 as \( n \to \infty \) for \( t \to T_{gel} \).

### 2.2 Introducing the smoothing kernel \( u_{n,m} \) and its properties

In Chapter 1 we described a collection of functions \( u_{n,m} \) which were related to the microscopic coagulation kernels \( V_{n,m} = (r(n) + r(m))^{-2} V \left( \frac{r(n) + r(m)}{r(n) + r(m)} \right) \):
\[ -\Delta u_{n,m} = \frac{\alpha(n, m)}{d(n) + d(m)} V_{n,m}(1 - u_{n,m}) \]

What we saw in this analysis was that \( u_{n,m} \) could be understood as the probability of killing in the killed Brownian motion process with killing rate \( \frac{\alpha(n, m)}{d(n) + d(m)} V_{n,m} \). An absolutely essential part to proving that \( \tau_g < \infty \) will be a way to estimate the number of collisions between particles. The difficulty is, of course, that given a pair of particles \((x_i, m_i)\) and \((x_j, m_j)\) – even if they both lie in a mesoscopic neighborhood of radius \( \delta \) – there is no obvious way to systematically estimate the. Indeed, the microscopic collision kernel \( V^c_{n,m} \) has support
of radius $\epsilon (r(n) + r(m))$, and we assume $\epsilon \ll \delta$. To count collisions, then, it is probably hopeless to attempt to compute a quantity such as 1.24 directly:

$$Q_0(t) = \int_0^t 1/2 \sum_{i \neq j} \alpha(m_i, m_j)V_{m_i, m_j}^\epsilon(x_i - x_j)J(x_i)ds$$

Instead of estimating the time-average of this very singular kernel, we might more easily estimate a distance function:

$$Y_n(t) = \int_0^t \sum_{i \neq j} 1/|x_i - x_j|ds$$

excluding the set of measure zero where $x_i(s) = x_j(s)$ for some $s, i$ and $j$. The idea would be that distance is inversely related to the probability of a pair of particles colliding, so quantifying collisions might be possible by giving this functional a lower bound. Unfortunately, this has no simple relationship with $Q_0(t)$, specifically it is clearly not a lower bound, averaged over all particle trajectories. On the other hand, by the Feynman-Kac representation 1.19, the candidate correlation equations $u_{m,n}^\epsilon$ represent the long-time probabilities that two tagged independent Brownian particles will coagulate under Markov process, assuming they do not collide with something else first. This is a big assumption and the reason why, when $\chi > 1/(d-2)$, $Q_z(t)$ in 1.35 does not give a useful estimate for $Q_0(t)$, even when averaged over a mesoscopic region. The key ingredient to our proof of instantaneous gelation will give a mathematically precise modification of this estimate.

### 2.3 Strategy and outline of the proof

To show instantaneous gelation, we will draw inspiration from two proofs of instantaneous gelation in the Smoluchowski ODE and the Marcus-Lushnikov model. In [CC92], Carr and da Costa prove that for coagulation kernels $\alpha(m, n) \geq C(m^c + n^c)$, $c > 1$, there exist no mass conserving solutions for any $t_0 > 0$. To show this, the authors defined $p$-moments for integers $p > 1$:

$$S_p(t) = \sum_{k=1}^\infty k^p f(k, t)$$

As clusters collide, since $(n + m)^p > n^p + m^p$ is super-additive, $S_p(t)$ will grow. The authors show that if any such solution, if it exists, will have an instantaneous blow-up time for $S_p$. In a stochastic setting, recall that the Marcus-Lushnikov process [Rez13] is a stochastic model of coalescence that in the scaling limit converges to the deterministic Smoluchowski equation – under appropriate conditions on the rate of coalescence:

$$L(t) = \left( L_1(t), \cdots, L_N(t) \mid \sum_{k=1}^N kL_k = N \right)$$ (2.1)
CHAPTER 2. MAIN RESULT

\[ \lim_{n \to \infty} \left| \frac{L_n(t)}{N} - f_n(t) \right| = 0 \quad (2.2) \]

for some \( \{f_n\} \) satisfying 1.1. In this setting, Rezakhanlou\cite{Rez13} defines instead moments of large particles \( M_{p,l}(t) = \sum_{k=l}^{\infty} k^p f_k(t) \) where \( f_k = L_k(t)/N \) and uses an inductive argument to show that so long as there are sufficiently many small (of mass \( n \leq l \), which is true before a microscopic gelation time) particles, one can derive a lower bound on the formation of particles sized \( l+1 \) that is sufficiently large to show that we can obtain rapidly \( \tau_{gel} \sim \frac{1}{\log(N)c} \) for some power \( c \) large \( m_1, \ldots, m_k \) values for these moments, indicating many collisions: for some \( \beta > 0 \),

\[ \mathbb{L}(M_{p,l+1}(t)) \geq CM_{p,l}^{1+\beta}(t) \]

where \( M(t) \) is a martingale, and \( \mathbb{L} \) is the infinitessimal generator of the Marcus-Lushnikov system.

Our proof will draw on these constructs but we must find an appropriate modification for the discrete Brownian particle setting. Simply examining the moments

\[ M_p(t) = \sum_{x_i \in I(q)} m_i^p J(x_i) \]

for some smooth bounded \( J \) is hopeless, because we cannot readily estimate the number of collisions globally. The same goes for \( M_{p,l} \): the problem is that we have no guarantee of anything like a Stosszahlensatz due to the random diffusive nature of Brownian particles, so we are unable to estimate the number of collisions with particles of sized \( l \) or greater based on the number of smaller particles. We have discussed why being able to estimate the number of \( m \) and \( n \) sized particles with a Stosszahlensatz estimate of the form 1.35:

\[ Q_0(t) = \beta(m, n) \int_0^t \int_{\mathbb{R}^d} f_{m,\delta}(x,t) f_{n,\delta}(x,t) + Err(\epsilon, \delta) \quad (2.3) \]

with \( \Gamma = B(0, \delta) \) would be mathematically incompatible with our goal of showing instantaneous gelation. Therefore we should look for a more qualified replacement. The first thing to note is that we may be able to tame the difficulties of the diffusive nature of Brownian particles by focusing on a smaller scale. Given a mesoscopic neighborhood of radius \( \delta \), for Brownian particles with diffusion at most \( \bar{d} \), under any time scale on the order of \( o(\delta^2) \) we will likely see almost all of the particles remaining in the neighborhood. On the other hand, since we expect gelation, one problem with a the estimate 1.35 is that over the long run it underestimates the number of collisions. And since our system is stochastic, there should be some uncertainty for how long this local Stosszahlensatz lowerbound will work. Our strategy to show large particles form very quickly can be summarized as follows:

**Step 1.** Formulate a version of the Stosszahlensatz 1.35 that works as a lower bound for the change in the functional \( M_{p,\delta,x}(t) \) – a localized, short-time version.

**Step 2.** Show that, given a series of concentric spheres of radius \( \delta_1, \delta_2, \ldots, \delta_L \), with very high probability our high moments \( M_{p,\delta_k,x}(t) \) very quickly exceed \( m_k \). Drawing on \cite{Rez13} for
inspiration, we will show that these rapidly growing high moments imply the formation of a large particle.

**Step 3.** Show that this large particle (the *gel*) gobbles up everything else.

### 2.4 Proof of the main result

Before beginning the proof, let us first set up some notation.

- Let $Y(z)$ be any smooth bump function whose support is contained in the unit ball, and for which $|\Delta Y(x)| \leq C(d)$. $Y_{\delta,x}(z) = Y_\delta(z) = Y(\frac{x_i-x}{\delta})$ is the scaled version.

- We will be principally concerned with the growth of $p$-moments
  $$M_{p,\delta,x}(t) = \sum_{i \in I(q(t))} m_i(t)^p Y_\delta(x_i)$$
  as a way of demonstrating by proxy that heavy particles have formed.

- Define $\delta = \left(\frac{1}{\log(N)}\right)^b$, where $b = \frac{1}{8}$.

- Ensure, without loss of generality, that if $h(x)$ is the particles’ initial position density, $x$ is a point of arbitrary positive density, and set $m_1 = M_{1,\delta,x}(0)$.

- Set $p = 8\delta^{-2}m_1^{-1}$.

- We define a sequence of radii $\delta = r_1 > r_2 > \cdots > r_{L'} = \epsilon$, where
  $$\frac{r_k}{2 \log(N)^t} \leq r_{k+1} \leq \frac{r_k}{\log(N)^t}$$
  for any $0 < t < 1/16$, and satisfying
  $$(r_k)^d \leq L'K(r_{k+1})^{d-2}$$
  for $1 \leq k \leq L' - 1$, where $K = \left(\frac{1}{\log(N)}\right)^{c_1}$, with $c_1 = 1 + c$. We aim to make $M_{p,x,\delta}$ roughly double every $\left(\frac{1}{\log(N)}\right)^c$ units of time. To this end, we relate $c = \frac{1}{3}$. Because $t$ is a constant, we know $L'$ is of the order log log $N$, so equation 2.4 is satisfied.

- We denote by $\{C_k\}$ a series of weights such that $1/2 = C_1 > C_2 \cdots > C_L$ and $\sum_{k=1}^L C_k = 1$.

We also have a number of stopping times which we will use to delineate either key progress markers or the occurrence of an exceptional event that indicates.
• The formation of a mesoscopic gel

$$\tau_M = \inf \{ t > 0 \mid \exists (x_i(t), m_i(t)), \text{ such that } er(m_i) > \delta/2 \}.$$ 

• For each particle \((x_i(t), m_i(t)) \in I(q(t))\), define the distance \(D_i(t) = |x_i(t) - x_i(0)|\). The path of each surviving particle is just a piecewise independent Brownian motions moving at diffusion rates bounded by \(\bar{d}\). We have by the Law of Iterated Logarithm [PM10] almost surely that

$$\limsup_{t \to 0^+} \frac{B(t)}{\sqrt{2t \log \log(t)}} = 1.$$ 

We will therefore define a stopping time representing unexpectedly high dispersion:

$$\tau_D = \inf \{ t > 0 \mid \exists (x_i(t), m_i(t)) \text{ s.t. } D_i(t) \geq K' \sqrt{t} \}$$

where \(K' = \left(\frac{1}{\log(N)}\right)^b\), and \(b > 1/2\).

• The stopping time representing complete gelation:

$$\tau_C = \inf \{ t > 0 \mid M_p(t) = N^p \}.$$ 

The critical lemma at the heart of our discussion, which will help us show that in a mesoscopic region the number of collisions can be bounded from below – more directly, showing how quickly the \(p\)-moments \(M_{p,\delta,x}\) grow – is this:

**Lemma 1.** Suppose \(J(x_i, x_j, m_i, m_j) = p(m_i m_j^{p-1} + m_j m_i^{p-1})Y(x_i)\). Let \(\tau_a \leq \tau_b\) be two stopping times. Then there exists some \(s\) such that \(s \leq \tau_b + \frac{2}{\log(N)^b}\) and

$$\int_{\tau_a}^{s} \sum_{i \neq j} \alpha(m_i, m_j) V_c(x_i - x_j) J(x_i, x_j, m_i, m_j) \geq$$

$$\int_{\tau_a}^{s} \sum_{i \neq j} C_k \sum_{i \neq j} \text{Avg}_{z \in B(0, r_k)} - \Delta \alpha (m_i, m_j) (x_i - x_j + z) J(x_i, x_j, m_i, m_j) + M(s) - M(\tau_a)$$

where \(M(t)\) is a martingale.

**Remark.** What is critical here is the use of averaging over several scales \(B(0, r_k)\), weighted by \(C_k\). Each averaged term in the sum helps counteract the error generated from the correlation functional \((X_z - X_0)(t)\) at the previous scale.

For a fixed \(i \in I_q(t)\), for each \(s > t\) let us define \((\tilde{x}(s), \tilde{m}(s))\) to be the unique particle whose genealogy of collisions can be traced to \((x_i(t), m_i(t))\). In other words, for all \(s\) if there is a
sequence of times $t_{j_1}, \cdots, t_{j_n} = s$ and particles such that $(x_{j_k}, m_{j_k})$ collides with $(x_{j_k+1}, m_{j_k+1})$ at time $t_{j_k}$, and $(x_{j_k}, m_{j_k}) = (\bar{x}(t'), \bar{m}(t'))$ for $t_{j_k} < t' < t_{j_k+1}$. For all $t_k < t' < t_{k+1}$. Put simply, it keeps track of the particle that carries the mass that began with $(x_i(t), m_i(t))$ all the way to $(x_i(s), m_i(s))$.

**Lemma 2.** Suppose that the (possibly empty) set of collision times experienced by $(\bar{x}, \bar{m})$ during $(t, s)$ is given by $t_{j_1}, \cdots, t_{j_N}$. Recall that in between collisions, $(\bar{x}, \bar{m})$ is an independent Brownian motion with diffusion rate $d(\bar{m}) \leq \bar{d}$. Now suppose that $x_i(t) \in B(x, \delta_k)$ while $\bar{x}(s)$ is not in $B(x, \delta_{k+1})$. Then at least one of the following must have occurred:

1. The distance traveled by $(\bar{x}, \bar{m})$ in between collisions during the time interval $(t, s)$ was greater than $\frac{\delta_{k+1} - \delta_k}{2}$
2. $\epsilon r(\bar{m}(t)) \geq \delta_{k+1}/4$.

The next lemma uses Lemma 1 to allow us to show that either $M_{p, \delta_k}$ grows very quickly or a mesoscopic particle forms. For the following lemma, assume we are given a sequence $\{m_k\}$ with $3m_k > m_{k+1} > 2m_k$. Using these $m_k$, set $\tau_1 = 0$ and inductively define

$$\tau_{k+1} = \max \{ \tau_k, \inf \{ t > 0 \mid M_{p, x, \delta_{k+1}}(t) \geq m_{k+1} \} \}.$$

**Lemma 3.** With the sequences $\{\tau_k\}, \{m_k\}$ defined as above, then

$$\mathbb{E} [\min \{\tau_{k+1}, \tau_M\} - \tau_k] \leq 4\delta_k^{1-\beta} (m_{k+1} - m_k) \frac{1+\beta}{m_k^{1+\beta} p} + \left( \frac{2}{\log(N)} \right)^c.$$

**Lemma 4.** If a mesoscopic gel forms, the rest of the mass instantaneously clumps together. In other words,

$$\mathbb{E} [\tau_C - \tau_M] \leq \frac{2}{\log(N)^c}.$$

Before collecting our work from the previous into a result showing complete instantaneous gelation asymptotically almost surely, let us summarize the ways in which the procedure may fail to explain the “asymptotically” qualifier. Some particles may drift away so fast that the mixing process has no chance to produce collisions between particles. As we have shown, a necessary condition for this is for some $k$, a particle that begins in $B(0, \delta_k)$ after $\tau_k$ must end outside $B(0, \delta_{k+1})$. Here is the main result:

**Theorem 8.** As $N \to \infty$, we have complete gelation asymptotically almost surely: $\mathbb{P} [\tau_C < \frac{C_2}{\log(N)^c}] \to 1$, where $C_2 = (2 \log \log(N) + \log(p) + 1)$. 
CHAPTER 2. MAIN RESULT

Proof. The proof will proceed in two stages. In the first step, we show that our previous work allows a functional of the form

\[ M_{p,\delta,x}(t) = \sum m_i^p Y_{x,\delta}(x_i) \]

grow so quickly that we must conclude by Lemma 3 that a mesoscopic sized gel has formed. After that, we cite use the result from Lemma 4 to show that a macroscopic size gel forms.

We begin by fixing \( x \), a point of positive density in the initial macroscopic distribution of particles. We also fix a sequence of \( \delta \) scales:

\[ \delta_1 < \delta_2 < \ldots < \delta_L = 2\delta_1 \]

satisfying the condition that

\[ \delta_{k+1} - \delta_k \geq \left( \frac{1}{\log(N)} \right)^b \]

The point of these increasing concentric spheres is to estimate the collisions in some interval of time (denoted by stopping times \( \tau_k \) and \( \tau_{k+1} \)) so that with very high probability no particle travels from inside \( B(x,\delta_k) \) to the outside of \( B(x,\delta_{k+1}) \) from the Brownian motion portion of its trajectroy. The particle may leave \( B(x,\delta_{k+1}) \) without traveling far through the Brownian motion by colliding with larger particles; this alternative scenario will also be considered by using Lemma . Now let us define our stopping times

\[ \tilde{\tau}_k = \inf \left\{ t > 0 \mid M_{p,x,\delta_k}(t) \geq m_k \right\} \]

and

\[ \tau_k = \max \{ \tau_1, \ldots, \tau_{k-1}, \tilde{\tau}_k \} \]

for \( k > 1 \), or just \( \tilde{\tau}_1 \) if \( k = 1 \). We have assumed that \( M_{p,x,\delta}(0) \geq \frac{1}{\log(N)^c} \). Therefore, let us form a sequence

\[ m_1 = \frac{1}{\log(N)^c} < m_2 < \ldots < m_L = N^p \]

and stipulate that \( 3m_k \geq m_{k+1} \geq 2m_k \). This allows us to assume that \( L \leq C\log\log(N) \) for some small constant \( C \). From Lemma 3, we have that

\[
\mathbb{E} [\min \{ \tau_{k+1}, \tau_M \} - \tau_k] \leq 4\delta_k^{(1-\beta)d} \frac{(m_{k+1} - m_k)}{m_k^{1+\beta} p} + \log(N)/K
\]

where \( K \geq \log(N)^c \). Our choice of \( \delta_k \) and \( p \), in addition to the fact that

\[ \frac{m_{k+1} - m_k}{m_k^{1+\beta}} \leq 3 \]

ensure that

\[
\mathbb{E} [\min \{ \tau_{k+1}, \tau_M \} - \tau_k] \leq C \left( \frac{1}{\log(N)} \right)^c \tag{2.5}
\]
Our goal is to show that a gel forms by bounding $\tau_M$ and then using Lemma 4. Consider the events which record the existence of a particle that starts in $B(x, \delta_k)$ but leaves the larger sphere $B(x, \delta_{k+1})$:

$$E_k = \{ \exists t \in [\tau_k, \tau_{k+1}], (x_i, m_i) \text{ s.t. } x_i(\tau_k) \in B(x, \delta_k), \quad x_i(t) \notin B(x, \delta_{k+1}) \}$$

Precisely because $\max \{ \tau_D, \tau_M \} = 1$,

$$\frac{\delta_{k+1} - \delta_k}{2} \geq \frac{1}{\log(N)^b},$$

we see that the stopping time representing the extraordinary particle dispersion, $\tau_D$ is triggered in the event of $E_k$:

$$\bigcup_{k=1}^{L} \{ \tau_D \leq \tau_{k+1} \} \supseteq \bigcup_{k=1}^{L} E_k.$$

Now, note that $\min \{ \tau_M, \tau_L \} = \tau_M$ since if $M_{p, \delta_L}(\tau_L) = N^p$, clearly a mesoscopic gel has already formed. Repeated application and summing of 2.5 therefore gives us:

$$\mathbb{E}[\tau_M \mathbb{1}_{\{ \tau_M < \tau_D \}}] \leq LC \left( \frac{1}{\log(N)} \right)^c \quad (2.6)$$

and therefore by Lemma 4

$$\mathbb{E}[\tau_G \mathbb{1}_{\{ \tau_G < \tau_D \}}] \leq (L + 1)C \left( \frac{1}{\log(N)} \right)^c \quad (2.7)$$

Since $L \leq C (\log(p) + \log \log(N))$, to complete the proof it suffices to show that

$$\mathbb{P}(\tau_D < 1) = o(1)$$

as $N \to \infty$. For this statement it is sufficient to show that for a Brownian motion

$$\mathbb{P} \left( \sup_{t \in [0,1]} \frac{B(t)}{\sqrt{t}} \geq K \right) = o(1/N)$$

which is a straightforward consequence of the Law of Iterated Logarithm and the standard estimate on large deviations of the normal distribution. \qed

## 2.5 Proofs of Lemmas

We will need the following facts about the smoothed kernel $u_{m,n}$.

**Lemma 5.** Suppose $x_i - x_j = z \in B(0, r_2) \setminus B(0, r_1)$, and that $r(m_i) + r(m_j) \leq \frac{r_2}{4}$. Then
1. There exists a constant $C(d)$ which depends only on the dimension and not $m, n$ such that for $|z| \leq 2c(r(m) + r(n))$, $u^e_{m,n}(x_i - x_j) \geq C(d)\epsilon^{2-d}$.

2. \[ \text{Avg}_{z \in B(0, 2r_2)} - \Delta u^e_{n,m}(x_i - x_j + z) \geq (m^q + n^q) r_2^{-d} \]

3. \[ u^e_{n,m}(x_i - x_j) \leq (r(m) + r(n))^{d-2} r_2^{-d} \]

4. If $y \geq r(m) + r(n)$,
   \[ u_{m,n}(2y) \geq C_2 u_{m,n}(y). \]

5. If $m \geq k$, there is a $C_3$ such that
   \[ u_{m,n}(y) \geq C_3 u_{k,n}(y) \]

Proof. 1. We will use the interpretation of $u_{m,n}(x)$ as the probability of a Brownian trajectory killed at rate
   \[ V_{m,n}(x) = \frac{\alpha(m, n)}{d(n) + d(m)}(r(m) + r(n))^{-2} V\left(\frac{\cdot}{r(m) + r(n)}\right). \]

   To show that $u_{m,n}$ is bounded from below, it is sufficient to bound the probability that the integral
   \[ \int_0^t V_{m,n}(x + B(s))ds \]

   is greater than one from below for $|x| \leq 2(r(m) + r(n))$. Recalling that by assumption there is some $K$ such that
   \[ \alpha(m, n) \geq K (d(m) + d(n)) \]

   we may just estimate the probability that a $d$-dimensional Brownian trajectory travels in the region where
   \[ V_{m,n} \geq (K/2)(r(m) + r(n))^{-2} \]

   for $1/2K \times (r(m) + r(n))^2$ units of time. By assumption on $V$, this region contains the set
   \[ \{ z : |z| \leq 3/4(r(m) + r(n)) \} . \]

   Define two stopping times:
   \[ \tau_1 = \inf \left\{ t > 0 \mid |z| = \frac{1}{2}(r(m) + r(n)) \right\} \]
\[ \tau_2 = \inf \{ t > 0 \mid |z| = 4(r(m) + r(n)) \}. \]

It is a standard fact\textsuperscript{[PM10]} that for a \( d \)-dimensional Brownian motion starting at \( z \),
\[
P(\tau_1 < \tau_2) = \frac{(4(r(m) + r(n))^{2-d} - |z|^{2-d}}{(4^{2-d} - 1^{2-d}) (r(m) + r(n))^{2-d}} \geq C_1(d) \tag{2.8} \]

If \( |z| \leq 2(r(m) + r(n)) \) we can compute a lower bound for the probability that
\[
\int_0^t V_{m,n}(z + B(s))ds \geq 1
\]
as the product of \( P(\tau_1 < \tau_2) \) and \( P(E) \) where
\[
E = \left\{ \sup_{0 \leq s \leq (r(m) + r(n))^2} |B(s)| < \frac{1}{4}(r(m) + r(n)) \right\}
\]

We know from an application of Brownian scaling and the reflection principle that there is some \( C \) such that \( P(E) \geq C \) independently of \( m \) and \( n \), because up to some constant factor this is just
\[
P\left( \max_{0 \leq s \leq r^2} B(s) \leq r \right) \geq C_2(d).
\]

This completes the proof of part 1.

2. Note that
\[
-\Delta u_{n,m}^\epsilon(z) = \epsilon^{-d}V_{n,m}(z/\epsilon)(1 - u_{n,m}(z/\epsilon))
\]
Noting that \( V_{n,m}(z) \geq 1/2 \) for all \( \epsilon r(n) + r(m) \)/2 \( \geq |z| \geq 0 \), and that \( u_{n,m}(z) \geq 1/2 \) in this region as well. Given that \( r(m) + r(n) \leq r_2/4 \), the radius of the sphere over which we are averaging \( z \), the result follows from the fact that \( (r(m) + r(n))^{d-2} \geq m^q + n^q \) if we can show that for
\[
3/2(r(m) + r(n)) < |z| \leq 2(r(m) + r(n))
\]
we can bound
\[
0 < C_1(d) \leq u_{m,n}(z) \leq C_2(d) < 1. \tag{2.9}
\]
The lower bound was part 1 of this Lemma, and we can show the upper bound in a similar way. Using the same construction as before, let
\[
\tau_1 = \inf \left\{ t > 0 \mid |z| = \frac{3}{2}(r(m) + r(n)) \right\}
\]
and
\[ \tau_M = \inf \{ t > 0 \mid |z| = M(r(m) + r(n)) \} . \]

As \( M \to \infty \), we see that
\[ P(\tau_1 < \infty) = \left( \frac{3|z|}{2(r(m) + r(n))} \right)^{d-2} \]

This implies the existence of \( C_2(d) \) in 2.9.

3. To deduce this, again recalling the killed Brownian motion interpretation, we may consider an alternative function \( u_{m,n} \) where \( V_{m,n} \) is replaced by a function \( \tilde{V}_{m,n} \) where for all \( x \),
\[ \tilde{V}_{m,n}(x) \geq V_{m,n}(x) \]

From the representation
\[ \tilde{u}_{m,n}(y) = 1 - \mathbb{E} \left[ \exp \left( - \int_0^\infty \tilde{V}_{m,n}(y + B(s))ds \right) \right] \]

it is clear that \( \tilde{u}_{m,n} \geq u_{m,n} \). Now, if we choose
\[ \tilde{V}_{m,n}(x) = K1_{x \in B(0, r(m) + r(n))} \]

for some large integer \( K \) (or rather a smooth version of this \( \tilde{V}_{m,n}(x) \)), it is obvious that as \( K \) approaches infinity, we get a pointwise convergence
\[ u_{m,n}(x) \to u(x) = \mathbb{P}(\tau_3 < \infty) \]

where
\[ \tau_3 = \inf \{ t > 0 \mid x + B(t) \in B(0, (r(m) + r(n))) \} . \]

Again, it is well known [PM10] that
\[ \mathbb{P}(\tau_3 < \infty) = \left( \frac{|x|}{r(m) + r(n)} \right)^{2-d} \]

Since \( x_i - x_j = z \in B(0, r_2) \setminus B(0, r_1) \), we have \( |x_i - x_j|^{2-d} \leq r_2^{2-d} \). Using the fact that \( r(m_i) + r(m_j) \leq r_2/4 \), we are done.

4. To prove this statement, we can use the fundamental solution of the partial differential equation: since
\[ -\Delta u_{m,n}(y) = V_{m,n}(y)(1 - u_{m,n}(y)) \]
we know that
\[ u_{m,n}(y) = \int C(d)V_{m,n}(z)(1 - u_{m,n}(z))|y - z|^{2-d}dz \]
If we want to compare \( u_{m,n}(2y) \) to \( u_{m,n}(y) \) to we may compute
\[ u_{m,n}(2y) = \int C(d)V_{m,n}(z)(1 - u_{m,n}(z))|2y - z|^{2-d}dz. \]
Since
\[ |2y - z| \leq |y| + |y - z| \leq 3|y - z| \]
if \( |z| \leq |y|/2 \), it would be prudent to split up the integral into \( U_1 = B(0, |y|/2) \) and \( U_2 = \mathbb{R}^d/U_1 \). The support of \( V_{m,n} \) is contained entirely in the former, and we know that
\[ \int_{U_1} V_{m,n}(z)(1 - u_{m,n}(z))|2y - z|^{2-d}dz \geq 3^{2-d} \int_{U_1} V_{m,n}(z)(1 - u_{m,n}(z))|y - z|^{2-d}dz. \]
This shows that if \( y \geq \epsilon(r(m) + r(n)) \), \( u^{\epsilon}_{m,n}(2y) \geq Cu^{\epsilon}_{m,n}(y) \) where \( C = 3^{2-d} \).

5. This is also easiest proven using the fundamental solution. We have that
\[ u_{m,n}(y) = \int V_{m,n}(z)(1 - u_{m,n}(z))|y - z|^{2-d}dz \]
and
\[ u_{m,k}(y) = \int V_{m,n}(z)(1 - u_{m,k}(z))|y - z|^{2-d}dz \]
Note that by 2.9 for \( z \) in the support of \( V_{m,n} \), it suffices by a change of variables, \( z_1 = (r(m) + r(n))z \) and \( z_2 = (r(m) + r(k))z \) to show that
\[ \int V(z_1)|y - (r(m) + r(n))z|^{2-d}dz_1 \geq C \int V(z_2)|y - (r(m) + r(k))z|^{2-d}dz_2 \quad (2.10) \]
because
\[ \frac{\alpha(m, n)}{d(n) + d(m)} \geq K \frac{\alpha(m, k)}{d(k) + d(m)}. \]
Now, since we assumed \( |y| \geq 2(r(m) + r(n)) \geq 2(r(m) + r(k)) \),
\[ |y - (r(m) + r(k))z| \geq |y - (r(m) + r(n))z| \]
so we are done.

Proof of Lemma 1:
For any $b$ and $z$ we have the almost sure identity

$$X_z(b) - X_z(a) = \int_a^b \mathbb{L}X_z(s)ds + M(b) - M(a)$$

Taking the infinitessimal generator $\mathbb{L}$ of $X_z$, we find that

$$X_z(t) - X_z(s) = \int_s^t \sum_{i \neq j} (d(m_i) + d(m_j)) \Delta u_{m_i,m_j}^\epsilon (x_i - x_j + z)J(x_i, x_j, m_i, m_j) d\theta$$

$$+ \int_s^t \sum_{i \neq j} \alpha(m_i, m_j)V(x_i - x_j)J(x_i, x_j, m_i, m_j) ds$$

$$+ \int_s^t [G_z(1) + G_z(2) - G_0(1)] d\theta$$

$$+ \int_s^t (R_{z,i}(\theta) + R_{z,j}(\theta) - R_{0,i}(\theta) - R_{0,j}(\theta)) d\theta$$

where $M(t)$ is a martingale and,

$$G_z(1) = 1/2 \sum_{i \neq j} \alpha(m_i, m_j)V(x_i - x_j)\epsilon^{2(d-2)} \sum_k$$

$$H(m_i, m_j) \left[ u_{m_i,m_k}^\epsilon(x_i - x_k + z)J(x_i, x_j, m_i + m_j, m_k) + u_{m_i,m_j}^\epsilon(x_i - x_k + z)J(x_i, x_j, m_k, m_j + m_i) \right]$$

$$+ H(m_j, m_i) \left[ u_{m_i,m_j}^\epsilon(x_j - x_k + z)J(x_i, x_j, m_i + m_j, m_k) + u_{m_i,m_j}^\epsilon(x_k - x_j + z)J(x_i, x_j, m_k, m_i + m_j) \right]$$

$$- \left[ u_{m_i,m_k}^\epsilon(x_i - x_k + z)J(x_i, x_j, m_k, m_j) + u_{m_i,m_k}^\epsilon(x_k - x_i + z)J(x_i, x_j, m_i + m_j) \right]$$

$$- \left[ u_{m_j,m_k}^\epsilon(x_j - x_k + z)J(x_i, x_j, m_j, m_k) + u_{m_j,m_k}^\epsilon(x_k - x_j + z)J(x_i, x_j, m_i + m_j) \right]$$

and

$$R_{z,i}(\theta) = \nabla_{x_i}J(x_i, x_j, m_i, m_j) \cdot \nabla_{x_i}u_{m_i,m_j}^\epsilon(x_i - x_j + z)$$

and

$$G_z(2) = -\epsilon^{2(d-2)} \sum_{i \neq j} \alpha(m_k, m_j)V(x_i - x_j)u_{m_i,m_j}^\epsilon(x_i - x_j + z)J(x_i, x_j, m_i, m_j)$$
where \( H(m_i, m_j) = \frac{m_i}{m_j + m_i} \) when \( m_j \leq m_i \leq 2m_j \) and \( H(m_i, m_j) = 1_{m_i \geq m_j} \) otherwise. Let us explain the form of these rather complicated expressions. When the coagulation part of the infinitesimal generator is applied to the double sum \( X_z \), we are summing over 4-tuples \((i, j, k, l)\) of particles; the only relevant terms come from when the first two indices \((i, j)\) overlap with the last two \((k, l)\). When this overlap is just one particle, we arrive at \( G_z(1) \), and when they are identical, we get the negative contribution of \( G_z(2) \). Crucially, because of the carefully chosen form of \( u_{m,n} \), since

\[-(d(n) + d(m)) \Delta u_{m,n} = V_{n,m}(1 - u_{n,m}) = V_{n,m} - V_{n,m} u_{n,m},\]

we get a cancellation of \( G_0(2) \) with the other term from the free motion part of the infinitesimal generator applied to \( u_{m_i,m_j}^\epsilon J \):

\[-\Delta (d(n) + d(m)) V_{n,m}^\epsilon (x_i - x_j) u_{m,n}^\epsilon (x_i - x_j) J(x_i, x_j, m_i, m_j)\]

and sum to zero. Moving all the terms except for the coagulation to the left, in order to prove the lemma we first aim to show the non-negativity of the expression

\[\int_s^t \left[ G_0(1) - G_z(1) - G_z(2) \right] ds\]

up to some small error. Since \(-G_z(2)\) is positive we therefore focus on \( G_0(1) - G_z(1) \). For triplets \((i, j, k)\) such that \( m_j \leq m_i \leq 2m_j \), we note that

\[J_1 = \frac{m_i}{m_i + m_j} J(x_i, x_j, m_i + m_j, m_k) - J(x_i, x_j, m_i, m_k) > 0\]

Since \( u_{m_i,m_k}^\epsilon \) is superharmonic (i.e. \(-\Delta u_{m_i,m_k}^\epsilon \geq 0\) everywhere), we know [Eva98] that

\[J_1(u_{m_i,m_k}^\epsilon (x_i - x_j) - \text{Avg}_{z \in B(0,r)} u_{m_i,m_k}^\epsilon (x_i - x_k + z)) \geq 0\]

we may repeat the above argument for the four pairs of terms in \( G_z(1) \) and \( G_0(1) \), grouped by the argument of the mass pair in \( u^\epsilon \). When \( m_i > 2m_j \), we must make an alternative argument. First, note that \( 1 = \frac{m_j}{m_i + m_j} + \frac{m_j}{m_i + m_j} \), so to repeat our earlier argument, all we have to do is show that

\[\frac{m_j}{m_i + m_j} [u_{m_i,m_k}^\epsilon (x_i - x_k + z) J(x_i, x_k, m_i + m_j, m_k) + u_{m_i,m_k}^\epsilon (x_i - x_k + z) J(x_i, x_k, m_k, m_j + m_i)]\]

\[\geq [u_{m_i,m_k}^\epsilon (x_i - x_k + z) J(x_i, x_k, m_k, m_j) + u_{m_i,m_k}^\epsilon (x_k - x_i + z) J(x_i, x_j, m_i, m_k)]\]

To show this, we will use the fact that since \((x_i, m_i)\) and \((x_j, m_j)\) are colliding, we will show that there is a constant \( C < 1 \) not depending on \( N, i, j, k \) or \( z \) such that

\[u_{m_i,m_k}^\epsilon (x_i - x_k + z) \geq C u_{m_j,m_k}^\epsilon (x_j - x_k + z)\]
and use the radial symmetry of \( u^\epsilon \) to argue similarly for the other two terms. This constant comes from the fact that in our work above, we demonstrated that there is a constant \( C_2 \) such that
\[
u_{m,n}^\epsilon(y) \geq C_2
\]
for \( |y| \leq 2 (r(m) + r(n)) \) independently of \( m \) and \( n \). In the case that \( |x_i - x_k + z| \) lies outside of this range, we know from the fact that \( u_{m,n} \) decreases radially and is radially symmetric that
\[
u_{m_i,m_k}^\epsilon(x_i - x_k + z) \geq C_3 \nu_{m_i,m_k}^\epsilon(x_j - x_k + z) \geq C_4 \nu_{m_j,m_k}^\epsilon(x_j - x_k + z)
\]
The first inequality comes from the fact that since \( (x_i, m_i) \) and \( (x_j, m_j) \) are colliding, \( |x_i - x_j| \leq |x_i - x_k + z| \), so
\[|x_j - x_k + z| \leq 2|x_i - x_k + z|
\]
This combined with the fact that \( \nu_{m_i,m_k}^\epsilon \) is radially decreasing proves the first inequality. The second inequality follows from part 5 of the Lemma. To bound the \( R_{z,t} \) we need to use the difference bound derived in Lemma 6.3 of [Ham12]:
\[
\|
abla \nu_{m,n}^\epsilon(x + z) - \nabla \nu_{m,n}^\epsilon(x) \| \leq (r(m) + r(n))^{d-2} \frac{|z|}{|x|^d}.
\]
Averaging this \( z \) over the scales \( B(0, r_k) \) gives us a term that is dominated by the time integral of \( X_z \) described below. Another term, given by the free motion part of the infinitesimal operator
\[
\frac{1}{N^2} \int_s^t \sum_{i \neq j} -d(m_i) \Delta x_i + d(m_j) \Delta x_j J(x_i, x_j, m_i, m_j) \left[ \nu_{m_i,m_j}^\epsilon(x_i - x_j + z) - \nu_{m_i,m_j}^\epsilon(x_i - x_j) \right]
\]
can be bounded by the fact that 1) \( \Delta x_j = 0 \) and 2) \( d(n) \leq C \) for all \( n \) and \( -\Delta J \leq \frac{1}{g^2} \). We will bound this term with the computation below, which gives a lower bound on \( X_z(t) \). Instead of bounding \( X_z(t) \) directly, we will find some \( b \) where \( \int_s^b X_z(y)dy \) dominates the term \( X_z(b) \) and then show that the former term is dominated by
\[
- \int_s^b \sum_{i \neq j} (d(m_i) + d(m_j)) \Delta u_{m_i,m_j}^\epsilon(x_i - x_j + z) J(x_i, x_j, m_i, m_j) ds.
\]
In other words, we would like to find a \( b \) such that
\[
\sum_{k=1}^L C_k \int_{\tau_a}^b \text{Avg}_{z \in B(0,r_k)} \sum_{i \neq j} -\Delta u_{m_i,m_j}^\epsilon(x_i - x_j + z) J(x_i, x_j, m_i, m_j) ds
\]
dominates (is more than twice)
\[
\sum_{i \neq j} u_{m_i,m_j}(x_i(b) - x_j(b)) J(x_i, x_j, m_i, m_j).
\]
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For each pair of particles let us define the average killed probability in the ball $B(x_i - x_j, r_n)$:

$$R_n(t, x_i - x_j) = \text{Avg}_{z \in B(0, r_n)} - \Delta u^\varepsilon_{m_i, m_j}(x_i - x_j + z)$$

and the error term from such an estimate:

$$E_n(t, x_i - x_j) = \text{Avg}_{z \in B(0, r_n)} \left[ u^\varepsilon_{m_i, m_j}(x_i(t) - x_j(t) + z) - u^\varepsilon_{m_i, m_j}(x_i(t) - x_j(t)) \right]$$

and further define

$$R(t, x_i - x_j) = V_{\epsilon, m_i, m_j}(x_i - x_j) + \sum_{n=0}^{M} C_n R_n$$

$$E(t, x_i - x_j) = \sum_{n=0}^{M} C_n E_n$$

with $C_n$ chosen so that $\sum_{n=0}^{M} C_n = 1$. Intuitively, for each pair $(x_i, m_i)$ and $(x_j, m_j)$, $R(t, x_i - x_j)$ records the increase in the propensity of the particles to collide, averaged over several scales, while $E(t, x_i - x_j)$ records the potential error in this measurement. Our goal, roughly speaking, is to show that $R(t, x_i - x_j)$ is the dominant term so that we may safely ignore $E(t, x_i - x_j)$. It is easy to see, from the fact that of

$$-\Delta u^\varepsilon_{m_i, m_j} = 0$$

outside of $B(0, (r(m_i) + r(m_j))\epsilon)$, that $E_n = 0$ if $x_i - x_j \notin B(0, r_n)$, which follows from the mean value property of harmonic functions. This implies that $R(t, x_i - x_j) \geq \sum_{n=0}^{l} C_n R_n$ and $E(t, x_i - x_j) \leq \sum_{n=0}^{l} C_n E_n$ where $l$ is the integer such that $x_i - x_j \in B(0, r_l), x_i - x_j \notin B(0, r_{l+1})$. We would like to show that for every particle pair,

$$R(t, x_i - x_j) \geq K E(t, x_i - x_j).$$

To see this, we first examine the case where $|x_i - x_j| \leq \epsilon(r(m_i) + r(m_j))$. In this case,

$$V^\varepsilon_{m_i, m_j}(x_i - x_j) \geq K \epsilon^{2-d} \geq E(t, x_i - x_j).$$

In the case that $|x_i - x_j| > \epsilon(r(m_i) + r(m_j))$, we need to do a more careful accounting. Let us denote by $l$ is the largest integer such that $r_l > 2|x_i - x_j|$. We know that

$$R_l(t, x_i - x_j) \geq r_l^{-d} (r(m_i) + r(m_j))^{d-2}$$

and

$$E_l(t, x_i - x_j) \leq |x_i - x_j|^{2-d} (r(m_i) + r(m_j))^{d-2} \leq r_{l+1}^{-d} (r(m_i) + r(m_j))^{d-2}.$$  

Combining these inequalities, we have that

$$R_l(t, x_i - x_j) \geq \frac{r_l^{-d}}{r_{l+1}^{2-d}} E_l(t, x_i - x_j) \geq K M E_l(t, x_i - x_j)$$
Because \( u_{m,n} \) decreases radially, \( E_k(t, x_i - x_j) \) is increasing in the index \( k \), so we can see that

\[
C_l R_l(t, x_i - x_j) \geq KME_l(t, x_i - x_j),
\]

from which the desired inequality

\[
R(t, x_i - x_j) \geq KE(t, x_i - x_j)
\]

follows. Summing over the pairs, we also get that

\[
R(t) \geq KE(t)
\]

Since we are concerned with comparing the time-average of \( R(t) \) with \( E(t) \) at a particular stopping time \( b \), we will actually show that for some sufficiently small \( b \), the time-average \( \int_s^b KE(t) \) dominates \( E(s) \). In other words, we must show that there is some \( b \) such that

\[
\int_s^b R(t)dt \geq \int_s^b KE(t) \geq E(s)
\]

To this end, note that for any positive function \( F(t) \), if \( F(s) \geq \int_a^t KF(t)dt \) for all \( t \in [s,b] \), then \( F(t) \geq F(s) e^{K(t-s)} \) for the same interval. The proof follows that of Gronwall’s inequality, with the inequality sign reversed. For the particular case of the error function \( E(t) \), we know that for each pair of particles

\[
C \epsilon^{2-d} \geq u_{m_i,m_k}(x_i - x_j) \geq r_1^{2-d}
\]

where the second inequality comes from the fact that the particles under consideration are always in the bounded region of \( B(0, r_1) \) unless the stopping time \( \tau_D \) has been reached. If \( b \) is defined as the first time that

\[
\int_s^b KE(t)dt \geq E(s),
\]

then

\[
\frac{E(b)}{E(s)} \geq e^{K(b-s)},
\]

or

\[
\log(N) + (d-2) \log(r_1) \geq K(b-s),
\]

which shows that \( b - s \leq 2 \log(N)/K \).

\[\square\]

Proof of Lemma 2:
Proof. The movement of this tagged is a series of independent $d$-dimensional Brownian motions and jumps due to collisions with particles not smaller than a factor of $C = 2$. If in some future time $t$ we have that $(\tilde{x}, \tilde{m}) \notin B(x, \delta_{k+1})$, then if the particle has not traveled a distance of at least $\delta_{k+1} - \delta_k$, its center must have jumped at least that much owing to collisions with larger particles. Let us suppose that over the course of time from $s$ to $t$, $\tilde{m}(s) = m_{i_1}$ collides with particles of size $m_{i_2}, \ldots, m_{i_R}$. Since we are analyzing the case of collisions where the center jumps, by our modeling assumptions on the collision operator (i.e., the form of $H(m_i, m_j)$), we may also assume $m_{i_{k+1}} \geq \frac{1}{C} m_{i_k}$ because of the way we model collisions. In fact, this condition implies something stronger:

$$m_{i_{k+1}} \geq \frac{1}{C} \sum_{j=1}^{k} m_{i_j}$$

which follows easily from the assumption and induction. To bound the distance travelled by $(\tilde{x}, \tilde{m})$ as a result of center of mass jumps, we can bound the jumps by the effective radii of the two particles. For example, if two particle $m_i$ and $m_j$ collide, the center of either could have jumped by at most $r(m_i) + r(m_j)$. In this way, a tagged particle which experiences collisions with particles of mass $m_{i_2} < \ldots < m_{i_R}$ will have jumped at most

$$r(m_1) + r(m_2) + r(m_1 + m_2) + r(m_3) + \ldots + r(m_1 + \ldots + m_{i_{R-1}}) + r(m_{i_R}) = \sum_{k=1}^{R} \left[ r(m_k) + \sum_{j=1}^{k} m_j \right]$$

where $\tilde{m}(t) = m_1 + \ldots + m_{i_R}$. To show that $\epsilon r(\tilde{m}(t)) \geq \delta_{k+1}/4$, we consider the cases that $\chi \geq 1$ and $\chi < 1$ separately. In the former case, we know that

$$r(\sum_{j=1}^{k} m_j) \geq r(\sum_{j=1}^{k-2} m_j) + r(m_{i_{k-1}}) + r(m_{i_k}) \geq (1 + \frac{2}{C}) r(\sum_{j=1}^{k-2} m_j)$$

Assuming without loss of generality that $k$ is even, we can then sum even indices

$$\sum_{k \text{ even}}^{R} r(\sum_{j=1}^{k} m_j) \leq \left[ \sum_{n=1}^{\infty} 1/3^n \right] \left[ \sum_{j=1}^{R} r(\sum_{j=1}^{\sum_{j=1}^{R}} m_j) \right]$$

Together with a similar calculation for the odd indices gives us that

$$\sum_{k=1}^{R} \left[ r(m_k) + r(\sum_{j=1}^{k} m_j) \right] \leq 3 r(\sum_{j=1}^{R} m_j) = 3 r(\tilde{m}(t))$$

For $\chi < 1$, we note that

$$\sum_{k=1}^{R} \left[ r(m_k) + r(\sum_{j=1}^{k} m_j) \right] \leq r(\sum_{k=1}^{R} (R-j)m_{i_k})$$
Since $m_{i+2} \geq 2m_i$, a similar estimate as above gives us that
\[ r\left(\sum_{k=1}^{R}(R-k)m_{ik}\right) \leq \left[\sum_{n=1}^{\infty} \frac{n}{2^n}\right] r\left(\sum_{k=1}^{R} m_{ik}\right) = 2r(\tilde{m}(t)). \]

The alternative conclusion then follows, $C = 2$ for $H(m_i, m_j)$ as we have defined it. \qed

The next lemma uses Lemma 1 to allow us to show that either $M_{p,\delta_k}$ grows very quickly or a mesoscopic particle forms. For the following lemma, assume we are given a sequence \{\(m_k\)\} with $3m_k > m_{k+1} > 2m_k$. Using these $m_k$, set $\tau_1 = 0$ and inductively define
\[ \tau_{k+1} = \min\{\tau_k, \inf\{t > 0 \mid M_{p,x,\delta_{k+1}}(t) \geq m_{k+1}\}\}. \]

We also choose $p$ large enough so that $p > 8\delta^3 m_1^{-1}$.

**Proof of Lemma 3:**

Proof. If $\tau_{k+1} < \tau_M$, then no particle , then no particle beginning in $B(x, \delta_k)$ at $\tau_k$ has left $B(x, \delta_{k+1})$ by $\tau_{k+1}$. Applying the infinitessimal operator to $M_p$, we know from the strong Markov property that
\[ E[M_p(\tau_{k+1})] = E[M_p(\tau_k)] + E[\int_{\tau_k}^{\tau_{k+1}} LM_p(s)ds] \]

The free motion operator applied to $M_p$ is non-trivial: by definition
\[ |(d(m_i) + d(m_j))\Delta Y(x_i)| \leq C\delta^{-2}, \]
where $C \geq 2\max_n d(n)$, so that
\[ \mathbb{E} \left[ \int_s^t A_0 M_p(s)ds \right] \geq \mathbb{E} \left[ -\frac{C}{\delta^2} \int_s^t M_p(s)ds \right] \]
for any $s, t$. Therefore for the purposes of our analysis, we must show that the dominant term comes from the coagulation term in the infinitesimal operator. We compute that $A_c M_p(s) = \sum_{i \neq j} \alpha(m_i, m_j) V_{m_i, m_j}(x_i - x_j)$
\[ ((H(m_i, m_j)Y(x_i) + H(m_j, m_i)Y(x_j))(m_i + m_j)^p - m_i^p Y(x_i) - m_j^p Y(x_j)) \]
Since the function $Y$ decays slowly and $(x_i, m_i)$ and $(x_j, m_j)$ are colliding,
\[ \frac{1}{2} Y(x_j) \leq Y(x_i) \leq 2Y(x_j). \]

This means that
\[ A_c M_p(s) \geq \frac{1}{N} \sum_{i \neq j} \alpha(m_i, m_j) V_{m_i, m_j}(x_i - x_j) \left( \frac{p}{2} m_i^{p-\epsilon} m_j^{1+\epsilon} Y(x_i) \right). \]
Written in this form, with
\[ J(x_i, x_j, m_i, m_j) = \frac{p}{2} m_{i}^{p-1-c} m_{j}^{1+c} Y(x_i) \]
we may apply the sub-Stosszahlensatz lemma. For some \( b \leq \tau_{k+1} + \log(N)/K \), we have that
\[
\mathbb{E} \left[ \int_{\tau_k}^{b} \mathbb{A}_c M_p(s) ds \right] 
\geq 
\frac{1}{N^2} \int_{\tau_k}^{b} \sum_{i \neq j} \sum_{l} C_l(d(m_i) + d(m_j)) \text{Avg}_{z \in B(0,r_i)} - \Delta u_{m_i,m_j}^\varepsilon(x_i - x_j + z) J(x_i, x_j, m_i, m_j) \]
\[
\geq 
\frac{1}{N^2} \sum_{i \neq j} p/2 (r(m_i) + r(m_j))^{d-2} m_{i}^{1+c} m_{j}^{p-1-c} Y(x_i) \]
where the second inequality comes from the fact that \( r_1 = \delta_{k+1} \), and that the integral of \(-\Delta u_{m_i,m_j}^\varepsilon(x_i - x_j + z)\) over the ball \( B(0, r_1) \) is \((r(m_i) + r(m_j))^{d-2}\) assuming that they both lie within the ball. We can split the double sum into a product of sums: if \( q = \chi(d-2) \),
\[
\frac{1}{N^2} \sum_{i \neq j} p/2 m_{i}^{p-1+q} Y(x_i) Y(x_j) = \frac{1}{N^2} \sum_{i} m_{i}^{Y(x_i)} \frac{1}{N} \sum_{j} m_{j}^{p-1+b} Y(x_j) \]
\[
- \frac{1}{N^2} \sum_{i} m_{i}^{p+b} Y(x_i)^2 \]
with the latter term negligible in comparison to \( M_p(s) \) since \( \frac{1}{N} \sum_i m_i 1(m_i \geq R) \leq 1/2 \). Therefore we may conclude that
\[
\mathbb{E} \left[ \int_{\tau_k}^{b} \mathbb{A}_c M_p(s) ds \right] \geq \mathbb{E} \left[ \int_{\tau_k}^{b} \frac{p}{4\delta^d} M_{1,\delta_{k+1}}(s) M_{p+q-1,\delta_{k+1}}(s) ds \right] .
\]
By Holder’s inequality,
\[ M_{p+q-1,\delta_{k+1}} \geq M_{p,\delta_{k+1}}^{1+\beta} M_{1,\delta_{k+1}}^{1-\beta}, \]
where \( \beta = \frac{q}{p-1}, q = \chi(d-2) > 1 \). Now using the fact that for \( t < \min \{ \tau_M, \tau_D \} \), \( M_{p,\delta_{k+1}}(t) \geq M_{p,\delta_{k}^c}(\tau_k) \), and also that \( M_1(t) \geq C\delta_1^d \),
\[
\mathbb{E} \left[ \int_{\tau_k}^{b} \mathbb{A}_c M_p(s) ds \right] \geq \mathbb{E} \left[ \int_{\tau_k}^{b} \frac{p}{4} M_{1,\delta_{k+1}}^{1+\beta} M_{p,\delta_{k+1}}^{1-\beta} \right] ds \]
Since \( p = 8\delta^{-2} m_{1}^{-1} \), the free motion is negligible, and combining our equations we get that
\[
\mathbb{E} [\tau_{k+1} - \tau_k] \leq \frac{4\delta^{-2} (1-\beta)(m_{k+1} - m_k)}{m_k^{1+\beta} p} + \log(N)/K
\]
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Proof of Lemma 4:

Proof. For some $R$ big enough that $\epsilon r(R) = \delta/4$, we would like to show that the many small particles in the surrounding area quickly collide with the gel. Therefore one quantity of interest will be

$$M_{x,\delta,R}(t) = \sum_{m_i \geq R} m_i^3 Y_{\delta,x}(x_i).$$

As particles collide with the gel, a lower bound for the action $A_c$ on such a functional will then be given by summing

$$J(x_i, y, m_i, m_j) = m_i^{3/2} m_j^{3/2} \mathbb{1}(m_j \geq R) Y_{x,\delta}(x_j) Y_{x,\delta}(x_i).$$

We will repeat with minor modifications the same argument as the sub-Stossahlensatz 1 to show that in a very short time we expect one particle to grow to a macroscopic size – in other words, for some particle with mass $m_i$ such that $r(m_i)\epsilon > 1$. Since $\chi > 1/(d-2)$, note that this is true if $m_i = \frac{N}{\log(N) c}$ for any $c$. Just like before, we begin by defining a functional

$$X(t) = \sum_{i,j} \left\{ \frac{u^{\epsilon}_{m_i,m_j}(x_i - x_j + z)}{u^{\epsilon}_{m_i,m_j}(x_i - x_j)} \right\} J(x_i, x_j, m_i, m_j)$$

where $-\Delta u_{m,n} = V_{m,n}(1 - u_{m,n})$ and $u^{\epsilon}_{m,n}(y) = \epsilon^{2-d} u_{m,n}(y/\epsilon)$. The form of $J$ however now requires a different argument. In this case, the first mass variable is also strictly superlinear in the sense that

$$J(x, y, m + k, n) > J(x, y, m, n) + J(x, y, k, n).$$

We must show that when $(x_i, m_i)$ and $(x_j, m_j)$, $z \in B(0, 2\delta)$,

$$u^{\epsilon}_{m_i,m_k}(x_i - x_k + z) J(x_i, x_j, m_i, m_k + m_j)$$

$$\geq u^{\epsilon}_{m_i,m_k}(x_i - x_k + z) J(x_i, x_j, m_k, m_i + m_j) + u^{\epsilon}_{m_j,m_k}(x_j - x_k + z) J(x_i, x_j, m_k, m_j)$$

Note that the very last term is 0 unless $m_j \geq R$ as well. In other words, this would require two gels in the same space. But by definition, if $H(m_i, m_j) = 1(m_i \geq C m_j)$, and both $m_i, m_j \geq R$, then $m_i \geq C m_j$. In this case, our conclusion follows if

$$u^{\epsilon}_{m_i,m_k}(x_i - x_k + z) J(x_i, x_j, m_k, m_i) \geq C_2 u^{\epsilon}_{m_j,m_k}(x_j - x_k + z) J(x_i, x_j, m_k, m_j)$$

, where $C_2 \leq (1 + C)^{3/2} - 1$. Because the two particles $(x_i, m_i)$ and $(x_j, m_j)$ are colliding and $m_i \geq C m_j \geq R$, $|z| \leq 2\delta$, we see by Lemma 2.1 that

$$u^{\epsilon}_{m_i,m_k}(x_i - x_k + z) \geq C_3 u^{\epsilon}_{m_j,m_k}(x_j - x_k + z).$$

The other part of the argument that must be modified is handling the error from $X(t)$ itself. In the proposition, we averaged pairs of particles over several scales. For this situation, since
one of the particles is already assumed to be very large, it is sufficient to average over just
one scale. In other words, we define
\[ R(t, x_i - x_j) = V_{\epsilon, m_i, m_j}(x_i - x_j) + Avg_{\mathcal{B}(0, r)}(d(m_i) + d(m_j)) u_{m_i, m_j}^\epsilon(x_i - x_j + z) \]
and
\[ E(t, x_i - x_j) = Avg_{\mathcal{B}(0, r)} u_{m_i, m_j}^\epsilon(x_i - x_j + z) \]
Again, we use the crude bound for \( E \):
\[ E(t, x_i - x_j) \leq \epsilon^{2-d} \]
which comes from a crude bound on \( u_{m_i, m_j} \leq 1 \) and scaling. If \((x_i, m_i)\) and \((x_j, m_j)\) are not
colliding, then the average in the term for \( R \) has a lower bound of
\[ \delta - d(r(m_i) + r(m_j)) \geq \delta - d(r(m_i)) \geq K \epsilon^{2-d}, \]
the second inequality coming from the fact that \( r(\mathcal{R}) \geq \delta N^{(d-2)x} \). Following the argument
in Lemma 1, if \( q = \chi(d-2) \),
\[ \frac{1}{N^2} \sum_{i \neq j} p \frac{m_i^{1+c} m_j^{p+q+c}}{2} 1(m_j \geq R) Y_{2\delta}(x_i) Y_{2\delta}(x_j) \geq M_{3, R, 2\delta}(t)^{1+c} M_{1, 2\delta}(t) \]
so long as
\[ \frac{1}{N} \sum_{i \in I(q)} m_i 1(m_i \geq R) \leq 1/2 \]
where
\[ M_{3, R, 2\delta}(t) = \frac{1}{N} \sum_{i \in I(q)} m_i^3 1(m_i \geq R) Y_{2\delta}(x_i). \]
As a result for \( t \leq \tau_D \), we have the bound
\[ A_{\epsilon} M_{3, 2\delta}(t) \geq R^\alpha M_{3, 2\delta}(t), \]
and since the free motion operator is negligible as before, there is a \( b \leq \log(N)/K \) such that
\[ \mathbb{E} \left[ \int_{\tau_R}^b A_{\epsilon} M_{x, \delta, R} \right] \geq \mathbb{E} \left[ \int_{\tau_R}^b 1/N \sum_{i \neq j} r(m_j)^{d-2} m_i^3 1(m_j \geq R) \right] \]
Since \( N^2 \geq M_{3, 2\delta}(\tau_M) \geq N^t \) for some \( t > 0 \), we may repeat our procedure with concentric
spheres a constant number of times (not dependent on \( N \), each taking an expected \( \log(N)/K \)
to achieve complete gelation.
Chapter 3

Instantaneous Gelation of Smoluchowski PDE

3.1 Existence and Previous Results

As discussed in Chapter 1, the Smoluchowski PDE is a coupled system of equations modeling the coagulation of discretely sized clusters.

\[
\frac{\partial}{\partial t} f_n(x,t) = d(n) \Delta f_n(x,t) + Q_n(f)(x,t)
\]  

(3.1)

where \( Q_n(f)(x,t) = Q_n^+(f)(x,t) - Q_n^-(f)(x,t) \),

\[
Q_n^+(f)(x,t) = \frac{1}{2} \sum_{m=1}^{n-1} \alpha(m,n-m) f_m(x,t) f_{n-m}(x,t)
\]  

(3.2)

and

\[
Q_n^-(f)(x,t) = f_n(x,t) \sum_{m=1}^{\infty} \alpha(m,n) f_m(x,t).
\]  

(3.3)

We interpret the solution in the weak sense, i.e. \( Q_n \in L^1(\mathbb{R}^d \times [0,T]) \) for a given \( T > 0 \):

\[
f_n(x,t) = S_t^{d(n)} f_n(x,0) + \int_0^t S_{t-s}^{d(n)} Q_n(f)(x,s) \, ds
\]  

(3.4)

We define the mass or \( M_1 \)-moment at a point to be

\[
M(x,t) = \sum_{n=1}^{\infty} n f_n(x,t)
\]

and the global mass

\[
M(t) = \int_{\mathbb{R}^d} \sum_{n=1}^{\infty} n f_n(x,t) \, dx.
\]
In general, it is true that $M(t) \leq M(0)$. To see this, we start by grouping the like terms from $Q_n$:

$$
\sum_{n=1}^{k} nQ_n(x,t) = \frac{1}{2} \sum_{m,n} [(m+n)1(m+n \leq k) - m1(m \leq k) - n1(n \leq k)] \alpha(m,n)f_m(x,t)f_n(x,t).
$$

Defining $M_k(t) = \int_{\mathbb{R}^d} \sum_{n=1}^{k} nf_n(x,t)dx$,

$$
\frac{\partial}{\partial t} M_k(t) = \int_{\mathbb{R}^d} \sum_{n=1}^{k} nd(n)\Delta f_n(x,t)dx + \int_{\mathbb{R}^d} \sum_{n=1}^{k} nQ_n(x,t)dx.
$$

As $f_n \in H^2(\mathbb{R}^d \times [0,T])$, we can test it against a bump function which is equal to a constant over a large radius to show that the first integral is zero. Therefore, to show $M(t) \leq M(0)$, it suffices to show that $\sum_{n=1}^{k} nQ_n(x,t) \leq 0$ for each $k$. This easily follows from the fact that

$$
1(m+n \leq k) \leq \min\{1(n \leq k), 1(m \leq k)\}.
$$

For this chapter, we are interested in the phenomenon of gelation, namely whether

$$
\tau_g = \inf \{t > 0 \mid M(t) < M(0)\}
$$

is finite. (Note that although we are using the same notation as in Chapter 2, this is not a probabilistic stopping time since this is a deterministic PDE.)

**Remark.** For $t < \tau_g$, it must be that

$$
\sum_{n=1}^{\infty} nQ_n = 0.
$$

This is a consequence of the fact that $\sum_{n=1}^{\infty} nQ_n$ is non-positive, so if it were strictly negative at any point, the integral over all of $\mathbb{R}^d$ would be, too. For our model we will make some assumptions:

- The initial distribution of particles $\{f_n(x,0)\}$ are smooth functions in $C_c^\infty$ and their support is contained in the unit ball in $\mathbb{R}^d$.

- The diffusion coefficients $d(n)$ are uniformly bounded by $\bar{d}$.

- The initial mass $M(0)$ is finite.

**Lemma 6.** Suppose $\alpha(m,n) \geq K(mn)^c$ for some constant $K$ and $c > 1/2$. Then for any $\theta \geq 1/2$ and $k$ even, there is some $C_1(\theta,c) = C_1 > 0$ not dependent on $k$ such that

$$
\frac{\partial}{\partial t} \int_{\mathbb{R}^d} \sum_{j=1}^{k} n^\theta f_n(x,t) \leq -C_1 \int_{\mathbb{R}^d} \left( \sum_{j=1}^{k/2} n f_n(x,t) \right)^2 dx
$$

(3.5)
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Remark. The idea behind this is that for $0 < \theta < 1$, the $\theta$-moments

$$M_\theta(t) = \int_{\mathbb{R}^d} M_\theta(x,t) dx = \int_{\mathbb{R}^d} \sum_{k=1}^\infty k^\theta f_k(x,t) dx$$

decrease after collisions precisely because $(a + b)^\theta \leq a^\theta + b^\theta$ for $a, b > 0$.

Proof. First, we note that

$$\int_{\mathbb{R}^d} d(n) \Delta_x f_n(x,t) = 0.$$

Now, by summing and grouping terms from $Q_n$ together, we see that

$$\sum_{n=1}^k n^\theta Q_n = \sum_{m,n} [m^\theta 1(m \leq k) + n^\theta 1(n \leq k) - (m + n)^\theta 1(m + n \leq k)] \alpha(m,n) f_m(x,t) f_n(x,t)$$

(3.6)

Since

$$\min \{1(m \leq k), 1(n \leq k)\} \geq 1(m + n \leq k) \geq 1(m, n \leq k/2)$$

and

$$(m + n)^\theta - m^\theta - n^\theta \leq (2 - 2^\theta) \min \{n^\theta, m^\theta\},$$

we have that 3.6 is bounded from below by

$$\sum_{m,n=1}^{k/2} K(2 - 2^\theta) \min(m,n)^\theta m^\epsilon n^\epsilon f_m(x,t) f_n(x,t),$$

(3.7)

where $\alpha(m,n) \geq K(mn)^\epsilon$. Given that for all $y \in \mathbb{R}$,

$$\sum_{n=2}^k n^y \left( \frac{1}{1+y} \right) k^{1+y}$$

(3.8)

we see that a lower bound for 3.7 is

$$\geq \sum_{m,n=1}^{k/2} K(2 - 2^\theta) \theta \left( \sum_{l=2}^{\min(m,n)} l^{\theta-1} \right) m^\epsilon n^\epsilon f_m(x,t) f_n(x,t)$$

$$= K(2 - 2^\theta) \theta \sum_{l=2}^{k/2} l^{\theta-1} \left( \sum_{n=l}^{k/2} n^\epsilon f_n(x,t) \right)^2$$

$$= K(2 - 2^\theta) \theta \sum_{l=2}^{k/2} l^{\theta-1-r} \left( l^{r/2} \sum_{n=l}^{k/2} n^\epsilon f_n(x,t) \right)^2$$
Now by Hölder’s inequality,
\[
\left( \sum_{k=2}^{k/2} \frac{t^{\theta/2 - 1/2 - r/2}}{f_n(x, t)} \right)^{2/\theta - 1/2} \left( \frac{t^{r/2}}{\sum_{n=1}^{k/2} n^c f_n(x, t)} \right)^{1/2} \leq \left( \sum_{l=2}^{k/2} t^{\theta - 1 - r} \right)^{1} \left( \sum_{l=2}^{k/2} t^{r/2} \frac{t^{r/2}}{\sum_{n=2}^{k/2} n^c f_n(x, t)} \right)^{2}
\]
So we have the lower bound
\[
\geq K(2 - 2^\theta) \theta \left( \sum_{l=2}^{k/2} t^{\theta - 1 - r} \right)^{-1} \left( \sum_{l=2}^{k/2} t^{r/2} \frac{t^{r/2}}{\sum_{n=2}^{k/2} n^c f_n(x, t)} \right)^{2} \quad (3.9)
\]
Since \( \theta - 1 - r < 0 \), we have by 3.8 that \( \left( \sum_{l=2}^{k/2} t^{\theta - 1 - r} \right)^{-1} \geq (\theta - r) \). Moreover, we can apply 3.8 inequality again to the squared term to find that 3.9 has the lower bound
\[
C(r, \theta) \sum_{n=2}^{k/2} n^c f_n(x, t)
\]
with
\[
C(r, \theta) = C(\theta) = K \theta (\theta - r)(\theta - r/2)^{-1}
\]
for \( r = 2(c + 1 - \theta) \). To find the constant \( C_1 \) in the statement of the Lemma, we simply set \( C_1 = 2C(\theta) \), since \(-Q_1(x, t) \geq f_1^2(x, t) \) and
\[
\left( \sum_{n=1}^{k/2} n f_n(x, t) \right)^{2} \leq 2f_1(x, t)^2 + \left( \sum_{n=2}^{k/2} n f_n(x, t) \right)^{2}.
\]
Lemma 7. If \( d(n) \leq \bar{d}, \delta > 0 \) and \( t < \tau_g \), then there is a \( C \) which depends only on the dimension \( d \) and \( \delta \) such that
\[
\int_{B(x, \delta)} M(y, t) dy \geq \exp \left( -C(\bar{d}, \delta) t \right) \int_{B(x, \delta/2)} M(y, 0) dy.
\]
Proof. Let \( Y \) be a bump function with \( \int Y = 1 \), support contained in \( B(0, \delta/2) \) and \( |\Delta Y| < C(\delta) \). Define the modified mass
\[
\tilde{M}(x, t) = \sum_{n=1}^{\infty} n f_n \ast Y(x, t).
\]
Note that
\[
\frac{\partial}{\partial t} \tilde{M}(x, t) = \sum_{n=1}^{\infty} d(n) \Delta f_n \ast Y(x, t) + \sum_{n=1}^{\infty} n Q_n \ast Y(x, t)
\]
Since $t < \tau_g$, the latter term is zero everywhere. By construction, $\Delta Y(x) > -C(\delta)$, so by integration by parts, and $\widetilde{d} \geq d(n)$,

$$\frac{\partial}{\partial t} \int_{B(0,\delta)} \tilde{M}(x,t)dx > -C(\delta)\widetilde{d} \int_{B(0,\delta)} \tilde{M}(x,t)dx.$$  

By Grönwall’s inequality and Young’s convolution inequality, we have

$$\int_{B(0,\delta)} M(x,t) \geq \exp \left( -C(\widetilde{d},\delta) t \right) \int_{B(x,\delta/2)} \tilde{M}(y,0)dy.$$  

**Theorem 9.** Given the hypotheses above, there exists some initial mass threshold $C(d)$ such that, if $M(0) \geq C(d)$, $\tau_g < K(\widetilde{d},d)$.

**Proof.** Let

$$M_{\theta,k}(x,t) = \int_{\mathbb{R}^d} \sum_{n=1}^{k} n^\theta f_n(x,t)dx.$$  

Denote by $Y(x)$ a bump function whose support is contained in $B(0,2)$ and which is equal to the constant 1 on the ball $B(0,1)$. We may assume that

$$||\Delta Y(x)||_{\infty} \leq C$$

for some $C = C(d)$ dependent on the dimension. Taking the product of $M_{\theta,k}$ and $Y$ and differentiating in the time variable, we have,

$$\frac{\partial}{\partial t} (M_{1,k}(x,t)Y(x)) = \sum_{n=1}^{k} Y(x)d(n)n^\theta \Delta f_n(x,t) + \sum_{n=1}^{k} nQ_n(x,t)Y(x).$$

By Lemma 6, we see that for each $k$, and $t < \tau_g$,

$$\int_{\mathbb{R}^d} M_{\theta,k}(x,t)Y(x)dx = \int_{\mathbb{R}^d} M_{\theta,k}(x,0)Y(x)dx + \int_{0}^{t} \int_{\mathbb{R}^d} \sum_{n=1}^{k} Y(x)d(n)n^\theta \Delta f_n(x,t)dxds \quad (3.10)$$

$$+ \int_{0}^{t} \int_{\mathbb{R}^d} \sum_{n=1}^{k} n^\theta Q_n(x,t)Y(x)dxds$$

Using integration by parts and the bound on $\Delta Y$ we see that

$$\int_{\mathbb{R}^d} \sum_{n=1}^{k} Y(x)d(n)n^\theta \Delta f_n(x,t)dx = \int_{\mathbb{R}^d} \Delta Y(x) \left( \sum_{n=1}^{k} d(n)n^\theta f_n(x,t) \right) dx \geq -C\widetilde{d} \int_{\mathbb{R}^d} M_{k,\theta}(x,t)dx \quad (3.11)$$
Using 3.11 on 3.10, the fact that \( M \geq M_{k,\theta} \geq 0 \), re-arranging terms, and finally Lemma 6, we get

\[
\int_0^t \int_{B(0,1)} (C_1 M(x,t)^2 - C\bar{d}M(x,t)) \, dx \, dt \leq \int_{\mathbb{R}^d} M(x,0) \, dx \tag{3.12}
\]

Define \( M_S(t) = \int_{B(0,2)} M(x,t) \), and letting \( C(d) \) be the volume of the \( d \)-dimensional sphere, by Hölder’s inequality, for \( t < \tau_g \),

\[
\int_0^t \left( C\bar{d}M_S(s) + \frac{C_1}{2^d C(d)} M_S(s)^2 \right) \, ds \leq M_S(0).
\]

Note by 7, since the the support of \( M(x,0) \) is contained in the unit ball,

\[
M_S(t) \geq M_S(0) \exp (-Ct)
\]

Using this bound, we compute that there is a mass threshold \( C \) which depends on \( \bar{d} \) and dimension \( d \) such that if \( M(0) \geq C \), \( \tau_g \leq K(\bar{d},d) \).

The next thing to prove is that this mass threshold cannot be completely eliminated as a requirement. We will use a theorem about the blowup time for the nonlinear heat equation.

**Theorem 10.** (Fujita, 1966). There exists some \( \delta > 0 \) such that if \( a(x) \in C^2(\mathbb{R}^d) \) for \( d \geq 3 \) satisfies

\[
0 \leq a(x) \leq H(\gamma, x),
\]

where

\[
H(t, x) = \frac{1}{(4\pi t)^{d/2}} \exp \left(-|x|^2/4t\right)
\]

is the standard \( d \)-dimensional heat kernel, then there is a global, continuous infinite time solution to the non-linear heat equation

\[
u_t = \Delta u + u^2 \tag{3.13}
\]

with \( u(x, 0) = a(x) \) satisfying \( u(x,t) \leq KH(t + \gamma, x) \) for all \( t > 0 \).

Using this theorem as an ingredient, we may demonstrate the following:

**Theorem 11.** Let \( \alpha(m,n) = mn \) and \( d(n) = d \) for all \( n \) and assume \( M(0,t) \leq \delta H(0,x) \) with \( \delta \) and \( H \) from Theorem 10. Then \( M(t) = M(0) \) for all \( t > 0 \).

**Proof.** Set

\[
M_2(x,t) = \sum_{k=1}^{\infty} n^2 f_n(x,t).
\]
Taking the derivative in the time variable, we see that,
\[
\frac{\partial}{\partial t} M_2(x,t) = \sum_{k=1}^{\infty} n^2 d(n) f_n(x,t) + \sum_{k=1}^{\infty} n^2 Q_n(x,t)
\]
\[
= d\Delta \sum_{k=1}^{\infty} n^2 f_n(x,t) + 2 \sum_{n,m} (nm)^2 f_n(x,t) f_m(x,t)
\]
\[
= d\Delta M_2(x,t) + 2M_2(x,t)^2
\]
where the second equality comes from 3.6 with \(\theta = 2\), and the third comes from factoring the sum of products into a product of sums. If we scale \(M_2\):
\[
\tilde{M}_2(x,t) = M_2\left(\frac{x}{2d}, \frac{t}{2}\right),
\]
then \(\tilde{M}\) solves 3.13. Defining \(M_{>k}(x,t) = \sum_{n>k}^{\infty} n f_n(x,t)\), it is clear that
\[
\frac{1}{k} M_{>k}(x,t) \leq M_2(x,t).
\]
Since Theorem 10 shows \(M_2(t)\) is uniformly bounded in \(L_1\), this proves
\[
\lim_{k \to \infty} \sum_{n=k}^{\infty} \int_{\mathbb{R}^d} n f_n(x,t) dx = 0
\]
for all \(t > 0\), which is what we wanted to show.

We now have the tools to show that the story for coagulation kernels with \(\alpha(m,n) \geq K(m^b + n^b)\), \(b > 1\) We are now able to adapt a lemma from Carr and da Costa [CC92]:

**Lemma 8.** (Finite p-moment property) Suppose \(\alpha(m,n) \geq K(m^b + n^b)\) where \(b > 1\), and assume \(x\) is a point of positive initial density: \(M(x,0) > 0\). If \(\{f_n\}\) is a mass-conserving solution of the Smoluchowski PDE 3.1, and \(t < \min \{\tau_g, 1\}\), then there is a \(\delta\) such that a neighborhood around \(x\) of radius \(\delta\) has finite p-moment.

**Proof.** Let \(Y(x,t)\) be a bump function (constant in time) as before with \(\int Y = 1\) and support contained in \(B(0,\delta)\). Define the mass *above* \(k\)
\[
\tilde{M}_{>k}(x,t) = \sum_{n=k}^{\infty} X f_n(x,t)
\]

By differentiating in time, we have that
\[
\frac{\partial}{\partial t} \tilde{M}_{>k}(x,t) \geq \sum_{m=1}^{k-1} \sum_{n=k}^{\infty} K(m+n)(n^b + m^b) f_m(x,t) f_n Y(x,t)
\]
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\[ \geq k^{b-1} \left( \sum_{n=1}^{k-1} n f_n(x,t) \right) \left( \sum_{m=k}^{\infty} m f_m * Y(x,t) \right). \]

For any \( 0 < a < c < t \) we have by a convolution inequality

\[ \int_{B(x,\delta)} M_{\geq k}(x,c) dx - \int_{B(x,\delta)} M_{\geq k}(x,a) dx \geq C k^{b-1} \left( \inf_{r \in [a,c]} \int_{B(x,\delta)} M(x,r) dx \right) \int_{a}^{c} \int_{B(x,\delta)} M_{\geq k}(x,s) dxds. \]

In our previous Lemma 7, we showed that if \( c < 1 \),

\[ \int_{B(x,\delta)} M(x,c) dx \geq C(d,x,\delta). \] (3.14)

We may conclude by Gronwall’s inequality that for sufficiently large \( k \), \( \bar{M}_k(t) = \int_{B(x,\delta)} M_{\geq k}(x,t) dx \),

\[ \bar{M}_k(t) \leq C_1 \exp(-C_2 k^{b-1} t). \] (3.15)

Since \( m f_m(x,t) \leq \sum_{n=m}^{\infty} n f_n(x,t) \),

\[ \int_{B(x,\delta)} \sum_{n=k}^{\infty} n^p f_n(x,t) dx \leq \sum_{n=k}^{\infty} n^{p-1} \int_{B(x,\delta)} M_{\geq k}(x,t) dx \]

we may sum using 3.15 to show that the \( p \)-moments are finite. \( \square \)

To prove instantaneous gelation, and therefore to show no positive time solution to 3.1 exists, it therefore suffices to show if one did exist, for any \( 1 > t > 0 \) a \( p \)-moment is infinite.

**Theorem 12.** Suppose \( M(x,0) \) is smooth and compactly supported, \( d(n) \leq \bar{d} \), and \( \alpha(m,n) \geq K(m^{b} + n^{b}) \) for \( b > 1 \). Then no positive time solution to 3.1 exists.

**Proof.** Given some \( 1 > t > 0 \), fix a small \( \delta > 0 \), \( Y \) a bump function as before, and set \( \bar{M}_p(s) = \int_{B(x,\delta)} \sum_{n=1}^{\infty} n^p f_n * Y(x,s) dx \). By 8 and 3.11,

\[ \frac{d}{dt} \bar{M}_p(s) \geq \int_{B(x,\delta)} \sum_{n=1}^{\infty} p(m^{p-1} n^{p-1} m^{b} + n^{b+1}) f_m(x,s) f_n * Y(x,s) dx - \int_{B(x,\delta)} \bar{d} \delta^{-2} M_{\geq k}(x,s) dx \]

Choosing \( p \) and \( \delta \) so that \( p > 2 \delta^{-2} \bar{d} C(d,\delta) \), where \( C(d,\delta) \) is given by 3.14,

\[ \geq \int_{B(x,\delta)} \frac{p}{2} \left( \sum_{n=1}^{\infty} m^{p-1+b} f_m(x,s) \right) \left( \sum_{n=1}^{\infty} n f_n * Y(x,s) \right) dx. \]

By Hölder’s inequality,

\[ \sum_{n=1}^{\infty} m^{p-1+b} f_m(x,s) \geq \left( \sum_{m=1}^{\infty} m^p f_m(x,s) \right)^{1+\gamma} \left( \sum_{n=1}^{\infty} n f_n * Y(x,s) \right)^{-\gamma} \]
with $\gamma = (b - 1)/(p - 1)$. Therefore,

$$\frac{d}{dt} \bar{M}_p(s) \geq pC(d, \delta)^{1-\gamma} \int_{B(x, \delta)} \left( \sum_{m=1}^{\infty} m^p f_m(x, s) \right)^{1+\gamma} dx$$

$$\geq (pC(d)\delta^{1-\gamma-d}) \bar{M}_p(s)^{1+\gamma}$$

the second inequality coming from Jensen’s inequality. We may therefore choose a $t_1 < t$ such that $1 \leq pC(d)\delta^{1-\gamma-d}\bar{M}_p(t_1) < \infty$, we could choose an infinite sequence $t_1 < t_2 < \cdots < t$ such that $t_{k+1} - t_k \leq \frac{C}{pk^2}$ and

$$\bar{M}_p(t_{k+1}) \geq k^{2-\gamma}.$$ 

This shows that $\bar{M}_p$ has blowup time $t < \frac{C}{p}$. By picking $p$ sufficiently large, we have proven there is a moment with blowup time smaller than any arbitrarily small $t > 0$. \qed
Bibliography


