Title
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NOTE ON COORDINATE TRANSFORMATIONS AND SOLID ANGLE JACOBIAN FOR CHARGED PARTICLES IN A MAGNETIC FIELD

Walter H. Barkas

March 10, 1953

Berkeley, California
Memorandum

To: Recipients of UCRL-2126
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From: Information Division

Subject: Errata to UCRL-2126

Will you please make the following changes in your copy of UCRL-2126.

Page 2. Second paragraph, line 7. Should read "The x axis is then"

Page 2. Second paragraph, line 12. Should read "The element of solid angle in the direction $\theta_1$, $\phi_1$ is: $d\mu_1, d\phi_1$, where $\mu_1 = \cos \theta_1$".

Page 2. Third paragraph, line 3. Should read "$p = p/qH$, where $p$ is the$".

Page 3. Equation 1a. Should read, $x_2 = 2p\sqrt{1 - \mu_1^2} \sin z_2/2p \mu_1 \cos(\phi_1 - z_2/2p \mu_1)$

Page 3. Equation 1b. Should read, $y_2 = 2p\sqrt{1 - \mu_1^2} \sin z_2/2p \mu_1 \sin(\phi_1 - z_2/2p \mu_1)$

Page 3. Equation 2a. Should read, $\rho = \rho$ instead of $p = \rho$.

Page 4. Equation 3. $J = \int \left( \frac{x_2, y_2, \beta}{\rho, \mu_1, \phi_1} \right)$

Page 5. Equation 9. Should read, $|J| = 2 \left[ x_2^2 (\pi/2 - \beta)^2 + z_2^2 \cos^2 \beta \right] / |z_2|$
NOTE ON COORDINATE TRANSFORMATIONS AND SOLID ANGLE JACOBIAN FOR CHARGED PARTICLES IN A MAGNETIC FIELD

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March 10, 1953

The path of a charged particle in a uniform magnetic field is well known to be a helix. The geometry of this type of orbit is sufficiently complicated that coordinate and solid angle calculations for bundles of particles leaving a source in a magnetic field are troublesome, and errors are likely to be made. In this memorandum the solutions of some of the common geometrical problems of this type are given. The equations are limited in application to magnetic fields which are nearly uniform. For non-uniform fields the analytic solid angle and coordinate transformations become too complicated to be useful.

Choice of Coordinate Frame

A coordinate system with its origin at the source of particles or target is indicated. In all problems there is one principal direction, namely the direction of the magnetic field vector. In a cyclotron the direction of the beam is also important, for one expects symmetry about this axis for the emission of reaction products. The importance of the magnetic field direction is, however, overriding, and it is chosen for the z axis. In a cyclotron, the beam is directed along the y axis. The x axis is then determined for a right-handed system of coordinates. In the cyclotron it lies in the median plane extending from the target toward the cyclotron center. The initial direction cosines (l, m, n) of an ion path starting in a direction defined by spherical coordinates \( \theta_1 \) and \( \phi_1 \) then are: 

\[
\ell = \sin \theta_1 \cos \phi_1, \quad m = \sin \theta_1 \sin \phi_1, \quad n = \cos \theta_1.
\]

The element of solid angle in the direction \( \theta_1, \phi_1 \) is \( \sin \theta_1 \, d\mu_1, \, d\phi_1 \), where \( \mu_1 = \cos \theta_1 \). The definitions are illustrated in Figure 1.

The Particle Momentum and The Orbit Equations

An ion moving in a magnetic field will traverse a helical path with constant speed. The projection of its path on the xy plane will be a circle of radius 

\[
\rho \sqrt{1 - \mu_1^2}.
\]

\( \rho = p/qH_1 \) where \( p \) is the momentum of the ion, \( q \) its charge in emu, and \( H \) the magnetic field intensity in gauss. The three quantities \( p, \mu_1 \) and \( \phi_1 \) at the origin suffice to describe completely the ion path. Using the definitions above the path is given by:
The direction of the particle path at any point is defined by a constant angle, arc cos $\mu_1$, between the orbit and the z direction, and by the angle $\beta = -\arccot(dy/dx)_z$. $\beta$ is also given by:

$$\beta = \phi - \frac{z}{\rho \mu_1} + \frac{\pi}{2}$$  (1c)

The point with subscript 2 is any point on the orbit; in particular the point at which it enters the emulsion.

**Inverse Coordinate Transformation**

Using nuclear track plates one is able to observe the point of arrival of a charged particle, and he can also measure its direction in the plane of the emulsion. It is less feasible to measure dip angles in the emulsion. One is therefore led to detect particles by exposing plates with the z axis normal to the plane of the film and with the particles entering through the free surface of the emulsion. Then $x_2$, $y_2$, and $z_2$ are obtained from the position of the plate and the position of the track on the plate. $\beta$ (which is the angle the track makes with the negative direction of the y axis measured counter-clockwise in the xy plane) is also normally measured in the microscope. We wish to use the measured quantities $x_2$, $y_2$, and $\beta$ (with $z_2$ a constant) to determine $\rho$, $\mu_1$ and $\phi_1$. This is accomplished by the transformations inverse to Eqs. (1), namely:

$$p = \sqrt{\frac{\left(\frac{x_2^2}{\omega^2} + \frac{y_2^2}{\omega^2}\right)\omega^2 + z_2^2 \sin^2 \omega}{2 \omega \sin \omega}}$$  (2a)

$$\mu_1 = \frac{z_2 \sin \omega}{\sqrt{\left(\frac{x_2^2}{\omega^2} + \frac{y_2^2}{\omega^2}\right)\omega^2 + z_2^2 \sin^2 \omega}}$$  (2b)

$$\phi_1 = \omega + \arctan\frac{y_2}{x_2}$$  (2c)

where $\omega = \frac{\pi}{2} - \beta + \arctan\frac{y_2}{x_2}$

$$= \frac{z_2}{\rho \mu_1} - \phi_1 + \arctan\frac{y_2}{x_2}$$
Transformation Jacobian

One scans an element of area $\Delta A$ between $x_2$ and $x_2 + \Delta x_2$ and $y_2$ and $y_2 + \Delta y$. In this area he finds a number of tracks, $\Delta N$, with measured angles lying between $\beta$ and $\beta + \Delta \beta$. These data provide information on the number of particles leaving the origin between $\rho$ and $\rho + \Delta \rho$, $\mu_1$ and $\mu_1 + \Delta \mu_1$; and $\phi_1$ and $\phi_1 + \Delta \phi_1$. We wish to calculate the number of particles leaving the target with coordinates $\rho$, $\mu_1$ and $\phi_1$ per unit solid angle, per unit radius of curvature interval. The transformation is:

$$\frac{\Delta N_1}{\Delta \rho \Delta \mu_1 \Delta \phi_1} = J \left( \frac{x_2', \mu_2', \beta}{\rho, \mu_1, \phi_1} \right) \frac{\Delta N}{\Delta A \Delta \beta}$$

(3)

where $J \left( \frac{x_2', y_2', \beta}{\rho, \mu_1, \phi_1} \right) = 
\begin{vmatrix}
\frac{\partial x_2}{\partial \rho} & \frac{\partial x_2}{\partial \mu_1} & \frac{\partial x_2}{\partial \phi_1} \\
\frac{\partial y_2}{\partial \rho} & \frac{\partial y_2}{\partial \mu_1} & \frac{\partial y_2}{\partial \phi_1} \\
\frac{\partial \beta}{\partial \rho} & \frac{\partial \beta}{\partial \mu_1} & \frac{\partial \beta}{\partial \phi_1}
\end{vmatrix}$

From Eq. (1) we evaluate the determinant.

$$J \left( \frac{x_2', y_2', \beta}{\rho, \mu_1, \phi_1} \right) = -2 \left[ \frac{(x_2^2 + y_2^2) \omega^2 + z_2^2 \sin^2 \omega}{z_2^2} \right]$$

(4)

Eq. (3) can now be stated as follows:
The number of particles per unit radius of curvature interval per unit solid angle interval equals:

$$2 \left[ (x_2^2 + y_2^2) \omega^2 + z_2^2 \sin^2 \omega \right] / z_2^2$$

times the number of particles per unit area per unit angular interval.

Special Cases

Special Case I

In the vicinity of $y = \beta = 0$ (180° focussing)

$$|J| \approx \frac{(4z_2^2 + \pi^2 x_2^2)}{2 |z_2|}$$

(5)
Special Case II
When \( z_2^2 \sin^2 \omega \gg (x_2^2 + y_2^2) \omega^2 \) (360° focussing),
\[
|J| \approx 2 \left| z_2 \right| \sin^2 \omega.
\]  

Special Case III
An interesting case arises in connection with a two body disintegration. Then \( \rho = \rho \left( \sin \theta_1 \sin \phi_1 \right) = \rho \left( m \right) \).
In the \((x, z)\) plane
\[
\omega = \phi_1 = \frac{\pi}{2} - \beta = z_2/2\rho \mu_1 = \arctan m/\ell.
\]
Eq. (1a) then becomes simply:
\[
x_2 = 2m \rho(m).
\]  

All particles on the cone of direction cosine \( m \) pass through the \((x, z)\) plane at the same value of \( x \), and, therefore, the locus of such points is a straight line. The value of \( z \) at which a particular orbit intersects the \( zz \) plane can be easily calculated.
It is:
\[
z_2 = 2n \omega \rho \left( m \right)
\]  
The Jacobian is:
\[
|J| = \frac{2 \left[ x_2^2 \left( \pi/2 - \beta \right)^2 + z_2^2 \cos^2 \rho \right]}{|z_2|}
\]  

Acknowledgement
Mr. Charles Gilbert has aided in the preparation of this note by checking the calculations, by preparing the figure, and by first pointing out the existence of relation (7).