Title
Explicit solutions of the Kapchinskij-Vladimirskij equations for quadrupoles with unequal drift lengths, arbitrary field strength, and undepressed tune

Permalink
https://escholarship.org/uc/item/28c813hx

Author
Anderson, Oscar

Publication Date
2014-04-21
Explicit solutions of the Kapchinskij-Vladimirskij equations for quadrupoles with unequal drift lengths, arbitrary field strength, and undepressed tune

O. A. Anderson
LBNL, Berkeley, CA 94720, USA

L. L. LoDestro
LLNL, Livermore, CA 94551, USA

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor The Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or The Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or The Regents of the University of California.
Explicit solutions of the Kapchinskij-Vladimirskij equations for quadrupoles with unequal drift lengths, arbitrary field strength, and undepressed tune

O. A. Anderson  
LBNL, Berkeley, CA 94720, USA

L. L. LoDestro  
LLNL, Livermore, CA 94551, USA  
(Dated: September 10, 2012)

In 1958, Courant and Snyder analyzed alternating-gradient beam transport using a paraxial model without focusing gaps or space charge. Recently we revisited their work and found the exact solution for matched-beam envelopes in a linear quadrupole lattice [O.A. Anderson and L.L. LoDestro, Phys. Rev. ST Accel. Beams, 2009]. We extend that work here to include the effect of drift spaces, not necessarily of equal length. We calculate the exact envelopes and show results for a wide range of field strengths, occupancies, and drift-length ratios. We obtain an exact equation for the peak envelope excursion, a critical parameter in machine design. We discuss how this result can be used to find a favorable operating point (in terms of focus strength or phase advance $\sigma_0$) for an arbitrary lattice configuration. The smallest excursions always occur in the first stable band ($0^\circ < \sigma_0 < 180^\circ$) in a region near its midpoint; and their amplitude is remarkably insensitive to the drift-space geometry. The exact solutions for the second stable band ($180^\circ < \sigma_0 < 360^\circ$) exhibit the extreme beam-compression effect discussed in the above reference. Practical difficulties discussed there for possible applications are seen to be exacerbated by the inclusion of drift spaces. Finally, we show how to scale our exact results to approximate the effect of strong space charge.

I. INTRODUCTION

In their classic paper, Courant and Snyder [1] studied paraxial beam-envelope dynamics in a circular machine with negligible space charge, using the piecewise-constant focus-defocus (FD) model. They obtained an approximate solution for the envelope, using an expansion in focusing strength. For a straight machine, the same case was recently analyzed and an exact solution in explicit form was obtained [2], [3]. In the present paper we extend that work to include focusing gaps, which are allowed to have unequal lengths (syncopation). Particular cases with unequal gaps have long been studied via computer simulations; an especially thorough study of the KV equations [4] (including space charge) was published by Lund and Bukh [5].

Our three main motivations for presenting the explicit analytic envelope function for this case are: (1) ease of performing parametric studies such as those shown in Fig. 2, Fig. 5, and App. D, using simple spreadsheets; (2) ability to analyze solution properties such as extrema and limits (Sec. VIII); and (3) facilitating the study of envelope functions in the higher stable bands (Figs. 4 and 6), where approximation methods fail and simulations become difficult. In particular, we are interested in the effect of drift spaces and asymmetry on the pronounced second-band beam compression effect found earlier for the FD model.

Our piece-wise constant focusing model, while not realistic, can represent physical models fairly well if $\eta$ is appropriately chosen [5], [6]. In any case, our studies here should give good insight into the trends in physical models.

In this paper, instead of using the direct approach of Ref. [3], we obtain the exact envelope functions and phase advances by using the linear single-particle equation and the phase-amplitude method (Apps. A and B). A condensed version of this work was presented in Ref. [7].

Our exact results do not include space charge, but its effect can be approximated by scaling (Sec. VII D). The scaled results could provide good starting seeds for difficult simulations with strong asymmetry and space charge.

To indicate briefly that our doublet model includes piecewise-constant focusing and unequal drift spaces, we introduce the abbreviation FoDO.

II. FoDO FOCUSING MODEL

We assume a transverse linear focusing function $\kappa(z)$ that is periodic over a lattice with period $2L$, so that $\kappa(z + 2L) = \kappa(z)$. We take $\kappa(z)$ to be piecewise constant with value $+\kappa_{\text{max}}$ in the focus and $-\kappa_{\text{max}}$ in the defocus sections. These sections have length $\eta L$, where $\eta$ is the occupancy factor. The intervening drift sections have lengths $d_1$ and $d_2$, where $d_2 \geq d_1$ is assumed. For convenience throughout, we define

\[ k \equiv \sqrt{\kappa_{\text{max}}}. \]  

(1)

Our model is then described for the $x$-$z$ plane by Eqs. (2) and Fig. 1:

\[
\kappa(z) \equiv \begin{cases} 
+k^2, & 0 < z < \eta L; \\
0, & \eta L < z < \eta L + d_1; \\
-k^2, & \eta L + d_1 < z < 2\eta L + d_1; \\
0, & 2L - d_2 < z < 2L.
\end{cases} \]  

(2)
Drift centers $z_c$:

Since the FoDO lattice cell has equal focus and defocus lengths, the fields have antisymmetry about each drift center $z_c$. These centers have spacing $L$. For a matched beam, this antisymmetry yields a relationship between the envelopes $a(z)$ and $b(z)$ in the $x$-$z$ and $y$-$z$ planes, respectively. Using arguments similar to those in Ref. [6], we find that

$$b(z) = a(2z_c - z),$$

where $z_c$ is any drift center, so that we only need to solve one differential equation [for $a(z)$] in what follows.

Symmetries and initial conditions:

Section VII, along with the appendices, derives the formula for the exact envelope function. There are three types of cases for our model: (i) no drift spaces, i.e., \( \eta = 1 \); (ii) equal drift spaces, i.e., \( \mu = 0 \) [see Eq. (4)]; and (iii) unequal drift spaces. In the first two cases, in addition to the above antisymmetry, there is symmetry about the quadrupole midpoints. It follows that, for these two cases, $a(z)$ and $b(z)$ are even about those midpoints. All three cases are illustrated in Figs. 3 and 4.

From Eq. (3), the initial conditions for a matched beam, starting at a drift center, are

$$a(z_c) = b(z_c)$$

and

$$a'(z_c) = -b'(z_c).$$

III. DEFINITIONS

A. Drift asymmetry parameter $\mu$

We define the drift asymmetry parameter $\mu$ (where \( 0 \leq \mu \leq 1 \)):

$$\mu \equiv \frac{d_2 - d_1}{2d}, \quad (4)$$

where $d$ is the average drift length:

$$d \equiv \frac{d_2 + d_1}{2} = (1 - \eta)L. \quad (5)$$

Then

$$d_1 = d(1 - \mu), \quad d_2 = d(1 + \mu). \quad (6)$$

The normalized drift lengths, for use in Sec. V, are

$$\nu_1 \equiv kd_1 = \nu(1 - \mu), \quad (7)$$

$$\nu_2 \equiv kd_2 = \nu(1 + \mu), \quad (8)$$

$$\nu \equiv kd = k(1 - \eta)L = \frac{1 - \eta}{\eta} \theta, \quad (9)$$

where $\theta$ is defined by Eq. (10) below.

B. The FoDO focus parameter $\theta$

The FoDO focus parameter $\theta$, used throughout this paper, is defined by

$$\theta \equiv \eta kL. \quad (10)$$

We introduce the following quantities that depend on $\theta$:

$$sn \equiv \sin \theta, \quad cs \equiv \cos \theta, \quad (11)$$

$$sh \equiv \sinh \theta, \quad ch \equiv \cosh \theta.$$

Note: Ref. [3], which considered $\eta = 1$ only, used the symbol $\theta$ to stand for $kL$. Our new definition is an extension of that usage. All the results in this paper reduce to those in [3] in the FD full-occupancy limit. [See note after Eq. (32).]

IV. SOLUTION OF THE ENVELOPE EQUATION

The discussion in this section applies to the general case of an arbitrary periodic focus function $f(z)$. For a beam with the KV distribution, emittance $\xi$, and space-charge term $Sp$, the $x$-$z$ plane envelope function $a(z)$ is determined by [4]:

$$a''(z) + f(z) a - \frac{\xi^2}{a^3} + Sp = 0 \quad (12)$$

along with initial or periodic conditions for $a$ and $b$.

In this paper we assume $\xi_x = \xi_y = \xi$. We neglect the space-charge term $Sp$ — except for the discussion in Sec. VII D. (We also note that in the absence of space charge, the KV distribution can be replaced by a class of physically realistic distributions.)

For a matched beam without space charge, it is unnecessary to solve the nonlinear Eq. (12) directly, as we did in Ref. [3]. Instead, one can begin with the linear single-particle equation

$$x''(z) + f(z) x(z) = 0 \quad (13)$$

and use the phase-amplitude method [1] to find the envelopes. We review this in Apps. A and B.
The appendices give an elementary derivation (not requiring Twiss parameters or the Courant-Snyder invariant) of the well-known envelope solution, Eq. (B4). We repeat that result here for convenience:

\[
\frac{1}{a^2(z)} = \frac{M_{12}(z)}{P \sqrt{1 - \left(\frac{1}{2} \text{Tr} \ M\right)^2}},
\]

where \( M \) is the advance matrix for one cell of an arbitrary periodic lattice and \( P \) is the sign function, defined in Eq. (B6). For any focus parameter \( \theta \), \( P \) provides the correct sign for the radical.

For the particular cell model in which the focus function \( f(z) \) consists of segments having constant focus strength, the matrix \( M \) is the product of the transfer matrices for the individual segments. Figure 1 shows the FoDO model. The four segments—taken in the order shown in the figure—have transfer matrices given by [1], [8]:

\[
M_F = \begin{pmatrix}
\cos \theta & \frac{1}{k} \sin \theta \\
-k \sin \theta & \cos \theta
\end{pmatrix},
\]

\[
M_{O1} = \begin{pmatrix}
1 & d_1 \\
0 & 1
\end{pmatrix},
\]

\[
M_D = \begin{pmatrix}
\cos \theta & \frac{1}{k} \sin \theta \\
-k \sin \theta & \cos \theta
\end{pmatrix},
\]

\[
M_{O2} = \begin{pmatrix}
1 & d_2 \\
0 & 1
\end{pmatrix}.
\]

The matrix for the entire cell (Fig. 1), starting at \( z = 0 \), is

\[
M(0) = M(2L) = M_{O2} M_D M_{O1} M_F.
\]

The ranges of \( z \) for the four individual segments are indicated in Fig. 1. If \( z \) does not fall on a segment boundary, then the segment splits into two subranges—for example, \( z \) and \( (\eta L - z) \) if \( z \) is in the first segment. There are then five component matrices, as in Sec. VII A.

V. PHASE ADVANCE AND STABILITY

Reference [1] shows that a single-particle orbit is stable if

\[
|\text{Tr} \ M| < 2
\]

and that \( \text{Tr} \ M \) is independent of \( z \). (Cf. App. A, Eq. A11.) We calculate the trace at \( z = 0 \), using \( M(0) \) from Eq. (19). First, we introduce a matrix that will be used again in Sec. VII A:

\[
M_{III} = M_{O2} M_D M_{O1} = \begin{pmatrix}
A_1 & \frac{2B + sh}{k} \\
k \sin \theta & A_2
\end{pmatrix}.
\]

with \( A_1 \) and \( A_2 \) defined by

\[
A_1 \equiv \cos \theta + \mu \sin \theta, \quad A_2 \equiv \cos \theta + \nu \sin \theta,
\]

and \( B \) by

\[
B \equiv \nu \cos \theta + \frac{1 - \mu^2}{2} \nu^2 \sin \theta.
\]

Then

\[
M(0) = \begin{pmatrix}
A_1 & \frac{2B + sh}{k} \\
k \sin \theta & A_2
\end{pmatrix} = \begin{pmatrix}
\cos \theta & \frac{1}{k} \sin \theta \\
-k \sin \theta & \cos \theta
\end{pmatrix}.
\]

Phase advance \( \sigma_0 \) for a whole period

It is unnecessary to write all the components of the full matrix in order to find the phase advance. Only \( M_{11} \) and \( M_{22} \) are needed, according to Eq. (B2). Thus,

\[
\cos \sigma_0 = \frac{\text{Tr} \ M}{2} = (\cos \theta + \nu \sin \theta) \cos \theta - B \sin \theta,
\]

which agrees with the result given by Lund and Bukh [5].

Stability

The envelope solution will be stable for all values of \( \theta \) for which the right-hand side of Eq. (24) lies within the range \([-1, 1]\). Such regions of \( \theta \) are referred to as stable bands or passbands. Reference [3] shows how these bands are related to the branches of \( \cos \sigma_0 \). Using appropriate branches of Eq. (24), the phase advance is plotted as a function of \( \kappa L / \pi \) in Fig. 2 for various values of occupancy \( \eta \) and drift asymmetry \( \mu \). Drift spaces provide no focusing, so reducing \( \eta \) requires increasing \( \kappa L \) to achieve a given phase advance. Figure 2 shows this and shows the effect to be stronger in the second passband. Another effect of reducing \( \eta \) is seen: In the first panel of Fig. 2, the fractional width \((\Delta \kappa L / \kappa L)\) of the second passband at full occupancy is 0.763%. This fraction is reduced significantly for half occupancy, becoming 0.263% and 0.279% in the second and third panels, respectively. Related results are discussed in Sec. VII C.

VI. PASSBAND MIDPOINTS

We define the midpoint of any passband as the point where \( \text{Tr} \ M = 0 \), i.e., where

\[
\sigma_0 = \sigma_{0n} = 90^\circ + (n - 1)180^\circ
\]

with \( n \) the passband number.

We define the midpoint focus parameter \( \theta_n \) of the \( n \)th passband as the value of \( \theta \) that satisfies

\[
\text{Tr} \ M(\theta_n) = 0.
\]

An examination of Eq. (24) provides insight into the effects of varying \( \eta \) and \( \mu \). In the special case \( \eta = 1 \), Eq. (24) gives \( \cos \sigma_0 = \cos \theta \sin \theta \); thus \( \theta_n = \sigma_{0n} = (n -}
Both $\theta$ and $\sigma_0$ advance by $\pi$ between midpoints of successive passbands—see the top row of Fig. 2.

For general $\eta$, Eq. (24) gives

$$\tan \theta_n = \frac{(ch + \nu sh)}{B} \bigg|_{\theta = \theta_n},$$

where the right-hand side is a positive single-valued function of $\theta_n$. Thus, for general $\eta$ and $\mu$, there continues to be one passband for each interval of $\pi$ in $\theta$.

Equation (27) is useful for finding the location of the narrow second passband in $\theta$ space, given $\eta$ and $\mu$ (see App. F). Note that although $\theta \approx \theta_n$ in the higher passbands, $\sigma_0$ still varies over the range of $180^\circ$ and the solution still changes drastically near the band edges.

\[\text{FIG. 2. (a) Phase advances calculated from Eq. (24) for the first two stable bands. (b) The second band repeated, with the } kL \text{ axis magnified. Occupancies are } \eta = 1.0 \text{ for the top panel and } \eta = 0.5 \text{ for the others. The bottom panel has drift-space asymmetry } \mu = 0.8. \text{ Note that introducing drift spaces narrows the second passband (see text).} \]

\[\text{VII. EXPLICIT ENVELOPE FOR FoDO STABLE BANDS} \]

\[\text{A. Focus segment} \]

Now we are ready to use Eq. (14) to find the envelope function. We begin with the first (focus) segment of the lattice. For an arbitrary point $z$ in this segment, the full-period transfer matrix is obtained from $M_{\text{III}}$ after pre- and post-multiplying by the two subunits of $M_F$ mentioned at the end of Sec. IV. Using Eq. (21), we have

$$M^F_z = \left( \begin{array}{cc} \cos k z & \frac{1}{k} \sin k z \\ -k \sin k z & \cos k z \end{array} \right) \left( \begin{array}{cc} A_1 & 2B + sh \\ k sh & A_2 \end{array} \right).$$

The superscript $F$ signifies that $z$ is restricted here to the focusing segment of the quadrupole cell. The matrix multiplications are a bit tedious, but to find $a(z)$ we only need the element $M^F_{12}$. It is:

$$2kM^F_{12} = (A_1 + A_2)sn + (A_1 - A_2) \sin[k(\eta L - 2z)] + 2B cs + (2B + 2sh) \cos[k(\eta L - 2z)].$$

Finally, we define

$$F^F(z) \equiv kM^F_{12}$$

and use Eqs. (10), (14) and (22) to get the exact result for the focus-segment envelope. It is

$$a^2(z) = \frac{F^F(z)}{\mathcal{P} \theta \sqrt{1 - (\frac{1}{2} \text{Tr } M)^2}}$$

with Tr $M$ from Eq. (24) and

$$F^F(z; \theta, \mu, \nu) = (ch + \nu sh) sn + \mu \nu sh \sin[\theta(1 - 2z/\eta L)] + B cs + (B + sh) \cos[\theta(1 - 2z/\eta L)].$$

We have introduced the dimensionless quantity $F^F$—rather than using $M^F_{12}$ as in Eq. (14)—in order to clarify the dependence on the focus parameter $\theta$.

When $\nu = 0$, then $\eta = 1$ (i.e., there is full occupancy) and $B = 0$. In this case Eq. (31) reduces to the previous result for FD focusing [3], which was derived by a different method. (See also the note at the end of Sec. III.)

\[\text{B. Drift and defocus segments} \]

The beam envelopes for the three other segments in the FoDO lattice cell are given in App. C. These exact results for $a(z)$ [Eqs. (C8), (C9), and (C10)] were used,
along with Eq. (31), for the figures in this section. The transverse envelope \( b(z) \) was easily obtained from \( a(z) \) using Eq. (3).

Figures 3 and 4 plot the normalized envelope functions

\[
a_{\text{norm}} \equiv a(z)/\sqrt{\varepsilon L} \quad \text{and} \quad b_{\text{norm}} \equiv b(z)/\sqrt{\varepsilon L}
\]

for several combinations of occupancy and drift asymmetry. In these figures, \( z = 0 \) at the beginning of the focus section as in Fig. 1, but the plots are shifted in order to display the matched-beam symmetry described in Sec. II. The plots start at the center of the second drift space.

C. Variability of peak envelope

First passband:

The peak envelope amplitudes (1.740, 1.776, 1.761, and 1.817) in Fig. 3 hardly differ in spite of widely differing lattice configurations. However, comparing data for the first and second panels of Fig. 3, the lower occupancy in the latter requires its field-strength parameter \( k^2 L^2 \) to be larger by the ratio 1.409 in order to maintain the phase advance of 80°. This ratio becomes 1.616 for the third panel and reaches 5.941 for the low-occupancy case \( \eta = 0.1 \).

The exact results in Fig. 3 show, for a few examples, the insensitivity of peak beam excursion to changes in \( \eta \) and \( \mu \) while \( \sigma_0 \) is held constant near the first-band midpoint. This insensitivity holds for wide-ranging values of \( \eta \) and \( \mu \), and can be predicted from Eq. (36) below. See Fig. 7 and Table I in App. D and the discussion of limiting cases in App. E.

Second passband:

Figure 4 shows a different situation in the second passband. The focus parameter \( \theta \) has now been adjusted to give phase advance \( \sigma_0 = 270° \) (the midpoint of the second band) in each case. Here, the peak radius varies considerably with the lattice parameters. The radii are 4.860, 7.386, and 6.742, respectively. The field-strength parameter \( k^2 L^2 = 2.25 \pi^2 \) required for the top panel must be increased by the factor 2.2985 when the occupancy is reduced by half and by the factor 2.5820 with occupancy 0.5 and asymmetry \( \mu = 0.8 \).

Note that the envelope minima in the focusing sections have very small values, which will be discussed in Sec. IX.

It is interesting that large envelope fluctuations in cases of large phase advance were found for a completely different case (solenoid focusing, zero emittance) by Lee and Briggs using both iterative analysis and simulation [9].

D. Close approximation for space charge effect

For the second and third panels in Fig. 3, we have chosen the same occupancies, drift asymmetries, and undepressed phase advances as those used in the simulations of Lund, Chilton, and Lee [10], Figs. 2b and 2c. Their

![FIG. 3. Exact normalized envelope functions from Eqs. (31), (C8), (C9), and (C10). The normalized radii are defined by Eq. (33). Lattice parameters are as in Fig. 2. The value of \( kL \) for each panel is adjusted to give \( \sigma_0 = 80° \), which is near the midpoint of each passband. The required values are \( kL = 0.47642\pi, 0.56546\pi, 0.60565\pi, \) and \( 1.16121\pi \), respectively. The field strengths, which are proportional to \( k^2 \), thus change considerably while the peak envelopes remain nearly constant. See the contour plots and discussion in App. D. Note: These exact results are plotted conventionally, starting and ending at envelope crossover points. However, to agree with Fig. 1, \( z = 0 \) at start of focus section.](image-url)
simulations used the charge parameter $Q = 4 \times 10^{-4}$, which depressed the phase advance from 80° to 24.74°. Their envelope shapes appear very close to ours, indicating that their simulation results might be approximated by scaling our analytic results. Indeed, we find that the accuracy is of the order of 1%.

Scaling factor for space charge

To obtain a scaling factor for the case of the second panel of Fig. 3, we first note that its symmetries are the same as those assumed for the iterative analysis in Ref. [6]. We use formulae from Ref. [6] to calculate the beam envelope with and without space charge to about 0.5% accuracy. We take the ratio of the peak values to obtain the scaling factor and use that ratio to find the peak envelope with space charge for the case of the second panel. The result agrees with the simulation from Ref. [6]. We use formulae from Ref. [6] to calculate the same as those assumed for the iterative analysis in Ref. [10]. All the simulation results—although utilizing an unsyncopated model—give accuracies to within 0.5% and 3.6% for the strongly syncopated, strongly depressed case of Fig. 3, third panel. Using these seeds with the ESQPER code [11] gives convergence to $10^{-6}$ accuracy in about 20 iterations. Scaled solutions can also serve as the initial guess for iterative solutions of the “IM” type. Reference [10] describes various schemes iterating on numerically converged solutions to find solutions with given phase advance and emittances, current, or depressed phase advance. Our scaled results could be used to provide improved seeds for schemes with convergence difficulties, such as IM Case 0. See Sec. VB in Ref. [10].

VIII. PEAK ENVELOPE ANALYSIS

The peak envelope values (beam excursions) always occur in the focus segment. In the higher stable bands, minimum values occur there as well. For a given $\theta$ in the $n^{th}$ passband, we determine the locations $z_m$ of the envelope extrema by differentiating Eq. (32) with respect to $z$ and setting the result equal to zero. We find the roots

$$\left(1-2z_m/\eta L\right)\theta = \tan^{-1}\frac{\mu s h}{B+sh} +$$

$$0, \pm \pi, \pm 2\pi, \pm \cdots, \pm (n-1)\pi, \quad (34)$$

where $m = 1, \cdots, 2n-1$ and the principal value is used for $\tan^{-1}$. Note that the passband number is determined, given $\mu$, $\eta$, and $kL$. One sees by inspection of Eq. (34) [or, indeed, of (32)] that the extrema are always spaced at equal $z$ intervals $\Delta z_m/L = \pi/2kL$.

In the first passband focus segment, there is a single maximum; in the second, there is a minimum between two maxima, and so forth. Since, for any band $n$, all the maxima have equal value and since the first extremum is always a maximum, we need consider only the first of

![FIG. 4. Exact solutions of Eqs. (31), etc., with $kL$ adjusted to give phase advance $\sigma_0 = 270^\circ$, the midpoint of the narrow second passband seen in Fig. 2. Values of $kL$ are 1.5$\pi$, 2.274106$\pi$, and 2.410273$\pi$, respectively. The effect of drift spaces on peak radius is larger here than for the first band (Fig. 3). See the contour plots and discussion in App. D.](image-url)
the extrema. Then
\[ z_1 = \frac{\eta L}{2} \left[ 1 - \frac{1}{\theta} \left( \tan^{-1} \frac{\mu \nu \tan \theta}{B + \tan \theta} + (n - 1)\pi \right) \right] \] (35)

locates the maximum excursion of the envelope in the \( n \)th passband, and we get

\[ \sigma_{\text{max}}^2(\theta) = \varepsilon \eta L \frac{F_{\text{max}}^F}{P \theta \sqrt{1 - \left( \frac{1}{2} \text{Tr} M \right)^2}}, \] (36)

where

\[ F_{\text{max}}^F = F^F(z_1; \theta, \mu, \nu), \] (37)

with \( F^F \) given by Eq. (32).

Results for the first two passbands are plotted in Fig. 5. The lattice parameters are the same as for Figs. 2–4. Note the previous discussion of peak values in connection with those figures in Sec. VII C.

The minima of \( \sigma_{\text{max}} \) with respect to \( kL \) (seen in Fig. 5) are plotted for the entire ranges of \( \eta \) and \( \mu \) in App. D.

Limiting cases for \( n = 1 \) are analyzed in App. E, where the small variability of \( \sigma_{\text{max}} \) with respect to \( \eta \) and \( \mu \) in the vicinity of the first passband-midpoint is deduced from Eq. (36). Cases with \( n > 1 \) are analyzed in App. F.

IX. SECOND-BAND BEAM COMPRESSION

First we discuss the fully symmetric case (\( \mu = 0 \)). Using differentiation of Eqs. (31) and (C9) or symmetry arguments, it is easy to see that for the even passbands, \( a(z) \) has a minimum at the center of the focus section and again a minimum at the center of the defocus section. The same is true for \( b(z) \) [Eq. (3)], so that the beam area \( A(z) = \pi a(z)b(z) \) can become very small at those points. (See Fig. 4.) We define the compression ratio

\[ R \equiv \frac{\max(A)}{\min(A)}. \]

In general, to calculate the beam area in the focus segment, we first find \( a(z) \) from Eq. (31) and then \( b(z) \) from Eq. (3). The latter requires \( a(z) \) in the defocus segment, i.e., Eqs. (C9).

The maximum beam area occurs at the midpoint of the narrower of the drift spaces. It is found from Eqs. (C8) or Eqs. (C10).

Figure 4 shows exact beam envelopes for the midpoint of the second passband. In the first panel, the ratio \( R \) of maximum to minimum beam areas is \( 0.59 \times 10^3 \). In the second panel, with \( \eta \) reduced from 1.0 to 0.5, \( R \) increases to \( 2.68 \times 10^3 \). In the third panel, with large drift-length asymmetry (\( \mu = 0.8 \)), \( R \) is \( 2.44 \times 10^3 \), only slightly less than for the second panel. The value of the first \( a \) minimum depends strongly on \( \eta \) and weakly on \( \mu \). Additional results and discussion are presented in App. D (see Fig. 9).

Previously, Ref. [3] showed that, for the FD model, the compression effect is greatly augmented for phase advances near a passband edge (Fig. 9)—and noted that caveats apply. Then, Ref. [7] treated the FoDO case. It compared compressions at half occupancy near the band edge for symmetric and asymmetric drift spaces. The phase advances \( \sigma_0 \) were 356.75º and 356.6º, respectively. A large change in \( R \) was noted. Here, we present the same comparison, holding both phase advances to 356.58º. For the symmetric case (\( \mu = 0.0 \)), the nominal compression ratio \( R = 1.06 \times 10^6 \) is large because the first \( b \) minimum is very small. In the second case, drift asymmetry (\( \mu = 0.8 \)) shifts the minimum of \( a(z) \) away from this point and reduces \( R \) to \( 6.93 \times 10^4 \) (see Fig. 6).

Space charge would also affect the beam compression. The effect on the beam waist could be small because emittance pressure scales more rapidly with radius than does space-charge pressure.

We note again that the strong focusing fields needed for the second passband violate the KV paraxial model. The WARP particle code [12] could determine how severe the effect is.
We have obtained exact matched-beam envelopes for the FoDO lattice model, which specifies focus occupancy \( \eta \) and drift-space asymmetry \( \mu \). From these solutions we have derived an exact equation for the peak envelope excursion as a function of \( \eta, \mu \), and the field-strength parameter \( k^2L^2 \).

First passband

Our primary finding for the first passband is the remarkable insensitivity of the peak envelope excursion to variations of the lattice parameters \( \eta \) and \( \mu \) over their entire ranges. We find that the corresponding values of \( kL \) depend weakly on \( \mu \) and increase by about 30% as \( \eta \) decreases from unity to 0.5 (see App. D). Thus, the present paper shows that the practical necessity of occupancies \( \eta \) smaller than one does not drastically increase \( kL \).

Second passband

We begin our discussion of the second passband by recalling our earlier study of the FD model [3], where we showed why the second band is impractical for beam transport and how it could be useful for novel applications. We found that the field strength required for minimum beam size is about an order of magnitude larger than for the first passband and that the beam excursions swell by a factor \( \sim 2.5 \). Furthermore, the width of the second passband is so narrow that it would take great effort to set and maintain the correct field strength. The strong fields would be difficult to apply and would likely violate the paraxial assumption of the KV equations.

For the FoDO case, the present paper shows that the required field strength again depends only weakly on \( \mu \). The dependence on \( \eta \), however, is stronger in the second band: the \( kL \) required for minimum beam size with \( \eta = 0.5 \) exceeds that for \( \eta = 1 \) by about 70% (see App. D). The above practical problems of the second passband are, then, exacerbated. Nevertheless, we have presented second-band quantitative results here in some detail, because although the existence of higher passbands is well known and they have attracted some interest (see, e.g., Ref. [13]), we have not found their solutions explored elsewhere.

We have also analyzed the second passband to investigate the possibility of applications beyond ordinary beam transport. Such applications could take advantage of the small envelope minima in both focusing planes at certain \( z \) locations (for even bands).

Note that these minima, though small, are unequal in size. For the case \( \eta = 0.5, \mu = 0 \), the beam has an elliptical waist with axis ratios ranging from 20.07 at \( \sigma_0 = 270^\circ \) to 14.33 at \( \sigma_0 = 346.6^\circ \). In this range the compression ratio rises from \( 2.7 \times 10^3 \) to \( 69.4 \times 10^3 \) but at the expense of doubling the already large peak envelope amplitude. Therefore, any attempt to observe or utilize the second passband in practice should try to operate at the passband center.

Conceivably, one could utilize the periodicity of the double beam-waists to provide powerful differential pumping of the beam line using a series of small elliptical apertures. Another possibility is the direct production of an external beam with small spot size at the end of a quadrupole-focused accelerator. Much research would be needed first to resolve the practical issues observed in Ref. [3].

Our exact results throughout this paper apply to the zero-space-charge limit. For cases where space charge dominates, Sec. VIID describes, for the first passband, a method for finding good approximations for beam-envelope amplitudes and angles. (Note that it utilizes the formulae of Ref. [6], which were developed only for the first passband.) This method could be useful for surveying large regions of parameter space and finding initial values for simulations.

**ACKNOWLEDGMENTS**

We thank Steve Lund and Ed Lee for suggesting the transfer-matrix method of solution and reading a draft of this paper. We also thank Dr. Lund for having generously provided data connected with Fig. 2 in Ref. [10] along with much useful advice. This work was supported in part by the Director, Office of Science, Office of Fusion
Appendix A: PHASE-AMPLITUDE METHOD

In their 1958 paper [1], Courant and Snyder described the phase-amplitude method for dealing with equations having the form of Eq. (13) [or (A1)] and applied it to obtain an approximate solution to the alternating-gradient problem. Their paper introduced a number of powerful concepts — beta function, Courant-Snyder invariant, etc.—which have been used by many authors, in various ways, to derive the envelope function for a given focusing function. The Courant-Snyder concepts are not needed for our present work. We use a somewhat simplified treatment to derive the solution [Eq. (B4)] in this appendix and the next one. These make this paper fairly self-contained and, we hope, accessible to the non-specialist. One may also refer to standard textbooks, such as Wiedemann’s [14].

Without space charge, the transverse position \( x(z) \) of a single particle obeys the linear equation

\[
x''(z) + f(z)x(z) = 0. \tag{A1}
\]

(We assume here that \( f(z) \) is a periodic function.) One can verify by substitution that

\[
x = \frac{a(z)}{\sqrt{\varepsilon}}(C_1 \cos \psi + C_2 \sin \psi) \tag{A2}
\]

is the general solution of Eq. (A1), provided that \( a(z) \) satisfies Eq. (12) and that

\[
\psi' = \frac{\varepsilon}{a^2}. \tag{A3}
\]

The quantities \( \psi(z) \) and \( a(z) \) are known as the phase and amplitude. If we differentiate Eq. (A2) and use Eq. (A3), we get

\[
\sqrt{\varepsilon}\psi' = (C_1 a' + C_2 \varepsilon/a) \cos \psi + (C_2 a' - C_1 \varepsilon/a) \sin \psi. \tag{A4}
\]

Set \( \psi(z_0) = 0. \) Then

\[
C_1 = \sqrt{\varepsilon} x_0/a_0; \tag{A5}
\]
\[
C_2 = (a_0 x'_0 - a'_0 x_0)/\sqrt{\varepsilon}. \tag{A6}
\]

Inserting these into Eq. (A2) gives

\[
x(z) = g_{11}(z, z_0) x_0 + g_{12}(z, z_0) x'_0; \tag{A7a}
\]
\[
x'(z) = g_{21}(z, z_0) x_0 + g_{22}(z, z_0) x'_0, \tag{A7b}
\]

where the coefficients \( g_{ij} \) can be deduced from Eqs. (A2), (A4), (A5), and (A6). In App. B we will need the coefficient \( g_{12} \), which is

\[
g_{12}(z, z_0) = \frac{1}{a_0 a(z)} \sin \psi. \tag{A8}
\]

At this point, we specify that the beam is matched, i.e., \( a(z) \) is periodic with the same period, \( 2L \), as the lattice. Then \( a(z_0 + 2L) = a(z_0) \). We define

\[
\psi(z_0 + 2L) \equiv \sigma_0, \tag{A9}
\]

the phase advance for a whole period, and observe from Eq. (A3) that, for the matched beam, \( \sigma_0 \) is independent of the choice of \( z_0 \). Then, writing Eqs. (A7) in matrix form for the case \( z = z_0 + 2L \), we have

\[
\begin{pmatrix} x(z + 2L) \\ x'(z + 2L) \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} x(z) \\ x'(z) \end{pmatrix}. \tag{A10}
\]

We drop the subscript on \( z_0 \) because the location of \( z_0 \) is arbitrary. Note that the matrix elements here are periodic but \( x(z) \) is not. If one writes out \( g_{11} \) and \( g_{22} \), one sees that the trace has the value

\[
g_{11} + g_{22} = 2 \cos \sigma_0, \tag{A11}
\]

which is independent of \( z \).

Appendix B: MATCHED ENVELOPE SOLUTION

Now we calculate the matched beam envelope for a machine with periodic focusing function \( f(z) \). In general, there is a machine matrix \( M(z) \), obtainable in principle by integrating Eq. (A1). (Analytic results are easily obtainable if \( f(z) \) is piece-wise constant, but not in most other cases.) The result gives the phase-space change over a whole machine period (for the specific FoDO example, see Secs. V and VII A):

\[
\begin{pmatrix} x(z + 2L) \\ x'(z + 2L) \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} x(z) \\ x'(z) \end{pmatrix}. \tag{B1}
\]

Comparing Eqs. (B1) and (A10) and using Eq. (A11) shows that

\[
2 \cos \sigma_0 = M_{11} + M_{22} = \text{Tr} M \tag{B2}
\]

and

\[
M_{12} = \frac{1}{\varepsilon} a^2(z) \sin \sigma_0 \tag{B3}
\]

or

\[
\frac{1}{\varepsilon} a^2(z) = \frac{M_{12}(z)}{\text{P} \sqrt{1 - (1/2 \text{Tr} M)^2}}. \tag{B4}
\]

where

\[
\text{P}(\sigma_0) \equiv \text{sign}(\sin \sigma_0) \tag{B5}
\]

gives the correct branch of the radical and ensures that the right-hand side of Eq. (B4) is always positive.

Various versions of Eq. (B4) have appeared in the literature — for example, Eq. (19) in Ref. [10]. The criterion for single-particle stability [Eq. (20), Sec. V] is \( |\text{Tr} M| < 2 \). Notice from Eq. (B4) that the
criterion for existence of a real solution for the envelope equation is the same. Therefore, existence implies stability and vice versa.

In the stable regions, specializing to FoDO, Eq. (B5) is equivalent to

$$P(\theta) \equiv \text{sign}(\sin \theta).$$  \hspace{1cm} (B6)

Equation (B6), which was noted in Ref. [3], follows easily from Eq. (24) if $\eta = 1$. A brief calculation shows that Eq. (B6) applies without modification for general $\eta$ and $\mu$.

**Appendix C: EXACT ENVELOPES IN THE DRIFT AND DEFOCUS SEGMENTS**

In the main text, we gave details of the envelope calculation for the focus segment. Here, we briefly describe our method for the three remaining segments and give the results.

The envelope calculation for any segment is simplified by moving the origin to the beginning of the segment. Thus, the matrices that will yield the envelopes for the three remaining segments are

$$M^{O_1}(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} M_F M_{O_2} M_D \begin{pmatrix} 1 & d_1 - z \\ 0 & 1 \end{pmatrix},$$  \hspace{1cm} (C1)

$$M^D(z) = \begin{pmatrix} \cosh kz & 1 \\ k \sinh kz & \cosh kz \end{pmatrix} M_{O_1} M_F M_{O_2} \times \begin{pmatrix} \cosh k(\eta L - z) & 1 \\ k \sinh k(\eta L - z) & \cosh k(\eta L - z) \end{pmatrix},$$  \hspace{1cm} (C2)

$$M^{O_2}(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} M_D M_{O_1} M_F \begin{pmatrix} 1 & d_2 - z \\ 0 & 1 \end{pmatrix}.$$  \hspace{1cm} (C3)

Note that the cyclic order is maintained in all cases.

After performing the matrix multiplications, one obtains envelopes from Eq. (B4), using $M^{O_1}_2$, $M^D_2$, and $M^{O_2}_2$.

First, as in Sec. VII A, we define dimensionless quantities

$$F^{O_1} \equiv k M^{O_1}_2,$$  \hspace{1cm} (C4)

$$F^D \equiv k M^D_2,$$  \hspace{1cm} (C5)

$$F^{O_2} \equiv k M^{O_2}_2.$$  \hspace{1cm} (C6)

We write Eq. (31) as

$$a^2(z) = \frac{\epsilon \eta L}{P} \frac{F^j(z)}{\theta \sqrt{1 - (\frac{1}{2} \text{Tr} M)^2}},$$  \hspace{1cm} (C7)

where the superscript $j$ represents $F, O_1, D$, or $O_2$.

For the first drift segment, the exact envelope is obtained from Eq. (C7) and

$$F^{O_1} = G_1 - 2 H_1 k z + J_1 k^2 z^2.$$  \hspace{1cm} (C8a)
[Fig. 5 ($a_{\text{max}}$ vs. $\theta$) had shown these minimum values for just three combinations of $\eta$ and $\mu$.]

In the first passband (upper panel of Fig. 7), $a_{\text{min}}^\text{max}$ exhibits non-trivial structure, approaching a saddle point at $\eta = 0.6, \mu = 1$; but the numerical value of $a_{\text{min}}^\text{max}$ changes very little over the entire range of $\eta$ and $\mu$ (see the contour levels in Table I). The second passband (lower panel), on the other hand, shows a weak variation of $a_{\text{min}}^\text{max}$ with respect to drift asymmetry (except at very small occupancy); but a strong variation with occupancy, diverging as $\eta \to 0$; it lacks the saddle point but is ill behaved at $\eta = 1, \mu = 1$.

The focusing-field strength—proportional to $k^2L^2$—is a critical parameter in machine design. In Fig. 8, we plot the $kL$ required for minimum peak-envelope amplitude at given $\eta$ and $\mu$; i.e., we plot $kL$ at the minimizing $\theta$ found for Fig. 7. For both passbands, the dependence on $\mu$ is weak; $kL$ increases as the occupancy decreases and

![FIG. 7. Contour plots of $a_{\text{min}}^\text{max}/\sqrt{\varepsilon L}$ as a function of $\eta$ and $\mu$. The contour levels are given in Table I, which shows a weak variation for the first passband (upper panel) and a strong variation for the second passband (lower panel).](image_url)

![FIG. 8. Contour plots of $kL$ at which $a_{\text{max}} = a_{\text{min}}^\text{max}$ (see Fig. 7), as a function of $\eta$ and $\mu$ for the first passband (upper panel) and second passband (lower panel).](image_url)

| Table I. Table showing values of contours in Figs. 7–9 |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Fig. 7a | Fig. 7b | Fig. 8a | Fig. 8b | Fig. 9 |
| K | 1.6958 | C | 8.118 | C | 3.235 | C | 14.09 | C | 8.244E+3 |
| L | 1.7156 | D | 10.29 | D | 4.726 | D | 23.89 | D | 2.862E+4 |
| N | 1.7460 | F | 16.52 | F | 10.08 | F | 68.70 | F | 3.451E+5 |
| O | 1.7557 | G | 20.93 | G | 14.73 | G | 116.5 | G | 1.198E+6 |
| P | 1.7595 | H | 26.52 | H | 21.51 | H | 197.5 | H | 4.160E+6 |
| Q | 1.7600 | I | 33.61 | I | 31.42 | I | 335.0 | I | 1.445E+7 |
| R | 1.7761 | J | 42.59 | J | 50.16E+7 |
| S | 1.7967 | K | 53.96 | K | 1.742E+8 |
but the bands. For the first passband, it is also true: Our results from an examination of limiting cases.

\[ \theta_1 \text{ can never be large.} \] Nor, again from Eq. (27), when \( \theta_1 \sim \mathcal{O}(1) \) can \( \nu \) be very large. So we choose the following expansions:

\[
\frac{\theta_1}{\pi/2} \quad (\nu \ll 1)
\]

1. \( 1 - \frac{1 - \eta}{\eta} + \left( \frac{1 - \eta}{\eta} \right)^2 \left[ 1 + \frac{\pi}{2} \tanh \left( \frac{\pi}{2} \right) \frac{1 + \mu^2}{2} \right] \)

and

\[
\theta_1 \quad (\theta_1 \ll 1)
\]

\[
\left[ \frac{\eta(1 - \eta)}{2 + \frac{1 - \mu^2}{1 - \eta}} \right]^{1/4}
\]

1. \( \left( \frac{3 - \eta}{2(1 - \eta)} \right)^{1/2} \), \( \theta_1 \ll 1, \frac{1 - \mu^2}{1 - \eta} \ll \frac{2}{3} \);

3. \( \left( \frac{1 - \eta}{1 - \eta} \right)^{1/2} \left( \frac{2 - \mu}{1 - \mu} \right)^{1/2}, \theta_1 \ll 1, \frac{1 - \mu^2}{1 - \eta} \gg \frac{2}{3} \).

In each region, we have calculated only through the terms needed to obtain the lowest-order variations of \( a_{\text{max}}^\min \) below. Note the weak dependence on parameters: weaker upon \( \mu \) than \( \eta \) in region I; and weak in regions II and, except at very small \( \eta \), III. The only strong variation occurs as \( \eta \to 0 \), when \( \theta_1 \propto \sqrt{\eta} \to 0 \). Note also that \( \theta_1 \) spans the full range where \( \tan \theta \) in Eq. (27) is positive in the first passband, from \( \theta_1 = 0 \), at \( \eta = 0 \), to \( \pi/2 \) at \( \eta = 1 \).

The boundary between regions II and III \( \left( \frac{1 - \mu^2}{1 - \eta} = \frac{2}{3} \right) \) is a line that extends from \( \mu = 1, \eta = 0 \) to \( \mu = 0, \eta = 3/7. \) \( \theta_1 = 1 \) restricts region II to the left of \( \eta = 0.4. \) (The general small-\( \theta_1 \) expansion is valid at slightly larger \( \eta \) for small \( \mu \).) Region I extends from \( \eta = 1/2 \) [where \( 1 - \eta) / (1 - \eta) = 1 \)] to \( \eta = 1 \). Note that there is a region at intermediate \( \eta \), between regions I and II, where \( \theta_1 \) and \( \eta \) are \( \mathcal{O}(1) \) and none of the above expansions is valid. One could of course expand around some intermediate \( \eta \); but since the variations for intermediate \( \eta \) remain weak and we are concerned here only with finding the major trends, we do not carry that out.

Next, we evaluate \( a_{\text{max}}^2 \) [Eq. (36)] at \( \theta = \theta_1 \), in the same limits as above, to obtain an approximate \( a_{\text{max}}^\min \). We find:

\[
a_{\text{max}}^\min^2 \underset{\in L}{\sim} \begin{cases}
I: & \frac{\cosh \left( \frac{\pi}{2} + \sinh \left( \frac{\pi}{2} \right) \right)}{\pi/2} \left[ 1 + \frac{(1 - \eta^2)}{\eta} \right] (1 + \mu^2) \\
II: & 2 + \frac{3}{2} \left( \frac{1 + \eta}{\eta^2} \right)^{3/2} (1 - \mu^2) \\
III: & 2 + (1 - \eta) \left[ 2(1 - \mu^2) \right]^{1/2}
\end{cases}
\]

At \( \eta = 1 \), this gives \( a_{\text{max}}^\min^2 / \underset{\in L}{\sim} 1.75 \), which agrees well with Table I. \( a_{\text{max}}^\min \) decreases with \( \eta \) and increases with \( \mu \) as \( \eta \to 1 \); but the variation is very weak, appearing only at second order in \( 1 - \eta \) (the first-order terms cancelled). In regions II and III, the variations with parameters are again weak (see the expansion parameters given for \( \theta_1 \)). In region II, \( a_{\text{max}}^\min \) decreases with \( \eta \), more weakly as \( \eta \) and \( \mu \) increase; and \( a_{\text{max}}^\min \) decreases with \( \mu \), more weakly as \( \eta \to 1 \). In region III, \( a_{\text{max}}^\min \) decreases with \( \eta \), more weakly as \( \eta \) and \( \mu \to 1 \); and \( a_{\text{max}}^\min \) decreases with \( \mu \), more weakly as \( \eta \) and \( \mu \to 1 \).
weakly as $\eta$ increases and $\mu$ decreases. Thus the trends and structure observed in Fig. 7—the insensitivity in the first passband of $a^\text{min}_\text{max}$ to parameters and the presence of the saddle point—are found within this simple analysis.

**Appendix F: LIMITING CASES — HIGHER PASSBANDS**

We mentioned earlier that, for $n > 1$, the minimizing $\theta$ is obviously very near $\theta_n$ because of the narrowness in $\theta$ of the higher stable bands. We note from Sec. VI and Eq. (27) that $(n-1)\pi < \theta_n < (n-\frac{1}{2})\pi$. Here, then, we expand Eq. (27) to lowest order around large $\theta_n$, so that $\tanh \theta_n \to 1$. We find to first order in the remaining small parameters:

$$\theta_n = \begin{cases} 
I: (n-\frac{1}{2})\pi - \nu_n, & \nu \ll 1; \\
II: (n-\frac{1}{2})\pi + \frac{\eta \nu_n}{\pi} - \frac{1-\mu^2 \nu}{2}, & \nu \gg 1, \frac{1-\mu^2}{2} \nu \ll 1; \\
III: (n-1)\pi + \frac{\eta}{2} \frac{1}{1-\mu^2 \nu_n}, & \nu \gg 1, \frac{1-\mu^2}{2} \nu \gg 1. 
\end{cases}$$

Again, regions I, II, and III refer, roughly, to the right, central, and left portions of Fig. 7.

$\nu_n$ stands for $\nu$ with $\theta$ evaluated at the lowest-order $\theta_n$, i.e., at the leading term in each of the three cases above. For $n = 2$, the border between cases I and II ($\nu_n = 1$) occurs at $\eta \sim 0.8$. The border between cases II and III $(1 - \mu^2 = 2\nu_n)$ runs from $\mu = 1$, $\eta = 0$ to $\mu = 0$, $\eta \sim 0.64$. For case II, note also that the two first-order terms cancel on a line above and to the right of the II/III border, excepting $\mu = 0$ at $\eta \sim 0.74$. The limiting cases thus cover the entire space (although of course more terms would be needed for accurate expressions near the expansion limits).

From these results we see that $\theta_n$ spans its full possible range, going from $(n-1)\pi$ at $\eta = 0$ to $(n-\frac{1}{2})\pi$ at $\eta = 1$. To lowest order, $\theta_n$ is constant in each of the three regions; the first-order terms vary with parameters in a direction to make $\theta_n$ approach the neighboring region’s constant. For the second passband, there is only a 50% variation in $\theta_n$ over the entire range of $\eta$ and $\mu$. The $\mu$-dependence is weak: it enters only at first order, and only for cases II and III, and is appreciable there only for $\mu$ near unity. Fig. 8 together with Table I confirms these results. At the minimizing $\theta$, then, $kL \sim (n-\frac{3}{2})\pi/\eta$—the focus-parameter strength varies approximately inversely with the occupancy.

For $a^\text{min}_\text{max}^2$ at $\theta = \theta_n$, we find to lowest order:

$$a^\text{min}_\text{max}^2 \in L \sim \begin{cases} 
I: \frac{\eta}{(n-\frac{3}{2})\pi}; \\
II: \frac{\eta}{(n-\frac{3}{2})\pi} \frac{\mu}{2} \left(\sqrt{2 + \frac{1}{1+\mu^2}}\right); \\
III: \frac{\eta}{(n-\frac{3}{2})\pi} \frac{1-\mu^2}{4} \nu_n^3. 
\end{cases}$$

Note that the $\mu$-dependence is again weak, but the trends are opposite for cases II and III, as born out by the lower panel of Fig. 7; and that, from case III, $a^\text{min}_\text{max} \propto 1/\eta$ as $\eta \to 0$.

---