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Essays on Contests, Identification, and Agglomeration

A dissertation submitted in partial satisfaction
of the requirements for the degree
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by

Gregory Scott Kubitz

2016


ABSTRACT OF THE DISSERTATION

Essays on Contests, Identification, and Agglomeration

by

Gregory Scott Kubitz

Doctor of Philosophy in Economics
University of California, Los Angeles, 2016
Professor Jernej Copic, Chair

This manuscript consists of three essays divided into three chapters. The second chapter is an essay co-authored with Tiago Caruso, and the third chapter is a version of a collaborative project with Jernej Copic, Robert Sherman, and Omer Tamuz.

The first chapter studies repeated contests with private information. In these contests, weak contestants prefer to appear strong while strong contestants prefer to appear weak. In contrast to a single contest, this leads to an equilibrium where effort is not strictly monotonic in ability and allows for a less able contestant to win against a contestant of higher ability. While the aggregate payoffs of contestants are higher per contest than in the single contest benchmark, aggregate output per contest is lower. Depending on the economic setting, the presence of private information can lead to productive or allocational inefficiencies.

In the second chapter we study a binary choice model where an agent makes a decision that is informed by his beliefs after observing a public signal. This model generalizes to a wide range of economic environments which require econometricians to estimate the beliefs of agents. With minimal structure imposed on the agent’s utility function, we characterize the structure of information needed to identify the beliefs of the agent after observing both signals and decisions. We find that the information must be sufficiently convincing and dense for the agent’s beliefs to be point identified. When the full range of information is relaxed, we show how the agents beliefs can be partially identified. Additionally, we explicitly show how the econometrician can construct the sharpest boundaries around the agents beliefs, as she observes signals and decisions.
In the third chapter we propose a simple model of agglomeration of some particles (or agents) when there is no growth in the number of agents. In many periods, countably many agents move freely (randomly) along a line until they encounter other agents, in which case they form a community and stop moving. We show that as time goes to infinity, the distribution of sizes of communities is exponential. When agents can also detach (leave) a community, we show that when the probability of leaving a community decreases sufficiently fast with the community size, there is no steady-state distribution of community sizes: as time goes to infinity, community sizes tend to infinity.
The dissertation of Gregory Scott Kubitz is approved.

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University of California, Los Angeles
2016
To my Mom and Dad
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CHAPTER 1

Repeated Contests with Private Information

1.1 Introduction

Contests are frequently used to stimulate effort from economic agents. These contests are often dynamic and offer multiple prizes, as in the case, for example, of innovation races and employee competitions. In both of these settings, there is extensive literature discussing how to best design contests to maximize the output of the contestants. However, the behavior of economic agents in repeated contests is not fully characterized in situations where agents have private information about ability or the cost of production.

In this paper, we study repeated contests in a framework designed to capture both moral hazard (hidden effort choice) and adverse selection (privately known abilities). That is, when contestants’ abilities are private information, the contestants must consider the signaling effect that exerting effort in early contests will have in future contests. Contrary to the conventional wisdom that all contestants want to appear strong to their opponents, countervailing incentives emerge in this setting and strategies are non-monotonic. These incentives can create multiple economic inefficiencies, depending on the economic context. First, overall output in repeated contests is lower relative to the single contest benchmark. In the contests described above, this reduction in output is a negative welfare result. Second, a player with low ability may beat a player with high ability in the first contest or both contests. In multi-stage tournaments, this may prevent the best contestant from winning the tournament, or even making the later rounds.

We consider the simplest setting that captures the signaling incentives of repeated contests:

---

1Gallini and Scotchmer 2002 survey the discussion about the optimal “effective time” for the length of intellectual property rights that provides incentives for initial innovations without stifling subsequent innovation. In labor market competitions, see Ridlon and Shin 2013, Ederer 2010, and Aoyagi 2010.
two contestants, who have either low or high ability, competing in two successive contests. In each contest, the contestants exert effort with the goal of producing the most output. The player who does so wins a prize.\footnote{For a general description of static (one-shot) games of this kind, see Siegel 2009.} The amount of effort it takes to produce output depends on individual ability, which is privately known by each contestant. After the first contest, the output of each contestant is publicly observed and players can update their beliefs about their opponents’ ability. Given this additional information, contestants choose a new level of effort for the second contest. The contestants choose their effort levels to maximize their total payoff over the two contests.

We show there is a unique symmetric equilibrium for this repeated contest game. To derive equilibrium strategies, we utilize the findings in the single all-pay contest with asymmetric information presented by Siegel 2014. In particular, we use his equilibrium construction to solve for the optimal strategies and expected payoffs in the second contest for any set of beliefs. From these payoffs, we show that a contestant with high ability will always prefer to have his opponent believe they have low ability. Likewise, a contestant with low ability wants to appear to have high ability. While uniqueness of equilibrium in dynamic games with signaling is not common, the incentives to misrepresent type in this setting rule out the possibility of different off-path beliefs which would be necessary to form any other equilibria. Additionally, these incentives lead to an equilibrium that has partial pooling in the first contest, i.e. there are outputs which can be produced by either low ability or high ability contestants. Low ability players who produce output in this range are \textit{bluffing} while high ability players who do so are \textit{sandbagging}.\footnote{The terms \textit{sandbag} and \textit{bluff} are used in the literature to describe a player signaling to his opponent that he is weak when he is actually strong and strong when he is actually weak, respectively. These terms originate from the game of poker. In poker, \textit{sandbagging} is when a player calls or does not increase the pot when he believes he has the better hand. \textit{Bluffing} is when a player bids up the pot when he does not think he has the best hand.}

Bluffing is used to discourage an opponent by appearing to be strong. In our setting, this means having high ability. Avery 1998 shows that this type of behavior can emerge in a single dynamic contest where bidders submit jump bids. Additionally, in dynamic contests with hidden actions and incomplete information, contestants bluff by signal-jamming their opponents. This is found in models of labor market contests (Ederer 2010), all-pay auctions (Ortega Reichert 2000), and duopoly competition (Mirman, Samuelson, and Urbano 1993). Sandbagging, as described in
Rosen 1986, is used to lull opponents into a false sense of security. In a framework similar to ours, but with only one type of active contestant, Münster 2009 shows that the active contestant may sandbag by sitting out of the first contest in order to win the second contest more easily.

Both bluffing and sandbagging are present when bidders are allowed to send costly signals before an auction as in Hörner and Sahuguet 2007. If the bidder makes a sunk investment in the form of a jump bid before the auction, then the other bidder must match it to enter the auction. Bidders with moderate values may use a jump bid to keep other competitors out of the auction while bidders with high values may allow opponents to enter the auction freely in order to appear weak and face less competition in the auction itself.

In our setting of repeated contests, both tactics are used because each contestant is concerned with how opponents of similar ability react to the outcomes of the first contest. For example, low ability contestants would be discouraged facing a strong opponent while a high ability contestant would react by increasing effort. On the other hand, a low ability contestant would be encouraged by a weak opponent and increase effort while a high ability contestant would reduce effort, thinking he could win with ease. As in Hörner and Sahuguet 2007, this leads to non-monotonicities in the equilibrium, where a contestant with low ability may beat one with high ability in either one, or both of the contests. In elimination tournaments and multi-stage auctions, designers often prefer to have the best contestants in the final round. However, if the actions of the first round are publicly observable, top contestants would be worried about revealing information about their ability to their future opponents. This may cause them to lose to a lesser opponent in the first round, leading to a less competitive final round.

Lastly, we consider the effect of multiple contests on the aggregate output of the contestants. The consequence of bluffing and sandbagging is a decrease in aggregate output in the first contest when the difference between high and low ability contestants is large enough. In the setting of Münster 2009, the one active type has an incentive to sandbag, and output in the first contest is always reduced compared to a single contest benchmark. On the other hand, the effects are the opposite in Ederer 2010 before the midterm evaluation. All contestants have the incentive to bluff.

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4 Additionally, sandbagging can be used to take advantage of a tournament structure as described by Kräkel 2014.

5 See Moldovanu and Sela 2006 and Fullerton and McAfee 1999, respectively.
in order to discourage their opponents after the evaluation. This increases aggregate output before the evaluation relative to the setting where no evaluation was given.

Despite the incentives to hide true ability, partial pooling in the first contest leads to asymmetries in the second contest. These asymmetries lead to reduced aggregate output in the second contest. This is consistent with the results of Che and Gale 2003 who show that asymmetries between contestants reduces total sunk expenditures in the contest. Combining the effects of the first and second contests, we show that the reduction of output in the second contest always outweighs the potential increase in output in the first contest. Therefore the aggregate output of contestants in repeated contests is lower than the output of a single contest benchmark.6

The paper is organized as follows. In section 2 we introduce the model of the contest played in each stage. In section 3, we characterize the equilibrium of a single contest and discuss the payoffs to the contestants in terms of their ability and their opponent’s perception of their ability. In section 4 we solve for the unique equilibrium of the successive contest. In section 5 we discuss welfare implications. Section 6 concludes.

1.2 Stage Game

We first introduce the fundamentals of the contest that is played in each stage of the game of successive contests. Two contestants, player 1 and player 2, are independently endowed with ability, \( a_i \) for \( i = 1, 2 \). Ability can either be low, \( a = a_L \), or high, \( a = a_H \). The probability of each player having high ability is given by \( \Pr(a_1 = a_H) = \mu_1 \) and \( \Pr(a_2 = a_H) = \mu_2 \). The endowment of ability is private information for each player.

Other than probability of having high ability, the two contestants are ex-ante identical. Players compete by producing output, \( x \), which is a function of their ability and effort, \( e \). We assume the output function takes the form \( x(a, e) = a \cdot e \). The player that produces the most output receives a prize. If the two players produce the same output, then the prize is given randomly with each

---

6In dynamic contests with complete information and uncertain outcomes, contestants who fall behind in early stages will put in less effort moving forward (e.g. Harris and Vickers 1987). Because each period is a separate contest in our model, reduced output is not due to this discouragement effect. See Konrad 2012 for a detailed survey of dynamic contests under complete information.
contestant winning with equal probability. The prize has the same value for each contestant which is normalized to one.

We define $c(e)$ to be the cost function of effort for each player, regardless of ability. This function is assumed to be twice differentiable on the non-negative reals, increasing and weakly convex, with the cost of zero effort being zero.

We normalize ability so that $a_\ell = 1$ and $a_h > 1$. Then the marginal cost of output for the high and low ability workers are $\frac{1}{a_h}c'(\frac{x}{a_h})$ and $c'(x)$ respectively. The payoffs of each agent are

$$\tilde{\pi}_i(a_i, e_i) = \begin{cases} 1 - c(e_i), & x(a_i^{-i}, e_i^{-i}) < x(a_i, e_i) \\ 1/2 - c(e_i), & x(a_i^{-i}, e_i^{-i}) = x(a_i, e_i) \\ -c(e_i), & x(a_i^{-i}, e_i^{-i}) > x(a_i, e_i) \end{cases}$$

Given a strategy of their opponent, the expected payoffs of each contestant is equal to the probability that the contestant wins less his cost of effort. Here we abuse notation and let $x_i = x(a_i, e_i)$ for $i = 1, 2$. Then the expected payoffs for each player are

$$E[\tilde{\pi}_i(a_i, e_i)] = \Pr(x_i^{-i} < x_i) + \frac{1}{2} \Pr(x_i^{-i} = x_i) - c(e_i).$$

Since players know their own ability and the relationship between effort and output is deterministic, players choosing their effort level is equivalent to them choosing their output. Therefore, we will write the strategies of players in terms of output to ease comparisons of players with different abilities. Additionally, it puts players’ strategies in terms of what their opponents will observe. With this in mind, we write contestants’ payoffs in terms of output.

$$E[\pi_i(x_i, a_i)] = \Pr(x_i^{-i} < x_i) + \frac{1}{2} \Pr(x_i^{-i} = x_i) - c(x_i/a_i), \text{ for } i = 1, 2.$$

In the following sections, we will define players’ strategies in terms of output and describe the effort of players only in the context of providing intuition for the results. In the section immediately

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7The convexity of the cost function amplifies abilities effect on the marginal cost of output as $\frac{d}{dx} c'(\frac{x}{a}) = \frac{1}{a} c'(\frac{x}{a})$, where $c'(\frac{a}{a}) \leq c'(x)$.

8Equivalent to the notion of private information about ability is private information about the cost of output. Additionally, if the cost of effort is linear, then this framework is equivalent to an all-pay auction where values are private information and bids are observed.
following, players will maximize payoffs above in a single contest. In section 4, which contains the main model of successive contests, abilities for each player stay the same for both contests, and players will be maximizing the sum of payoffs for two contests, without discounting.

1.3 Single Contest

We first find the equilibrium strategies of players engaged in a single contest described in the previous section. This will serve two purposes when we analyze repeated contests. First, the payoffs of the contestants and the output they produce in a single contest will serve as a benchmark to better understand the strategic effects of an additional contest. Second, the equilibrium payoffs will be used to calculate continuation payoffs in the repeated contest setting. After the first of two contests, each player will believe that their opponent has high ability with some probability. For each set of these probabilities, the single contest equilibrium characterized in this section will be played in the second contest. Therefore, the expected payoffs of contestants in this section will be equal to the continuation payoffs of the second contest in the next section.

For the remainder of this section, we name our two players the strong player and the weak player, so that $i = s, w$ where $\mu_s \geq \mu_w$. This implies that, the strong player, player $s$, is at least as likely to have high ability as the weaker player, player $w$, ex-ante. However, this does not rule out the possibility of the weaker player having high ability or the stronger player having low ability, or both.

1.3.1 Strategies

The strategies of each player consist of output distributions for both high and low ability realizations. We define $L_i(x|\mu_i, \mu_{-i}) \equiv \Pr(x \leq x|a^l = a, \mu_i, \mu_{-i})$ and $H_i(x|\mu_i, \mu_{-i}) \equiv \Pr(x \leq x|a^h = a, \mu_i, \mu_{-i})$ which denote the distribution of output of player $i$ given he has low ability and high ability respectively. Additionally, we define $F_i(x|\mu_i, \mu_{-i}) \equiv \Pr(x \leq x|\mu_i, \mu_{-i})$ to be the ex-ante output distribution of player $i$. This is also the output distribution of player $i$ from the perspective of player $-i$. For these distributions to be consistent with the information sets of the
contestants, we must have \( F_i(x|\mu_i, \mu_{-i}) = (1 - \mu_i)L_i(x|\mu_i, \mu_{-i}) + \mu_iH_i(x|\mu_i, \mu_{-i}) \). Additionally, we let \( \ell_i(x|\mu_i, \mu_{-i}), h_i(x|\mu_i, \mu_{-i}) \) and \( f_i(x|\mu_i, \mu_{-i}) \) be the densities induced from \( L_i(x|\mu_i, \mu_{-i}), H_i(x|\mu_i, \mu_{-i}) \) and \( F_i(x|\mu_i, \mu_{-i}) \). For simplicity, we suppress the probabilities, \((\mu_i, \mu_{-i})\), from the notation of the output distributions for the remainder of this section.

We denote support of the strategies of each type of player by \( X_i^\ell \equiv \{x : \ell_i(x) > 0\} \) and \( X_i^h \equiv \{x : h_i(x) > 0\} \). For a given expected output distribution of their opponent, the best response set for a given contestant and given ability is defined as

\[
\text{BR}_i(a^i) \equiv \{x : E[\pi^i(x^i, a^i)] \geq E[\pi^i(\tilde{x}^i, a^i)], \forall \tilde{x}^i \geq 0\}.
\]

An equilibrium is a set of output distributions, \((L_s(x), H_s(x), L_w(x), H_w(x))\), such that \( X_i^\ell \subseteq \text{BR}_i(a^\ell) \), and \( X_i^h \subseteq \text{BR}_i(a^h) \) for \( i = s, w \). The general properties of an equilibrium are outlined in the following lemma; the proof is in the appendix.

**Lemma 1.3.1.** In any equilibrium, players’ distributions of output, \( H_s(x), L_s(x), H_w(x), \) and \( L_w(x) \), are continuous on \((0, x^*)\), where

\[
x^* \equiv \sup\{\text{BR}_s(a^\ell) \cup \text{BR}_s(a^h)\} = \sup\{\text{BR}_w(a^\ell) \cup \text{BR}_w(a^h)\}
\]

and \( \inf\{\text{BR}_s(a^\ell) \cup \text{BR}_s(a^h)\} = \inf\{\text{BR}_w(a^\ell) \cup \text{BR}_w(a^h)\} = 0 \).

From these properties, it must be that for both the strong and weak contestants, the combined best response sets of the low ability and high ability type must be an interval. Moreover, this interval for each contestant is the same. Intuitively, if the supremum of the interval was smaller for one of the contestants, then the other contestant would be wasting effort by sometimes producing more output than would ever be necessary to win the contest. Moreover, if there are gaps of positive measure in this combined interval for either contestant, then the opponent would have no incentive to produce output in the interior of the gap. However, this leads to a gap in the best response intervals for both players, but this cannot happen. By this argument, which is formalized in the proof of Lemma 3.1, the infimum of the best response interval of each player must be 0. Lastly,

\footnote{Here we use the extended definition of density using Dirac-delta functions where necessary to properly define these densities when their corresponding distributions have mass points.}
if either player played a positive output with positive probability, this implies that their opponent must have a gap in their combined best response interval behind this output. We argued that this gap cannot exist.

While the fundamentals of this model are somewhat different to those studied by Siegel 2014, the properties of the equilibrium strategies above are the same. Moreover, he shows that when types are independently drawn and value of winning the contest increases in type, then there is a unique equilibrium which must be monotonic, i.e., for both contestants, all actions of the high type are at least as high as all actions of the low type. These properties hold in our setting where types are abilities and bids are outputs. This implies that there is a unique equilibrium that is monotonic, so that for \( i = s, w \) and any \( x \in BR_i(a_h) \) and \( x' \in BR_i(a_{i'}) \) it must be that \( x' < x \). This fact, combined with the previous lemma implies that \( x_i^* \equiv \sup\{BR_i(a_{i'})\} = \inf\{BR_i(a_h)\} \), for \( i = s, w \).

**Proposition 1.3.2 (Unique Equilibrium - Single Contest).** There is a unique equilibrium, \( (L_s^*(x), H_s^*(x), L_w^*(x), H_w^*(x)) \), where \( \overline{X}_i^s = BR_i(a_i) = [0, x_i^*] \), \( \overline{X}^i_s = BR_i(a_h) = [x_i^*, x^*] \) for \( i = s, w \) and \( 0 \leq x_i^* \leq x_w^* \leq x^* \). These output distributions are given by

\[
L_s^*(x) = \begin{cases} 
\frac{c(x)}{\mu_s} & 0 \leq x \leq x_s^*, \\
1 & x_s^* \leq x \leq x^*
\end{cases},
H_s^*(x) = \begin{cases} 
0, & 0 \leq x \leq x_s^*, \\
\frac{c(x)}{\mu_s} - \frac{c(x)}{\mu_w} & x_s^* \leq x \leq x_w^*, \\
1 + \frac{c(x/a_h)}{\mu_w} - \frac{c(x/a_h)}{\mu_w} & x_w^* \leq x \leq x^*
\end{cases},
\]

\[
L_w^*(x) = \begin{cases} 
\frac{c(x)}{1-\mu_w} + L_w^*(0), & 0 \leq x \leq x_s^*, \\
1 + \frac{c(x/a_h)}{1-\mu_w} - \frac{c(x/a_h)}{1-\mu_w} & x_s^* \leq x \leq x_w^*, \\
1 & x_w^* \leq x \leq x^*
\end{cases},
H_w^*(x) = \begin{cases} 
0, & 0 \leq x \leq x_w^*, \\
1 + \frac{c(x)}{\mu_w} - \frac{c(x)}{\mu_w} & x_w^* \leq x \leq x^*
\end{cases},
\]

where \( x_s^* = c^{-1}(1-\mu_s) \), \( x_w^* = c^{-1}(1-\mu_w) \), and \( x^* = a_h c^{-1} \left( \mu_w + c \left( \frac{c^{-1}(1-\mu_s)}{a_h} \right) \right) \), and \( L_w^*(0) = \frac{1}{1-\mu_w} \left[ \mu_s - \mu_w - c \left( \frac{c^{-1}(1-\mu_w)}{a_h} \right) - c \left( \frac{c^{-1}(1-\mu_s)}{a_h} \right) \right] \).

Here we highlight the important details of the construction of the equilibrium; see the appendix for the technical details.

First from Lemma 3.1, the combined best response sets for the strong player and the weak player are the same, which we denote by the interval, \([0, x^*]\). Second, since the equilibrium is

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10While, in this section, we borrow heavily from the properties of Siegel 2014 and follow his construction to characterize the unique equilibrium, the setting is slightly different and therefore we cannot directly use his results. His contestants differ on value of the prize rather than ability and the cost of output, which is a simple bid, is linear. Monotonicity holds in the current model as a player’s marginal cost for a given output is ranked by type, where his contestants marginal value is ranked by type for all bids. A simple transformation makes these two settings isomorphic.
monotonic, the best response sets of each ability are disjoint for each player, with the set for high ability ranging over larger outputs than the set for low ability. Third, since the strong player is more likely to be high ability, the length of the best response set of high ability for is longer for the strong player. We depict the basic structure of these best response sets in Figure 1.

\[
\begin{array}{c|c|c}
\text{Strong:} & BR_s(a_\ell) & BR_s(a_h) \\
& x_s^* & x^* \\
\text{Weak:} & BR_w(a_\ell) & BR_w(a_h) \\
& x_w^* & x^* \\
\end{array}
\]

Figure 1: Representation of best response sets of the strong and weak players.

To characterize the output distributions of each contestant for each ability level, we first find the expected output distributions that each contestant must face to be indifferent between output levels when each best response set. We start from \( x^* \) and work backwards toward zero. We will be able to pin down the value of \( x^* \) and subsequently \( x_w^* \) and \( x_s^* \), using that fact that only one player can have a mass point at \( x = 0 \), \( F_i(0) > 0 \), and for the other \( F_{-i}(0) = 0 \) and \( F_{-i}(x) > 0 \) for \( x > 0 \). Therefore the expected output distribution that hits zero first will pin down the \( x^* \).

For output levels between \( x^* \) and \( x_w^* \), the high ability type of each contestant must be indifferent. This means that the marginal benefit of increasing output must equal the marginal cost, i.e. \( f_i(x) = c'(x/a_h) \) for \( x \in (x_w^*, x^*) \) and \( i = s, w \). This implies that the output distributions for this range of output are the same for both contestants. For output between \( x_w^* \) and \( x_s^* \), \( f_s(x) \) must equal the marginal cost of the weak contestant and \( f_w(x) \) is equal to the marginal cost of the strong contestant. Since this range is a best response of the low ability type of the weak player and the high ability type of the strong player, this implies that \( f_s(x) = c'(x/a_\ell) = c'(x) \) and \( f_w(x) = c'(x/a_h) \). Lastly, for the output range of \( x_s^* \) to \( x = 0 \), the low ability type of each player must be indifferent, and therefore \( f_i(x) = c'(x) \).

Given the densities for all levels of output, we can compute the ex-ante expected output distributions for each player given the condition that \( F_i(0) = 0 \) for one contestant. This must be the strong contestant as \( f_s(x) \geq f_w(x) \) for all output levels between 0 and \( x^* \). Intuitively, it is the weaker contestant that has a mass point at zero, i.e. if this player draws low ability, they may not put any
effort into the contest.

The expected output distribution of the weak and strong players are

\[
F_s^*(x) = \begin{cases} 
    c(x), & 0 \leq x \leq x_w^* \\
    1 - \mu_w - c \left( \frac{c^{-1}(1-\mu_w)}{a_h} \right) + c \left( \frac{x}{a_h} \right), & x_w^* \leq x \leq x^* 
\end{cases}
\]

\[
F_w^*(x) = \begin{cases} 
    F_w(0) + c(x), & 0 \leq x \leq x_s^* \\
    1 - \mu_w - c \left( \frac{c^{-1}(1-\mu_w)}{a_h} \right) + c \left( \frac{x}{a_h} \right), & x_s^* \leq x \leq x^* 
\end{cases}
\]

An example of these distributions with \(a_h = 2\) and \(c(e) = \frac{1}{2}e^2\) is depicted below.

![Graph showing c.d.f. distributions](image-url)

Figure 2: Expected output distributions of contestants in a single contest.

Because the best response sets of low and high ability types are disjoint for each contestant, we can recover the output distributions of both high and low ability types of both contestants.
1.3.2 Payoffs

From the equilibrium of the single contest, our main objects of interest are the payoffs of the contestants. Given the uniqueness of this equilibrium for any pair of probabilities \((\mu_s, \mu_w)\), these payoffs will equal the expected payoffs the contestants expect to recieve in the second contest, given the beliefs that result from the first contest. We denote the payoffs, which are functions only of the contestant’s ability and the ex-ante probabilities of each contestant being high ability as \(v_i(\mu_i, \mu_{-i}, a^i) = E[\pi^i(\hat{x}^i, a^i)]\) where \(\hat{x}^i \in BR_i(a^i)\).

**Corollary 1.3.3.** The ex-interim expected payoff of each contestant is

\[
v_s(\mu_s, \mu_w, a_h) = v_w(\mu_w, \mu_s, a_h) = 1 - \mu_w - c \left( \frac{c^{-1}(1 - \mu_w)}{a_h} \right) \]

\[
v_s(\mu_s, \mu_w, a_\ell) = \mu_s - \mu_w - \left[ c \left( \frac{c^{-1}(1 - \mu_w)}{a_h} \right) - c \left( \frac{c^{-1}(1 - \mu_s)}{a_h} \right) \right] \]

\[
v_w(\mu_w, \mu_s, a_\ell) = 0.\]

**Proof.** Each type of each contestant is indifferent between all outputs in their best response set. In particular, because \(x^*\) is in the best response set of high ability contestants, their expected payoffs are equal to the value of winning less the cost of producing output \(x^*\), since, if they produce \(x^*\),
they will win with certainty.

\[ v_s(\mu_s, \mu_w, a_h) = v_w(\mu_w, \mu_s, a_h) = 1 - c(x^*/a_h) = 1 - \mu_w - c \left( \frac{c^{-1}(1 - \mu_w)}{a_h} \right) \]

Similarly, since \( x = 0 \) is in the best response set of a low ability contestant, the expected payoffs of low ability contestants is probability they win, given they exert no effort. This is the probability that your opponent puts in no effort.\(^{11}\)

\[ v_s(\mu_s, \mu_w, a_L) = (1 - \mu_w)L_w(0) = \mu_s - \mu_w - \left[ c \left( \frac{c^{-1}(1 - \mu_w)}{a_h} \right) - c \left( \frac{c^{-1}(1 - \mu_s)}{a_h} \right) \right] \]

\[ v_w(\mu_w, \mu_s, a_L) = (1 - \mu_s)L_s(0) = 0 \]

For both the strong and weak contestants who are high ability, the expected payoff is entirely determined by the value of \( x^* \). This value is pinned down by the ex-ante expected output distribution of the stronger contestant which is constructed by making the weaker player indifferent. Therefore, \( x^* \) is a determined entirely by \( \mu_w \), the probability that the weaker player has high ability. Intuitively, high ability players are confident they can win, but the overall competitiveness of the contest will determine how much effort they need to exert to do so. This payoff decreases as \( \mu_w \) increases, implying that increased competition will increase the effort of high ability players, decreasing their expected payoff.

For contestants that are low ability, expected payoffs are determined by how often they can freely when a contest. The strong contestant will exert no effort with probability zero, while the weak contestant will exert no effort with a probability that increases with the strength of their opponent. Intuitively, a low ability player becomes discouraged when he believes that his opponent is likely to have high ability. Therefore, the low ability type of the weaker contestant will never when a contest when they exert no effort leading to an expected payoff of zero. The stronger contestant who has low ability will have positive expected payoffs which increase with the contestant’s relative strength.

\(^{11}\)Here we technically are assuming the contestant wins all ties at zero, but if the agent exerts a tiny amount of effort and we let that effort shrink to zero, then this is the payoff of the agent in the limit. Since the payoffs are continuous, these limits must be the payoffs of the low ability contestants.
The comparative statics of these payoffs are important for the analysis of the two contest model as the payoffs in the second contest are the same as in the single contest given the strength of each player. The contestants can effect their perceived strength in this second contest through their choice of output in the first contest. For a contestant with high ability, payoffs decrease when the contest appears more competitive. Therefore, they may prefer to look weak entering the second contest in order to reduce the perceived level of competition. On the other hand, the payoffs of low ability contestants increase when they appear strong to their opponent. These countervailing incentives, which will be formalized in the following section, are a significant strategic force in the first contest of the two contest model.

### 1.3.3 Output

The second outcome of interest in the single contest is the expected total output of the contestants. We will use this output as a benchmark to compare with per-period output in the repeated contest model. In order to have a closed form solution for expected output, we impose a parametric structure to the cost function.

**Corollary 1.3.4.** If we let \( c(e) = ke^\alpha \), with \( k > 0 \) and \( \alpha \geq 1 \), then \( E[x_s(\mu_s, \mu_w, a^s) + x_w(\mu_w, \mu_s, a^w)] \), the ex-ante expected aggregate output is

\[
\left( \frac{\alpha}{\alpha + 1} \right) \left( \frac{1}{k} \right)^{1/\alpha} \left[ (1 - \frac{1}{a_h^{\alpha/\alpha}}) \left( 1 - \mu_w \right)^{\frac{\alpha+1}{\alpha}} + (1 - \mu_s)^{\frac{\alpha+1}{\alpha}} \right] + 2a_s \left( \mu_w + \frac{1 - \mu_w}{a_h^{\alpha/\alpha}} \right)^{\frac{\alpha+1}{\alpha}}.
\]

A significant determinant of this total output is \( \mu_w \), or the probability the weaker contestant has high ability. Additionally, while an increase in \( \mu_w \) will increase output, an increase in \( \mu_s \) will have the opposite effect.

**Corollary 1.3.5.** For a fixed value of \( \mu_w \), an increase in \( \mu_s \) leads to a reduction in expected aggregate output.

**Proof.**

\[
\frac{\partial}{\partial \mu_s} E[x_s(\mu_s, \mu_w, a^s) + x_w(\mu_w, \mu_s, a^w)] = - \left( \frac{1}{k} \right)^{1/\alpha} \left( 1 - \frac{1}{a_h^{\alpha/\alpha}} \right) (1 - \mu_s)^{\frac{1}{\alpha}} < 0.
\]

These are implied by the assumptions of a cost function that is strictly increasing and weakly convex.
This reflects the decrease in the overall competition of the contest when one contestant is stronger than the other. Therefore the overall competitiveness of a contest, in the sense of expected output produced, is effected both by the absolute strength of the two contestants and their strengths relative to each other. Intuitively, if a very strong contestant, \( \mu_s \approx 1 \), and a very weak contestant, \( \mu_w \approx 0 \), compete against each other, the weak person would likely put in little or no effort, thinking that winning is very unlikely. Additionally, the strong contestant would know the weak contestant is putting in low effort and reduce their effort accordingly.

### 1.4 Repeated Contests

We now turn to the main model of the paper, a game of two repeated contests where agents are privately informed about their ability. We assume the contestants are symmetric, ex-ante, and they are equally likely to have low or high ability. The contestants maximize the sum of expected payoffs in each contest, and their abilities do not change after the initial draw of types.

After realizing their respective abilities, contestants play the first contest. Once the first contest ends, the outputs of each contestant become public information. Contestants use this information to update their beliefs about their opponent’s ability prior to competing in the second contest. Because these outputs are commonly observed, first order beliefs are sufficient for characterizing optimal strategies. In particular, beliefs which are consistent with the first period equilibrium strategy will have the same effect on the strategies of the second contest as the ex-ante probabilities of being high ability have in the single contest model.

Given a strategy of player \( -i \), we denote expected payoffs of player \( i \) over the two contests as

\[
E[\pi^i(x^1_i, x^2_i, a^i)] = E[\pi^i_1(x^1_i, a^i)] + E[\pi^i_2(x^2_i, a^i)|\mu(x^1_i)]
\]

\[
= \Pr(x^{-i}_1 < x^1_i) + \frac{1}{2} \Pr(x^{-i}_1 = x^1_i) - c(x^1_i/a^i)
\]

\[
+ \Pr(x^{-i}_2 < x^1_i|\mu(x^1_i)) + \frac{1}{2} \Pr(x^{-i}_2 = x^2_i|\mu(x^1_i)) - c(x^2_i/a^i) \text{ for } i = 1, 2.
\]
1.4.1 Strategies

For each player $i = 1, 2$, we let $L_i^1(x)$ and $H_i^1(x)$ denote the first period output distributions of a contestant with low ability and high ability respectively. Then the ex-ante expected output distribution is $F_i^1(x_1) = \frac{1}{2}L_i^1(x_1) + \frac{1}{2}H_i^1(x_1)$, for $i = 1, 2$. Additionally, we denote $f_i^1$, $\ell_i^1$ and $h_i^1$ as the densities that are induced from the distribution functions $F_i^1$, $L_i^1$ and $H_i^1$. Lastly, define $X_i^{h,i} = \{x|h_i^1(x) > 0\}$ and $X_i^{\ell,i} = \{x|\ell_i^1(x) > 0\}$. Since contestant’s are symmetric, we will restrict attention to equilibria that are symmetric.

A set of output distributions $\{H_i^1(x_1), L_i^1(x_1), H_i^2(x_2|\mu_i, \mu_{-i}), L_i^2(x_2|\mu_i, \mu_{-i})\}$ for $i = 1, 2$ form a symmetric perfect Bayesian equilibrium (SPBE) for two successive contests if

1. strategies are symmetric: $H_i^1(x) = H_i^2(x)$, $L_i^1(x) = L_i^2(x)$, $H_i^1(x|\mu_1, \mu_2) = H_i^2(x|\mu_1, \mu_1)$, and $L_i^1(x|\mu_1, \mu_2) = L_i^2(x|\mu_2, \mu_1)$.

2. contestants play the unique Bayes Nash equilibrium in the second contest: for $i = 1, 2$ and for every $(\mu_i, \mu_{-i})$,

$$(H_i^2(x|\mu_i, \mu_{-i}), L_i^2(x|\mu_i, \mu_{-i})) = \begin{cases} \left(H^*_w(x|\mu_i, \mu_{-i}), L^*_w(x|\mu_i, \mu_{-i})\right), & \text{if } \mu_i \leq \mu_{-i} \\ \left(H^*_a(x|\mu_i, \mu_{-i}), L^*_a(x|\mu_i, \mu_{-i})\right), & \text{if } \mu_i > \mu_{-i} \end{cases}$$

3. players update beliefs according to Bayes rule when feasible:

$\mu_i = \mu(x_i) = \frac{h_1(x_i)}{h_1(x_i) + \ell_1(x_i)}$, for $i = 1, 2$, and

4. given (2) and (3), contestants always choose an optimal output in the first contest: for $i = 1, 2$, for every $x_1^i \in X_i^{\ell,i}$ player $i$ chooses an

$$x_1^i \in \arg\max_{x_1^i} E[\pi(x_1^i, x_2^i(\mu(x_1^i), \mu(x_1^{-i}), a_\ell), a_2)] \equiv BR_i(a_\ell),$$

and for every $x_1^i \in X_i^{h,i}$ player $i$ chooses an

$$x_1^i \in \arg\max_{x_1^i} E[\pi(x_1^i, x_2^i(\mu(x_1^i), \mu(x_1^{-i}), a_h), a_h)] \equiv BR_i(a_h).$$

---

13 Again we are using the extended definition of density using Dirac-delta functions where necessary.

14 Using the extended definition of density allows agents to update their beliefs even when they see their opponent produce an output where the distribution has a mass point. For example, if the $H_1$ has a mass point at $x$, while $L_1$ does not, this definition implies $\mu(x_1) = 1$. 

---
In the single contest section, we found the unique strategies that satisfy condition (2). To find the strategies of each type of player in the first period that satisfy (4), we first examine how output in the first contest affects expected payoffs in the second contest. From condition (2), for a given set of beliefs that arise from outputs in the first period, the expected payoffs for each player are $v_i(\mu(x^i_1), \mu(x^{-i}_1), a^i)$. Therefore the payoffs to player $i$ for the two contests can be written in terms of output in the first contest.

$$E[\pi^i(x^i_1, x^i_2(\mu(x^i_1), \mu(x^{-i}_1), a^i), a^i)] = E[\pi^i_1(x^i_1, a^i)] + E[v_i(\mu(x^i_1), \mu(x^{-i}_1), a^i)]$$

Previous analysis showed that for a given belief of the opponent, a high ability contestant will have higher expected payoffs if his opponent believes he is low type with high probability. Furthermore, it showed that a low ability contestant will have higher expected payoffs if his opponent believes he is high type with high probability. The following proposition shows that in expectation, high ability players always prefer to look weaker entering the second contest, while low ability players always prefer to look stronger. The proof of this proposition and other results of this section are relegated to the appendix.

**Proposition 1.4.1** (Countervailing Incentives). For all $\mu_i \in (0, 1)$, expected payoffs in the second contest decrease for high ability players as $\mu_i$ increases, $\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_h)] < 0$, and increase with $\mu_i$ for low ability players, $\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_l)] > 0$.

In particular, the marginal effect of beliefs on payoffs in the second contest is given by

$$\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_h)] = d(\mu_i)(F_{\mu_{-i}}(\mu_i) - 1) < 0$$

and

$$\frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_l)] = d(\mu_i)F_{\mu_{-i}}(\mu_i) > 0,$$

where $d(\mu_i) \equiv \left[1 + \frac{\partial}{\partial \mu_i} c \left(\frac{c^{-1}(1 - \mu_i)}{a_h}\right)\right]$ and $F_{\mu_{-i}}(\mu_i) = \Pr(\mu_{-i} \leq \mu_i)$. The convexity of the cost of effort implies that $d(\mu_i) \in \left[\frac{a_h}{a_h}, 1\right]$ for all $\mu_i$, which guarantees that these incentives are strict.

Differences in marginal cost of effort for the first contest, and the incentives stemming from the second contest combine to require that the belief function, $\mu(x)$, increases in first period output, in
equilibrium. If it does not, a higher output would result in a lower belief about the ability of the contestant. From condition (3), this implies that the higher output must be in the $BR(a_\ell)$ and the lower output will be in $BR(a_h)$. However, if the lower output $\in BR(a_h)$, then the low ability player must strictly prefer the lower output to the higher output, as the marginal cost of this contestant for the high output is larger and the expected payoffs in the second contest would be higher for the lower output.

**Corollary 1.4.2** (Monotonic Beliefs). In every SPBE, $\mu(x)$ is weakly increasing in $x$ for all $x \in X_1 = X_1^h \cup X_1^\ell$.

In addition to restricting the belief function on the equilibrium path, the countervailing incentives also eliminate multiplicity of equilibria. Therefore, in this setting, there is a unique symmetric equilibrium even without additional equilibrium refinements.

**Theorem 1.4.3** (Uniqueness of Equilibrium). There is a unique symmetric perfect Bayes Nash equilibrium \{ $(L_1^*(x_1), L_2^*(x_2|\mu_i, \mu_{-i}))$, $(H_1^*(x_1), H_2^*(x_2|\mu_i, \mu_{-i}))$ \}.

The following lemmas show that equilibrium strategies in the first contest have no atoms and there are no gaps in the best response sets.

**Lemma 1.4.4.** There is no output that is played with positive probability and $\Pr(win|x) = F_1(x)$ is continuous.

**Lemma 1.4.5.** $BR(a_\ell)$ and $BR(a_h)$ are intervals where $0 = x_{\ell,*} \leq x_{h,*} < x^*_\ell \leq x^*_h$ and we define $x_{\ell,*} = \inf\{BR(a_\ell)\}$, $x^*_\ell = \sup\{BR(a_\ell)\}$, $x_{h,*} = \inf\{BR(a_h)\}$ and $x^*_h = \sup\{BR(a_h)\}$.

Lemma 4.4 implies that first period payoffs are continuous in output. Furthermore, there can be no gaps in the best responses of each type of contestant. In other games with signaling, gaps may exists when off path beliefs prevent players from choosing actions. However, any pathological belief will benefit at least one type of contestant due to their countervailing incentives. Therefore, if there were gaps in best response sets, and therefore outputs where density is zero for contestants of both high and low ability, then one of the player types would benefit from deviating to an output in the gap.
Additionally, the countervailing incentives imply that \( \text{BR}(a_\ell) \cap \text{BR}(a_h) \), the intersection of the best response sets of the low and high ability players, is non trivial. In contrast to the equilibrium properties of a single contest, this overlap shows that there are outputs that both high and low ability contestants could optimally choose.

\[
\begin{align*}
0 & \quad x_{h,*} \quad x_{\ell,*} \quad x_h^* \\
\text{BR}(a_\ell) & \quad \text{BR}(a_h)
\end{align*}
\]

Figure 4: Representation best response sets of the high ability and low ability players in the first contest.

The lemmas above show that there are three distinct intervals in each equilibrium. These intervals are partitioned by the best response sets of the high and low ability players. The first is the set of outputs where only low ability players are optimizing: \( [0, x_{h,*}) = \{ \text{BR}(a_\ell) \setminus \text{BR}(a_h) \} \). Next is the set of outputs where both high and low ability players are optimizing \( [x_{h,*}, x_\ell^*] = \{ \text{BR}(a_\ell) \cap \text{BR}(a_h) \} \). Lastly is the set of outputs where only high ability players are optimizing: \( (x_\ell^*, x_h^*) = \{ \text{BR}(a_h) \setminus \text{BR}(a_\ell) \} \). For each output where \( x \in \text{BR}(a_\ell) \), the low ability player’s first order condition must hold and likewise for each \( x \in \text{BR}(a_h) \) the high ability player’s first order condition must hold.

Continuous output distribution functions and cost functions along with indifference over best response sets imply that the belief function must also be continuous. Therefore, on the three intervals, the belief function must be 0, weakly increasing, and 1 respectively.

**Corollary 1.4.6.** The belief function and the distribution functions of output are continuous in output on \([0, x_h^*]\). Additionally, the belief function is given by \( \mu(x) = 0 \) for all \( x \in [0, x_{h,*}] \), \( \mu(x) = 1 \) for all \( x \in [x_{\ell,*}, x_h^*] \) and is weakly increasing on \((x_{h,*}, x_\ell^*)\).

Conditions for \( x \) being in \( \text{BR}(a_h) \) and \( \text{BR}(a_\ell) \) are

\[
\text{BR}(a_h) : F_1^*(x) + E[v_i(\mu(x), \mu(x^{-i}), a_h)] - c(x) \left( \frac{x}{a_h} \right) = k_h
\]
\[
\text{BR}(a_\ell) : F_1^*(x) + E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] - c(x) = k_\ell = 0
\]
• For the range of $0 \leq x < x_{h,*}$ we have that $E[v_i(\mu(x), \mu(x^{-i}), a_i)] = 0$ as $\mu(x) = 0$. Therefore we have that $F_1^*(x) = c(x)$ for all $x \in [0, x_{h,*}]$.

• For the range $x_{\ell}^* < x \leq x_{h}^*$, $E[v_i(\mu(x), \mu(x^{-i}), a_h)] = E_{x_j}[v_i(1, \mu(x_j), a_h)] \equiv v_h$. Then we have $F_1^*(x) + v_h = c(x/a_h) + k_h$, for all $x \in [x_{\ell}^*, x_{h}^*]$.

• For the range $x_{h,*} \leq x \leq x_{\ell,*}$, for all $x \in \{X_{\ell}^1 \cup X_{h}^1\}$ we know $x \in \{X_{\ell}^1 \cap X_{h}^1\}$. Therefore, both low and high ability players are indifferent between all outputs in this range. Because the marginal cost of the low ability player is always more than the marginal cost of the high ability player, this can only be true if increasing output benefits the low ability player more than the high ability player. This indifference condition determines the belief function over this interval and as the difference in marginal benefits of increasing output for the high ability and low ability players must equal the difference in marginal costs that they face today.

$$\mu'(x)d(\mu(x)) = c'(x) - \frac{1}{a_h}c'(x/a_h)$$

Combining the condition on the belief function with the best response conditions of both the high and low ability contestants gives us the a condition on the density function over this interval.

$$f_1^*(x) = \frac{\partial}{\partial x}c(x)(1 - F_1^*(x)) + \frac{\partial}{\partial x}c\left(\frac{x}{a_h}\right)F_1^*(x) \quad (\dagger)$$

Given $f_1(x)$ on $[0, x^*]$, the endpoints $x_{h,*}, x_{\ell}^*$, and $x_{h}^*$ can be solved for using the following conditions, $\mu(x_{\ell}^*) = 1$, $L_1(x_{\ell}^*) = 1$, continuity of $F_1$ at $x_{\ell}^*$ and $F_1(x_{h}^*) = 1$. Additionally, the equilibrium strategies of high ability and low ability players are determined by $f_1(x)$ and $\mu(x)$. 

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1.4.2 Discussion

Observed output in the first contest from each contestant will land in one of three intervals. If the output is between 0 and $x_{h,*}$, then the player is revealed to have low ability, while output between $x_{h}^*$ and $x_{h}^*$ must have been produced by a player with high ability. Output between these ranges may have been produced by a player with either high or low ability.

Low ability contestants either choose a low effort which produces an output that reveals their ability to their opponent, or they decide to bluff by choosing a higher effort which produces an output that may have been produced by a high ability contestant. This higher effort level will provide a lower expected payoff in the first contest as the additional cost of effort will exceed the benefit of increasing the probability of winning this contest. These contestants are willing to put in the additional effort to have a stronger position entering the second contest, as they benefit from appearing to have high ability. Therefore ex-interim expected payoffs for a low ability player are negative in the first contest and positive in the second contest.

High ability contestants either choose a high effort which produces an output that reveals their ability to their opponent, or they decide to sandbag by choosing a lower effort which produces an output that may have been produced by a low ability player. Since effort is less costly to high...
ability players, the lower cost of reduced effort will not offset the lower winning probability of the first contest. These players are willing to produce the lower output in the first contest as they benefit from appearing to have a low ability entering the second contest. The per period expected payoffs are depicted in Figure 6.

![Figure 6: Beliefs and expected payoffs as a function of first period output](image)

Strategic incentives will reduce the expected output of high ability players and increase the expected output of low ability players in the first contest as compared to the expected output each type of player in a single contest. These distortions in effort in the first contest effect the aggregate output and payoffs as well as the potential outcomes of each contest.

1.5 Welfare

We now analyze the welfare effects of the incentives in the first round of players to hide their ability and the information released between contests. We compare the outcomes, expected payoffs and the expected output of the contestants in the repeated contest model to a benchmark where players compete with the same payoff structure but the possibility of signaling private information through actions is suppressed.
The benchmark can be thought of in two different ways. First, there are two separate contests as before, but the output (and winner) is not revealed after the first contest. Winners of both contests are revealed after the second contest. Second, you can think of it as one longer contest, where either the cost function is stretched by factor of two, or cost is a function of the intensity of effort of the contestant over the two periods. In the later case, contestants would choose an intensity level that they would maintain over the two periods.

For the benchmark, we can take the strategies from the single contest analysis. To compare this model to the repeated contest model, we assume the contestants are symmetric ex-ante, i.e. initial beliefs are taken to be $\mu_s = \mu_w = \frac{1}{2}$. Additionally, to compute expected aggregate output, we assume that the cost function of effort is given by $c(e) = ke^{\alpha}$. The distribution function of output for each contestant is

$$F(x) = \begin{cases} 
  kx^{\alpha}, & 0 \leq x \leq \left(\frac{1}{2k}\right)^{1/\alpha} \\
  \frac{1}{2A} + \frac{k}{a_h^{\alpha}}x^{\alpha}, & \left(\frac{1}{2k}\right)^{1/\alpha} \leq x \leq \left(\frac{a_h^{\alpha} + 1}{2k}\right)^{1/\alpha}
\end{cases}$$

### 1.5.1 Outcomes

In the equilibrium for two successive contests, overlapping best response sets give a low ability player a positive probability beating a high ability player in the first contest. Additionally, if the low ability player enters the second contest in a stronger position, which is always the case when they win the first contest, they may also win the second contest.

**Corollary 1.5.1** (Surprise Victories). A low ability player has a positive probability of winning each contest, even if they are competing against a high ability player.

In contrast to this, in the benchmark model, the best response sets for high and low ability players are disjoint, implying that a high ability player will always win a contest against a low ability player.
1.5.1.1 Application to Multi-Stage Tournaments

This fact is used to motivate a multi-stage tournament by both Moldovanu and Sela 2006 and Fullerton and McAfee 1999. However, as shown in Ye 2007, efficient entry into later rounds can not be guaranteed when contestants have private information that is preserved between rounds.

Our results show that when private information about ability is preserved and contestants can learn about future opponents through their past output, then the best player may not win the tournament, and in fact, may not make it to the final round. To see this connection, consider a four player, two stage tournament where the payoffs of each stage is identical to each contest in the current model. Output from the first stage is observed by all four players before the second stage. Because these outputs are sufficient to characterize the second round strategies, the countervailing incentives in the first round will be consistent with the current model despite the fact that a first round winner will be playing a different opponent in the second round. The difference from the current model stems from the fact that the loser of the first stage does not compete in the second stage. This would alleviate some of the distortion as high ability players would now be more motivated to win the first stage, but given the fact that they make the final round, they would still prefer to appear weak. This would leave the possibility of high ability players sandbagging in the first round and therefore not making the final round.

1.5.2 Payoffs

**Theorem 1.5.2 (Increased Aggregate Payoffs).** The expected payoff for the low ability player is the same per contest as the single contest benchmark, while the high ability player receives a higher expected payoff per contest.

**Proof.** Equilibrium payoffs of a low ability player in the two contest model are 0. This is equal to the expected payoff in the single contest benchmark.

Payoffs of the high type in the benchmark game are given by $\frac{1}{2A}$. Payoffs for the low type are zero. If no information is revealed, then over two periods (cost functions are stretched by 2), the expected payoffs for a high ability player are $\frac{1}{A}$. If we compare this to the two period payoff of the
high type in successive contests where information is revealed, then we see that it is higher as
\[
k_h - \frac{1}{A} = \frac{1}{2A} + \frac{(2A - 1)(1 - a_h^\alpha e^{-1/A})}{2Aa_h^\alpha (1 - e^{-1/A})} = \frac{1}{2A} \left( 1 - \frac{(2A - 1)(e^{-1/A} - 1/a_h^\alpha)}{1 - e^{-1/A}} \right) > 0
\]
Therefore, two period payoffs of the high ability player are higher than one long contest.

High ability players benefit from the compression of potential outputs that arise in the first contest from players hiding information about their ability. Therefore in successive contests, we expect to see lower overall output and a higher reward for players with higher ability who are benefiting from the reduction in effort levels.

![Figure 1.6: Payoffs of high ability player in terms of ability ratio, cost: c(e) = e and c(e) = e^2.](image)

### 1.5.3 Output

**Theorem 1.5.3 (Reduced Aggregate Output).** *Ex-ante expected aggregate effort of the players in each of the two contests is less than in the single contest.*

*Proof.* Ex-ante payoffs for the players are \( k_h \frac{1}{2} \) in the two period game, and \( \frac{1}{2A} \) in the benchmark. Since the players are symmetric, then ex-ante, each will win each game with one half chance in both the two period and in the benchmark game. Therefore, expected payoffs can be written as

\[
E[\pi_1 + \pi_2] = 1 - E[c(x_1) + c(x_2)] = \frac{k_h}{2} > \frac{1}{2A} = 1 - E[2c(x)] = E[2\pi_b]
\]

This implies that \( E[c(x_1) + c(x_2)] < E[2c(x)] \). Also, since \( c(\cdot) \) is weakly convex, then

\[
E \left[ 2c \left( \frac{x_1 + x_2}{2} \right) \right] \leq E[c(x_1) + c(x_2)] < E[2c(x)].
\]
Because \( c(\cdot) \) is strictly increasing, this implies that
\[
E[x_1 + x_2] < E[2x]
\]
and therefore output in the two period game is lower than the two period benchmark where no information is revealed after the first round.

From the incentives driven by the continuation values, it is clear that in the first contest, high ability players have an incentive to reduce effort and appear weaker. These players have a lower expected output in the first of two contests than in the single contest benchmark. On the other hand, low ability players have an incentive to appear stronger, which increases their expected output above the level of the benchmark. When the abilities of players are sufficiently different, players’ ex-ante expected outputs are lower than in the benchmark as the effect of the high ability players outweighs that of the low ability players. Intuitively, since the output of high ability players is higher for a given level of effort relative to a low ability player, then changes in their effort result in a greater change in output.

The reduction of effort in the second contest stems from increased differentiation of players. With a high probability, one player will enter the second contest in a stronger position than his opponent. The difference in positions will reduce competition and on average, less total output will be produced. The weaker player faces a negative motivation effect, while the stronger player will not compete as aggressively against a weaker opponent.

1.5.3.1 Application to Performance Evaluations

Ederer 2010 and Aoyagi 2010 both discuss the potential merits of performance evaluations in a single contest between two employees. Performance evaluations can be thought of as dividing a contest into two separate contests, where agents choose a level of effort before and after the evaluation. Aoyagi 2010 shows that when output is a noisy signal of effort and abilities do not effect output, performance evaluations reduce the expected output of the workers if the cost of effort is convex. On the other hand, Ederer 2010 shows that when ability affects output and the contestants do not know their ability, there are two competing effects of performance evaluations: the “eval-
Figure 1.7: Output in terms of ability ratio, cost: $c(e) = e$ and $c(e) = e^2$.

The "evaluation effect" which stems from relative position in the contest and the "motivation effect" which encourages the contestant who appears more productive. Strategically, the evaluation effect discourages the employee who is further behind while the motivation effect discourages employees who think they are less productive. This motivation effect also provides additional incentive for effort before the performance evaluation is administered as the employee wants to appear more productive to his opponents. It is shown that this additional effort before the performance evaluation may outweigh the loss in output from the decreased competition after the evaluation.

Our results indicate that when employees have private information about their abilities, that the effect in the first period is not one directional. After the midterm evaluation a high ability employee would actually prefer to look weaker, and therefore will produce less effort before the evaluation. This would effectively counteract any additional effort exerted by low ability employees before the midterm evaluation. Additionally, after the evaluation, both differentiation between employees abilities and the discouragement effect stemming from one employee falling behind will combine...
to reduce expected output. Therefore, in this setting, performance evaluations would not encourage additional effort from employees.

### 1.6 Conclusion

While competing in two repeated contests with asymmetric information, contestants have an incentive to give up potential profits in the first contest to prevent revealing their private information. This leads to both bluffing and sandbagging in the first contest and can cause the following inefficiencies as compared to the single contest benchmark. First, a low ability player has a positive probability of winning both contests against a player who has high ability. Second, repeated contests have a lower expected output than the single contest, and additionally, the expected output of the second contest is lower than that of the first. While the results may seem overwhelmingly negative, in settings where the payoffs of competitors are of interest, our results are positive as ex-ante expected payoffs are higher for the contestants. Additionally, we feel that the intuitions developed here apply in more general dynamic settings where private information is valuable, and we leave that for future work.

### 1.7 Appendix

#### 1.7.1 Equilibrium Construction

**Single Contest**

Because the equilibrium is monotonic, we know $BR_i(a_{ih}) = x^*$ for $i = s, w$. Additionally, each contestant must be indifferent between all $x \in (x^*_s, x^*)$ when they have high ability. Given high ability these contestants have the same marginal cost of output, and therefore the density of the expected output of their opponents must also be the same for both indifference conditions to hold. Therefore, $f_s^*(x) = f_w^*(x)$ for $x \in (\max\{x^*_s, x^*_w\}, x^*)$ and $F_s^*(x^*) = F_w^*(x^*) = 1$. Since $f_i^*(x) = \mu_i h_i^*(x)$ for all $x \in (x^*_i, x^*)$, then $h_s^*(x) \leq h_w^*(x)$ for all $x \in (\max\{x^*_s, x^*_w\}, x^*)$. Also, $H_i(x^*_i) = 0$, which implies that $x_s^* \leq x_w^*$. Therefore, for the remainder of the construction, there are three intervals to consider:
the best response set of the low types of both the stronger and the weaker players, \(0 \leq x \leq x_s^*\), the best response set of the low type of the weaker player and the high type of the strong type, \(x_s^* \leq x \leq x_w^*\), and best response set of the high types of each player, \(x_w^* \leq x \leq x^*\).

Within their best response sets, players must be indifferent between all output levels. For example, player \(s\) given that he has ability of \(a_h\), must be indifferent to picking all outputs between \(x_s^*\) and \(x^*\). Then, for any for any \(x\) and \(x'\) in this interval the payoffs for the strong contestant must be the same. This puts a condition on \(H_w(x)\), the output distribution of the weak contestant with high ability on the interval \([x_w^*, x^*]\), as the indifference for the strong contestant implies

\[
\mu_w H_w^*(x') - c\left(\frac{x}{a_h}\right) = \mu_w H_w^*(x') - c\left(\frac{x'}{a_h}\right).
\]

Rearranging and taking the limit as \(x \to x'\),

\[
\lim_{x \to x'} \frac{H_w^*(x) - H_w^*(x')}{x - x'} = \frac{\partial H_w^*}{\partial c}\left(\frac{x}{a_h}\right) = \frac{1}{\mu_w}.
\]

We use this to calculate the output density of the weak contestant on this interval.

\[
\lim_{x \to x'} \frac{H_w^*(x) - H_w^*(x')}{x - x'} = \lim_{x \to x'} \frac{H_w^*(x) - H_w^*(x')}{c\left(\frac{x}{a_h}\right) - c\left(\frac{x'}{a_h}\right)} \frac{c\left(\frac{x}{a_h}\right) - c\left(\frac{x'}{a_h}\right)}{a_h (x - x')} = \frac{\partial H_w^*}{\partial c}\left(\frac{x}{a_h}\right) c'\left(\frac{x}{a_h}\right) \frac{1}{a_h} = \frac{c'(x'/a_h)}{a_h \mu_w}.
\]

A similar calculation on each interval for each player allows us to characterize the densities of the output on each of the intervals below.

- \(x_w^* \leq x \leq x_s^*: \quad h_w^*(x) = \frac{c'(x/a_h)}{a_h \mu_w}, \quad h_s^*(x) = \frac{c'(x/a_h)}{a_h \mu_w}, \quad f_s^*(x) = f_w^*(x) = \frac{c'(x/a_h)}{a_h}.

- \(x_s^* \leq x \leq x_w^*: \quad h_s^*(x) = \frac{c'(x)}{\mu_w}, \quad f_s^*(x) = \frac{c'(x)}{a_h (1 - \mu_w)} = \frac{c'(x/a_h)}{a_h (1 - \mu_w)}, \quad f_w^*(x) = c', \quad f_w^*(x) = \frac{c'(x/a_h)}{a_h}.

- \(0 \leq x \leq x_s^*: \quad f_s^*(x) = \frac{c'(x)}{1 - \mu_w}, \quad f_w^*(x) = \frac{c'(x)}{1 - \mu_w}, \quad f_s^*(x) = f_w^*(x) = c'.

From the definition of the best response sets and the consistency of player’s information sets, the distribution of output for each player must satisfy

\[
L_i^*(x_i^*) = 1, \quad H_i^*(x_i^*) = 0, \quad F_i^*(x_i^*) = 1 - \mu_i, \quad F_i^*(x^*) = 1.
\]

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To find the endpoints we look at the stronger player’s distribution of strategies. The stronger contestant does not choose zero effort with positive probability, and therefore \( L_s^*(0) = 0 \). Using \( L_s^*(x_s^*) = 1 \) and the definition of \( \ell_s^*(x) \) on \([0,x_s^*] \), we calculate \( x_s^* \).

\[
\int_0^{x_s^*} \ell_s^*(x) \, dx = L_s^*(x_s^*) - L_s(0) = \frac{c(x_s^*)}{1 - \mu_s} = 1
\]

Then \( c(x_s^*) = 1 - \mu_s \), so that \( x_s^* = c^{-1}(1 - \mu_s) \). Similarly, \( x_w^* = c^{-1}(1 - \mu_w) \). From these endpoints we can calculate \( x^* \).

\[
\int_{x_w^*}^{x_s^*} h_s^*(x) \, dx = \frac{c(x_w^*) - c(x_s^*)}{\mu_s} = \frac{(1 - \mu_w) - (1 - \mu_s)}{\mu_s} = \frac{\mu_s - \mu_w}{\mu_s}
\]

\[
\int_{x_w^*}^{x_s^*} h_w^*(x) \, dx = 1 - \frac{\mu_s - \mu_w}{\mu_s} = \frac{\mu_w}{\mu_s}
\]

\[
\int_{x_w^*}^{x_s^*} f_s^*(x) \, dx = c \left( \frac{x^*}{a_h} \right) - c \left( \frac{c^{-1}(1 - \mu_w)}{a_h} \right) = \mu_w
\]

\[
x^* = a_h c^{-1} \left( \mu_w + c \left( \frac{c^{-1}(1 - \mu_w)}{a_h} \right) \right)
\]

Lastly, we can pin down the probability that the weaker player exerts no effort.

\[
\int_{x_w^*}^{x_s^*} \ell_w^*(x) \, dx = \frac{1}{1 - \mu_w} \left[ c \left( \frac{c^{-1}(1 - \mu_w)}{a_h} \right) - c \left( \frac{c^{-1}(1 - \mu_s)}{a_h} \right) \right]
\]

\[
\int_0^{x_s^*} \ell_w^*(x) \, dx = \frac{c(c^{-1}(1 - \mu_s))}{1 - \mu_w} - 0 = \frac{1 - \mu_s}{1 - \mu_w}
\]

\[
L_w^*(0) = 1 - \frac{1 - \mu_s}{1 - \mu_w} - \frac{1 - \mu_s}{1 - \mu_w} \left[ c \left( \frac{c^{-1}(1 - \mu_w)}{a_h} \right) - c \left( \frac{c^{-1}(1 - \mu_s)}{a_h} \right) \right] = \frac{\mu_s - \mu_w}{1 - \mu_w}
\]

The endpoints of the best response sets for each type of each player, and characterizations of the density functions within these best response sets characterize the unique equilibrium.
Repeated Contests

We solve for the equilibrium of successive contest given a parameterization of the cost function, \( c(x) = kx^\alpha \). The original assumptions on the cost function imply that \( \alpha \geq 1 \) and \( k > 0 \).

For the range of \( 0 \leq x < x_{h,*} \) we have \( F_1^*(x) = kx^\alpha \), and for the range \( x_{h}^* < x \leq x_{h}^* \), we have \( F_1^*(x) + v_h = kx^\alpha \).

For the range \( x_{h,*} \leq x \leq x_{h}^* \), the solution to (\( \dagger \)) is

\[
F_1^*(x) = B e^{c(x/a_h) - c(x)} + \frac{\partial}{\partial x} c(x) - \frac{\partial}{\partial x} c(x/a_h), \quad \text{with } F_1(x_{h,*}) = kx_{h,*}^\alpha.
\]

Solving for, \( B \), the ex-ante distribution function of each player on \( [x_{h,*}, x_{h}^*] \) is

\[
F_1^*(x) = \frac{a_h^\alpha}{a_h^\alpha - 1} - \left( \frac{a_h^\alpha}{a_h^\alpha - 1} - kx_{h,*}^\alpha \right) e^{\frac{k(1-a_h^\alpha)}{a_h^\alpha} (x-a_h^\alpha)}.
\]

The belief function must satisfy

\[
d(\mu(x))\mu'(x) = c'(x) - \frac{1}{a_h} c'(x/a_h) \quad \text{for } x \in [x_{h,*}, x_{h}^*] \text{, with } \mu(x_{h,*}) = 0.
\]

Therefore, on this interval, the belief function is \( \mu(x) = k(x^\alpha - x_{h,*}^\alpha) \), and the distribution function can be written as

\[
F_1^*(x) = \frac{a_h^\alpha}{a_h^\alpha - 1} - \left( \frac{a_h^\alpha}{a_h^\alpha - 1} - kx_{h,*}^\alpha \right) e^{\frac{k(1-a_h^\alpha)}{a_h^\alpha} \mu(x)}.
\]

Given \( F_1^*(x) \) and \( \mu(x) \) we can also calculate the output distribution of the both the high and low ability players on \( [x_{h,*}, x_{h}^*] \), using \( 2F_1^*(x) = L_1^*(x) + H_1^*(x) \) and \( \mu(x) = \frac{h_1^v(x)}{2f_1^v(x)} \).

\[
H_1^*(x) = H_1^*(x_{h,*}) + 2\int_{x_{h,*}}^x \mu(t)f_1^v(t)dt
\]

\[
= 2\mu(t)F_1^*(t)|_{x_{h,*}}^x - 2\int_{x_{h,*}}^x \mu'(t)F_1^*(t)dt
\]

\[
= 2 \left( \frac{a_h^\alpha}{a_h^\alpha - 1} - \left( \frac{a_h^\alpha}{a_h^\alpha - 1} + \mu(x) \right) e^{\frac{1-a_h^\alpha}{a_h^\alpha} \mu(x)} \right) \left( \frac{a_h^\alpha}{a_h^\alpha - 1} - kx_{h,x}^\alpha \right)
\]

\[
L_1^*(x) = 2F_1^*(x) - H_1^*(x)
\]

\[
= \frac{2a_h^\alpha}{a_h^\alpha - 1} + 2 \left( \mu(x) + \frac{a_h^\alpha}{a_h^\alpha - 1} e^{\frac{1-a_h^\alpha}{a_h^\alpha} \mu(x)} - \frac{a_h^\alpha}{a_h^\alpha - 1} \right) \left( \frac{a_h^\alpha}{a_h^\alpha - 1} - kx_{h,x}^\alpha \right)
\]
Let \( A = \frac{a_h^\alpha}{a_h^\alpha - 1} \). Ex ante and ex-interim strategies in the first contest are

\[
F_1^*(x) = \begin{cases} 
  kx^\alpha, & 0 \leq x \leq x_{h^*} \\
  A - (A - kx_{h^*}^\alpha)e^{-\mu(x)/A}, & x_{h^*} \leq x \leq x_{h^*}^* \\
  k(x/a_h)^\alpha + k_h - v_h, & x_{h^*}^* \leq x \leq x_h^*
\end{cases}
\]

\[
L_1^*(x) = \begin{cases} 
  2kx^\alpha, & 0 \leq x \leq x_{h^*} \\
  2A + 2((\mu(x) + A - 1)e^{-\mu(x)/A} - A)(A - kx_{h^*}^\alpha), & x_{h^*} \leq x \leq x_{h^*}^* \\
  1, & x_{h^*}^* \leq x \leq x_h^*
\end{cases}
\]

\[
H_1^*(x) = \begin{cases} 
  0, & 0 \leq x \leq x_{h^*} \\
  2(A - (A + \mu(x))e^{-\mu(x)/A}(A - kx_{h^*}^\alpha), & x_{h^*} \leq x \leq x_{h^*}^* \\
  2k(x/a_h)^\alpha + 2(k_h - v_h) - 1, & x_{h^*}^* \leq x \leq x_h^*
\end{cases}
\]

We use the following conditions to find the unknowns, \( x_{h^*}, x_{h^*}^*, x_h^*, v_h \) and \( k_h \):

1. Continuity of the belief function implies that \( \mu(x_{h^*}^*) = 1 \) and \( k(x_{h^*}^* - x_{h^*}^\alpha) = 1 \).

2. Since \( x_{h^*}^* = \sup\{BR(a_{h^*})\} \), then \( L_1^*(x_{h^*}^*) = 1 \).

\[
L_1^*(x_{h^*}^*) = 2A + 2((\mu(x_{h^*}^*) + A - 1)e^{-\mu(x_{h^*}^*)/A} - A)(A - kx_{h^*}^\alpha) = 1
\]

\[
\Rightarrow kx_{h^*}^\alpha = A - \frac{2A - 1}{2A(1 - e^{-1/A})}, \quad kx_{h^*}^\alpha = 1 + A - \frac{2A - 1}{2A(1 - e^{-1/A})}
\]

3. Continuity of \( F_1^*(x) \) at \( x_{h^*}^* \) gives

\[
A - (A - kx_{h^*}^\alpha)e^{-1/A} = \frac{kx_{h^*}^\alpha}{a_h^\alpha} + k_h - v_h.
\]

Substituting from above we get the two period payoff of the high type player

\[
k_h = v_h + \frac{1}{A} + \frac{(2A - 1)(1 - a_h^\alpha e^{-1/A})}{2Aa_h^\alpha(1 - e^{-1/A})}.
\]

4. \( v_h \) is the expected payoff in the second period of a player with high ability who reveals that he is high type in the first period.

\[
v_h = E[v_i(1, \mu(x^{-i}), a_h)] = \frac{1}{A}E[1 - \mu(x^{-i})] = \frac{1}{2A}
\]

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\[ k_h = \frac{3}{2A} + \frac{(2A - 1)(1 - a_h^\alpha e^{-1/A})}{2Aa_h^\alpha(1 - e^{-1/A})} \]

5. The last condition is using that \( x_h^* = \sup BR(a_h) \), so that \( F_1^+(x_h^*) = 1 \).

\[ F_1^+(x_h^*) = \frac{kx_h^*}{a_h^\alpha} + k_h - v_h = 1 \]

\[ \Rightarrow kx_h^* = 1 - \frac{(2A - 1)(1 - a_h^\alpha e^{-1/a})}{2A(1 - e^{-1/A})} \]

1.7.2 Proofs

**Lemma 3.1** In any equilibrium, players’ distributions of output, \( H_s(x) \), \( L_s(x) \), \( H_w(x) \), and \( L_w(x) \), are continuous on \((0, x^*)\), where

\[ x^* \equiv \sup \{ BR_s(a_{\ell}) \cup BR_s(a_h) \} = \sup \{ BR_w(a_{\ell}) \cup BR_w(a_h) \} \]

and \( \inf \{ BR_s(a_{\ell}) \cup BR_s(a_h) \} = \inf \{ BR_w(a_{\ell}) \cup BR_w(a_h) \} = 0. \)

**Proof.** The proof follows in four steps:

1. There is no \( x \) at which both players both have an atom.

   If both players played some \( x \) with positive probabilities given by \( p_1 \) and \( p_2 \). Then either player can increase output slightly above \( x \), to \( x + \epsilon \). This would increase the payoff of that player since the cost of effort is continuous and we can pick \( \epsilon \) such that \( c(x + \epsilon) - c(x) < p_2 \). However, this implies that \( x \) is not a best response of that player, a contradiction.

2. If a player has an atom, then it is at zero.

   Assume that player \( i \) has an atom at \( x > 0 \) where \( x \) is played is probability \( p > 0 \). Then by the continuity of the cost function in output, there is a \( \delta > 0 \) such that \( \hat{x} \in (x - \delta/2, x) \), \( \hat{x} \notin BR_{-i}(a^{-i}) \). This implies, that player \( i \) would do better by playing \( x - \delta/4 \), and therefore \( x \notin BR_i(a^i) \). This is a contradiction. Therefore the output distribution functions of each type of each player is continuous on \((0, \infty)\). This implies that \( F_s(x), F_w(x) \)

3. If \( x > 0 \) is not a best response for any ability of one of the contestants, then for all \( x' > x \), \( x' \) is not a best response for either type of either player.
Step (2) implies that payoffs are continuous, since both the cost function and the probability of winning are continuous in \( x \). Now, since \( x \not\in \{ BR_i(a_\ell) \cup BR_i(a_h) \} \), for some \( i = s, w \), \( \exists \bar{x}(a_h), \bar{x}(a_\ell) \) for which \( \pi^i(\bar{x}(a_h), a_h) > \pi^i(x, a_h) + \epsilon \) and \( \pi^i(\bar{x}(a_\ell), a_\ell) > \pi^i(x, a_\ell) + \epsilon \). Then, there is a \( \delta > 0 \) for which \( \pi^i(\bar{x}(a_h), a_h) > \pi^i(\hat{x}, a_h) \) and \( \pi^i(\bar{x}(a_\ell), a_\ell) > \pi^i(\hat{x}, a_\ell) \), \( \forall \hat{x} \in (x, x + \delta) \). Therefore every \( \hat{x} \) in this neighborhood cannot be a best response of either type of player \( i \). Additionally, no \( \hat{x} \) in this neighborhood can be a best response for any type of player \( -i \), as they could improve utility by lowering output to \( x \). Therefore there is an interval with positive measure for which there is no best responses for either player for either type. Let \( X^* \) be the set of all outputs that are greater than \( x \) and are a best response for some player of either type. Let \( x_* = \inf \{ X^* \} \). Then, necessarily \( x_* \) has a gap \( (x_* - \delta', x_*], \delta' > 0 \) for which there are no best responses. However, since there is an \( x \in X^* \) such that \( x - x_* < \delta \), \( x \) cannot be a best response. Therefore, \( x_* \) does not exist and \( X^* \) is empty. This implies that \( \sup \{ BR_s(a_\ell) \cup BR_s(a_h) \} = \sup \{ BR_w(a_\ell) \cup BR_w(a_h) \} \). We call this output level \( x^* \).

4. Each player has a type who has best response that is arbitrarily close to 0.

If this were not true, then there is a player and an \( x > 0 \) such that all \( \hat{x} \leq x \) are not a best response for any type of that player. Then from step (3), that player has no best responses. This cannot be true in equilibrium.

\[ \square \]

**Proposition 4.1**

(Countervailing Incentives) \( \frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_h)] < 0 \) and \( \frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_\ell)] > 0 \) for all \( \mu_i \in (0, 1) \).

**Proof.** In the second contest, for a given pair of beliefs, players will expect the following payoffs:

\[
v_i(\mu_i, \mu_{-i}, a_h) = 1 - \min \{ \mu_i, \mu_{-i} \} - c \left( \frac{c^{-1}(1 - \min \{ \mu_i, \mu_{-i} \})}{a_h} \right)
\]

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\[ v_i(\mu_i, \mu_{-i}, a_\ell) = \begin{cases} 
\mu_i - \mu_{-i} - \left[ c \left( \frac{c^{-1}(1-\mu_i)}{a_h} \right) - c \left( \frac{c^{-1}(1-\mu_{-i})}{a_h} \right) \right] & \text{if } \mu_i \geq \mu_{-i} \\
0 & \text{otherwise} 
\end{cases} \]

For a high ability contestant whose opponent has belief \( \mu_i \), the expected payoff in the second contest is given by

\[ E[v_i(\mu_i, \mu_{-i}, a_h)] = \int_0^1 \left( 1 - \min\{\mu_i, \mu_{-i}\} - c \left( \frac{c^{-1}(1-\min\{\mu_i, \mu_{-i}\})}{a_h} \right) \right) dF_{\mu_{-i}}(\mu_{-i}) \]

As the opponent believes the contestant is stronger, the change in expected payoff is

\[ \frac{\partial}{\partial \mu_i} E_{\mu_{-i}}[v_i(\mu_i, \mu_{-i}, a_h)] = \left( 1 + \frac{\partial}{\partial \mu_i} c \left( \frac{c^{-1}(1-\mu_i)}{a_h} \right) \right) (F_{\mu_{-i}}(\mu_i) - 1) \]

For a low ability contestant, the expected payoff given \( \mu_i \) is:

\[ E[v_i(\mu_i, \mu_{-i}, a_\ell)] = \int_0^{\mu_i} \left( \mu_i - \mu_{-i} + c \left( \frac{c^{-1}(1-\mu_i)}{a_h} \right) - c \left( \frac{c^{-1}(1-\mu_{-i})}{a_h} \right) \right) dF_{\mu_{-i}}(\mu_{-i}) \]

The effect of his opponent’s beliefs is

\[ \frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_\ell)] = \left( 1 + \frac{\partial}{\partial \mu_i} c \left( \frac{c^{-1}(1-\mu_i)}{a_h} \right) \right) F_{\mu_{-i}}(\mu_i) \]

Given the assumptions on the cost of effort, \( c'(e) > 0 \) and \( c''(e) \geq 0 \),

\[ \frac{\partial}{\partial \mu_i} c \left( \frac{c^{-1}(1-\mu_i)}{a_h} \right) = -\frac{1}{a_h} c' \left( \frac{c^{-1}(1-\mu_i)}{a_h} \right) c' \left( \frac{1}{c^{-1}(1-\mu_i)} \right) \in \left( -\frac{1}{a_h}, 0 \right) \]

If we define,

\[ d(\mu_i) \equiv 1 + \frac{\partial}{\partial \mu_i} c \left( \frac{c^{-1}(1-\mu_i)}{a_h} \right) \]

where \( d(\mu_i) \in \left[ \frac{a_h - 1}{a_h}, 1 \right] \) for all \( \mu_i \),

then it is clear that

\[ \frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_h)] = d(\mu_i)(F_{\mu_{-i}}(\mu_i) - 1) < 0 \]

\[ \frac{\partial}{\partial \mu_i} E[v_i(\mu_i, \mu_{-i}, a_\ell)] = d(\mu_i)F_{\mu_{-i}}(\mu_i) > 0. \]
Corollary 4.2 In every SPBE, $\mu(x)$ is weakly increasing in $x$ for all $x \in X_1 = X_1^H \cap X_1^L$.

Proof. Assume otherwise, namely that for a given $x$ and $y$ which are best responses for some ability level we have that $x < y$ and $\mu(x) > \mu(y)$. This implies that $0 \leq \mu(y) < \mu(x) \leq 1$. Then by Bayes’ Rule, $h_1(x) > 0$ and $h_1(y) < f_1(y)$, which implies that $\ell_1(y) > 0$. For the strategies to be optimal it must be that $x \in BR(a_h)$ and $y \in BR(a_\ell)$. Then we know that

$$\Pr(\text{win}|y) - c(y) + E[v_i(\mu(y), \mu(x^{-i}), a_\ell)] \geq \Pr(\text{win}|x) - c(x) + E[v_i(\mu(x), \mu(x^{-i}), a_\ell)]$$

$$\Pr(\text{win}|y) - c(y/a_h) + E[v_i(\mu(y), \mu(x^{-i}), a_h)] \leq \Pr(\text{win}|x) - c(x/a_h) + E[v_i(\mu(x), \mu(x^{-i}), a_h)]$$

This implies that

$$\Pr(\text{win}|y) - \Pr(\text{win}|x) + E[v_i(\mu(y), \mu(x^{-i}), a_\ell)] - E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] \geq c(y) - c(x)$$

$$\Pr(\text{win}|y) - \Pr(\text{win}|x) + E[v_i(\mu(y), \mu(x^{-i}), a_h)] - E[v(\mu(x), \mu(x^{-i}), a_h)] \leq c(y/a_h) - c(x/a_h)$$

The expected payoff in the second contest increases for a low ability contestant as $\mu$ increases and for a high ability contestant decreases as $\mu$ increases. Then $\mu(x) > \mu(y)$ implies

$$E[v_i(\mu(y), \mu(x^{-i}), a_\ell)] - E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] < 0,$$

and $E[v_i(\mu(y), \mu(x^{-i}), a_h)] - E[v(\mu(x), \mu(x^{-i}), a_h)] > 0$.

Combine with previous inequalities:

$$c(y) - c(x) < \Pr(\text{win}|y) - \Pr(\text{win}|x) < c(y/a_h) - c(x/a_h)$$

However, since $c''(x) \geq 0$ and $c'(x) > 0$, we must have that

$$c(y/a_h) - c(x/a_h) \leq c(y) - c(y - (y/a_h - x/a_h))$$

$$= c(y) - c\left(\frac{x + (a_h - 1)y}{a_h}\right) < c(y) - c(x).$$

This is a contradiction.
Lemma 4.4 There is no output that is played with positive probability and \( \Pr(\text{win}|x) = F(x) \) is continuous.

Proof. In a symmetric equilibrium, if an output is played with positive probability by one type of player, then it must be played with positive probability by this type of both players. Let \( \hat{x} \in \{X^f_1 \cup X^h_1\} \) be played with probability \( p > 0 \). Then

\[
\Pr(\text{win}|\hat{x}) + \frac{p}{2} \leq \Pr(\text{win}|x) \text{ for all } x > \hat{x}.
\]

Since for some \( a, \hat{x} \in BR(a) \), then \( \pi(\hat{x}|a) \geq \pi(x|a) \) for all \( x \). This implies that

\[
\Pr(\text{win}|\hat{x}) - c(\hat{x}/a) + E[v_i(\mu(\hat{x}), \mu(x^{-i}), a^i)] \\
\geq \Pr(\text{win}|x) - c(x/a^i) + E[v_i(\mu(x), \mu(x^{-i}), a^i)]
\]

Combing the above inequalities we have

\[
\frac{p}{2} \leq \Pr(\text{win}|x) - \Pr(\text{win}|\hat{x}) \\
\leq E[v_i(\mu(\hat{x}), \mu(x^{-i}), a^i)] - E[v_i(\mu(x), \mu(x^{-i}), a^i)] + c(x/a^i) - c(\hat{x}/a^i)
\]

By continuity of the cost function, \( \exists \varepsilon > 0 \) such that for all \( x \in (\hat{x}, \hat{x} + \varepsilon) \), we have \( c(\frac{\hat{x} + \varepsilon}{a^i}) - c(\frac{\hat{x}}{a^i}) < \frac{p}{2} \). Then for each \( x \) in this range we know

\[
E[v_i(\mu(\hat{x}), \mu(x^{-i}), a^i)] - E[v_i(\mu(x), \mu(x^{-i}), a^i)] > 0.
\]

If \( a^i = a_f \), then \( \mu(\hat{x}) > \mu(x) \) and therefore \( \hat{x} \in \{X^f_1 \cap X^h_1\} \). Similarly, if \( a^i = a_h \), then \( \mu(\hat{x}) < \mu(x) \) and \( \hat{x} \in \{X^f_1 \cap X^h_1\} \). In either case, \( \hat{x} \in \{BR(1) \cap BR(a_h)\} \). However, the inequality cannot hold for both \( a^i = a_f \) and \( a^i = a_h \) at the same time, so we have a contradiction. \( \square \)

We now can use the fact that \( F_1(x) \) is continuous in \( x \) and we have that \( \Pr(\text{win}|x) = \Pr(x < x^{-i}) = \Pr(x \leq x^{-i}) = F_1(x) \). Combined with Lemma 4.2, we have \( \Pr(\mu(x) < \mu(y)) \leq \Pr(\text{win}|y) = F(y) \leq \Pr(\mu(x) \leq \mu(y)) \).

Lemma 4.5 \( BR(a_f) \) and \( BR(a_h) \) are intervals where \( 0 = x_{f*,*} \leq x_{h*,*} < x_{f*} \leq x_{h*} \) and we define \( x_{f*,*} = \inf\{BR(a_f)\}, x_{f*} = \sup\{BR(a_f)\}, x_{h*,*} = \inf\{BR(a_h)\} \) and \( x_{h*} = \sup\{BR(a_h)\} \).

Proof. The proof follows in four steps.
1. We first show that \( x_{\ell,*} = 0 \). We do this by first showing that \( x_{\ell,*} \leq x_{h,*} \), and then showing that \( x_{\ell,*} \) cannot be larger than zero.

Let \( x_{h,*} < x_{\ell,*} \). Since \( x_{h,*} = \inf\{X_i^h, \forall \varepsilon > 0, \exists x_\varepsilon \text{ such that } x_{h,*} \leq x_\varepsilon < x_{h,*} + \varepsilon \text{ and } x_\varepsilon \in X_i^h \). In particular, this holds for \( \varepsilon^* = x_{\ell,*} - x_{h,*} \). Then \( x_{\varepsilon^*} \in \{X_i^h \setminus X_i^\ell \} \) and \( \mu(x_{\varepsilon^*}) = 1 \). However, from Lemma 3.1 we would have \( \mu(x) = 1 \) for all \( x \in X_i^\ell \), which cannot hold. Therefore \( x_{h,*} \leq x_{\ell,*} \).

If \( 0 < x_{\varepsilon^*} < x_{h,*} \), then let \( x_{h,*} - x_{\varepsilon^*} = \delta_1 \). Since \( F_1 \) is continuous from Lemma 3.2, then \( \exists \delta_2 \) with \( 0 < \delta_2 < \delta_1 \) such that \( \forall x \in (x_{\varepsilon^*}, x_{h,*} + \delta_2) \) we have \( |F_1(x) - F_1(0)| = |F_1(x) - F_1(x_{\varepsilon^*})| < c(x_{\varepsilon^*}) < c(x_{\delta_2}) \). Let \( x_{\delta_2} \in X_i^\ell \cap (x_{\varepsilon^*}, x_{h,*} + \delta_2) \). Then \( \mu(x_{\delta_2}) = 0 \) and

\[
E[\pi'(0)|a_\ell] = F_1(0) + E[v_i(\mu(0), \mu(x^{-i}), a_\ell)] \\
> F_1(x_{\delta_2}) + E[v_i(\mu(x_{\delta_2}), \mu(x^{-i}), a_\ell)] - c(x_{\delta_2}) \\
= E[\pi'(x_{\delta_2})|a_\ell]
\]

Then \( x_{\delta_2} \notin BR(a_\ell) \), a contradiction.

If \( 0 < x_{\ell,*} = x_{h,*} \), then \( \exists x_\ell, x_h \text{ such that } x_\ell \leq x_h, x_\ell \in X_i^\ell, x_h \in X_i^h, \text{ and } F_1(x_\ell) - F_1(x_{\ell,*}) = F_1(x_\ell) < c(x_{\ell,*}) < c(x_\ell) \) and \( F_1(x_h) - F_1(x_{h,*}) = F_1(x_h) < c(x_{h,*}/a_h) < c(x_h/a_h) \), by the continuity of \( F_1 \).

\( x_\ell \in X_i^\ell \) implies that

\[
E[\pi'(x_\ell)|a_\ell] = F_1(x_\ell) - c(x_\ell) + E[v_i(\mu(x_\ell), \mu(x^{-i}), a_\ell)] \\
\geq F_1(0) - c(0) + E[v_i(\mu(0), \mu(x^{-i}), a_\ell)] = E[\pi'(0)|a_\ell]
\]

This can hold only if \( E[v_i(\mu(x_\ell), \mu(x^{-i}), a_\ell)] > E[v_i(\mu(0), \mu(x^{-i}), a_\ell)] \), which implies that \( \mu(x_\ell) > \mu(0) \).
$x_h \in X^h_1$ implies that

$$E[\pi^i(x_2)|a_h] = F_1(x_h) - c(x_h/a_h) + E[v_i(\mu(x_h), \mu(x^{-i}), a_h)]$$

$$\geq F_1(0) - c(0) + E[v_i(\mu(0), \mu(x^{-i}), a_h)] = E[\pi^i(0)|a_h]$$

This can hold only if $E[v_i(\mu(x_h), \mu(x^{-i}), a_h)] > E[v_i(\mu(0), \mu(x^{-i}), a_h)]$, which implies that $\mu(x_h) < \mu(0)$.

Combining these two inequalities leads to $\mu(x_h) < \mu(x_\ell)$. This contradicts Lemma 4.2.

Therefore we must have $0 = x_{\ell, *} \leq x_{h,*}$.

2. We next show that $x_{h,*} \leq x^*_\ell$.

If $x^*_\ell > x_{h,*}$, then $\forall x \in (x^*_\ell, x_{h,*})$, $x \not\in \{X^*_\ell \cap X^h_1\}$. Let $x' = \frac{x^*_\ell + x_{h,*}}{2}$ and $\varepsilon = c(x_{h,*}/a_h) - c(x'/a_h)$.

There is a $\delta > 0$ such that $\forall x \in (x_{h,*}, x_{h,*} + \delta)$, $F(x) - F(x_{h,*}) < \varepsilon$. Pick an $x_\delta$ such that $x_\delta \in (x_{h,*}, x_{h,*} + \delta)$ and $x_\delta \in X^h_1$. Then $F_1(x_\delta) - F_1(x_{h,*}) = F_1(x_\delta) - F_1(x') < \varepsilon$, $c(x_\delta/a_h) - c(x'/a_h) > \varepsilon$, and

$$E[v_i(\mu(x_\delta), \mu(x^{-i}), a_h)] \leq E[v_i(\mu(x'), \mu(x^{-i}), a_h)].$$

Therefore

$$E[\pi^i(x')|a_h] = F_1(x') + E[v_i(\mu(x'), \mu(x^{-i}), a_h)] - c\left(\frac{x'}{a_h}\right)$$

$$> F(x_\delta) + E[v_i(\mu(x_\delta), \mu(x^{-i}), a_h)] - c\left(\frac{x_\delta}{a_h}\right) = E[\pi^i(x_\delta)|a_h],$$

a contradiction. So we can conclude that $x^*_\ell \leq x_{h,*}$.

Also note that we must have $x^*_\ell \leq x^*_h$. If we assume otherwise, then we can find $x \in \{X^\ell_1 \setminus X^h_1\}$ where $x > x^*_h$ and $\mu(x) = 0$. Lemma 4.2 rules out this possibility.

We have shown so far that $0 = x_{\ell, *} \leq x_{h,*} \leq x^*_\ell \leq x^*_h$.

3. We next will show that for all $x \in (x_{\ell,*}, x_{h,*})$, $x \in BR(a_\ell)$ and for all $x \in (x^*_\ell, x^*_h)$, $x \in BR(a_h)$.

If $x_{\ell,*} < x_{h,*}$, then let $X^\ell_1 = \{x \in \{(x_{\ell,*}, x_{h,*}) \setminus BR(a_\ell)\}\}$. If $x \in X^\ell_1$, then $\exists \varepsilon > 0$ such that

$$E[\pi^i(x)|a_\ell] < E[\pi^i(x')|a_\ell] - \varepsilon$$

for all $x' \in \{(x_{\ell,*}, x_{h,*}) \cap X^\ell_1\}$. This implies that:

$$F_1(x) + E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] - c(x) < F_1(x') + E[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] - c(x') - \varepsilon,$$
where \( E[v_i(\mu(x), \mu(x^{-i}), a_t)] \geq E[v_i(\mu(x'), \mu(x^{-i}), a_t)] \) as \( \mu(x') = 0 \). Therefore \( F_1(x) - c(x) < F_1(x') - c(x') - \varepsilon \), and for all \( x' > x \) with \( x' \in \{(x_{t, i}, x_{h, i}) \cap X_c^f\} \), \( F_1(x') - F_1(x) > c(x') - c(x) - \varepsilon \).

Since \( F_1 \) and \( c \) are continuous, then there is a \( \delta(\varepsilon) > 0 \) such that for all \( x' \in X_c^f \), \(|x' - x| \geq \delta(\varepsilon)\). This implies that \( x \) is contained in an interval which is a subset of \( X_c^f \). Let \( a \) and \( b \) be the infimum and supremum of this interval respectively.

- If \( b < x_{h, i} \), then \( \exists x' < x_{h, i}, x' \in X_c^f \) where \(|x' - b| < \delta, \forall \delta > 0\). Then, by the continuity of \( F \), \( \exists x' \in X_c^f \) and \( F(x') - F(b) < c(b) - c(\frac{a + b}{2}) \). Then we know that
  \[
  F_1(x') - F_1\left(\frac{a + b}{2}\right) < c(b) - c\left(\frac{a + b}{2}\right)
  \]
  \[
  E[v_i(\mu(x'), \mu(x^{-i}), a_t)] \leq E\left[v_i\left(\mu\left(\frac{a + b}{2}\right), \mu(x^{-i}), a_t\right)\right].
  \]
  This implies that \( E[\pi^f(x')|a_t] \leq E[\pi^f(\frac{a + b}{2})|a_t] \) which contradicts \( x' \in BR(a_t) \).

- If \( b = x_{h, i} \), then \( \forall \delta > 0, \exists x' \in X_c^f \), s.t. \(|x' - b| < \delta\). We again can take \( x' \in X_c^f \) such that
  \[
  F_1(x') - F_1(b) < c\left(\frac{b}{a_h}\right) - c\left(\frac{a + b}{2a_h}\right).
  \]
  * If \( x' \notin X_c^f \) then \( \mu(x') = 1 \), but since \( E[v_i(\mu(x'), \mu(x^{-i}), a_h)] \leq E\left[v_i\left(\mu\left(\frac{a + b}{2}\right), \mu(x^{-i}), a_h\right)\right] \), then this contradicts \( x' \in BR(a_h) \).
  * If \( x' \in X_c^f \) then \( \mu(x') \in (0, 1) \). If \( \mu(x') \leq \mu\left(\frac{a + b}{2}\right) \), then this contradicts \( x' \in BR(a_t) \), but if \( \mu(x') \geq \mu\left(\frac{a + b}{2}\right) \), this contradicts \( x' \in BR(a_h) \).

Therefore \( X_c^f \) must be empty.

If \( x_t^* < x_h^* \), then let \( X_c^h = \{x | x \in \{(x_t^*, x_h^*) \setminus BR(a_h)\} \}. If x \in X_c^h, \) then \( \exists \varepsilon > 0 \) such that \( E[\pi(x)|a_h] \leq E[\pi(x')|a_h] - \varepsilon \) for all \( x' \in X_c^f \). This implies that
\[
F_1(x) - c\left(\frac{x}{a_h}\right) + E[v_i(\mu(x), \mu(x^{-i}), a_h)] < F_1(x') - c\left(\frac{x'}{a_h}\right) + E[v_i(\mu(x'), \mu(x^{-i}), a_h)] - \varepsilon,
\]
where \( E[v_i(\mu(x), \mu(x^{-i}), a_h)] \geq E[v_i(\mu(x'), \mu(x^{-i}), a_h)] \) as \( \mu(x') = 1 \). Therefore \( F_1(x) - c\left(\frac{x}{a_h}\right) < F_1(x') - c\left(\frac{x'}{a_h}\right) - \varepsilon \). Since \( F_1 \) and \( c \) are continuous, then this holds only if \(|x' - x| \geq \delta(\varepsilon) > 0, \forall x' \in BR(a_h) \). We take \( a \) and \( b \) to be the infimum and supremum respectively of the interval of \( X_c^h \) containing \( x \). Note that \( b < x_h^* \), by the definition of \( x_h^* \).
Now, there is an $x' \in X^h_i$ where $|x' - b| < \delta$ for all $\delta > 0$. Therefore there is an $x' \in BR(a_h)$ such that $F_1(x') - F_1(b) < c\left(\frac{b}{a_h}\right) - c\left(\frac{b+a}{2a_h}\right)$. Note that this implies that $F_1(x') - F_1(\frac{b+a}{2}) < c\left(\frac{x'}{a_h}\right) - c\left(\frac{b+a}{2a_h}\right)$. However, this implies that

$$E\left[\pi^i\left(\frac{b+a}{2}\right)|a_h\right] = F_1\left(\frac{b+a}{2}\right) - c\left(\frac{b+a}{2a_h}\right) + E\left[v_i\left(\frac{\mu\left(\frac{b+a}{2}\right)}{a_h}, \mu(x^{-i}), a_h\right)\right]
\geq F_1(x') - c\left(\frac{x'}{a_h}\right) + E[v_i(\mu(x'), \mu(x^{-i}), a_h)] = E[\pi^i(x')|a_h].$$

This contradicts $x' \in BR(a_h)$, and therefore $X^h_c$ must be empty.

4. Lastly, we show that $x_{h,*} < x_{l,*}^c$, and for all $x \in (x_{h,*}, x_{l,*}^c)$, $x \in \{BR(a_l) \cap BR(a_h)\}$.

If $x_{l,*} = x_{h,*}$, then $\forall \delta > 0$, there is $x_{l} \in BR(a_l)$ and $x_h \in BR(a_h)$ where $|x_h - x_{l}| < \delta$. Therefore, by the continuity of $F_1$ and $c$, there is $x_h$ and $x_{l}$ for which

$$F_1(x_h) - c\left(\frac{x_h}{a_h}\right) - \left(F_1(x_{l}) - c\left(\frac{x_{l}}{a_h}\right)\right)
< E[v_i(\mu(x_{l}), \mu(x^{-i}), a_h)] - E[v_i(\mu(x_h), \mu(x^{-i}), a_h)]
= E[v_i(0, \mu(x^{-i}), a_h)] - E[v_i(1, \mu(x^{-i}), a_h)],$$

since $E[v_i(0, \mu(x^{-i}), a_h)] - E[v_i(1, \mu(x^{-i}), a_h)] > 0$. This implies that

$$E[\pi^i(x_{l})|a_h] = F_1(x_{l}) - c\left(\frac{x_{l}}{a_h}\right) + E[v_i(\mu(x_{l}), \mu(x^{-i}), a_h)]
\geq F_1(x_h) - c\left(\frac{x_{h}}{a_h}\right) + E[v_i(\mu(x_h), \mu(x^{-i}), a_h)] = E[\pi^i(x_h)|a_h],$$

which cannot be true as $x_h \in BR(a_h)$.

Now define $X_c = \{x| x \in (x_{h,*}, x_{l,*}^c) \setminus (BR(a_l) \cup BR(a_h))\}$. From Lemma 3.1, we know that for all $x' \in \{(x_{h,*}, x_{l,*}^c) \cap (X^c \cup X^h)\}$, $\mu(x') \in (0, 1)$ and therefore $x' \in \{X^c \cap X^h\}$. If $\mu(x') = 1$, we must have $x_{l,*}^c \geq x'$, a contradiction. Similarly if $\mu(x') = 0$, then we must have $x_{h,*} \geq x'$ which is also a contradiction.

Let $x \in X_c$ be given. Then for all $x', x'' \in \{(x_{h,*}, x_{l,*}^c) \cap (X^c \cap X^h)\}$ such that $x' < x < x''$ we must have $\mu(x') \leq \mu(x'')$. Let $\mu^* \in [\sup\{\mu(x')\}, \inf\{\mu(x'')\}]$. These are well-defined as there is at least one such $x'$ and $x''$. 

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If \( \mu(x) \geq \mu^* \) then \( E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] \geq E[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] \) for all \( x' \) as defined above. Therefore

\[
F_1(x') - c(x') + E[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] - \varepsilon_1 > F_1(x) - c(x) + E[v_i(\mu(x), \mu(x^{-i}), a_\ell)]
\]

\[
\Rightarrow F_1(x') - c(x') - \varepsilon_1 > F_1(x) - c(x)
\]

Then, by continuity of \( F_1 \) and \( c \), \( \exists \delta_1 > 0 \) such that \( \forall x', |x' - x| > \delta_1 \). Then \( [x - \delta_1, x] \subset X_c \).

If \( \mu(x) < \mu^* \), then \( E[v_i(\mu(x), \mu(x^{-i}), a_h)] \geq E[v_i(\mu(x''), \mu(x^{-i}), a_h)] \) for all \( x'' \) as defined above. Therefore

\[
F_1(x'') - c \left( \frac{x''}{a_h} \right) + E[v_i(\mu(x''), \mu(x^{-i}), a_h)] - \varepsilon_2
\]

\[
> F_1(x) - c \left( \frac{x}{a_h} \right) + E[v_i(\mu(x), \mu(x^{-i}), a_h)]
\]

\[
\Rightarrow F_1(x'') - c \left( \frac{x''}{a_h} \right) - \varepsilon_2 > F_1(x) - c \left( \frac{x}{a_h} \right)
\]

Then, by continuity, \( \exists \delta_2 > 0 \) such that \( \forall x'', |x'' - x| > \delta_2 \). Then \( [x, x + \delta_2] \subset X_c \).

In either case, if \( x \in X_c \), then there is an interval with some supremum \( b \) and infimum \( a \) such that \( x \in (a, b) \subset X_c \).

If \( b < x^*_\ell \), then there is an \( x' \in \{ (x_{h^*}, x^*_\ell) \cap X_1^f \cap X_1^h \} \) where \( |x' - b| < \delta \) for all \( \delta > 0 \). Therefore there is an \( x' \in \{ (x_{h^*}, x^*_\ell) \cap X_1^f \cap X_1^h \} \) such that \( F(x') - F(b) < c(b/a_h) - c \left( \frac{b + a}{2a_h} \right) \). Note that this implies that \( F_1(x') - F_1 \left( \frac{b + a}{2a_h} \right) < c(x'/a_h) - c \left( \frac{b + a}{2a_h} \right) \) and \( F_1(x') - F_1 \left( \frac{b + a}{2a_h} \right) < c(x') - c \left( \frac{b + a}{2a_h} \right) \).

If \( \mu((b + a)/2) < \mu(x') \) then

\[
E \left[ \pi^i \left( \frac{b + a}{2} \right) | a_h \right] = F_1 \left( \frac{b + a}{2} \right) - c \left( \frac{b + a}{2a_h} \right) + E \left[ v_i \left( \mu \left( \frac{b + a}{2} \right), \mu(x^{-i}), a_h \right) \right]
\]

\[
> F_1(x') - c \left( \frac{x'}{a_h} \right) + E[v_i(\mu(x'), \mu(x^{-i}), a_h)] = E[\pi^i(x')|a_h].
\]

If \( \mu((b + a)/2) \geq \mu(x') \) then

\[
E \left[ \pi^i \left( \frac{b + a}{2} \right) | a_\ell \right] = F_1 \left( \frac{b + a}{2} \right) - c \left( \frac{b + a}{2} \right) + E \left[ v_i \left( \mu \left( \frac{b + a}{2} \right), \mu(x^{-i}), a_\ell \right) \right]
\]

\[
> F_1(x') - c(x') + E[v_i(\mu(x'), \mu(x^{-i}), a_\ell)] = E[\pi^i(x')|a_\ell].
\]
In either case, this contradicts \( x' \in \{X_1^e \cap X_1^h \} \).

If \( b = x^*_h \), then there is an \( x' \in X_1^h \), such that \( |x' - b| < \delta \), and \( \mu(x') = 1 \). This implies that

\[
F_1(x') - F_1\left(\frac{b + a}{2}\right) < c(x'/a_h) - c\left(\frac{b + a}{2a_h}\right),
\]

and

\[
E\left[\pi_i\left(\frac{b + a}{2}\right)|a_h\right] = F_1\left(\frac{b + a}{2}\right) - c\left(\frac{b + a}{2a_h}\right) + E\left[v_i(I(\mu(x'), x^{-i}), a_h)\right]
\]

\[
> F_1(x') - c\left(\frac{x'}{a_h}\right) + E[v_i(\mu(x'), x^{-i}, a_h)] = E[\pi'(x')|a_h].
\]

This contradicts \( x' \in X_1^h \). Therefore \( X_\varepsilon \) must be empty and for all \( x \in (x^*_h, x^*_e) \), we must have \( x \in \{BR(a_\varepsilon) \cap BR(a_h)\} \).

\[\square\]

**Corollary 4.6** The belief function and the distribution functions of output are continuous in output on \( (0, x^*_h) \). Additionally, the belief function is given by \( \mu(x) = 0 \) for all \( x \in [0, x^*_h] \), \( \mu(x) = 1 \) for all \( x \in [x^*_e, x^*_h] \) and is weakly increasing on \( (x^*_h, x^*_e) \).

**Proof.** By definition, distribution functions are right continuous. Lemma 4.3 shows that there no output is played with positive probability by either low or high ability players. This implies that the right limit of the distribution function is equal to the left limit at every point. Therefore \( H_1 \) and \( L_1 \) are continuous and \( F_1 = \frac{1}{2}L_1 + \frac{1}{2}H_1 \) is also continuous.

To show that \( \mu(x) \) is continuous on \( (0, x^*_h) \), note that \( E[\pi'(x)|a_\varepsilon] \) is constant for all \( x \in BR(a_\varepsilon) \) and \( E[\pi'(x)|a_h] \) is constant for all \( x \in BR(a_h) \). Since both \( F_1(x) \) and \( c(x) \) are continuous on \( (0, \infty) \) and \( E[v_i(\mu(x), x^{-i}, a_\ell)] = c(x, 1) - F_1(x) + k_\ell \) on \( [0, x^*_e] \) for some constant \( k_\ell \), then \( E[v_i(\mu(x), x^{-i}, a_\ell)] \) must be continuous on this interval. Also, \( E[v_i(\mu(x), x^{-i}, a_h)] = c_\ell\left(\frac{a}{a_h}\right) - F_1(x) + k_h \) on \( [x^*_h, x^*_h] \) for some constant \( k_h \), then \( E[v_i(\mu(x), x^{-i}, a_h)] \) is continuous on this interval. Since \( E[v_i(\mu(x), x^{-i}, a_h)] \) is strictly decreasing in \( \mu(x) \), and \( E[v_i(\mu(x), x^{-i}, a_\ell)] \) is strictly increasing in \( \mu(x) \), then \( \mu(x) \) must also be continuous on \( BR(a_\ell) \cup BR(a_h) = [0, x^*_h] \).

Using the above, we now show that the set \( [0, x^*_h] \setminus X_1 \) has no interior, i.e. there can be no interval \( [a, b] \subset [0, x^*_h] \) where for all \( x \in [a, b], x \notin X_1 \). This implies that \( X_1 \) is dense in \( [0, x^*_h] \).

If we let \( [\bar{a}, \bar{b}] \subset [0, x^*_h] \setminus X_1 \) be given, then define \( a \) and \( b \) to be the infimum and supremum
respectively of the interval in $[0,x^*_h) \setminus X_1$ which contains $[\tilde{a}, \tilde{b}]$. Neither $x_{h,*}$ nor $x^*_f$ can be contained in the interval as they are the limit point of a subset of $X_1$. Then the interval $[a,b]$ must be contained within either $[0,x_{h,*}]$, $[x_{h,*}, x^*_f]$, or $[x^*_f, x^*_h]$.

1. If $[a,b] \subset [0,x_{h,*}]$, then for all $x \in [a,b]$, $f_1(x) = 0$ which implies that $F_1(x) = F_1(a)$ and $x \in BR(a_\ell)$. Therefore,

$$E[v_i(\mu(b), \mu(x^{-i}), a_\ell)] - c(b) = E[v_i(\mu(a), \mu(x^{-i}), a_\ell)] - c(a),$$

and $\mu(b) > \mu(a)$. Then for all $\delta > 0$, there is an $x \in X_1$ such that $|x-b| < \delta$. If $x \in X_1^f$, we must have $\mu(x) = 0$ or $x \in X_1^h$. Since $\mu(x)$ is continuous, then $\mu(x) \neq 0$, so $x \in X_1^h$. Also if $x \in X_1^f$, then $x \in X_1^h$. Either way, for all $\delta > 0$, there must be an $x \in X_1^h$ for which $|x-b| < \delta$. If $x \in X_1^h \cap X_1^f$, then $\mu(x) = 1$, and $E[\pi'(a+b)|a_h] > E[\pi'(x)|a_h]$, a contradiction. If $x \in X_1^h \cap X_1^f$ then either $E[\pi'(a+b)|a_\ell] > E[\pi'(a)|a_\ell]$ or $E[\pi'(a+b)|a_\ell] > E[\pi'(x)|a_h]$, again a contradiction. Therefore $[a,b] \not\subset [0,x_{h,*}]$.

2. If $[a,b] \subset [x_{h,*}, x^*_f]$, then for all $x \in [a,b], x \in \{BR(a_\ell) \cap BR(a_h)\}$ which implies

$$E[v_i(\mu(b), \mu(x^{-i}), a_\ell)] - c(b) = E[v_i(\mu(a), \mu(x^{-i}), a_\ell)] - c(a),$$

$$E[v_i(\mu(b), \mu(x^{-i}), a_h)] - c(b/a_h) = E[v_i(\mu(a), \mu(x^{-i}), a_h)] - c(a/a_h)$$

This gives

$$E[v_i(\mu(b), \mu(x^{-i}), a_\ell)] - E[v_i(\mu(a), \mu(x^{-i}), a-\ell)] = c(b) - c(a) > 0,$$

$$E[v_i(\mu(b), \mu(x^{-i}), a_h)] - E[v_i(\mu(a), \mu(x^{-i}), a_h)] = c(b/a_h) - c(a/a_h) > 0.$$ 

However, these inequalities cannot hold at the same time, so $[a,b] \not\subset [x_{h,*}, x^*_f]$.

3. If $[a,b] \subset [x^*_f, x^*_h]$, then for all $x \in [a,b], x \in BR(a_h)$ and therefore,

$$E[v_i(\mu(b), \mu(x^{-i}), a_h)] - c(b/a_h) = E[v_i(\mu(a), \mu(x^{-i}), a_h)] - c(a/a_h),$$

and $\mu(b) < \mu(a) \leq 1$. Then for all $\delta > 0$, there is an $x \in X_1^h$ such that $|x-b| < \delta$ and $\mu(x) = 1$. However, this contradicts the continuity of $\mu(x)$. Therefore $[a,b] \not\subset [x^*_f, x^*_h]$. 

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Therefore the interior of $[0,x_h^*] \setminus X_1$ is empty, and $X_1$ is dense on $[0,x_h^*]$.

Since $X_1$ is dense on $[0,x_h^*]$ we can now show that $\mu(x) = 0$ for any $x \in [0,x_h^*)$. If $\mu(x) = \varepsilon > 0$, then by the continuity of $\mu(x)$, $\exists \delta > 0$ where $\forall x', |x' - x| < \delta$, $\mu(x) > \varepsilon/2$. However for all $\delta > 0$ there is an $x' \in X_1 \setminus X_1^0$ for which $\mu(x') = 0$, a contradiction. Therefore $\mu(x) = 0$ for all $x \in [0,x_h^*)$. Note that $\mu(x_h^*) = 0$ which follows from a similar argument of continuity from the left. Additionally, $\mu(x) = 1$ for all $x \in [x_\ell^*, x_h^*].$

Lastly we show that $\mu(x)$ is weakly increasing on $[x_{h,*}, x_\ell^*]$. Let $x,y \in [x_{h,*}, x_\ell^*]$ be such that, $\mu(x) > \mu(y)$ and $x < y$. Then there is an $x'$ and $y'$ arbitrarily close to $x$ and $y$ respectively, where $x',y' \in X_1$ and therefore $\mu(x') \leq \mu(y')$. This is not consistent with $\mu(\cdot)$ being continuous, a contradiction. □

**Theorem 4.3**

*Proof.* The lemmas above show that there are three distinct intervals in each equilibrium. We will show that the endpoints of these intervals and the distribution functions on the intervals are completely determined by the first order conditions of players.

The three intervals we investigate are partitioned by the best response sets of the high and low ability players. The first is the set of outputs where only low ability players are optimizing: $[0,x_{h,*}) = \{BR(a_\ell) \setminus BR(a_h)\}$. Next is the set of outputs where both high and low ability players are optimizing $[x_{h,*}, x_\ell^*] = \{BR(a_\ell) \cap BR(a_h)\}$. Lastly is the set of outputs where only high ability players are optimizing: $(x_\ell^*, x_h^*) = \{BR(a_h) \setminus BR(a_\ell)\}$. For each output where $x \in BR(a_\ell)$, the low ability player’s first order condition must hold and likewise for each $x \in BR(a_h)$ the high ability player’s first order condition must hold.

Conditions for $x$ being in $BR(a_h)$ and $BR(a_\ell)$ are
\[
BR(a_h) : F_1^*(x) + E[v_i(\mu(x), \mu(x^{-i}), a_h)] - c(\frac{x}{a_h}) = k_h
\]
\[
BR(x_\ell) : F_1^*(x) + E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] - c(x) = k_\ell = 0
\]

For the range of $0 \leq x < x_{h,*}$ we have that $E[v_i(\mu(x), \mu(x^{-i}), a_\ell)] = 0$ as $\mu(x) = 0$. Therefore we have that $F_1^*(x) = c(x)$ for all $x \in [0,x_{h,*}].$
For the range \( x^*_h < x \leq x^*_h \), \( E[v_i(\mu(x), \mu(x^{-i}), a_h)] = E_{x_i}[v_i(1, \mu(x_j), a_h)] \equiv v_h \). Then we have \( F^*_h(x) + v_h = c(x/a_h) + k_h \), for all \( x \in [x^*_h, x^*_h] \).

For the range \( x^*_h < x \leq x^*_h \), for all \( x \in \{X^h_1 \cup X^h_1\} \) we know \( x \in \{X^h_1 \cap X^h_1\} \). Therefore, both low and high ability players are indifferent between all outputs in this range. Because the marginal cost of the low ability player is always more than the marginal cost of the high ability player, this can only be true if increasing output benefits the low ability player more than the high ability player. Since the beliefs players have in the second period completely determine their expected payoffs, this indifference condition determines the belief function over this interval. The difference in marginal benefits of increasing output for the high ability and low ability players must equal the difference in marginal costs that they face today. To derive this, we subtract the condition for \( X^h_1 \) from the condition for \( X^h_1 \):

\[
E[v_i(\mu(x), \mu(x^{-i}), a_h)] - E[v_i(\mu(x), \mu(x^{-i}), a_h)] = c\left(\frac{x}{a_h}\right) + k_h - c(x)
\]

Taking the derivative of each side with respect to output,

\[
\frac{\partial}{\partial x}(E[v_i(\mu(x), \mu(x^{-i}), a_h)] - E[v_i(\mu(x), \mu(x^{-i}), a_h)]) = \frac{\partial}{\partial x}(c\left(\frac{x}{a_h}\right) - c(x))
\]

\[
\mu'(x) \left[d(\mu(x))(F^*_h(\mu(x)) - 1) - d(\mu(x))F^*_h(\mu(x))\right] = \frac{1}{a_h} c'\left(\frac{x}{a_h}\right) - c'(x)
\]

\[
\mu'(x)d(\mu(x)) = c'(x) - \frac{1}{a_h} c'(x/a_h)
\]

Note that on this interval, \( \mu'(x) > 0 \) and therefore, \( F^*_h(\mu(x)) = F^*_h(x) \) for all \( x \in (x^*_h, x^*_h) \).

We now take the derivative of the condition for \( X^h_1 \) and combine with the previous equality:

\[
f^*_h(x) + \mu'(x)d(\mu(x))F^*_h(x) = c'(x)
\]

\[
f^*_h(x) + \left(c'(x) - \frac{1}{a_h} c'(x/a_h)\right) F^*_h(x) = c'(x)
\]

\[
f^*_h(x) = \frac{\partial}{\partial x} c(x)(1 - F^*_h(x)) + \frac{\partial}{\partial x} c\left(\frac{x}{a_h}\right) F^*_h(x) \quad (\dagger)
\]

From continuity of \( F^*_h(x) \), we also have that \( F^*_h(x^*_h) = c(x^*_h) \). For a given \( x^*_h \), using the Picard - Lindelof Theorem\(^{15} \) we know that there is a unique solution for \( f^*_h(x) \) on \([x^*_h, x^*_h]\), and

\(^{15}\)The right hand side of \((\dagger)\) is continuous in \( x \) and uniformly Lipshitz continuous in \( F^*_h(x) \) on the interval of \([x^*_h, x^*_h]\), see Lindelof 1894
therefore \( F^*_1(x) \) is determined on this interval. Additionally, given \( f_1(x) \) on \([0,x^*_h]\), the endpoints \( x_{h,*}, x^*_h, \) and \( x^*_h \) can be solved for. With \( \mu(x) \) characterized, and the equilibrium strategies of high ability and low ability players can be calculated.

To see why only one such \( x_{h,*} \) can lead to an equilibrium, consider a different initial condition, 
\[ F^*(\tilde{x}_{h,*}) = c(\tilde{x}_{h,*}) \] where \( \tilde{x}_{h,*} > x_{h,*} \) and the associated \( \tilde{f}_1(x) \) on \([\tilde{x}_{h,*}, \tilde{x}^*_h]\). First note that \( L(\tilde{x}_{h,*}) = c(\tilde{x}_{h,*}) > c(x_{h,*}) = L(x_{h,*}) \). Also, from (†), for each \( x \in [\tilde{x}_{h,*}, \tilde{x}^*_h] \), \( \tilde{f}_1(x) > f_1(x) \). Lastly, \( \tilde{\mu}(\tilde{x}_{h,*}) < \mu(\tilde{x}_{h,*}) \). Since \( \mu(x) = 1 - \frac{\ell(x)}{2f(x)} \), then for all \( x \) where \( \tilde{\mu}(x) < \mu(x) \), we must have \( \tilde{\ell}(x) > \ell(x) \). Then we have that \( \tilde{\ell}(x) > \ell(x) \) and \( \tilde{\mu}(x) < \mu(x) \) for every \( x \in [\tilde{x}_{h,*}, \tilde{x}_{x,*} + \varepsilon] \). In order to get 
\[ \tilde{\ell}(\tilde{x}^*_h) = \tilde{\mu}(\tilde{x}^*_h) = 1, \] there must be an \( x \) such that \( \tilde{f}(x) = f(x) \) in \([\tilde{x}_{h,*}, \tilde{x}^*_h]\), but this can’t be true because \( \tilde{f}(x) \) and \( f(x) \) are different members of the same family of solutions, and cannot cross. Similarly, there cannot be an equilibrium where \( \tilde{x}_{h,*} < x_{h,*} \).

Therefore \( F^*_1(x) \) is uniquely characterized on \( X_1 \) where \( X_1 = [0,x^*_h] \). Then \( L^*_1(x) \) and \( H^*_1(x) \) are uniquely determined on this set. These distributions along with the second period output distributions \( L^*_2(x|\mu_i, \mu_{-i}) \) and \( H^*_2(x|\mu_i, \mu_{-i}) \) form the unique symmetric Bayes Nash equilibrium. \( \square \)
CHAPTER 2

Identification of Beliefs in Decision Making

2.1 Introduction

If we observe an individual making decisions and if we also observe the signals that he has received, what can we say about his beliefs? In this paper, we direct the tools of identification to address this question. Whereas the classical approach of decision theory assumes a rich set of choice problems, we work on a binary choice problem, assuming a potentially rich set of signals to achieve identification.

Recent literature has found heterogeneous beliefs to be associated with the formation of bubbles, Xiong 2013; motivation to commit crimes, Lochner 2003; and the choice of contraceptive methods, Delavande 2008; among others, see Manski 2004. In many decision problems, the agent’s beliefs cannot be disentangled from his utility. However in the above models, beliefs, not utilities, are the most interesting object, as they help to explain past actions and predict future behavior.

In our model, we address situations where beliefs are unknown, yet restrictions can be placed on agent’s utility. Our paper shows that in this setting, if the space of signals is sufficiently rich, an econometrician can precisely identify any belief the decision maker may have. We initially achieve this result when there are only two states in the world. Subsequently, we define richness of signals in multiple finite states, show that this condition is necessary for identification of beliefs, and we briefly discuss the sufficient conditions. Following the literature on partial identification developed in Manski 2003, we also describe what an econometrician can say when there is not sufficient information to point identify beliefs. In these cases, such as when the information has limited likelihood or when there is a finite number of signals, we construct the sharpest bounds an econometrician can form around the agent’s beliefs. We also analyze the case when the agent is
dynamically updating his beliefs in a sequence of decisions and show how to construct the sharpest boundaries around his priors.

We first direct our attention to equilibrium beliefs. In this setting, a single agent makes a binary decision after observing a signal which allows him to update his belief about his current state. The econometrician observes the signals and the decisions made repeatedly¹. Based on this information, she makes an inference about the agent’s belief. We show that if the state is relevant, the space of signals is rich enough, and the beliefs are non trivial, we can identify any equilibrium belief the agent might have in a model with two states. We generalize this result for the case of a finite number of states and discuss how rich the distribution of signals must be to achieve identification. In order to derive the above results we do not impose any meaningful assumptions on the agent’s utility function. If we consider that the econometrician also knows the decision rule, which is a function of the utilities in each state, we show that even if the space of signals is not rich, there will be a region where the beliefs are point identified and another region where they can be partially identified. Moreover, we show how one can impose non-trivial bounds on the agent’s beliefs even when there is a finite number of possible signals.

The equilibrium beliefs approach, although closely related to the identification literature, may be unsatisfactory in many economic situations. For this reason, our second approach addresses identification of priors that are updated after each signal. For a given sequence of signals, if we know the conditional distribution of signals, non-degenerate priors hold a one-to-one relation with posteriors. Using this relationship, we show that with any finite number of observations, we can impose non-trivial bounds on the agent’s priors. The bounds are constructed by looking at the decisions made when the agent is least sure about the optimal decision for their current state. Since we are looking at these marginal decisions, the sharpest bounds will depend on the entire history of signals and decisions. As such, to best infer the agent’s priors and current beliefs, the econometrician should be concerned with all past observations.

¹We focus our paper in the individual interpretation of the decision making process, when we observe an individual making the same decision repeatedly. However, the results of the paper, specifically those in the next section, can be understood in the context of a population of identical individuals receiving conditionally independent signals and the variable of interest would be the average belief in the population. Although this could be a a more credible data set, we refrain from getting back to this interpretation for the sake of having a coherent story.
This paper follows the tradition founded by Ramsey 1931 and Savage 1972, who describe probability as a subjective feature based on the actions they could generate. According to Savage’s view, if we could offer the agent a menu of lotteries, we could find the one which makes him indifferent between actions. From this it would be possible to recover the probability he assigned to each state of the world. We only offer the agent binary choices but infer these probabilities from how the choice changes as the agent receives information. In addition to these seminal works, Anscombe and Aumann 1963 provided a proof of the existence of these subjective probabilities and Gul 1992 showed that we can guarantee their existence in a setting with a finite number of states. More recently, Ross 2013 showed that observing prices from all possible states, we can recover both utilities and beliefs. Arieli and Mueller-Frank 2014 show that if the action set is uncountable, then there exists a continuous utility function such that actions reveal beliefs.

Our paper is also closely related to the literature on experts theory and scoring rules. Savage 1971 formalized the discussion on how to elicit probabilities. More recently, Dillenberger and Sadowski 2012 and Lu 2013 discussed identification on decision theory models when we can observe the individuals’ entire preference relation or the entire stochastic choice function respectively. Dillenberger and Sadowski 2012 describes conditions on utilities over menus of securities which correspond to an individual behaving as if they had a distribution over the probabilities of the outcomes and was choosing optimally. In Lu’s set up, the econometrician observes a probability distribution over the menu of choices and he determines whether these choices are consistent with the the realization of a random private signal. He shows that choices among binary decisions are sufficient to recover information received by the agent if we employ a test function, whose known payoff varies. In our model, the choice space is restricted to the simplest example of a binary decision where the payoff does not vary, so the identification question depends on the richness of the signal space.

An alternative approach to inferring beliefs is to directly elicit them from agents. Nyarko and Schotter 2002 show how to elicit beliefs in a game using proper scoring rules; Delavande 2008 survey sexually active women in Chicago and elicited expectations of alternative contraceptive methods; Lochner 2003 employ data about expectations of being arrested to construct a model of utility for criminal behavior; and Delavande, Giné, and McKenzie 2011 review the methods of
eliciting beliefs in developing countries and argue that probabilistic questions can be accurately answered even by individuals with very low education levels. Manski 2004 offers a comprehensive survey of this literature, defending the survey based approach for determining beliefs. This approach can suffer from misreporting due to inability or unwillingness to give correct answers. We believe that results should be complemented by and consistent with inference of beliefs from choice data.

The rest of the paper is organized as follows. Section 2 describes the general model and derive the results for the identification of beliefs. Section 3 uses a modified model to address the issue of identification of priors. Section 4 concludes.

2.2 Identification of Beliefs

The decision making problem

There is one agent making a binary decision in an uncertain world with two possible states. The agent does not know which world he is in, but has a belief about the likelihood of each state. Let the decision of the agent be whether or not to invest \( d = i \) or \( d = ni \) in a world that is either good or bad for investment \( s = h \) or \( s = ℓ \). Before making his decision, the agent receives a signal \( x \in X \), say available economic indicators, which allows him to update his beliefs from known conditional distributions, \( f(x|h) \) and \( f(x|ℓ) \).

We will incorporate the uncertainty in our model in two different ways. In the first one, which we consider throughout this section, there is an independent identically distributed move of nature in each period. Therefore a new state is drawn before each signal is received and decision is made. In this case the belief of the agent, denoted by \( p = (p(h), 1 - p(h)) \), can be interpreted as the agent’s belief about the distribution of this move of nature, and it remains fixed for each decision. Therefore the agent only considers the signal realized in the current decision process when deciding which action to take. In the next section we will allow the agents beliefs to be evolve as signals are aggregated.

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2 In a later subsection we will discuss our results for a finite number of states \( S \).

3 In this sense a binary decision is the hardest case since working with more decisions would make the identification strategies easier since every decision would provide more information.
We restrict the signal space to be countable but potentially infinite in every state of the world. Although informative about the state, the signal does not directly influence the agent’s payoff. After observing it, the agent updates his beliefs according to Bayes’ rule and his posterior probability is given by:

\[
p(h|x) = \frac{f(x|h)p(h)}{f(x)} = \frac{f(x|h)p(h)}{f(x|h)p(h) + f(x|\ell)(1 - p(h))}
\] (2.1)

His payoffs are characterized by the values, \(v(i, h), v(ni, h), v(i, \ell),\) and \(v(ni, \ell),\) the payoff of each decision in each state. We define \(\tilde{v}(s) \equiv v(i, s) - v(ni, s),\) the difference between investing and not investing in each state. In order to keep the decision problem from becoming trivial, we make an assumption on these payoffs.

**Assumption 2.2.1.** \(\tilde{v}(h) > 0 > \tilde{v}(\ell)\)

This condition implies that the agent wants to invest in the high state and does not want to invest in the low state. Since he does not know in which state he is, he will invest exactly when

\[
E_{p(s|x)}[\tilde{v}(s)] \geq 0.
\] (2.2)

**The econometrician problem** The econometrician sees the same signals as the agent and she observes his decisions. She also knows the conditional distribution of signals, \(f(x|s),\) so every observation can be summarized as a pair \((d_i, x_i).\) Her main objective is to identify the agent’s initial beliefs \(p.\) We follow the notation and definition of identification in Manski 2003.

**Definition 2.2.1.** The set of priors that is observationally equivalent to \(p\) is given by \(H[p] \equiv \{p' \in \Delta^{\mathbb{R}^S}: \{d(p, x_i)\}_{i=1}^\infty = \{d(p', x_i)\}_{i=1}^\infty\}.\)

Each \(p' \in H[p]\) is consistent with the decisions of an agent with prior \(p\) for any information that could be received. If \(H[p]\) is a strict subset of \(\Delta^{\mathbb{R}^S},\) \(p\) is partially identified, and if \(H[p]\) is a singleton, then we say that \(p\) is point identified. If \(H[p]\) is a singleton for each \(p,\) then we say the beliefs of the agent are point identified.
**Identification result** We can manipulate equation (2) to characterize when the agent will invest given his posteriors and payoffs.

\[ p(h|x) \geq \frac{\bar{v}(h)}{\bar{v}(h) - \bar{v}(\ell)} \equiv \bar{d} \]  

(2.3)

Hereafter \( \bar{d} \) can be interpreted as the threshold posterior of the decision rule, where the agent is indifferent between investing and not investing.\(^4\) Combining equations (1) and (3), we can write the decision rule in terms of the agents beliefs.

\[ \frac{f(x|h)p(h)}{f(x|h)p(h) + f(x|\ell)(1 - p(h))} \geq \bar{d} \]  

(2.4)

We will use the likelihood ratio as a summary statistic for the information that is observed by the agent and the econometrician. This likelihood ratio is defined by

\[ \gamma(x) \equiv \frac{f(x|h)}{f(x|\ell)} \text{ for each } x \in X. \]

We can now write the decision rule for the agent in terms of his prior belief about the distribution of states, our variable of interest. The agent chooses to invest given signal \( x \) exactly when

\[ p(h) \geq \frac{\bar{d}}{\bar{d} + \gamma(x)(1 - \bar{d})}. \]  

(2.5)

In order for the econometrician to point identify the agent’s beliefs there must be enough information available. Not only must there be information that can sway both an optimistic and pessimistic agent, this information must also be dense enough to differentiate beliefs that are relatively similar. These requirements are formalized in the following full range assumption on the likelihood ratio of the signals in the information set.

**Assumption 2.2.2.** For each \( y \in (0, \infty) \) and \( \varepsilon > 0 \) there exists a signal \( x \) such that \( \gamma(x) \in B_\varepsilon(y) \).\(^5\)

An information set which satisfies the full range assumption is sufficient for the econometrician to identify the agent’s beliefs. This is shown in the following result.

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\(^4\)Even if the econometrician does not know agent’s utilities, assumption 1.1 is sufficient to guarantee that \( \bar{d} \in (0, 1) \).

\(^5\)Since the rationals are countable and dense on the real line, any information set \( X \) which \( \gamma(\cdot) \) maps onto the rational numbers would satisfy this property.
Proposition 2.2.1. If the state is relevant and likelihood ratio of the signals has full range then the initial beliefs of the decision maker are point identified.

Proof. Let \( p = (p(h), 1-p(h)) \) and \( p' = (p'(h), 1-p'(h)) \) be two distinct beliefs. Without loss of generality assume that \( p(h) > p'(h) \). We define \( \hat{p} \) and \( \hat{y} \) such that

\[
\hat{p} = \frac{p(h) + p'(h)}{2} \quad \text{and} \quad \hat{y} = \frac{(1-\hat{p})\overline{d}}{\hat{p}(1-\overline{d})}
\]

Since \( \hat{p} \in (0,1) \) and \( \overline{d} \in (0,1) \) we have that \( \hat{y} \in (0,\infty) \). By the full range assumption, there exists an \( \hat{x}(\varepsilon) \) such that \( \gamma(\hat{x}(\varepsilon)) \in B_\varepsilon(\hat{y}) \), for all \( \varepsilon > 0 \). For small enough \( \varepsilon > 0 \), we have that

\[
p(h) > \frac{\overline{d}}{\overline{d} + \gamma(\hat{x}(\varepsilon))(1-\overline{d})} \approx \hat{p} > p'(h).
\]

With the signal of \( \hat{x}(\varepsilon) \), \( d(p, \hat{x}(\varepsilon)) = i \) and \( d(p', \hat{x}(\varepsilon)) = ni \). This shows that the two beliefs \( p \) and \( p' \) are not observational equivalent. Since \( p \) and \( p' \) are arbitrary beliefs of the agent, all beliefs are identified.

This result closely resembles the seminal result of Savage 1972. Here, instead of offering the agent a pool of lotteries between outcomes, the agent may observe a range of signals that allows the econometrician to pin down his initial belief over the distribution of the states.

Additionally, the full range assumption is necessary for point identification of the agent’s beliefs. If there is an interval on the positive real line for which there is no signal \( x \) whose likelihood ratio \( \gamma(x) \) lies within the interval (the range of \( \gamma(x) \) is not dense on the positive real line), then there will be two different beliefs \( p \) and \( p' \) which will be observationally equivalent. For any such interval, there is a pair or priors that can only be differentiated by a signal whose likelihood ratio falls within that interval.

In the following sections, we show that when the assumption of full range is relaxed the econometrician can still partially identify the priors of the agent.

2.2.1 Partial identification without full range

We now weaken the assumption of full range of information and consider partial identification when the range of the likelihood ratio is limited. Intuitively, this may happen when there is no
information that is strong enough to either convince very pessimistic agents to invest or keep very optimistic agents from investing. Specifically, for the given information set, the range of the likelihood function is bounded, either above, away from zero, or both.

\[
\text{Range}(\gamma(x)) \subset (\alpha, \overline{\alpha}) \subseteq \mathbb{R}_+
\]

Within these bounds, we still assume that information is sufficiently dense as in the full range assumption from the previous section. In order for the econometrician to know the range of prior beliefs that can be identified, she must have information about the agent’s utility function. It is sufficient for her to know the value of \(\overline{d}\), which determines the decision rule of the agent.

**Assumption 2.2.3.** The econometrician knows \(\overline{d}\).

With this assumption, the econometrician can point identify prior beliefs of an agent who can be persuaded by available information. The beliefs of agents who cannot be persuaded by any of the available information will be partially identified.

**Proposition 2.2.2.** All beliefs of the agent for which \(p(h)\) belongs to the interval

\[
\left(\frac{\overline{d}}{\overline{d} + \alpha (1 - \overline{d})}, \frac{\overline{d}}{\overline{d} + \alpha (1 - \overline{d})}\right)
\]

are point identified. If the belief is such that \(p(h)\) is the right of the interval, then

\[
H[p] = \left[\frac{\overline{d}}{d + \alpha (1 - \overline{d})}, 1\right],
\]

and if it is to the left of the interval

\[
H[p] = \left[0, \frac{\overline{d}}{d + \alpha (1 - \overline{d})}\right].
\]

This proof follows the steps of Proposition 1 for the beliefs contained within the point identified interval. If the agent’s beliefs are such that his decision does not change, his beliefs can still be partially identified. For example, if the agent always decides to invest, then his beliefs are such that \(p(h)\) is observationally equivalent to 1.

Note that without Assumption 3, or any knowledge about the agent’s utility function, \(\overline{d}\) can take on any value between 0 and 1. In this case, the econometrician would not know which beliefs are point identified and which are just partially identified. The region of priors that are point identified is determined by \(\overline{d}\).
2.2.2 Partial identification with finite information

Point identification, even over a small region of the agent’s priors, requires an information set that is infinite. Nevertheless, with assumptions 2.1 and 2.3, we can partially identify the prior beliefs of the agent with a finite information set. The tightness of these bounds will depend on how dense and how persuasive these observations are.

We now define $\gamma^*(p)$ as the likelihood ratio that would make the agent indifferent between investing and not investing given beliefs $p$.

$$p(h) = \frac{\bar{d}}{d + \gamma^*(p)(1 - \bar{d})} \Rightarrow \gamma^*(p) = \frac{\bar{d}(1 - p(h))}{1 - \bar{d}}$$

Note that the agent will decide to invest after receiving signal $x$ if and only if $\gamma(x) > \gamma^*(p)$. Using this, we can impose bounds on $p(h)$.

Assume we have $M$ possible signals given the information set $X$, $|X| = M$. We can order the $M$ potential observations according to the size of their likelihood ratio, i.e. $\gamma(x_{m-1}) \leq \gamma(x_m)$, for all $1 \leq m \leq M$. Then let $\tilde{m}$ be the smallest $m$ for which the agent decides to invest after receiving signal $x_{\tilde{m}}$. This would imply (for $\tilde{m} > 1$) that $\gamma(x_{\tilde{m}}) > \gamma^*(p) > \gamma(x_{\tilde{m}-1})$ and the identification region is given by

$$H_M[p(h)] = \left[\frac{\bar{d}}{d + \gamma(x_{\tilde{m}})(1 - \bar{d})}, \frac{\bar{d}}{d + \gamma(x_{\tilde{m}-1})(1 - \bar{d})}\right]. \quad (2.6)$$

If $\tilde{m} = 1$, then the agent always invests, and the identification region is

$$H_M[p_h] = \left[\frac{\bar{d}}{d + \gamma(x_1)(1 - \bar{d})}, 1\right].$$

On the other hand, if the agent never decides to invest, then the identification region is

$$H_M[p_h] = \left[0, \frac{\bar{d}}{d + \gamma(x_M)(1 - \bar{d})}\right].$$

2.2.3 Multiple States

We now consider a situation where the agent is making a binary decision, but there are more than two possible states of the world. This allows us to capture a situation where the agent has multiple reasons for why he would, or would not, want to invest. The decision rule is now a function of the
agents posterior beliefs, \( p(s|x) \), and gain from investing, \( \tilde{v}(s) \), for each state \( s \in \{1, \ldots, S\} \). From Bayes’ rule, these posterior beliefs are given by

\[
p(s|x) = \frac{f(x|s)p(s)}{\sum_{s' \in S} f(x|s')p(s')} \quad \text{for all } s \in S. \tag{2.7}
\]

Without loss, we order the states according to the gain in deciding to invest such that: \(\tilde{v}(S) > \tilde{v}(S-1) > \cdots > \tilde{v}(1)\). As in the two state model, we require the agent to prefer investing in at least one state and to prefer not investing in at least one different state.

**Assumption 2.2.4.** There exists a state \( s^* \) such that \( \tilde{v}(s^*) > 0 > \tilde{v}(s^*-1) \).

For a given signal \( x \), the agent will decide to invest exactly when the expected gain from investing is positive. Thus the agent invests when

\[
E_{p(s|x)}[\tilde{v}(s)|x] = \sum_{s \in S} \tilde{v}(s)p(s|x) = \sum_{s \in S} \tilde{v}(s)p(s)f(x|s) \geq 0. \tag{2.8}
\]

In order for the decision rule to be non-trivial, we assume that the agent is uncertain about whether the state of the world is good for investment. In this case, we say the agent has a prior belief that may be identified by the econometrician for a given information set. An agent who is certain that the state is good, or certain that it is bad, will always make the same decision, and the econometrician will not be able to determine why this decision is being made.

**Definition 2.2.2.** A prior \( p \in \Delta(\mathbb{R}^S) \) is relevant if \( p(r) > 0 \) and \( p(t) > 0 \) for some states \( r < s^* \leq t \).

Let the set of relevant priors be denoted by \( P \subset \Delta(\mathbb{R}^S) \).

For the econometrician to identify the beliefs of the agent in this broader setting, there must be information that not only separates beliefs by how optimistic they are, but also tell us why the agent is optimistic or pessimistic. The full range assumption which gives the econometrician sufficient information for identification is similar to the corresponding assumption when there are only two states. For each pair of states where one state is good for investing, and the other bad, we must have information that can sway both optimistic and pessimistic individuals, and be dense enough to separate beliefs that are similar, *in this particular dimension*. This assumption is formalized below.
Assumption 2.2.5. For each \( y \in \mathbb{R}_+ \), \( \epsilon > 0 \), \( r < s^* \) and \( t \geq s^* \) there is an \( x \in X \), such that \( f(x|s) = 0 \) for all \( s \neq r,t \) and \( \frac{f(x|r)}{f(x|r)} = \gamma_{ir}(x) \in B_\epsilon(y) \).

The following lemma shows that the econometrician can point identify the priors of the agent when information satisfies full range and we can find two states, one good for investing, where the agents have different beliefs. The argument is similar to the proof of identification when there are two states. The proof is relegated to the appendix.

Lemma 2.2.3. For any \( p, p' \in P \), if there are two states \( r < s^* \leq t \) such that \( p'(t) \) and \( p'(r) \) are non zero and \( \frac{p'(t)}{p'(r)} \neq \frac{p(r)}{p(r)} \), then under the assumption of full range in multiple states, \( p' \notin H[p] \).

To show identification of the set of relevant priors, it is now enough to show that any two distinct, relevant priors have the stated property of Lemma 1. Again the proof is in the appendix.

Theorem 2.2.4. The set of relevant priors is point identified if \( X \) satisfies full range in multiple states.

While the full range in multiple states assumption appears strong, it cannot be weakened significantly. To see this, consider two priors that are indexed by 3 parameters, \( k_1 > 1, k_2 > 1, n \geq 3 \). Let \( p(s; k_1, k_2, n) = 1 - \frac{1}{(k_1 + 1)n} - \frac{1}{k_2 n}, p(r; k_1, k_2, n) = \frac{1}{(k_1 + 1)n}, p(t; k_1, k_2, n) = \frac{1}{k_2 n}, \) and \( p(s'; k_1, k_2, n) = 0 \) for all \( s' \neq r, s, t \), where \( r < s^* \) and \( t \geq s^* \). Let \( p'(s; k_1, k_2, n) = 1 - \frac{1}{(k_1 + 1)n} - \frac{1}{(k_2 + 1)n}, p'(r; k_1, k_2, n) = \frac{1}{k_1 n}, p'(t; k_1, k_2, n) = \frac{1}{(k_2 + 1)n}, \) and \( p'(s'; k_1, k_2, n) = 0 \) for all \( s' \neq r, s, t \), where \( r < s^* \) and \( t \geq s^* \). Note that a decision maker with a prior of \( p \) is more optimistic than one with a prior of \( p' \) for all \( k_1 > 1, k_2 > 1, \) and \( n \geq 3 \). In order for \( p'(k_1, k_2, n) \notin H[p(k_1, k_2, n)] \), there must be an \( x \in X \) such that the following inequalities hold.

\[
0 \leq f(x|s) \left[ \frac{(k_1 + 1)k_2 n - (k_1 + 1) - k_2}{(k_1 + 1)k_2 n} \right] \tilde{v}(s) + f(x|r) \frac{1}{(k_1 + 1)n} \tilde{v}(r) + f(x|t) \frac{1}{k_2 n} \tilde{v}(t) \tag{2.9}
\]

\[
0 > f(x|s) \left[ \frac{k_1 (k_2 + 1)n - k_1 - (k_2 + 1)}{k_1 (k_2 + 1)n} \right] \tilde{v}(s) + f(x|r) \frac{1}{k_1 n} \tilde{v}(r) + f(x|t) \frac{1}{(k_2 + 1)n} \tilde{v}(t) \tag{2.10}
\]
If $s \geq s^*$, then (10) implies that
\[
\gamma_{sr}(x) < \left[ \frac{k_2 + 1}{k_1(k_2 + 1)n - k_1 - k_2 - 1} \right] \frac{-\bar{v}(r)}{\bar{v}(s)}.
\]
For large $n$, both decision makers are almost certain that they are in a state where investing is the optimal choice. In order to differentiate between the two, they must receive a piece of information that almost never occurs in this state. Condition (10) implies that the more pessimistic player must decide not to invest. Therefore $\gamma_{sr}(x) \to 0$ as $n \to \infty$.

Also, if $\gamma_{sr}(x) = 0$. Then condition (9) and (10) combine to require a piece of information for which the optimistic player invests, while the pessimistic player does not
\[
\frac{-\bar{v}(r)}{\bar{v}(t)} \frac{k_2}{(k_1 + 1)} < \gamma_{sr}(x) < \frac{-\bar{v}(r) k_2 + 1}{k_1}.
\]
Therefore $\gamma_{sr}(x)$ must have full range while $\gamma_{sr}(x) = 0$. This argument can be made for each triple of states, $(s, r, t)$, where $r < s^*$ and $t \geq s^*$.

### 2.3 Identification of priors

For many economic applications, the idea that we observe an individual making the same decision repeatedly maybe of little practical purpose. In this section, we depart from the idea of identifying fixed, steady state beliefs and focus on the situation where there is only one move of nature that describes the state of the world and a series of signals about this underlying state. Provided that individuals have strictly positive priors and that we understand how they update their beliefs, there is a one to one mapping between priors and posteriors.\(^6\) Thus although we will focus on estimating priors, both objects are equivalent for predicting the agent’s behavior.

We return to the binary world and binary decision framework. Now the agent observes a sequence of signals, $X_M = \{x\}_{m=1}^M$, and after each signal he updates his posteriors and makes a decision, but he does not observe payoffs. In our investment problem, this assumption would

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\(^6\)Acemoglu, Chernozhukov, and Yildiz 2009 showed that if individuals disagree or are uncertain about the interpretation of the signals, there will be no asymptotic learning and no asymptotic agreement. Rather, even if individuals have the same initial prior and observe the same infinite sequence of signals, individuals can reach different posterior beliefs.
correspond to the idea that the agent observes $M$ signals and makes $M$ subsequent decisions, before
the state of the economy is revealed. We assume that the signals are independent, conditional on
the state of the world.

The econometrician observes the sequence of signals $X_M$ and the sequence of decisions $D_M = \{d\}_{m=1}^M$. From these observations, she creates the sharpest bounds on the agent’s prior. The sharpest bounds constitute the smallest identification region for the given sequence of observations. After each signal, the agent will invest exactly when the expected value of investment is positive, given his current posterior beliefs:

$$E_{p(s|X_m)}[\tilde{v}(s)|X_m] \geq 0.$$ 

This holds whenever $p(h|X_m) \geq d$.

In order to highlight the one-to-one relationship between posteriors and priors, we take the log
of the ratio of the posterior. Additionally, we can separate the effect that each signal has on the
posterior of the agent. The assumption that the signals are conditionally independent given the
state allows us to describe this relationship in the following equation:

$$\ln \left( \frac{p(h|X_M)}{p(\ell|X_M)} \right) = \ln \left( \frac{f(x_1|h) \cdots f(x_M|h)p(h)}{f(x_1|\ell) \cdots f(x_M|\ell)p(\ell)} \right) = \ln \left( \prod_{m=1}^M \gamma(x_m) \left( \frac{p(h)}{p(\ell)} \right) \right)$$

$$= \sum_{m=1}^M \ln(\gamma(x_m)) + \ln \left( \frac{p(h)}{p(\ell)} \right).$$  \hspace{1cm} (2.11)

The following proposition provides the sharpest bounds the econometrician can impose on the
agent for a given sequence of signals and decisions.

**Proposition 2.3.1.** The sharpest bounds in the region of identification of priors are given by

$$H \left[ \ln \left( \frac{p(h)}{p(\ell)} \right) \right] = \left[ \max_{1 \leq m \leq M} \ln \left( \frac{\tilde{d}}{1 - \tilde{d}} \right) - \sum_{m=1}^M \ln(\gamma(x_m)) , \min_{1 \leq m \leq M} \ln \left( \frac{\tilde{d}}{1 - \tilde{d}} \right) - \sum_{m=1}^M \ln(\gamma(x_m)) \right].$$

**Proof.** In every period $j$, the agent will decide to invest if and only if

$$\sum_{m=1}^j \ln(\gamma(x_m)) + \ln \left( \frac{p(h)}{p(\ell)} \right) \geq \tilde{d}.$$ 

Thus whenever the agent chooses to invest, given observation pair $(\{x\}_{m=1}^j, d_j)$, we can impose a
lower bound for $\ln(p(h)/p(\ell))$. The sharpest lower bound is derived from the period where the
agent has the lowest posterior but still decides to invest. This is exactly

$$\max_{1 \leq m \leq M} \ln \left( \frac{\bar{d}}{1 - \bar{d}} \right) - \sum_{m=1}^{M} \ln(\gamma(x_m)).$$

Similarly the sharpest upper bound is formed from the period where the agent has the highest posteriors given that he decides not to invest.

Figure 1 depicts an example of the construction of these bounds. The econometrician knows $\bar{d}$, which is denoted as the horizontal line, and she also knows the summation of the log likelihood ratio of the sequence of signals. On the figure, this is the vertical distance between the log of the initial beliefs and the log of the current beliefs. In this example, the upper bound is derived from observation 4, which is the highest posterior for which the agent chooses not to invest. The lower
bound is constructed from observation 7. We construct each bound by subtracting this vertical distance from the \( \bar{d} \) in each key observation. After finding the bounds of \( \ln(p(h)/p(\ell)) \) we can use them to derive bounds for \( p(h) \) and \( p(\ell) \).

There is no reason to believe that the inference about the prior will necessarily improve asymptotically. However, since we are working with conditionally independent signals, the posterior of the agent approaches the true state of the world, so identification becomes less important.

From this example it is clear that all observations can be important in forming bounds on the priors of the agent. The econometrician should be concerned with all the past decisions, not because the present world mechanically depends on the past, but because past observations tighten the bounds on the agent’s beliefs.

### 2.4 Concluding remarks

In this paper we direct our attention to a classical question in economics: what can we infer about an agent’s beliefs about the state of the world if we observe his decisions? We believe that there is a growing class of problems where economists are interested in an agent’s beliefs and are comfortable making assumptions about his utility function, or at least about his decision rule. We show that even in binary decision making with multiple states, agents beliefs can be precisely identified if we observe a range of signals that are sufficiently rich. We hope that these positive theoretical results will lead to further research agendas, specifically in experimental economics and structural economics where identifying beliefs is the main goal.

### 2.5 Appendix

**Lemma 1**

For any \( p, p' \in P \), if there are two states \( r < s^* \leq t \) such that \( p'(t) \) and \( p'(r) \) are non zero and \( \frac{p(t)}{p'(t)} \neq \frac{p(r)}{p'(r)} \), then under the assumption of full range in multiple states, \( p' \not\in H[p] \).
Proof. We take \( \frac{p(r)}{p'(r)} > \frac{p(t)}{p'(t)} \) (the argument is symmetric for the case where \( \frac{p(r)}{p'(r)} < \frac{p(t)}{p'(t)} \)). Then,

\[
\frac{p(r)}{p'(r)} > \frac{p(t)}{p'(t)} \Rightarrow \frac{\tilde{v}(r)p(r)}{\tilde{v}(t)p'(r)} > \frac{\tilde{v}(t)p(t)}{\tilde{v}(t)p'(t)}
\]

\[
\Rightarrow \frac{\tilde{v}(r)p(r)}{\tilde{v}(t)p'(r)} < \frac{\tilde{v}(r)p'(r)}{\tilde{v}(t)p'(t)} \Rightarrow \frac{-\tilde{v}(r)p(r)}{\tilde{v}(t)p'(r)} > \frac{-\tilde{v}(r)p'(r)}{\tilde{v}(t)p'(t)}
\]

For \( p' \not\in H[p] \), we must have that \( d(p, x) \neq d(p', x) \) for some \( x \in X \). Consider \( X_{rt} = \{ x \in X : f(x|s) = 0 \text{ for } s \neq r, t \} \). Then for \( x \in X_{rt}, d(p, x) \neq d(p', x) \) exactly when

\[
\tilde{v}(t)p(t)f(x|t) + \tilde{v}(r)p(r)f(x|r) < 0 \text{ and } \tilde{v}(t)p(t)f(x|t) + \tilde{v}(r)p'(r)f(x|r) \geq 0.
\]

Dividing by \( f(x|r) \)

\[
\tilde{v}(t)p(t)\gamma_r(x) + \tilde{v}(r)p(r) < 0 \text{ and } 0 \leq \tilde{v}(t)p'(t)\gamma_r(x) + \tilde{v}(r)p'(r)
\]

Combining the inequalities,

\[
-\frac{\tilde{v}(r)p'(r)}{\tilde{v}(t)p'(t)} \leq \gamma_r(x) < -\frac{\tilde{v}(r)p(r)}{\tilde{v}(t)p(t)}
\]

Now let

\[
y = \frac{1}{2} \left( -\tilde{v}(r)p'(r) - \frac{-\tilde{v}(r)p(r)}{\tilde{v}(t)p(t)} \right), \text{ and } \varepsilon = \frac{1}{3} \left( -\tilde{v}(r)p(r) - \frac{-\tilde{v}(r)p'(r)}{\tilde{v}(t)p'(t)} \right)
\]

Then there is an \( x \in X_{rt} \subset X \) such that \( \gamma_r(x) \in B_{\varepsilon}(y) \) and for this \( x \), \( d(p, x) \neq d(p', x) \) and therefore \( p' \not\in H[p] \). \qed

Theorem 1

The set of relevant priors is point identified if \( X \) satisfies full range in multiple states.

Proof. Let \( p, p' \in P \) where \( p \neq p' \). Then there is an \( s \in S \), where \( p(s) \neq p'(s) \). Without loss of generality, we take \( 0 \leq p(s) < p'(s) \).

If \( s < s^* \) then there is a state \( t \geq s^* \) where \( p'(t) > 0 \), which implies \( \frac{p(s)}{p'(s)} \) and \( \frac{p(t)}{p'(t)} \) are non-negative numbers. If \( \frac{p(s)}{p'(s)} \neq \frac{p(t)}{p'(t)} \), then by Lemma 2.2 we know that \( p' \not\in H[p] \). If \( \frac{p(s)}{p'(s)} = \frac{p(t)}{p'(t)} \), then since \( \sum_{i=1}^{S} p(i) = \sum_{i=1}^{S} p'(i) = 1 \), there must be an \( s' \) where \( 0 \leq p'(s') < p(s') \). If \( s' < s^* \) then
\[ \frac{p'(s)}{p'(s')} \neq \frac{p(t)}{p'(t)}, \] and we can use Lemma 2.2 to show that \( p' \not\in H[p] \). If \( s' \geq s^* \) then \( \frac{p(s)}{p'(s)} \neq \frac{p(s)}{p'(s')} \), and we can again use Lemma 2.2.

Now if \( s \geq s^* \), then there is a state \( r < s^* \) where \( p'(r) > 0 \). We can now repeat the argument above to show that \( p' \not\in H[p] \).

Since \( p \) and \( p' \) were arbitrary, then \( H[p] \) is a singleton for all \( p \in P \), and \( P \) is point identified. \( \square \)
CHAPTER 3

Community formation, agglomeration, and size distribution

3.1 Introduction

The distribution of a particular quantity of interest comes up in many economic contexts. For example, the distribution of resources, or wealth, across a set of agents; or the distribution of sizes of firms, e.g., how a number of agents are allocated between different firms, or the distribution of price changes of a stock, city sizes, and so on. In these settings, either power laws or log-normal (or exponential) distributions have been proposed either on the basis of estimation, or on the basis of the description of a specific generative process.\footnote{See literature review below.} However, some concerns have been raised as to the usefulness of power-law statements without any theory to guide such statements, see, e.g., R. J. Barro 1996, Brock 1999, and Durlauf 2005. In a nutshell, a power law may be difficult to distinguish from a log-normal distribution and these authors argue that it would be important for the power law/scaling literature to develop formal statistical methodologies for model comparison. One possibility is to investigate properties of a specific class of generating processes that may lead to the power law or a log-normal law. These properties may then be relevant for counterfactual predictions, e.g., the impact of different policies.

We propose an intuitive class of generative models of agglomeration and study the resulting limiting distribution of community sizes. A distinctive property of our model is that the number (or density) of particles remains constant through time, that is, the agglomeration is in our model not a result of some aggregate growth process, but rather the result of pure redistribution of existing particles. In our model, there are countably many particles (or agents), which may during many time periods move in some space to encounter and then attach or leave existing communities of...
such agents. It has the additional flexibility of allowing agents to attach and leave communities at different rates which may or may not depend on the size of the community itself. The main question is under what circumstances will the limiting distribution of community sizes exist, and given its existence does it have fat “tails” or does is follow an exponential law. In particular, we look at the possibility that the probability of leaving a community varies with the size of the community. We are able to solve for an explicit bound when the probability of leaving a community is zero, and we show the distribution of community sizes in the limit follows an exponential law. When the probability of leaving a community is positive, simulations show that if this probability decreases sufficiently fast with the community size, as time goes to infinity, the community sizes grow beyond all bounds, i.e., there does not exist a stationary size distribution and community sizes explode. When the probability decreases at a slower rate, or actually increases in the size of communities then the simulations converge to a exponential distributions. The simple description of the generative process allows, at least in principle, to take our model to specific environments and relate these attachment probabilities to considerations of agents’ incentives or some other equilibrium or feasibility considerations.

We explicitly describe the process by which particles move in a space and attach or leave existing communities. Our basic model is very simple. Initially, particles are dispersed on an infinite line at equal distances. In each period, a particle can move left or right, and when it collides with another particle or agglomeration (community), it attaches to that agglomeration and stops moving. This basic model is somewhat akin to a random walk, and we show that the limiting distribution of community sizes is bounded above by exponential law: a simple argument yields a simple exponential bound, and a slightly more sophisticated argument yields a bound on the tail of the distribution that is exponential in size squared. Therefore, in the absence of any detachment and re-attachment (or death and birth), the limiting distribution of sizes is clearly exponential.

We then modify the basic model by assuming that once an agent attaches to a community, it may in subsequent periods leave the community with some probability $q$. Therefore, rather than the agents being attached to a community once and for all, they may continue moving around.\footnote{Alternatively, one could imagine this as a balanced birth-death process where some number of agents in a community die off and an equal number of agents are born, and these newly born agents may then leave the community.}
Any agent that has left a community then moves freely in subsequent periods until it attaches to another community (which the agent can then leave with some probability in the future periods). In this case, we are able to show that when the probability of leaving $q(n)$ decreases faster than $n^{-1}$, where $n$ is the size of the community, then as time goes by, the communities (on average) become larger and larger. In the limit, there does not exist a steady state distribution of community sizes. The reason is that on average, the number of agents leaving a small community is larger than the number of agents leaving a large community but since the agents move randomly the probability of an agent attaching to any given community is approximately the same across communities. Moreover, as a community becomes very large, it becomes very likely that none of the agents detach from the community in a given period. ³

Power laws,⁴ as opposed to log-normal distributions, have been identified in a variety of economic settings. Champernowne 1953 identified a power law in the income distribution, along with Herbert A Simon 1955 and Wold and Whittle 1957, who argued that wealth distribution followed a Pareto distribution. These findings are robust across countries, c.f., Solomon and Richmond 2001, with the tail exponent in the range between 1 and 3. In the context of firms, power laws have been identified for firm growth, Steindl 1965, and in the case of firm sizes, Ijiri and Herbert Alexander Simon 1977 and Axtell 2001 identified a Zipf’s law. Zipf’s law has been identified for city sizes and growth, Gabaix 1999b, Gabaix 1999a. In the context of financial markets, the power laws have first been studied by Mandelbrot 1997. More recently, power laws have been identified in more specific market settings: large price changes on short time scales are distributed according to a power law, c.f., Fama 1965, Officer 1972, Koedijk, Schafgans, and De Vries 1990, Loretan and Phillips 1994, Mantegna, Stanley, et al. 1995, Longin 1996, Lux 1996, Ghashghaie et al. 1996, Müller, Dacorogna, and Pictet 1998, Gopikrishnan et al. 1999; the autocorrelation of the absolute value of price changes is a long-memory process whose autocorrelation function decays according to a power law, c.f., Ding, Granger, and Engle 1993, Beran 1994, Mantegna and Stanley 1999, with some probability.

³For the intermediate cases, when $q(n)$ decreases less fast than $n^{-1}$ our simulations suggest that there exists a steady state probability distribution over community sizes, which is exponential. However, we have so far not been able to prove (or disprove) this analytically.

⁴in particular the Zipf distribution, Zipf 1949
Despite this growth in the literature, there have recently been some concerns raised as to the usefulness of power-law statements without any theory to guide such statements, see R. J. Barro 1996, Brock 1999, and Durlauf 2005. Firstly, power laws are statements about the stationary distributions of extreme-value distributions of stochastic processes so that different classes of stochastic processes can lead to the same power law in the limit. Moreover, a power law may be very difficult to distinguish from a log-normal distribution. Since the generating process is relevant for counterfactual predictions (e.g., the impact of different policies), these papers argue that it would be important for the power law/scaling literature to develop formal statistical methodologies for model comparison. In response to such criticisms, there has been a surge in work trying to identify more detailed descriptions of processes generating power laws stemming from theoretical foundations, see Mitzenmacher 2004 for a good overview.  

What these literatures have in common is that, in essence, there is a given growth process, and the question is the impact on the size distribution of some variable of interest. For example, there is a growth rate of some basic resource or particles and the question is the size distribution of agglomerations when these particles attach preferentially. Our intention is to complement this by investigating a purely redistributive process, whereby there is no growth, but the particles instead reallocate according to some (random) process. Our results reported here seem to suggest that the steady-state distribution that obtains is either exponential; or there is no steady state distribution and the agglomerations become larger and larger. Adding a growth component seems natural and we leave that to the future. By investigating the attachment probabilities and growth rates of communities, the goal is to then study the incentives of agents to attach to such communities i.e., to mesh our model with an equilibrium model.  

---

5For example, an assignment model explains the power law for the distribution of CEO earnings, which was first observed by Rosen 1981, and then J. R. Barro and R. J. Barro 1990, Sattinger 1993, Gabaix and Landier 2008, and Terviö 2008. Another important mechanism generating power laws are the birth-death processes, see Reed 2001, Malevergne, Saichev, and Sornette 2008, and Gabaix 2009.

6Our interest in these questions stems from studies of biofilms (surface-associated integrated bacterial communities) of the bacterium P. aeruginosa responsible for cystic fibrosis. See e.g., Gibiansky et al. The life-cycle of such biofilms is not well understood. The process here, while stylized, is motivated by the movement and attachment process of bacteria. An important aspect of biofilms is their growth rates, and to apply our model to such clinical settings,
The rest of the paper is organized as follows. In Section 2 we give the basic model. In Section 3 we give the two upper bounds on the distribution of community sizes in the basic model. In Section 4 we extend the basic model to the case where particles can also leave communities and sketch an example of distribution of firm sizes. In Section 5 we give the proofs of our results.

### 3.2 Community Formation without Detachment

There are infinitely many agents populating the line, where agent \( i \) is initially, at time \( t = 0 \), at location \( 2i + 1 \), \( i \in \mathbb{Z} \). Time is discrete and denoted by \( t \in \{0, 1, ...\} \). Denote by \( \ell_i(t) \) the location of agent \( i \) at time \( t \).

We say that agent \( i \) is free at \( t \), if \( i \) is the only agent at his location at that time. That is, if,

\[
\ell_i(t) \neq \ell_j(t), \forall j \neq i.
\]

Therefore, at \( t = 0 \), all agents are free. At each \( t \geq 0 \), each free agent moves left or right, and the probability of a free agent moving one way or the other is determined by a coin toss: \( p \) is the probability of moving left, \( 1 - p \) is the probability of moving right. For now, assume that \( p = \frac{1}{2} \), so that, \( P(\ell_i(1) = 2i) = P(\ell_i(1) = 2i + 2) = \frac{1}{2} \), where \( P(\cdot) \) denotes the probability of an event.

In this section, an agent that is not free does not move. This implies that once an agent is in a community with at least one other agent, then it will never detach from that community. Denote by \( c_t(i, j, l) \) the event that agents \( i \) and \( j \) have formed a community at time \( t \) at location \( l \), that is, the event \( \{\ell_i(t) = \ell_j(t) = l\} \); denote by \( c_t(i, j) \) the event that agents \( i \) and \( j \) have formed a community at \( t \) at some location, by \( \bar{c}_t(i, j) \) the event that agents \( i, j \) have formed a community at some time \( t' \leq t \) at some location, by \( c_t(l) \) the event that there was an community at location \( l \) at time \( t \), and by \( \bar{c}_t(l) \) the event that there was a community at location \( l \) at some time \( t' \leq t \). Therefore, in the basic model, agent \( i \) is in a community at \( t > 0 \) in the event

\[
\bigcup_l \{c_{t'}(l), t' \leq t \} \cap \{\ell_i(t) = l\}.
\]

It might be useful to add a process by which agents in communities grow (or multiply) within the community. While such settings might not be directly relevant to large-scale economic environments they have an advantage of generating large amount of controlled data. More importantly, biofilms can be 1000-times more resistant to antibiotic treatments than single bacteria leading to high mortality rates in such patients. The hope is that understanding the incentives of bacteria to attach to communities might lead to more effective treatment of such infections.
3.2.1 Two bounds

Our main object of interest is the size distribution of communities as \( t \to \infty \). Denote by \( \bar{F} \) the cumulative probability distribution of sizes of communities as \( t \to \infty \), i.e., \( \bar{F}(n) = \lim_{t \to \infty} P(|C_t| \leq n) \), \( n \geq 1 \), where \( C_t \) is a randomly chosen community at time \( t \). In this section we derive upper bounds on the limiting size density of communities, \( \bar{f}(n) = \lim_{t \to \infty} \Pr(|C_t| = n) \).

We derive the first bound by first considering the distribution of communities at time \( t = 1 \). If any two agents form a community in the first period, then they no longer move, and any additional agents that join the community stop moving as well. Therefore these communities act as barriers which agents cannot cross. That is, if at some \( t \geq 0 \), there are \( l, l', l' < l \), such that \( \bar{c}_t(l), \bar{c}_t(l') \), and \( l \leq \ell_i(t) \leq l' \), then,

\[
l \leq \ell_i(t') \leq l', \forall t' \geq t.
\]

The largest size that a community can get is limited by the number of free agents on either side of a community before the next community is reached. To quantify this bound on size, we define members of a semi-community to be those in a community in addition to all of the free agents between that community and the next community to the left.

**Definition 3.2.1.** Fix a time \( t > 0 \) and let \( \bar{c}_t(l), \bar{c}_t(l'), l, l', l' < l \), be two adjacent communities. The agents between such adjacent communities at time \( t \) including the agents at the community on the right are called a semi-community at \( t \) and denoted by \( \gamma_t(l, l') \).

Then, the probability that a community is of a given size \( n \) in the limit, as \( t \) tends to infinity, is bounded by the probability that the sum of the sizes of the two adjacent semi-communities is \( n + 2 \). That is, by the likelihood that all the free agents in the two adjacent semi-communities eventually end up in the common boundary between the two semi communities (the two-rightmost agents in the left semi-community).

Let \( \gamma_1 \) be a semi-community around some given location at \( t = 1 \), for instance take \( \gamma_1 \) to be the semi-community around point 0, so that \( \gamma_1 = \gamma_1(l', l) \), where \( l' < 0 \leq l \), and let \( \gamma'_1 \) be the semi-community to the right of \( \gamma_1 \).
Proposition 3.2.1. The distribution of sizes of $\gamma_1$ is given by:

$$P(|\gamma_1| = n) = (n-1)2^{-n}, n \geq 2. \quad (3.1)$$

Additionally, the sizes of $\gamma_1$ and $\gamma'_1$ are i.i.d.

A heuristic proof of Proposition 3.2.1 is as follows. The likelihood of two agents at $t = 1$ forming a community is $(\frac{1}{2})^2$. Fixing the left and right communities, which form the boundary for a semi-community, there are then $(n-1)$ ways in which the rest of the agents in the semi-community do not collide: all the agents move in the same direction, or there is some location, such that agents to the left of it move to the left and agents to the right of it move to the right. For each of these ways, all remaining $n-2$ agents in the semi-community (the additional 2 agents are in the right-hand boundary) must move in a specified direction, which happens with the likelihood of $(\frac{1}{2})^{n-2}$. From Theorem 3.2.1 we can derive the following upper bound on the distribution of sizes of communities at $t = \infty$. Denote by $\gamma_{\infty}$ the limiting community at or immediately to the right of location 0 as $t \to \infty$.

Theorem 3.2.2. The size distribution of communities in the limit is bounded by an exponential function. In particular,

$$\bar{f}(n) = \lim_{t \to \infty} P(|C_t| = n) \leq \frac{(n-1)(n+3)^2}{2^{n+3}}.$$

Proof. For a $C_t$ to be of size $n$ at $t = \infty$, then at $t = 1$ the sum of the sizes of the two semi-communities $\gamma_1$ and $\gamma'_1$ must be at least $n+2$. Additionally, all free agents of $\gamma_1$ must eventually hit the right-most boundary of $\gamma_1$ and all free agents in $\gamma'_1$ must also hit that same boundary; the two agents in the right most-boundary of $\gamma'_1$ are not free. The sizes of the two semi-communities are independent. Therefore, if $\Gamma$ is the sum of the sizes of two semi-communities at $t = 1$, then the
probability of a given community being of size $n$ is bounded by

$$f(n) = \leq P(\Gamma = n + 2) = \sum_{k=2}^{n} P(|\gamma_1| = n + 2 - k)P(|\gamma_1'| = k)$$

$$= \sum_{k=2}^{n} ((n - k + 1)2^{-(n+2-k)})((k - 1)2^{-k})$$

$$= \frac{1}{2^{n+2}} \sum_{k=2}^{n} (n - k + 1)(k - 1)$$

$$\leq \frac{(n - 1)(n + 3)^2}{2^{n+3}}$$

Theorem 3.2.2 shows that in this simple model, the community sizes obey an exponential law in the sense that the distribution is bounded by an exponential function. By considering only the semi-communities at $t = 1$ that bound is in fact somewhat crude. By considering more explicitly the process by which the communities are formed, we can obtain the following bound: the tail of the distribution of community sizes is bounded by an exponential in the community sizes squared.

Consider a set $S$ of $n$ agents who are initially in consecutive locations. Let $C_{S(n)}$ be the event that all of these agents eventually join the same community, as $t \to \infty$.

**Theorem 3.2.3.** For any such set $S$ of size $n$ large enough,

$$2^{-n^2/2} \leq P(C_{S(n)}) \leq 2^{-n^2/160}.$$ 

Because no two agents can cross each other, a community of size $n$ must be formed by $n$ adjacent agents. Therefore $P(|C_t| = n) = P(C_{S(n)})$, and we have a tighter bound on the tail of the distribution of community sizes. The additional bite for this shaper bound stems from the fact that none of the $n$ agents can collide with each other except at the location where the community will be formed. Each additional time period, there is a restriction on the possible moves to avoid an additional community from being formed out of the group of $n$ agents. These restrictions were ignored when calculating the previous bound.
3.3 Community Formation with Detachment

In this section we modify our basic model. In particular, we assume that agents in an existing community may in each period leave the community with some probability. Therefore, at each period \( t \), any free agent moves as before, with equal probabilities to the left and to the right. Meanwhile any agent who is in a community of size \( n \) leaves the community with some probability \( q(n) \), and remains in the community with the complementary probability \( 1 - q(n) \). Conditional on moving out of the community, the agent then moves left or right with equal probabilities.

Our main objective is to investigate under what conditions there exists a steady-state distribution of community sizes, as \( t \to \infty \), and what form this distribution takes on. From the previous section, we showed that there is an exponential steady state distribution when \( q(n) = 0 \). The other two cases we explore in this section include when the probability of a community decreasing in size is not effected by the size of the community, and when this probability decreases rapidly as the community size grows.

To analyse these situations, let \( q(n) = kn^{-\alpha} \), where \( \alpha \geq 0 \) and \( 1 \geq k > 0 \). If \( \alpha \to \infty \), then we are in the case of the previous section. If \( \alpha = 1 \), then the probability of an agent leaving a community will always be \( k \). If \( \alpha > 1 \) then this probability decreases with the community’s size. The section includes simulations of all three of these cases in Figures 3.1 and 3.2.

In order to gain intuition for the case of \( \alpha > 1 \), first consider the following simple scenario where the probability of leaving a community drops to zero when the community reaches a certain size, \( \bar{n} \), which is some given positive integer. In this setting, agents that enter a community large enough will never leave. Since each agent will encounter a community of that size at some finite time, the likelihood of encountering a community of a size smaller than \( \bar{n} \) must vanish as \( t \to \infty \).

**Proposition 3.3.1.** Fix a positive integer \( \bar{n} > 1 \), let \( \alpha > 0 \) and let \( q_{\bar{n}}(n) \) be given by,

\[
q_{\bar{n}}(n) = \begin{cases} 
n^{-\alpha} & n < \bar{n} \\
0 & n \geq \bar{n} 
\end{cases}
\tag{3.2}
\]

Then \( P(|C_\infty| < n) = 0 \), for every \( n \leq \bar{n} \).
Proof. Take an agent, which is in a community of a size smaller than \( \bar{n} \). By the law of large numbers, the agent will with probability 1 leave this community, as \( t \) becomes large, and will eventually, again by the law of large numbers, agglomerate into a community of a size greater than \( \bar{n} \). Hence, in the limit, as \( t \to \infty \), there will be no communities of a size smaller than \( \bar{n} \).

Consider now the case where \( q(n) \) does not vanish at any finite \( \bar{n} \). Since \( q_{\bar{n}}(n) \) becomes arbitrarily close to \( q(n) \) as \( \bar{n} \) becomes large, Proposition 3.3.1 suggests that there will be no limiting distribution of the community size as \( t \to \infty \). Simulations confirm that this is indeed the case, when \( q(n) \) tends to 0 sufficiently fast as \( n \) tends to infinity. Specifically, there is no limiting distribution when the probability of leaving a community of a given size decreases at a rate faster than the inverse of the size of the community.

**Proposition 3.3.2.** Let \( q(n) \leq kn^{-\alpha} \), where \( \alpha \geq 1 \) and \( k \) is some positive constant. Then there does not exist a steady-state distribution of community sizes, and in particular, as \( \lim_{t \to \infty} P(|C_t| \geq n) \to 1 \), for all \( n > 0 \).

When \( q(n) = kn^{-\alpha} \), then the probability that none of the agents leave the community as a function of the size of the community is give by

\[
\Pr(\text{no agents detach}) = \left( 1 - \frac{k}{n^\alpha} \right)^n
\]

For values of \( \alpha > 1 \), or when detachment is much less likely in larger communities than smaller communities, this probability approaches 1 as the community size gets large

\[
\lim_{n \to \infty} \left( 1 - \frac{k}{n^\alpha} \right)^n = 1
\]

This implies that once a community becomes large enough, the likelihood of any agents leaving becomes small enough, so that the expected arrival rate of the agents will exceed the expected departure. Therefore the size of this community would continue to increase indefinitely. This is simulated and Figure 3.2 shows the community size distribution for a large \( t \). From this simulation, it is clear that there is a significant number of very large communities, and simulations where \( t \) is larger, shows that the average community size continually increases.
Figure 3.1: Simulation of the log of frequency of communities on community size for $\alpha = 2$ for large $t$.

For $\alpha \leq 1$ the probability that none of the agents leave the community is bounded away from 1. In particular, when $\alpha = 1$, then this probability converges to $1/e^k$. When $\alpha < 1$, then it converges to 0. While we have not been able to solve for analytical solutions, the simulations pictured in Figure 3.1 suggest that there is a limiting distribution of community sizes when $\alpha \leq 1$.

We conclude with an example of firm sizes where workers are matched to firms in a (locally) random search.

**Example 3.3.1.** Think of each agent as a worker. The agents’ initial location determines some relevant characteristics of the worker, e.g., the initial location corresponds to an industry index so that it can be interpreted as the set of skills relevant to that industry. In each period, the interpretation of movement of a free agent is that the agent engages in a local random search. Think of an agglomeration of two or more agents as a firm. When a free agent encounters an agglomeration (or another free agent), he can join that firm (or initiate a firm/partnership in case of encountering a single free agent). In general, the incentives of a free agent to join a firm might depend on the location, e.g., specific location (industry) might be more profitable than another. However, in our stylized model, we abstract from such considerations and we consider all locations as equal, on average. Next, the incentives of an agent will also depend on the size of the firm that the agent

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When $\alpha = 0$ and $k = 1$, then the communities have no effect on the movement of the agents. Therefore the distribution of agents will be determined by a random walk and the limiting distribution would be $P(|C_\infty|) = 2^{-n}$.
encounters, as well as what other opportunities might be available to the agent at other firms or by remaining free. On average, at a given firm, the agents’ opportunities are a function of the firms’ size (ex ante, prior to joining a firm, the agents are all the same). The opportunities available to the agent at other firms are summarized by the distribution of sizes of all firms. Therefore, if in period $t$ the distribution of firm sizes is given by $f_t$, and a worker encounters a firm of size $n$, the likelihood that the worker joins it is given by some function $\beta(n, f_t)$. On the other hand, a worker who is currently in a firm might decide to leave it in a given period and let the likelihood of that event be given by $\bar{\beta}(n, f_t)$. For example, one could assume $\beta(n, f_t) = 1 - \bar{\beta}(n, f_t)$, i.e., that the probability that a worker joins a firm is the same as the probability that the worker remains in a firm.\footnote{For most firms, the probability that the worker remains in a firm might be higher than the probability of a worker joining the firm. This is roughly consistent with the probability of leaving a firm of a larger size decreasing in the size of the firm.} More generally, $q(\cdot)$ will be bounded above by $\max \beta, \bar{\beta}$. Therefore, when independently of $f_t$, we have that $\beta, \bar{\beta} \leq n^{-\alpha}$, by Proposition 3.3.2 the firm sizes will be arbitrarily large, as $t \to \infty$. 

Figure 3.2: Simulations of the log of frequency of communities on community size for $\alpha = \infty$ and $\alpha = 1$ and large $t$. 
3.4 Appendix

Proof of Theorem 3.2.1

In the case of the simpler boundary we go through an argument with \( N \) agents, and there is a boundary at 0. That is, there is already at time \( t = 0 \) a fixed boundary at position 0 over which the agents can’t move. With the above notation we can by convention denote this by \( c_0(0) \), e.g., one could imagine that there used to be agents at locations \(-1, 1\) that have collided at some prior time \( t \leq 0 \).

Let \( \bar{k}(t) \) be the number of collisions by time \( t \geq 0 \), that is, \( \bar{k}(t) = |\{ l \mid \bar{c}_t(l) \}| \). For \( k \in \{0, \ldots, \bar{k}(t)\} \), we denote by \( \bar{c}_t^k \) the \( k \)-th collision (by location) by time \( t \), where \( c(0) \) is the “zero”-th collision. Hence,

\[
c_0^0 = 0, \quad t \geq 0, \quad \text{and} \quad \bar{c}_t^k = l, \quad \text{such that,} \quad l = \min_{l > \bar{c}_t^{k-1}} \bar{c}_t(l), \quad 0 < k \leq \bar{k}(t).
\]

There is some probability that at \( t = 1 \) there are no semi-communities, so that if \( \tau^1(1) \) is formally the first semi-community from the left at time \( t = 1 \), there is some probability that there is no such semi-community, i.e., \( P(\bar{k}(1) = 0) \), and if there is at least one semi-community, then its size will take different values with some probabilities. We adopt the convention that if there is no semi-community then the size of \( \tau^1(1) \) is 1, i.e., if \( \bar{k}(1) = 0 \), then we formally define \( |\tau^1(1)| = 1 \).

We now have the following proposition.

**Proposition 3.4.1.** Fix \( N \) and suppose there is a boundary at 0, i.e., \( c^0(0) \). The probability that the size of the first semi-community at time \( t = 1 \) takes different values is given by

\[
P(|\tau^1(1)| = n) = \begin{cases} (N + 1)2^{-N}, & n = 1 \\ (n - 1)2^{-n}, & 2 \leq n \leq N \\ 0, & n > N \end{cases}
\]

**Proof.** In order for there to be no semi-community, if the first agent moved right, then all the remaining agents had to move right; if the first agent moved left, then the second agent could move...
right or left. When the first agent moved left and the second agent moved right then there is a gap between agents 1 and 2, i.e., a space that is of size 4. In general, if the gap is between agents \( i \) and \( i + 1 \), then in order for there to be no semi-community, all the agents to the left of the gap had to move to the left, and all the agents to the right of the gap had to move to the right, which happens with a probability \( 2^{-N} \). If all agents moved to the left then we can think of the gap being to the right of agent \( N \), and if all agents moved to the right, then we can think of the gap being to the left of agent 1. Once the location of the gap is determined, all the \( N \) agents had to move in a specific way, which happens with a probability \( 2^{-N} \). There are \( N + 1 \) possible locations for the gap so that the probability of there being no semi-community is \( (N + 1)2^{-N} \).

If the first collision (location wise, i.e., the left-most collision) is between agents \( n - 1 \) and \( n \), then these two agents had to move in a specific way (\( n - 1 \) to the right and \( n \) to the left), which happens with a probability \( 2^{-2} \). Agents to the right of agent \( n \) can move in any way. In order for the collision between \( n - 1 \) and \( n \) to have been the left-most one, there are \( n - 1 \) possible locations for the gap to the left of agent \( n - 1 \). Given the location of the gap, all the agents to the left of agent \( n - 1 \) had to move in a specific way. Hence, the probability that the first collision was between agents \( n - 1 \) and \( n \) is \( (n - 1)2^n \). This is evidently the probability the first semi-community at time \( t = 1 \) is of size \( n \).

Since there are \( N \) agents, the probability of the first semi-community being of a size \( n \) greater than \( N \) is 0.

For a given \( N \), if there is more than one semi-community, then the probability distribution of \( k \)-th semi-community being of a given size is given by a similar expression, conditional on the sum of the sizes of all semi-communities to the left of \( k \)-th semi-community. We again adopt the convention that if there are \( \bar{k}(1) \) semi-communities, then the size of the \( \bar{k}(1) + 1 \)-st semi-community is 1, if there are any remaining agents to the right of \( \bar{k}(1) \)-th semi-community, and is 0 if there are no remaining agents to the right of \( \bar{k}(1) \)-th semi-community.

**Corollary 3.4.2.** Fix \( N \), suppose there is a boundary at 0, i.e., \( c^0(0) \), let \( k \leq \bar{k}(1) + 1 \), denote
\[ N(k) = N - \sum_{k' < k} |\tau^{k'}(1)| \] and take the event where \( N(k) \geq 1 \). Then,

\[
P(|\tau^k(1)| = n | N(k) \geq 1) = \begin{cases} 
(N(k) + 1)2^{-N(k)}, & n = 1 \\
(n - 1)2^{-n}, & 2 \leq n \leq N(k) \\
0, & n > N(k) 
\end{cases}
\] (3.4)

Proof. There are \( k - 1 \) semi-communities to the left of the \( k \)-th semi-community, and in the event that the sum of their sizes is \( \sum_{k' < k} |\tau^{k'}(1)| < N \), there are \( \bar{N}(k) \) points to the right of the \( k - 1 \)-st collision (location wise) by \( t = 1 \), i.e., to the right of \( \bar{c}_1^{k-1} \). This \( k - 1 \)-st collision now takes the role of the left boundary of the \( k \)-th semi-community (probability distribution of the size of a semi-community is translation invariant with respect to its left boundary). Hence, conditional on the event \( \bar{N}(k) = N' \), the probability distribution of the size of \( k \)-th semi-community is computed as in Proposition 3.4.1, with \( N' \) taking the place of \( N \).

Up to this point we have that the semi-community sizes are conditionally independent. From (3.3) we obtain the probability distribution of the size of the first semi-community when \( N \to \infty \). Second, we obtain the likelihood of the event that \( \bar{k}(1) \leq K \) for any \( K > 0 \). In particular, as \( N \to \infty \), we have that \( \bar{k}(1) \to \infty \). Third, we obtain for each \( \bar{N}, N, N' < N \) and \( k > 0 \), the likelihood of the event that there are \( N' \leq \bar{N} \) points to the right of the \( k \)-th semi-community at time \( t = 0 \), and for a fixed \( k \) and \( \bar{N} \), as \( N \to \infty \), this likelihood goes to 0. That is, as \( N \to \infty \), the probability that there are at least \( N' \) agents to the right of the \( k \)-th semi-community tends to 1 for every \( k \) and \( N' \).

Combining all these arguments, we therefore have that when agents are initially located on the interval \([-2N + 1, 2N + 1]\), as \( N \to \infty \), the size distribution of semi-communities at \( t = 1 \) tends to

\[(n - 1)2^{-n}.
\]

If there is no boundary at 0, but the \( N \) agents are instead initially equidistant (separated by a distance of 2) on the interval \([-2N + 1, 2N + 1]\), then, as \( N \to \infty \) the argument is no different, as the probability that there are no collisions at \( t = 1 \) then vanishes. A collision at \( t = 1 \) any point then acts as a left boundary to all agents to the right of it, and the number of those agents tends to \( \infty \), which proves Theorem 3.2.1.
Proof of Theorem 3.2.3

Next we prove our Theorem 3.2.3.

If the agents in $S$ all join the same community, then there is a (random) location $\ell$ in which this community forms. Note that at each time period at most 2 agents can arrive at $\ell$. Hence at time period $t$ there are at most $2t$ agents in $\ell$, and none of the remaining agents in $S$ (of which there are at least $n - 2t$) have yet to experience any collisions. Denote by $S_t$ the subset of agents in $S$ who have yet to collide by time $t$.

Let $S_t$ be the disjoint union of $S_{1t}, S_{2t}, \ldots, S_{Kt}$, where each $S_{it}$ is a maximal group of agents in $S_t$ who are in consecutive (i.e., distance two) locations, and $K$ is the (random) number of such subsets. Denote by $n_t$ the size of $S_{it}$.

Lemma 3.4.3. The probability that the agents in $S_{it}$ experience at most one collision at time $t + 1$ is at most

$$\frac{2(n_t + 1)}{2^{n_t - 1}}.$$

Proof. To choose a way in which the agents in $S_{it}$ can move without experiencing more than one collision, one can choose some location $m$ that is adjacent to some $S_{it}$ (of which there are $n_t + 1$), and then either (1) have all of those on the right of $m$ move to the right and all those on the left move to the left - in which case there is no new collision - or (2) have all of those on the right of $m$ move to the left and all those on the left move to the right - in which case there is one new collision. Of course, one of these two options might not be possible, in the case that one of the extremal agents in $S_{it}$ is on the boundary of the interval. But in either case the number of ways the agents can move without experiencing more than one collision is at most $2(n_t + 1)$.

Since the total number of ways that the agents can move is at least $2^{n_t - 1}$ ($n_t - 1$ and not $n_t$ because of the boundary case again) we arrive at the expression above. \hfill \Box

As noted above, in order for $C_S$ to occur the size of $S_t$ cannot decrease by more than two at every period. Accordingly, let $E_{S,t}$ be the event that $|S_{t+1}| \geq |S_t| - 2$, and let

$$F_{S,t} = \bigcap_{t' = 1}^{t-1} E_{S,t'}$$

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be the intersection of all the $E_{S,t'}$'s in the periods prior to $t$. Note that $F_{S,t}$ implies that $|S_t| \geq n - 2t$. Note also that the event $C_S$ is contained in $F_{S,t}$ for all $t \leq n/2$.

We show that conditioned on $F_{S,t}$, the number $K$ of subsets $S_i^t$ is small.

**Lemma 3.4.4.** $F_{S,t}$ implies that $K \leq 3t + 1$.

**Proof.** When $t$ is even then every agent in $S_t$ is on an odd location, since none have yet to experience a collision. Likewise, when $t$ is odd then every agent in $S_t$ is on an even location. We assume (without loss of generality, as will become apparent) that $t$ is odd; the argument below goes through when “odd” is replaced with “even” and vice versa.

At period $t$ the distance between the left-most agent in $S_t$ and the right-most agent in $S_t$ is less than $2(n + t)$, since the initial distance is $2(n - 1)$ and each agent moves a distance of one at each period. It follows that the number of unoccupied even locations is at most $n + t - |S_t|$. Since $F_{S,t}$ implies $|S_t| \geq n - 2t$, the number of unoccupied even locations is at most $n + t - n + 2t = 3t$. It follows that $K$ is at most $3t + 1$.

The next claim bounds the probability of $E_{S,t}$, conditioned on $F_{S,t}$ and for low enough $t$.

**Proposition 3.4.5.** Let $t \leq n/79$. Conditioned on $F_{S,t}$, the probability of $E_{S,t}$ is at most $2^{-n/2}$.

**Proof.** By Lemma 3.4.3, the probability that there is at most one collision in a particular $S_i^t$ at time $t + 1$ is at most $(n_i + 1)/2^{n_i - 2}$. Note that conditioning on the past event $F_{S,t}$ does not change this probability. To facilitate our calculations, we note that this is at most $n_i/2^{n_i - 3}$ and use the latter expression instead.

It follows that the probability that in every subset $S_i^t$ there is at most one collision at time $t + 1$ is at most

$$p := \prod_{i=1}^{K} \frac{n_i}{2n_i - 3},$$

where, to remind the reader, $K$ is the number of subsets $S_i^t$. Now, $\sum_i n_i = |S_t| \geq n - 2t$, and $K \leq 4t$, by Lemma 3.4.4. Hence

$$p = \frac{1}{2^{\sum_i n_i - 12t}} \prod_{i=1}^{K} n_i \leq \frac{1}{2^{n - 14t}} \prod_{i=1}^{K} n_i. \quad (3.5)$$
Now, by the arithmetic-mean-geometric-mean inequality, the product of \( k \) numbers that sum to \( m \) is at most \((m/k)^k\), and so
\[
\prod_{i=1}^{K} n_i \leq (|S_t|/K)^K.
\]
The function \( f(k) = (|S_t|/k)^k \) has a unique maximum at \( k = |S_t|/e \). But in our case \( K \leq 4t \), and so, when \( 4t \leq |S_t|/e \), the maximum of \( f(k) \) is achieved at \( k = 4t \). Since \( |S_t| \geq n - 2t \) and \( t \leq n/79 \) then this is indeed the case, and so
\[
\prod_{i=1}^{K} n_i \leq (|S_t|/(4t))^{4t} \leq (n/(4t))^{4t}.
\]
Let \( M = n/t \) so that \( t = n/M \) and \( M \geq 79 \). Then
\[
\prod_{i=1}^{K} n_i \leq (|S_t|/(4t))^{4t} \leq (M/4)^{4n/M} \leq M^{4n/M} = 2^{4n \log_2(M)/M}.
\]
Substituting this back into (3.5), we arrive at
\[
p \leq 2^{4n \log_2(M)/M} = 2^{2n+14t+4n \log_2(M)/M}.
\]
Again applying \( t = n/M \) yields
\[
p \leq 2^{-n(1 - 14/M - 4 \log_2(M)/M)}.
\]
Finally, since \( M \geq 79 \) then
\[
1 - 14/M - 4 \log_2(M)/M \geq 1/2
\]
and so
\[
p \leq 2^{-n/2}.
\]
Hence, conditioned on \( |S_t| \geq n - 2t \), the probability that there are at most \( k \) collisions in \( S_t \) at time \( t + 1 \) is at most \( 2^{-n/2} \). It follows that the probability that there is at most one collision is also bounded from above by \( 2^{-n/2} \). But this event is equal to the event that \( |S_{t+1}| \geq |S_t| - 2 \).

We are now ready to prove our theorem.
Proof of Theorem 3.2.3. As we observe above, $F_{S,\lfloor n/79 \rfloor}$ contains $C_S$ and so establishing an upper bound on $P(F_{S,\lfloor n/79 \rfloor})$ shows the same upper bound for $P(C_S)$.

Now, by the definition of $E_{S,t}$ and $F_{S,t}$, it holds for all $t > 0$ that

$$P(F_{S,t}) = P(E_{S,t-1}|F_{S,t-1}) \cdot P(F_{S,t-1}),$$

where $E_{S,0}$ and $F_{S,1}$ are understood to be full-measure events. Hence by induction on $t$ we have that

$$P(F_{S,\lfloor n/79 \rfloor}) = \prod_{t=1}^{\lfloor n/79 \rfloor} P(E_{S,t-1}|F_{S,t-1}).$$

By Proposition 3.4.5 each of these multiplicands is at most $2^{-n/2}$, and so

$$P(F_{S,\lfloor n/79 \rfloor}) \leq \left(2^{-n/2}\right)^{\lfloor n/79 \rfloor},$$

which, for $n$ large enough, is at most

$$\left(2^{-n/2}\right)^{n/80}.$$

Thus we have shown that

$$P(C_S) \leq 2^{-n^2/160}.$$

For the other side, consider the event that each of the left $\lfloor n/2 \rfloor$ of the agents in $S$ move to the right at each period until they collide, and the right $\lceil n/2 \rceil$ agents move to the left. The probability of this event is at least $2^{-n^2/2}$, and this event in contained in $C_S$. Hence

$$P(C_S) \geq 2^{-n^2/2}.$$
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