Lawrence Berkeley National Laboratory
Recent Work

Title
GLUEBALL MASS FROM THE Nc = oo STRONG COUPLING

Permalink
https://escholarship.org/uc/item/2fk97349

Author
Cristofano, G.

Publication Date
1983-03-01
GLUEBALL MASS FROM THE $N_c = \infty$ STRONG COUPLING EXPANSION

Gerardo Cristofano

March 1983
DISCLAIMER

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.
GLUEBALL MASS FROM THE $N_c = \infty$ STRONG COUPLING EXPANSION *

Gerardo Cristofano
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

ABSTRACT

We perform the strong coupling expansion for the glueball mass to the 8th order in $b = 1/g_o^2 N_c$ for the Euclidean Yang Mills theory $SU(N_c)$ on the lattice at $N_c = \infty$. The ratio of the glueball mass $m_g$ to the square root of the string tension $\sqrt{\kappa}$ for the $SU(\infty)$ theory is also studied.

* This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098.

† Participating guest at LBL during 1982/83 academic year. On leave of absence from Istituto di Fisica Teorica, Mostra d’Oltremare Pad. 19, 90125 Napoli (Italy).

I.

The lattice approach to quantum chromodynamics (QCD) \(^1\) has become in recent years a useful framework toward a better understanding of the problem of quark confinement.\(^2,3\) It has also provided an estimate of physical quantities such as the string tension $\kappa$\(^2,4,5,6\) which is directly related to the phenomenology of the Regge trajectories. The lattice formulation of QCD is simply a gauge theory with a gauge invariant but Lorentz non-invariant cutoff, which hopefully reduces to the usual Yang Mills theory of the continuum in the limit of the lattice spacing “$\alpha$” going to zero. There are several advantages of the lattice theory. For example one can perform Monte Carlo calculations\(^2\) and analytic expansions at large bare coupling $g_o$ for physical quantities like the string tension $\kappa$\(^2,4,5,6\) or the glueball mass $m_g$.\(^7\) Analytic expansions in the small $g_o$ region are not possible for $\kappa$ and $m_g$ due to the presence of an essential singularity at $g_o = 0$, as predicted from the renormalization group equations.\(^2\)

During the last few years extensive calculations of the string tension in the strong coupling expansion framework have been done for the theories $SU(2)$,\(^4\) $SU(3)$\(^5\) and $SU(\infty)$\(^8,9\) in the Euclidean as well as in the Hamiltonian lattice formulation.\(^6\) The authors in references (4,5,6,8) were able to determine the ratio $\sqrt{\kappa / \Lambda_L}$ for the $SU(N_c)$ theories (where $\Lambda_L = \exp(3\pi^2/11 N_c)\Lambda_{\overline{\text{MS}}}/39.0$ is the lattice $\Lambda$-parameter) by using the “matching procedure”. The agreement with Creutz’s Monte Carlo estimates\(^4\) for $\sqrt{\kappa / \Lambda_L}$ for the SU(2) and SU(3) lattice gauge theories is encouraging. Recently Munster\(^7\) has done an analogous calculation for the glueball mass $m_g$ for the SU(2) and SU(3) lattice theories determining the ratio $m_g / \Lambda_L$ from the matching condition and also the
ratios $m_g/\kappa$ by using the string tension results. It is then interesting to evaluate $m_g/\kappa$ for the SU($\infty$) gauge theory.

In this letter we perform the strong coupling expansion for $m_g$ to the $g^8$ order, where $g = 1/g_0^2 N_c$ is the expansion parameter, $g_0$ the bare coupling constant and $N_c$ is the number of colors. The ratio $m_g/\kappa$ is also studied in two different ways. By using the "matching procedure" the value $m_g/\kappa = 1.1$ is found. By simply using the strong coupling series for $m_g$ and $\kappa$, the ratio $m_g/\kappa$ is shown to have a slowly varying dependence on $g$, for values of $g$ extending throughout the transition region to the estimated first order phase transition point, at $g = 0.395$ and beyond. The rough estimate $m_g/\kappa = 2.0$ is given in this case.

The theory is defined by the partition function

$$Z = \int dU_L e^{i g \sum \left( g^{a} U_p + g^{a} U_p^+ \right)}$$

where the link variable $(U_L)$ is a $N \times N$ special unitary matrix, the plaquette variable $(U_p)$ is the ordered product of four link variables $U_L$ around a plaquette and $\sum$ indicates a sum over all oriented plaquettes in the $d$-dimensional Euclidean lattice.

Of special interest is the connected plaquette-plaquette correlation function

$$\langle T_c U_p(0) T_c U_p(L) \rangle_c \leq \langle T_c U_p(0) T_c U_p(L) \rangle - \langle T_c U_p(0) \rangle \langle T_c U_p(L) \rangle$$

(1)

between two plaquettes at Euclidean time separation $L$. For large coupling $g_0$ and for large Euclidean time separation $L$ the correlation function (1) between the two plaquettes is expected to decay exponentially as $L \to \infty$.

$$\langle T_c U_p(0) T_c U_p(L) \rangle_c \sim e^{-a m_g L}$$

(2)

In Eq. (2) $L$ is expressed in units of the lattice spacing "a" and $m_g$ is the characteristic mass gap or glueball mass of the physical system. The behavior described in Eq. (2) for the plaquette-plaquette correlation
function is the analogue of the exponential decay with distance of the spin-spin correlation function at high temperature. In the spin system the thermal fluctuations, at high temperature, dominate over the tendency of distant spins to align and the correlation function decays exponentially with the distance.

By making use of the strong coupling expansion it is possible to determine the correlation in Eq. (1) and consequently the glueball mass \( m_g \) through Eq. (2). We sum over all "spatial" positions of the two plaquettes at fixed Euclidean time separation \( L \) in order to project out zero momentum eigenstates and sum over all space-like orientations of the two plaquettes independently to project out positive parity scalars. For large \( g_o^2 N_c \) one can expand the exponential in Eq. (1) and get in lowest order in \( 1/g_o^2 N_c \)

\[
\langle T \rho \rho^{(0)} T \rho \rho^{(L)} \rangle \simeq \left( \frac{1}{g_o^4} \right)^L \left( \frac{1}{N_c} \right)^L \frac{4L \text{ sgn} \left( \frac{1}{g_o^2 N_c} \right)}{4L \text{ sgn} \left( \frac{1}{g_o^2 N_c} \right)}
\]

The diagram corresponding to Eq. (3) is a tube of plaquettes, of area \( 4L \), connecting the plaquette at the origin with the plaquette at distance \( L \) (see Fig. 1). In higher order one has in general

\[
\langle T \rho \rho^{(0)} T \rho \rho^{(L)} \rangle \simeq 4L \left\{ \ln \left( \frac{1}{\beta} \right) - P(\beta) \right\}
\]

gets for the glueball mass the following expression

\[
m_g \approx \frac{4}{a} \left\{ \ln \left( \frac{1}{\beta} \right) - P(\beta) \right\}
\]

The further determination of \( P(\beta) \) is done by using the standard diagrammatic expansion technique for strong coupling for lattice gauge theories. In particular, for the SU(3) theory, very useful diagrammatic rules are found in Ref. (8). The final results are

for \( d = 3 \)

\[
a \times m_g \approx -4 \left\{ \ln \beta + 2 \beta^4 + 8 \beta^6 + \beta^8 \right\}
\]

for \( d = 4 \)

\[
a \times m_g \approx -4 \left\{ \ln \beta + 8.5 \beta^4 + 4 \beta^6 + 209 \beta^8 \right\}
\]

where \( P(\beta) \) is a polynomial in \( \beta = 1/g_o^2 N_c \). From Eqs. (4) and (2) one
III.

Recently several authors\textsuperscript{12,10,13} have been arguing about the existence of phase transitions for the SU($N_c$) Wilson theory at large $N_c$, by studying the bulk properties of the theory. First Gross and Witten\textsuperscript{12} showed the two dimensional SU($\infty$) Wilson theory to present a third order phase transition and argued about its persistence in the four dimensional world. Later Green and Samuel\textsuperscript{10} and Creutz,\textsuperscript{13} by using different methods, strongly argued for the existence of a first order phase transition for the SU($N_c$) Wilson theory at large $N_c$. The location of the phase transition was estimated to be at $\beta = 0.395$.\textsuperscript{10}

Although no deconfinement phenomenon is believed to take place for the SU($N_c$) theory at large $N_c$, the existence of such phase transitions might spread some doubts about the extrapolation to the continuum of the strong coupling calculations. At the moment it is not clear the precise relationship between the behavior of bulk quantities and the properties of the string tension or of the glueball mass. In fact a recent investigation\textsuperscript{14} shows that the crossover in the string tension for SU(2) Wilson theory does not track the peak found for the specific heat.\textsuperscript{15} In this section we will present two ways of learning about the behavior of the ratio $m_g/\Lambda$ in the crossover region for the SU($\infty$) Wilson theory and comment about the results.

Since the glueball mass $m_g$ has the dimensions of the inverse of a length, the renormalization group equations predict for the dimensionless quantity $a \times m_g^2$ the following behavior for large $\beta$

\begin{equation}
    a \times m_g^2 \approx C \left( \frac{\gamma_o}{g_o^2} \right)^{1/2} \frac{\gamma}{\gamma_o^2} \lesssim \frac{1}{2} \frac{\gamma}{\gamma_o^2}
\end{equation}

where $\gamma_o = 11 N_c^2 / 48 \pi^2$, $\gamma / \gamma_o^2 = 102 / 121$ and $C$ is a dimensionless constant. The constant $C$ can be expressed in terms of $m_g$ and the lattice cutoff $\Lambda$ as $C = m_g / \Lambda$ and can be determined numerically by the "matching procedure" used by many authors.\textsuperscript{4,5,6,8} In Fig. 2 ($\gamma_0 g_o^2)^{1/12} \times a \times m_g$ is plotted on a log scale versus $\beta$. The continuous curve represents the strong coupling series, the dashed line represents the asymptotic freedom prediction. The matching requirement of these two curves determines the value of $C = m_g / \Lambda$.

The result is

\begin{equation}
    m_g / \Lambda \approx 2.54 \times 10^3
\end{equation}

to the $\beta^5$ order in the strong coupling expansion, and the crossover point is $\beta_c = 0.391$. A completely analogous determination of the ratio $\gamma / \Lambda$ has been done for the SU($\infty$) theory.\textsuperscript{8} The result was (see Fig. 3)

\begin{equation}
    \frac{\gamma}{\Lambda} \approx 2.321 \times 10^3
\end{equation}

to the $\beta^{12}$ order in the strong coupling expansion, and the crossover point was $\beta_c = 0.440$. It should be mentioned that Munster and Weisz\textsuperscript{9} found from a Pade analysis of the width of the string the location of the roughening transition at $\beta_R = 0.38$ for the SU($\infty$) Wilson theory (see
Table I. Then $\beta_R$ is a natural boundary for the strong coupling expansion unless the strength of the nonanalyticity is negligibly small.

By eliminating the $\lambda_L$ parameter between equations (8) and (9) we get

$$m_q/\sqrt{\lambda} \approx 1.1 \quad (10)$$

The above result is smaller than the corresponding estimates for the SU(2) and SU(3) Wilson theories by about a factor of two.$^{52}$

The ratio $m_q/\sqrt{\lambda}$ can also be studied by simply considering the strong coupling series for $m_q$ and for $\lambda$, given in equation (6) and in references (8, 9), respectively. As we can see in Table II, $m_q/\sqrt{\lambda}$ has a slowly varying dependence on $\beta$, for values of $\beta$ extending throughout the transition region to the estimated first order phase transition point, located at $\beta = 0.395,^{15}$ and beyond. If we assume that scaling already sets in the vicinity of $\beta = 0.395$ and that the crossover region is narrow$^{33}$ we conclude from Table II that a rough estimate for the ratio is

$$m_q/\sqrt{\lambda} \approx 2.0 \quad (11)$$

At the moment it is not clear to us how to account for the discrepancy between the results in equations (10) and (11). The persistence of lattice artifacts at the values of $\beta$, at which we extrapolated the properties of the continuum in equations (8 + 10), with the first and/or second method, could explain it. The validity of our results in equation (10) and/or equation (11) would then be very questionable. The solution of the above problem needs further investigation and certainly is beyond the scope of this letter. We last observe that the smooth behavior in $\beta$ of the ratio $m_q/\sqrt{\lambda}$ in Table II around the predicted value of the first order phase transition might suggest that dimensionless ratios of physical quantities (like for example $m_q/\sqrt{\lambda}$) are more suitable to investigation by using strong coupling methods.

For completeness we summarize in Table III some Monte Carlo estimates of the ratio $m_q/\sqrt{\lambda}$ in the SU(2) Wilson theory.

ACKNOWLEDGMENTS

I would like to thank R. Brower, J. Primack and M. Sher for useful discussions and comments on the manuscript and G. Giberti for helping me with the computer while visiting the Istituto di Fisica Teorica in Napoli. This work was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098.
REFERENCES


FOOTNOTES

F1. For detailed calculations see G. Cristofano, Ph. D. Thesis Santa Cruz, June 1982.
F2. For SU(2) we get \( m_g/\sqrt{\kappa} = 2.0 \pm 0.8 \) by taking \( m_g/\Lambda_L = (193 + 127 + 92)/3 = 137.3 \) from Ref. (7) and \( \sqrt{\kappa}/\Lambda_L = 69 \) from Ref. (8). For SU(3) we get \( m_g/\sqrt{\kappa} = 2.4 \pm 0.5 \) by taking \( m_g/\Lambda_L = (786 + 490)/2 = 638 \) from Ref. (7) and \( \sqrt{\kappa}/\Lambda_L = 267.2 \) from Ref. (8).
F3. Strong evidence of a narrow crossover region comes from Monte Carlo calculations of the string tension \( \tilde{\kappa} \) and of the glueball mass \( m_0 \) for the SU(2) and SU(3) lattice theories.
TABLE CAPTIONS

Table I. The location $\beta_0$ of the roughening transition for the SU($\infty$) theory is given for Pades of order $[M,N]$.  
Table II. The ratio $m_0/\sqrt{E}$ is given for different values of $\beta$.  
Table III. The Monte Carlo results for the ratio $m_0/\sqrt{E}$ in SU(2) are given for different references.

<table>
<thead>
<tr>
<th>Pade</th>
<th>[4,4]</th>
<th>[2,4]</th>
<th>[2,6]</th>
<th>[0,8]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_0$</td>
<td>0.365</td>
<td>0.366</td>
<td>0.383</td>
<td>0.393</td>
</tr>
<tr>
<td>( \frac{m_b}{\sqrt{F}} )</td>
<td>Reference</td>
<td>Brower et al.</td>
<td>Engel et al.</td>
<td>Berg et al.</td>
</tr>
<tr>
<td>----------------</td>
<td>----------</td>
<td>--------------</td>
<td>-------------</td>
<td>------------</td>
</tr>
<tr>
<td>1.5</td>
<td>1.7 ( \pm ) 0.2</td>
<td>2.4 ( \pm ) 0.6</td>
<td>3.0 ( \pm ) 1.0</td>
<td>2.35 ( \pm ) 0.30</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( m_g \sqrt{F} )</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.81</td>
<td>0.385</td>
</tr>
<tr>
<td>2.79</td>
<td>0.386</td>
</tr>
<tr>
<td>2.76</td>
<td>0.387</td>
</tr>
<tr>
<td>2.73</td>
<td>0.388</td>
</tr>
<tr>
<td>2.68</td>
<td>0.389</td>
</tr>
<tr>
<td>2.65</td>
<td>0.390</td>
</tr>
<tr>
<td>2.62</td>
<td>0.391</td>
</tr>
<tr>
<td>2.57</td>
<td>0.392</td>
</tr>
<tr>
<td>2.54</td>
<td>0.393</td>
</tr>
<tr>
<td>2.51</td>
<td>0.394</td>
</tr>
<tr>
<td>2.45</td>
<td>0.395</td>
</tr>
<tr>
<td>2.42</td>
<td>0.396</td>
</tr>
<tr>
<td>2.38</td>
<td>0.397</td>
</tr>
<tr>
<td>2.35</td>
<td>0.398</td>
</tr>
<tr>
<td>2.38</td>
<td>0.399</td>
</tr>
<tr>
<td>2.38</td>
<td>0.400</td>
</tr>
</tbody>
</table>
FIGURE CAPTIONS

Fig. 1.  Lowest order diagram in the strong coupling expansion of $<\text{Tr} \, U_p(0) \, \text{Tr} \, U_p(L)>$, the plaquettes cover the surface of the shown parallelepiped.

Fig. 2.  $(\gamma_0 g_o)^{2^{1/21}} \times a \times m_g$ is plotted on a log scale versus $\beta$. The solid line represents the strong coupling series, the dashed straight line represents the continuum theory. Indicated is also the crossover point $\beta_c = 0.391$.

Fig. 3.  $(\gamma_0 g_o)^{3^{102/21}} \times a^2 \times \kappa$ is plotted on a log scale versus $\beta$. The solid line represents the strong coupling series, the dashed straight line represents the continuum theory. Indicated is also the crossover point $\beta_c = 0.440$. 
Fig. 1
\[ \frac{10^2}{(\chi_0 g_0^2)^{\frac{1}{2}1} \times a^2 \times k} \]

\[ \beta = \frac{1}{g_0^2 N_c} \]

\[ \beta_c = 0.440 \]

Fig. 3
This report was done with support from the Department of Energy. Any conclusions or opinions expressed in this report represent solely those of the author(s) and not necessarily those of The Regents of the University of California, the Lawrence Berkeley Laboratory or the Department of Energy.

Reference to a company or product name does not imply approval or recommendation of the product by the University of California or the U.S. Department of Energy to the exclusion of others that may be suitable.