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Krull dimensions of rings of holomorphic functions

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Abstract. We prove that the Krull dimension of the ring of holomorphic functions of a connected complex manifold is at least the cardinality of continuum iff it is $> 0$.

Let $R$ be a commutative ring. Recall that the Krull dimension $\dim(R)$ of $R$ is the supremum of cardinalities lengths of chains of distinct proper prime ideals in $R$. Our main result is:

Theorem 1. Let $M$ be a connected complex manifold and $H(M)$ be the ring of holomorphic functions on $M$. Then the Krull dimension of $H(M)$ either equals 0 (iff $H(M) = \mathbb{C}$) or is infinite, iff $M$ admits a nonconstant holomorphic function $M \to \mathbb{C}$. More precisely, unless $H(M) = \mathbb{C}$, $\dim H(M) \geq c$, i.e., the ring $H(M)$ contains a chain of distinct prime ideals whose length has cardinality of continuum.

Our proof of this theorem mostly follows the lines of the proof by Sasane [S], who proved that for each nonempty domain $M \subset \mathbb{C}$ the Krull dimension of $H(M)$ is infinite (he did not prove that $\dim H(M) \geq c$).

Remark 2. We note that Henricksen [H] was the first to prove that the Krull dimension of the ring of entire functions on $\mathbb{C}$ has cardinality at least continuum.

In our proof we will use the Axiom of Choice in two ways: (a) to establish existence of certain maximal ideals and (b) to get existence of a nonprincipal ultrafilter $\omega$ on $\mathbb{N}$ and, hence of the ordered field $^*\mathbb{R}$ of nonstandard real (or, surreal) numbers. The field $^*\mathbb{R}$ contains $^*\mathbb{N}$, the nonstandard natural (or surrenal) numbers.

The field $^*\mathbb{R}$ is a certain quotient of the countable direct product $\prod_{k \in \mathbb{N}} \mathbb{R}$; we will denote the equivalence class (in $^*\mathbb{R}$) of a sequence $(x_k)$ in $\mathbb{R}$ by $[x_k]$. Accordingly, $^*\mathbb{N}$ consists of equivalence classes $[n_k]$ of sequences of natural numbers. Roughly speaking, we will use $^*\mathbb{N}$ and certain order relation on it to compare rates of growth of sequences of natural numbers.

Definition 3. A commutative unital ring $R$ is ample if there exists a sequence of valuations $\nu_k$ on $R$ such that for each $\beta \in ^*\mathbb{N}$, there $a = a_\beta \in R$ with the property

$$[\nu_k(a)] = \beta.$$
The main technical result of this paper is:

**Theorem 4.** For each ample ring $R$, $\dim(R) \geq c$. In particular, $R$ has infinite Krull dimension.

This theorem and its proof are inspired by Theorem 2.2 of [S], although some parts of the proof resemble the ones of [H].

We will verify, furthermore, that whenever $M$ is a connected complex manifold which has a nonconstant holomorphic function, the ring $H(M)$ is ample. This, combined with Theorem 4, will immediately imply Theorem 1.

**Remark 5.**

1. We refer the reader to Section 5.3 of [Cla] for further discussion of algebraic properties of rings of holomorphic functions.
2. Theorem 1 shows that for every Stein manifold $M$ (of positive dimension), the ring $H(M)$ has infinite Krull dimension. In particular, this applies to any noncompact connected Riemann surfaces (since every such surface is Stein, [BS]).
3. Noncompact connected complex manifolds $M$ of dimension $> 1$ can have $H(M) = \mathbb{C}$; for instance, take $M$ to be the complement to a finite subset in a compact connected complex manifold (of dimension $> 1$).

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1. **Surreal numbers**

We refer the reader to [Go] for a detailed treatment of surreal numbers, below is a brief introduction. A nonprincipal ultrafilter on $\mathbb{N}$ can be regarded as a finitely-additive probability measure on $\mathbb{N}$ which vanishes on each finite subset and takes the value 0 or 1 on each subset of $\mathbb{N}$. The existence of nonprincipal ultrafilters (the ultrafilter lemma) follows from the Axiom of Choice. Subsets of full measure are called $\omega$-large. Using $\omega$ one defines the following equivalence relation on the product $\prod_{k \in \mathbb{N}} \mathbb{R}$.

Two sequences $(x_k)$ and $(y_k)$ are equivalent if $x_k = y_k$ for an $\omega$-all $k$, i.e. the set $\{k : x_k = y_k\}$ is $\omega$-large. The quotient by this equivalence relation, denoted $^*\mathbb{R} = \prod_{k \in \mathbb{N}} \mathbb{R}/\omega$, is the set of surreal numbers. Let $[x_k]$ be the equivalence class of the sequence $(x_k)$.

The binary operations on sequences of real numbers project to binary operations on $^*\mathbb{R}$ making $^*\mathbb{R}$ a field. The total order $\leq$ on $^*\mathbb{R}$ is defined by $[x_k] \leq [y_k]$ iff $x_k \leq y_k$ for an $\omega$-all $k \in \mathbb{N}$. With this order, $^*\mathbb{R}$ becomes an ordered field.

The set of surreal numbers embeds into $^*\mathbb{R}$ as the set of equivalence classes of constant sequences; the image of a real number $x$ under this embedding is still denoted $x$. We set $^*\mathbb{R}_+ := \{\alpha \in ^*\mathbb{R} : \alpha > 0\}$.
The projection of 
\( \prod_{k \in \mathbb{N}} \mathbb{N} \subset \prod_{k \in \mathbb{N}} \mathbb{R} \)
to \( \ast \mathbb{R} \) is denoted \( \ast \mathbb{N} \), this is the set of surnatural numbers. We define a further equivalence relation \( \sim_u \) on \( \ast \mathbb{R} \) by:

\[ \alpha \sim_u \beta \]

if there exist positive real numbers \( a, b \) such that

\[ a\alpha \leq \beta \leq b\alpha. \]

The equivalence class \( (\alpha) \) of \( \alpha \in \ast \mathbb{R} \) (for this equivalence relation) is a multiplicative analogue of the galaxy \( \text{gal}(\alpha) \) of \( \alpha \), see [Go]:

**Definition 6.** The galaxy \( \text{gal}(\alpha) \) of a surreal number \( \alpha \in \ast \mathbb{R} \) is the union

\[ \bigcup_{n \in \mathbb{N}} [\alpha - n, \alpha + n] \subset \ast \mathbb{R}. \]

In other words, \( \beta \in \text{gal}(\alpha) \) iff there exist a real number \( a \) such that \( \alpha - a \leq \beta \leq \alpha + a \).

The next lemma is immediate:

**Lemma 7.** For \( \alpha \in \ast \mathbb{R}_+ \), the equivalence class \( (\alpha) \) of \( \alpha \) equals \( \exp(\text{gal}(\log(\alpha))) \).

We let \( \ast \mathbb{R} \) denote the quotient \( \ast \mathbb{R}/ \sim_u \) and \( \ast \mathbb{N} \) the projection of \( \ast \mathbb{N} \) to \( \ast \mathbb{R} \).

Define the total order \( \gg \) on \( \ast \mathbb{R} \) by

\[ (\beta) \gg (\alpha) \]

if for every real number \( c \), \( c\alpha < \beta \). By abusing the notation, we will simply say that \( \beta \gg \alpha \), with \( \alpha, \beta \in \ast \mathbb{R} \).

For the reader who prefers to think in terms of sequences of (positive) real numbers, the relation \( (\beta) \gg (\alpha) \) is an analogue of the relation

\[ (a_n) = o((b_n)), \quad n \to \infty. \]

**Remark 8.** The equivalence relation \( \sim_u \) and the order \( \gg \) are similar to the ones used by Henricksen in [H].

**Proposition 9.** The set \( \ast \mathbb{N} \) has the cardinality of continuum.

*Proof. Note first, that \( \ast \mathbb{R} \) has cardinality of continuum, hence, the cardinality of \( \ast \mathbb{N} \) is at most \( c \). The proof of the proposition then reduces to two lemmata.*

**Lemma 10.** The set \( \text{gal}(\ast \mathbb{R}_+) \) of galaxies \( \{ \text{gal}(\alpha) : \alpha \in \ast \mathbb{R}_+ \} \) has the cardinality of continuum.

*Proof. For each \( \alpha = [a_k] \in \ast \mathbb{R}_+ \), the galaxy \( \text{gal}(\alpha) \) contains the surnatural number \( [\alpha] = [b_k] \), where \( b_k = [a_k] \). For each surnatural number \( \beta \in \ast \mathbb{N} \), and natural number \( n \in \mathbb{N} \), the intersection

\[ [\beta - n, \beta + n] \cap \ast \mathbb{N} \]

is finite, equal \( \{ \beta - n, \ldots, \beta + n \} \). Therefore, \( \text{gal}(\beta) \cap \ast \mathbb{N} = \{ \beta \} + \mathbb{Z} \). It follows that the map

\[ \ast \mathbb{N} \to \text{gal}(\ast \mathbb{R}_+), \quad \beta \mapsto \text{gal}(\beta) \]

is a bijection modulo \( \mathbb{Z} \). Lastly, the set of surnatural numbers \( \ast \mathbb{N} \) has the cardinality of continuum. \( \square \)
Lemma 11. The map \( \lambda : \ast \mathbb{N} \to \text{gal}(\ast \mathbb{R}_+), \lambda : \beta \mapsto \text{gal}(\log(n)) \), is surjective.

Proof. For each \( \alpha \in \ast \mathbb{R}_+ \) let \( \beta = [\exp(\alpha)] \in \ast \mathbb{N} \). Since \( \log(x + 1) - \log(x) \leq 1 \) for \( x \geq 1 \), we have that

\[
\log(\beta) \in \text{gal}(\alpha).
\]

Now, we can finish the proof of the proposition. The map \( \lambda : \ast \mathbb{N} \to \text{gal}(\ast \mathbb{R}_+) \) descends to a map \( \mu : u \mathbb{N} \to \text{gal}(\ast \mathbb{R}_+) \). According to Lemma 11 the map \( \mu \) is surjective. By Lemma 10 the set \( \text{gal}(\ast \mathbb{R}_+) \) has the cardinality of continuum. □

We will prove Theorem 4 in the next section by showing that for each ample ring \( R \), the ordered set \( (u \mathbb{N}, \gg) \) embeds into the poset of prime ideals in \( R \) reversing the order:

\[
(\beta) \gg (\alpha) \Rightarrow P_\beta \subset P_\alpha
\]

for certain prime ideals \( P_\gamma \subset R \) determines by \( (\gamma) \in u \mathbb{N} \). Proposition 9 will then imply that the Krull dimension of \( R \) is at least \( c \).

2. Krull dimension of ample rings

Recall that a valuation on a unital ring \( R \) is a map \( \nu : R \to \mathbb{R}_+ \cup \{\infty\} \) such that:

1. \( \nu(a + b) \geq \min(\nu(a), \nu(b)) \),
2. \( \nu(ab) = \nu(a) + \nu(b) \),
3. \( \nu(a) = \infty \iff a = 0 \).
4. \( \nu(1) = 0 \).

For the following lemma, see Theorem 10.2.6 in [Coh] (see also Proposition 4.8 of [Cla] or Theorem 1 in [K]).

Lemma 12. Let \( I \) be an ideal in a commutative ring \( A \) and \( M \subset A \setminus I \) be a subset closed under multiplication. Then there exists an ideal \( J \subset A \) containing \( I \) and disjoint from \( M \), so that \( J \) is maximal with respect to this property. Furthermore, \( J \) is a prime ideal in \( A \).

Let \( R \) be an ample ring and \( \nu_k \) the corresponding sequence of valuations on \( R \). For each \( \beta \in \ast \mathbb{N} \) we define

\[
I_\beta := \{ a \in R | [\nu_k(a)] \gg [\beta] \} \subset R.
\]

Lemma 13. Each \( I_\alpha \) is an ideal in \( R \).

Proof. We will check that \( I_\alpha \) is additive since it is clearly closed under multiplication by elements of \( R \). Take \( p', p'' \in I_\alpha \),

\[
[\nu_k(p')] \gg \alpha, [\nu_k(p'')] \gg \alpha.
\]

By the definition of a valuation,

\[
n_k := \nu_k(p' + p'') \geq \min(\nu_k(p'), \nu_k(p'')),
\]

for each \( k \in \mathbb{N} \). For \( m \in \mathbb{N} \), define the \( \omega \)-large sets

\[
A' = \{ k : \nu_k(p') \geq ma \}, \quad A'' = \{ k : \nu_k(p'') \geq ma \}.
\]

Therefore, their intersection \( A = A' \cap A'' \) is \( \omega \)-large as well, which implies that

\[
\forall m \in \mathbb{N}, [n_k] \geq ma \Rightarrow [n_k] \gg \alpha. \quad \Box
\]
Then for each \( \gamma \gg \beta \), the element \( a_\gamma \) as in Definition 3 belongs to \( I_\beta \). It follows that \( I_\beta \neq 0 \) for every \( \beta \). Define the subsets

\[
M_\beta := \{ a \in R \mid \exists n \in \mathbb{N}, [\nu_k(a)] \leq n\beta \} \subset R;
\]

each \( M_\beta \) is closed under the multiplication. It is immediate that whenever \( \alpha \leq \beta \), we have the inclusions

\[
I_\beta \subset I_\alpha, \quad M_\alpha \subset M_\beta.
\]

It is also clear that \( I_\beta \cap M_\beta = \emptyset \). At the same time, for each \( \beta \gg \alpha \),

\[
a_\beta \in I_\alpha \cap M_\beta.
\]

For each \( \alpha \) we let \( J_\alpha \) denote the set of ideals \( P \subset R \) such that \( I_\alpha \subset P, P \cap M_\alpha = \emptyset \).

**Lemma 14.** Every \( J_\alpha \) contains unique maximal element, which we will denote \( P_\alpha \) in what follows.

**Proof.** Suppose that \( P', P'' \) are two maximal elements of \( J_\alpha \). We define the ideal \( P = P' + P'' \). Clearly, \( P \) contains \( I_\alpha \). To prove that \( P \) is disjoint from \( M_\alpha \), take \( p' \in P', p'' \in P'' \), since \( p' \notin M_\alpha, p'' \notin M_\alpha \). Then the same proof as in Lemma 13 shows that \( [\nu_k(p' + p'')] \gg \alpha \) which means that \( p' + p'' \notin M_\alpha \). Thus, \( P \in J_\alpha \) and, in view of maximality of \( P', P'' \), we obtain 

\[
P' = P = P''. \quad \square
\]

For each \( \beta \gg \alpha \) we define the ideal \( Q_{\alpha\beta} := I_\alpha + P_\beta \).

**Lemma 15.** \( Q_{\alpha\beta} \cap M_\alpha = \emptyset \).

**Proof.** The proof is similar to the one of the previous lemma. Let \( q = c + p, c \in I_\alpha, p \in P_\beta \). Since \( p \notin M_\beta, p \notin M_\alpha \) as well. Therefore,

\[
[\nu_k(p)] \gg \alpha.
\]

Since \( c \in I_\alpha, [\nu_k(c)] \gg \alpha.
\]

Hence,

\[
[\nu_k(c + p)] \gg \alpha
\]

as well. Thus, \( q \notin M_\alpha \). \( \square \)

**Corollary 16.** \( Q_{\alpha\beta} \in J_\alpha \). In particular, \( Q_\alpha \subset P_\alpha \).

**Proof.** It suffices to note that \( I_\alpha \subset Q_{\alpha\beta} \) according to the definition of \( Q_{\alpha\beta} \). \( \square \)

**Lemma 17.** The inequality \( \beta \gg \alpha \) implies \( P_\beta \subset P_\alpha \) and this inclusion is proper.

**Proof.** By the definition of \( Q_{\alpha\beta} \) and Corollary 16 we have the inclusions

\[
P_\beta \subset Q_\alpha \subset P_\alpha.
\]

We now claim that \( P_\beta \neq Q_{\alpha\beta} = I_\alpha + P_\beta \). Recall that \( a_\alpha \in I_\alpha \subset Q_{\alpha\beta} \) and \( a_\alpha \in M_\beta \), while \( M_\beta \cap P_\beta = \emptyset \). Thus, \( a_\alpha \in Q_{\alpha\beta} \setminus P_\beta \). \( \square \)
According to Proposition 9, the set \(*N\) of surnatural numbers contains a subset \(S\) of cardinality continuum such that for all \(\alpha < \beta\) in \(S\), we have \(\beta \gg \alpha\). The map 
\[ \alpha \mapsto P_{\alpha} \]
sends each \(\alpha \in S\) to a prime ideal in \(R\); \(\alpha < \beta\) implies that \(P_{\beta} \subseteq P_{\alpha}\).

We conclude that the ring \(R\) contains the (descending) chain of distinct prime ideals \(P_{\alpha}, \alpha \in S\); the length of this chain has the cardinality of continuum. In particular, \(\text{dim}(R) \geq c\). Theorem 4 follows. \(\square\)

3. Ampleness of rings of holomorphic functions

We will need the following classical result, see e.g. [Con Ch. VII, Theorem 5.15]:

**Theorem 18.** Let \(D \subset \mathbb{C}\) be a domain, and let \(c_k \in D\) be a sequence which does not accumulate anywhere in \(D\) and let \(m_k\) be a sequence of natural numbers. Then there exists a holomorphic function \(g\) in \(D\) which has zeroes only at the points \(c_k\) and such that \(m_k\) is the order of zero of \(g\) at \(c_k\), \(k \in \mathbb{N}\).

**Corollary 19.** If \(M\) is a connected complex manifold which admits a non-constant holomorphic function \(h : M \to \mathbb{C}\), then the ring \(H(M)\) is ample.

**Proof.** We let \(D\) denote the image of \(h\). Pick a sequence \(c_k \in D\) which converges to a point in \(\mathbb{C} \setminus D\) and which consists of regular values of \(h\). (Here \(\mathbb{C}\) is the Riemann sphere.) For each \(c_k\) the preimage \(C_k := h^{-1}(c_k)\) is a complex submanifold in \(M\); in each \(C_k\) pick a point \(b_k\). Define valuations
\[ \nu_k : H(M) \to \mathbb{Z}_+ \cup \{\infty\} \]
by \(\nu_k(f) := \text{ord}_{b_k}(f)\), the total order of \(f\) at \(b_k\), cf. [Gu Chapter C, Definition 1].

Now, given \(\beta \in \ast \mathbb{N}, \beta = [m_k]\), we let \(g = g_{\beta}\) denote a holomorphic function on \(D\) as in Theorem 18. Define \(a = a_{\beta} := g \circ h \in H(M)\). Then \(\nu_k(a) = m_k\), which implies that the ring \(H(M)\) is ample. \(\square\)

Ampleness of \(H(M)\) together with Theorem 4 imply Theorem 1.

References


