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The Term Structure of Interest Rates, Monetary Policy, and Macroeconomy

A dissertation submitted in partial satisfaction of the requirements for the degree
Doctor of Philosophy

in

Economics

by

Fan Xia

Committee in charge:

Professor James Hamilton, Chair
Professor Davide Debertoli
Professor Jun Liu
Professor Harry Markowitz
Professor Giacomo Rondina
Professor Allan Timmermann

2014
The dissertation of Fan Xia is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

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Chair

University of California, San Diego

2014
DEDICATION

To

my grandfather Yaoting Shen.
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With permission from the coauthor Jing Cynthia Wu, chapters 2 of this dissertation contains work from a manuscript that has been submitted for publication with the title, Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bound 2014, Jing Cynthia Wu and Fan Dora Xia.
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ABSTRACT OF THE DISSERTATION

The Term Structure of Interest Rates, Monetary Policy, and Macroeconomy

by

Fan Xia

Doctor of Philosophy in Economics

University of California, San Diego, 2014

Professor James Hamilton, Chair

This dissertation studies the relationship between the term structure of interest rates, monetary policy, and macroeconomy.

The first chapter, “A Parsimonious No-Arbitrage Term Structure Model that is Useful for Forecasting,” offers a solution to a well-known puzzle in the term structure literature. The puzzle is that while the level, slope and curvature (or the first three principal components of yields) can quite accurately summarize the cross-section of yields at any point in time, different functions of interest rates and other macroeconomic variables appear to be helpful when the goal is to predict future interest rates. My paper proposes a parsimonious representation to capture this feature in a large dataset. In the first step, I run reduced rank regressions
of one-year excess returns on a panel of 131 macroeconomic variables and initial forward rates from 1964 to 2007. I find that a single linear combination of macroeconomic variables and forward rates can predict excess returns on two- to five-year maturity bonds with R-squared up to 0.71. The forecasting factor subsumes the tent-shaped linear combination of forward rates constructed by Cochrane and Piazzesi (2003) and explains excess returns better. In the second step, I estimate a restricted Gaussian Affine Term Structure Model (GATSM) with the level, slope and curvature commonly used by most term structure models along with the forecasting factor. Restrictions are derived based on the fact that while cross-sectional information in yields is spanned by the level, slope and curvature, cross-sectional information in expected excess returns is spanned by the forecasting factor. Compared with a conventional GATSM only including the level, slope and curvature, the restricted four-factor GATSM generates plausible countercyclical term premia.

The second and third chapter focus on the recent zero lower bound (ZLB) period. In the second chapter, “Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bound” coauthored with Cynthia Wu, we employ an approximation that makes a nonlinear shadow rate term structure model (SRTSM) extremely tractable for analysis of an economy operating near the zero lower bound for interest rates. We show that such a model offers a better description of the data compared to the widely used GATSM. Moreover, the model can be used to summarize the macroeconomic effects of unconventional monetary policy at the ZLB. Using a simple factor-augmented vector autoregression (FAVAR), we show that the shadow rate calculated by our model exhibits similar dynamic correlations with macro variables of interest in the period since 2009 as the fed funds rate did in data prior to the Great Recession. This result gives us a tool for measuring the effects of monetary policy under the ZLB, using either historical estimates based on the fed funds rate or less precisely measured estimates inferred solely from the new data for the shadow rate alone. We show that the Fed has used unconventional policy measures to successfully lower the shadow rate. Our estimates imply that the Fed’s efforts to stimulate the economy since 2009 have succeeded in lowering the unemployment rate by 0.13% relative to where it would have been in the absence
of these measures.

The third chapter, “Effects of Unconventional Monetary Policies on the Term Structure of Interest Rates,” offers a complete characterization of effects of unconventional monetary policies on interest rates by examining policies’ impacts on the whole yield curve. I make use of the SRTSM to summarize all interest rates with factors of lower dimension so that I can capture responses of all interest rates in a parsimonious way. By investigating how policy announcements affect the three factors and then the whole forward curve accordingly, I find that during the ZLB period, forward rate with short maturities are constrained, while forward rates with long maturities still respond to policy announcements. Following each easing (tightening) policy announcement, long forward rates would decrease (increase) by 10 basis points on average.
Chapter 1

A Parsimonious No-Arbitrage Term Structure Model that is Useful for Forecasting

Abstract

I propose a parsimonious Gaussian Affine Term Structure Model (GATSM) to reconcile empirical findings that while the level, slope and curvature (or the first three principal components of yields) can quite accurately describe the cross-section of yields, different linear combinations of interest rates and other macro variables are useful to predict excess returns. I introduce a forecasting factor, which compactly summarizes rich information in expected excess returns, to a conventional three-factor (the level, slope and curvature) GATSM. This fourth factor is constructed by reduced rank forecasting regression with a large predictor set, and it can explain one-year excess returns of two- to five-year maturity bonds from 1964 to 2007 with $R^2$ up to 0.71. Considering the fact that the forecasting factor and the first three principal components span the cross-section of expected excess returns and that of yields, respectively, I restrict parameters of the four-factor GATSM. In contrast with the conventional three-factor GATSM, the restricted four-factor GATSM generates plausible countercyclical term premia.
1.1 Introduction

Back in the 1990s, Litterman and Scheinkman (1991) found that more than 99% of the movements of bond yields with various maturities are captured by the first three principal components of yields. These are commonly described as level, slope, and curvature based on how shocks to these components affect the yield curve. Quite interestingly, different linear combinations of interest rates and other macro variables are revealed to be useful when the goal is to predict excess returns. Cochrane and Piazzesi (2005) documented a tent-shaped linear combination of forward rates that predicts excess bond returns. Cooper and Priestly (2009) found that the output gap is helpful for forecasting excess returns in stock and bond markets. Ludvigson and Ng (2011) demonstrated that factors constructed from a large panel of macroeconomic variables have significant predictive power for excess bond returns. Dahlquist and Hasseltoft (2012) identified local and global factors that jointly predict returns in international bond markets.

Factor structure exhibited in the cross-section of yields has been explored by many term structure models. An extensively employed approach is the Gaussian Affine Term Structure Model (GATSM). It not only condenses the rich term structure of interest rates with factors of lower dimension but also restricts the mapping between factors and yields to preclude arbitrage opportunities. Among many other applications, the model has been used to forecast yields, characterize time variations of risk premia, and assess monetary policies’ effect on the yield curve.\(^1\) In most of these studies, researchers implicitly assume that only factors contributing to explain the cross-section of yields are relevant. This assumption, however, contradicts the empirical evidence noted above that variables other than those summarizing the cross-section of yields predict excess returns. An apparent solution is to include all variables with predictive power as factors. But this approach would result in a large number of extra factors and make overfitting unavoidable.

In this paper I propose a parsimonious GATSM to circumvent inadequate

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\(^1\)Piazzesi (2010), Gürcaynak and Wright (2012a), and Duffee (forthcoming) provided extensive relevant literature reviews.
factor specification in commonly used GATSMs without exploding the parameter space. First, I augment a conventional three-factor (the level, slope and curvature) GATSM with a forecasting factor, which compactly summarizes information in expected excess returns. The forecasting factor is constructed using reduced rank forecasting regression of excess returns on a large predictor set, including a panel of 131 macroeconomic variables and initial forward rates. The comprehensive predictor set makes use of rich information in the macroeconomy and allows relatively thorough extraction of predicted components from realized excess returns. Reduced rank forecasting regression serves the purpose of finding minimal sufficient linear combinations of regressors to predict excess returns with different maturities. This forecasting factor can explain one-year excess returns of two- to five-year maturity bonds from 1964 to 2007 with $R^2$ up to 0.71. Second, I restrict parameters of the four-factor GATSM to further shrink parameter space. Factors are constructed in such a way that while cross-sectional information in yields is spanned by the first three principal components, cross-sectional information in expected excess returns is spanned by the forecasting factor, thus leading to restrictions on parameters. To assess economic significance of the proposed model, I apply the model to decompose the yield curve. The restricted four-factor GATSM generates countercyclical term premium as economic theories suggest, while the conventional three-factor GATSM implies almost acyclical term premium.

Similar models have been proposed in other studies. Cochrane and Piazzesi (2009) used the tent-shaped linear combination of forward rates from Cochrane and Piazzesi (2005) as a fourth factor in addition to the first three principal components. They put a restriction that only level risk is priced by the fourth factor. Joslin et al. (2010) studied a restricted GATSM with the national activity index, the inflation rate along with the level, slope and curvature. The first set of their restrictions, which is similar to a subset of restrictions derived in this paper, is imposed so that two macro factors do not help explaining contemporaneous yields conditional on the first three principal components. Other restrictions are selected out of $2^{19}$ possibilities by model selection technique. Compared with previous research, this paper is different in the following two aspects. First, I use a consid-
erably larger predictor set to extract more predictable components from realized excess returns. Second, restrictions are derived from factors’ information-spanning properties instead of previous experience or computationally expensive model selection procedures.

Alternative approaches without specifying factors as observables are taken by Kim and Wright (2005) and Duffee (2011). Kim and Wright (2005) specified a GATSM with three latent factors to explain both yields and survey data on expected interest rates. Duffee (2011) estimated a restricted GATSM with five latent factors, in which two additional factors allows both information in the cross-section of yields and that in the dynamics of yields to be filtered out. Compensations for risks are restricted so that price for level risk is time-varying, price for slope risk is constant and other risks are not priced. As Duffee (2011) pointed out, latent factor approach avoids the risk of misspecifying the relation between the yield curve and the macroeconomy, while models using other macroeconomic variables have more precise estimates.

The rest of the paper is organized as follows. Section 1.2 constructs the forecasting factor by reduced rank forecasting regression. Section 1.3 derives restrictions. I estimate the model in Section 1.4 and examine the model’s implications for the term premium in Section 1.5. Section 1.6 concludes.

1.2 Forecasting factor construction

1.2.1 GATSM

Before showing how to construct the forecasting factor, I review the GATSM briefly.

**Notation** I denote the log price of an $n$-period zero-coupon bond at time $t$ by $p_t^{(n)}$. The corresponding yield is

$$y_t^{(n)} = -\frac{1}{n}p_t^{(n)}.$$
The $m$-period forward rate between time $t + n$ and $t + n + m$ is defined as
\[ f_t^{(n,m)} = \frac{1}{m} \left( (n + m) y_t^{(n+m)} - ny_t^{(n)} \right). \]

The holding period return on buying an $n$-period zero-coupon bond at time $t$ and then selling it as an $(n - m)$-period zero-coupon bond at time $t + m$ is given by
\[ hpr_{t,t+m}^{(n)} = \frac{1}{m} \left( ny_t^{(n)} - (n - m) y_{t+m}^{(n-m)} \right). \]

The difference between $hpr_{t,t+m}^{(n)}$ and $y_t^{(m)}$ is the $m$-period excess return of an $n$-period bond:
\[ exr_{t,t+m}^{(n)} = \frac{1}{m} \left( ny_t^{(n)} - (n - m) y_{t+m}^{(n-m)} - my_t^{(m)} \right). \]

**Model specification** The GATSM assumes that there are $K$ factors, denoted by $X_t$, relevant for bond pricing. These factors follow a first order vector autoregressive process (VAR(1)) under the physical $(\mathbb{P})$ measure:
\[ X_{t+1} = \mu + \rho X_t + \Sigma \varepsilon_{t+1}, \varepsilon_{t+1} \sim N(0, I). \tag{1.1} \]

Log stochastic discount factor is essentially affine as in Duffee (2002)
\[ m_{t+1} = -r_t - \frac{1}{2} \lambda_t' \lambda_t - \lambda_t' \varepsilon_{t+1}, \tag{1.2} \]

where $r_t$ is the short rate, and $\lambda_t$ is the price of risk which is linear in the factors, i.e.
\[ \lambda_t = \lambda_0 + \lambda_1 X_t. \tag{1.3} \]

With assumptions (3.2)-(1.3), it can be shown that any assets with payoff being a function of the factors $g(X_{t+1})$ can be priced in the following way (see Appendix 1.7 for the derivation):
\[ \text{Price} \left( X_t \right) = \exp \left( -r_t \right) \mathbb{E}_t^Q \left( g \left( X_{t+1} \right) \right). \]
The expectation is taken under the risk neutral (Q) measure, in which $X_t$ follows a VAR(1) as well:

$$X_{t+1} = \mu^Q + \rho^Q X_t + \Sigma \varepsilon_{t+1}^Q \quad \text{where} \quad \varepsilon_{t+1}^Q \sim N(0, I),$$

with

$$\mu^Q = \mu - \Sigma \lambda_0 \quad \text{and} \quad \rho^Q = \rho - \Sigma \lambda_1.$$

If we further assume that the short rate is an affine function of the factors, i.e.

$$y_t^{(1)} = r_t = \delta_0 + \delta_1' X_t,$$  \hfill (1.4)

then we can derive analytical expressions for yields of zero-coupon bonds with all maturities. It turns out that they are affine functions of the factors as well (see Appendix 1.7 for the derivation), i.e.

$$y_t^{(n)} = \frac{a_n}{n} + \frac{b_n'}{n} X_t,$$  \hfill (1.5)

where $a_n$ and $b_n$ can be calculated recursively for $n \geq 2$ as follows:

$$a_n = a_{n-1} + \delta_0 + b'_{n-1} \mu^Q - \frac{1}{2} b'_{n-1} \Sigma \Sigma' b_{n-1},$$

$$b_n' = b'_{n-1} \rho^Q + \delta_1'.$$  \hfill (1.6)

The recursion starts with initial conditions: $a_1 = \delta_0$ and $b_1' = \delta_1'$.

### 1.2.2 Reduced rank forecasting regression

In the GATSM, $X_t$ determines both yields and expected excess returns. When taking the model to data, most researchers focus solely on variables that explain the cross-section of yields as if only they are relevant. This common practice leaves out factors that are not spanned by the cross-section of yields but can predict excess returns. To correct model misspecification mentioned above, we need
to augment conventional GATSMs with factors conveying information in expected excess returns. At the same time, we do not want to increase the number of factors substantially due to overfitting concern. Therefore, it is desirable to use factors of low dimension to summarize the cross-section of expected excess returns in a similar way the level, slope and curvature achieve for the cross-section of yields. Principal component analysis, through which the level, slope and curvature are obtained, can not be applied since only realized excess returns are observed. For unobserved expected excess returns, fitted values from forecasting regressions are often used as proxies. Most researchers focus on a handful of predictor variables to limit parameter space. But this approach increases the chance of missing predictable components in realized returns. To address this concern, Ludvigson and Ng (2011) introduced a comprehensive panel of 131 macroeconomic variables and used dynamic factor analysis to mitigate overparameterization. In their paper, they first constructed eight factors from the panel by principal component analysis or Gibbs sampling. Then they chose macroeconomic predictors from these eight factors, their square and cubic terms by an extensive model selection process. Note that they selected different sets of predictors for excess returns with different maturities. Though replacing 131 macroeconomic variables with eight factors greatly reduces the dimension of predictor set, these eight factors were constructed solely from macroeconomic series to describe the associations among them rather than to predict excess returns. Also, studying excess returns individually does not fulfill the aim of consolidating cross-sectional information in expected excess returns. In order to extract more predictable components from realized returns, I include the large panel of macroeconomic variables in my predictor set as well. Different from Ludvigson and Ng (2011), I employ reduced rank regression to shrink the parameter space. Reduced rank regression uses low dimension linear combinations of regressors to model the variation in regressands by making use of the fact that regressands are likely to be correlated.

\footnote{Anderson (1951) first introduced reduced rank regression, and Reinsel and Velu (1998) offered a comprehensive review.}
Regression specification  The reduced rank forecasting regression is specified as follows

\[
\begin{equation}
\text{exr}_{t,t+12} = c_r + \alpha_r \beta_r \left( Z_t' F_t \right)' + e_{r,t}.
\end{equation}
\]

(1.7)

The dependent variable \(\text{exr}_{t,t+12}\) is the \(4 \times 1\) column vector of standardized one-year excess returns of two- to five-year maturity bonds, i.e.

\[
\text{exr}_{t,t+12} = \begin{pmatrix}
exr_{t,t+12}^{24} \\
exr_{t,t+12}^{36} \\
exr_{t,t+12}^{48} \\
exr_{t,t+12}^{60}
\end{pmatrix}'.
\]

For the predictors, \(Z_t\) collects 131 macroeconomic variables, and \(F_t\) includes the one-year forward rates of one- to five-year maturity bonds, i.e.

\[
F_t = \begin{pmatrix}
f_t^{0,12} \\
f_t^{12,12} \\
f_t^{24,12} \\
f_t^{36,12} \\
f_t^{48,12}
\end{pmatrix}'.
\]

The matrices \(\alpha_r\) and \(\beta_r\) are of dimension \(4 \times r\) and \(r \times 136\), respectively, where \(r\) is the rank to be specified. By using the product of \(\alpha_r\) and \(\beta_r\) as the coefficient, equation (1.7) assumes that the variation of excess returns is predicted by \(r\) linear combinations of regressors, i.e. \(\beta_r \left( Z_t' F_t \right)'.\) Compared with unrestricted regressions, reduced rank regression shrinks the number of parameters by \((4 - r)(136 - r)).^3\) The constant term and the forecasting error are represented by \(c_r\) and \(e_{r,t}\), respectively. Both \(c_r\) and \(e_{r,t}\) are \(4 \times 1\) column vectors. I specify \(e_{r,t} \sim N(0, \sigma^2_r I)\) with two considerations. First, as pointed out in Reinsel and Velu (1998), when the number of observations is not large relative to the dimensions of regressors and regressands, estimates allowing general error covariance matrix may not be very accurate and specifying \(\text{Cov}(e_{r,t}) = \sigma^2_r I\) with standardized response variables is preferred.\(^4\) Also, my purpose here is to extract factors to explain the total variance of excess returns instead of capturing their covariance. I thus impose diagonal error covariance matrix. With returns being brought to same scale.

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\(^3\)Though the parameter space is still large compared with that in Ludvigson and Ng (2011) even for the lowest possible rank \(r = 1\), reduced rank regression can be quite powerful in applications with a large cross-section of excess returns.

\(^4\)In my case, I have 136 predictors with only 528 observations.
I further assume that diagonal elements are identical.

For estimation purpose, $\alpha_r, \beta_r, c_r$ and $\sigma^2_r$ are chosen to maximize the log likelihood. Corresponding expressions can be found in Reinsel and Velu (1998) and are given by

$$\hat{\alpha}_r = \left( \hat{V}_1 \ldots \hat{V}_r \right),$$

$$\hat{\beta}_r = \begin{pmatrix} \hat{V}_1' \\
\vdots \\
\hat{V}_r' \end{pmatrix} \hat{\Sigma}_{\text{exr},zf}^{-1} \hat{\Sigma}_{zf,zf} \hat{\Sigma}_{\text{exr},zf}' ,$$

$$\hat{c}_r = \frac{1}{T} \sum_{t=1}^{T} \text{EXR}_{t,t+12} - \frac{1}{T} \sum_{t=1}^{T} \hat{\alpha}_r \hat{\beta}_r \left( Z_t' F_t' \right)' ,$$

$$\hat{\sigma}^2_r = \frac{1}{T} \sum_{t=1}^{T} \left( \text{EXR}_{t,t+12} - \hat{c}_r - \hat{\alpha}_r \hat{\beta}_r \left( Z_t' F_t' \right)' \right)' \left( \text{EXR}_{t,t+12} - \hat{c}_r - \hat{\alpha}_r \hat{\beta}_r \left( Z_t' F_t' \right)' \right) .$$

In the above expressions, $T$ is the number of observations, $\hat{\Sigma}_{zf,zf}$ and $\hat{\Sigma}_{\text{exr},zf}$ are respectively the sample variance of the predictors and the sample covariance between the predictors and the standardized excess returns, and $\hat{V}_j$ is the eigenvector of $\hat{\Sigma}_{\text{exr},zf} \hat{\Sigma}_{zf,zf}^{-1} \hat{\Sigma}_{\text{exr},zf}'$ associated with the $j$th largest eigenvalue. The corresponding maximized log likelihood is $\mathcal{L}_r = -\frac{T \times 4}{2} \log (2\pi) - \frac{T \times 4}{2} \log (\hat{\sigma}^2_r) - \frac{T \times 4}{2} .

**Data** I use the end-of-month Fama-Bliss data of one- through five-year zero coupon bond prices from January 1964 to December 2007, and compute yields, forward rates and excess returns accordingly.\(^5\) The Fama-Bliss data is not smoothed across maturities and has been widely used to study excess bond returns. For $Z_t$, I use the panel of 131 monthly macroeconomic variables in Ludvigson and Ng (2011).\(^6\) The panel represents eight broad categories of macroeconomic time series, including output and income, labor market, housing, consumption, orders

\(^5\)The data can be obtained from the Center for Research in Security Prices (CRSP).

\(^6\)The data can be downloaded from Serena Ng’s website: http://www.columbia.edu/~sn2294/pub.html. Thanks to Sydney C. Ludvigson and Serena Ng for making their data public available.
and inventories, money and credit, bond and exchange rates, prices and stock market. It offers a comprehensive description of the macroeconomy. All series are stationarized and standardized before regressions. The sample starts from 1964 due to the availability of bond price data and ends before the Zero Lower Bound (ZLB) period, during which the GATSM is not appropriate to apply as Wu and Xia (2014) pointed out.

**Rank selection** How many factors we use to model the variation in expected returns depends on our choice of rank $r$. I let the data decide using likelihood ratio tests. First I test $H_0 : \text{rank} = r$ over alternative full rank hypothesis. I compute $-2(\mathcal{L}_r - \mathcal{L}_4)$ for $r = 1, 2, \text{and } 3$, and I reject $H_0$ if the test statistic is greater than the critical value determined by the $\chi^2_{(4-r)(136-r)}$ distribution. The smallest value of $r$ for which $H_0$ is not rejected provides a good candidate. Respective test statistics for $r = 1, 2, \text{and } 3$ are 76.86, 13.27, and 4.14. Corresponding critical values with 95% level of confidence are 452.92, 307.18, and 160.91, respectively. Thus this test proposes $r = 1$. Another possible guidance can be obtained by considering the following hypothesis test $H_0 : \text{rank} = r$ v.s. $H_A : \text{rank} = r + 1$ for $r = 1, 2, \text{and } 3$. The null hypothesis is rejected if $-2(\mathcal{L}_r - \mathcal{L}_{r+1})$ is above the critical value from $\chi^2_{(4+136-(2r+1))}$. And the smallest $r$ that $H_0$ is not rejected gives another choice. Respective test statistics for $r = 1, 2, \text{and } 3$ are 63.60, 9.12, and 4.14, while critical values with 95% level of confidence are 165.32, 163.12, and 160.91, respectively. The second test recommends $r = 1$ as well. Both tests suggest that all we need is one factor to capture the cross-section of expected excess returns. Compared with the most flexible forecasting regression, reduced rank regression with $r = 1$ decreases the number of parameters by 74%.

**Forecasting factor** With $r = 1$, I calculate $\hat{\beta}_1$ and compute $H_t = \hat{\beta}_1 \left( Z_t' F_t \right)'$. Since reduced rank forecasting regression suggests that this particular linear combination of regressors is sufficient to predict excess returns, I call $H_t$ the forecasting factor. Before introducing the forecasting factor to a GATSM, it is important

---

7A detailed description of each series and its corresponding transformation to ensure stationarity can be found in Data Appendix of Ludvigson and Ng (2011).
to check how well it predicts individual excess returns. I run the following OLS forecasting regressions for \( n = 24, 36, 48 \) and 60:

\[
exr_{t,t+12}^{(n)} = \alpha_n h_t + \beta_n H_t + e_{n,t}^{(n)}. \quad (1.8)
\]

\( R^2 \)s are 0.70, 0.71, 0.71, and 0.68, respectively. I also plot expected excess returns predicted by \( H_t \) with realized returns in Figure 1.1. It can be seen that predicted excess returns track the dynamics of realized returns reasonably well across all maturities. Indeed, the constructed forecasting factor well summarizes the cross-section of expected excess returns.

A well known factor that can predict excess bond returns is the tent-shaped linear combination of forward rates proposed by Cochrane and Piazzesi (2005), denoted by \( CP_t \). It is interesting to make a comparison between \( H_t \) and \( CP_t \). I construct \( CP_t \) by first running a OLS forecasting regression of the average excess return on all forward rates:

\[
\begin{align*}
\frac{1}{4}(exr_{t,t+12}^{(24)} + exr_{t,t+12}^{(36)} + exr_{t,t+12}^{(48)} + exr_{t,t+12}^{(60)}) &= \gamma_0 + (\hat{\gamma}_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5) F_t + +e_{cp,t}.
\end{align*}
\]

Estimated coefficients \( \hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4, \hat{\gamma}_5 \) are plotted in Figure 1.2. The tent shape remains robust with the longer sample.\(^8\) Then \( CP_t \) can be computed by \( CP_t = (\hat{\gamma}_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5) F_t \). \( R^2 \)s from regressing individual excess returns on \( CP_t \) with a constant term are 0.26, 0.27, 0.30, and 0.28 for \( n = 24, 36, 48 \) and 60, respectively. Though \( CP_t \) stays a quite useful predictor in the extended sample, \( H_t \) demonstrates more in-sample predictive power with higher \( R^2 \)s. Next I examine whether \( CP_t \) remains significant conditional on \( H_t \) by the following augmented forecasting regressions

\[
exr_{t,t+12}^{(n)} = \alpha_n hcp + \beta_n hcp H_t + \gamma_n hcp CP_t + e_{hcp,t}^{(n)}. \quad (1.10)
\]

for \( n = 24, 36, 48, \) and 60. Respective \( R^2 \)s from OLS regressions specified in equa-
tion (1.10) are 0.70, 0.71, 0.71, and 0.68, the same with regressions without $CP_t$ included in equation (1.8). And $\gamma_{hec}^{(n)} = 0$ can not be rejected at any conventional significance level for all $n$ since corresponding t-statistics with Newey-West standard errors are 0.62, 0.21, 0.56, and 0.18, respectively. The above exercise shows that $H_t$ subsumes $CP_t$’s role in in-sample prediction. If $CP_t$ instead of $H_t$ is used as the forecasting factor, certain predictable information in realized excess returns is neglected.

1.3 Restrictions derivation

Section 1.2 constructs the forecasting factor with which we can efficiently incorporate information in expected excess returns to a GATSM. This section aims for further model simplification by putting restrictions on parameters. It is not uncommon to restrict parameters of GATSMs. Restrictions can be set either by zeroing out insignificant parameters from estimating more flexible models or by researchers’ economic intuition on risk pricing, with Duffee (2002), Ang and Piazzesi (2003), Cochrane and Piazzesi (2009), and Duffee (2011) as examples. In this paper, I take a less ad hoc approach and impose restrictions based on factors’ information-spanning properties.

In this paper, I add the forecasting factor to the conventional three-factor GATSM. Therefore, $X_t$ consists of two sets of factors, i.e. $X_t = \left( P_t^T \ H_t^T \right)^T$. The first set of factors $P_t$ includes the level, slope and curvature, which are constructed from principal component analysis of yields. Together the level, slope and curvature explain $98.61\% + 1.33\% + 0.03\% = 99.97\%$ of the total variance of yields. The second set of factor $H_t$ is the forecasting factor constructed in Section 1.2. It spans the cross-section of expected excess returns. The time series for these factors are plotted in Figure 1.3. Note that conditional on $P_t$, $H_t$ does not offer additional explanation power on current yields. On the other hand, conditional on $H_t$, $P_t$ does not help to explain expected one-year excess returns. As a result, corresponding factor loadings must be zero. And restrictions can be derived accordingly.
1.3.1 Restriction I

\[ \delta_{1,H} = 0 \quad \text{and} \quad \rho^Q_{PH} = 0. \]  

(1.11)

This set of restrictions is based on the idea that the factor loadings of yields on the forecasting factor should be zero.

The short rate loads on the factors in the following way:

\[ y_t^{(1)} = \delta_0 + \delta'_1 X_t \]
\[ = \delta_0 + \left( \begin{array}{ccc} \delta'_{1,P} & \delta'_{1,H} \end{array} \right) \begin{pmatrix} P_t \\ H_t \end{pmatrix} \]
\[ = \delta_0 + \delta'_{1,P} P_t + \delta'_{1,H} H_t. \]

It is straightforward to see that the factor loading on the forecasting factor of the short rate is zero if and only if \( \delta_{1,H} = 0 \). Assuming \( b_{n,H} = 0 \), we have

\[ b'_{n+1} = b'_n \rho^Q + \delta'_1 \]
\[ = \left( b'_{n,P}, b'_{n,H} = 0 \right) \begin{pmatrix} \rho^Q_{PP} & \rho^Q_{PH} \\ \rho^Q_{HP} & \rho^Q_{HH} \end{pmatrix} + \left( \delta'_{1,P}, \delta'_{1,H} = 0 \right) \]
\[ = \left( b'_{n,P} \rho^Q_{PP}, b'_{n,H} \rho^Q_{PH} \right) + \left( \delta'_{1,H} = 0 \right). \]

It is not hard to see that \( \rho^Q_{PH} = 0 \) is the necessary and sufficient condition to get the induction \( b_{n,H} = 0 \Rightarrow b_{n+1,H} = 0 \) going through.

Intuitively, the restriction \( \rho^Q_{PH} = 0 \) can be understood by the following argument. The yield of an \( n \)-period zero-coupon bond is the average of expected future short rates between \( t \) and \( t + n - 1 \) under the Q measure plus a constant, i.e.

\[ y_t^{(n)} = \frac{1}{n} \left( r_t + E^Q_t (r_{t+1}) + \ldots + E^Q_t (r_{t+n-1}) \right) + \text{constant}. \]

Since short rates only depend on \( P_t \) (guaranteed by \( \delta_{1,H} = 0 \)), if and only if \( H_t \) does not affect time evolution of \( P_t \) under the Q measure, it would have no effect on yields with arbitrary maturity. The irrelevance of \( H_t \) on dynamics of \( P_t \) under the Q measure can be achieved by setting \( \rho^Q_{PH} \) to zero. Thus, only the upper left block of \( \rho^Q \) is
relevant for bond pricing. The upper right block is all zeros and lower blocks are not identified from observed yields, i.e. $\rho^Q = \begin{pmatrix} \rho_{PP}^Q & 0 \\ 0 & \text{not identified} \end{pmatrix}$.

1.3.2 Restriction II

$\left( (\rho^Q)^m \right)_{PP} = (\rho^m)_{PP}$.  \hspace{1cm} (1.12)

This restriction is based on the idea that the factor loadings on $P_t$ of expected $m$-period excess returns should be zero $^9$.

Expected excess returns depend on the factors and their expectations in the following way:

$$mE_t \left( exr_{t,t+m}^{(n,m)} \right) = ny_t^{(n)} - (n - m) E_t \left( y_{t+m}^{(n-m)} \right) - my_t^{(m)}$$

$$= a_n - a_{n-m} - a_m + (b'_n - b'_m)X_t - b'_{n-m}E_t \left( X_{t+m} \right).$$

Since

$$E_t \left( X_{t+m} \right) = \left( I + \rho + \ldots + \rho^{m-1} \right) \mu + \rho^m X_t,$$

$$b'_n - b'_m = b'_{n-m} \left( \rho^Q \right)^m,$$

we have

$$mE_t \left( exr_{t,t+m}^{(n,m)} \right) = a_n - a_{n-m} - a_m - b'_{n-m} \left( I + \rho + \ldots + \rho^{m-1} \right) \mu$$

$$+ b'_{n-m} \left( \left( \rho^Q \right)^m - \rho^m \right) X_t.$$  \hspace{1cm} (1.13)

$^9$Since I focus on one-year excess returns, $m$ is fixed to be 12 for the rest of the paper.
Define $\Lambda_m \equiv (\rho^Q)^m - \rho^m$, then

$$b'_{n-m} \Lambda_m = \begin{pmatrix} b'_{n-m,P} & b'_{n-m,H} = 0 \end{pmatrix} \begin{pmatrix} \Lambda_{m,PP} & \Lambda_{m,PH} \\ \Lambda_{m,HP} & \Lambda_{m,PP} \end{pmatrix} = \begin{pmatrix} b'_{n-m,P} \Lambda_{m,PP} & b'_{n-m,P} \Lambda_{m,PH} \end{pmatrix}.$$ 

The factor loading of interest is $b'_{n-m,P} \Lambda_{m,PP}$. It is a zero vector if and only if $\Lambda_{m,PP} = 0$. By the definition of $\Lambda_m$, restriction 1.12 can be obtained.

There is an alternative derivation. Note that

$$n y_{t}^{(n)} - m y_{t}^{(m)} = E_t^Q (r_{t+m}) + \ldots + E_t^Q (r_{t+n-1}) + \text{constant}$$

$$= (n - m) E_t^Q (y_{t+m}^{(n-m)}) + \text{constant}.$$ 

Then the expected $m$-period excess return is

$$m E_t (e x r_t^{(n),t+m}) = (n - m) (E_t^Q - E_t) (y_{t+m}^{(n-m)}) + \text{constant}$$

$$= b'_{n-m} (E_t^Q - E_t) (X_{t+m}) + \text{constant}$$

$$= b'_{n-m} ((\rho^Q)^m - \rho^m) X_t + \text{constant.} \quad (1.14)$$

Except for the unspecified constant part, the resulting expression in equation (1.14) is the same as that in equation (1.13). The restriction 1.12 can then be derived in the same fashion.

1.4 Empirical results

The proposed restricted four-factor GATSM presents a parsimonious Gaussian Affine Term Structure Model (GATSM) incorporating both cross-sectional information in yields and that in expected excess returns. The restricted four-factor model can be summarized by the following two equations:

$$X_{t+1} = \mu + \rho X_t + \Sigma \varepsilon_{t+1}, \varepsilon_{t+1} \sim N(0, I),$$

$$Y_t^o = A + B X_t + e_t, e_t \sim N(0, \Omega).$$
The factor is \( X_t = \left( P_t' \ H_t' \right)' \). Respectively, \( \mu, \rho \) and \( \Sigma_{\varepsilon_{t+1}} \) are constant, autoregressive coefficients and shock in the factor’s VAR(1) process. The vector of observed yields with maturities ranging from one to five years is denoted by \( Y_t^o \), i.e. \( Y_t^o = \left( y_{t}^{(12)} \ y_{t}^{(24)} \ y_{t}^{(36)} \ y_{t}^{(48)} \ y_{t}^{(60)} \right)' \). Matrices \( A \) and \( B \) are obtained by stacking corresponding \( a_n s \) and \( b_n s \) as follows

\[
A = \begin{pmatrix} a_{12} & a_{24} & a_{36} & a_{48} & a_{60} \end{pmatrix}' , \\
B = \begin{pmatrix} b_{12} & b_{24} & b_{36} & b_{48} & b_{60} \end{pmatrix}'.
\] (1.15)

Measurement error is represented by \( e_t \) and is assumed to follow a normal distribution with mean zero and covariance \( \Omega \). I assume that \( \Omega \) is diagonal following the literature. Let \( \Theta \) collect all parameters, i.e. \( \Theta = (\mu, \rho, \mu^Q, \rho^Q, \delta_0, \delta_1, \Sigma, \Omega) \). The log likelihood is given by

\[
\mathcal{L}(Y_T, Y_{T-1}, ..., Y_1, X_T, X_{T-1}, ...X_2 | X_1; \Theta) = \\
- \frac{(T-1) \times 4}{2} \log{(2\pi)} - \frac{T-1}{2} \log(|\Sigma\Sigma'|) \\
- \frac{1}{2} \sum_{2}^{T} (X_t - \mu - \rho X_{t-1})' (\Sigma\Sigma')^{-1} (X_t - \mu - \rho X_{t-1}) \\
- \frac{T \times 5}{2} \log{(2\pi)} - \frac{T}{2} \log(|\Omega|) \\
- \frac{1}{2} \sum_{2}^{T} (Y_t^o - A - BX_t)' \Omega^{-1} (Y_t^o - A - BX_t). 
\] (1.16)

Note that \( A \) and \( B \) only depend on subsets of \( \Theta \), i.e. \( A = A(\delta_0, \delta_1, \mu^Q, \rho^Q, \Sigma) \) and \( B = B(\delta_1, \rho^Q) \).

**Concentrated MLE** Before estimating the model by MLE, I demean both \( Y_t^o \) and \( X_t \). Instead of estimating \( a_n s \) and \( \mu \), I use sample averages. Consequently, \( \Sigma \) does not enter the last line of equation (1.16). Then estimation can be done by concentrated MLE, which makes optimization algorithm significantly faster to converge. By using sample averages, restrictions between \( a_n s \) imposed by no-arbitrage are removed. While we may lose efficiency, we decrease the risk of misspecifying
restrictions. Also, Hamilton and Wu (2014) showed that $a_n$s from the model are poorly estimated, which provides another justification for demeaning. The log likelihood function with demeaned variables can be written as follows:

$$
L(Y_t, Y_{t-1}, X_t, X_{t-1}, \ldots | \bar{X}_t; \rho, \delta_1, \Sigma, \Omega) =
$$

$$
-\frac{(T-1) \times 4}{2} \log(2\pi) - \frac{T-1}{2} (\log|\Sigma|) - \frac{T \times 5}{2} \log(2\pi) - \frac{T}{2} (\log|\Omega|) - \frac{1}{2} \sum_{1}^{T} (Y_t^\sigma - B\bar{X}_t)^\prime \Omega^{-1} (Y_t^\sigma - B\bar{X}_t),
$$

where $Y_t^\sigma$ and $\bar{X}_t$ are demeaned yields and factors. As we can see, given any $\rho$, $\rho^Q$ and $\delta_1$, the values of $\hat{\Omega}$ and $\hat{\Sigma}'$ that maximize the log likelihood function would just be the sample covariance matrix of residuals, i.e.

$$
\hat{\Sigma}'(\rho, \rho^Q, \delta_1) = \frac{1}{T-1} \sum_{2}^{T} (X_t - \rho X_{t-1})(X_t - \rho X_{t-1})^\prime,
$$

$$
\hat{\Omega}(\rho, \rho^Q, \delta_1) = \text{diag}\left(\frac{1}{T} \sum_{1}^{T} (Y_t - B\bar{X}_t) \odot (Y_t - B\bar{X}_t)\right),
$$

where $\odot$ is element-by-element multiplication and $\text{diag}(.)$ is a diagonal matrix using input vector as its diagonal. The corresponding log likelihood function would take the value

$$
-\frac{(T-1) \times 4}{2} \log(2\pi) - \frac{T-1}{2} (\log|\hat{\Sigma}'|) - \frac{(T-1) \times 4}{2} - \frac{T \times 5}{2} \log(2\pi) - \frac{T}{2} (\log|\hat{\Omega}|) - \frac{T \times 5}{2}.
$$

Thus maximizing the log likelihood function is equivalent to finding $\hat{\rho}$, $\hat{\rho}^Q$ and $\hat{\delta}_1$ so that expression 1.19 is maximized with expressions for $\hat{\Omega}$ and $\hat{\Sigma}'$ in equations (1.17) and (1.18) with $B$ given by equations (1.15) and (1.6) subject to the further restrictions derived in Section 1.3, i.e. $\hat{\delta}_{1,H} = 0, \hat{\rho}_{PH}^Q = 0$ and $\left((\hat{\rho}^Q)^m\right)_{PP} = (\hat{\rho}^m)_{PP}$. Note that $\rho^Q$ is identified up to $(\hat{\rho}^Q)^{12}$ since only yields of one through
five years are used. $\Sigma$ is assumed to be lower triangular for identification purpose. The optimization problem is solved by the MATLAB function “fminunc” with 500 different initial values.

**Estimates** The estimated $\rho$ with robust standard errors (see Hamilton (1994) p.145) in parentheses is given below:

$$
\hat{\rho} = \begin{pmatrix}
1.0058 & -0.3166 & 1.0722 & 0.2217 \\
(0.0053) & (0.0298) & (0.2286) & (0.0164) \\
0.0036 & 0.9237 & 0.5234 & 0.0217 \\
(0.0006) & (0.0059) & (0.0344) & (0.0054) \\
-0.0013 & 0.0059 & 0.8709 & -0.0031 \\
(0.0002) & (0.0021) & (0.0134) & (0.0020) \\
-0.0065 & 0.1849 & -0.5476 & 0.7840 \\
(0.0077) & (0.0460) & (0.3216) & (0.0217)
\end{pmatrix}.$$

The first two elements of $\hat{\rho}_{PH}$, i.e. the first two elements of the last column of $\hat{\rho}$, are significantly different from zero (with t-statistic larger than 1.96). It follows that the forecasting factor has significant impacts on the dynamics of level and slope under the $\mathbb{P}$ measure. Therefore, the conventional GATSM only including the first three principal components misspecifies the dynamics of yields. Recall that $\hat{\rho}_{PH}^Q = \rho - \hat{\rho}_{PH}^Q$. Thus the statistical significance of $\hat{\rho}_{PH}$ means that both level and slope risk are priced by the forecasting factor. This finding differs from restrictions imposed in Cochrane and Piazzesi (2009) and Duffee (2011) that only price for level risk is time-varying, but agrees with Joslin et al. (2010). Estimates of other parameters are not of direct interest and thus not shown.

I also estimate the conventional three-factor GATSM via concentrated MLE after demeaning variables. Further computation simplification can be achieved for the unrestricted three-factor model by recognizing that $\rho$ can be directly estimated with OLS as Joslin et al. (2011) pointed out.
1.5 Implications for the term premium

In this section, I assess economic significance of the model by examining its implications. In particular, I focus on employing the model to decompose the yield curve. The yield curve can be decomposed into two components, expected future yields and term premia. While the former depends on markets' projection of future monetary policy, the latter reflects compensations for bearing interest rate risk. Identifying each component’s contribution is of great interest to both market practitioners and monetary authorities. Theoretical studies, Campbell and Cochrane (1999), Bansal and Yaron (2004) and Wachter (2006) for examples, mainly support countercyclical term premia, and they present a challenge for the conventional GATSM. The model generates almost acyclical term premia, which is pointed by Bauer et al. (2012) among others and is reproduced in Figure 1.6. In contrast to the conventional GATSM, which use the level, slope and curvature both to explain contemporaneous yields and to predict future yields, my restricted four-factor GATSM provides an extra factor to forecast future yields. Because decomposing the yield curve hinges on how to form expectations of future yields, one possible source accounting for the failure of the conventional GATSM to generate countercyclical term premia could be inadequate factor specification. This is plausible because the term premia estimated by Kim and Wright (2005) were countercyclical when they filtered out information not only from the cross-section of yields but also the survey data on expected interest rates. Another possible explanation is the small-sample bias argued by Bauer et al. (2012). In the remaining of the section, I decompose the yield curve with my model and compare the result with that of the conventional three-factor GATSM.

The term premium for a \( n \)-period bond is defined as follows

\[
TP_t^{(n)} = y_t^{(n)} - \frac{1}{n/12}E_t \left( y_t^{(12)} + \ldots + y_{t+n-12}^{(12)} \right) .
\]  (1.21)

This is the excess return that investors required as compensation for holding a long term bond instead of a series of short ones. Since expected yields can not be observed, different models lead to distinct term premia measures. With es-
timates obtained in Section 1.4, I can compute the term premia for 2- through 5-year bonds implied by both the restricted four-factor GATSM and the conventional three-factor GATSM. They are displayed in Figure 1.4. Not surprisingly, the two models disagree on their term premia measures because they have different specifications for the dynamics of yields. Since economic theories suggest term premia are countercyclical, it is worthwhile to examine how these two measures change over the business cycle. In Figures 1.5 and 1.6, I plot the term premia with the growth rate of the Industrial Production Index (IPI) for the two models, respectively. The risk premia from the restricted four-factor GATSM display marked cyclical variation with their values rising significantly during recessions, consistent with economic theories. The correlations between the term premia and the IPI growth rate are around $-0.15$. On the other hand, the term premia from the conventional three-factor GATSM are almost acyclical. Their correlations with the IPI growth rate are around $-0.04$. This comparison suggests that the inability of the conventional three-factor GATSM to generate countercyclical risk premia may originate from factor misspecification, i.e. only using factors that span the cross-section of yields and omitting factors that drive their dynamics.

1.6 Conclusion

I propose a parsimonious GATSM exploiting information in expected excess returns, which has been ignored by commonly used GATSMs. I construct a single forecasting factor to summarize the cross-section of expected excess returns by reduced rank forecasting regression with a large predictor set. This factor can explain realized excess returns with $R^2$ up to 0.71 and subsumes the role of the tent-shaped combination of forward rates in Cochrane and Piazzesi (2005) in in-sample prediction. I include the forecasting factor along with the commonly used level, slope and curvature in a GATSM, and I impose restrictions considering the fact that the forecasting factor and the three principal components span the cross-section of expected returns and the cross-section of yields, respectively. In contrast to the traditional three-factor (the level, slope and curvature) GATSM,
the resulting restricted four-factor GATSM generates countercyclical term premia as economic theories suggest.

1.7 Appendix

Derivation of bond pricing in the GATSM

Under no-arbitrage restrictions, an asset with payoff $g(X_{t+1})$ is priced by

$$\text{Price}(X_t) = E_t (\exp (m_{t+1}) g (X_{t+1})) .$$

Plug in equations (3.2) and (1.2),

$$E_t (\exp (m_{t+1}) g (X_{t+1})) = \exp (-r_t) \exp \left( -\frac{1}{2} \lambda_t' \lambda_t \right) E_t \left( \exp \left( -\lambda_t' \Sigma^{-1} (X_{t+1} - \mu_t) \right) g (X_{t+1}) \right) ,$$

where $\mu_t \equiv \mu + \rho X_t$. Since $X_{t+1}$ follows a conditional normal distribution with conditional mean $\mu_t$ and conditional variance $\Sigma \Sigma'$,

$$E_t \left( \exp \left( -\lambda_t' \Sigma^{-1} (X_{t+1} - \mu_t) \right) g (X_{t+1}) \right) = (2\pi)^{-K/2} |\Sigma \Sigma'|^{-1/2} \int \exp \left( -\frac{1}{2} (X_{t+1} - \mu_t)' (\Sigma \Sigma')^{-1} (X_{t+1} - \mu_t) - \lambda_t' (X_{t+1} - \mu_t) \right) g (X_{t+1}) \, dX_{t+1}$$

$$= \exp \left( \frac{1}{2} \lambda_t' \lambda_t \right) (2\pi)^{-K/2} |\Sigma \Sigma'|^{-1/2} \int \exp \left( -\frac{1}{2} (X_{t+1} - \mu_t^Q)' (\Sigma \Sigma')^{-1} \left( X_{t+1} - \mu_t^Q \right) \right) g (X_{t+1}) \, dX_{t+1} ,$$

where $\mu_t^Q = \mu_t - \Sigma \lambda_t$. Plug in equation (1.3),

$$\mu_t^Q = \left( \mu - \Sigma \lambda_0 \right) + \left( \rho - \Sigma \lambda_1 \right) X_t .$$
Therefore,

\[ \text{Price} \left( X_t \right) = E_t \left( \exp \left( m_{t+1} \right) g \left( X_{t+1} \right) \right) \]

\[ = \exp \left( -r_t \right) \left( 2\pi \right)^{-K/2} |\Sigma \Sigma'|^{-1/2} \]

\[ \int \exp \left( -\frac{1}{2} \left( X_{t+1} - \mu_t^Q \right)^T \left( \Sigma \Sigma' \right)^{-1} \left( X_{t+1} - \mu_t^Q \right) \right) g \left( X_{t+1} \right) dX_{t+1} \]

\[ = \exp \left( -r_t \right) E_t^Q \left( g \left( X_{t+1} \right) \right). \]

Assume \( y_t^{(n)} = \frac{a_n}{n} + \frac{b_n'}{n} X_t \). From \( \exp \left( p_t^{(n)} \right) = \exp \left( -r_t \right) E_t^Q \left( \exp \left( p_{t+1}^{(n-1)} \right) \right) \), we have

\[ \exp \left( -a_n - b_n' X_t \right) = \exp \left( -\delta_0 - \delta_1' X_t \right) E_t^Q \left( \exp \left( -a_{n-1} - b_{n-1}' X_{t+1} \right) \right). \]

Since \( X_{t+1} \) is conditionally normally distributed with mean \( \mu^Q + \rho^Q X_t \) and variance \( \Sigma \Sigma' \) under the \( Q \) measure,

\[ E_t^Q \left( \exp \left( -b_{n-1}' X_{t+1} \right) \right) = \exp \left( -b_{n-1}' \left( \mu^Q + \rho^Q X_t \right) + \frac{1}{2} b_{n-1}' \Sigma \Sigma' b_{n-1} \right). \]

Matching constant terms and coefficients in front of \( X_t \) on both sides of the equation, we can derive the recursive formulation for \( a_n, b_n \) for \( n \geq 2 \):

\[ a_n = \delta_0 + a_{n-1} + b_{n-1}' \mu^Q - \frac{1}{2} b_{n-1}' \Sigma \Sigma' b_{n-1}, \]

\[ b_n' = \delta_1 + b_{n-1}' \rho^Q. \]

The recursion starts from \( a_1 = \delta_0, b_1' = \delta_1' \) by equation (1.4).
1.8 Figures

**Figure 1.1**: Realized and predicted excess returns
Blue lines are realized excess returns. Red lines are predicted excess returns from forecasting regressions specified by equation (1.8). Shaded areas are recession periods. Four panels are for different maturities.

**Figure 1.2**: Forecasting regression coefficients on forward rates
Figure 1.3: Time series for factors
Four panels plot time series for level, slope, curvature, and the forecasting factor, respectively. Shaded areas are recession periods.

Figure 1.4: Term premia: the restricted four-factor GATSM v.s. the conventional three-factor GATSM
Blue lines are term premia defined in equation (1.21) for 2- through 5-year bonds computed with the restricted four-factor GATSM. Green lines are computed with the conventional three-factor GATSM. Shaded areas are recession periods.
Figure 1.5: Term premia and the IPI growth rate: the restricted four-factor GATSM
Standardized 12-month moving average for term premia from the restricted four-factor GATSM (blue lines) and for the IPI growth rate (red lines) are plotted. Shaded areas are recession periods. Four panels correspond to four different maturities.

Figure 1.6: Term premia and the IPI growth rate: the conventional three-factor GATSM
Standardized 12-month moving average for term premia from the conventional three-factor GATSM (green lines) and for the IPI growth rate (red lines) are plotted. Shaded areas are recession periods. Four panels correspond to four different maturities.
Chapter 2

Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bound

Abstract

This paper employs an approximation that makes a nonlinear term structure model extremely tractable for analysis of an economy operating near the zero lower bound for interest rates. We show that such a model offers an excellent description of the data and can be used to summarize the macroeconomic effects of unconventional monetary policy at the zero lower bound. Our estimates imply that the efforts by the Federal Reserve to stimulate the economy since July 2009 succeeded in making the unemployment rate in December 2013 0.13% lower than it otherwise would have been.

2.1 Introduction

Historically the Federal Reserve has used the federal funds rate as the primary instrument of monetary policy, lowering the rate to provide more stimulus and raising it to slow economic activity and control inflation. But since December 2008, the fed funds rate has been near zero, so that lowering it further to produce
more stimulus has not been an option. Consequently, the Fed has relied on unconventional policy tools such as large-scale asset purchases and forward guidance to try to affect long-term interest rates and influence the economy. Assessing the impact of these measures or summarizing the overall stance of monetary policy in the new environment has proven to be a big challenge. Previous efforts include Gagnon et al. (2011), Hamilton and Wu (2012), Krishnamurthy and Vissing-Jorgensen (2011), D'Amico and King (2013), Wright (2012), Bauer and Rudebusch (forthcoming), and Swanson and Williams (forthcoming). However, these papers only focused on measuring the effects on the yield curve. Our interest in this paper is the more important goal of assessing the effects on the real economy.

A related challenge has been to describe the relations between the yields on assets of different maturities in the new environment. The workhorse model in the term structure literature has been the Gaussian affine term structure model (GATSM); for surveys, see Piazzesi (2010), Duffee (forthcoming), Gürkaynak and Wright (2012b), and Diebold and Rudebusch (2013). However, because this model is linear in Gaussian factors, it potentially allows nominal interest rates to go negative and faces real difficulties in the zero lower bound (ZLB) environment. One approach that could potentially prove helpful for both measuring the effects of policy and describing the relations between different yields is the shadow rate term structure model (SRTSM) first proposed by Black (1995). This model posits the existence of a shadow interest rate that is linear in Gaussian factors, with the actual short-term interest rate the maximum of the shadow rate and zero. However, the fact that an analytical solution to this model is known only in the case of a one-factor model makes using it more challenging.

In this paper we propose a simple analytical representation for bond prices in the multi-factor SRTSM that provides an excellent approximation and is extremely tractable for analysis and empirical implementation. It can be applied directly to discrete-time data to gain immediate insights into the nature of the SRTSM predictions. We demonstrate that this model offers an excellent empirical description of the recent behavior of interest rates.

More importantly, we show using a simple factor-augmented vector autore-
gression (FAVAR) that the shadow rate calculated by our model exhibits similar
dynamic correlations with macro variables of interest in the period since July 2009
as the fed funds rate did in data prior to the Great Recession. This result gives
us a tool for measuring the effects of monetary policy at the ZLB, and offers an
important insight to the empirical macro literature where people use the effective
federal funds rate in vector autoregressive (VAR) models to study the relation-
ship between monetary policy and the macroeconomy. Examples of this literature
include Christiano et al. (1999), Stock and Watson (2001), and Bernanke et al.
(2005). The evident structural break in the effective fed funds rate prevents re-
searchers from getting meaningful information out of a VAR during and even post
the ZLB. In contrast, the continuation of our series allows researchers to update
their favorite VAR using the shadow rate for the ZLB period.\footnote{Our shadow rate data with monthly update is available at http://faculty.chicagobooth.edu/jing.wu/research/data/WX.html.}. Using the series
combining the historical effective fed funds rate with the shadow rate at the ZLB,
we show that the Fed has used unconventional policy measures to successfully
lower the shadow rate. Our estimates also imply that the Fed’s efforts to stimulate
the economy since July 2009 have succeeded in lowering the unemployment rate
by 0.13% relative to where it would have been in the absence of these measures.

The SRTSM has been used to describe the recent behavior of interest rates
and monetary policy by Kim and Singleton (2012) and Bauer and Rudebusch
(2013), but these authors relied on simulation methods to estimate and study
the model. Krippner (2013) proposed a continuous-time analog to our solution,
where he added a call option feature to derive the solution. Ichiue and Ueno
(2013) derived similar approximate bond prices by ignoring Jensen’s inequality.
Both derivations are in continuous time, which requires numerical integration when
applied to discrete-time data.

Our paper also contributes to the recent discussion on the usefulness of
the shadow rate as a measure for the monetary policy stance. Christensen and
Rudebusch (2013) and Bauer and Rudebusch (2013) pointed out that the estimated
advocated the potential of the shadow rate to describe the monetary policy stance.
Our results provide further empirical evidence to support the latter view, and
demonstrate that the shadow rate is a powerful tool to summarize information at
the ZLB.

The rest of the paper proceeds as follows. Section 2.2 describes the SRTSM.
Section 2.3 proposes a new measure for monetary policy at the ZLB and demon-
strates its advantage over the effective federal funds rate. Section 2.4 summarizes
the implication of unconventional monetary policy on the macroeconomy using
historical data from 1960 to 2013, and Section 2.5 zooms in on the ZLB period.
Section 3.5 concludes.

2.2 Shadow rate term structure model

2.2.1 Shadow rate

Similar to Black (1995), we assume that the short term interest rate is the
maximum of the shadow rate $s_t$ and a lower bound $r$:

$$r_t = \max(r, s_t). \quad (2.1)$$

If the shadow rate $s_t$ is greater than the lower bound, then $s_t$ is the short rate. Note
that when the lower bound is binding, the shadow rate contains more information
about the current state of the economy than does the short rate itself. Since the
end of 2008, the Federal Reserve has paid interest on reserves at an annual interest
rate of 0.25%, proposing the choice of $r = 0.25$.\footnote{Our results are robust if we estimate $r$ as a free parameter.}

2.2.2 Factor dynamics

We assume that the shadow rate $s_t$ is an affine function of some state
variables $X_t$,

$$s_t = \delta_0 + \delta_1' X_t. \quad (2.2)$$
The state variables follow a first order vector autoregressive process (VAR(1)) under the physical measure ($P$):

\[ X_{t+1} = \mu + \rho X_t + \Sigma \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, I). \]  

(2.3)

The log stochastic discount factor is essentially affine as in Duffee (2002)

\[ M_{t+1} = \exp \left(-r_t - \frac{1}{2} \lambda_t' \lambda_t - \lambda_t' \varepsilon_{t+1} \right), \]  

(2.4)

where the price of risk $\lambda_t$ is linear in the factors

\[ \lambda_t = \lambda_0 + \lambda_1 X_t. \]

This implies that the risk neutral measure ($Q$) dynamics for the factors are also a VAR(1):

\[ X_{t+1} = \mu^Q + \rho^Q X_t + \Sigma^Q \varepsilon_{t+1}^Q, \quad \varepsilon_{t+1}^Q \sim N(0, I). \]  

(2.5)

The parameters under the $P$ and $Q$ measures are related as follows:

\[ \mu - \mu^Q = \Sigma \lambda_0, \]

\[ \rho - \rho^Q = \Sigma \lambda_1. \]

### 2.2.3 Forward rates

Equation (2.1) introduces non-linearity into an otherwise linear system. A closed-form pricing formula for the SRTSM described in Sections 3.3 - 2.2.2 is not available beyond one factor. In this section, we propose an analytical approximation for the forward rate in the SRTSM, making the otherwise complicated model extremely tractable. Our formula is simple and intuitive, and we will compare it to the solution in a Gaussian model in Section 2.2.4 to gain some intuition. A simulation study in Section 2.2.6 demonstrates that the error associated with our approximation is only a few basis points.
Define \( f_{n,n+1,t} \) as the forward rate at time \( t \) for a loan starting at \( t+n \) and maturing at \( t+n+1 \). The forward rate in the SRTSM described in equations (2.1) to (3.3) can be approximated by

\[
\begin{align*}
    f_{n,n+1,t}^{\text{SRTSM}} = \tau + \sigma_n^Q g \left( \frac{a_n + b_n' X_t - \tau}{\sigma_n^Q} \right),
\end{align*}
\]

where \((\sigma_n^Q)^2 \equiv \text{Var}_t Q(s_{t+n})\). The function \( g(.) \) is defined as \( g(z) \equiv z \Phi(z) + \phi(z) \), where \( \Phi(.) \) and \( \phi(.) \) are the cumulative distribution function and probability density function of a standard normal distribution. The expressions for \( a_n \) and \( b_n \) as well as the derivation are in 2.7.1. Equation (2.6) implies time-varying factor loadings:

\[
\frac{\partial f_{n,n+1,t}^{\text{SRTSM}}}{\partial X'_t} = \Phi \left( \frac{a_n + b_n' X_t - \tau}{\sigma_n^Q} \right) \times b'_n.
\]

To our knowledge, we are the first in the literature to propose an analytical approximation for the forward rate in the SRTSM that can be applied to discrete-time data directly. For example, Bauer and Rudebusch (2013) used a simulation-based method. Krippner (2013) proposed an approximation for the instantaneous forward rate in continuous-time. To apply his formula to the one-month ahead forward rate in the data, a researcher needs to numerically integrate the instantaneous forward rate over that month. Conversely, our discrete-time formula can be applied directly to the one-month ahead forward rate. In summary, our approximation is free of any simulation error associated with simulation methods and numerical integration.

2.2.4 Relation to Gaussian Affine Term Structure Models

If we replace equation (2.1) with

\[ r_t = s_t, \]
the SRTSM becomes a GATSM, the benchmark model in the term structure literature. The forward rate in the GATSM is an affine function of the factors:

\[ f_{n,n+1,t}^{GATSM} = a_n + b'_n X_t, \]  

(2.8)

where \( a_n \) and \( b_n \) are the same as in equation (2.6), and the derivation is in 2.7.1.

The GATSM’s linear Gaussian feature makes it more tractable than its competitors, and hence extremely popular in the literature. The same feature makes GATSM undesirable when the economy gets closer to the ZLB, because it permits negative interest rates. We are interested in the following questions: when the economy is normal, is the GATSM a close description of the yield curve? Is its potential issue outweighed by its benefits?

The difference between equations (2.6) and (2.8) is that equation (2.6) adds non-linearity through the function \( g(\cdot) \). To appreciate this difference better, we plot \( g(z) \) as a function of \( z \) in Figure 2.1 together with the 45 degree line. It is a non-linear and increasing function in \( z \). It is indistinguishable from the 45 degree line for inputs greater than 2, and is practically zero for \( z \) less than \(-2\). The fact that \( g(z) \approx z \) for \( z > 2 \) demonstrates that the GATSM is a simple and close approximation for the SRTSM, when the economy is away from the ZLB. It justifies people’s intuition about the GATSM. The intuition is simple: when the current short rate is sufficiently positive, the expected future short rate will most likely stay positive, because of the highly persistent feature of the data. In this scenario, the lower bound introduced by equation (2.1) becomes irrelevant.

In contrast to equation (2.7), the factor loadings in the GATSM are constant as usual:

\[ \frac{\partial f_{n,n+1,t}^{GATSM}}{\partial X'_t} = b'_n. \]  

(2.9)

Equations (2.7) and (2.9) provide a nice contrast, and help us to better appreciate the difference between the GATSM and SRTSM. Besides the \( b'_n \) in the GATSM, the SRTSM has an additional term. This term is between 0 and 1, and depends on where the economy is expected to be in the future. If the economy is expected
to be far away from the lower bound, the factor loadings are practically $b'_n$. But when the economy is close to the ZLB, the factor loadings are attenuated by the additional term. They are essentially zero and the forward rate does not respond to any news when we expect the economy to stay at the ZLB for a very long period of time.

2.2.5 Estimation

State space representation for the SRTSM We write the SRTSM as a nonlinear state space model. The transition equation for the state variables is equation (3.2). From equation (2.6), the measurement equation relates the observed forward rate $f_{n,n+1,t}$ to the factors as follows:

$$ f_{n,n+1,t} = r + \sigma_n^Q g \left( \frac{a_n + b'_n X_t - r}{\sigma_n^Q} \right) + \eta_{nt}, \tag{2.10} $$

where the measurement error $\eta_{nt}$ is i.i.d. normal, $\eta_{nt} \sim N(0, \omega)$. The observation equation is not linear in the factors. We use the extended Kalman filter for estimation, which applies the Kalman filter by linearizing the nonlinear function $g(.)$ around the current estimates. See 2.7.2 for details. The extended Kalman filter is extremely easy to apply due to the closed-form formula in equation (2.6). We take the observation equation (2.10) directly to data without any further numerical approximation, necessary for pricing formula derived in the continuous time. The likelihood surface behaves similarly to a GATSM, because the function $g(.)$ is monotonically increasing. These features together make our formula very appealing.

State space representation for the GATSM For the GATSM described in Section 2.2.4, equation (3.2) is still the transition equation. Equation (2.8) implies the measurement equation:

$$ f_{n,n+1,t} = a_n + b'_n X_t + \eta_{nt}, \tag{2.11} $$
with \( \eta_{nt} \sim N(0, \omega) \). We apply the Kalman filter for the GATSM, because it is a linear Gaussian state space model. See 2.7.2 for details.

**Data** We construct one-month forward rates for maturities of 3 and 6 months, 1, 2, 5, 7 and 10 years from the Gürkaynak et al. (2007) dataset, using observations at the end of the month.\(^3\) Our sample spans from January 1990 to December 2013.\(^4\) We plot the time series of these forward rates in Figure 2.2. In December 2008, the Federal Open Market Committee (FOMC) lowered the target range for the federal funds rate to 0 to 25 basis points. We refer to the period from January 2009 to the end of the sample as the ZLB period, and highlight with shaded area. For this period, forward rates of shorter maturities are essentially stuck at zero, and do not display meaningful variation. Those with longer maturities are still far away from the lower bound, and display significant variation.

**Normalization** The consensus in the term structure literature is that three factors are sufficient to account for almost all of the cross-sectional variation in yields. Therefore, we focus our discussions on three factor models.\(^5\) The collection of parameters we estimate include \((\mu, \mu^Q, \rho, \rho^Q, \Sigma, \delta_0, \delta_1)\). For identification, we impose normalizing restrictions on the \(Q\) parameters similar to Joslin et al. (2011) and Hamilton and Wu (2014): (i) \(\delta_1 = [1, 1, 0]'\); (ii) \(\mu^Q = 0\); (iii) \(\rho^Q\) is in real Jordan form with eigenvalues in descending order; and (iv) \(\Sigma\) is lower triangular.

**Repeated eigenvalues** Estimation assuming that \(\rho^Q\) has three distinct eigenvalues produces two smaller eigenvalues almost identical to each other, with the difference in the order of \(10^{-3}\). This evidence points to repeated eigenvalues. Creal and Wu (2013) have documented a similar observation using a different dataset.

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\(^3\)As a robustness check, we also estimate the SRTSM and extract the shadow rate with Fama and Bliss (1987) zero coupon bond data from CRSP and a dataset kindly provided to us by Ahn Le. We get similar results.

\(^4\)Starting the sample from 1990 is standard in the GATSM literature, see Wright (2011) and Bauer et al. (2012) for examples.

\(^5\)All of our main results relating to the macroeconomy, from Section 2.3 onward, are robust to two-factor models. But for the term structure models themselves, two-factor models perform worse than three-factor models in terms of fitting the data.
and a different model. With repeated eigenvalues, the real Jordan form becomes

\[
\rho^Q = \begin{bmatrix}
\rho_1^Q & 0 & 0 \\
0 & \rho_2^Q & 1 \\
0 & 0 & \rho_2^Q
\end{bmatrix}.
\]

**Model comparison**  Maximum likelihood estimates, and robust standard errors (See Hamilton (1994) p. 145) are reported in Table 3.2. The log likelihood value is 755.46 for the GATSM, and 855.57 for the SRTSM. The superior performance of the SRTSM comes from its ability to fit the short end of the forward curve when the lower bound binds. In Figure 2.3, we plot average observed (red dots) and fitted (blue curves) forward curves in 2012. The left panel illustrates that the SRTSM fitted forward curve flattens at the short end, because the \( g(\cdot) \) function is very close to zero when the input is sufficiently negative. This is consistent with the feature of the data. In contrast, the GATSM in the right panel has trouble fitting the short end. Instead of having a flat short end as the data suggest, the GATSM generates too much curvature. That is the only way it can approximate the yield curve at the ZLB.

As demonstrated in Section 2.2.4, the GATSM is a good approximation for the SRTSM when forward rates are sufficiently higher than the lower bound. We illustrate this property using the following numerical example. When both models are estimated over the period of January 1990 to December 1999, the maximum log likelihood is 475.71 for the SRTSM, and 476.69 for the GATSM. The slight difference in the likelihood comes from the linear approximation of the extended Kalman filter.

### 2.2.6 Approximation error

An alternative to equation (2.6) to compute forward rates or yields is simulation. In Table 2.2, we compare forward rates and yields implied by equation (2.6) and by an average of 10 million simulated paths to measure the size of the approximation error of equation (2.6). The details of simulation are explained in Table 2.2. The approximation errors grow with the time to maturity for both
forward rates and yields. We focus on the longest end to report the worst case scenario. The average absolute approximation error of the 24 Januaries between 1990 and 2013 for the 10-year ahead forward rate is 2.3 basis points, about 0.36% of the average forward rate for this period (6.37%). The average number is 0.78 basis points for the 10 year yield with an average level of 5.29%. The ratio is 0.14%. The approximation errors for long term forward rates are larger than those for yields, because yields factor in the smaller approximation errors of short term and medium term forward rates. Regardless, the approximation errors are at most a few basis points, orders of magnitude smaller than the level of interest rates. The approximation errors in Table 2.2 contain simulation errors. With the large number of draws (10 million), the simulation errors are negligible. To calculate the simulation error, we compare the analytical solution in equation (2.8) for the GATSM and simulation with the same parameters and state variables as before. The average absolute simulation errors are 0.1 basis points for the 10 year ahead forward rate and 0.04 for the 10 year yield.

2.3 Policy rate

The effective federal funds rate has served as the conventional policy rate to measure the monetary policy stance in the literature, and provided the basis for most empirical studies of the interaction between monetary policy and the economy. However, since 2009, the effective federal funds rate has been stuck at the lower bound, and no longer conveys any information due to its lack of variability. How do we summarize the effects of monetary policy in this situation? More importantly, what should economists use to measure the entire history of monetary policy when the short rate exits the ZLB and researchers include this period in their study? We aim to bridge this gap by proposing a new policy rate consistent across both the non-ZLB and ZLB periods. The shadow rate from the SRTSM is a natural candidate. We construct the new policy rate \( s_t \) by splicing together the effective federal funds rate before 2009 and the estimated shadow rate since 2009. This combination makes the most use out of both series. In
Sections 2.3.2 and 2.5.1, we will elaborate on how the information summarized in the shadow rate is relevant for the economy.

We plot the model implied shadow rate (in blue) and the effective federal funds rate (in green) in Figure 2.4. Before 2009, the ZLB was not binding, the model implied short rate was equal to the shadow rate. The difference between the two lines in Figure 2.4 reflects measurement error, in units of basis points. The two rates have diverged since 2009. The effective federal funds rate has been stuck at the ZLB. In contrast, the shadow rate has become negative and still displays meaningful variation. We update our shadow rate monthly at http://faculty.chicagobooth.edu/jing.wu/research/data/WX.html.

2.3.1 Factor augmented vector autoregression

We use the FAVAR model proposed by Bernanke et al. (2005) to study the effects of monetary policy. The basic idea of the FAVAR is to compactly summarize the rich information contained in a large set of economic variables $Y_t^m$ using a low-dimensional vector of factors $x_t^m$. This model allows us to study monetary policy’s impact on any macroeconomic variable of interest. The factor structure also ensures that the parameter space does not explode.

Model Following Bernanke et al. (2005), we use 3 factors, and assume that the factors $x_t^m$ and the policy rate $s_t^o$ jointly follow a VAR(13):

$$
\begin{bmatrix}
  x_t^m \\
  s_t^o
\end{bmatrix} = \begin{bmatrix}
  \mu^x \\
  \mu^s
\end{bmatrix} + \rho^m \begin{bmatrix}
  X_{t-1}^m \\
  S_{t-1}^o
\end{bmatrix} + \Sigma^m \begin{bmatrix}
  \varepsilon_t^m \\
  \varepsilon_t^{MP}
\end{bmatrix}, \quad \begin{bmatrix}
  \varepsilon_t^m \\
  \varepsilon_t^{MP}
\end{bmatrix} \sim N(0, I),
$$

(2.12)

where we summarize the current value of $x_t^m$ (and $s_t^o$) and its 12 lags using a capital letter to capture the state of the economy, $X_t^m = [x_t^{m'}, x_{t-1}^{m'}, \ldots, x_{t-12}^{m'}]$ (and $S_t^o = [s_t^o, s_{t-1}^o, \ldots, s_{t-12}^o]$). Constants $\mu^x$ and $\mu^s$ are the intercepts, and $\rho^m$ is the autoregressive coefficient. The matrix $\Sigma^m$ is the cholesky decomposition of the covariance matrix. The monetary policy shock is $\varepsilon_t^{MP}$. We identify the monetary

\footnote{Our results hold with different numbers of factors (3 or 5) and with different lag lengths (6, 7, 12 or 13).}
policy shock through the recursiveness assumption as in Bernanke et al. (2005); for details see 2.7.3. Observed macroeconomic variables load on the macroeconomic factors and policy rate as follows:

\[
Y_t^m = a_m + b_x x_t^m + b_s s_t^o + \eta_t^m, \quad \eta_t^m \sim N(0, \Omega),
\]

where \(a_m\) is the intercept, and \(b_x\) and \(b_s\) are factor loadings.

**Data**  Similar to Bernanke et al. (2005), \(Y_t^m\) consists of a balanced panel of 97 macroeconomic time series from the Global Insight Basic Economics, and our data spans from January 1960 to December 2013.\(^7\) We have a total of \(T = 635\) observations. We apply the same data transformations as in the original paper to ensure stationarity. See Table 2.3 for detailed data description.

**Estimation**  First, we extract the first three principal components of the observed macroeconomic variables over the period of January 1960 to December 2013, and take the part that is orthogonal to the policy rate as the macroeconomic factors. Then, we estimate equation (2.13) by ordinary least squares (OLS). See 2.7.3 for details. Next, we estimate equation (2.12) by OLS.

**Macroeconomic variables and factors**  The loadings of the 97 macro variables on the factors are plotted in Figure 2.5. Real activity measures load heavily on factor 1, price level indexes load more on factor 2, and factor 3 contributes primarily to employment and prices. For the contemporaneous regression in equation (2.13), more than one third of the variables have an \(R^2\) above 60%, which confirms the three-factor structure. Besides the policy rate, we focus on the following five macroeconomic variables: industrial production, consumer price index, capacity utilization, unemployment rate and housing starts. They represent the three factors, and cover both real activities and price levels. The \(R^2\)s for these

---

\(^7\)Global Insight Basic Economics does not maintain all 120 series used in Bernanke et al. (2005). Only 97 series are available from January 1960 to December 2013. The main results from Bernanke et al. (2005) can be replicated by using the 97 series in our paper for the same sample period.
macroeconomic variables are 73%, 89%, 64%, 64% and 67% respectively.

2.3.2 Measures of monetary policy

The natural question is whether the shadow rate could be used in place of the fed funds rate to describe the stance and effects of monetary policy under the ZLB. We first approach this using a formal hypothesis test - can we reject the hypothesis that the parameters relating the shadow rate to macroeconomic variables of interest under the ZLB are the same as those that related the fed funds rate to those variables in normal times?

We begin this exercise by acknowledging that we do not attempt to model the Great Recession in our paper, because it was associated with some extreme financial events and monetary policy responses. For example, Ng and Wright (2013) provided some empirical evidence to show that the Great Recession is different in nature from other post-war recessions. Instead, we are interested in the behavior of monetary policy and the economy in the period following the Great Recession, when policy returned to a new normal that ended up being implemented through the traditional 6-week FOMC calendar but using the unconventional tools of large scale asset purchases and forward guidance. We investigate whether a summary of this new normal based on our derived shadow rate shows similar dynamic correlations as did the fed funds rate in the period prior to the Great Recession.

We modify the first block in equation (2.12) by allowing the coefficient in front of the lagged policy rate to be different before, during, and after the Great Recession, and we also allow for different measures of the policy rate:

\[
x_t^m = \mu^x + \rho^{xx} X_{t-1}^m + 1_{(t < \text{December 2007})} \beta_1^{xs} \tilde{S}_t^o + 1_{(\text{December 2007} \leq t \leq \text{June 2009})} \beta_2^{xs} \tilde{S}_t^o + 1_{(t > \text{June 2009})} \beta_3^{xs} \tilde{S}_t^o + \sum \varepsilon_t^m, (2.14)
\]

where \( \tilde{S}_t^o = [\tilde{s}_t^o, \tilde{s}_{t-1}^o, ..., \tilde{s}_{t-12}^o]' \), and \( \tilde{s}_t^o \) is represented by one of two different mea-
sures: our new policy rate \( \tilde{s}_t^o = s_t^o \), or the effective federal funds rate \( \tilde{s}_t^o = r_t^o \). The null hypothesis is that the coefficient \( \rho^{x,s} \) is the same before and after the Great Recession:

\[
H_0 : \rho_1^{x,s} = \rho_3^{x,s}.
\]

We construct the likelihood ratio statistic as follows (see Hamilton (1994) p. 297):

\[
(T - k)(\log|\Sigma^{xx}_{\tilde{U}}\Sigma^{xx}_{\tilde{U}}| - \log|\Sigma^{xx}_{\tilde{R}}\Sigma^{xx}_{\tilde{R}}|),
\]

where \( T \) is the sample size, \( k \) is the number of regressors on the right hand side of equation (2.14), \( \Sigma^{xx}_{\tilde{U}}\Sigma^{xx}_{\tilde{U}} \) is the estimated covariance matrix, and \( \Sigma^{xx}_{\tilde{R}}\Sigma^{xx}_{\tilde{R}} \) is the estimated covariance matrix with the restriction imposed by the null hypothesis. The likelihood ratio statistic has an asymptotic \( \chi^2 \) distribution with 39 degrees of freedom. The \( p \)-value is 0.29 for our policy rate \( s_t^o \). We fail to reject the null hypothesis at any conventional significance level. This is consistent with the claim that our proposed policy rate impacts the macroeconomy the same way at the ZLB as before. If we use the effective federal funds rate instead, the \( p \)-value is 0.0007, and we would reject the null hypothesis at any conventional significance level. Our results show that there is a structural break if one tries to use the conventional monetary policy rate. Using a similar procedure for the coefficients relating lagged macro factors to the policy rate, the \( p \)-values are 1 for both our policy rate and the effective fed funds rate. In summary, our policy rate exhibits similar dynamic relations to key macro variables before and after the Great Recession, and appears to capture meaningful information missing from the effective federal funds rate after the economy reached the ZLB.

### 2.4 Macroeconomic implications

After the Great Recession the Federal Reserve implemented a sequence of unconventional monetary policy measures including large-scale asset purchases and forward guidance. The literature has thus far focused on large-scale asset purchases, and its effects on the yield curve. In contrast to previous studies, here
we attempt to answer some more fundamental questions: what is the overall impact of these new unconventional policy tools on the real economy? Is the Fed able to achieve its stated goal of lowering the unemployment rate?

2.4.1 Historical decomposition

In this section, we attempt to assess the effect of the various unconventional policy measures adopted by the Federal Reserve after the Great Recession with a historical decomposition. The basic idea is that we can write each variable in equation (2.12) as a sum of past shocks and its initial condition. Specifically, the contribution of monetary policy shocks after the Great Recession (between \([t_1 = July 2009, t_2 = December 2013]\)) to an individual economic variable \(Y_{t}^{m,i}\) can be summarized by

\[
\max(t,t_2) \sum_{\tau=t_1}^{\max(t,t_2)} \Psi_{t-\tau}^{MP,i} \varepsilon_{\tau}^{MP},
\]

(2.15)

where \(\Psi_{j}^{MP,i}\) is the impulse response

\[
\Psi_{j}^{MP,i} = \frac{\partial Y_{t}^{m,i}}{\partial \varepsilon_{t}^{MP}} = b_{x,i} \frac{\partial x_{t}^{m,j}}{\partial \varepsilon_{t}^{MP}} + b_{s,i} \frac{\partial s_{t}^{o,j}}{\partial \varepsilon_{t}^{MP}},
\]

(2.16)

for variable \(i\) after \(j\) periods in response to a one unit shock in \(\varepsilon_{t}^{MP}\), and the derivatives on the right hand side are the impulse responses from a standard VAR.

In Figure 2.6, we plot the observed time series for the six variables in blue, and counterfactual paths in red dashed lines for an alternative world where all the monetary policy shocks at the ZLB were zero. In the top left panel, we show the difference between the realized and counterfactual policy rates. Without any deviation from the traditional monetary policy rule, the shadow rate would have been about -1% in December 2013, whereas the actual shadow rate then was about -2%. On average, the shadow rate would have been 0.4% higher between 2011 and 2013 if the monetary policy shocks were set to zero. These results indicate that unconventional monetary policy has been actively lowering the policy rate, and the Federal Reserve has employed an expansionary monetary policy since 2011.

Next consider implications for the real economy. In the absence of ex-
pansionary monetary policy, in December 2013, the unemployment rate would be 0.13% higher at the 6.83% level rather than 6.7% in the data. The industrial production index would have been 101.0 rather than 101.8, and capacity utilization would be 0.3% lower than what we observe. Housing starts would be 11,000 lower (988,000 vs. 999,000). These numbers suggest that unconventional monetary policy achieved its goal of stimulating the economy. Interestingly, the accommodative monetary policy during this period has not boosted real activities at the cost of high inflation. Instead, monetary policy shocks have contributed to decreasing the consumer price index by 1. Our result exhibits the same price puzzle that has been discussed in earlier macro studies.\footnote{Examples include Sims (1992) and Eichenbaum (1992).}

The historical decomposition exercise calculates the contribution of monetary policy shocks defined as deviations of the realized shadow rate from the policy rate implied by the historical monetary policy rule. Another question of interest is what would happen if the Fed had adopted no unconventional monetary policy at all. This question is more difficult to answer, because it is not clear what the counterfactual shadow rate would be. One possible counterfactual to consider would be what would have happened if the shadow rate had never fallen below the lower bound $\bar{r}$. Specifically, we replace the realized monetary policy shock ($\varepsilon_{\tau}^{MP}$) in equation (2.15) with the counterfactual shocks, $\varepsilon_{\tau}^{MP,II}$, such that these shocks would have kept the shadow rate at the lower bound. One might view the difference between the actual shadow rate and this counterfactual as an upper bound on the contribution of unconventional monetary policy measures. If instead of the realized shadow rate, monetary policy had been such that the shadow rate never fell below 0.25%, the result would have been an unemployment rate 1% higher than observed.

Our estimated effect of unconventional monetary policy on the unemployment rate is smaller than the ones found in Chung et al. (2012) and Baumeister and Benati (2013). This is primarily because they assumed that unconventional monetary policy had a big impact on the yield curve. For example, Chung et al. (2012) assumed that the large-scale asset purchases reduced the long term interest
rates by 50 basis points, and then translated this number into a 1.5% decrease in the unemployment rate. If we were to use Hamilton and Wu (2012)’s estimate of 13 basis-point decrease in the 10 year rate, a simple linear calculation would translate this number into a 0.39% reduction in the unemployment rate. This is comparable to our estimate.

2.4.2 Impulse responses

What would happen to the unemployment rate one year later if the Fed decreases the policy rate by 25 basis points now? An impulse response function offers a way to think about questions as such by describing monetary policy’s dynamic impact on the economy.

We compute the impulse responses using equation (2.16) and plot them in Figure 2.7 for six economic variables (the policy rate, industrial production, consumer price index, capacity utilization, unemployment rate and housing starts) to a loosening monetary policy shock with a size of 25 basis points ($\Sigma^{ss} e_{t}^{MP} = -25$ bps). The 90% confidence intervals are in the shaded areas.\textsuperscript{9} With an expansionary monetary policy shock, real activity increases as expected: industrial production, capacity utilization and housing starts increase while the unemployment rate decreases. The impacts peak after about a year. Specifically, one year after a -25 basis-point shock to the policy rate, industrial production is 0.5% higher than its steady state level, capacity utilization increases by 0.2%, the unemployment rate decreases by 0.06%, and housing starts is 1.3% above its steady state level. After the peak, the effects die off slowly, and they are eventually gone in about 8 years.

2.5 Macroeconomic impact at the ZLB

The above measures assumed a constant structure before and after Great Recession. The ZLB period (July 2009 to December 2013) draws attention in its own right. Ideally, we want to repeat the FAVAR (13) exercise with the data only

\textsuperscript{9}Confidence intervals are constructed by bootstrapping.
from the ZLB. However, with a sample size of 53 months, we use a 1-lag FAVAR to get some inference instead.

### 2.5.1 New vs. conventional policy rates

Consider first an attempt to estimate a first-order FAVAR for data at the ZLB period in which the effective fed funds rate is used as the policy rate. We plot impulse responses to an expansionary policy shock of 25 basis points in Figure 2.8. The turquoise lines are median responses, and 90% confidence intervals are in the turquoise areas. For comparison, we also plot the impulse responses for the full sample with our policy rate in blue. These are identical to the impulse responses presented in Figure 2.7. For the ZLB subsample, the impulse responses to a shock to the effective federal funds rate are associated with huge uncertainty, with the confidence intervals orders of magnitude bigger than those for the full sample. This indicates that the effective federal funds rate does not carry much information at the ZLB. The reason is simple: it is bounded by the lower bound, and does not display any meaningful variation. We can also see this from Figure 2.4.

By contrast, Figure 2.9 plots the ZLB impulse-response functions in turquoise with our policy rate introduced in Section 2.3. Again, we compare them with full sample impulse responses in blue. Overall, the subsample impulse responses are qualitatively the same as those for the full sample. Specifically, an expansionary monetary policy shock boosts real economic activity. The impulse responses for the subsample and full sample also look quantitatively similar. The point estimates and confidence intervals have the same orders of magnitude. Therefore, at the ZLB, our new policy rate conveys important and economically meaningful information; while the conventional policy rate gets stuck around zero.

### 2.5.2 Forward guidance

Since December 2008, the federal funds rate has been restricted by the ZLB. The conventional monetary policy is no longer effective, because the Federal Reserve cannot further decrease the federal funds rate below zero to boost the econ-
omy. Consequently, the central bank has resorted to a sequence of unconventional monetary policy tools. One prominent example is forward guidance, or central bank communications with the public about the future federal funds rate. In particular, forward guidance aims to lower the market’s expectation regarding the future short rate. Market expectations about future short rates feed back through the financial market to affect the current yield curve, especially at the longer end. Lower long term interest rates in turn stimulate aggregate demand. The Federal Reserve has made considerable use of forward guidance since the federal funds rate first hit the ZLB. In Table 2.4, we summarize a list of forward guidance quotes, when the Fed expected a different lift-off date or condition for the ZLB. Some of these dates overlap with Woodford (2012). The wording focuses either on (i) the length of the ZLB, or (ii) the target unemployment rate. Section 2.5.2 compares the length of the ZLB prescribed by forward guidance and the market’s expectation from our model. Section 2.5.2 studies the impact of forward guidance on the unemployment rate.

**ZLB duration**

One focus of forward guidance is for the Federal Reserve to implicitly or explicitly communicate with the general public about how long it intends to keep the federal funds rate near zero, as demonstrated in Table 2.4. For example, in the earlier FOMC statements in late 2008 and early 2009, they used phrases such as “some time” and “an extended period”. Later on, starting from late 2011, the Federal Reserve decided to be more transparent and specific about forward guidance. In each statement, they unambiguously revised the date, on which they expected the ZLB to end, according to the development of the overall economy.

Our model implies a closely related concept: the ZLB duration. It measures the market’s perception of when the economy will finally escape from the ZLB. This is a random variable defined as

$$\tau_t \equiv \inf\{\tau_t \geq 0 | s_{t+\tau} \geq r\}.$$
Thus $\tau_t$ represents how much time passes before the shadow rate first crosses the lower bound from below. At time $t$, $s_{t+\tau}$ is unknown. We simulate out $N = 10000$ paths of the future shadow rate given the information at time $t$. Every simulated path generates an estimate of $\tau_t$. Therefore, we have a distribution of $\tau_t$, and we take the median across $N$ simulations as our measure of the market’s expected ZLB duration.

We summarize the time series of the market’s expected ZLB duration in Figure 2.10 as the difference between the blue dots and dashed 45 degree line. The duration increased since early 2009 and kept above the two-and-a-half-year level from late-2011 to mid-2013, when it plummeted to around one year and a half. Since then, it has been between one and a half to two years. We highlight four different months: August 2011, January 2012, September 2012 and June 2013. They correspond to those dates when the Fed explicitly spelled out the ZLB lift-off dates (see Table 2.4). On August 9, 2011, the Federal Reserve promised to keep the rate low “at least through mid-2013”. The market anticipated this development one month ahead. When the lift-off date was postponed to “at least through late 2014” on January 25, 2012, the market expected the ZLB to last another three years. The two expectations overlap each other. On September 13, 2012, the forward guidance further extended the lift-off date to “at least through mid-2015”, the market’s expected duration increased to three and a half years. On June 19, 2013, Federal Reserve Board Chairman Ben Bernanke expressed in a press conference the Federal Reserve’s plan to maintain accommodative monetary policy until 2015 based on the economic outlook at that time. Following his remarks, the market’s expected lift-off date jump right on top of Bernanke’s expectation.11

Overall, evidence suggests that forward guidance and the market’s expectation align well. The market seems to adjust towards the Fed’s announcements ahead of time. For multiple occasions, the two expectations overlapped each other. In the next section, we will use the expected ZLB duration as a proxy for forward

---

10 Similar to Bauer and Rudebusch (2013), we use the $Q$ parameters for simulation, because (i) $Q$ is the probability measure reflected in assets price, and (ii) $Q$ parameters are estimated with much more precision than $P$ parameters (see discussion in Creal and Wu (2013) for example).

11 The results look very similar if we use real time duration instead, i.e., compute the ZLB duration at time $t$ using only data up to $t$. 
guidance, and study its impact on the real economy, especially the unemployment rate.

**Impact on unemployment**

We have demonstrated that forward guidance has achieved a great success in guiding the market’s expectation and influencing the yield curve. The ultimate question central bankers and economists care about is whether forward guidance is as successful in terms of its impact on the real economy, especially unemployment. We phrase this question in a FAVAR(1) framework with the expected ZLB duration measuring the monetary policy, and use this tool to study the transmission mechanism of forward guidance. For the macroeconomic factors, we keep them as they were. Figure 2.11 shows the impulse responses to a shock to the expected ZLB duration of one year for the same set of variables. Overall, in response to an easing of monetary policy, the economy starts to expand. Most interestingly, a one year increase in the expected ZLB duration translates into a 0.25% decrease in the unemployment rate, although the impulse response is not statistically significant at 10% level.

A simple calculation suggests that a one year increase in the expected ZLB duration has roughly the same effect on the macroeconomy as a 35 basis-point decrease in the policy rate. The visual comparison is in Figure 2.12, where the blue part is identical to Figure 2.11, and the turquoise portion is 35/25 times the turquoise in figure 2.9. Figure 2.12 suggests that in response to a one year shock to the expected ZLB duration, or a negative 35 basis-point shock to the policy rate, capacity utilization goes up by 0.6%, unemployment rate decreases by 0.25% and housing starts is about 5% over its steady state.

### 2.6 Conclusion

We have developed an analytical approximation for the forward rate in the SRTSM, making the otherwise complicated model extremely tractable. The SRTSM is an excellent description of the data especially when the economy is
at the ZLB, with the approximation error being only a couple of basis points. We used the shadow rate from the SRTSM to construct a new measure for the monetary policy stance when the effective federal funds rate is bounded below by zero, and employed this measure to study unconventional monetary policy’s impact on the real economy. We have found that our policy rate impacts the real economy since July 2009 in a similar fashion as the effective federal funds rate did before the Great Recession. An expansionary monetary policy shock boosts the real economy. More specifically, at the ZLB, in response to a negative 35 basis-point shock to the policy rate, the unemployment rate decreases by 0.25%. This quantity is equivalent to a one year extension of the expected ZLB period, prescribed by forward guidance. Our historical decomposition has found that the efforts by the Federal Reserve to stimulate the economy since July 2009 succeeded in making the unemployment rate in December 2013 0.13% lower than it otherwise would have been. The continuation in our policy rate series provides empirical researchers, who used effective federal funds rate in a VAR to study monetary policy in the macroeconomy, a tool to update their historical analysis. It also has potential applications in other areas in macroeconomics, such as dynamic stochastic general equilibrium models.

2.7 Appendix

2.7.1 Approximation to Forward rates

Define

\[ a_n \equiv \delta_0 + \delta'_1 \left( \sum_{j=0}^{n-1} (\rho Q)^j \right) \mu Q, \]

\[ a_n \equiv \bar{a}_n - \frac{1}{2} \delta'_1 \left( \sum_{j=0}^{n-1} (\rho Q)^j \right) \Sigma \Sigma' \left( \sum_{j=0}^{n-1} (\rho Q)^j \right)' \delta_1, \]

\[ b'_n \equiv \delta'_1 (\rho Q)^n. \]
Shadow rate  The shadow rate is affine in the state variables. Under the risk neutral measure, it is conditionally normally distributed. The conditional mean is

$$E_t^Q [s_{t+n}] = \bar{a}_n + \beta'_n X_t,$$

the conditional variance is

$$\text{Var}_t^Q [s_{t+n}] \equiv (\sigma_n^Q)^2 = \sum_{j=0}^{n-1} \delta_j^1 (\rho_j^Q)^j \Sigma_j \Sigma_j' (\rho_j^Q)^j \delta_1,$$

and

$$\frac{1}{2} \left( \text{Var}_t^Q \left[ \sum_{j=1}^{n} s_{t+j} \right] - \text{Var}_t^Q \left[ \sum_{j=1}^{n-1} s_{t+j} \right] \right) = \bar{a}_n - a_n.$$

SRTSM  We start the derivation of equation (2.6) with the following approximation: \( \log \left( E [e^Z] \right) \approx E [Z] + \frac{1}{2} \text{Var} [Z] \) for any random variable \( Z \). This approximation uses Taylor series expansions for the exponential and natural logarithm functions. For the special case of a Gaussian random variable \( Z \), this approximation is exact. Then the forward rate between \( t+n \) and \( t+n+1 \) can be approximated as follows:

$$f_{n,n+1,t}^{SRTSM} = (n+1)y_{n+1,t} - ny_{nt}$$

$$= -\log \left( e^{-r_{n} E_t^Q \left[ e^{-\sum_{j=1}^{n} r_{t+j}} \right]} \right) + \log \left( e^{-r_{n} E_t^Q \left[ e^{-\sum_{j=1}^{n-1} r_{t+j}} \right]} \right)$$

$$\approx E_t^Q \left[ \sum_{j=1}^{n} r_{t+j} \right] - \frac{1}{2} \text{Var}_t^Q \left[ \sum_{j=1}^{n} r_{t+j} \right] - E_t^Q \left[ \sum_{j=1}^{n-1} r_{t+j} \right] + \frac{1}{2} \text{Var}_t^Q \left[ \sum_{j=1}^{n-1} r_{t+j} \right]$$

$$= E_t^Q \left[ r_{t+n} \right] - \frac{1}{2} \left( \text{Var}_t^Q \left[ \sum_{j=1}^{n} r_{t+j} \right] - \text{Var}_t^Q \left[ \sum_{j=1}^{n-1} r_{t+j} \right] \right). \tag{2.17}$$
We calculate the first term $E_t^Q [r_{t+n}]$ analytically:

$$
E_t^Q [r_{t+n}] \\
= E_t^Q [\max (r, s_{t+n})] \\
= P_t^Q [s_{t+n} < r] \times r + P_t^Q [s_{t+n} \geq r] \times E_t^Q [s_{t+n} | s_{t+n} \geq r] \\
= r + \sigma_n^Q \left( \left( \frac{a_n + b'_n X_t - r}{\sigma_n^Q} \right) \Phi \left( \frac{a_n + b'_n X_t - r}{\sigma_n^Q} \right) + \phi \left( \frac{a_n + b'_n X_t - r}{\sigma_n^Q} \right) \right) \\
= r + \sigma_n^Q g \left( \frac{a_n + b'_n X_t - r}{\sigma_n^Q} \right). \quad (2.18)
$$

Using the second moments for the truncated normal distribution, we have the following approximations for the conditional variance and covariance (see details in 2.7.1):

$$
\text{Var}_t^Q [r_{t+n}] \\
\approx P_t^Q [s_{t+n} \geq r] \text{Var}_t^Q [s_{t+n}], \quad (2.19) \\
\text{Cov}_t^Q [r_{t+n-j}, r_{t+n}] \\
\approx P_t^Q [s_{t+n-j} \geq r, s_{t+n} \geq r] \text{Cov}_t^Q [s_{t+n-j}, s_{t+n}], \quad \forall j = 1, ..., n - 1. \quad (2.20)
$$

Next, we take the approximation

$$
P_t^Q [s_{t+n-j} \geq r | s_{t+n} \geq r] \approx 1,
$$

using the fact that the shadow rate is very persistent. Equation (2.20) becomes

$$
\text{Cov}_t^Q [r_{t+n-j}, r_{t+n}] \approx P_t^Q [s_{t+n} \geq r] \text{Cov}_t^Q [s_{t+n-j}, s_{t+n}].
$$
Then, the second term in equation (2.17) is

\[
\frac{1}{2} \left( \text{Var}_t^Q \left[ \sum_{j=1}^n r_{t+j} \right] - \text{Var}_t^Q \left[ \sum_{j=1}^{n-1} r_{t+j} \right] \right) \\
\approx \ \text{Pr}_t^Q (s_{t+n} \geq r) \times \frac{1}{2} \left( \text{Var}_t^Q \left[ \sum_{j=1}^n s_{t+j} \right] - \text{Var}_t^Q \left[ \sum_{j=1}^{n-1} s_{t+j} \right] \right)
\]

\[
= \Phi \left( \frac{\bar{a}_n + b'_n X_t - r}{\sigma_n^Q} \right) \times (\bar{a}_n - a_n).
\]

(2.21)

Plug equations (2.18) and (2.21) to (2.17), we conclude our derivation for equation (2.6) with another first-order Taylor approximation:

\[
f_{n,n+1,t}^{\text{SRTSM}} \approx r + \sigma_n^Q g \left( \frac{\bar{a}_n + b'_n X_t - r}{\sigma_n^Q} \right) + \Phi \left( \frac{\bar{a}_n + b'_n X_t - r}{\sigma_n^Q} \right) \times (\bar{a}_n - a_n)
\]

\[
= r + \sigma_n^Q g \left( \frac{\bar{a}_n + b'_n X_t - r}{\sigma_n^Q} \right) + \sigma_n^Q \frac{\partial g}{\partial \bar{a}_n} \left( \frac{\bar{a}_n + b'_n X_t - r}{\sigma_n^Q} \right) \times (\bar{a}_n - a_n)
\]

\[
\approx r + \sigma_n^Q g \left( \frac{a_n + b'_n X_t - r}{\sigma_n^Q} \right).
\]

(2.22)

**GATSM** In the GATSM, the forward rate between \( t + n \) and \( t + n + 1 \) is priced as follows

\[
f_{n,n+1,t}^{\text{GATSM}}
= (n + 1)y_{n+1,t} - ny_{nt}
\]

\[
= \log \left( e^{-s_t E_t^Q \left[ e^{-\sum_{j=1}^n s_{t+j}} \right]} \right) + \log \left( e^{-s_t E_t^Q \left[ e^{-\sum_{j=1}^{n-1} s_{t+j}} \right]} \right)
\]

\[
= E_t^Q \left[ \sum_{j=1}^n s_{t+j} \right] - \frac{1}{2} \text{Var}_t^Q \left[ \sum_{j=1}^n s_{t+j} \right] - E_t^Q \left[ \sum_{j=1}^{n-1} s_{t+j} \right] + \frac{1}{2} \text{Var}_t^Q \left[ \sum_{j=1}^{n-1} s_{t+j} \right]
\]

\[
= E_t^Q \left[ s_{t+n} \right] - \frac{1}{2} \left( \text{Var}_t^Q \left[ \sum_{j=1}^n s_{t+j} \right] - \text{Var}_t^Q \left[ \sum_{j=1}^{n-1} s_{t+j} \right] \right)
\]

\[
= \bar{a}_n + b'_n X_t + a_n - \bar{a}_n
\]

\[
= a_n + b'_n X_t.
\]
Approximations to variance and covariance

Define

\[ \tilde{s}_{t+n} = \frac{s_{t+n} - \mathbb{E}_t^Q [s_{t+n}]}{\sigma_n^Q} \quad \text{and} \quad \alpha_{nt} = \frac{r - \mathbb{E}_t^Q [s_{t+n}]}{\sigma_n^Q}, \]

then \( r_{t+n} = \sigma_n^Q \tilde{r}_{t+n} + \mathbb{E}_t^Q [s_{t+n}] \), where \( \tilde{r}_{t+n} \equiv \max (\tilde{s}_{t+n}, \alpha_{nt}) \).

**Variance** Standard results for the truncated normal distribution states that if \( x \sim N(0, 1) \), then (i) \( \text{Pr} [x \geq \alpha] = 1 - \Phi (\alpha) \), (ii) \( \text{Pr} [x \geq \alpha] \mathbb{E} [x | x \geq \alpha] = \phi (\alpha) \), and (iii) \( \text{Pr} [x \geq \alpha] \mathbb{E} [x^2 | x \geq \alpha] = 1 - \Phi (\alpha) + \alpha \phi (\alpha) \). Since \( \tilde{s}_{t+n} \) is conditionally normally distributed with mean 0 and variance 1 under the \( Q \) measure,

\[
\mathbb{E}_t^Q [\tilde{r}_{t+n}] = \text{Pr}_t^Q [\tilde{s}_{t+n} \geq \alpha_{nt}] \mathbb{E}_t [\tilde{s}_{t+n} | \tilde{s}_{t+n} \geq \alpha_{nt}] + \text{Pr}_t^Q [\tilde{s}_{t+n} < \alpha_{nt}] \alpha_{nt} \\
= \phi (\alpha_{nt}) + \alpha_{nt} \Phi (\alpha_{nt}), \quad (2.23)
\]

\[
\mathbb{E}_t^Q [\tilde{r}_{t+n}^2] = \text{Pr}_t^Q [\tilde{s}_{t+n} \geq \alpha_{nt}] \mathbb{E}_t [\tilde{s}_{t+n}^2 | \tilde{s}_{t+n} \geq \alpha_{nt}] + \text{Pr}_t^Q [\tilde{s}_{t+n} < \alpha_{nt}] \alpha_{nt}^2 \\
= 1 - \Phi (\alpha_{nt}) + \alpha_{nt} \phi (\alpha_{nt}) + \alpha_{nt}^2 \Phi (\alpha_{nt}).
\]

Accordingly,

\[
\text{Var}_t^Q [r_{t+n}] = (\sigma_n^Q)^2 \text{Var}_t^Q [\tilde{r}_{t+n}] = (\sigma_n^Q)^2 \left( \mathbb{E}_t [\tilde{r}_{t+n}^2] - (\mathbb{E}_t [\tilde{r}_{t+n}])^2 \right) \\
= (\sigma_n^Q)^2 \left( 1 - \Phi (\alpha_{nt}) + \alpha_{nt} \phi (\alpha_{nt}) + \alpha_{nt}^2 \Phi (\alpha_{nt}) - (\phi (\alpha_{nt}) + \alpha_{nt} \Phi (\alpha_{nt}))^2 \right), \quad (2.24)
\]

Comparing the exact formula in equation (2.24) with the approximation in equation (2.19), or \( \text{Var}_t^Q (r_{t+n}) \approx \text{Pr}_t^Q [s_{t+n} \geq r] \text{Var}_t^Q [s_{t+n}] = (\sigma_n^Q)^2 (1 - \Phi (\alpha_{nt})) \), the approximation error is

\[
(\sigma_n^Q)^2 \times \left\{ (1 - \Phi (\alpha_{nt}) + \alpha_{nt} \phi (\alpha_{nt}) + \alpha_{nt}^2 \Phi (\alpha_{nt}) - (\phi (\alpha_{nt}) + \alpha_{nt} \Phi (\alpha_{nt}))^2 \right) \\
- (1 - \Phi (\alpha_{nt}))) \}
= - (\sigma_n^Q)^2 g (\alpha_{nt}) g (-\alpha_{nt}) \equiv (\sigma_n^Q)^2 D (\alpha_{nt}).
\]
The first derivative of $D(\alpha_{nt})$ is $D'(\alpha_{nt}) = -g'(\alpha_{nt})g(-\alpha_{nt}) + g(\alpha_{nt})g'(-\alpha_{nt})$, and $D'(\alpha_{nt})|_{\alpha_{nt}=0} = 0$. Therefore $D(0)$ is a local maximum/minimum. From Figure 2.13, $D(\cdot)$ is bounded by 0 from above and achieves the global minimum at $\alpha_{nt} = 0$. Therefore, the absolute approximation error is bounded by a small number $(\sigma_{Qn})^2 \phi(0)^2$.

**Covariance** Standard results for the multivariate truncated normal distribution states that if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \sim N \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right)$, then

(i) $\Pr[x_1 \geq \alpha_1, x_2 \geq \alpha_2] = F(-\alpha_1, -\alpha_2; \rho)$,

(ii) $\Pr[x_1 \geq \alpha_1, x_2 \geq \alpha_2] \mathbb{E}[x_1|x_1 \geq \alpha_1, x_2 \geq \alpha_2] = h(\alpha_1, \alpha_2, \rho) + \rho h(\alpha_2, \alpha_1, \rho)$,

(iii) $\Pr[x_1 \geq \alpha_1, x_2 \geq \alpha_2] \mathbb{E}[x_1x_2|x_1 \geq \alpha_1, x_2 \geq \alpha_2]
\begin{align*}
&= \rho (\alpha_1 h(\alpha_1, \alpha_2; \rho) + \alpha_2 h(\alpha_2, \alpha_1; \rho) + F(-\alpha_1, -\alpha_2; \rho)) \\
&+ (1 - \rho^2) f(\alpha_1, \alpha_2; \rho),
\end{align*}$

where

\[ f(x_1, x_2; \rho) \equiv \lambda (2\pi)^{-1} \exp \left\{ -\frac{1}{2} \lambda^2 (x_1^2 - 2\rho x_1 x_2 + x_2^2) \right\}, \]

\[ F(\alpha_1, \alpha_2; \rho) \equiv \int_{-\infty}^{\alpha_1} \int_{-\infty}^{\alpha_2} f(x_1, x_2; \rho) dx_1 dx_2, \]

\[ h(\alpha_1, \alpha_2; \rho) \equiv \phi(\alpha_1) \Phi(\lambda (\rho \alpha_1 - \alpha_2)), \]

\[ \lambda \equiv (1 - \rho^2)^{-\frac{1}{2}}. \]
Let $\rho_{mnt}$ be the correlation between $\tilde{s}_{t+m}$ and $\tilde{s}_{t+n}$ under the Q measure, then,

$$
\mathbb{E}_t^Q[\tilde{r}_{t+m}\tilde{r}_{t+n}]
= \mathbb{E}_t^Q[\tilde{s}_{t+m}\tilde{s}_{t+n}] \mathbb{E}_t^Q[\tilde{r}_{t+m}] \mathbb{E}_t^Q[\tilde{r}_{t+n}]
= (\phi(\alpha_{mt}) + \alpha_{m}\phi(\alpha_{mt})) (\phi(\alpha_{nt}) + \alpha_{n}\phi(\alpha_{nt})) .
$$

With the identity $h(\alpha_1, \alpha_2; \rho) = h(-\alpha_1, \alpha_2; -\rho)$, we simplify the expression above as follows:

$$
\mathbb{E}_t^Q[\tilde{r}_{t+m}\tilde{r}_{t+n}]
= \rho_{mnt} F(-\alpha_{mt}, -\alpha_{nt}; \rho_{mnt}) + (1 - \rho_{mnt}^2) f(\alpha_{mt}, \alpha_{nt}; \rho_{mnt})
+ \alpha_{mt} h(-\alpha_{nt}; -\rho_{mnt}) - \rho_{mnt} h(-\alpha_{mt}, \alpha_{nt}; -\rho_{mnt})
+ \alpha_{nt} h(-\alpha_{mt}; -\rho_{mnt}) - \rho_{mnt} h(-\alpha_{nt}, \alpha_{mt}; -\rho_{mnt})
+ \alpha_{mt}\alpha_{nt} F(\alpha_{mt}, \alpha_{nt}; \rho_{mnt}) .
$$

From equation (2.23), we have

$$
\mathbb{E}_t^Q[\tilde{r}_{t+m}] \mathbb{E}_t^Q[\tilde{r}_{t+n}]
= \left( \phi(\alpha_{mt}) + \alpha_{m}\phi(\alpha_{mt}) \right) \left( \phi(\alpha_{nt}) + \alpha_{n}\phi(\alpha_{nt}) \right) .
$$

Accordingly,

$$
\text{Cov}_t^Q[r_{t+m}, r_{t+n}]
= \sigma_m^Q \sigma_n^Q \text{Cov}_t^Q[\tilde{r}_{t+m}, \tilde{r}_{t+n}]
= \sigma_m^Q \sigma_n^Q \left( \mathbb{E}_t^Q[\tilde{r}_{t+m}\tilde{r}_{t+n}] - \mathbb{E}_t^Q[\tilde{r}_{t+m}] \mathbb{E}_t^Q[\tilde{r}_{t+n}] \right) .
$$

(2.25)
Comparing the exact formula in equation (2.25) with the approximation in equation (2.20), the approximation error is

\[
\sigma_m^Q \sigma_n^Q \times \left\{ \left(1 - \rho_{mnt}^2\right) f(\alpha_{mt}, \alpha_{nt}; \rho_{mnt}) + \alpha_{mt} h(\alpha_{nt}, -\alpha_{mt}; -\rho_{mnt}) + \alpha_{nt} h(\alpha_{mt}, -\alpha_{nt}; -\rho_{mnt}) + \alpha_{mt}\alpha_{nt} F(\alpha_{mt}, \alpha_{nt}; \rho_{mnt}) - (\phi(\alpha_{mt}) + \alpha_{mt}\Phi(\alpha_{mt}))(\phi(\alpha_{nt}) + \alpha_{nt}\Phi(\alpha_{nt})) \right\} \\
\equiv \sigma_m^Q \sigma_n^Q D(\alpha_{mt}, \alpha_{nt}; \rho_{mnt}).
\]

The first derivative of \(D(\alpha_{mt}, \alpha_{nt}; \rho_{mnt})\) with respect to \(\alpha_{mt}\) is

\[
\frac{\partial D(\alpha_{mt}, \alpha_{nt}; \rho_{mnt})}{\partial \alpha_{mt}} = - (\alpha_{mt} - \rho_{mnt}\alpha_{nt}) f(\alpha_{mt}, \alpha_{nt}; \rho_{mnt}) \]

\[
+ h(\alpha_{nt}, -\alpha_{mt}; -\rho_{mnt}) + \lambda_{mnt} \alpha_{mt} \phi(\alpha_{nt}) \phi(\lambda_{mnt} (-\rho_{mnt}\alpha_{nt} + \alpha_{mt})) \]

\[- \alpha_{nt} \alpha_{mt} \Phi(\alpha_{mt}) \Phi(\lambda_{mnt} (-\rho_{mnt}\alpha_{nt} + \alpha_{mt})) \]

\[- \lambda_{mnt} \rho_{mnt} \alpha_{mt} \phi(\alpha_{mt}) \phi(\lambda_{mnt} (-\rho_{mnt}\alpha_{nt} + \alpha_{nt})) \]

\[+ \alpha_{nt} F(\alpha_{mt}, \alpha_{nt}; \rho_{mnt}) + \alpha_{mt} \alpha_{nt} h(a_{mt}, -\alpha_{nt}; -\rho_{mnt}) \]

\[ - \Phi(\alpha_{mt}) (\phi(\alpha_{nt}) + \alpha_{nt} \Phi(\alpha_{nt})) \],

where

\[
\lambda_{mnt} = \left(1 - \rho_{mnt}^2\right)^{-\frac{1}{2}}.
\]

And

\[
\frac{\partial D(\alpha_{mt}, \alpha_{nt}; \rho_{mnt})}{\partial \alpha_{mt}} \bigg|_{\alpha_{mt} = 0, \alpha_{nt} = 0} = \phi(0) \Phi(0) - \phi(0) \Phi(0) = 0.
\]

Since

\[
D(\alpha_{mt}, \alpha_{nt}; \rho_{mnt}) = D(\alpha_{nt}, \alpha_{mt}; \rho_{mnt}),
\]

we have

\[
\frac{\partial D(\alpha_{mt}, \alpha_{nt}; \rho_{mnt})}{\partial \alpha_{nt}} \bigg|_{\alpha_{mt} = 0, \alpha_{nt} = 0} = 0 \text{ as well.}
\]

Thus, \(D(0, 0; \rho_{mnt})\) is a local maximum/minimum. We plot \(D(\alpha_{mt}, \alpha_{nt}; \rho_{mnt})\) for \(\rho_{mnt} = -0.9, -0.8, \ldots, 0.8, 0.9\) in Figure 2.14, and \(D(\alpha_{mt}, \alpha_{nt}; \rho)\) is bounded by 0
from above and achieves the global minimum at $\alpha_{nt} = 0$. Therefore, the absolute approximation error is bounded by a small number, $\sigma_m^Q \sigma_n^Q \left( 1 - (1 - \rho_{mnt}^2)^{\frac{1}{2}} \right) \phi^2(0)$.

2.7.2 Kalman filters

Extended Kalman filter for the SRTSM The transition equation is in (3.2).

Stack the observation equation in (2.10) for all 7 maturities, we get the following system:

$$F_{t+1}^0 = G(X_{t+1}) + \eta_{t+1} \quad \eta_{t+1} \sim N(0, \omega I_7).$$

Approximate the conditional distribution of $X_t$ with $X_t | F_{1:t}^0 \sim N(\hat{X}_t|t, P_t|t)$. Update $\hat{X}_{t+1|t+1}$ and $P_{t+1|t+1}$ as follows:

$$\hat{X}_{t+1|t+1} = \hat{X}_{t+1|t} + K_{t+1}(F_{t+1}^0 - \hat{F}_{t+1|t}^0),$$
$$P_{t+1|t+1} = (I_3 - K_{t+1} H_{t+1}') P_{t+1|t},$$
$$\hat{X}_{t+1|t} = \mu + \rho \hat{X}_{t|t},$$
$$P_{t+1|t} = \rho P_{t|t} \rho' + \Sigma \Sigma'.$$

with the matrices defined as

$$\hat{F}_{t+1|t}^0 = G(\hat{X}_{t+1|t}),$$
$$H_{t+1} = \left( \frac{\partial G(X_{t+1})}{\partial X_{t+1}'} \bigg|_{X_{t+1} = \hat{X}_{t+1|t}} \right)',$$
$$K_{t+1} = P_{t+1|t} H_{t+1} \left( H_{t+1}' P_{t+1|t} H_{t+1} + \omega I_7 \right)^{-1},$$

where we can obtain $H_{t+1}'$ by stacking $\Phi \left( \frac{a_n + b_n' \hat{X}_{t+1|t} - \mu}{\sigma_n^2} \right) \times b_n'$ for the 7 maturities. Given the initial values $\hat{X}_{0|0}$ and $P_{0|0}$, we can update $\{\hat{X}_{t|t}, P_{t|t}\}_{t=1}^T$ recursively with
the above algorithm. The log likelihood is

\[
\mathcal{L} = -\frac{7T}{2}\log 2\pi - \frac{1}{2} \sum_{t=1}^{T} \log |H_{t}^r P_{t+1} P^{-1}_{t} + \omega I_7| \\
- \frac{1}{2} \sum_{t=1}^{T} (F_t^o - G(\hat{X}_{t|t-1}))' (H_{t}^r P_{t+1} P^{-1}_{t} H_t + \omega I_7)^{-1} (F_t^o - G(\hat{X}_{t|t-1})).
\]

**Kalman filter for the GATSM** The GATSM is a linear Gaussian state space model. The \(G(.)\) function stacks the linear function in equation (2.11). The matrix \(H_{t+1}^r\) stacks \(b_{n}'\) for the 7 maturities. The algorithm described above collapses to a Kalman filter.

### 2.7.3 Factor construction for the FAVAR

This appendix illustrates how to construct the macro factors. First, extract the first 3 principal components \(\hat{p}c_t\) from \(Y_t^m\). Then extract first 3 principal components \(\hat{p}c_t^*\) from the slowing-moving variables indicated with “*” in Table 2.3. Normalize them to unit variance. Next, run the following regression \(\hat{p}c_t = b_{pc} \hat{p}c_t^* + b_{pc,s} s_t^o + \eta_{pc}^t\), and construct \(\hat{x}_t^m\) from \(\hat{p}c_t - \hat{b}_{pc,s} s_t^o\). We then estimate equation (2.13) as follows. If \(Y_t^{m,i}\) is among the slow-moving variables, we regress \(Y_t^{m,i}\) on a constant and \(\hat{x}_t^m\) to obtain \(\hat{a}_{m,i}\) and \(\hat{b}_{x,i}\) and set \(\hat{b}_{s,i} = 0\). For other variables, we regress \(Y_t^{m,i}\) on a constant, \(\hat{x}_t^m\) and \(s_t^o\) to get \(\hat{a}_{m,i}\), \(\hat{b}_{x,i}\) and \(\hat{b}_{s,i}\).
2.8 Figures

**Figure 2.1:** The function $g(.)$
Blue curve: the function $g(z) = z\Phi(z) + \phi(z)$. Red dashed line: the 45-degree line.

**Figure 2.2:** One-month forward rates
Sample: January 1990 to December 2013. Rates are measured in annualized percentage points. Maturities are 3 and 6 months, 1, 2, 5, 7 and 10 years. The gray area marks the ZLB period from January 2009 to December 2013.
Figure 2.3: Average observed and fitted forward curves in 2012
Blue: fitted forward curves, from the SRTSM in the left panel and the GATSM in the right panel. Red dots: observed data. X-axis: maturity in years.

Figure 2.4: The estimated shadow rate v.s. the effective fed funds rate
Figure 2.5: Loadings of standardized economic variables $Y^m_t$ on the three macroeconomic factors and the standardized policy rate
X-axis: identification number for economic variables in Table 2.3.

Figure 2.6: Observed and counterfactual macroeconomic variables
Blue lines: observed economic variables between July 2009 and December 2013.
Red dashed lines: what would have happened to these macroeconomic variables, if all the monetary policy shocks were shut down. Green dashed lines: what would have happened if the shadow rate was kept at $r$. 
Figure 2.7: Impulse responses to a -25 basis-point shock on monetary policy 90% confidence intervals are shaded. Sample: January 1960 - December 2013. Model: FAVAR with 3 macro factors and 13 lags. X-axis: response time in months. The policy rate is measured in annualized percentage; the industrial production index, consumer price index and housing starts are measured in percentage deviation from the steady state; the capacity utilization and unemployment rate are measured in percentage point.
Figure 2.8: Impulse responses to a -25 basis-point shock on monetary policy 90% confidence intervals are shaded. Blue: full sample from January 1960 to December 2013 with the policy rate in FAVAR (13). Turquoise: ZLB from July 2009 to December 2013 with the effective federal funds rate in FAVAR (1). X-axis: response time in months. The policy rate is measured in annualized percentage; the industrial production index, consumer price index and housing starts are measured in percentage deviation from the steady state; the capacity utilization and unemployment rate are measured in percentage point.
Figure 2.9: Impulse responses to a -25 basis-point shock on monetary policy 90% confidence intervals are shaded. Blue: full sample from January 1960 to December 2013 with the policy rate in FAVAR (13). Turquoise: ZLB from July 2009 to December 2013 with the policy rate in a FAVAR (1). X-axis: response time in months. The policy rate is measured in annualized percentage; the industrial production index, consumer price index and housing starts are measured in percentage deviation from the steady state; the capacity utilization and unemployment rate are measured in percentage point.
Figure 2.10: The market’s expected vs. Fed’s announced ZLB lift-off dates
Blue dots: the market’s expected lift-off dates from January 2009 to December 2013. Four green vertical lines mark the following months when forward guidance specified explicit lift-off dates for the ZLB: August 2011, January 2012, September 2012 and June 2013. The corresponding lift-off dates are in red dots. Black dashed line: the 45 degree line.
Figure 2.11: Impulse responses to a one year shock to expected ZLB duration 90% confidence intervals are shaded. Sample: ZLB from July 2009 to December 2013. Model: FAVAR (1) with the ZLB duration as the monetary policy measure. X-axis: response time in months. The expected duration is measured in year; the industrial production index, consumer price index and housing starts are measured in percentage deviation from the steady state; the capacity utilization and unemployment rate are measured in percentage point.
Figure 2.12: Impulse responses at ZLB (policy rate v.s. ZLB duration)
Turquoise: impulse responses to a -35 basis-point shock on the policy rate. Blue: impulse responses to a one year shock on the ZLB duration. 90% confidence intervals are shaded. Sample: ZLB from July 2009 to December 2013. Model: FAVAR (1). X-axis: response time in months. The policy rate is measured in -35 basis points; the expected duration is measured in year; the industrial production index, consumer price index and housing starts are measured in percentage deviation from the steady state; the capacity utilization and unemployment rate are measured in percentage point.

Figure 2.13: $D(\alpha_{nt})$
Figure 2.14: \( D(\alpha_{mt}, \alpha_{nt}; \rho_{mnt}) \)
## 2.9 Tables

**Table 2.1: Maximum likelihood estimates with robust standard errors**

Maximum likelihood estimates for the three-factor SRTSM and the three-factor GATSM with robust standard errors in parentheses. Sample: January 1990 to December 2013.

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<th>SRTSM</th>
<th>GATSM</th>
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<td>(0.1464)</td>
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<td></td>
<td>(0.0202)</td>
<td>(0.0185)</td>
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<tr>
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<td>0.0030</td>
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<tr>
<td></td>
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<td>(0.0015)</td>
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<tr>
<td>eig(ρ)</td>
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<td>0.9870</td>
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<tr>
<td></td>
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<td>(0.0015)</td>
</tr>
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<td>ρQ</td>
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<tr>
<td></td>
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<td>(0.0003)</td>
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<tr>
<td></td>
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<td>0.9503</td>
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<tr>
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<td>(0.0012)</td>
</tr>
<tr>
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<td>11.6760</td>
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Log likelihood value | 855.5743 | 755.4587
Table 2.2: Approximation error

Differences in forward rates and yields implied by equation (2.6) and by simulation for the 24 Januaries between 1990 and 2013. At time $t$, we simulate 10 million paths of $s_{t+j}$ for $j = 1, \ldots, 120$ with the estimated factors $X_t$ and $Q$ parameters, and compute $r_{t+j}$ based on equation (2.1). Then we compute the corresponding 10 million $y_{nt} = \frac{1}{n}\log \left( E_t^Q[\exp(-r_t - r_{t+1} - \ldots - r_{t+n-1})]\right)$ and $f_{n,n+1,t} = (n + 1)y_{n+1,t} - ny_{nt}$. We take the average of the 10 million draws as the simulated yield or forward rate. All numbers are measured in basis points.

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Table 2.2: Approximation error, continued

Differences in forward rates and yields implied by equation (2.6) and by simulation for the 24 Januaries between 1990 and 2013. At time $t$, we simulate 10 million paths of $s_{t+j}$ for $j = 1, ..., 120$ with the estimated factors $X_t$ and $Q$ parameters, and compute $r_{t+j}$ based on equation (2.1). Then we compute the corresponding 10 million $y_{nt} = -\frac{1}{n} \log \left( E_t^Q [\exp(-r_t - r_{t+1} - ... - r_{t+n-1})] \right)$ and $f_{n,n+1,t} = (n + 1)y_{n+1,t} - ny_{nt}$. We take the average of the 10 million draws as the simulated yield or forward rate. All numbers are measured in basis points.

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Table 2.3: Macroeconomic data

This table lists the mnemonics, short names and transformations for the 97 macroeconomic series used in the paper. All series are from the Global Insights Basic Economics Database. Slow-moving variables are marked with *.

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<th>Transformation</th>
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Table 2.3: Macroeconomic data, continued

This table lists the mnemonics, short names and transformations for the 97 macroeconomic series used in the paper. All series are from the Global Insights Basic Economics Database. Slow-moving variables are marked with ∗.

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| Stock prices                      |                                                |
| 59  | FSPCOM.M   | S&P’S COMMON STOCK PRICE INDEX, COMPOSITE (1941-43=10) | Δ/ln |
| 60  | FSPIN.M    | S&P’S COMMON STOCK PRICE INDEX, INDUSTRIALS (1941-43=10) | Δ/ln |

| Exchange rates                     |                                                |
| 61  | EXRUK.M    | FOREIGN EXCHANGE RATE. UNITED KINGDOM (CENTS PER POUND) | Δ/ln |
| 62  | EXRCAN.M   | FOREIGN EXCHANGE RATE. CANADA (CANADIAN $ PER U.S.$) | Δ/ln |

| Interest rates                     |                                                |
| 63  | FYFF.M     | INTEREST RATE. FEDERAL FUNDS (EFFECTIVE) (% PER ANNUM, NSA) |                |
| 64  | FYGM3.M    | INTEREST RATE. U.S. TREASURY BILLS, SEC Mkt. 3-MO ( % PER ANN, NSA) |                |
| 65  | FYGM6.M    | INTEREST RATE. U.S. TREASURY BILLS, SEC Mkt. 6-MO ( % PER ANN, NSA) |                |
| 66  | FYGT1.M    | INTEREST RATE. U.S. TREASURY CONST MATURITIES 1-YR, ( % PER ANN, NSA) |                |
| 67  | FYGT3.M    | INTEREST RATE. U.S. TREASURY CONST MATURITIES 3-YR, ( % PER ANN, NSA) |                |
| 68  | FYGT10.M   | INTEREST RATE. U.S. TREASURY CONST MATURITIES 10-YR. ( % PER ANN, NSA) |                |
| 71  | FYGT1.M-FYFF.M | SPREAD: FYGT1.M-FYFF.M |                |

| Money and credit quantity aggregates |                                                |
| 74  | ALCIBL00.M | COML & IND LOANS OUTST IN 2009 $, SA-US | Δ/ln |
| 75  | CCINRV.M   | CONSUMER CREDIT OUTSTANDING - NONREVOLVING(G19) | Δ/ln |
| 76  | FM1.M      | MONEY STOCK: M1(CURR,TRAV.CKS,DEM DEP,OTHER C/CABLE DEP) (BILLS,SA) | Δ/ln |
| 77  | FM2.M      | MONEY STOCK: M2(M1+O/NITE RPS,EURO$,G/P&B/D MAMF'S/SAV&SM TIME DEP) (BILLS,SA) | Δ/ln |
| 78  | MBASE.M    | REVISED MONETARY BASE-ADJUSTED. (FED RESERVE BANK-SAINT LOUIS), SA-US | Δ/ln |
| 79  | MNY2.M     | M2 - MONEY SUPPLY - M1 + SAVINGS DEPOSITS, SMALL TIME DEPOSITS, & MMMFS [H6], SA-US | Δ/ln |

| Price indexes                       |                                                |
| 80  | PSXCP.M    | NAPM COMMODITY PRICES INDEX (PERCENT)         | Δ/ln |
| 81  | PWYSA.M    | PRODUCER PRICE INDEX. FINISHED GOODS (82–100.SA) | Δ/ln |
| 82  | PWFCSA.M   | PRODUCER PRICE INDEX. CONSUMER GOODS (92–100.SA) | Δ/ln |
| 83  | PWDMSA.M   | PRODUCER PRICE INDEX. INTERMED MAT SUPPLIES & COMPONENTS (82–100.SA) | Δ/ln |
| 84  | PWCDSA.M   | PRODUCER PRICE INDEX. CRUDE MATERIALS (82–100.SA) | Δ/ln |
| 85  | PUNEW.M    | CPLU. ALL ITEMS (82–84–100.SA) | Δ/ln |
| 86  | PUS8.M     | CPLU. APPAREL & UPKEEP (82–84–100.SA) | Δ/ln |
| 87  | PUS4.M     | CPLU. TRANSPORTATION (82–84–100.SA) | Δ/ln |
| 88  | PUS5.M     | CPLU. MEDICAL CARE (82–84–100.SA) | Δ/ln |
| 89  | PUC.M      | CPLU. COMMODITIES (82–84–100.SA) | Δ/ln |
| 90  | PUCD.M     | CPLU. DURABLES (82–84–100.SA) | Δ/ln |
| 91  | PUSM.M     | CPLU. SERVICES (82–84–100.SA) | Δ/ln |
| 92  | PUXF.M     | CPLU. ALL ITEMS LESS FOOD (82–84–100.SA) | Δ/ln |
| 93  | PUXHS.M    | CPLU. ALL ITEMS LESS SHELTER (82–84–100.SA) | Δ/ln |
| 94  | PUXM.M     | CPLU. ALL ITEMS LESS MEDICAL CARE (82–84–100.SA) | Δ/ln |

| Average hourly earnings              |                                                |
| 95  | CES277.M   | AVG HRLY EARNINGS, PROD WKRS, NONFARM - CONSTRUCTION | Δ/ln |
| 96  | CES278.M   | AVG HRLY EARNINGS, PROD WKRS, NONFARM - MFG | Δ/ln |

| Miscellaneous                        |                                                |
| 97  | U0M083.M   | BUSINESS CYCLE INDICATORS. CONSUMER EXPECTATIONS, NSA |
Table 2.4: Forward guidance quotes

This table summarizes a list of forward guidance quotes, when the Fed expected a different lift-off date or condition for the ZLB. All quotes except the one on 6/19/2013 are from FOMC statements. The quote on 6/19/2013 is from Chairman Bernanke’s press conference. Asterisks mark the statements with explicit lift-off dates, with the corresponding lift-off dates in red. Source: http://www.federalreserve.gov/monetarypolicy/fomccalendars.htm.

<table>
<thead>
<tr>
<th>Date</th>
<th>Quotes</th>
</tr>
</thead>
<tbody>
<tr>
<td>12/16/2008</td>
<td>“...anticipates that weak economic conditions are likely to warrant exceptionally low levels of the federal funds rate for some time.”</td>
</tr>
<tr>
<td>03/18/2009</td>
<td>“...anticipates that economic conditions are likely to warrant exceptionally low levels of the federal funds rate for an extended period.”</td>
</tr>
<tr>
<td>08/09/2011*</td>
<td>“...anticipates that economic conditions – including low rates of resource utilization and a subdued outlook for inflation over the medium run – are likely to warrant exceptionally low levels for the federal funds rate at least through mid-2013.”</td>
</tr>
<tr>
<td>01/25/2012*</td>
<td>“...decided today to keep the target range for the federal funds rate at 0 to 1/4 percent and currently anticipates that economic conditions – including low rates of resource utilization and a subdued outlook for inflation over the medium run – are likely to warrant exceptionally low levels for the federal funds rate at least through late 2014.”</td>
</tr>
</tbody>
</table>
Table 2.4: Forward guidance quotes, continued

This table summarizes a list of forward guidance quotes, when the Fed expected a different lift-off date or condition for the ZLB. All quotes except the one on 6/19/2013 are from FOMC statements. The quote on 6/19/2013 is from Chairman Bernanke’s press conference. Asterisks mark the statements with explicit lift-off dates, with the corresponding lift-off dates in red. Source: http://www.federalreserve.gov/monetarypolicy/fomccalendars.htm.

<table>
<thead>
<tr>
<th>Date</th>
<th>Quote</th>
</tr>
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<tbody>
<tr>
<td>09/13/2012*</td>
<td>“...decided today to keep the target range for the federal funds rate at 0 to 1/4 percent and currently anticipates that exceptionally low levels for the federal funds rate are likely to be warranted at least through mid-2015.”</td>
</tr>
<tr>
<td>12/12/2012</td>
<td>“...decided to keep the target range for the federal funds rate at 0 to 1/4 percent and currently anticipates that this exceptionally low range for the federal funds rate will be appropriate at least as long as the unemployment rate remains above 6-1/2 percent, inflation between one and two years ahead is projected to be no more than a half percentage point above the Committee’s 2 percent longer-run goal, and longer-term inflation expectations continue to be well anchored.”</td>
</tr>
<tr>
<td>06/19/2013*</td>
<td>“...14 of 19 FOMC participants indicated that they expect the first increase in the target for the federal funds rate to occur in 2015, and one expected the first increase to incur in 2016.”</td>
</tr>
<tr>
<td>12/18/2013</td>
<td>“...anticipates, based on its assessment of these factors, that it likely will be appropriate to maintain the current target range for the federal funds rate well past the time that the unemployment rate declines below 6-1/2 percent, especially if projected inflation continues to run below the Committee’s 2 percent longer-run goal.”</td>
</tr>
</tbody>
</table>
2.10 Acknowledgment

With permission from the coauthor Jing Cynthia Wu, chapter 2 of this dissertation contains work from a manuscript that has been submitted for publication with the title, Measuring the Macroeconomic Impact of Monetary Policy at the Zero Lower Bound 2014, Jing Cynthia Wu and Fan Dora Xia.
Chapter 3

Effects of Unconventional Monetary Policies on the Term Structure of Interest Rates

Abstract

I make use of a non-linear term structure model to capture impacts of unconventional monetary policies on the whole yield curve in a parsimonious way. I find that, at the Zero Lower Bound (ZLB), while forward rates with short maturities are constrained, forward rates with long maturities are still responsive to monetary policy announcements. Each easing (tightening) policy announcement would decrease (increase) long forward rates by around 10 basis points on average.

3.1 Introduction

Since the end of 2008, the fed funds rate, a primary tool for the Fed to conduct monetary policy prior to the Great Recession, has been lowered to essentially zero. Unable to decrease the fed funds rate further, the Fed has resorted to unconventional tools, including large-scale asset purchases (LSAP) and forward guidance, to provide more monetary policy accommodation at the zero lower bound (ZLB). By purchasing large quantities of long-term securities and sending signals
about the future path of the fed funds rate, the Fed intends to put downward pressure on long-term interest rates. A question of great interest to both researchers and policy makers is how these policies affect the term structure of interest rates.

One approach in the literature to isolate effects of these unconventional monetary policies on interest rates is event studies. They examine how interest rates change over short time intervals around policy announcements assuming that they are the only sources of the changes. Previous efforts include Gagnon et al. (2011), Krishnamurthy and Vissing-Jorgensen (2011), Neely (2011), Christensen and Rudebusch (2012), and Woodford (2012). They select interest rates of certain maturities, and study how these particular interest rates move after relevant policy announcements. They all find that specified yields decrease on these announcements as the Fed intended.

One limitation of their approach is that only responses of a narrow set of yields are considered. In order to provide a more complete characterization of unconventional monetary policies’ effects, response of the whole yield curve should be examined. The challenge of studying the whole yield curve lies in the fact that its movement is an intractable object with infinite dimensions. For tractability, we need a parsimonious representation of responses of all interest rates. One way to achieve this goal is using linear term structure models as in Bauer (forthcoming). These models posit that the whole yield curve is fully spanned by linear combinations of a few factors. This allows us to summarize responses of all interest rates in terms of corresponding movements in a small number of factors, which is a tractable object with finite dimensions.

But traditional linear summaries cannot do a satisfactory job of representing the yield curve under the ZLB, under which a long stretch of the curve has been essentially flat. In this paper I employ a non-linear term structure model, namely the shadow rate term structure model (SRTSM). This model maps the whole yield curve into factors of lower dimension, and the nonlinear mapping between factors

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1Another relevant question is how these policies affect the macroeconomy in the U.S. This question has been explored in Chung et al. (2012), Baumeister and Benati (2013), and Wu and Xia (2014).
and yields is restricted to exclude arbitrage opportunities and the possibility of negative nominal interest rates. As Krippner (2013), Wu and Xia (2014), and Christensen and Rudebusch (forthcoming) point out, this model provides a good description of the term structure of interest rates at the ZLB.

I estimate a three-factor SRTSM with daily data, and then examine how the factors move after key unconventional monetary policy announcements. Compared with previous studies, I expand the set of events by including the third round LSAP and developments in forward guidance since 2012. Regression results suggest that the changes of all the three factors are significantly different over dates with policy news relative to dates without policy news. In terms of forward rates, following an easing (tightening) policy announcement, short forward rates does not change much, and long forward rates would decrease (increase) by around 10 basis points.

The rest of the paper is structured as follows: Section 3.2 summarizes dates of key unconventional monetary policy announcements. Section 3.3 details the specification and estimation of the SRTSM. In Section 3.4, I examine how the yield curve change in response to unconventional monetary policies. Section 3.5 concludes.

3.2 Event dates

To identify event dates with key policy announcements, I follow the previous literature, and focus on official communications\(^2\) that contain surprise information regarding (i) potential and actual LSAPs in terms of programs’ timing, size, and asset composition, and (ii) ZLB duration. These dates are listed below by the following five categories: dates related to LSAP1, LSAP2, Maturity Extension Program, LSAP3, and forward guidance.

**LSAP1** The first round LSAP was announced in November 2008, and completed in March 2010. In total, the Fed bought $1.25 trillion agency Mortgage-Backed

---

Security (MBS), $175 billion agency debt securities, and $300 billion Treasury securities. For event dates related to LSAP1, I adopted the following eight dates, which are identified by Gagnon et al. (2011), and used by Krishnamurthy and Vissing-Jorgensen (2011), Neely (2011), and Christensen and Rudebusch (2012).

- **On November 25, 2008**, the press release from the Fed stated that the Fed would purchase “*up to $100 billion in GSE direct obligations*” and “*up to $500 billion in MBS*”.

- **On December 01, 2008**, Fed Chairman Bernanke’s speech stated the Fed’s intention to purchase “*longer-term Treasury or agency securities on the open market in substantial quantities*”.

- **On December 16, 2008**, the FOMC statement indicated that the Fed would expand purchases of agency securities: “*over the next few quarters the Federal Reserve will purchase large quantities of agency debt and mortgage-backed securities to provide support to the mortgage and housing markets, and it stands ready to expand its purchases of agency debt and mortgage-backed securities as conditions warrant*”.

- **On January 28, 2009**, the FOMC statement indicated that the Fed would initiate purchase on longer-term Treasury securities: “*the Committee also is prepared to purchase longer-term Treasury securities if evolving circumstances indicate that such transactions would be particularly effective in improving conditions in private credit markets*”.

- **On March 18, 2009**, the FOMC statement announced that the Fed would purchase “*up to $300 billion of longer-term Treasury securities over the next six months*” and to expand purchase of agency debt “*up to $1.25 trillion*” and agency MBS “*up to $200 billion*”.

- **On August 12, 2009**, the FOMC statement dropped the “*up to*” in the language to quantify the amount of longer-term Treasury securities to be purchased.
• On September 23, 2009, the FOMC statement dropped the “up to” in the language to quantify the amount of agency MBS to be purchased.

• On November 04, 2009, the FOMC statement stated that the exact amount of agency debt to be purchased would be “about $175 billion”, which is less than the previously announced maximum of $200 billion.

**LSAP2**  The second round LSAP was announced in November 2010, and completed in June 2011. In total, the Fed bought $600 billion Treasury securities. The announcement of LSAP2 on November 3, 2010 was widely anticipated. According to Bloomberg News, “fifty-three of 56 economists surveyed by Bloomberg News last week predicted the central bank would announce asset purchases today, with 29 forecasting a pledge to buy $500 billion or more”. But two previous announcements in August 2010 and September 2010 changed the market’s expectation that the Fed would let its portfolio runoff, and made it clear that the Fed intended to shift toward purchasing bonds but not other securities. Following Krishnamurthy and Vissing-Jorgensen (2011), I include these prior announcements dates as relevant event dates:

• On August 10, 2010, the FOMC statement stated that the Fed would “keep constant the Federal Reserve’s holdings of securities at their current level by reinvesting principal payments from agency debt and agency mortgage-backed securities in longer-term Treasury securities. The Committee will continue to roll over the Federal Reserve’s holdings of Treasury securities as they mature”.

• On September 21, 2010, the FOMC statement reiterated the Fed’s intention to “maintain its existing policy of reinvesting principal payments from its securities holdings”.

**Maturity Extension Program**  The Maturity Extension Program was announced in September 2011, extended in June 2012, and completed in December 2012. In total, the Fed swapped $700 billion Treasury securities of 1-3 years maturities with
the same amount of Treasury securities of 6-30 years maturities. Since the program extension was “a widely anticipated move” according to CNN Money, I only include the date on which the Fed first announced the Maturity Extension Program as relevant event date:

- On **September 21, 2011**, the FOMC statement stated that the Fed would “purchase, by the end of June 2012, $400 billion of Treasury securities with remaining maturities of 6 years to 30 years and to sell an equal amount of Treasury securities with remaining maturities of 3 years or less”.

**LSAP3** Fed Chairman Bernanke hinted the third round of LSAP in August 2012. The actual programs were announced in September 2012 for $40 billion per month purchase of agency MBS, and in December 2012 for $45 billion per month purchase of Treasury securities. Bernanke told Congress that the Fed may cut the pace of bond purchases at the next few meetings in May 2013. An actual $5 billion per month cut for both agency MBS and Treasury securities was announced in December 2013. And two further reduction announcement were made in January 2014 and March 2014. Since the last two tapering announcement were “expected” according to CNN Money, they are excluded in event studies. The event dates used in this study are as follows:

- On **August 31, 2012**, Fed Chairman Bernanke’s speech in Jackson Hole indicated the Fed’s willingness to provide more monetary policy accommodation: “the Federal Reserve will provide additional policy accommodation as needed to promote a stronger economic recovery and sustained improvement in labor market conditions in a context of price stability”.

- On **September 13, 2012**, the FOMC statement stated that the Fed would “increase policy accommodation by purchasing additional agency mortgage-backed securities at a pace of $40 billion per month”.

- On **December 12, 2012**, the FOMC statement stated that the Fed would “purchase longer-term Treasury securities after its program to extend the
average maturity of its holdings of Treasury securities is completed at the end of the year, initially at a pace of $45 billion per month”.

- On **May 22, 2013**, Fed Chairman Bernanke’s testimony before the Joint Economic Committee indicated that the central bank may reduce its stimulus at some point: “If we see continued improvement, and we have confidence that that is going to be sustained, in the next few meetings we could take a step down in our pace of purchases”.

- On **December 18, 2013**, the FOMC statement stated that the Fed would “add to its holdings of agency mortgage-backed securities at a pace of $35 billion per month rather than $40 billion per month, and will add to its holdings of longer-term Treasury securities at a pace of $40 billion per month rather than $45 billion per month”.

**Forward guidance** In addition to bond purchases, the Fed also actively communicates with the market regarding for how long the fed funds rate will remain close to zero. Language in the FOMC statement changes from unspecified length to calendar dates, then to economic conditional thresholds. I used the set of event dates identified in Wu and Xia (2014), which is expanded from Woodford (2012):

- On **December 16, 2008**, the FOMC statement stated that the Fed “anticipates that weak economic conditions are likely to warrant exceptionally low levels of the federal funds rate for some time”.

- On **March 18, 2009**, the FOMC statement stated that “economic conditions are likely to warrant exceptionally low levels of the federal funds rate for an extended period”.

- On **August 09, 2011**, the FOMC statement stated that “economic conditions—including low rates of resource utilization and a subdued outlook for inflation over the medium run—are likely to warrant exceptionally low levels for the federal funds rate at least through mid-2013”.
• On **January 25, 2012**, the FOMC statement stated that “economic conditions—including low rates of resource utilization and a subdued outlook for inflation over the medium run—are likely to warrant exceptionally low levels for the federal funds rate at least through late 2014”.

• On **September 13, 2012**, the FOMC statement stated that “exceptionally low levels for the federal funds rate are likely to be warranted at least through mid-2015”.

• On **December 12, 2012**, the FOMC statement stated that “this exceptionally low range for the federal funds rate will be appropriate at least as long as the unemployment rate remains above 6-1/2 percent, inflation between one and two years ahead is projected to be no more than a half percentage point above the Committee’s 2 percent longer-run goal, and longer-term inflation expectations continue to be well anchored”.

• On **June 19, 2013**, Fed Chairman Bernanke’s press conference speech after the FOMC meeting stated that “in the projections submitted for this meeting, 14 of 19 FOMC participants indicated that they expect the first increase in the target for the federal funds rate to occur in 2015”.

• On **December 18, 2013**, the FOMC statement stated that “it likely will be appropriate to maintain the current target range for the federal funds rate well past the time that the unemployment rate declines below 6-1/2 percent, especially if projected inflation continues to run below the Committee’s 2 percent longer-run goal”.

Out of all of above policy news, only four of them represent policy tightening: (i) the amount of agency debt to be purchased in LSAP1 was revised down to $175 billion on November 4, 2009; (ii) tapering was hinted on May 22, 2013; (iii) the lift-off date was revised down to 2015 on June 19, 2013; and (vi) tapering was announced on December 18, 2013. All other announcements indicate policy easing.

Note that there are five dates that fall into more than one categories, including December 16, 2008, March 18, 2009, September 13, 2012, December 12,
2012, and December 18, 2013. For the first four, all news on these dates represents easing monetary policy. Therefore we can keep them in the list as long as we are not trying to distinguish difference between bond purchasing programs and forward guidance. For December 18, 2013, news on the LSAP3 and the duration of ZLB indicate tightening and easing policy stances respectively. Therefore I exclude this date from the list. Table 3.1 provides a summary of the event dates.

3.3 Shadow rate term structure model

Most event studies pick a few yields with particular maturities, and examine how these interest rates change in response to policy news. A disadvantage of this approach is that it fails to provide a complete picture of policies’ impact on all interest rates. To fully characterize responses of the whole yield curve and maintain the problem tractable at the same time, one solution suggested by Bauer (forthcoming) is making use of term structure models. By using a few factors to describe the term structure of interest rates, responses of the whole yield curve can be summarized by responses of these lower-dimensional factors. Bauer (forthcoming) used the Gaussian affine term structure model (GATSM), the workhorse model in term structure literature.\(^3\) However, the GATSM fails to respect the zero lower bound on nominal interest rates, and would predict negative interest rates at the ZLB. As an alternative, I employ the SRTSM, which ensures non-negativity of nominal interest rates. A number of studies, including Krippner (2013), Wu and Xia (2014), and Christensen and Rudebusch (forthcoming), show that the SRTSM outperforms the GATSM at the ZLB in describing the term structure of interest rates.

3.3.1 Model specification

Set up The SRTSM is first proposed by Black (1995). The model assumes that the short term interest rate is the maximum of a shadow rate $s_t$, which is an affine

\(^3\)For surveys on the GATSM, see Piazzesi (2010), Duffee (forthcoming), Gürkaynak and Wright (2012b), and Diebold and Rudebusch (2013).
function of some factors $X_t$, and a non-negative constant $r$. Mathematically,

$$r_t = \max(s_t, r),$$  \hspace{1cm} (3.1)

where $s_t = \delta_0 + \delta_1 X_t$, $r \geq 0$.

The model further assumes that the factors follow a first order vector autoregressive process (VAR(1)) under both the physical and the risk-neutral measures:

$$X_{t+1} = \mu + \rho X_t + \Sigma \varepsilon_{t+1}, \quad \varepsilon_{t+1} \sim N(0, I), \hspace{1cm} (3.2)$$

$$X_{t+1} = \mu^Q + \rho^Q X_t + \Sigma^Q \varepsilon_{t+1}^Q, \quad \varepsilon_{t+1}^Q \sim N(0, I). \hspace{1cm} (3.3)$$

**Solution**  To preclude arbitrage, the one-period forward rate between $t + n$ and $t + n + 1$ should satisfy

$$f_{n,n+1,t} = -\log \left( e^{-\gamma_{n}^Q} E_t^Q \left[ e^{-\sum_{j=1}^{n} r_{t+j}} \right] \right) + \log \left( e^{-\gamma_{n}^Q} E_t^Q \left[ e^{-\sum_{j=1}^{n-1} r_{t+j}} \right] \right). \hspace{1cm} (3.4)$$

As a result of the nonlinearity introduced by the max function in equation (3.1), an analytical solution to this model, i.e. an explicit function that maps $f_{n,n+1,t}$ to $X_t$, is known only in the case of a one-factor model. For multi-factor models, researchers relies on either numerical simulations or analytical approximations. I use the approximate analytical solution proposed by Wu and Xia (2014), which both offers an excellent approximation and is easy to implement empirically. Wu and Xia (2014) shows that $f_{n,n+1,t}$ can be well approximated by

$$f_{n,n+1,t} \approx r + \sigma_n^Q g \left( \frac{a_n + b_n^Q X_t - r}{\sigma_n^Q} \right), \hspace{1cm} (3.5)$$
\[
\begin{align*}
\mathbf{a}_n & = \delta_0 + \delta'_1 \left( \sum_{j=0}^{n-1} (\rho^Q)^j \right) \mu^Q - \frac{1}{2} \delta'_1 \left( \sum_{j=0}^{n-1} (\rho^Q)^j \right) \Sigma^Q \left( \sum_{j=0}^{n-1} (\rho^Q)^j \right)' \\
\mathbf{b}'_n & = \delta'_1 (\rho^Q)^n, \\
\sigma^Q_n & = \sqrt{\sum_{j=0}^{n-1} \delta'_1 (\rho^Q)^j \Sigma^Q (\rho^Q)^j} \delta_1.
\end{align*}
\]

The function \( g(\cdot) \) is defined as \( g(z) \equiv z\Phi(z) + \phi(z) \), where \( \Phi(\cdot) \) and \( \phi(\cdot) \) are the cumulative distribution function and probability density function of a standard normal distribution.

From equation 3.5, we can write changes in any forward rates in terms of changes in the factors as follows:

\[
\begin{align*}
\mathbf{f}_{n,n+1,t} - \mathbf{E}_{t-1} (\mathbf{f}_{n,n+1,t}) & \approx \sigma^Q_n \left( g \left( \frac{a_n + b'_n X_t - r}{\sigma^Q_n} \right) - \mathbf{E}_{t-1} \left[ g \left( \frac{a_n + b'_n X_t - r}{\sigma^Q_n} \right) \right] \right) \\
& \approx \phi \left( \frac{a_n + b'_n X_{t-1} - r}{\sigma^Q_n} \right) \times b'_n (X_t - \mathbf{E}_{t-1} \mathbb{[}X_t]) .
\end{align*}
\]  

(3.6)

### 3.3.2 Estimation

To see how the factors change in response to policy news over event dates, we need to estimate the SRTSM, and filter the latent factors first. Following Wu and Xia (2014), we can write the SRTSM as a nonlinear state space model. The transition equation for the factors is equation (3.2). From equation (3.5), the measurement equation relates the observed forward rate \( \mathbf{f}_{n,n+1,t}^o \) to the factors as follows:

\[
\begin{align*}
\mathbf{f}_{n,n+1,t}^o & = r + \sigma^Q_n g \left( \frac{a_n + b'_n X_t - r}{\sigma^Q_n} \right) + \eta_{nt},
\end{align*}
\]  

(3.7)

where the measurement error \( \eta_{nt} \) is i.i.d. normal, \( \eta_{nt} \sim N(0, \omega) \). With the state space representation, we can use the Extended Kalman filter (EFK) for estimation and filtering.

To conduct event studies, it is desirable to work with the daily SRTSM
so that daily series of the filtered factors can be obtained. However, with large sample size of daily data, it takes a long time for the EFK to estimate parameters. As an alternative, I first estimate the monthly SRTSM. Then I convert monthly parameters into daily parameters. Finally I filter daily factors with the daily parameters, which takes little time. Details for each step are provided below.

**Estimation of the monthly SRTSM**

To estimate the monthly SRTSM, I follow Wu and Xia (2014) with the only difference that I estimate the lower bound \( r \) instead of imposing its value to be 25 basis points per annum. While results in Wu and Xia (2014) are robust to this restriction, daily series of the filtered factors are more sensible without it.

**Data** I used one-month forward rates for maturities of 3 and 6 months, 1, 2, 5, 7 and 10 years from Wu and Xia (2014). They are end-of-month observations from January 1990 to December 2013.

**Normalization** As in Wu and Xia (2014), I estimate a three-factor SRTSM and impose the following normalizing restrictions: (i) \( \delta_1 = [1, 1, 0]' \); (ii) \( \mu^Q = 0 \); (iii) \( \rho^Q \) is in real Jordan form with eigenvalues in descending order;\(^4\) and (iv) \( \Sigma \) is lower triangular.

**Maximum likelihood estimation** The observation equation in (3.7) for all 7 maturities can be stacked as follows

\[
F_{t+1}^o = G(X_{t+1}) + \eta_{t+1}, \quad \eta_{t+1} \sim N(0, \omega I_7).
\]

\(^4\)The specification assuming distinct eigenvalues for \( \rho^Q \) generates two smaller eigenvalues very close to each other. Thus I further restrict these two eigenvalues to be the same. This restriction is imposed in Wu and Xia (2014) as well.
The EKF approximates the conditional distribution of $X_t$ with $X_t|F_{t|t} \sim N(\hat{X}_{t|t}, P_{t|t})$, and updates $\hat{X}_{t+1|t+1}$ and $P_{t+1|t+1}$ as follows:

$$
\hat{X}_{t+1|t+1} = \hat{X}_{t|t} + K_{t+1}(F_{t+1} - \hat{F}_{t+1|t}), \\
P_{t+1|t+1} = (I_3 - K_{t+1}H_{t+1}) P_{t|t}, \\
\hat{X}_{t+1|t} = \mu + \rho \hat{X}_{t|t}, \\
P_{t+1|t} = \rho P_{t|t}\rho' + \Sigma\Sigma',
$$

where

$$
\hat{F}_{t+1|t} = G(\hat{X}_{t+1|t}), \\
H_{t+1} = \left( \frac{\partial G(X_{t+1})}{\partial X_{t+1}} \right)_{X_{t+1}=\hat{X}_{t+1|t}}, \\
K_{t+1} = P_{t+1|t}H_{t+1}(H'_{t+1}P_{t+1|t}H_{t+1} + \Omega I_T)^{-1},
$$

and $H'_{t+1}$ is obtained by stacking $\Phi\left(\frac{a_n + b_n' \hat{X}_{t+1|t} - \xi}{\sigma_n^2}\right) \times b_n'$ for the 7 maturities. The EKF updates $\{\hat{X}_{t|t}, P_{t|t}\}_{t=1}^T$ recursively with the above algorithm provided the initial values $\hat{X}_{0|0}$ and $P_{0|0}$. And the corresponding log likelihood is $\mathcal{L} = -\frac{7T}{2} \log 2\pi - \frac{1}{2} \sum_{t=1}^{T} \log |H_{t+1|t}H_{t} + \Omega I_T| - \frac{1}{2} \sum_{t=1}^{T} (F_{t} - G(\hat{X}_{t|t-1}))' (H'_{t+1}P_{t+1|t}H_{t} + \Omega I_T)^{-1} (F_{t} - G(\hat{X}_{t|t-1})).$ Maximum likelihood estimates with robust standard errors (See Hamilton (1994) p. 145) are reported in Table 3.2.

**Conversion from monthly parameters to daily parameters**

To filter daily series of the factors, I need to obtain daily parameters that describe the factor dynamics in daily frequency. To convert monthly parameters to daily parameters, I make use of the multivariate Ornstein-Uhlenbeck (OU) process, continuous time analogue of the discrete-time VAR(1) process. The OU process is described by $dX_t = \kappa(\theta - X_t)dt + \Omega dW_t$, where $W_t$ is the standard Brownian motion. The conditional mean and variance of $X_t$ implied by the OU process are
as follows:

\[
E_t[X_{t+\Delta}] = (I_3 - \exp(-\kappa \Delta)) \theta + \exp(-\kappa \Delta) X_t,
\]

\[
\text{Var}_t[X_{t+\Delta}] = V_x, \text{ which is the solution to the equation: }
\]

\[
\text{vec}(V_x) = [(I_3 \otimes \kappa) + (\kappa \otimes I_3)]^{-1} [I_3 - (\exp(-\kappa \Delta) \otimes \exp(-\kappa \Delta))] \text{ vec}(\Omega') \text{.}
\]

Recall that the monthly VAR(1) would imply the conditional mean and variance of \( X_t \) as follows:

\[
E_t[X_{t+1}] = \mu + \rho X_t, \text{ and } \text{Var}_t[X_{t+1}] = \Sigma \Sigma'.
\]

By matching the first and second moments, I can calculate parameters of the OU process from the estimated parameters to describe the factor dynamics in monthly VAR(1):

\[
\hat{\kappa} = -\log(\hat{\rho}) \text{, }
\hat{\theta} = (I_3 - \hat{\rho})^{-1} \hat{\mu} \text{, }
\text{vec} \left( \hat{\Omega}' \hat{\Omega} \right) = (I_3 - (\hat{\rho} \otimes \hat{\rho}))^{-1} ((I_3 \otimes \hat{\kappa}) + (\hat{\kappa} \otimes I_3)) \text{ vec}(\hat{\Sigma} \hat{\Sigma}') \text{.}
\]

Let \( D_t \) denotes the number of days within month \( t \). Then the daily VAR(1) would imply the conditional mean and variance of \( X_t \) as follows:

\[
E_t \left[ X_{t+\frac{1}{D_t}} \right] = \mu_d + \rho_d X_t, \text{ and } \text{Var}_t \left[ X_{t+\frac{1}{D_t}} \right] = \Sigma_d \Sigma_d'.
\]

Again, by matching the first and second moments, I can back out daily VAR(1) parameters during month \( t \) from parameters of OU process as follows:

\[
\hat{\rho}_{d,t} = \exp \left( \frac{-\hat{\kappa} \frac{1}{D_t}}{D_t} \right) \text{, }
\hat{\mu}_{d,t} = (I_3 - \hat{\rho}_{d,t}) \hat{\theta} \text{, }
\text{vec} \left( \hat{\Sigma}_{d,t} \hat{\Sigma}'_{d,t} \right) = ((I_3 \otimes \hat{\kappa}) + (\hat{\kappa} \otimes I_3))^{-1} (I_3 - (\hat{\rho}_{d,t} \otimes \hat{\rho}_{d,t})) \text{ vec}(\hat{\Omega} \hat{\Omega}') \text{.}
\]

\(^5\text{Thanks to Drew Creal for sharing his notes.}\)
Filtering daily series of the factors

After obtaining \( \Theta_{d,t} = (\hat{\rho}_{d,t}, \hat{\mu}_{d,t}, \hat{\Sigma}_{d,t}) \), I can filter daily factors via the EKF assuming that the mapping between the factors and forward rates in the daily model is the same as in the monthly model. The EKF approximates the conditional distribution of the factors at month \( t + 1 \) and day \( \tau, X_{t+\tau+1/2} \), with \( X_{t+\tau+1/2} \) as follows:

\[
X_{t+\tau+1/2} | \begin{cases} F_{t+\tau+1/2}^o; F_{t+\tau+1/2}^f \end{cases} \sim N(\hat{X}_{t+\tau+1/2}, P_{t+\tau+1/2}), \quad \text{and updates}
\]

\[
\hat{X}_{t+\tau+1/2} \quad \text{and} \quad P_{t+\tau+1/2}
\]

\[
\hat{X}_{t+\tau+1/2} = \hat{X}_{t+\tau+1/2} + K_{t+\tau+1/2} (F_{t+\tau+1/2}^o - \hat{F}_{t+\tau+1/2}^o),
\]

\[
P_{t+\tau+1/2} = (I - K_{t+\tau+1/2} H_{t+\tau+1/2}^t) P_{t+\tau+1/2},
\]

where

\[
\hat{F}_{t+\tau+1/2}^o = G(X_{t+\tau+1/2}),
\]

\[
H_{t+\tau+1/2} = \left( \frac{\partial G(X_{t+\tau+1/2})}{\partial X_{t+\tau+1/2}} \right)_{X_{t+\tau+1/2} = \hat{X}_{t+\tau+1/2}},
\]

\[
K_{t+\tau+1/2} = P_{t+\tau+1/2} H_{t+\tau+1/2} (H_{t+\tau+1/2} P_{t+\tau+1/2} H_{t+\tau+1/2} + \omega I_7)^{-1},
\]

and \( H_{t+\tau+1/2} \) is obtained by stacking \( \Phi \left( a_n b_n \hat{X}_{t+\tau+1/2} \right) \times b_n' \) for the 7 maturities. The EKF updates \( \{\hat{X}_{t+\tau+1/2} | P_{t+\tau+1/2} \} \) recursively with the above algorithm provided the initial values \( \hat{X}_{0/0} = 0 \) and \( P_{0/0} = 0 \).

For daily forward rates, I construct daily one-month forward rates for maturities of 3 and 6 months, 1, 2, 5, 7 and 10 years between Jan 1990 and April 2014.

---

\(^6\)Recall that \( X_t \) represents factors at the end of month \( t \).
from Gürkaynak et al. (2007). The filtered daily factors are displayed in Figure 3.1. Note that the above algorithm updates $X_{t+\tau \tau_{t+1}}$ without using the information between $t + \frac{1}{D_{t+1}}$ and $t + \frac{\tau - 1}{D_{t+1}}$ so that the daily SRTSM and the monthly SRTSM generate identical filtered factors at the end of each month. Compared with estimating the daily SRTSM directly, the three-step approach proposed above is less efficient since it does not utilize the information on forward rates between $t + \frac{1}{D_{t+1}}$ and $t + \frac{\tau - 1}{D_{t+1}}$. However, as mentioned before, an advantage of using the three-step approach is that it takes less computation time.

### 3.3.3 Fitting error

To make sure that the estimated model well represents the observed daily forward rates with the three factors, I calculate root mean square fitting errors over the full sample from January 1990 to April 2014. The numbers are tabulated in Table 3.3. As we can see, all the root mean square fitting errors are around a few basis points. Indeed, the estimated three-factor SRTSM provides a satisfactory description of observed daily interest rates. \(^7\)

To further ensure that the model can capture changes of forward rates over event dates, I plot fitted and observed changes in Figure 3.2. It can be seen that the estimated model is able to capture observed movements in forward rates very well.

### 3.4 Response of the forward curve on event dates

With the estimated three-factor SRTSM, we are ready to examine how forward rates with any maturities respond to policy announcements. The calculation can be proceeded in two steps. First, we need to compute the changes of the three factors in response to policy announcements. Second, we can compute corresponding changes in any forward rates through the mapping implied by the SRTSM as in equation (3.6).

\(^7\)This result also validates the assumption that the mapping between the factors and forward rates in the daily SRTSM is the same as in the monthly SRTSM.
In Table 3.4, I list the changes of the three factors over the event dates identified in Section 3.2. To see how easing and tightening unconventional monetary policy announcements affect the changes of the three factors, I run following regressions:

\[
X_{i,j} - E_{j-1} [X_{i,j}] = \beta_{0}^{i,e} \\
+ \beta_{1}^{i,e} \times 1_{\{\text{easing unconventional monetary policy announcements were made on date j}\}} \\
+ e_j^{i,e}, \text{ where } e_j^{i,e} \sim N(0, (\sigma_e^{i,e})^2) \tag{3.12}
\]

and

\[
X_{i,j} - E_{j-1} [X_{i,j}] = \beta_{0}^{i,t} \\
+ \beta_{1}^{i,t} \times 1_{\{\text{tightening unconventional monetary policy announcements were made on date j}\}} \\
+ e_j^{i,t}, \text{ where } e_j^{i,t} \sim N(0, (\sigma_t^{i,t})^2) \tag{3.13}
\]

for \(i = 1, 2,\) and 3. The dependent variables are unexpected changes of the three factors. The regressors include a constant term and the dummy variable that takes the value of 1 if easing (tightening) unconventional monetary policy was announced in equation (3.12) ((3.13)). The regression results are tabulated in Table 3.5. We can see that estimates for \(\beta_{1}^{i,e}\) and \(\beta_{1}^{i,t}\) are significant different from 0 at 99% confidence interval for all the three factors. It follows that the changes of the three factors in response to easing or tightening policy announcements are significantly different from the changes over dates without policy news. On average, the relative change in the three factors are -0.1182, 0.1138, and -0.0101 following each easing policy announcement, and 0.1377, -0.1367 and 0.0154 following each tightening policy announcement. Notice that none of the estimates for \(\beta_{0}^{i,e}\) and \(\beta_{0}^{i,t}\) is significant. This result is consistent with our intuition that the factors do not change significantly over dates without policy news.

Now let’s turn our focus to translating changes of the three factors to changes in forward rates using equation (3.6). The equation states that the change in the forward rate \(f_{n,n+1,t}\) is a linear combination of the changes in the
three factors with weight $\phi \left( \frac{a_n + b'_n X_{i-1} - r}{\sigma_n^2} \right) \times b_n(i)$ for the $i$-th factor. The value of $\phi \left( \frac{a_n + b'_n X_{i-1} - r}{\sigma_n^2} \right)$ changes over time, and is common to the factors. The value of $b_n(i)$ is time-invariant and factor-specific. In Figure 3.3, I plot $\phi \left( \frac{a_n + b'_n X_{i-1} - r}{\sigma_n^2} \right)$ since 2009 (ZLB period), and also from 2003 to 2008 for comparison. Before the ZLB, $\phi \left( \frac{a_n + b'_n X_{i-1} - r}{\sigma_n^2} \right)$ is essentially one across all maturities through out the sample period. During the ZLB period, $\phi \left( \frac{a_n + b'_n X_{i-1} - r}{\sigma_n^2} \right)$ generally increases with $n$: close to 0 for short maturities, and close to 1 for long maturities. In other words, forward rates with short maturities are constrained by the lower bound, and do not respond to changes in any factors; while forward rates with long maturities are not much affected by the lower bound, and still respond to changes in the factors in the similar fashion as they do before the ZLB. To investigate impacts of each easing and tightening policy announcement on the forward curve quantitatively, I compare the forward curve observed in data and the forward curve in counterfactual scenarios where the factors had not been changed by $\hat{\beta}^i_1$ and $\hat{\beta}^t_1$. I plot the average forward curve during the ZLB period observed in data and the counterfactual scenarios in Figure 3.4. As we can see that while forward rates with short maturities do not move much, forward rates with long maturities would decrease (increase) by around 10 basis points on average following each easing (tightening) policy announcements.

3.5 Conclusion

To capture the effects of unconventional monetary polices on all interest rates in a parsimonious fashion, I employ the three-factor SRTSM to summarize the forward yield curve with three factors. The estimated model provides a good description of forward rates observed in data. By examining how policy announcements would affect the three factors and then the whole forward curve accordingly, I find that during the ZLB period, forward rate with short maturities are constrained, while forward rates with long maturities still respond to policy announcements. Following each easing (tightening) policy announcement, long forward rates would decrease (increase) by 10 basis points on average.
3.6 Figures

Figure 3.1: The filtered daily factors from January 1990 to December 2013
Figure 3.2: Changes in forward rates over event dates
Green line stands for model-implied changes in forward rates; and red crosses are observed changes. X-axis stands for maturity in unit of month.
Figure 3.3: \( \phi \left( \frac{a_n + b'_n X_{t-1} - r}{\sigma_n^Q} \right) \)

\( \phi \left( \frac{a_n + b'_n X_{t-1} - r}{\sigma_n^Q} \right) \) from January 2009 to April 2014 (upper panel), and from January 2003 to December 2008 (lower panel).

Figure 3.4: Observed and counter-factual average forward curves
Black line is the average forward curve during the ZLB observed in data. Red line is the average forward curve during the ZLB in the counter-factual that the factors had not been changed by \( \hat{\beta}_1 \). Green line is the average forward curve during the ZLB in the counter-factual that the factors had not been changed by \( \hat{\beta}_1 \).
3.7 Tables

Table 3.1: List of event dates

List of event dates relevant for LSAPs and forward guidance. Details of these events are provided in Section 3.2.

<table>
<thead>
<tr>
<th>Date</th>
<th>Categories</th>
<th>Tightening or easing</th>
</tr>
</thead>
<tbody>
<tr>
<td>November 25, 2008</td>
<td>LSAP1</td>
<td>easing</td>
</tr>
<tr>
<td>December 01, 2008</td>
<td>LSAP1</td>
<td>easing</td>
</tr>
<tr>
<td>December 16, 2008</td>
<td>LSAP1 and forward guidance</td>
<td>easing</td>
</tr>
<tr>
<td>January 28, 2009</td>
<td>LSAP1</td>
<td>easing</td>
</tr>
<tr>
<td>March 18, 2009</td>
<td>LSAP1 and forward guidance</td>
<td>easing</td>
</tr>
<tr>
<td>August 12, 2009</td>
<td>LSAP1</td>
<td>easing</td>
</tr>
<tr>
<td>September 23, 2009</td>
<td>LSAP1</td>
<td>easing</td>
</tr>
<tr>
<td>November 4, 2009</td>
<td>LSAP1</td>
<td>tightening</td>
</tr>
<tr>
<td>August 10, 2010</td>
<td>LSAP2</td>
<td>easing</td>
</tr>
<tr>
<td>September 21, 2010</td>
<td>LSAP2</td>
<td>easing</td>
</tr>
<tr>
<td>August 9, 2011</td>
<td>forward guidance</td>
<td>easing</td>
</tr>
<tr>
<td>September 21, 2011</td>
<td>Maturity Extension Program</td>
<td>easing</td>
</tr>
<tr>
<td>January 25, 2012</td>
<td>forward guidance</td>
<td>easing</td>
</tr>
<tr>
<td>August 31, 2012</td>
<td>LSAP3</td>
<td>easing</td>
</tr>
<tr>
<td>September 13, 2012</td>
<td>LSAP3 and forward guidance</td>
<td>easing</td>
</tr>
<tr>
<td>December 12, 2012</td>
<td>LSAP3 and forward guidance</td>
<td>easing</td>
</tr>
<tr>
<td>May 22, 2013</td>
<td>LSAP3</td>
<td>tightening</td>
</tr>
<tr>
<td>June 19, 2013</td>
<td>forward guidance</td>
<td>tightening</td>
</tr>
</tbody>
</table>
Table 3.2: Maximum likelihood estimates with robust standard errors

Maximum likelihood estimates with robust standard errors in parentheses for the monthly SRTSM.

<table>
<thead>
<tr>
<th></th>
<th>1200μ</th>
<th>1200ρ</th>
<th>1200Σ</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-0.3058</td>
<td>-0.2272</td>
<td>0.0275</td>
</tr>
<tr>
<td></td>
<td>(0.1867)</td>
<td>(0.1829)</td>
<td>(0.0159)</td>
</tr>
<tr>
<td>ρ</td>
<td>0.9629</td>
<td>-0.0039</td>
<td>0.3780</td>
</tr>
<tr>
<td></td>
<td>(0.0201)</td>
<td>(0.0179)</td>
<td>(0.4647)</td>
</tr>
<tr>
<td></td>
<td>-0.0218</td>
<td>0.9435</td>
<td>0.9696</td>
</tr>
<tr>
<td></td>
<td>(0.0205)</td>
<td>(0.0213)</td>
<td>(0.4915)</td>
</tr>
<tr>
<td></td>
<td>0.0036</td>
<td>0.0029</td>
<td>0.8900</td>
</tr>
<tr>
<td></td>
<td>(0.0018)</td>
<td>(0.0019)</td>
<td>(0.0370)</td>
</tr>
<tr>
<td>eig(ρ)</td>
<td>0.9861</td>
<td>0.9648</td>
<td>0.8455</td>
</tr>
<tr>
<td>ρQ</td>
<td>0.9977</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(0.003)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0.9501</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(0.0011)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0.9501</td>
</tr>
<tr>
<td></td>
<td>(0.0011)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1200δ₀</td>
<td>13.2210</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(1.0079)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1200Σ</td>
<td>0.4177</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(0.0390)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.4077</td>
<td>0.2496</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>(0.0367)</td>
<td>(0.0240)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-0.0121</td>
<td>0.0022</td>
<td>0.0388</td>
</tr>
<tr>
<td></td>
<td>(0.0068)</td>
<td>(0.0034)</td>
<td>(0.0029)</td>
</tr>
<tr>
<td>1200√ω</td>
<td>0.0875</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0027)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1200τ</td>
<td>0.1883</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.0090)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Log Likelihood value | 877.9281
Table 3.3: Root mean square fitting errors

Root mean square fitting errors over the full sample between January 1990 to April 2014 for the following maturities, 3 and 6 months, 1,2,5,7 and 10 years. Numbers are in unit of basis-point.

<table>
<thead>
<tr>
<th></th>
<th>3-month</th>
<th>6-month</th>
<th>1-year</th>
<th>2-year</th>
<th>5-year</th>
<th>7-year</th>
<th>10-year</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4.95</td>
<td>5.28</td>
<td>5.32</td>
<td>5.71</td>
<td>9.25</td>
<td>5.81</td>
<td>11.25</td>
</tr>
</tbody>
</table>

Table 3.4: Changes of the factors

Unexpected changes of the three filtered factors on the event dates.

<table>
<thead>
<tr>
<th>Date</th>
<th>change in $X_1$</th>
<th>change in $X_2$</th>
<th>change in $X_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>November 25, 2008</td>
<td>-0.2275</td>
<td>0.2597</td>
<td>-0.0162</td>
</tr>
<tr>
<td>December 1, 2008</td>
<td>-0.2401</td>
<td>0.2788</td>
<td>-0.0181</td>
</tr>
<tr>
<td>December 16, 2008</td>
<td>-0.2172</td>
<td>0.1981</td>
<td>-0.0099</td>
</tr>
<tr>
<td>January 28, 2009</td>
<td>0.1951</td>
<td>-0.2542</td>
<td>0.0062</td>
</tr>
<tr>
<td>March 18, 2009</td>
<td>-0.6734</td>
<td>0.6602</td>
<td>-0.0166</td>
</tr>
<tr>
<td>August 12, 2009</td>
<td>0.1585</td>
<td>-0.1423</td>
<td>-0.0148</td>
</tr>
<tr>
<td>September 23, 2009</td>
<td>0.0152</td>
<td>-0.0417</td>
<td>-0.0081</td>
</tr>
<tr>
<td>August 12, 2009</td>
<td>0.1421</td>
<td>-0.1423</td>
<td>-0.0148</td>
</tr>
<tr>
<td>September 23, 2009</td>
<td>0.0152</td>
<td>-0.0417</td>
<td>-0.0081</td>
</tr>
<tr>
<td>November 4, 2009</td>
<td>0.1421</td>
<td>-0.1642</td>
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<td>August 10, 2010</td>
<td>-0.0706</td>
<td>0.0915</td>
<td>-0.0087</td>
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<tr>
<td>September 21, 2010</td>
<td>-0.1460</td>
<td>0.1518</td>
<td>-0.0093</td>
</tr>
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<td>August 9, 2011</td>
<td>-0.2520</td>
<td>0.2296</td>
<td>-0.0208</td>
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<td>September 21, 2011</td>
<td>-0.3017</td>
<td>0.3798</td>
<td>0.0349</td>
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<td>January 25, 2012</td>
<td>-0.0549</td>
<td>0.0052</td>
<td>-0.0292</td>
</tr>
<tr>
<td>August 31, 2012</td>
<td>-0.1091</td>
<td>0.0403</td>
<td>-0.0147</td>
</tr>
<tr>
<td>September 13, 2012</td>
<td>0.0064</td>
<td>-0.0106</td>
<td>-0.0189</td>
</tr>
<tr>
<td>December 12, 2012</td>
<td>0.1467</td>
<td>-0.1444</td>
<td>-0.0064</td>
</tr>
<tr>
<td>May 22, 2013</td>
<td>0.1715</td>
<td>-0.1303</td>
<td>0.0117</td>
</tr>
<tr>
<td>June 19, 2013</td>
<td>0.0981</td>
<td>-0.1153</td>
<td>0.0430</td>
</tr>
</tbody>
</table>
Table 3.5: Estimated coefficients for equation (3.12) and (3.13)

OLS estimates with t-statistics in parentheses for equation (3.12) and (3.13). * indicates that estimates are significantly different from zero at 99% confidence interval.

<table>
<thead>
<tr>
<th>$i$</th>
<th>$\hat{\beta}_{0}^i$</th>
<th>$\hat{\beta}_{1}^i$</th>
<th>$\hat{\beta}_{0}^{i,t}$</th>
<th>$\hat{\beta}_{1}^{i,t}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i = 1$</td>
<td>-0.0001 (0.0744)</td>
<td>-0.1180* (5.4761)</td>
<td>-0.0004 (0.4105)</td>
<td>0.1377* (2.8557)</td>
</tr>
<tr>
<td>$i = 2$</td>
<td>-0.0003 (0.2709)</td>
<td>0.1138* (5.1103)</td>
<td>0.0000 (0.0448)</td>
<td>-0.1367* (2.7440)</td>
</tr>
<tr>
<td>$i = 3$</td>
<td>0.0000 (0.4378)</td>
<td>-0.0101* (4.9906)</td>
<td>0.0000 (0.1132)</td>
<td>0.0154* (3.4128)</td>
</tr>
</tbody>
</table>
Bibliography


Chung, Hess, Jean philippe Laforte, David Reifschneider, and John C. Williams, “Have We Underestimated the Likelihood and Severity of Zero Lower Bound Events?,” Journal of Money, Credit and Banking, 2012, 44, 47–82.


