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Publication Date
1964-11-09
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Berkeley, California
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A VARIATIONAL PRINCIPLE FOR THE FADDEEV EQUATIONS

Jon Wright and Roland Omnes

November 9, 1964
Several satisfactory integral formulations of the nonrelativistic scattering of three particles have been proposed recently.\textsuperscript{1-3} The simplest one has been given by Faddeev, who writes the off-the-energy-shell scattering amplitude $T$ as a sum

$$T(z) = T_1(z) + T_2(z) + T_3(z),$$

where $z$ is the total energy (eventually complex). The operators $T_i(z)$ must satisfy a set of coupled integral equations:

$$
\begin{bmatrix}
T_1(z)
\end{bmatrix}
\begin{bmatrix}
t_1(z)
\end{bmatrix}
= 
\begin{bmatrix}
0 & t_1(z) & t_1(z)
\end{bmatrix}
\begin{bmatrix}
T_1(z)
\end{bmatrix},
$$

$$
\begin{bmatrix}
T_2(z)
\end{bmatrix}
\begin{bmatrix}
t_2(z)
\end{bmatrix}
= 
\begin{bmatrix}
t_2(z) & 0 & t_2(z)
\end{bmatrix}
\begin{bmatrix}
G_0(z)
\end{bmatrix}
\begin{bmatrix}
T_2(z)
\end{bmatrix},
$$

$$
\begin{bmatrix}
T_3(z)
\end{bmatrix}
\begin{bmatrix}
t_3(z)
\end{bmatrix}
= 
\begin{bmatrix}
t_3(z) & t_3(z) & 0
\end{bmatrix}
\begin{bmatrix}
T_3(z)
\end{bmatrix},
$$

or

$$T = t - KG_0 T. \quad (3)$$

Here $t_i(z)$, for instance, is the off-the-energy-shell scattering amplitude of particles 2 and 3 and $G_0(z)$ is the three-particle free Green's function. These equations can also be used in relativistic situations where three-particle unitarity is important and crossing is neglected.\textsuperscript{4,5} In any case a systematic attempt at solving these equations under various assumptions
for the two-body scattering amplitudes will considerably increase our understanding of the three-body problem.

A practical limitation to that program is due to the large number of variables which, even after separation of angular momentum and parity, makes it difficult to get any accuracy in the resolvent of Eq. (2).\(^6\)

However, when computing bound states and resonances, it is only necessary to find a value of \(z\) for which \(-\mathcal{K}_0\) has an eigenvalue unity.\(^7\)

The eigenvalues of \(\mathcal{K}_0\) can be found by introducing the functional

\[
\eta = \langle \phi | K | \psi \rangle \left[ \langle \phi | \mathcal{G}_0^{-1} | \psi \rangle \right]^{-1}.
\]

The variational principle \(\delta \eta = 0\) leads \(\mathcal{G}_0^{-1} \psi\) and \(\phi\) to be respectively the right and left eigenvectors of \(\mathcal{K}_0\) associated with the eigenvalue \(\eta\).

Owing to the non-Hermiticity of \(K\), there are two important practical difficulties in this variational approach:

(i) The right and left eigenfunctions are different.

(ii) Some eigenvalues are complex, even when \(z\) is chosen in the bound-state region and they happen sometimes to have larger absolute values than the real eigenvalues.\(^5\) These two effects would make the variational approach useless in its simple form, so that it is important to eliminate the spurious eigenvalues by reducing the arbitrariness of the trial functions.

When \(z\) is in the bound-state region, the \textit{H}ermicity of \(t_4(z)\) gives a relation between \(\phi(z)\) and \(\psi(z)\), valid when \(\eta\) is real:

\[
\phi_1(z) = \frac{1}{2} [\psi_2(z) + \psi_3(z)], \quad \psi_1(z) = \phi_2(z) + \phi_3(z) - \phi_1(z).
\]

(5)
The factorizability of the residues of the resolvent of Eq. (3), combined with Eq. (1) shows that an eigenvalue \( \eta \) contributes to \( T \) a term

\[
\frac{1}{1 + \eta(z)} \mathcal{G}_0^{-1} |\psi_1 + \psi_2 + \psi_3 \rangle \langle \varphi_1^* + \varphi_2^* + \varphi_3^*|.
\]

(6)

When \( \eta \) is real, the Hermiticity of \( T \) entails that the bra and ket vectors in Eq. (5) are proportional, i.e., taking Eq. (5) into account, one must have

\[
\varphi_1 + \varphi_2 + \varphi_3 = \lambda (\varphi_1^* + \varphi_2^* + \varphi_3^*) \mathcal{G}_0
\]

(7)

where \( \lambda \) is a real number and \( \psi = \psi_1 + \psi_2 + \psi_3 = \varphi_1 + \varphi_2 + \varphi_3 \) is the bound-state wave function.

Equation (7), which is a consequence of Eq. (2), is an integral relation (and in fact is satisfied with \( \lambda = \eta/2 \)). However, when any one of the amplitudes \( t_i(z) \) is approximated by

\[
t_i(\vec{p}, \vec{q}, z) = \sum_n g_n(\vec{p}) g_n(\vec{q}) \ t_i(z)
\]

(8)

Eq. (7) expresses \( \varphi_i \) explicitly in terms of the two other \( \varphi \)'s. The approximation in Eq. (8) is justified when \( t_i \) is dominated by bound states, resonances, and virtual states, i.e., it is in fact fairly general.

En resume, we propose to compute three-body bound states by the variational principle of Eq. (4), the trial functions being restricted by the conditions of Eqs. (5) and (7) (i.e., \( \lambda \) real but otherwise arbitrary under the restriction of Eq. (8) for one of the two-body amplitude.
The extension of these results from bound states to resonances goes along well-known lines. Note, however, that the subsidiary condition (7) is correct only at the location of the resonance.
FOOTNOTES AND REFERENCES

1. This work was performed under the auspices of the U.S. Atomic Energy Commission.


6. A. Ahmadzadeh and R. Omnes, Calculation of the $\omega^0$ Characteristics, Lawrence Radiation Laboratory Report UCRL-11479, November 1964 (to be submitted to Phys. Rev.)


8. L.D. Faddeev, Mathematical Problems of the Quantum Theory of Scattering for a Three-Particle System (Publication of the Steklov Mathematical Institute, Leningrad, 1963, No. 69). We are grateful to Dr. J. B. Sykes for sending us a copy of his translation of that paper. [English transl: H. M. Stationery Office (Harwell, 1964).]

9. Note that other subsidiary conditions could be used. For instance, one could choose to use the first of Eqs. (2). In completely symmetric problems like the $\omega^0$ system, condition (8) cannot be used. However, an explicit form of the trial functions satisfying $\psi(\hat{z}^*) = \psi^*(z)$ will also suppress the complex eigenvalues.
9. Such a variational formulation for a non-Hermitian kernel was found to be very accurate in the two-body case by Jon Wright and Michael Scadron. A Variational Method for Finding Eigenvalues of the Lippmann-Schwinger Equation and the Corresponding Regge Trajectories, Lawrence Radiation Laboratory Report UCRL-11397, (April 1964), Nuovo Cimento (to be published).
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