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Determinacy in Infinite Horizon Exchange Economies

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Abstract

Infinite horizon economies and other models which naturally require an infinite number of commodities, such as product differentiation models and financial markets models with an infinite number of states, have become increasingly important for studying a wide array of economic problems. Although the existence and optimality of equilibria in such models have been studied in depth, almost nothing is known about qualitative properties of equilibria such as determinacy in general infinite horizon or infinite-dimensional economies. Moreover, resolving issues such as determinacy is crucial before such models can be used as the basis for any robust comparative statics analysis or for drawing any meaningful policy conclusions, as if equilibria are indeterminate, slight measurement error or variations in initial conditions can result in drastically different equilibria, and hence drastically different conclusions about the effects of changes in parameters or policies. This paper provides a framework for establishing the determinacy of equilibria in general infinite horizon models, establishes a meaningful notion of regular economy for such models, and gives sufficient conditions for regular infinite horizon economies to have a finite number of equilibria, each of which is locally stable with respect to perturbations in exogenous parameters, as well as for regular economies to be generic.
1 Introduction

Dynamic infinite horizon general equilibrium models provide the foundation for analyzing a diverse array of important issues in economics, issues ranging from the volatility of asset prices to the effects of government policies for deficit reduction like taxation or subsidy schemes, or the effects of various monetary policies such as the "optimum quantity of money" doctrine or interest rate pegging, among many others. In these models, as in standard finite horizon or finite-dimensional general equilibrium models, the principal issues to resolve involve establishing the existence of equilibria and characterizing the qualitative behavior of equilibria in terms of optimality, the uniqueness or local uniqueness of equilibria, and the dependence of equilibrium prices or allocations on exogenously specified parameters or policies in the model. Moreover, as long as equilibria exist, the most important issues to resolve in order to meaningfully answer these questions and predict, for example, the effect of implementing an interest rate peg or a particular taxation scheme, are these issues concerning the determinacy of equilibria. Of course the strongest predictions result from models with unique equilibria which vary in a stable manner in response to shocks or fluctuations in underlying parameters; however, as in economies with a finite horizon or a finite number of commodities, the conditions under which equilibria can be shown to be unique in these infinite horizon models are quite restrictive. Conditions such as the existence of a representative agent and the absence of any market distortions or nonconvexities, or global gross substitutability of all commodities (see Dana (1993)). Without such restrictive assumptions, local uniqueness or finiteness of the set of equilibria and determinate local comparative statics are the strongest qualitative properties which could obtain in a broad class of infinite horizon models with heterogeneous households.

Furthermore, determinacy is typically the minimum robustness criterion necessary in order to have confidence in calculations or comparative statics analysis performed with such models. Indeed, as a number of authors have discussed, models in which equilibria are not locally unique have quite troubling implications for theories built upon such models, as agents who know the exact characteristics of all other agents including endowments and preferences and the precise workings of markets still could not predict the possible equilibrium outcomes with perfect accuracy. Although such indeterminacies or "sunspot equilibria" are prevalent in many infinite horizon economies with uncertainty and market distortions such as externalities or incomplete markets, such a feature is even more disturbing in a model with no distortions or measurement error. Moreover, resolving the issue of determinacy is crucial before such models can be used as the basis for any robust comparative statics analysis or for drawing any meaningful policy conclusions, as if equilibria are indeterminate, slight measurement error or variations in initial conditions can result in drastically different equilibria, and hence drastically different conclusions about the effects of changes in parameters or policies.

The importance of infinite horizon general equilibrium models, as well as other models such as commodity differentiation models and financial markets models with an infinite state space which naturally give rise to an infinite-dimensional commodity space, has led to a substantial literature on equilibrium analysis with infinite-dimensional commodity spaces (see Aliprantis, Brown, and Burkinshaw (1989) or Mas-Colell and Zame (1991) for surveys). This burgeoning literature has produced a number of general results concerning the existence and optimality of competitive equilibria in infinite economies. However, in contrast with the
Walrasian economy with a finite number of commodities, which has been shown to have a rich and detailed structure even in the presence of a variety of market distortions such as increasing returns to scale or incomplete markets, a structure including generic finiteness of the set of equilibria and smooth dependence of equilibria on exogenous parameters such as initial endowments, almost nothing is known about qualitative features of equilibria such as local uniqueness or comparative statics in general infinite horizon models, even in the simplest case of a pure exchange economy with a finite number of infinitely lived households.

A notable exception to this claim is one of the canonical examples of infinite horizon models, the exchange economy with a finite number of infinitely lived consumers in which the consumer’s utility function is additively separable. In a discrete time, infinite horizon model with a finite number of infinitely lived households in which each consumer’s utility function is additively separable, Kehoe and Levine (1985) show that equilibria are generically locally unique. The standard approach to studying local uniqueness in economies with a finite number of goods involves characterizing equilibria as the prices which clear all markets, and using tools from differential topology to make precise Walras’s original idea of counting equations and unknowns by observing that there are as many independent prices to be determined as there are independent markets to clear. Using such an approach to study determinacy in an infinite horizon economy would result in an infinite system of equations in an infinite number of variables. An alternative characterization of equilibria which simplifies the problem is Negishi’s approach, relying on the welfare theorems to characterize equilibria as the prices and Pareto optimal allocations at which each consumer’s budget constraint is exactly satisfied (see Negishi (1960)). If applicable, this method of using Pareto optimality in an exchange economy with a finite number of households results in a characterization of equilibria in terms of a finite number of equations and unknowns, a number which depends only on the number of consumers and is independent of the number of goods in the economy. Indeed, the Negishi approach, first applied to demonstrate the existence of equilibria in infinite-dimensional models by Bewley (1969), has been central to much of the work on the existence of equilibria in economies with an infinite-dimensional commodity space (see, e.g., Mas-Colell (1986), Mas-Colell and Zame (1991)) for precisely this reason, and is the approach used by Kehoe and Levine to study determinacy in the additively separable model.

Moreover, because consumers’ preferences are additively separable in this model, consumption decisions in one period have no effect on marginal utility in any other period, and hence the infinite-dimensional problem of characterizing Pareto optimal allocations as functions of the welfare weights assigned to each consumer becomes simply a countable sequence of independent finite-dimensional problems, finding the Pareto optimal allocations in a sequence of independent, standard finite-dimensional economies. Standard finite-dimensional tools such as the implicit function theorem can be applied to each of these single period, finite economies to demonstrate that the Pareto optimal allocations and supporting prices are smooth functions of the welfare weights in each period. Once this has been established, each of the \( m \) consumers’ budget equations can be written as a smooth function of these welfare weights, and equilibria are solutions to the \( m - 1 \) independent budget equations in the \( m - 1 \) independent welfare weights. Since the equilibria are the solutions to a smooth finite system of equations, the standard techniques of differential topology such as the implicit function theorem and Sard’s theorem immediately yield the conclusion that equilibria are generically locally unique in this model.
Although the additively separable model yields sharp and straightforward results concerning determinacy of equilibria, the assumption of additive separability is quite restrictive as a model of intertemporal preferences for consumption over an infinite horizon. For example, the additively separable model implies that consumers' marginal rates of substitution between periods depend only on the amount of the goods consumed in each period, and are independent of consumption in any other period, which is contradicted by experimental evidence and casual empiricism (see, e.g., Peleg and Yaari (1973), Strotz (1956), Thaler (1990) or Laibson (1993)). Similarly, consumers are required to discount future utility at a constant rate which is independent of consumption in any period, and in models with uncertainty, additive separability requires that the rate of intertemporal substitution and the rate of risk aversion cannot be disentangled (see Epstein and Zin (1989)). As soon as one moves away from this model to one which allows for more interaction across time or between commodities in consumers' utility, such as models incorporating habit formation or recursive preferences, the argument used to establish determinacy in the additively separable case breaks down. No longer can the characterization of Pareto optimal allocations and supporting prices as functions of the welfare weights be decomposed into a sequence of independent finite-dimensional problems; such a characterization is inherently and inextricably an infinite-dimensional problem, and the difficulty of showing that these allocations and prices can be expressed as smooth functions of the welfare weights in this more general setting, and hence that the equilibrium equations are smooth, has been well-documented (see, e.g., Mas-Colell (1992) or Mas-Colell and Zame (1991)). Indeed, as is discussed in more detail throughout this paper, this model is one in which the nonsmoothness of the equilibrium equations is inherent in the problem; no simple assumptions such as differential convexity or that the indifference curves do not intersect the axes are at once consistent with the other requirements of the model and sufficient to guarantee smoothness.

The only other work on determinacy in general infinite horizon or infinite-dimensional exchange economies is the work of Kehoe, Levine, Mas-Colell, and Zame (1989), who study large square economies with a continuum of consumers. Returning to the standard approach of characterizing equilibria as the prices at which excess demand is equal to zero, they show in their model that equilibria are generically determinate as long as agents' characteristics are not too dissimilar, a special case of which is the model with a finite number of agents, by taking as primitives smooth demand functions for all consumers. Although such an approach is standard in models with a finite number of commodities, and reasonable assumptions on preferences such as differential convexity and strong survival conditions like requiring that indifference curves not intersect the axes will ensure the existence of smooth demand in finite economies (see Debreu (1972)), this approach is much more problematic in models with an infinite number of commodities. As has been well-documented in the literature on infinite economies, for many reasonable preferences over infinite-dimensional commodity spaces, demand functions are not defined for all prices, and in almost all infinite-dimensional spaces, including the Hilbert spaces used by Kehoe, Levine, Mas-Colell and Zame, the positive cone, which is the natural domain for consumption bundles and prices, has an empty interior. If the consumption set and price set have empty interiors, then to assume that utility functions and demand functions are smooth is to assume that utility functions are well-defined for both positive and negative quantities of goods and that demand functions are well-defined for both positive and negative prices.
Studying issues of determinacy in infinite horizon models allowing for more complicated interactions between consumption in different periods or for a more complicated relationship between time and uncertainty will then require a different mode of analysis, one not predicated on the finite-dimensional structure of the additively separable model or on the notion of excess demand, and one not necessarily requiring smoothness. This paper provides such a framework for establishing the determinacy of equilibria in general infinite horizon models with a finite number of consumers by defining a meaningful notion of regular economy for such models which is independent of smoothness, and by giving sufficient conditions for regular infinite horizon economies to have a finite number of equilibria, each of which is locally stable with respect to perturbations in exogenous parameters, as well as for regular economies to be generic.

There are two major steps in developing this framework. The first of these is the observation that smoothness is not necessary for studying local uniqueness and stability of solutions to a finite system of equations. Shannon (1994a) shows that there are natural notions of regular and critical value for nonsmooth equations for which analogues of all of the powerful results for smooth equations carry over: solutions to a system of equations at a regular value are locally unique, degree theory can be used in the same manner to get an estimate of the number of solutions, and if the equations are Lipschitz continuous, then the solutions vary with the parameter values in a Lipschitzian manner, so that small changes in the exogenous parameters yield only small changes in the resulting solutions. Finally, Rader (1973) has shown that an analogue of Sard's theorem holds for a broad class of nonsmooth functions including Lipschitz and locally Lipschitz functions. By using these results on nonsmooth equations together with the Negishi approach, this paper shows that equilibria in general infinite horizon exchange economies with a finite number of consumers are the solutions to a finite family of equations, and that even if the equations defining equilibria are not smooth, there is a natural notion of regular economy in this setting. Moreover, this paper shows that such regular economies possess all of the strong properties which hold for regular economies in finite-dimensional models. More precisely, this paper shows that if equilibria in infinite horizon models can be characterized as solutions to a finite system of Lipschitz equations, then regular economies have an odd number of equilibria, degree theory can be used to give a rough estimate of the number of equilibria, and each equilibrium varies locally with initial endowments in a Lipschitzian manner.

The second step is to establish reasonable conditions on the economic primitives of the model, preferences and endowments, under which these determinacy results apply, and in particular under which the equations describing equilibria are Lipschitz continuous. In addition to discussing several classes of examples including habit formation and recursive preferences, the paper introduces in the final section the main technique for applying these determinacy results, a Lipschitz implicit function theorem which holds for infinite-dimensional spaces.

The remainder of the paper is organized as follows. As a foundation for using the Negishi approach to characterize equilibria, section 2 discusses the structure of the set of Pareto optimal allocations in these infinite economies, and briefly reviews the main results concerning the determinacy of solutions to nonsmooth equations. Regular economies are defined in section 2, and the generic determinacy of equilibria is established for a broad class of economies. A simple application to an economy with recursive preferences is discussed in section 3, and
more general conditions on preferences under which these determinacy results hold are developed in section 4 by building on the lipschitz implicit function theorem presented there. Section 5 concludes.

2 Regular Infinite Horizon Economies

The models studied in this paper are exchange economies with a finite number of consumers in which the commodity space is $\ell_\infty$, the space of all bounded real-valued sequences. These models can be interpreted as, for example, discrete-time infinite horizon economies in which commodities are time-dated consumption and consumers are infinitely lived, or as economies with uncertainty over a countable number of exogenous states of the world in which commodities are state-contingent consumption. The first step in the Negishi method for characterizing and studying equilibria using the welfare theorems is to study the structure of the set of Pareto optimal allocations in these infinite horizon economies. In a model with a finite number of commodities and consumers. Smale (1976) shows that if each consumer's utility function is monotone, $C^2$, and differentiably strictly convex, then the set of Pareto optimal allocations is an $m - 1$ dimensional $C^1$ manifold, where $m$ is the number of consumers. One of the most important consequences of this work is that the dimension of the Pareto manifold depends only on the number of consumers and is independent of the number of goods in the economy, so long as the economy has a finite number of goods. By appealing to the First and Second Welfare theorems, equilibria can then be characterized as the prices and Pareto optimal allocations on this $m - 1$ dimensional manifold at which each of the $m - 1$ independent budget constraints is satisfied. Standard results from differential topology can then be used to argue that generically these equations have a finite number of solutions. In keeping with the nonsmooth nature of the infinite horizon economies studied in this paper, Shannon (1994b) shows that as long as consumers are sufficiently myopic in these infinite horizon models, the set of Pareto optimal allocations in these models will in general be a $C^0$ or topological manifold of dimension $m - 1$, rather than a $C^1$ manifold.1

Given the aggregate endowment vector $\omega$ and the consumers' utility functions $U_i: \ell_\infty^+ \to \mathcal{R}$ for $i = 1, \ldots, m$, where $\ell_\infty^+ = \{ x \in \ell_\infty : x_i \geq 0 \ \forall i \}$ is the positive cone in $\ell_\infty$, we can define the Pareto maximization problem:

$$\hat{U}_1(U_2, \ldots, U_m) = \max \ U_1(x_1)$$
subject to $U_j(x_j) \geq U_j, \ j = 2, \ldots, m$,

$$\sum_{j=1}^m x_j \leq \omega,$$
$$x_j \in \ell_\infty^+, \ j = 1, \ldots, m.$$ 

If it is well-defined, $\hat{U}_1(U)$ describes the maximum amount of utility that the first consumer can obtain in any feasible allocation in which each other consumer attains a utility level at least as high as the constraint level $U_j$. The graph of $\hat{U}_1$, denoted $\hat{U}$, is the utility possibility

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1 A set $\mathcal{M}$ is an $n$ dimensional $C^0$ or topological manifold if it is a Hausdorff topological space such that for every $x \in \mathcal{M}$, there exists an open neighborhood $W_x$ of $x$ and a homeomorphism $\phi_x: W_x \to \mathcal{R}^n$. 
or Pareto frontier, which is a subset of \( \mathcal{R}^m \) even though the commodity space is infinite-dimensional. Shannon (1994b) shows that as long as each consumer’s utility function is strictly concave, strictly monotone, and Mackey continuous for the pairing \((\ell_\infty, \ell_1)\),\(^2\) then \( \hat{U}_1 \) is not only well-defined, but strictly concave and continuous as well.

The assumption of Mackey continuity, which carries the behavioral interpretation of impatience or discounting, is important to ensure that consumers are discounting sufficiently for there to exist solutions to the Pareto optimization problem, yet the assumption of Mackey continuity also implies that consumers’ indifference curves will intersect the boundary of the positive cone (see Brown and Lewis, 1982). For example, if \( U_i(\cdot) \) is Mackey continuous and \( U_i(x) > U_i(y) \), then there exists \( n \) such that \( U_i(x_1, \ldots, x_n, 0, 0, \ldots) > U_i(y) \), so that preferences between bundles do not rely on consumption “at infinity”. Any assumption prohibiting the indifference curves from intersecting the boundary of the consumption set, assumptions standard in models with a finite number of commodities to guarantee the existence of smooth demand functions, smooth solutions to the Pareto maximization problem, or interior equilibria, will then be incompatible with the assumption of Mackey continuity.

Since the Pareto maximization problem is well-defined if each consumer’s utility function is strictly concave, strictly monotone and Mackey continuous, the function

\[
x(U) = \arg \max \ U_1(x_1) \\
\text{s.t. } U_j(x_j) \geq U_j, \quad j = 2, \ldots, m \\
\sum_{j=1}^{m} x_j \leq \omega
\]

is also well-defined on the domain of \( \hat{U}_1 \), which is the set

\[
D = \{ U \in \mathcal{R}_+^{m-1} : \exists \text{ an allocation } x \text{ such that } U_j(x_j) \geq U_j, \quad j = 2, \ldots, m \}.
\]

Moreover, if \( U \in \mathcal{U},^3 \) then \( x(U) \) is a Pareto optimal allocation, and thus \( x \) is a map from the Pareto frontier \( \mathcal{U} \) into the set of Pareto optimal allocations, which will be denoted \( \Theta \). Shannon (1994b) shows that \( x(U) \) is a homeomorphism between \( \mathcal{U} \) and \( \Theta \), which suffices to show that \( \Theta \) is a topological manifold of dimension \( m - 1 \). This result is weaker than the analogous result obtained by Smale (1976) in a model with a finite number of commodities: Smale showed that the set of Pareto optimal allocations is in fact a \( C^1 \) manifold. Of course Smale’s result follows from stronger assumptions than those made thus far concerning consumers’ utility functions, and in particular, Smale’s results follow from the assumptions that consumers’ utility functions are \( C^2 \) and differentiably strictly convex. No assumptions regarding smoothness of consumers’ utility functions have been made thus far here, so it is not surprising that the conclusion reached is that the Pareto set is a \( C^0 \) manifold rather than a \( C^1 \) or smooth manifold.

In order to guarantee that the Pareto map has this additional structure, we would clearly have to place more restrictions on consumers’ preferences than we have so far, including

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\(^2\)The Mackey topology for the pairing \((\ell_\infty, \ell_1)\), denoted \( \tau(\ell_\infty, \ell_1) \), is the locally convex topology on \( \ell_\infty \) of uniform convergence on \( \text{weak}^* (\sigma(\ell_\infty, \ell_1)) \) compact, convex, balanced sets. A set \( S \) is balanced if \( x \in S \) and \( |\lambda| \leq 1 \) implies that \( \lambda x \in S \). See Brown and Lewis (1982) for a discussion of the relationship between topologies on \( \ell_\infty \) and behavioral assumptions.

\(^3\)I am abusing notation a bit. I should say \((\hat{U}_1(U), U) \in \mathcal{U} \) since \( \mathcal{U} \subset \mathcal{R}^m \).
restrictions such as smoothness. One of the biggest obstacles to utilizing calculus techniques and results in infinite-dimensional economies is that in most infinite-dimensional spaces, the positive cone, which is the natural consumption set and the natural domain of both utility functions and prices, has an empty interior, which precludes the use of calculus on the consumption set or on the price space unless we assume that utility functions are defined for negative quantities of goods and that demand functions are defined for negative prices. For instance, of the sequence spaces $\ell_p$, $1 \leq p \leq \infty$, the only space in which the positive cone has nonempty interior is $\ell_\infty$. If the commodity space is $\ell_\infty$ as in the models in this paper, and each household's consumption set is $\ell_{\infty+}$, which does have a nonempty interior, then calculus assumptions and methods are not useless a priori. However, unlike finite-dimensional models, in which the interior of the consumption set $\mathcal{R}_+^i$ is simply the set $\mathcal{R}_{++}^i$ of all strictly positive bundles, the set of strictly positive vectors $\ell_{\infty++}$ is not open. It would be excessively restrictive and counter to the intuition that consumers are impatient over the infinite horizon to limit attention to utility functions defined only on the interior of $\ell_{\infty+}$, so the only meaningful assumption of smoothness in these models is the assumption that the utility function is $C^2$ on the interior of $\ell_{\infty+}$. Such an assumption means that while calculus results such as the implicit function theorem can be used in analyzing this model, they can only be used at interior points at which derivatives can be defined. In models with a finite number of commodities, it is easy to ensure that individual demand or Pareto optimal allocations are interior simply by assuming that indifference curves do not intersect the axes, i.e., lie entirely in the interior of the positive cone. An assumption like that here would mean that consumers cannot be impatient and must instead be infinitely farsighted. At the end of this section, I will return to this issue and discuss sufficient conditions to guarantee that all individually rational Pareto optimal allocations are interior allocations which are still consistent with the assumption of impatience.

Motivated by this discussion, the major assumptions on preferences which will be maintained throughout the paper are contained in the following definition.

**Definition 2.1.** Let $\omega = (\omega_1, \ldots, \omega_m)$ denote the vector of initial endowments. An exchange economy $E_\omega$ is an economy with smooth myopic preferences if the following conditions hold:

1. $\omega_i \in \text{int } \ell_{\infty+}$ for $i = 1, \ldots, m$.

2. $U_i$ is Mackey continuous, strictly concave, and strictly monotone on $\ell_{\infty+}$, and $U_i(0) = 0$ for $i = 1, \ldots, m$.

3. $U_i$ is $C^2$ on $\text{int } \ell_{\infty+}$ for $i = 1, \ldots, m$.

4. For $i = 1, \ldots, m$, $DU_i(x) \in \ell_{1++} \equiv \{ a \in \ell_1 : a_t > 0 \ \forall t \}$ for every $x \in \text{int } \ell_{\infty+}$.

5. For $i = 1, \ldots, m$, if $\{x^n\} \subseteq \text{int } \ell_{\infty+}$ and there exists $b > 0$ such that $\|x^n\| \leq b$ for all $n$, then $DU_i(x^n) \not\rightarrow 0$ pointwise.

6. For $i = 1, \ldots, m$, $D^2U_i(x)$ is negative definite for all $x \in \text{int } \ell_{\infty+}$.

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*Indeed, an element $x \in \ell_{\infty++}$ is an interior element if and only if it is bounded away from 0, so that $\inf_n x_n > 0$: every strictly positive sequence in $\ell_{\infty++}$ which converges to 0 is not an interior element.*
Most of these assumptions are self-explanatory. Following the previous discussion, the third condition is the most meaningful smoothness assumption which can be imposed in this model without violating myopia, and the sixth condition is the analogue of differentiably strictly convex preferences in this model. Condition (5) is a natural Inada-type condition in this setting, which says that as long as the consumption stream is bounded, then marginal utility for consumption does not go to zero in each period. The fourth condition calls for a bit of discussion. The assumption that \( DU_i(x) \in \ell_{1+} \) for every interior bundle \( x \) is really two assumptions: first, that \( DU_i(x) \in \ell_1 \), and second, that the derivative is in fact a strictly positive vector in \( \ell_1 \). The latter requirement is the natural analogue of differential monotonicity in this model, given the assumption that the derivative is indeed a sequence of partial derivatives. To understand the meaning of the first requirement, recall that by definition the derivative at each point is a continuous linear functional on \( \ell_\infty \), i.e., an element of the dual space of \( \ell_\infty \), which is \( \ell_1 \). To say that in fact the derivative lies in \( \ell_1 \) means that we can think of the derivative as a sequence of marginal utilities or generalized discount factors, and that as time goes to infinity, marginal utility of consumption in period \( t \) goes to 0. Then to say that the derivative lies in \( \ell_1 \) is to say that when calculating the marginal utilities of goods across time, consumers do not put much weight on consumption arbitrarily far in the future, and don't place any weight on consumption “at infinity”. As such, this requirement is a differential form of impatience or myopia, and so it should not be surprising that this property will always hold if the utility function is Mackey continuous, as the following result shows. Thus the only assumption made in (4) over and above the assumptions made previously is that marginal utility for each good is strictly positive.

**Theorem 2.1.** If \( U : \ell_\infty^+ \to \mathcal{R} \) is strictly concave, strictly monotone, continuously Fréchet differentiable on int \( \ell_\infty^+ \), and Mackey continuous, then \( DU(x) \in \ell_1 \) for every \( x \in \text{int} \ell_\infty^+ \).

**Proof:** Let \( x \in \text{int} \ell_\infty^+ \) be given. Since \( U \) is Fréchet differentiable at \( x \), \( DU(x) \in \ell_1 \), and \( DU(x) = \partial U(x) \), the set of subgradients of \( U \) at \( x \). By the Hewitt-Yosida theorem, \( p \equiv DU(x) = p_c + p_f \), where \( p_c \in \ell_1 \) and \( p_f \) is purely finitely additive. Then it suffices to show that \( p_f = 0 \). To see this, we will show that \( p_c \) is also a subgradient of \( U \) at \( x \). Suppose not, so that there exists \( z \) such that \( U(z) > U(x) \) and \( p_c z < p_c x \). Since \( U \) is Mackey continuous, there exists \( n \) such that \( U(z_n) > U(x) \), where \( z_n = (z_1, \ldots, z_n, 0, 0, \ldots) \), and \( p_c z_n < p_c x \). Thus

\[
p \cdot z_n = p_c \cdot \dot{z}_n + p_f \cdot \dot{z}_n = p_c \cdot \dot{z}_n < p_c \cdot x + p_f \cdot x = p \cdot x
\]

which is a contradiction, since \( p \) is a subgradient of \( U \) at \( x \). Thus \( DU(x) = p_c \in \ell_1 \).

For example, if \( U_i(x) = \sum_{i=0}^\infty \beta^i u_i(x_i) \), where for each \( i = 1, \ldots, m \), \( 0 < \beta_i < 1 \), and \( u_i : \mathcal{R}_+ \to \mathcal{R} \) is \( C^2 \) and satisfies the conditions \( u_i'(r) > 0 \) and \( u_i''(r) < 0 \) for every \( r \in \mathcal{R}_+ \), then for any given initial endowment vectors \( \omega \in \text{int} \ell_\infty^+ \), the economy \( E_\omega \) is an economy with smooth myopic preferences, as the following theorem shows.

**Theorem 2.2.** If \( U(x) = \sum_{i=0}^\infty \beta^i u(x_i) \), where \( 0 < \beta < 1 \) and \( u : \mathcal{R}_+ \to \mathcal{R} \) is \( C^2 \), \( u'(r) > 0 \) and \( u''(r) < 0 \) for every \( r \in \mathcal{R}_+ \), then \( U(x) \) is strictly monotone, strictly concave, Mackey
continuous, and $C^2$ on $\text{int } \ell_{\infty+}$. Moreover, $DU(x) \in \ell_{1+}$, if $\{x^n\} \subset \text{int } \ell_{\infty+}$ is bounded, then $DU(x^n) \not\rightarrow 0$ pointwise. and $D^2U(x)$ is negative definite for every $x \in \text{int } \ell_{\infty+}$.

Proof: See the appendix.

Now let $E_\omega$ be an economy with smooth myopic preferences. Equilibria of the exchange economy $E_\omega$ are prices and Pareto optimal allocations at which each consumer's budget constraint is satisfied, or solutions $(p,U)$ to the $m-1$ independent budget equations

$$p \cdot (x_2(U) - \omega_2) = 0$$

$$\vdots$$

$$p \cdot (x_m(U) - \omega_m) = 0.$$

The zeros of the function $E_\omega(p,U) : \ell_1 \times \mathcal{U} \rightarrow \mathbb{R}^{m-1}$ defined by

$$E_\omega(p,U) = (p \cdot (x_2(U) - \omega_2), \ldots, p \cdot (x_m(U) - \omega_m))$$

then define the equilibria of the exchange economy. Moreover, by using the nature of Pareto optimality, the possible equilibrium prices can also be given a finite-dimensional parameterization, at least for interior equilibria. If $x(U)$ is an interior allocation for some $U \in \mathcal{U}$, then prices $p$ support $x(U)$ as an equilibrium if and only if $p$ is proportional to $DU_i(x_i(U))$ for $i = 1, \ldots, m$. Moreover, if $x(U)$ is not an interior allocation but there exists some consumer $j$ whose bundle in this Pareto optimal allocation is an interior bundle, then $p$ will support $x(U)$ as an equilibrium if and only if $p$ is proportional to $DU_j(x_j(U))$. Let $\Theta^o = \Theta \cap \text{int } \ell_{\infty+}^m$, so that $\Theta^o$ is the set of interior Pareto optimal allocations. There is an open subset of utility levels corresponding to the set of interior Pareto optimal allocations which will then allow us to parameterize the possible equilibrium prices over $\Theta^o$. To define this subset, let $\Omega = \{U \in \mathbb{R}_+^{m-1} : x(U) \in \Theta^o\}$. The following lemma demonstrates that $\Omega$ is in fact an open subset of $\mathbb{R}_+^{m-1}$.

**Lemma 2.3.** $\Omega$ is an open subset of $\mathbb{R}_+^{m-1}$.

**Proof:** Define $U : \ell_{\infty+}^m \rightarrow \mathbb{R}^m$ by $U(x_1, \ldots, x_m) = (U_1(x_1), \ldots, U_m(x_m))$. Shannon (1994b) shows that $U(x)$ is a homeomorphism between $\Theta$ and $\mathcal{U}$. Let $D^o$ be the relative interior of $D$; Shannon (1994b) shows that $D^o$ is open. Then note that $\Theta^o$ is an open subset of the manifold $\Theta$, and hence $U(\Theta^o)$ is an open subset of $\mathcal{U}$ because $U(\cdot)$ is a homeomorphism. The function $g : D^o \rightarrow \mathcal{U}$ defined by $g(U) = (\hat{U}_1(U), U_2, \ldots, U_m)$ is continuous, so $\Omega = g^{-1}(U(\Theta^o))$ is an open subset of $D^o$.

As the previous argument shows, on $\Omega$ equilibria are solutions to the equations

$$DU_1(x_1(U)) \cdot (x_2(U) - \omega_2) = 0$$

$$\vdots$$

$$DU_1(x_1(U)) \cdot (x_m(U) - \omega_m) = 0.$$

These $m-1$ equations depend only on the $m-1$ variables $U_2, \ldots, U_m$, and completely characterize the interior equilibria. In fact, if we define $f : \Omega \rightarrow \mathbb{R}^{m-1}$ by

$$f(U) = (DU_1(x_1(U)) \cdot (x_2(U) - \omega_2), \ldots, DU_1(x_1(U)) \cdot (x_m(U) - \omega_m)),$$
then on this open set $\Omega$, the zeros of the map $f(U)$ define the equilibria.

If the Pareto map $x(U)$ is $C^1$, then the equilibrium equations will be $C^1$ and the tools of differential topology can be used to study determinacy in such models in a relatively straightforward manner. As noted above, the Pareto optimal allocations $x(U)$ are the solutions to the Pareto maximization problem, and assuming for the moment that problems regarding derivatives have been solved, either by restricting attention to interior Pareto optimal allocations, or by making assumptions sufficient to guarantee that all Pareto optimal allocations are interior, the (interior) Pareto optimal allocations $x(U)$ will be the solutions to the first order conditions for this problem:

$$ F(x, \lambda, U) \equiv \begin{pmatrix} DU_1(x_1) - \lambda_2 DU_2(x_2) \\ \vdots \\ DU_1(x_1) - \lambda_m DU_m(x_m) \\ \sum_{j=1}^{m} x_j - \omega \\ U_2(x_2) - U_2 \\ \vdots \\ U_m(x_m) - U_m \end{pmatrix} = 0. $$

To use the implicit function theorem to show that the Pareto map is smooth, we must show that $D_{(x, \lambda)} F(x, \lambda, U)$ is an isomorphism. However, by Theorem 2.1, since consumers are discounting in these models, $DU_i(x) \in \ell_1$ for every $x \in \text{int} \, \ell_\infty$. Thus $D_{(x, \lambda)} F(x, \lambda, U)$ maps $\ell_\infty^m$ into $\ell_1^m$, and could never be an isomorphism. The economic requirements of the problem thus bring about a fundamental failure of the conditions necessary to employ the implicit function theorem in infinite horizon models.

If the Pareto manifold and equilibrium equations are not smooth, the methodology of differential topology and global analysis so crucial to the study of regular economies with a finite number of commodities cannot be used to study issues of local uniqueness and stability of equilibria in infinite-dimensional models. However, Shannon (1994a) shows that there is a natural and powerful analogue to this smooth methodology for nonsmooth equations, and that notions of regular and critical value for nonsmooth functions introduced by Rader (1973) can be used to establish local uniqueness and sensitivity results, as well as to form the basis for degree theory, even in the absence of smoothness. I will briefly discuss these results, which will provide the foundation for studying determinacy in infinite horizon models in the rest of the paper.

The main idea behind these results is to generalize the notions of regular and critical value to a nonsmooth setting. As in the standard smooth case, given a function $f$ which is not necessarily smooth, we will want all values in the range to be either regular values or critical values of $f$, and thus meaningful or useful notions of regular and critical values for nonsmooth functions must differ somewhat from the standard definitions for smooth functions. The basic idea behind these new definitions for nonsmooth functions is to enlarge the set of critical points to include points where the function fails to be differentiable, as well as points where the derivative exists but is singular. Consequently, the set of critical values will be enlarged to include any value for which the function is not differentiable at

---

\[ \text{Since } \ell_1 \text{ is separable and } \ell_\infty \text{ is nonseparable, they could never be isomorphic.} \]
each preimage. More precisely, if \( f : A \to \mathcal{R}^n \) is continuous and \( A \subseteq \mathcal{R}^n \) is open, \( y \in \mathcal{R}^n \) is said to be a regular value of \( f \) if for every \( x \in f^{-1}(y) \), \( Df(x) \) exists and is nonsingular. Hence \( y \in \mathcal{R}^n \) is said to be a critical value of \( f \) if for some \( x \in f^{-1}(y) \), either \( f \) is not differentiable at \( x \), or \( f \) is differentiable at \( x \) but \( Df(x) \) is singular. Similarly, \( x \in A \) is a regular point of \( f \) if \( Df(x) \) exists and is nonsingular; \( x \in A \) is a critical point of \( f \) if either \( f \) is not differentiable at \( x \), or \( f \) is differentiable at \( x \) but \( Df(x) \) is singular. Note that if \( f \) is differentiable everywhere, these definitions agree with the standard ones.

These notions are the appropriate analogues of regular and critical values for studying the nonsmooth equations that arise in infinite horizon models. To substantiate this claim, first note that if \( y \) is a regular value of \( f \), then the solutions to the equation \( f(x) = y \) are well-behaved; in particular, if \( y \) is a regular value, each such solution is locally unique.

**Theorem 2.4** (Shannon (1994a), Theorem 1, Corollary 2). Let \( f : A \to \mathcal{R}^n \) be continuous, \( A \subseteq \mathcal{R}^n \) be open, and \( x \in A \) be a regular point of \( f \), so that \( Df(x) \) exists and is nonsingular. Let \( f(x) = y \). Then there exists a neighborhood \( U \) of \( x \) such that \( f^{-1}(y) \cap U = \{ x \} \); i.e., \( x \) is locally unique. If \( y \) is a regular value of \( f \) and \( y \notin f(\partial A) \), then \( f^{-1}(y) \) is a finite set.

As in the smooth case, the usefulness of this result depends on the size of the set of regular values. For an arbitrary continuous function, which may fail to be differentiable on a set of positive measure, and possibly be nowhere differentiable, this set may be very small. However, Rader (1973) has shown that for a particular class of nonsmooth functions, the set of critical values has measure 0.

**Theorem 2.5** (Rader (1973, Lemma 2)). Let \( f : A \to \mathcal{R}^n \) be differentiable almost everywhere, and have the property that if a set \( B \) has measure 0, so does the image \( f(B) \). Let \( A \subseteq \mathcal{R}^n \) be open, and \( S_f = \{ x \in A : x \) is a critical point of \( f \} \). Then \( f(S_f) \) has Lebesgue measure 0.

Such functions which are differentiable almost everywhere and map sets of measure 0 into sets of measure 0 include Lipschitz functions, locally Lipschitz functions, and pointwise Lipschitz functions.\(^6\)

In order to examine how equilibria vary as underlying parameters like endowments change, we will need results concerning sensitivity analysis for solutions to nonsmooth equations. In contrast with the smooth case, without stronger assumptions than the assumption that \( f \) is continuous and \( x \) is a regular point of \( f \), in general \( f \) will not even be locally invertible around \( x \),\(^7\) so that even if \( y \) is a regular value of \( f \) and we restrict attention to points arbitrarily close to \( x \) and \( y \), the solution set \( f^{-1}(y) \) may be multivalued as \( y \) varies. In order to answer sensitivity questions relating to the dependence of solutions to the equation \( f(x) = y \) on the value \( y \), one must then turn to recent work in nonsmooth analysis.

---

\(^6\)A function is said to be pointwise Lipschitz at \( x \) if there exists \( K > 0 \) and a neighborhood \( N \) of \( x \) such that for all \( \tilde{x} \in N \), \( \| f(x) - f(\tilde{x}) \| \leq K \| x - \tilde{x} \| \). That Lipschitz and locally Lipschitz functions \( f : \mathcal{R}^n \to \mathcal{R}^n \) are differentiable almost everywhere follows from Rademacher’s theorem (see, e.g., Federer (1969, 3.1.6)), and that such functions map sets of measure 0 into sets of measure 0 is well known (see, e.g., Federer (1969, 2.10.11)); for a proof that pointwise Lipschitz functions \( f : \mathcal{R}^n \to \mathcal{R}^n \) satisfy these properties, see Rader (1973, Lemma 3).

\(^7\)A counterexample, due to Andrew McLennan, is the function \( x^2 \sin(1/x) + x \), which is Lipschitz continuous and for which 0 is a regular point, but which is not one-to-one on any neighborhood of 0.
which has concentrated on developing generalized notions of derivatives for multifunctions, as well as for Lipschitz and other nonsmooth functions, and building a calculus around such derivatives, in part to answer such sensitivity questions when the function in question is not single-valued but multi-valued (see e.g., Clarke (1975), (1983); Rockafellar (1988); Aubin and Frankowska (1990)). By using results from nonsmooth analysis, Shannon (1994a) shows that if we restrict attention to Lipschitz functions, then in a neighborhood of a regular value, the solution set will be locally stable even if it is multi-valued. To make this statement precise requires a notion of stability for correspondences. A correspondence \( G : Y \Rightarrow X \) is said to be upper Lipschitzian at \( \bar{y} \in Y \) if there exists \( \lambda \geq 0 \) and a neighborhood \( \Omega \) of \( \bar{y} \) such that

\[
G(y) \subset G(\bar{y}) + \lambda \|y - \bar{y}\| B \quad \forall y \in \Omega,
\]

where \( B \) is the unit ball in \( X \). Note that this says that if \( x \in G(y) \), then there exists \( \bar{x} \in G(\bar{y}) \) such that \( \|x - \bar{x}\| \leq \lambda \|y - \bar{y}\| \), so that as \( y \) varies, the values of \( G(y) \) remain close in a Lipschtizian sense. This notion of stability turns out to be the appropriate notion when studying solutions of Lipschitz equations, as the following result indicates.

**Theorem 2.6** (Shannon (1994a), Theorem 8). Let \( f : A \to \mathcal{R}^n \) be locally Lipschitz, where \( A \subset \mathcal{R}^n \) is open. If \( y \) is a regular value of \( f \), then every \( x \in f^{-1}(y) \) is locally unique; that is, given \( x \in f^{-1}(y) \), there exists a neighborhood \( W \) of \( x \) such that \( W \cap f^{-1}(y) = \{x\} \). Moreover, \( W \cap f^{-1}(\cdot) \) is upper Lipschitzian at \( y \).

Thus if \( y \) is a regular value of the locally Lipschitz function \( f \), the solutions \( x \in f^{-1}(y) \) are both locally unique and locally stable. Moreover, since \( f^{-1}(y) \) is a discrete set if \( y \) is a regular value, the stability conclusion is actually stronger. Since there exists a neighborhood \( W \) of \( x \) such that \( W \cap f^{-1}(y) = \{x\} \) and \( W \cap f^{-1}(\cdot) \) is upper Lipschitzian at \( y \), for \( y' \) sufficiently close to \( y \), if \( f(x') = y' \) and \( x' \in W \), then \( \|x - x'\| \leq \lambda \|y - y'\| \).

Finally, if \( y \) is a regular value of \( f \), then the degree counts the solutions to the equation \( f(x) = y \) exactly as in the smooth case.

**Theorem 2.7** (Shannon (1994a), Theorem 10). Let \( f : \mathcal{A} \to \mathcal{R}^n \) be continuous, \( A \subset \mathcal{R}^n \) be open and bounded, and \( y \notin f(\partial A) \) be a regular value of \( f \). Then \( d(f,A,y) = \sum_{x \in f^{-1}(y)} \text{sgn} \det Df(x) \).

If equilibria in infinite horizon models can be characterized as the solutions to a system of Lipschitz equations \( f(x) = y \) and if \( y \) is a regular value, then the equilibria will be locally unique, the degree of \( f \) will give a lower bound on the number of equilibria, and each equilibrium will depend on the underlying parameters in a Lipschitzian manner, ensuring that small changes in these parameters will result in only small changes in equilibria.

These results concerning regularity of nonsmooth equations can then be applied to study the determinacy of equilibria in infinite horizon economies provided these economies can be meaningfully parameterized by some finite-dimensional set. Determinacy results for standard exchange economies with a finite-dimensional commodity space hold only generically, except for a set of economies of measure 0, where these economies are typically parameterized by their initial endowments. When the commodity space is infinite-dimensional, parameterizing
the family of economies by the initial endowment vector is no longer a finite-dimensional parameterization. In these infinite horizon economies we want to express equilibria as solutions to an equation of the form \( f(U) = y \), where the values of \( y \) lie in a finite-dimensional space and index the family of economies. To do this, prices can first be normalized by setting the price of the first good equal to 1, or equivalently, by dividing the equilibrium map \( f(U) \) by \( p_1(U) \equiv (DU_1(x_1(U)))_1 \). By letting \( \bar{p}(U) = \frac{DU_1(x_1(U))}{p_1(U)} \) and \( \bar{\omega}_j = (0, \omega_{j2}, \omega_{j3}, \ldots) \), the resulting equations are

\[
F(U) \equiv \begin{pmatrix}
\bar{p}(U) \cdot (x_2(U) - \bar{\omega}_2) \\
\vdots \\
\bar{p}(U) \cdot (x_m(U) - \bar{\omega}_m)
\end{pmatrix} = \begin{pmatrix}
\omega_{j1} \\
\vdots \\
\omega_{m1}
\end{pmatrix}.
\]

Then define

\[
\mathcal{W} \equiv \{ \omega^1 = (\omega_{21}, \ldots, \omega_{m1}) \in \mathcal{R}^{m-1}_+ : \sum_{j=2}^{m} \omega_{ji} < \omega_1 \}.
\]

The open set \( \mathcal{W} \) then gives the desired finite dimensional parameterization of these infinite exchange economies, since the interior equilibria are the solutions to the equation \( F(U) = \omega^1 \), where \( F : \Omega \rightarrow \mathcal{R}^{m-1} \) and \( \Omega \subset \mathcal{R}^{m-1} \) is open. The definitions of regular points and values for nonsmooth functions now lead to a natural and meaningful definition of regular economy in models with an infinite-dimensional commodity space. Given an economy \( \mathcal{E}_\omega \) with smooth myopic preferences, we will say that \( \mathcal{E}_\omega \) is a regular economy if \( \omega^1 \) is a regular value of \( F \). With this definition of regular economy, we can establish the following.

**Theorem 2.8.** If \( \mathcal{E}_\omega \) is a regular economy with smooth myopic preferences, then the interior equilibria are locally unique. If \( F : \Omega \rightarrow \mathcal{R}^{m-1} \) is differentiable almost everywhere and maps sets of measure 0 into sets of measure 0, then for almost every \( \omega^1 \in \mathcal{W} \), the economy \( \mathcal{E}_\omega \) is a regular economy.

**Proof:** This follows from Theorem 2.4 and Theorem 2.5. \(\blacksquare\)

In particular, if \( F \) is Lipschitz, locally Lipschitz, or pointwise Lipschitz on \( \Omega \), then \( F \) will be differentiable almost everywhere and map sets of measure 0 into sets of measure 0, and hence almost every economy \( \mathcal{E}_\omega \) will be regular.

Furthermore, the strongest results concerning regularity of nonsmooth equations are obtained in the case where the equations in question are locally Lipschitz continuous. In that case, almost every value \( \omega^1 \) is a regular value of \( F \), and the solutions to the equations at a regular value are locally unique and upper Lipschitzian, which implies that the solutions are stable with respect to perturbations in this value \( \omega^1 \). As the following lemma shows, whether the equilibrium equations are locally Lipschitz may depend, not surprisingly, on the properties of the Pareto map \( x(U) \): if \( \mathcal{E}_\omega \) is an economy with smooth myopic preferences and the Pareto map \( x(U) \) is locally Lipschitz continuous, then the equilibrium map \( F(U) \) will indeed be locally Lipschitz on \( \Omega \).

**Lemma 2.9.** If \( \mathcal{E}_\omega \) is an economy with smooth myopic preferences and the Pareto map \( x(U) \) is locally Lipschitz, then \( F(U) \) is locally Lipschitz on \( \Omega \).

**Proof:** See the appendix. \(\blacksquare\)

If the equilibrium equations are locally Lipschitz, the economy \( \mathcal{E}_\omega \) will be called a Lipschitz economy. As long as the economy is a Lipschitz economy, these strong determinacy
results described above will hold: regular economies will have locally unique equilibria which are upper Lipschitzian in ω₁, and regular economies will be generic. These results are summarized in Theorem 2.10. As we have seen, a sufficient condition for the economy to be a Lipschitz economy is that the Pareto map x(U) is locally Lipschitz. A natural question is then under what restrictions on preferences is the Pareto map x(U) locally Lipschitz continuous, or more generally, what conditions on preferences ensure that the equilibrium equations are locally Lipschitz? These questions are the focus of the final sections.

**Theorem 2.10.** If Eω is a Lipschitz economy with smooth myopic preferences, then for almost all ω₁ ∈ W, the economy Eω is a regular economy. Moreover, if Eω is a regular Lipschitz economy, then the interior equilibria are locally unique and upper Lipschitzian in ω₁.

**Proof:** This follows from Lemma 2.9, and Theorems 2.4, 2.5, and 2.6.  

If the equilibrium equations are locally Lipschitz and if in addition all of the Pareto optimal allocations at which each consumer has positive utility are interior allocations, then all of the equilibria will be interior, and in this case we can make a stronger statement about the equilibria in regular economies. In this case, not only will a regular economy have locally unique equilibria, but it will in fact have an odd number of equilibria, each of which is upper Lipschitzian in ω₁. To parameterize the set of Pareto optimal allocations at which each consumer has positive utility, let D₀ be the relative interior of D, so that D₀ = {U ∈ D : U_j > 0 ∀ j and U₁(U) > 0}. Let Θ⁺ = {x ∈ Θ : x_i ≠ 0 ∀ i = 1, . . . , m} = x(D₀), so that Θ⁺ is precisely the set of Pareto optimal allocations at which each consumer has positive utility.

**Theorem 2.11.** If Eω is a regular Lipschitz economy with smooth myopic preferences and Θ⁺ ⊂ int ℓ∞m, then the economy Eω has an odd number of equilibria which are upper Lipschitzian in ω₁. Moreover, for almost every ω₁ ∈ W, the economy Eω is a regular economy.

**Proof:** If Θ⁺ ⊂ int ℓ∞m, then on the entire set D₀, which is open by Shannon (1994b), the equilibria are the solutions to the equation F(U) = ω₁. Let ω₁ be a regular value of F. The theorem will be proven by constructing an appropriate homotopy between the equilibrium equations and a set of equations whose degree is known to be 1, and appealing to homotopy invariance. Choose ε > 0 such that ε < U_i(ω_i)/2 for every i = 1, . . . , m, and let

$$\Omega₁ = \{U ∈ D₀ : U_i > U_i(ω_i) - ε, i = 1, . . . , m\}.$$  

Since any equilibrium must be individually rational, if F(U) = ω₁ then U ∈ Ω₁. Note that Ω₁ is open, and Ω₁ ∩ ∂D₀ = ∅. Then by Urysohn's lemma there exists a smooth function χ : ∂D₀ → [0, 1] such that

$$χ(∂₁) = 1;$$
$$χ(∂D₀) = 0.$$  

Now choose U ∈ D₀ such that \(\sum_{j=2}^{m} U_j < \sum_{j=2}^{m} U_j\) for every U ∈ \{U ∈ ∂D₀ : U_j > 0, j = 2, . . . , m\}. Such a U exists as \{U ∈ ∂D₀ : U_j > 0, j = 2, . . . , m\} = graph U₂, where

$$U₂(U) = \max U₂(x_2)$$
s.t. \( U_j(x_j) \geq U_j, \; j = 3, \ldots, m \)

\[
\sum_{j=2}^{m} x_j \leq \omega
\]

\[
x_j \in \ell_{\infty+}, \; j = 1, \ldots, m,
\]

The set graph \( \hat{U}_2 \) is a compact set on which \( \sum_{j=2}^{m} U_j > 0 \), so \( \min_{\text{graph } \hat{U}_2} \sum_{j=2}^{m} U_j > 0 \). It is clear then that there exists \( \tilde{U} \in D^o \) such that \( \sum_{j=2}^{m} \tilde{U}_j < \sum_{j=2}^{m} U_j \) for every \( U \in \text{graph } \hat{U}_2 \). Define \( h(U) = (1 - \chi(U))\tilde{U} + \chi(U)U \), and note that if \( U \in \Omega_1 \), \( h(U) \equiv U \), and if \( U \in \partial D^o \), \( h(U) \equiv \tilde{U} \). Then \( h \) is clearly locally Lipschitz, and so is

\[
\hat{F}(U) = ((\hat{p}(h(U))) \cdot (x_2(U) - \omega_2), \ldots, \hat{p}(h(U)) \cdot (x_m(U) - \omega_m)).
\]

Moreover, \( \hat{F}(U) = \omega^1 \iff F(U) = \omega^1 \), as there exist prices supporting \( x(U) \) as an equilibrium if and only if \( U \) is individually rational, and hence an element of \( \Omega_1 \).

Now define \( H(U, t) : D^o \times [0, 1] \to \mathbb{R}^{m-1} \) by

\[
H(U, t) = (1 - t)(U - \tilde{U}) + t(\hat{F}(U) - \omega^1).
\]

\( H(\cdot) \) is locally Lipschitz, and \( H(U, 0) \equiv U - \tilde{U} \), so \( d(H(U, 0), D^o, 0) = 1 \). Then by homotopy invariance, to show \( d(\hat{F} - \omega^1, D^o, 0) = 1 \), it suffices to show that \( H(\partial D^o, t) \neq 0 \) for every \( t \in [0, 1] \). Clearly if \( t = 0 \) or if \( t = 1 \), \( H(\partial D^o, t) \neq 0 \), so suppose \( t \in (0, 1) \) and \( H(U, t) = 0 \). Then

\[
ti(\hat{F}(U) - \omega^1) = -(1 - t)(U - \tilde{U}),
\]

or

\[
t\hat{p}(h(U)) \cdot (x_j(U) - \omega_j) = -(1 - t)(U_j - \tilde{U}_j), \; j = 2, \ldots, m.
\]

(1)

If \( U \in \partial D^o \), then \( U_i = 0 \) for some \( i = 1, \ldots, m \), where \( U_1 = \hat{U}_1(U) \). If \( U_i = 0 \) for some \( i = 2, \ldots, m \), then \( x_i(U) = 0 \), and (1) implies

\[
0 > -t\hat{p}(h(U)) \cdot \omega_i = t\hat{p}(h(U)) \cdot (x_i(U) - \omega_i) = (1 - t)\hat{U}_i > 0,
\]

which is a contradiction. Suppose \( U_1 = 0 \). By (1),

\[
t\sum_{j=2}^{m} \hat{p}(h(U)) \cdot (x_j(U) - \omega_j) = -(1 - t)\sum_{j=2}^{m} (U_j - \tilde{U}_j),
\]

and thus

\[
-t\hat{p}(h(U)) \cdot (x_1(U) - \omega_1) = -(1 - t)\sum_{j=2}^{m} (U_j - \tilde{U}_j).
\]

(2)

Since \( U_1 = 0 \), \( x_1(U) = 0 \), and then \( U \in \partial D^o \Rightarrow U \in \text{graph } \hat{U}_2 \), as \( U_1 = 0 \). Then by choice of \( \tilde{U} \), \( \sum_{j=2}^{m} (U_j - \tilde{U}_j) > 0 \). Thus (2) implies that if \( U \in \partial D^o \) and \( H(U, t) = 0 \),

\[
0 < t\hat{p}(h(U)) \cdot \omega_1 = -t\hat{p}(h(U)) \cdot (x_1(U) - \omega_1) = -(1 - t)\sum_{j=2}^{m} (U_j - \tilde{U}_j) < 0,
\]

16
which is a contradiction. Hence $H(\partial D^o, t) \neq 0$ for every $t \in [0, 1]$. By homotopy invariance, $d(\tilde{F} - \omega^1, D^o, 0) = 1$. Clearly $\tilde{F}(U) - \omega^1 = 0 \Rightarrow U \in \Omega_1$, and $\tilde{F}|_{\Omega_1} = F|_{\Omega_1}$. Since $\omega^1$ is a regular value of $F$, it is thus a regular value of $\tilde{F}$ as well. By Theorem 2.7,

$$
1 = d(\tilde{F}, D^o, \omega^1) = \sum_{U \in F^{-1}(\omega^1)} \text{sgn} \det D\tilde{F}(U)
$$

$$
= \sum_{U \in F^{-1}(\omega^1)} \text{sgn} \det DF(U).
$$

Hence $F^{-1}(\omega^1)$ has an odd number of elements. The remaining statements follow immediately from Theorem 2.5.

After studying the equilibrium equations, it should be apparent that the entire Pareto optimal allocation $x(U)$ need not lie in the interior of $\ell_{\infty}^m$ in order to express equilibria as the solutions to such equations. For these results, the role of this interiority restriction is simply to ensure that the possible equilibrium prices can be given a finite-dimensional parameterization using the first order conditions for Pareto optimality. The entire Pareto optimal allocation need not be interior as long as we can find a sufficiently well-behaved parameterization of equilibrium prices in terms of the utility levels, or some other finite set of variables. For example, if $\Theta_i = \{ x \in \Theta : x_i \in \text{int} (\ell_{\infty}^m) \}$, then on $\Theta_i$, equilibria will be solutions to the equations

$$
DU_i(x_i(U)) \cdot (x_2(U) - \omega_2) = 0
$$

$$
\vdots
$$

$$
DU_i(x_i(U)) \cdot (x_m(U) - \omega_m) = 0,
$$

or letting $\tilde{p}^i(U) = \frac{1}{(DU_i(x_i(U)))} DU_i(x_i(U))$, equilibria in $\Theta_i$ are solutions to the equation

$$
F^i(U) = (\tilde{p}^i(U) \cdot (x_2(U) - \omega_2), \ldots, \tilde{p}^i(U) \cdot (x_m(U) - \omega_m)) = \omega^1.
$$

An analogous version of Theorem 2.11 holds provided only that there is one consumer, the interior consumer, who always consumes an interior bundle in any Pareto optimal allocation in which each consumer receives positive utility; i.e., provided $\Theta_i = \Theta_+$ for some $i = 1, \ldots, m$.

**Theorem 2.12.** If $E_\omega$ is a regular Lipschitz economy with smooth myopic preferences and there is an interior consumer, so that $\Theta_i = \Theta_+$ for some $i = 1, \ldots, m$, then the economy $E_\omega$ has an odd number of equilibria which are upper Lipschitzian in $\omega^1$. Moreover, for almost every $\omega^1 \in W$, the economy $E_\omega$ is a regular economy.

**Proof:** This is proven by the same argument which proves Theorem 2.11.

By now it should be clear that guaranteeing that Pareto optimal allocations are interior, at least for some consumer, is of fundamental importance for this approach to determinacy in infinite horizon economies. This section concludes by examining a series of successively stronger survival conditions which imply successively stronger interiority properties. The first of these conditions is the weakest, and will ensure that all consumers consume positive quantities of all goods at any Pareto optimal allocation which assigns positive utility to each consumer, and so in particular that all individually rational Pareto optimal allocations are strictly positive.
Definition 2.2. Consumer \( i \) satisfies the **weak survival condition** if for every \( x \in \ell_{\infty}^+ \) and for every \( t \), if \( x_t > 0 \) then \( \frac{\partial U}{\partial x_t}(x_t, x_{-t}) \) exists, and

\[
\frac{\partial U}{\partial x_t}(x_t, x_{-t}) \to \infty \text{ as } x_t \to 0
\]

for all fixed \( x_{-t} \) such that \( x_s > 0 \).\(^8\)

The weak survival condition says that if consumption of some good goes to zero, then the consumer's marginal rate of substitution for that good with respect to some other good he consumes in a fixed positive amount goes to infinity as the amount of the diminishing good goes to 0, and this must hold regardless of the (fixed) level of consumption of other goods. This condition, if satisfied by all consumers in the economy, means that consumption bundles are strictly positive in Pareto optimal allocations, as I first establish for finite-dimensional economies.

**Theorem 2.13.** If \( \mathcal{E}_\omega \) is a finite-dimensional economy in which each consumer's utility function satisfies the weak survival condition, then all positive utility Pareto optimal allocations are interior.

**Proof:** Suppose not, so suppose that there exists a Pareto optimal allocation \((x_1, \ldots, x_m)\) such that \( U_i(x_i) > 0 \) for all \( i \), and such that \( x_{1_j} = 0 \) for some \( j \). Since \( U_i(x_1) > 0 \), there exists \( \delta \) such that \( x_{1_k} > 0 \). Moreover, by feasibility there exists \( i \) such that \( x_{i_j} > 0 \). Let \( \epsilon, \delta > 0 \). Then letting \( e_i \) be the \( i^{th} \) unit vector,

\[
U_i(x_i - \epsilon e_j + \delta e_k) - U_i(x_i) = DU_i(\hat{x}_i) \cdot (\delta e_k - \epsilon e_j)
\]

\[
= \delta \frac{\partial U_i}{\partial x_k}(\hat{x}_i) - \epsilon \frac{\partial U_i}{\partial x_j}(\hat{x}_i)
\]

for some \( \hat{x}_i = \alpha(x_i - \epsilon e_j + \delta e_k) + (1 - \alpha)x_i \), where \( \alpha \in (0, 1) \). In particular, note that \( \hat{x}_{i_j} > 0 \).

So

\[
U_i(x_i - \epsilon e_j + \delta e_k) - U_i(x_i) \geq 0 \iff \frac{\delta}{\epsilon} \frac{\partial U_i}{\partial x_k}(\hat{x}_i).
\]

Moreover, if \( \epsilon > 0 \),

\[
U_i(x_i + \epsilon e_j - \delta e_k) - U_i(x_i) \geq U_i(x_i + \epsilon e_j - \delta e_k) - U_i(x_i + \frac{\epsilon}{c} \epsilon e_j)
\]

\[
= DU_i(\hat{x}_1) \cdot (\frac{\epsilon}{c} - 1) e_j - \delta e_k)
\]

for some \( \hat{x}_1 = (1 - \gamma)(x_1 + \epsilon e_j - \delta e_k) + \gamma(x_1 + \frac{\epsilon}{c} \epsilon e_j) \). In particular, note that \( \hat{x}_{1_j} = (1 - \gamma)\epsilon + \frac{\epsilon}{c - 1} \epsilon \) and \( \hat{x}_{1_k} = x_{1_k} - (1 - \gamma)\delta \). Moreover, note that if \( \frac{\epsilon}{c - 1} \epsilon < \frac{\partial U_i}{\partial x_k}(\hat{x}_1) \), then

\[
U_i(x_1 + \epsilon e_j - \delta e_k) - U_i(x_1) > 0.\]

Now we can find constants \( \delta, \epsilon, c \) such that

\[
\frac{\partial U_i}{\partial x_k}(\hat{x}_1) \leq \frac{\delta}{\epsilon} < \frac{c}{c - 1} \delta < \frac{\partial U_i}{\partial x_k}(\hat{x}_1)
\]

\(^8\)Here, \( x_{-t} \) denotes the elements of \( x \) in periods \( t \neq t \).
since the left hand side is bounded and the right hand side goes to infinity as \( \epsilon \to 0 \). This is a contradiction, since we have thus found a Pareto improving trade.

In a finite-dimensional model, strictly positive bundles are interior bundles, but since that is not true in infinite-dimensional models, stronger conditions will be necessary to ensure that Pareto optimal allocations lie in the interior.

**Definition 2.3.** In an economy \( \mathcal{E}_\omega \), let \( x \) be an allocation in which \( x_{i_r} \to 0 \) for some sequence \( r \to \infty \), and in which there exists a consumer \( j \) and a fixed constant \( c > 0 \) such that \( x_{j_r} \geq c \) for every \( r \). Consumer \( i \) satisfies the survival condition if he satisfies the weak survival condition, and if

\[
\lim r \sup \frac{\partial U_i}{\partial x_{i_r}}(x_i) = \infty.
\]

Let \( x^n \) be a sequence of allocations such that \( x^n_k \in \text{int } \ell_{\infty}^+ \) for each \( k \) and \( n \), and such that for some sequence \( \{r(n)\} \) such that \( r(n) \to \infty \) as \( n \to \infty \), \( x^n_{i_{r(n)}} \to 0 \) and for some consumer \( j \) and some fixed constant \( c > 0 \), \( x^n_{j_{r(n)}} \geq c \) for all \( n \). Consumer \( i \) satisfies the strong survival condition if in addition to satisfying the survival condition,

\[
\lim \sup \frac{\partial U_i}{\partial x_{i_{r(n)}}} \left( x^n_{i} \right) = \infty.
\]

These definitions embody several different conditions. First, marginal utility for a good must go to infinity as the quantity of the good consumed goes to zero, and consumers should be discounting at roughly the same rate, which guarantees that the ratio of marginal utilities of consumption in time \( t \) of two consumers, calculated at bundles in which one consumer's consumption in period \( t \) is bounded away from zero uniformly across time and the other consumer's consumption is going to zero as \( t \) goes to infinity, should converge to infinity with \( t \). The strong survival condition strengthens this notion by requiring this convergence to hold across sequences of allocations. These stronger conditions are sufficient to guarantee that the Pareto optimal allocations are interior, and the strong survival condition guarantees in addition that the individually rational Pareto optimal allocations are uniformly bounded away from 0.

**Theorem 2.14.** If each consumer satisfies the weak survival condition and some consumer \( j \) satisfies the survival condition, then consumer \( j \) is an interior consumer. If all consumers satisfy the survival condition, then all positive utility Pareto optimal allocations are interior.

**Proof:** I will prove the second claim; the proof of the first is the same. Let \( (x_1, \ldots, x_m) \) be a positive utility Pareto optimal allocation. First note that \( x_{i_t} \neq 0 \) for all \( i \) and \( t \). To see this, suppose not, so suppose there exists \( i, t \) such that \( x_{i_t} = 0 \). Since \( U_j(x_j) > 0 \) for all \( j \), there exists \( t' \) such that for each consumer \( j \), \( x_{j_{t'}} > 0 \) for some \( r \leq t' \). For each \( s \) and for each \( z \in \ell_{\infty} \), define \( \tilde{z} = (z_1, \ldots, z_s) \). Let \( T = \max(t, t') \), and consider the truncated \( T \) period economy in which consumer \( j \)'s utility function \( U_j : \mathcal{R}_+^T \to \mathcal{R} \) is given by \( U_j(y) \equiv U_j(y, x_{T+1}, x_{T+2}, \ldots) \) for each \( y \in \mathcal{R}_+^T \), and in which consumer \( j \)'s endowment is \( \tilde{w}_j \). Clearly \( (\tilde{x}_1^T, \ldots, \tilde{x}_m^T) \) is a Pareto optimal allocation in this economy, and each consumer satisfies the weak survival condition in this finite-dimensional economy, so by the previous
theorem, \( \tilde{x}^t_{j_k} = x_{j_k} > 0 \) for all \( j \) and \( k \), which is a contradiction. So \( x_i \in \ell_{\infty+} \) for all \( i \). Now suppose there exists \( i \) such that \( x_{j_i} \not\in \text{int } \ell_{\infty+} \). Then there exists a sequence \( r \to \infty \) such that \( x_{j_i} \to 0 \). Moreover, by feasibility there exists a consumer \( j \) such that, by passing to a subsequence and relabeling if necessary, for each \( r \), \( x_{j_i} \geq \frac{\omega}{m} \) where \( \omega = \inf \omega_i > 0 \), where \( \omega_i \) is the aggregate endowment of good \( t \). Define

\[
\tilde{p}^i = \left( \frac{\partial U_i}{\partial x_1}(x_j), \ldots, \frac{\partial U_i}{\partial x_t}(x_j) \right)
\]

for each \( t \). Since \( (\tilde{x}^t_1, \ldots, \tilde{x}^t_m) \) is Pareto optimal in the truncated \( t \) period economy for each \( t \), the price \( \tilde{p}^i \) supports \( \tilde{x}^t_i \) for each \( r \). In particular, for each \( r \),

\[
\frac{\partial U_i}{\partial x_1}(x_i) = \frac{\partial U_i}{\partial x_1}(x_j),
\]

Without loss of generality, assume \( r_0 = 1 \). Thus for each \( r \),

\[
\frac{\partial U_r}{\partial x_1}(x_j) = \frac{\partial U_r}{\partial x_1}(x_i),
\]

but this is a contradiction, since the terms on the left are unbounded as \( r \to \infty \), and the term on the right is bounded. Thus \( x_i \in \text{int } \ell_{\infty+} \) for all \( i \).

**Theorem 2.15.** Suppose that each consumer satisfies the strong survival condition. Then there exists \( \epsilon > 0 \) such that every individually rational Pareto optimal allocation is bounded below by \( \epsilon \), i.e., such that for each individually rational Pareto optimal allocation \( x \), \( \inf_i |x_{i_t}| \geq \epsilon \) for all \( i \).

**Proof:** First, by the previous theorem, all individually rational Pareto optimal allocations are interior. Now suppose the theorem is false, so that there exists a sequence of individually rational Pareto optimal allocations \( x^n \) and there exists \( i \) such that \( \inf_i |x^n_{i_t}| = b^n \to 0 \) as \( n \to \infty \). Since each \( x^n \) is interior, for every \( n \) there exist \( \lambda^n_2, \ldots, \lambda^n_m > 0 \) such that \( DU_1(x^n_j) = \lambda^n_j DU_j(x^n_j) \) for each \( j \). So for each \( j, t, n \),

\[
\frac{\partial U_i}{\partial x_1}(x^n_j) = \frac{\lambda^n_j}{\lambda^n_i}.
\]

Now I claim that there exist \( b, B > 0 \) such that \( b \leq \lambda^n_j \leq B \) for all \( j \) and \( n \). To see this, define

\[
\gamma^n_j = \frac{\lambda^n_j}{1 + \sum_{k=2}^m \lambda^n_k}
\]

for each \( j = 2, \ldots, m \), and define \( \gamma^n_1 = 1 - \sum_{j=2}^m \gamma^n_j \). Then it suffices to show that there exists \( b > 0 \) such that \( \gamma^n_j \geq b \) for all \( j \) and \( n \). However, note that \( x^n \) solves the Pareto maximization
problem with constraint \( U = (U_2(x_2^n), \ldots, U_m(x_m^n)) \) iff \( x^n \) solves the problem

\[
\max \sum_{j=1}^m \gamma_j U_j(x_j) \\
\text{s.t.} \sum_{j=1}^m x_j \leq \omega.
\]

By Theorem 2.2 of Shannon (1994b), \( \hat{U}_1 \) is strictly concave, where \( \hat{U}_1 \) is the value function for the Pareto maximization problem. Let \( \tilde{V} = \{ U = (U_1, U) \in \mathcal{U} : U_j(\omega_j) \leq U_j(\omega), j = 1, \ldots, m \} \). Let \( (\hat{U}_1, \hat{U}) \in \tilde{V} \), and let \( \partial \hat{U}_1(\hat{U}) \) denote the set of subgradients of \( \hat{U}_1 \) at \( \hat{U} \), which is nonempty since \( \hat{U}_1 \) is continuous. Then for \( y \in \partial \hat{U}_1(\hat{U}) \), let \( \hat{y} = (1, -y) \). Then (see, e.g., Rockafellar (1970), Theorem 23.5)

\[
(\hat{U}_1, \hat{U}) = \arg\max_{U \in \tilde{V}} \hat{y} \cdot U.
\]

Thus since \( U \gg \emptyset, \hat{y} \gg \emptyset \). Moreover, \( y \) is uniformly bounded, i.e., there exists \( M \) such that \( ||q|| < M \) for all elements \( q \in \partial \hat{U}_1(U) \) for all \( U \in \tilde{V} \) (see, e.g., Phelps. Prop. 1.11). Let 

\[
\gamma = \frac{1}{\sum_{i=1}^m \hat{y}_i} \hat{y}_i
\]

and note that \( \gamma_i > 0 \) for all \( i \) and \( \sum_{i=1}^m \gamma_i = 1 \). Then if

\[
\bar{x} = \arg\max_{(x_1, \ldots, x_m)} \sum_{i=1}^m \gamma_i U_i(x_i) \\
\text{s.t.} \sum_{i=1}^m x_i \leq \omega,
\]

then \( U(\bar{x}) = \hat{U} \), and hence \( \bar{x} = x(\hat{U}) \). Now it suffices to show that there exists \( \epsilon > 0 \) such that \( \gamma_i > \epsilon \) for every \( i \). If not, then there exists a sequence \( \{\gamma^n\} \) such that for some \( i, \gamma_i^n \to 0 \). Thus \( y_i^n \to 0 \). But \( \{y^n\} \) lie in a compact set, so there exists a convergent subsequence \( y^\ast \to y \), and \( y_i = 0 \). Let \( \hat{U}^\ast \) be the utility level corresponding to \( y^\ast \). Since \( \{U^n\} \subset \tilde{V} \), there exists a convergent subsequence \( U^n \to U \in \tilde{V} \), and since \( \partial \hat{U}_1 \) is upper semi-continuous (see, e.g., Rockafellar (1970), Theorem 24.4), \( y \in \partial \hat{U}_1(U) \). But this is a contradiction since \( U \gg \emptyset \). Thus there exist \( b, B > 0 \) such that \( b \leq \lambda_j^n \leq B \) for all \( j \) and \( n \).

Since \( \tilde{V} \) is compact, \( x(\tilde{V}) \) is weak* compact, and in particular, for every \( t \) there exists \( M_t > 0 \) such that \( |x_{jt}| \geq M_t \) for all \( j \) and all \( x \in x(\tilde{V}) \). Thus there exists a sequence \( r(n) \to \infty \) as \( n \to \infty \) such that \( x_{jt}^{r(n)} \to 0 \). Now by feasibility, passing to a subsequence and relabeling if necessary, there exists \( j \) such that \( x_{jt}^{r(n)} \geq \frac{\omega}{m} \) for all \( n \). Then by the strong survival condition,

\[
\limsup_n \lambda_j^n = \limsup_n \frac{\partial U_j}{\partial x_{jt}(x^n_j)}(x^n_j) = \infty.
\]

But this contradicts the fact that \( \lambda_j^n / \lambda_j^n \) is bounded. Thus the individually rational Pareto optimal allocations are bounded below. \( \square \)

Moreover, note that the same argument applies to any compact set of utility levels \( \tilde{V} \subset D^e \), i.e., on any set of utility levels bounded away from 0 for each consumer, the Pareto
optimal allocations will be uniformly bounded away from 0 if the strong survival condition holds.

Finally, this section closes by giving several examples of economies satisfying these various survival conditions. First, the exchange economy in which each consumer’s utility function is additively separable provides a range of examples in which, depending on the consumers’ discount factors, either the positive Pareto optimal allocations are all interior, or there exists an “interior consumer”. For these examples, suppose that for \( i = 1, \ldots, m \), \( U_i(x) = \sum_{t=0}^{\infty} \beta_t u_t(x_i) \), where \( 0 < \beta_t < 1 \), \( u_t : \mathcal{R}_+ \rightarrow \mathcal{R} \) is \( C^2 \), \( u_t'(c) > 0 \) and \( u_t''(c) < 0 \) for every \( c \in \mathcal{R}_+ \), and \( u_t'(c) \rightarrow \infty \) as \( c \rightarrow 0 \). Each consumer in this economy satisfies the weak survival condition, as for all \( t \) and \( s \),

\[
\frac{\partial U_i(x_i)}{\partial x_i} = \frac{\beta_t u_t'(x_i)}{\beta_s u_t'(x_s)} \rightarrow \infty
\]
as \( x_i \rightarrow 0 \) if \( x_i \) is fixed and positive. Similarly, if all consumers discount at the same rate, so there exists some \( \beta \in (0, 1) \) such that \( \beta = \beta_i \) for all \( i \), then all consumers satisfy the other survival conditions as well. as

\[
\frac{\partial U_i(x_i)}{\partial x_i} = \frac{u_i'(x_i)}{u_j'(x_j)}
\]
for all \( i, j, r \). Furthermore, if consumers discount at different rates, then those consumers with the greatest discount factor are interior consumers. To see this, note that if we choose \( k \) such that \( \beta_k \geq \beta_i \) for every \( i \), \( \beta_k \geq 1 \) and

\[
\frac{\partial U_i(x_k)}{\partial x_i} = \left(\frac{\beta_k}{\beta_i}\right)^{r} \frac{u_i'(x_i)}{u_j'(x_j)}.
\]

Other examples are also easy to construct. For example, suppose consumer’s utility functions exhibit habit formation, so that they have the form \( U_i(x) = \sum_{t=0}^{\infty} \beta_t u_t(x_{i-1}, x_t) \), where \( u_t : \mathcal{R}_+^2 \rightarrow \mathcal{R} \) satisfies the following condition: for any sequence \((x_n, y_n)\) in \( \mathcal{R}_+^2 \) such that \((x_n, y_n) \rightarrow (x, y)\), \( \frac{\partial u_t}{\partial x_t} (x_n, y_n) \rightarrow \infty \iff x_n \rightarrow 0 \) and similarly \( \frac{\partial u_t}{\partial y_t} (x_n, y_n) \rightarrow \infty \iff y_n \rightarrow 0 \). Then all consumers will satisfy the strong survival condition, and thus by Theorem 2.15, Pareto optimal allocations will be uniformly bounded below. If consumers have different discount factors \( \beta_i \), then again the most patient, those with the largest discount factors, will be interior consumers.

Similarly, if all consumers have additively separable utility functions with constant discount rate \( \beta \) as described above, and one consumer has recursive preferences generated by an aggregator of the form \( w(c, z) = u(c) + g(z) \), where \( u'(c) \rightarrow \infty \) as \( c \rightarrow 0 \) and \( \alpha \leq g'(z) \leq \gamma \) for some \( \alpha, \gamma \in [\beta, 1) \), then the recursive consumer will satisfy the strong survival condition, and hence by Theorem 2.15 she will be an interior consumer.

The remaining open question is then to find conditions on the economic primitives of the model, consumer preferences, which ensure that the economy is a Lipschitz economy. Answering this question is the focus of the final two sections.

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\(^9\)An example of such a function is \( u(x, y) = \sqrt{x} + xy + \sqrt{y} \).
3 A Simple Example

This section develops a simple example of a Lipschitz economy in which consumers have recursive preferences. To simplify the discussion as much as possible, the analysis of this example makes heavy use of special features of recursive models, and in particular, of dynamic programming techniques. More general techniques and sufficient conditions for demonstrating that economies are Lipschitz economies are developed in the next section.

For this example, consider an exchange economy with two consumers, or more generally, two types of consumers, in which both consumers have recursive preferences. Suppose that the first consumer’s utility function is generated by a recursive aggregator of the form \( w_1(c, z) = u_1(c) + g(z) \), and the second consumer’s utility function is additively separable, so that the aggregator generating his utility function has the form \( w_2(c, z) = u_2(c) + \beta z \) for some \( \beta \in (0, 1) \). Moreover, suppose that for each \( i, u_i : \mathcal{R}_+ \to \mathcal{R} \) is \( C^\infty \), \( u_i(0) = 0 \), \( u'_i(c) > 0 \) and \( u''_i(c) < 0 \) for all \( c \), and \( u'_i(c) \to \infty \) as \( c \to 0 \). Assume that \( g : \mathcal{R} \to \mathcal{R} \) is \( C^\infty \), \( g(0) = 0 \), \( \alpha \leq g'(z) \leq \beta \) for some \( \alpha \in (0, \beta) \), and \( g''(z) < 0 \) for all \( z \). Let the social endowment vector be \( \omega = (1, 1, 1, \ldots) \equiv 1 \). Let \( U_i(x) \) and \( U_2(x) \) be the consumers’ utility functions generated by such aggregators. The recursive nature of preferences means that for each \( x \in \ell_\alpha \), \( U_i(x) = w_i(x_1, U_i(x_2)) \), where for any \( t, x \equiv (x_t, x_{t+1}, \ldots) \).

As before, the Pareto optimal allocations will be the solutions \( x(U) \) to the Pareto maximization problem

\[
\begin{align*}
\max & \quad U_1(x) \\
\text{s.t.} & \quad U_2(1 - x) \geq U
\end{align*}
\]

where as in section 2, \( \hat{U}_1(U) \) will denote the corresponding value function for this problem. Because of the recursive nature of preferences in this model, solving this problem is actually equivalent to solving a series of finite-dimensional problems.\(^\text{10}\) To see this, given any bounded, continuous function \( f : \mathcal{R} \to \mathcal{R} \), define the operator \( T \) by

\[
(Tf)(U) = \max_{c \in [0, 1]} \max_{y \in \ell_2} w_1(c, f(y)) \quad \text{s.t.} \quad w_2(1 - c, y) \geq U
\]

where \( \ell_2 = [0, U_2(1)] \). A standard dynamic programming argument shows that \( T \) is a contraction mapping from the space of real-valued, bounded, continuous functions on \( \mathcal{R} \) into itself, and thus has a unique fixed point \( v(U) \). Moreover, \( v(U) \) is strictly concave, and for any starting point \( v_0 \), the sequence \( v_n = Tv_{n-1} \) converges uniformly to \( v \). Now I claim that the solutions to the recursive maximization problem

\[
\begin{align*}
\max & \quad w_1(c, v(y)) \\
\text{s.t.} & \quad w_2(1 - c, y) \geq U
\end{align*}
\]

\(^{10}\)This approach is taken from Exercise 5.11 in Stokey and Lucas (1989).
recursively define the Pareto optimal allocations. More precisely, if \((c(U), y(U))\) is the solution to the recursive maximization problem and \(x(U) = (x^1(U), x^2(U))\) solves the Pareto problem,\(^{11}\) then for every \(t\), \(x^t_1(U) = c(y^{t-1}(U))\), where \(y^0(U) \equiv U\) and for \(t > 0\),

\[
y^t(U) = \underbrace{y(y(\cdots y(U) \cdots))}_{t \text{ times}}.
\]

**Theorem 3.1.** For every \(U \in I_2\), if \(x(U)\) solves the Pareto maximization problem and \((c(U), y(U))\) solves the recursive maximization problem, then \(x^t_1(U) = c(y^{t-1}(U))\) for each \(t\).

**Proof:** See the appendix.

Since we can find the Pareto optimal allocations by successively solving the finite-dimensional recursive problems, studying features of the Pareto map and the equilibrium equations essentially amounts to studying features of the solutions \(c(U)\) and \(y(U)\) to this recursive problem. The following lemma establishes several important properties of these functions.

**Lemma 3.2.** The value function \(v(U)\) is \(C^1\) and strictly concave. The functions \(c(U)\) and \(y(U)\) are Lipschitz continuous, \(c(U)\) is nonincreasing, \(y(U)\) is nondecreasing, and for all \(t\) and \(U\), \(y^{t+1}(U) \geq y^t(U)\).

**Proof:** See the appendix.

Using these features of the recursive problem and the results of the previous section, we can show that this economy is a Lipschitz economy, and thus that regular economies have determinate equilibria and that regular economies are generic in this recursive setting.

**Theorem 3.3.** In this recursive model, \(E_{\omega}\) is a Lipschitz economy. If \(E_{\omega}\) is a regular economy, then it has an odd number of equilibria, each of which is locally upper Lipschitzian in \(\omega^1\). Moreover, \(E_{\omega}\) is a regular economy for almost all \(\omega^1 \in \mathcal{W}\).

**Proof:** We must show that the equilibrium equations are locally Lipschitz. To do this, first note that the second consumer is an interior consumer, as he satisfies the strong survival condition. Prices can then be taken to be the gradient of his utility function. Moreover, the pricing function for the first good, \(u_2'(1 - c(U))\), is Lipschitz and bounded away from 0 on any interval of the form \((\epsilon, U_2(1))\), which guarantees that the normalized budget equation is locally Lipschitz if and only if the original budget equation \(DU_2(x^2(U)) \cdot (x^2(U) - \omega_2)\) is locally Lipschitz. Then it suffices to show that \(DU_2(x^2(U)) \cdot (x^2(U) - \omega_2)\) is locally Lipschitz.

Let \(U\) be given, and let \((\underline{U}, \bar{U})\) be a neighborhood about \(U\) such that \(\underline{U} > 0\) and \(\bar{U} < U_2(1)\). First note that by strict monotonicity, for each \(U\)

\[
U \equiv \sum_{t=1}^{\infty} \beta^{t-1} u_2(1 - c(y^{t-1}(U))) = U_2(x^2(U)).
\]

So

\[
U - U' = U_2(x^2(U)) - U_2(x^2(U'))
\]

\[
= DU_2(\bar{x})(x^2(U) - x^2(U'))
\]

\(^{11}\)For this discussion, subscripts on consumption bundles refer to time periods and superscripts refer to consumers.
for some \( \hat{x} \). Without loss of generality, assume \( U' < U \). Then \( x^2(U) \geq x^2(U') \) and so \( \hat{x} \in (x^2(U'), x^2(U)) \). Thus
\[

\|U - U'\| = \|DU_2(\hat{x})(x^2(U) - x^2(U'))\| \\
\geq \|DU_2(x^2(U))(x^2(U) - x^2(U'))\|
\]
since \( DU_2(x^2(U)) \leq DU_2(\hat{x}) \leq DU_2(x^2(U')) \) and \( x^2(U) \geq x^2(U') \). That is,
\[
\|DU_2(x^2(U))(x^2(U) - x^2(U'))\| \leq \|U - U'\|.
\]
Moreover,
\[

DU_2(x^2(U')) \cdot (x^2(U') - \omega_2) - \quad DU_2(x^2(U')) \cdot (x^2(U') - \omega_2) = \\
DU_2(x^2(U))(x^2(U) - x^2(U')) + (DU_2(x^2(U)) - DU_2(x^2(U')))(x^2(U') - \omega_2).
\]
So
\[

\|DU_2(x^2(U)) \cdot (x^2(U) - \omega_2) - DU_2(x^2(U')) \cdot (x^2(U') - \omega_2)\| \\
\leq \|DU_2(x^2(U))(x^2(U) - x^2(U'))\| + \|DU_2(x^2(U)) - DU_2(x^2(U'))\| (x^2(U') - \omega_2)\| \\
\leq \|U - U'\| + 2\|\omega\| \|DU_2(x^2(U)) - DU_2(x^2(U'))\|.
\]
Now consider \( DU_2(x^2(U)) - DU_2(x^2(U')) \). For some \( \hat{x} \in (x^2(U'), x^2(U)) \),
\[

DU_2(x^2(U)) - DU_2(x^2(U')) = D^2U_2(\hat{x})(x^2(U) - x^2(U')) \\
= \{\beta^t u^1_2(\hat{x}_t)(x^2(U) - x^2(U'))\} \\
= \{\beta^t \frac{u^2_2(\hat{x}_t)}{u^1_2(x^2(U))} u^r_2(x^2(U))(x^2(U) - x^2(U'))\}.
\]
Thus
\[

\|DU_2(x^2(U)) - DU_2(x^2(U'))\| = \sum_{t=1}^{\infty} \left| \frac{u^2_2(\hat{x}_t)}{u^1_2(x^2(U))} \right| \beta^t \|u_2^2(\hat{x}_t)(x^2(U) - x^2(U'))\| \\
\leq K \sum_{t=1}^{\infty} \|u_2^2(\hat{x}_t)(x^2(U) - x^2(U'))\| \\
= K \|DU_2(x^2(U))(x^2(U) - x^2(U'))\| \\
\leq K \|U - U'\|
\]
where we can find \( K \) for all \( U \) and \( t \), since for all \( U, U' \) and \( t \), \( 1 \geq x^2(U) \geq \hat{x}_t \geq x^2(U') \geq 1 - o(U) \). Thus
\[

\|DU_2(x^2(U)) \cdot (x^2(U) - \omega_2) - DU_2(x^2(U')) \cdot (x^2(U') - \omega_2)\| \leq (2\|\omega\|K + 1) \|U - U'\| = K'\|U - U'\|.
\]
i.e., the budget equations are locally Lipschitz. ■

An interesting feature to note about this argument is that it establishes directly that the equilibrium equations are locally Lipschitz in this economy, rather than first showing that
the Pareto map is locally Lipschitz. Without the detailed information about preferences and
the structure of the Pareto map available in this example, showing that the economy is a
Lipschitz economy without showing that the Pareto map is locally Lipschitz may be quite
difficult, but as this argument plainly illustrates, Lipschitz continuity of the Pareto map
is a sufficient condition for the economy to be a Lipschitz economy, but is certainly not a
necessary condition. Indeed, lemma 3.2 shows that the Pareto map is locally Lipschitz in
each period, since \( x_t^i(U) = c(y_t^{-1}(U)) \) and both \( c \) and \( y \) are locally Lipschitz for all \( t \), but to
show that the Pareto map is locally Lipschitz we would have to show that these Lipschitz
constants are bounded across all periods, or alternatively that \( y \) is Lipschitz with constant
less than or equal to 1. Lemma 3.2 establishes a Lipschitz constant for \( y \) of \( \frac{1}{\beta} \), which is
greater than 1 except in the degenerate case when \( \beta = 1 \), so the results established thus
far are not sufficient to conclude that the Pareto map is locally Lipschitz. As this example
indicates, it may be possible and much easier in some important examples to show directly
that the economy is a Lipschitz economy, but in most examples the simplest way to show
that the economy is a Lipschitz economy is to show that the Pareto map is locally Lipschitz.

Another interesting feature to note about this example is that it is an example of a
Lipschitz economy in which one consumer is an interior consumer, but the consumption
of the other consumer both in Pareto optimal allocations and in equilibrium need not be
interior. In fact, for each Pareto optimal allocation in this example, the second consumer's
consumption increases monotonically over time as the first consumer’s decreases. Moreover,
it is not hard to see that starting from any utility level, the first consumer’s consumption
decreases to 0 as time goes to infinity. His consumption bundle is then never interior, and
yet we are able to show that equilibria are generically determinate in this economy.

4 Lipschitz Economies

The recursive example in the previous section highlights a number of important points con-
cerning determinacy in infinite horizon economies, and shows, among other things, that the
generic determinacy results that hold in additively separable economies can be extended to
recursive economies in a relatively straightforward manner by using dynamic programming
techniques together with the Lipschitz determinacy results in the first part of the paper.
Moreover, although this was a particularly simple example, it should be possible to adapt
this approach to analyze more complicated preferences or models with more than two types
of consumers. No matter how much this argument can be generalized, however, such an
approach will always be limited to preferences to which dynamic programming techniques
apply.

This section develops methods for verifying that an economy is a Lipschitz economy,
and thus has generically determinate equilibria, which are independent of stationarity of
preferences or the tools of dynamic programming. To show that an economy is a Lipschitz
economy without the sort of detailed knowledge about preferences and the structure of the
Pareto map and equilibrium equations available in a highly parametrically specified model,
the most broadly applicable approach would be to try to show that the Pareto map is locally
Lipschitz. Given that, at least for interior optima, the Pareto map is the implicit function
defined by the first order conditions of the Pareto problem, we would like a Lipschitz implicit
function theorem which would allow us to conclude that the Pareto map is locally Lipschitz. Furthermore, we need to determine conditions on preferences which ensure that such an implicit function theorem is applicable. This program is carried out in this section, and several examples are given to illustrate these ideas.

Since the interior Pareto optimal allocations are the solutions to the first order conditions for the Pareto problem, the Pareto map \( x(U) \) is the implicit function defined by the equations

\[
F(x, \lambda, U) = \begin{cases}
DU_1(x_1) - \lambda_2 DU_2(x_2) \\
\vdots \\
DU_1(x_1) - \lambda_m DU_m(x_m) \\
\sum_{j=1}^{m} x_j - \omega \\
U_2(x_2) - U_2 \\
\vdots \\
U_m(x_m) - U_m
\end{cases} = 0.
\]

To use this characterization of Pareto optimal allocations, we will implicitly assume that attention is restricted to interior Pareto optimal allocations, either by restricting attention to the set \( \Omega \) of utility levels generating interior Pareto optimal allocations, or by imposing assumptions like the survival conditions which imply that all individually rational Pareto optimal allocations are interior.\(^{12}\) Moreover, the economies considered in this section will be assumed to be smooth myopic economies.

As discussed in section 2, one of the fundamental difficulties involved in establishing local uniqueness or comparative statics results for infinite horizon models is the impossibility of appealing to standard implicit function theorems to show that the equations defining equilibria are smooth. In order to show that these equations are nonetheless locally Lipschitz for a broad class of economies, we will need a Lipschitz implicit function theorem; clearly the assumptions for this theorem must be weaker than the requirement of the classical implicit function theorem that \( D_{(x,\lambda)} F(x, \lambda, U) \) be nonsingular.

The appropriate generalization of nonsingularity for this purpose is the notion of bounded below. If \( X \) and \( Y \) are normed vector spaces, a continuous linear operator \( T : X \to Y \) is

\(^{12}\)Here the assumption that Pareto optimal allocations are interior is more important than it was in section 2; in section 2 the interiority of these allocations was simply an easy sufficient condition for ensuring that we could find a finite-dimensional parameterization for the equilibrium prices. Here, to show that the Pareto map is locally Lipschitz using some sort of implicit function theorem, we need to be able to argue that the Pareto map is the implicit function determined by some set of equations; that is, the first order conditions must give us a system of \textit{equations}. The juxtaposition of Lipschitz or nonsmooth methods with such interiority conditions may be confusing to some, since one intuition from the finite-dimensional literature would suggest that the possibility of boundary solutions is what leads to lack of differentiability of demand functions, for example, while such functions are still Lipschitz continuous (see e.g., Shannon (1994a)). The boundary of the positive cone in \( L_\infty \) is "fat", so this intuition suggests that we might expect the Pareto map to be Lipschitz continuous rather than smooth when such boundary solutions arise. Although this idea might prove fruitful for developing further results on determinacy, it is not pursued here; such an approach would require a Lipschitz implicit function theorem for systems of inequalities or inclusions. A more appropriate analogy for the approach pursued in this section is with finite-dimensional examples like those of Katzner (1968), in which demand functions fail to be differentiable at interior solutions when preferences fail to be differentiably strictly convex, i.e., when the Hessian of the consumer's utility function is singular at some point.
said to be bounded below if there exists $c > 0$ such that for every $x \in X$, $\|T x\| \geq c \|x\|$. If $X$ and $Y$ are finite-dimensional spaces of the same dimension, then nonsingularity and boundedness below are equivalent, and both are simply equivalent to the operator being one-to-one. However, in an infinite-dimensional setting, all three notions are distinct.\(^{13}\) As the examples in this section show, bounded below is precisely the right intermediate regularity condition between one-to-one and invertible needed to guarantee that the implicit function is locally Lipschitz continuous in these economies. In particular, this condition leads to the following Lipschitz implicit function theorem.

**Theorem 4.1.** Let $X, Y$ and $Z$ be Banach spaces, and let $F(x, y) : X \times Y \to Z$ be continuously Fréchet differentiable in $x$ and uniformly Lipschitz in $y$. Suppose that the equation $F(x, y) = 0$ has a continuous, single-valued solution $x(y)$, i.e., that there exists a continuous function $x : Y \to X$ such that $F(x(y), y) \equiv 0$, and $F(x, y) = 0$ if and only if $x = x(y)$. If $D_x F(x, y)$ is bounded below for each $(x, y) \in X \times Y$, then $x(y)$ is locally Lipschitz continuous.

**Proof:** Let $y \in Y$ be arbitrary. Since $D_x F(x(y), y)$ is bounded below, there exists $c > 0$ such that

$$\|D_x F(x(y), y) x\| \geq c \|x\| \quad \forall x \in X.$$ 

Since $D_x F(\cdot)$ is continuous, there exists a neighborhood $U \times W$ of $(x(y), y)$ such that for every $(x', y') \in U \times W$, $\|D_x F(x(y), y) - D_x F(x', y')\| < \frac{c}{2}$. If $(x', y') \in U \times W$, then

$$\|D_x F(x', y') x\| = \|D_x F(x(y), y) x - [D_x F(x(y), y) - D_x F(x', y')] x\|$$

$$\geq \|D_x F(x(y), y) x\| - \|D_x F(x(y), y) - D_x F(x', y')\| x\|$$

$$\geq c \|x\| - \frac{c}{2} \|x\| = \frac{c}{2} \|x\|.$$ 

Since $x(y)$ is continuous, there exists a neighborhood $\tilde{W}$ of $y$ such that $y' \in \tilde{W} \Rightarrow x(y') \in U$. Let $\tilde{W} = W \cap \tilde{W}$. Then pick $y', y'' \in \tilde{W}$. We will show that $x(\cdot)$ is Lipschitz on $\tilde{W}$. By definition of $x(\cdot)$,

$$F(x(y'), y') - F(x(y''), y'') = 0$$

$$\Rightarrow F(x(y'), y') - F(x(y''), y'') = F(x(y''), y'') - F(x(y'), y'')$$

$$= D_x F(\bar{x}, \bar{y})(x(y'') - x(y'))$$

for some $(\bar{x}, \bar{y}) \in U \times \tilde{W}$. Thus

$$\|F(x(y'), y') - F(x(y''), y'')\| = \|D_x F(\bar{x}, \bar{y})(x(y'') - x(y'))\|.$$ 

Since $F$ is uniformly Lipschitz in $y$, there exists $k > 0$ such that

$$k \|y' - y''\| \geq \|F(x(y'), y') - F(x(y''), y'')\|$$

$$= \|D_x F(\bar{x}, \bar{y})(x(y'') - x(y'))\|$$

$$\geq \frac{c}{2} \|x(y') - x(y'')\|.$$ 

\(^{13}\)For example, if $X = Y = \ell_\infty$, and $T$ is the shift operator, so that $T x = (0, x)$, then $T$ is bounded below with lower bound $c = 1$, but $T$ is certainly not onto. Similarly, if $T$ is the operator $T x = (x_1, \frac{1}{2} x_2, \frac{1}{3} x_3, \ldots)$, then $T$ is one-to-one but is neither bounded below nor nonsingular.

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by the previous argument. Thus \( \|x(y') - x(y'')\| \leq K\|y' - y''\| \), where \( K = \frac{2k}{c} > 0 \).

If it is not known a priori that the implicit function \( x(y) \) is continuous, then \( x(y) \) will still be locally Lipschitz as long as there is a uniform lower bound on the derivative of \( F \). More precisely, if \( \{T(w) : w \in W\} \) is a family of continuous linear operators from \( X \) to \( Y \), we will say that \( T(w) \) is uniformly bounded below if there exists \( c > 0 \) such that for all \( w \in W \), \( \|T(w)x\| \geq c\|x\| \) for every \( x \in X \).

**Theorem 4.2.** Let \( X, Y, Z \) be Banach spaces and let \( F : X \times Y \to Z \) be continuously Fréchet differentiable in \( x \) and uniformly Lipschitz in \( y \). Suppose that the equation \( F(x, y) = 0 \) has a single-valued solution \( x(y) \), i.e., that there exists a function \( x : Y \to X \) such that \( F(x(y), y) \equiv 0 \) and \( F(x, y) = 0 \) if and only if \( x = x(y) \). If \( D_x F(x, y) \) is uniformly bounded below, then \( x(y) \) is Lipschitz continuous.

**Proof:** Let \( y, y' \in Y \). Then

\[
F(x(y), y) - F(x(y'), y') = 0 \\
\Rightarrow F(x(y), y) - F(x(y), y') = F(x(y'), y') - F(x(y), y') \\
= D_x F(x, y)(x(y') - x(y))
\]

for some \((\tilde{x}, \tilde{y}) \in X \times Y\). Thus

\[
\|F(x(y), y) - F(x(y), y')\| = \|D_x F(x, y)(x(y') - x(y))\|.
\]

Since \( F \) is uniformly Lipschitz in \( y \) and \( D_x F(\cdot) \) is uniformly bounded below, there exist \( k, c > 0 \) such that

\[
k\|y - y'\| \geq \|F(x(y), y) - F(x(y), y')\| \\
= \|D_x F(x, y)(x(y') - x(y))\| \\
\geq c\|(y') - x(y))\|
\]

i.e., \( \|x(y') - x(y)\| \leq K\|y' - y\| \), where \( K = \frac{k}{c} > 0 \). Thus \( x(y) \) is Lipschitz on \( Y \).

Moreover, note that the situation described by these implicit function theorems is exactly the situation we have when considering the Pareto map: the Pareto map is implicitly defined by the first order equations \( F(x, \lambda, U) = 0 \), which are \( C^1 \) by assumption and are clearly uniformly Lipschitz in \( U \), and by Shannon (1994b, Thm. 2.1) we know that the Pareto map is a well-defined, single-valued function in smooth myopic economies. Thus to show that the Pareto map is locally Lipschitz continuous, it would suffice to show that the corresponding derivative \( D_{(x,\lambda)} F(x, \lambda, U) \) had the requisite bounded below properties as specified by one of these two results. Given the fundamental role these conditions will play in developing methods for showing that the Pareto map is locally Lipschitz, before discussing such methods we should first examine this property more closely.

First, note that if a continuous linear operator \( T : X \to Y \) is bounded below, then it is clearly one-to-one. Moreover, if \( X \) is a Banach space and \( T \) is one-to-one, then \( T \) is bounded below if and only if its range is closed (see, e.g., Deimling (1985), Prop. 7.9). Thus if \( X \) and \( Y \) are Banach spaces and \( T \) is bounded below, then \( T \) is an isomorphism between \( X \).
and Range(T).\(^{14}\) In light of this result, it may seem like these conditions are no more useful than the standard ones in studying the Pareto map, since for every \((x, \lambda, U)\), the domain of \(D_{(x, \lambda)}F(x, \lambda, U)\) is \(\ell_\infty\) and its range is still a subspace of \(\ell_1\), and as discussed above \(\ell_\infty\) cannot be isomorphic to any subspace of \(\ell_1\). However, as the results of the rest of the section show, this problem essentially arises because of consumer discounting, and renormalizing the problem to take into account consumers’ effective discount factors, in a way which will be formalized below, can eliminate this problem. In order to develop the intuition behind the results of the rest of the section, I will focus first on the simplest version of the model in which each consumer’s utility function is additively separable, and then turn to more general models.

To use the Lipschitz implicit function theorems, we will have to study properties of the operator \(D_{(x, \lambda)}F(x, \lambda, U)\). To simplify notation, let \(x = (x, \lambda)\); then this operator has the form \(D_x F(x, \lambda, U) =\)

\[
\begin{pmatrix}
D^2U_1(x_1) & -\lambda_2 D^2U_2(x_2) & \cdots & 0 & -DU_2(x_2)^T & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
D^2U_1(x_1) & 0 & \cdots & -\lambda_m D^2U_m(x_m) & 0 & \cdots & -DU_m(x_m)^T \\
I & I & \cdots & I & 0 & \cdots & 0 \\
0 & DU_2(x_2) & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & DU_m(x_m) & 0 & \cdots & 0 
\end{pmatrix}
\]

By examining the Lipschitz implicit function theorems, it should be clear, and it will be established formally in Theorem 4.4 below, that in order to show that the Pareto map is locally Lipschitz, it will suffice to show that this operator is either bounded below or uniformly bounded below on

\[\{(z, r) \in \ell_m^{\infty} \times \mathcal{R}^{m-1} : z = x^1 - x^2 \text{ where } x^1, x^2 \text{ are Pareto optimal allocations}\}\]

\[\subset \{(z, r) \in \ell_m^{\infty} \times \mathcal{R}^{m-1} : \sum_{i=1}^m z_i = 0\}.
\]

This observation simplifies the problem, and means that it suffices to show that the operator \(H(x, \lambda, U)\) is either bounded or uniformly bounded below, where \(H(x, \lambda, U) \equiv\)

\[
\begin{pmatrix}
D^2U_1(x_1) + \lambda_2 D^2U_2(x_2) & \cdots & D^2U_1(x_1) & DU_2(x_2)^T & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
D^2U_1(x_1) & \cdots & D^2U_1(x_1) + \lambda_m D^2U_m(x_m) & 0 & \cdots & DU_m(x_m)^T \\
DU_2(x_2) & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & DU_m(x_m) & 0 & \cdots & 0
\end{pmatrix}
\]

\(^{14}\)Since Range(T) is closed, it is complete, and clearly T is a one-to-one, onto mapping from X to Range(T), so it is an isomorphism by the open mapping theorem.
To further simplify notation, let $D^2 \mathbf{U}(x, \lambda) \equiv$

\[
\begin{pmatrix}
D^2 U_1(x_1) + \lambda_1 D^2 U_2(x_2) & D^2 U_1(x_1) & \cdots & D^2 U_1(x_1) \\
D^2 U_1(x_1) & D^2 U_1(x_1) + \lambda_2 D^2 U_3(x_3) & \cdots & D^2 U_1(x_1) \\
\vdots & \vdots & \ddots & \vdots \\
D^2 U_1(x_1) & D^2 U_1(x_1) & \cdots & D^2 U_1(x_1) + \lambda_m D^2 U_m(x_m)
\end{pmatrix}
\]

and let

\[
D \mathbf{U}(x) = \begin{pmatrix}
D U_2(x_2) & 0 & \cdots & 0 \\
0 & D U_3(x_3) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & D U_m(x_m)
\end{pmatrix}.
\]

Then using this notation,

\[
H(x, \lambda, U) = \begin{pmatrix}
D^2 \mathbf{U}(x, \lambda) & D \mathbf{U}(x)^T \\
D \mathbf{U}(x) & 0
\end{pmatrix}.
\]

Consider the special case of an economy with two consumers in which each consumer's utility function is additively separable with constant discount rate $\beta$, so that each consumer's utility function has the form $U_i(x) = \sum_{t=0}^{\infty} \beta^t u_i(x_t)$, where $0 < \beta < 1$, $u_i : \mathcal{R}_+ \to \mathcal{R}$ is $C^2$, $u'_i(c) > 0$ and $u''_i(c) < 0$ for all $c \in \mathcal{R}_+$, and $u'_i(c) \to \infty$ as $c \to 0$. In this model, this operator has a particularly nice structure, as the second derivative of each consumer's utility function is simply a diagonal matrix with diagonal $\{\beta^t u''_i(x_t)\}$. In this case, it is easy to see that the range of this operator will be a subspace of $\ell_1$; indeed, here this subspace is easy to identify. First, we need a bit of notation. For arbitrary elements $\alpha \in \ell_1$ and $y \in \ell_\infty$, define the weighting operation $\otimes$ by $\alpha \otimes y = (\alpha_1 y_1, \alpha_2 y_2, \ldots)$, and note that for any such $\alpha$ and $y$, $\alpha \otimes y \in \ell_\infty$. Then define $\vec{\beta} = (1, \beta, \beta^2, \beta^3, \ldots)$. Using this notation, any element of the range of $H(x, \lambda, U)$ in this additively separable case has the form $(\vec{\beta} \otimes w, r)$, where $w \in \ell_\infty$ and $r \in \mathcal{R}$. Thus rather than being arbitrary elements of $\ell_1$, elements in the range of $H(x, \lambda, U)$ are simply elements of $\ell_\infty$ weighted by the sequence of discount rates $\vec{\beta}$. Alternatively, rather than being an arbitrary subspace of $\ell_1$, the range of $H(x, \lambda, U)$ is contained in the ideal $\mathcal{A}_{\vec{\beta}}$ generated by the vector $\vec{\beta}$ of discount factors, where

\[
\mathcal{A}_{\vec{\beta}} = \{y \in \ell_1 : \exists \gamma > 0 \text{ s.t. } |y| \leq \gamma \vec{\beta}\}.
\]

To account for the fact that this operator acts by weighting with the vector of discount factors $\vec{\beta}$, we would like to ignore them or divide them out somehow. More precisely, rather than using the standard $\ell_1$ norm on the range, we should use a weighted norm to account for the discounting; equivalently, we should use the natural norm induced on the ideal $\mathcal{A}_{\vec{\beta}}$, where the weighted norm $\|\cdot\|_{\vec{\beta}}$ is defined by $\|x\|_{\vec{\beta}} = \sup_t |x_t|$ and the natural norm induced on the ideal $\mathcal{A}_{\vec{\beta}}$ is defined by $\|y\|_\infty = \inf \{\gamma \geq 0 : |y| \leq \gamma \vec{\beta}\}$ for $y \in \mathcal{A}_{\vec{\beta}}$. Considering the range with such a weighted norm to take discounting into effect will make it possible to apply the Lipschitz implicit function theorem, and more importantly, verifying that the operator

\[15\]Recall that given a Riesz space $E$, the absolute value on $E$ is defined by $|y| = y \vee (-y)$ for $y \in E$. 31
\( H(x, \lambda, U) \) is uniformly bounded below with respect to the weighted norm is equivalent to verifying that a natural and easily interpretable transformation of the original operator is uniformly bounded below in the original \( \ell_\infty \) norm.

To see this requires some additional notation. Let \( DU_i^{\tilde{\beta}}(x_i) \equiv (u_i'(x_i^1), u_i'(x_i^2), \ldots) \), and let

\[
D^2 U_i^{\tilde{\beta}}(x_i) \equiv \begin{pmatrix}
u'_i(x_i^1) & 0 & 0 & \cdots \\
0 & \nu'_i(x_i^2) & 0 & \cdots \\
0 & 0 & \nu'_i(x_i^3) & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

Then using the same notation as above, let \( D^2 U^\tilde{\beta}(x, \lambda) \equiv \)

\[
\begin{pmatrix}
D^2 U_1^{\tilde{\beta}}(x_1) + \lambda_2 D^2 U_2^{\tilde{\beta}}(x_2) & D^2 U_2^{\tilde{\beta}}(x_1) & \cdots & D^2 U_m^{\tilde{\beta}}(x_1) \\
D^2 U_1^{\tilde{\beta}}(x_1) & D^2 U_1^{\tilde{\beta}}(x_1) + \lambda_3 D^2 U_3^{\tilde{\beta}}(x_3) & \cdots & D^2 U_m^{\tilde{\beta}}(x_1) \\
\vdots & \vdots & \ddots & \vdots \\
D^2 U_1^{\tilde{\beta}}(x_1) & D^2 U_1^{\tilde{\beta}}(x_1) & \cdots & D^2 U_1^{\tilde{\beta}}(x_1) + \lambda_m D^2 U_m^{\tilde{\beta}}(x_m)
\end{pmatrix}
\]

and

\[
DU^{\tilde{\beta}}(x) = \begin{pmatrix}
DU_2^{\tilde{\beta}}(x_2) & 0 & \cdots & 0 \\
0 & DU_3^{\tilde{\beta}}(x_3) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & DU_m^{\tilde{\beta}}(x_m)
\end{pmatrix}.
\]

Then using this notation, define

\[
H^{\tilde{\beta}}(x, \lambda, U) \equiv \begin{pmatrix}
D^2 U^{\tilde{\beta}}(x, \lambda) & DU^{\tilde{\beta}}(x)^T \\
DU(x) & 0
\end{pmatrix}.
\]

Note that \( H^{\tilde{\beta}} \) is just the operator obtained from \( H \) by dividing the \( t \)th row by \( \beta^t \).

**Theorem 4.3.** In the additively separable case with constant discount factors, the Pareto map \( x(U) \) is locally Lipschitz if \( H^{\tilde{\beta}}(x, \lambda, U) \) is uniformly bounded below on \( [z, \bar{z}] \times [\Delta, \bar{\lambda}] \) for each \( z, \bar{z} \in \ell_\infty^m \) and \( \Delta, \bar{\lambda} \in R_{m+1}^m \).

**Proof:** Let \( U \in D^v \) be given, and choose a neighborhood \( V \) of \( U \) such that \( V \subset D^o \). By Theorem 2.15 there exists \( \Delta, \bar{\lambda} \in \ell_\infty^m \) such that \( x(V) \subset [z, \bar{z}] \). Similarly, \( \lambda(U) \subset [\Delta, \bar{\lambda}] \) for some \( \lambda, \bar{\lambda} \in \ell_\infty^m \), as \( \lambda(U) \) solves the equations \( u'_i(x_i(U)) = \lambda_i(u'_i)(x_i(U)) \). Then given \( (x, r) \in \ell_\infty^m \times \ell_{m-1}^m \), define \( \|z, r\|_\beta \) by \( \|z, r\|_\beta = \max(\|r\|, \|zi, \|_\beta) \), \( i = 1, \ldots, m-1 \), where \( \|zi, \|_\beta = \sup_{x_i} \|\frac{\partial U}{\partial x_i}\| \). For an arbitrary element \( y \in \ell_\infty \), \( \|y\|_\beta \) may not be defined, but note that for every element \( (x, r) \in \text{Range } H(x, \lambda, U) \), \( (x, r) = (\beta \otimes w_1, \ldots, \beta \otimes w_{m-1}, r) \), where \( w_i \in \ell_\infty \), so \( \|(x, r)\|_\beta < \infty \). Let \( x(U) = (x(U), \lambda(U)) \). Choose \( U', U'' \in V \). Then

\[
F(x(U'), U') = F(x(U'', U'') = 0 \Rightarrow \\
F(x(U'), U') - F(x(U''), U'') = F(x(U''), U'') - F(x(U'), U'') \\
= D_x F(x^*, U^*) \cdot (x(U') - x(U''))
\]

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for some \((x^*, U^*) \in [0, \tilde{z}] \times [\lambda, \tilde{\lambda}] \times V\). Then note that \(F(x(U'), U') - F(x(U'), U'') = (0, \ldots, 0, U' - U'')\), so \(\|F(x(U'), U') - F(x(U'), U'')\|_\beta = \|U' - U''\|\) for all \(U', U''\). Thus
\[
\|U' - U''\| = \|F(x(U'), U') - F(x(U'), U'')\|_\beta = \|D_x F(x^*, U^*) \cdot (x(U') - x(U''))\|_\beta.
\]
Then note that \(\sum_{i=1}^m (x_i(U') - x_i(U'')) = 0\) since \(x(U')\) and \(x(U'')\) are Pareto optimal allocations. Thus
\[
\|D_x F(x^*, U^*) \cdot (x(U') - x(U''))\|_\beta = \|H(x^*, U^*) \cdot (x_{-m}(U') - x_{-m}(U''))\|_\beta
\]
where \(x_{-m}(U) = (x_1(U), \ldots, x_{m-1}(U), \lambda(U))\). So
\[
\|U' - U''\| = \|H(x^*, U^*) \cdot (x_{-m}(U') - x_{-m}(U''))\|_\beta
= \|H(x^*, U^*) \cdot (x_{-m}(U') - x_{-m}(U''))\|
\geq c\|x_{-m}(U') - x_{-m}(U'')\|.
\]
Thus \(\|x(U') - x(U'')\| \leq \frac{c}{c^1} \|U' - U''\|\), i.e., the Pareto map \(x(U)\) is locally Lipschitz.

In general, without additively separable preferences, consumers will not discount at a constant rate but rather at a rate which may change over time, depend on consumption in previous periods or depend on the entire consumption bundle chosen. However, we may still be able to identify analogous generalized discount factors for consumers, and even though consumers may no longer be discounting with a constant factor \(\beta\), the same ideas used in the additively separable example can be used to develop sufficient conditions for the economy with generalized discount factors to be a Lipschitz economy. For example, by the assumption of impatience, each consumer's vector of marginal utilities \(DU_i(x_i)\) is a strictly positive element of \(\ell_1\) for any \(x_i \in \text{int} \ell_{\infty}\), which suggests that we can think of this vector of marginal utilities as determining a sequence of generalized discount factors for each consumer at each consumption vector. By the same argument used to prove the previous result, if one of these generalized sequences of discount factors \(\tilde{\beta}_i(x_i)\) generates an order ideal which forms the range of the operator \(H(x, \lambda, U)\), and if the normalized operator \(H^{\tilde{\beta}}(x, \lambda, U)\) is uniformly bounded below, then the economy will be a Lipschitz economy. More generally, let \(\rho(x) : \ell^m_{\infty} \rightarrow \ell^{m-1}_{1+}\) be a generalized discounting function which associates to every set of bundles \(x = (x_1, \ldots, x_m)\) a sequence of discount factors \(\rho_i(x)\) for each consumer. For example, in the additively separable case, we could define \(\rho_i(x) \equiv \tilde{\beta}\) for each \(i\) and for each \(x\). Although they will not be constant across consumers or consumption bundles, we can normalize by these generalized discount factors as in the additively separable case by defining
\[
D^2U^i_{t^i}(x_i) \equiv \text{diag} \left\{ \frac{1}{\rho^i_t(x)} \right\} D^2U_t(x_i),
\]
where given a vector \((y_1, y_2, \ldots)\), the operator \(\text{diag}\{y_i\}\) denotes the operator with diagonal \(y\), i.e., the operator such that \([\text{diag}\{y_i\}]z = \{y_iz_1\}\) for every \(z \in \ell_{\infty}\). Similarly, we can define
\[
DU^i_{t^i}(x_i) \equiv \left\{ \frac{1}{\rho^i_t(x)} \frac{\partial U_t}{\partial x_t}(x_i) \right\}.
\]
Then let $D^2U_\rho(x, \lambda) \equiv$

$$
\begin{pmatrix}
D^2U_{1}^{p_1}(x_1) + \lambda_2 D^2U_{2}^{p_1}(x_2) & D^2U_{1}^{p_1}(x_1) & \cdots & D^2U_{1}^{p_1}(x_1) \\
D^2U_{1}^{p_2}(x_1) & D^2U_{1}^{p_2}(x_1) + \lambda_3 D^2U_{3}^{p_2}(x_3) & \cdots & D^2U_{1}^{p_2}(x_1) \\
\vdots & \vdots & \ddots & \vdots \\
D^2U_{1}^{p_{m-1}}(x_1) & D^2U_{1}^{p_{m-1}}(x_1) & \cdots & D^2U_{1}^{p_{m-1}}(x_1) + \lambda_m D^2U_{m}^{p_{m-1}}(x_m)
\end{pmatrix}
$$

and

$$
DU_\rho(x) = \begin{pmatrix}
D^2U_{2}^{p_1}(x_2) & 0 & \cdots & 0 \\
0 & D^2U_{3}^{p_2}(x_3) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & D^2U_{m}^{p_{m-1}}(x_m)
\end{pmatrix}. 
$$

Even though consumers are not discounting at a constant rate, if there exists some generalized discounting function $\rho(x)$ such that $H_\rho$ is uniformly bounded below, where

$$
H_\rho(x, \lambda, U) \equiv \begin{pmatrix}
D^2U_\rho(x, \lambda) \\
DU_\rho(x)^T
\end{pmatrix},
$$

then the economy will still be a Lipschitz economy, and thus generically have determinate equilibria.

**Theorem 4.4.** If there exists $\rho : \ell^\infty_{\omega} \rightarrow \ell^{m-1}_{1+\iota}$ such that $H_\rho(x, \lambda, U)$ is uniformly bounded below, then $x(U)$ is locally Lipschitz. Thus the economy $E_\omega$ is a Lipschitz economy.

**Proof:** Let $x(U) = (x(U), \lambda(U))$. Choose $U', U'' \in V$. Then

\[
F(x(U'), U') = F(x(U''), U'') = 0 \Rightarrow
\]

\[
F(x(U'), U') - F(x(U''), U'') = \frac{D_x F(x^*, U^*) \cdot (x(U') - x(U''))}{\rho}
\]

for some $(x^*, U^*)$. Given a point $(z, r) \in \ell^\infty_{\omega} \times \mathbb{R}^{m-1}$, define $\| (z, r) \|_\rho = \max \|r\|, \|z\|_{\rho_i(x)}$, $i = 1, \ldots, m-1$. Then note that $F(x(U'), U') - F(x(U''), U'') = (0, \ldots, 0, U' - U'')$, so $\| F(x(U'), U') - F(x(U''), U'') \|_\rho = \| U' - U'' \|$ for all $U', U''$. Thus

\[
\| U' - U'' \| = \| F(x(U'), U') - F(x(U''), U'') \|_\rho = \| D_x F(x^*, U^*) \cdot (x(U') - x(U'')) \|_\rho.
\]

Then note that $\sum_{i=1}^m (x_i(U') - x_i(U'')) = 0$ since $x(U')$ and $x(U'')$ are Pareto optimal allocations. Thus

\[
\| D_x F(x^*, U^*) \cdot (x(U') - x(U'')) \|_\rho = \| H(x^*, U^*) \cdot (x_m(U') - x_m(U'')) \|_\rho
\]

where $x_m(U) = (x_1(U), \ldots, x_{m-1}(U), \lambda(U))$. So

\[
\| U' - U'' \| = \| H(x^*, U^*) \cdot (x_m(U') - x_m(U'')) \|_\rho = \| H(x^*, U^*) \cdot (x_m(U') - x_m(U'')) \|_\rho \geq c \| x_m(U') - x_m(U'') \|.
\]

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Thus \( \|x(U') - x(U'')\| \leq (m - 1)\|x_{-m}(U') - x_{-m}(U'')\| \leq \frac{m-1}{\varepsilon}\|U' - U''\| \), i.e., the Pareto map \( x(U) \) is locally Lipschitz.

At this point, it may not be clear that these results are any more useful than the original formulations of the implicit function theorems. However, while the implicit function theorems are clearly inapplicable a priori since the range of \( H \) is a subspace of \( \ell_1 \), it is possible that there exists a sequence of generalized discount factors \( \rho(x) \) such that after normalizing by this sequence of discount factors, \( H^\rho \) could have a range which is a "large" subspace of \( \ell_\infty \). Equivalently, it is possible that the range of \( H \) will be a particularly nice subspace of \( \ell_1 \), the order ideal generated by a generalized discounting function \( \rho \). Since the ideal \( \mathcal{A}_\rho \) under the naturally induced norm is isomorphic to \( \ell_\infty \), it is not impossible a priori for \( H^\rho \) to be bounded below if the range of \( H \) is contained in the ideal \( \mathcal{A}_\rho \). The remaining problem is then to find conditions on preferences that will ensure that a suitable generalized discount function can be identified, and that once normalized this operator will be uniformly bounded below.

First, note that since \( D^2U_i(x_i) \) is negative definite for each \( i \) and \( x_i \), \( H^\rho(x, \lambda, U) \) is one-to-one for any such discounting function \( \rho \). To show that this operator is bounded below, it then suffices to show that it has closed range. Furthermore, \( H^\rho \) essentially has the form of a bordered Hessian in which the "border" has finite-dimensional range, i.e.,

\[
H^\rho(x, \lambda, U) = \begin{pmatrix}
0 & DU^\rho(x)^T \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
D^2U^\rho(x, \lambda) & 0 \\
DU(x) & 0
\end{pmatrix}.
\]

Since the sum of a closed subspace and a finite-dimensional subspace is always closed, \( H^\rho(x, \lambda, U) \) will be bounded below if \( D^2U^\rho \) is bounded below; we can essentially ignore the "border". In order to guarantee that this bound is uniform, slightly stronger conditions are necessary. One such condition is a strengthening of the assumption of negative definiteness.

**Definition 4.1.** Let \( \{T(w) : w \in W\} \) be a family of continuous linear operators from \( X \) to \( X \), where \( X \) is a Banach space and \( \langle X, X' \rangle \) is a Riesz dual pairing. This family is called **uniformly negative definite** if \( T(w) \) is negative definite for each \( w \in W \), and if there exists \( b > 0 \) such that for any sequences \( \{w^n\} \subset W \) and \( \{z^n\} \subset X \) such that \( \|z^n\| \leq b \) and \( z^n \cdot T(w^n)z^n \rightarrow 0 \), \( T(w^n)z^n \rightarrow 0 \) weak* \( \langle \sigma(X, X') \rangle \). If a utility function \( U : \ell_\infty+ \rightarrow \mathcal{R} \) has a second derivative \( D^2U(x) \) which is uniformly negative definite over a set \( W \subset \text{int } \ell_\infty+ \), then the function \( U(x) \) will be called **uniformly concave over \( W \).**

Although this definition is formulated very generally, the relevant Riesz dual pair for this paper is \( (\ell_\infty, \ell_1) \), so when I use uniform concavity below it will always be with respect to this pairing.

As the following result shows, if each consumer's utility function is uniformly concave, then to show that \( H^\rho \) is uniformly bounded below, and hence that the Pareto map is locally Lipschitz, it suffices to show that \( D^2U^\rho \) is uniformly bounded below; intuitively, we can ignore the "border".

**Theorem 4.5.** Suppose that for each \( i = 1, \ldots, m \), \( U_i(x_i) \) is uniformly concave and that there exists a discounting function \( \rho \) such that \( DU_i^{\rho-1} \) is uniformly bounded for each \( i \). Then
if $D^2 U^o$ is uniformly bounded below, the Pareto map is locally Lipschitz, i.e., the economy $\mathcal{E}_o$ is a Lipschitz economy.

**Proof:** Let $U \in D^o$ be given and let $V$ be a neighborhood of $U$ such that $V \subset D^o$. Since $\lambda(U)$ is continuous (see Shannon (1994b)), there exist $\lambda, \bar{\lambda} \in \mathcal{R}^+_{++}$ such that $\lambda(V) \subset [\lambda, \bar{\lambda}]$. By Theorem 4.4, it suffices to show that $H^o$ is uniformly bounded below. I will prove the theorem for the case $m = 2$; the general case is analogous. Suppose $H^o$ is not uniformly bounded below. Then there exists sequences $\{(z^n, r^n)\}$ and $\{(x^n, \lambda^n)\}$ such that $|\|z^n, r^n\| = 1$ for all $n$ but

$$D^2 U^o(x^n, \lambda^n)z^n + r^n DU^o_2(x^n)z^n = w^n_1,$$

$$DU^o_2(x^n)z^n = w^n_2,$$

and $\|w^n\| \to 0$. Let $c$ be the uniform lower bound on $D^2 U^o(x)$, and note that by assumption there exists $M > 0$ such that $\|DU^o_2(x^n)\| < M$ for all $n$. Then without loss of generality, suppose $\|r^n\| > \frac{c}{2M} > 0$, since if $\|r^n\| < \frac{c}{2M} < 1$, $\|w^n\| \geq \|w^n_1\| \geq \|D^2 U^o(x^n)z^n\| - \|r^n\||||DU^o_2(x^n)\| \geq \frac{c}{2}$. Thus

$$(\rho(x^n) \otimes z^n)^T D^2 U^o(x^n, \lambda^n)z^n + r^n DU^o_2(x^n) \cdot (\rho(x^n) \otimes z^n) = w^n_1 \cdot (\rho(x^n) \otimes z^n)$$

$$\Rightarrow z^n^T D^2 U(x^n, \lambda^n)z^n + r^n DU^o_2(x^n) \cdot z^n = w^n_1 \cdot (\rho(x^n) \otimes z^n)$$

$$\Rightarrow z^n^T D^2 U(x^n, \lambda^n)z^n = w^n_1 \cdot (\rho(x^n) \otimes z^n) - r^n DU^o_2(x^n) \cdot z^n$$

$$= w^n_1 \cdot (\rho(x^n) \otimes z^n) - r^n w^n_2$$

$$\to 0.$$

Now since $D^2 U_1(x_1)$ and $D^2 U_2(x_2)$ are uniformly negative definite, so is the sum $D^2 U(x, \lambda) = D^2 U_1(x_1) + \lambda D^2 U_2(x_2)$ on sets of the form $[\underline{x}, \bar{x}] \times [\lambda, \bar{\lambda}]$, for $\lambda > 0$. To see this, suppose

$$y^n^T (D^2 U_1(x^n_1) + \lambda^n D^2 U_2(x^n_2))y^n \to 0.$$ 

Since $y^n^T D^2 U_1(x^n_1)y^n < 0$ and $\lambda^n y^n^T D^2 U_2(x^n_2)y^n < 0$ for each $n$, $y^n^T D^2 U_1(x^n_1)y^n \to 0$ and $\lambda^n y^n^T D^2 U_2(x^n_2)y^n \to 0$. Since $\bar{\lambda} \geq \lambda \geq \underline{\lambda} > 0$, $y^n^T D^2 U_2(x^n_2)y^n \to 0$. Thus $D^2 U_1(x^n_1)y^n \to 0$ weak* (as $\mathcal{L}_\infty^+$, $\mathcal{C}$) and $D^2 U_2(x^n_2)y^n \to 0$ weak*, i.e., $D^2 U(x^n, \lambda^n)y^n \to 0$ weak*.

Thus $D^2 U(x^n, \lambda^n)z^n \to 0$ weak*, which implies that $r^n DU^o_2(x^n_2) \to 0$ weak*, and hence pointwise. But $r^n$ is bounded away from 0, so this contradicts the Inada condition that $DU^o_2(x^n_2) \not\to 0$ pointwise. Thus $H^o$ is uniformly bounded below. $\blacksquare$

For example, additively separable utility functions are uniformly concave on any set of the form $[\underline{x}, \bar{x}]$, with $\bar{x}, \underline{x} \in \text{int} \mathcal{L}_\infty^+$, as are habit formation utility functions. The proof of this claim for additively separable utility functions is given below; the argument for habit formation models is given in theorem 4.15.

**Theorem 4.6.** If $U(x) = \sum_{i=0}^{\infty} \beta^i u(x_i)$, where $\beta \in (0, 1)$, $u: \mathcal{R} \to \mathcal{R}$ is $C^2$, $u'(c) > 0$ and $u''(c) < 0$ for all $c$, then $U(x)$ is uniformly concave on any interval of the form $[\underline{x}, \bar{x}]$ where $\bar{x}, \underline{x} \in \text{int} \mathcal{L}_\infty^+$.

\[\text{In particular, note that we have shown that the properties of uniform negative definiteness and uniform concavity are preserved by addition.}\]

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Proof: Let $\tilde{x} \geq x$, and without loss of generality, assume $\tilde{x} = (\tilde{c}, \tilde{c}, \ldots)$ and $x = (c, c, \ldots)$ for $\tilde{c} \geq c > 0$. Suppose there exists $z^n, x^n \in \ell_\infty$ and $b > 0$ such that $\|z^n\| \leq b$ for all $n$ and $z^n^T D^2 U(x^n) z^n \rightarrow 0$. Thus

$$z^n^T D^2 U(x^n) z^n = \sum_{t=0}^{\infty} \beta^t u''(x^n_t)(z^n_t)^2 \leq \sum_{t=0}^{\infty} \beta^t \tilde{d}(z^n_t)^2 \rightarrow 0$$

where $\tilde{d} = \max_{[\tilde{c}, c]} u''(c) < 0$. Thus $z^n_t \rightarrow 0$ for each $t$, which implies that $\beta^t u''(x^n_t) z^n_t \rightarrow 0$ for each $t$. Then $D^2 U(x^n) z^n \rightarrow 0$ pointwise. Since $\|z^n\| \leq b$ for each $n$, $\{D^2 U(x^n) z^n : n = 1, 2, \ldots\}$ lies in a bounded set, and weak* convergence is equivalent to pointwise convergence on bounded sets, i.e., $D^2 U(x^n) z^n \rightarrow 0$ weak*. Thus $U(x)$ is uniformly concave.

If each consumer’s utility function is uniformly concave, then to show that the Pareto map is locally Lipschitz it suffices to show that there exists a discounting function $\rho(x)$ such that $D^2 U^\rho$ is uniformly bounded below. One approach to this problem is to separate it into two steps, and to first find a discount function $\rho$ such that $D^2 U_i^{\rho_i}$ is uniformly bounded below and $D U_i^{\rho_i}$ is bounded for each $i$, and then to find conditions under which $D^2 U^\rho$ is also uniformly bounded below. If the first part of the problem has been solved, the second part amounts to finding conditions under which the sum of two operators, each of which is uniformly bounded below, is also uniformly bounded below.\(^{17}\) Although this is a difficult problem in general, we can identify several sets of sufficient conditions. First, to simplify the discussion, we make the following definition.

**Definition 4.2.** Consumers’ utility functions $U_i(x_i)$ will be called compatible if there exists a discounting function $\rho(x) : \ell_\infty^m \rightarrow \ell_1^{m-1}$ such that $D^2 U_i^{\rho_i}$ is uniformly bounded below and $D U_i^{\rho_i}$ is bounded for $i = 1, \ldots, m$.

Using this definition, we can give a set of restrictions across consumer preferences which ensure that the Pareto map is locally Lipschitz. To make the statement and proof of this result simpler, I will also assume that we can find a single discounting function $\rho : \ell_\infty^m \rightarrow \ell_1^{m-1}$ appropriate for all consumers, so that $\rho_i(x) \equiv \rho(x)$ is constant across consumers for each bundle $x$. The extension of this result to the general case where $\rho_i(x)$ may vary across consumers is straightforward, and is discussed following the theorem.

**Theorem 4.7.** Suppose that the economy is composed of compatible consumers with generalized discounting function $\rho(x) : \ell_\infty^m \rightarrow \ell_++$ and that $\text{Range}(D^2 U_1(x_1)) \subset \text{Range}(D^2 U_i(x_i))$.

\(^{17}\)In the two-consumer case this is clear, since $D^2 U(x, \lambda) = D^2 U_1(x_1) + \lambda D^2 U_2(x_2)$. With more than two consumers, the operator $D^2 U(x, \lambda)$ can be decomposed in several ways; for example, $D^2 U(x, \lambda) =$

$$
\begin{pmatrix}
D^2 U_1(x_1) + \lambda_2 D^2 U_2(x_2) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & D^2 U_m(x_m) + \lambda_m D^2 U_m(x_m)
\end{pmatrix}
+ \begin{pmatrix}
0 & D^2 U_1(x_1) & \cdots & D^2 U_i(x_1) \\
\vdots & \ddots & \vdots & \vdots \\
D^2 U_1(x_1) & \cdots & D^2 U_i(x_i) & 0
\end{pmatrix}.
$$
for \(i = 2, \ldots, m\). If \(S(x) \equiv I + (\lambda_2 D^2 U_2^\rho(x_2))^{-1} + \cdots + (\lambda_m D^2 U_m^\rho(x_m))^{-1} D^2 U_1^\rho(x_1)\) is uniformly bounded below, then \(D^2 U^\rho(x)\) is uniformly bounded below. Hence if \(U_i(x_i)\) is also uniformly concave, then the economy \(E_\omega\) is a Lipschitz economy.

**Proof:** First, note that although \(D^2 U_i^\rho(x_i)\) may not be invertible, since it is bounded below it has an inverse on \(\text{Range}(D^2 U_i^\rho(x_i))\), so that \((D^2 U_i^\rho(x_i))^{-1} D^2 U_1^\rho(x_1)\) is well-defined. Suppose \(\|z^n\| = 1\) and \(D^2 U^\rho(x^n)z^n = w^n\), so

\[
\begin{align*}
[D^2 U_1^\rho(x^n_1) + \lambda_2 D^2 U_2^\rho(x^n_2)]z^n_1 + D^2 U_1^\rho(x^n_1)(z^n_2 + \cdots + z^n_{m-1}) & = w^n_1 \\
\vdots \\
[D^2 U_1^\rho(x^n_1) + \lambda_m D^2 U_m^\rho(x^n_m)]z^n_m - 1 + D^2 U_1^\rho(x^n_1)(z^n_2 + \cdots + z^n_{m-2}) & = w^n_{m-1}
\end{align*}
\]

and suppose \(\|w^n\| \to 0\). Then

\[
\begin{align*}
\lambda_2^\rho D^2 U_2^\rho(x^n_2)z^n_1 & = w^n_1 - D^2 U_1^\rho(x^n_1)(z^n_2 + \cdots + z^n_{m-1}) \\
\vdots \\
\lambda_m^\rho D^2 U_m^\rho(x^n_m)z^n_{m-1} & = w^n_{m-1} - D^2 U_1^\rho(x^n_1)(z^n_2 + \cdots + z^n_{m-1})
\end{align*}
\]

So

\[
\begin{align*}
z^n_1 & = [\lambda_2^\rho D^2 U_2^\rho(x^n_2)]^{-1}[w^n_1 - D^2 U_1^\rho(x^n_1)(z^n_2 + \cdots + z^n_{m-1})] \\
\vdots \\
z^n_{m-1} & = [\lambda_m^\rho D^2 U_m^\rho(x^n_m)]^{-1}[w^n_{m-1} - D^2 U_1^\rho(x^n_1)(z^n_2 + \cdots + z^n_{m-1})]
\end{align*}
\]

Then adding these terms yields

\[
\begin{align*}
z^n_1 + \cdots + z^n_{m-1} \\
= [\lambda_2^\rho D^2 U_2^\rho(x^n_2)]^{-1}w^n_1 + \cdots + [\lambda_m^\rho D^2 U_m^\rho(x^n_m)]^{-1}w^n_{m-1} \\
- ([(\lambda_2^\rho D^2 U_2^\rho(x^n_2)]^{-1} + \cdots + [\lambda_m^\rho D^2 U_m^\rho(x^n_m)]^{-1}) D^2 U_1^\rho(x^n_1)(z^n_2 + \cdots + z^n_{m-1})
\end{align*}
\]

or, rearranging terms,

\[
\begin{align*}
\left[I + (\lambda_2^\rho D^2 U_2^\rho(x^n_2)]^{-1} + \cdots + [\lambda_m^\rho D^2 U_m^\rho(x^n_m)]^{-1}D^2 U_1^\rho(x^n_1)\right](z^n_2 + \cdots + z^n_{m-1}) \\
= [\lambda_2^\rho D^2 U_2^\rho(x^n_2)]^{-1}w^n_1 + \cdots + [\lambda_m^\rho D^2 U_m^\rho(x^n_m)]^{-1}w^n_{m-1}
\end{align*}
\]

If \(\|w^n\| \to 0\), then by the above equation,

\[
\begin{align*}
\left[I + (\lambda_2^\rho D^2 U_2^\rho(x^n_2)]^{-1} + \cdots + [\lambda_m^\rho D^2 U_m^\rho(x^n_m)]^{-1}D^2 U_1^\rho(x^n_1)\right](z^n_2 + \cdots + z^n_{m-1}) \to 0,
\end{align*}
\]

and since by assumption \(I + (\lambda_2^\rho D^2 U_2^\rho(x_2)]^{-1} + \cdots + [\lambda_m^\rho D^2 U_m^\rho(x_m)]^{-1}) D^2 U_1^\rho(x_1)\) is uniformly bounded below, \(z^n_1 + \cdots + z^n_{m-1} \to 0\). Thus \(z^n_i \to 0\) for every \(i\), which contradicts the fact that \(\|z^n\| = 1\) for every \(n\). Thus \(D^2 U^\rho(x)\) is uniformly bounded below. The final statement follows from Theorem 4.5.

\[\blacksquare\]
If the discounting function \( p(x) \) varies across consumers, then the same argument with more cumbersome notation shows that if consumers are compatible and have uniformly concave utility functions, and if the operator

\[
S(x) \equiv I + (\lambda_2 D^2 U_2^{\beta_1}(x_2))^{-1} D^2 U_1^{\alpha_1}(x_1) + \cdots + (\lambda_m D^2 U_m^{\alpha_m-1}(x_m))^{-1} D^2 U_1^{\alpha_m-1}(x_1)
\]

is uniformly bounded below, then the economy is also a Lipschitz economy.

If these conditions or the conditions of theorem 4.7 hold, we will say the preferences are strongly compatible. Again the additively separable economy provides an example in which these ideas apply. If all consumers have additively separable preferences, then their preferences are compatible since they all discount at the constant rate \( \beta \), so \( \tilde{\beta} = (1, \beta, \beta^2, \ldots) \) gives the constant discounting function applicable to all consumers. Moreover, it is straightforward to show that the other conditions of theorem 4.7 hold as well.

**Theorem 4.8.** If each consumer's utility function is additively separable with constant discount factor \( \beta \), then the consumers' preferences are strongly compatible on any set of the form \([\underline{\tilde{x}}, \tilde{\tilde{x}}] \times [\underline{\Delta}, \tilde{\Delta}]\), where \( \tilde{\tilde{x}} \geq \tilde{x}, \tilde{\tilde{x}}, \tilde{x} \in \text{int } \ell_\infty^+ \) and \( \lambda \geq \Delta \gg 0 \).

**Proof:** I will prove this claim for the case \( m = 2 \); the general case is analogous and is left to the reader. Both cases hinge on the observation that if \( D(x) \) is a family of continuous linear operators from \( \ell_\infty \) to \( \ell_\infty \) such that \( D(x) \) is a diagonal operator with diagonal \( \{d_t(x)\} \) which is uniformly bounded away from 0, then \( D(x) \) is uniformly bounded below. To see this, note that

\[
\|D(x)z\| = \sup_t |d_t(x)z_t| = \sup_t |d_t(x)||z_t| \geq d \sup_t |z_t| = d\|z\|
\]

where \( 0 < d \leq |d_t(x)| \) for all \( t \) and \( x \).

Then consumers in this additively separable economy are compatible, since for all \( i \), \( D^2 U_i^{\tilde{\beta}_i}(x_i) \) is a diagonal operator with diagonal which is uniformly bounded away from 0 on \([\underline{\tilde{x}}, \tilde{\tilde{x}}]\), and \( DU_i^{\tilde{\beta}_i}(x_i) \) is bounded on \([\underline{\tilde{x}}, \tilde{\tilde{x}}]\). Similarly, preferences are strongly compatible. First, \( \text{Range}(D^2 U_1^{\tilde{\beta}}(x_1)) = \ell_\infty = \text{Range}(D^2 U_2^{\tilde{\beta}}(x_2)) \) for all \( x \). Moreover, \( I + (\lambda D^2 U_2^{\tilde{\beta}}(x_2))^{-1}(D^2 U_1^{\tilde{\beta}}(x_1)) \) is a diagonal operator with diagonal \( \{1 + \frac{u''_1(x_1)}{\lambda u'_1(x_1)}\} \), which is uniformly bounded away from 0 on \([\underline{\tilde{x}}, \tilde{\tilde{x}}] \times [\underline{\Delta}, \tilde{\Delta}]\).

Then putting all of these results together, we have established that in the special case when each consumer's utility function is additively separable, the Pareto map is locally Lipschitz and the determinacy results of the previous section apply.

**Theorem 4.9.** Let \( \omega \in \text{int } \ell_\infty^+ \), and for \( i = 1, \ldots, m \) let \( U_i(x) = \sum_{t=1}^{\infty} \beta^t u_i(t_i), \) where \( 0 < \beta < 1, u_i : \mathcal{R}_+ \to \mathcal{R} \) is \( C^2 \), \( u'_i(c) > 0 \) and \( u''_i(c) < 0 \) \( \forall c \), and \( u'_i(c) \to \infty \) as \( c \to 0 \). Then the economy \( \mathcal{E}_\omega \) is a Lipschitz economy with smooth myopic preferences. If \( \mathcal{E}_\omega \) is a regular
economy, then there are an odd number of equilibria, each of which is upper Lipschitzian in \( \omega^1 \). Moreover, for almost every \( \omega^1 \in \mathcal{W} \), the economy \( \mathcal{E}_\omega \) is a regular economy.

**Proof:** It suffices by Theorem 2.11 to show that \( x(U) \) is locally Lipschitz, but this follows from Theorems 4.6, 4.7 and 4.8.

Of course the generic determinacy of equilibria in the additively separable model was already established by Kehoe and Levine (1985). However, their approach, while simpler than the results applied here, is limited to the additively separable case; the methods developed here are certainly applicable in models with non-separable preferences, at least in theory. One drawback of the results presented thus far is that the sufficient conditions like strong compatibility which I give for generic determinacy rely on joint restrictions across preferences. To a certain extent this is unavoidable when studying determinacy in infinite-dimensional models. We can avoid making joint restrictions across preferences in finite-dimensional models and still reach strong conclusions about determinacy by first assuming that indifference curves do not intersect the boundary of the positive cone, to guarantee interior equilibria, and then assuming preferences are differentiable strictly convex, which implies that the finite-dimensional analogue of \( D^2u(x,\lambda) \) is negative definite since it is the sum of negative definite matrices, which in turn implies that the analogue of the “bordered Hessian” \( H(x,\lambda,U) \) is one-to-one and thus nonsingular. The same conclusion that the sum of negative definite matrices is negative definite holds here, implying here that \( D^2u(x,\lambda) \) is also negative definite and \( H(x,\lambda,U) \) is also one-to-one, but this is simply no longer sufficient to reach the conclusion that \( H(x,\lambda,U) \) is nonsingular. Although joint restrictions are unavoidable to some degree, more desirable would be conditions involving restrictions on individual preferences alone. Again, even though determinacy in the additively separable case is well-understood, studying additively separable preferences suggests one such set of restrictions on individual preferences sufficient to yield generic determinacy.

Note that if preferences are additively separable, then the only thing influencing marginal utility for consumption in period \( t \) is consumption in period \( t \). So if the utility function \( U_t(x) \) is additively separable, then the second derivative \( D^2U_t(x) \) is a diagonal matrix with diagonal \( \{\beta_t u_t'(x_i)\} \), and thus has a dominant diagonal in a very strong sense: all of the off-diagonal elements are 0. Moreover, if we normalize by using this diagonal vector \( d_t(x) \) rather than by the vector of marginal utilities, the resulting matrix \( D^2U_t(x) \) is simply the identity, which is both bounded below and invertible. This observation suggests that rather than taking the vector \( \tilde{d} \) to represent the sequence of discount factors, we could have chosen these diagonal vectors \( d_t(x) \). If we consider more general preferences, consumption in periods other than \( t \) will affect marginal utility for consumption in period \( t \), so that the second derivative will no longer be a diagonal matrix. However, as long as the effect of consumption in period \( s \) on marginal utility for consumption in period \( t \) is small relative to the effect of consumption in the same period \( t \), the economy will still be a Lipschitz economy. That is, even if the off-diagonal elements are not zero, dominant diagonal matrices have enough structure to suggest the appropriate normalization of the problem.

In order to make sense of that claim, we must first define the notion of dominant diagonal in this setting. If \( A(x) \) is a continuous linear operator from \( \ell_\infty \) to \( \ell_\infty \) for each \( x \in V \subset \ell_\infty \), then \( A(x) \) has a **dominant diagonal** \( d(x) = \{d_t(x)\} \) if there exists \( c \in (0,1) \) such that
\[
\sum_{s \neq t} \left| \frac{a_{ts}(x)}{a_{tt}(x)} \right| \leq c < 1 \text{ for every } t. \quad \text{Moreover, if this bound holds for every } x \in V, \text{ then we say that } A(x) \text{ has a uniformly dominant diagonal over the set } V. \text{ Now define } A^d(x) \text{ to be the operator whose } t^s \text{th element is } \frac{a_{ts}(x)}{a_{tt}(x)} = \frac{z_t(x)}{d_t(x)}. \text{ As the following theorem shows, if } A \text{ has a dominant diagonal, then normalizing by this diagonal produces an operator which is bounded below.}
\]

**Theorem 4.10.** If \( A(x) : \ell_{\infty} \to \ell_{\infty} \) is a continuous linear operator for every \( x \in V \subseteq \ell_{\infty} \) and if \( A(x) \) has a dominant diagonal, then \( A^d(x) \) is bounded below. If \( A(x) \) has a uniformly dominant diagonal on \( V \), then \( A^d(x) \) is uniformly bounded below on \( V \).

**Proof:** Note that for \( z \in \ell_{\infty} \),

\[
\left\| A^d(x)z \right\| = \sup_t \left| \sum_{s=1}^{\infty} \frac{a_{ts}(x)}{a_{tt}(x)} z_s \right|
\]

\[
= \sup_t \left| z_t + \sum_{s \neq t} \frac{a_{ts}(x)}{a_{tt}(x)} z_s \right|
\]

\[
\geq \sup_t \left| z_t - \sum_{s \neq t} \frac{a_{ts}(x)}{a_{tt}(x)} z_s \right|
\]

\[
\geq \sup_t \left| z_t - \frac{a_{tt}(x)}{a_{tt}(x)} \right| |z_s|
\]

\[
\geq \sup_t |z_t| - c\|z\|
\]

\[
= (1 - c)\|z\|
\]

where \( 1 - c > 0 \). Thus \( A^d(x) \) is bounded below. Moreover, this argument clearly shows that if \( A \) has a uniformly dominant diagonal, then \( A^d(x) \) is uniformly bounded below. \( \blacksquare \)

Moreover, this result suggests a straightforward test for determining if an operator is uniformly bounded below: if \( A(x) \) has a uniformly dominant diagonal which is itself uniformly bounded away from 0, then \( A(x) \) is uniformly bounded below.

**Theorem 4.11.** Suppose \( A(x) : \ell_{\infty} \to \ell_{\infty} \) is a continuous linear operator for every \( x \in V \subseteq \ell_{\infty} \) and \( A(x) \) has a uniformly dominant diagonal \( \{a_{tt}(x)\} \) such that \( |a_{tt}(x)| \geq a > 0 \) for some \( a \) and for all \( t \) and \( x \). Then \( A(x) \) is uniformly bounded below.

**Proof:** Let \( O_A(x) = \{o_{ts}(x)\} \), where

\[
o_{ts}(x) = \begin{cases} 0 & \text{if } s = t; \\ \frac{a_{ts}(x)}{a_{tt}(x)} & \text{if } s \neq t. \end{cases}
\]

Then for \( x \),

\[
A(x) = \text{diag}\{a_{tt}(x)\}[I + O_A(x)].
\]

Since \( A(x) \) has a uniformly dominant diagonal, there exists \( c \in (0, 1) \) such that \( \|O_A(x)\| \leq c \) for all \( x \). Thus \( I + O_A(x) \) is uniformly bounded below, since for all \( x \) and for all \( z \in \ell_{\infty} \),

\[
\|[I + O_A(x)]z\| \geq \|z\| - \|O_A(x)z\| \geq (1 - c)\|z\|.
\]

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Since \(|a_{tt}(x)| \geq a\) for all \(x\) and for each \(t\), \(\text{diag}\{a_{tt}(x)\}\) is uniformly bounded below as well. Thus \(A(x)\) is the product of two operators which are uniformly bounded below, so \(A(x)\) is uniformly bounded below.

Furthermore, the property of having a dominant diagonal is preserved under addition of such operators, and the appropriate normalization of the sum is given by the sum of the diagonals.

**Theorem 4.12.** Let \(A(x), B(x) : \ell_\infty \to \ell_\infty\) be continuous, negative definite linear operators for each \(x \in V \subset \ell_\infty\). If \(A(x)\) and \(B(x)\) have dominant diagonals, then \((A(x) + B(x))^d\) is bounded below. If \(A(x)\) and \(B(x)\) have uniformly dominant diagonals, then \((A(x) + B(x))^d\) is uniformly bounded below on \(U\).

**Proof:** First, note that since \(A(x)\) and \(B(x)\) are negative definite, \(a_{tt}(x) < 0\) and \(b_{tt}(x) < 0\) for all \(i\) and \(x\). By assumption, there exists \(c \in (0,1)\) such that for every \(x \in V\) and for every \(t\), \(\sum_{s \neq t} |a_{ts}(x)| \leq c|a_{tt}(x)|\) and \(\sum_{s \neq t} |b_{ts}(x)| \leq c|b_{tt}(x)|\). Thus

\[
\sum_{s \neq t} |a_{ts}(x) + b_{ts}(x)| \leq \sum_{s \neq t} |a_{ts}(x)| + \sum_{s \neq t} |b_{ts}(x)|
\leq c|a_{tt}(x)| + c|b_{tt}(x)| = c|a_{tt}(x) + b_{tt}(x)|
\]

So

\[
\sum_{s \neq t} \left| \frac{a_{ts}(x) + b_{ts}(x)}{a_{tt}(x) + b_{tt}(x)} \right| \leq c < 1
\]

for every \(t\) and \(x \in V\). The results now follow from the previous theorem.

In a two-consumer economy, these results mean that if each consumer has preferences in which period \(t\) consumption has a dominant effect on marginal utility for consumption in period \(t\) so that \(D^2U_i(x_i)\) has a uniformly dominant diagonal, then the appropriate discounting function is given by the sum of the diagonals of these second derivatives, and \(D^2U^d(x)\) is uniformly bounded below. Provided consumers’ utility functions are also uniformly concave, such an economy is a Lipschitz economy. These observations are collected in the following theorem.

**Theorem 4.13.** In a two-consumer economy that satisfies the strong survival condition, if each consumer’s utility function \(U_i(x)\) is uniformly concave and if \(D^2U_i(x)\) has a uniformly dominant diagonal on \([\underline{z}, \bar{z}]\) for all \(\underline{z} \geq \bar{z}\) with \(\underline{z}, \bar{z} \in \text{int} \ell_\infty^+\), then the economy is a Lipschitz economy.

One example of a class of utility functions which are uniformly concave and for which the second derivative has a uniformly dominant diagonal is habit formation preferences of the form \(U(x) = v(x_0) + \sum_{t=1}^{\infty} \beta^t u(x_{t-1}, x_t)\), where \(0 < \beta < 1\), \(u, v\) are \(C^2\), \(v'(c) > 0, v''(c) < 0\) for every \(c \in \mathcal{R}_+\), \(D^2u(c_1, c_2)\) is negative definite for every \((c_1, c_2) \in \mathcal{R}_+^2\), and has a dominant diagonal. To establish this claim, first note that if all consumers have such habit formation preferences, then the economy is a smooth myopic economy.
Theorem 4.14. Suppose each consumer's utility function is of the form \( U_i(x) = u_i(x_0) + \sum \beta^t u_i(x_{t-1}, x_t) \), where \( 0 < \beta < 1 \), \( u_i, v_i \) are \( C^2 \), \( v_i'(c) > 0, v_i''(c) < 0 \) for every \( c \in R_+ \), \( D^2 u_i(c_1, c_2) \) is negative definite for every \( (c_1, c_2) \in R^2_+ \), and has a dominant diagonal. Then the economy has smooth myopic preferences.

Proof: This proof is a straightforward adaptation of the proof of Theorem 2.2; the details are left to the reader.

In order to ensure that such preferences will satisfy the strong survival conditions and thus that the individually rational Pareto optimal allocations will be interior, assume in addition that the functions \( u_i \) satisfy the boundary condition described in section 2, i.e., that for any sequence \( \{(x_n, y_n)\} \in R_+^2 \) such that \( (x_n, y_n) \to (x, y) \), \( \frac{\partial u_i}{\partial x}(x_n, y_n) \to \infty \iff x_n \to 0 \) and similarly \( \frac{\partial u_i}{\partial y}(x_n, y_n) \to \infty \iff y_n \to 0 \); I will call this the boundary condition for habit formation. This assumption means that the individually rational Pareto optimal allocations lie in some interval of the form \([\underline{x}, \bar{x}]\), where \( \underline{x} \leq \bar{x} \) and \( \underline{x}, \bar{x} \in int C_{\infty}^+ \) by theorem 2.15. Furthermore, under the assumptions of Theorem 4.14, habit formation preferences are uniformly concave and have second derivatives with uniformly dominant diagonals on such intervals.

Theorem 4.15. Suppose \( U(x) = v(x_0) + \sum_{t=1}^{\infty} \beta^t u_t(x_{t-1}, x_t) \), where \( 0 < \beta < 1 \), \( v, v' \) are \( C^2 \), \( v'(c) > 0, v''(c) < 0 \) for every \( c \in R_+ \), \( D^2 u(c_1, c_2) \) is negative definite for every \( (c_1, c_2) \in R^2_+ \) and has a dominant diagonal, and \( u \) satisfies the boundary condition for habit formation. Then for any \( \underline{x}, \bar{x} \in int C_{\infty}^+ \) such that \( \underline{x} \leq \bar{x} \), \( U(x) \) is uniformly concave and \( D^2 U(x) \) has a uniformly dominant diagonal on \([\underline{x}, \bar{x}]\).

Proof: Let \( \underline{x}, \bar{x} \in int C_{\infty}^+ \) be given such that \( \underline{x} \leq \bar{x} \). First, \( U(x) \) is uniformly concave on \([\underline{x}, \bar{x}]\). To see this, consider sequences \( \{x^n\} \subset [\underline{x}, \bar{x}] \) and \( \{z^n\} \) such that for all \( n \), \( ||z^n|| \leq b \) for some \( b > 0 \) and such that \( z^n T D^2 U(x^n) z^n \to 0 \). Then

\[
\begin{align*}
z^n T D^2 U(x^n) z^n &= v''(x^n_0)(z^n_0)^2 + \sum_{t=1}^{\infty} \beta^t (z^n_{-1}, z^n_t)^T D^2 u(x^n_{t-1}, x^n_t) (z^n_{-1}, z^n_t) \\
&\to 0.
\end{align*}
\]

Thus \( z^n_t \to 0 \) for each \( t \). To see this, suppose \( \exists \delta \) such that \( z^n_0 \neq 0 \). Then there exists a subsequence \( \{z^n_0\} \) such that \( ||z^n_0|| \geq d > 0 \) for all \( n \). Let \( b^n = z^n T D^2 U(x^n) z^n \). Then

\[
|b^n| \geq \beta^S \langle (z^n_{-1}, z^n_0) T D^2 u(x^n_{-1}, x^n_0) (z^n_{-1}, z^n_0) \rangle
\]

for every \( n \). However, \( (z^n_{-1}, z^n_0) T D^2 u(x^n_{-1}, x^n_0) (z^n_{-1}, z^n_0) \) \( < 0 \) since \( (z^n_{-1}, z^n_0) \neq 0 \), and over the compact set \( \{(r_1, r_2) : 1 \geq |r_2| \geq d > 0, |r_1| \leq 1\} \times [-\bar{b}, \bar{b}] \), the continuous function \( (r_1, r_2)^T D^2 u(c_1, c_2)(r_1, r_2) \) achieves its maximum, which is negative. Thus there exists \( a > 0 \) such that

\[
|b^n| \geq \beta^S \langle (z^n_{-1}, z^n_0) T D^2 u(x^n_{-1}, x^n_0) (z^n_{-1}, z^n_0) \rangle \geq \beta^S a > 0
\]

for all \( n \), which contradicts the fact that \( b^n \to 0 \). Thus \( z^n_t \to 0 \) for every \( t \). Hence \( D^2 U(x^n) z^n \to 0 \) pointwise, and thus weak*, since pointwise convergence and weak* convergence agree on bounded sets. Thus \( U(x) \) is uniformly concave.
Moreover, consider an arbitrary row of \( D^2U(x) \equiv \{d_{ts}(x)\} \). Letting subscripts on \( u \) denote partial derivatives,

\[
d_{ts}(x) = \begin{cases} 
\beta^t u_{21}(x_{t-1}, x_t) & \text{if } s = t - 1; \\
\beta^tu_{22}(x_{t-1}, x_t) + \beta^{t+1}u_{11}(x_t, x_{t+1}) & \text{if } s = t; \\
\beta^{t+1}u_{21}(x_t, x_{t+1}) & \text{if } s = t + 1; \\
0 & \text{otherwise.}
\end{cases}
\]

Since \( D^2u \) has a dominant diagonal and is continuous, there exists \( k \in (0, 1) \) such that

\[
|u_{21}(c_1, c_2)| \leq k|u_{11}(c_1, c_2)| \\
|u_{21}(\tilde{c}_1, \tilde{c}_2)| \leq k|u_{22}(\tilde{c}_1, \tilde{c}_2)|
\]

for all \((c_1, c_2), (\tilde{c}_1, \tilde{c}_2) \in [\bar{b}, \bar{b}] \times [\bar{b}, \bar{b}]\). Thus

\[
|u_{21}(x_{t-1}, x_t)| \leq k|u_{22}(x_{t-1}, x_t)| \\
|u_{21}(x_t, x_{t+1})| \leq k|u_{11}(x_t, x_{t+1})|.
\]

So

\[
\sum_{s \neq t} \left| \frac{d_{ts}(x)}{d_{tt}(x)} \right| = \frac{\beta^t u_{21}(x_{t-1}, x_t) + \beta^{t+1}u_{21}(x_t, x_{t+1})}{\beta^t |u_{22}(x_{t-1}, x_t)| + \beta^{t+1} |u_{11}(x_t, x_{t+1})|} \\
\leq \frac{\beta^t k|u_{22}(x_{t-1}, x_t)| + \beta^{t+1} k|u_{11}(x_t, x_{t+1})|}{\beta^t |u_{22}(x_{t-1}, x_t)| + \beta^{t+1} |u_{11}(x_t, x_{t+1})|} \\
= k
\]

for all \( x \in [\underline{x}, \bar{x}] \). Thus \( D^2U(x) \) has a uniformly dominant diagonal.

For an economy with two consumers, these assumptions on habit formation preferences are sufficient to imply that the economy is a Lipschitz economy and thus that the equilibria are generically determinate by Theorem 4.13. In economies with more than two consumers, slightly stronger conditions are required to guarantee that the economy is a Lipschitz economy.

As above, suppose that each consumer’s utility function \( U_i(x) \) has a second derivative \( D^2U_i(x) = \{d_{it}(x)\} \) which has a uniformly dominant diagonal on any interval of the form \([\underline{x}, \bar{x}] \subset \text{int } \ell_{\infty+}\). Let \( d_i(x) = \{d_{it}(x)\} \) be this diagonal vector, and define \( O_i(x) = \{o_{it}(x)\} \) to be the matrix of normalized off-diagonal elements, i.e.,

\[
o_{it}(x) = \begin{cases} 
0 & \text{if } s = t; \\
d_{it}(x) & \text{if } s \neq t.
\end{cases}
\]

Using this notation,

\[
D^2U_i(x) = \text{diag}\{d_{it}(x)\}[I + O_i(x)]
\]

and

\[
D^2U_i^{ii}(x) = I + O_i(x).
\]

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The assumption that $D^2 U_i(x)$ has a uniformly dominant diagonal on $[\underline{x}, \bar{x}]$ means that there exists $k \in (0, 1)$ such that $\|O_i(x)\| \leq k$ for all $x \in [\underline{x}, \bar{x}]$. With only two consumers this condition was sufficient to ensure generic determinacy: with more than two consumers a slight strengthening of this bound together with a consistency condition across consumers will imply that equilibria are generically determinate.

I will say that utility functions $U_1(x_1), \ldots, U_m(x_m)$ are consistent if for any interval $[\underline{x}, \bar{x}] \subset \text{int } \ell^m_{\infty+}$, \( \left\{ \frac{d^2 U_i(x_i)}{d^2 U_j(x_j)} \right\} \) is uniformly bounded and uniformly bounded away from 0 for all $i \neq j$. This condition requires a consistency across consumers in the rate of change of their marginal utilities with respect to consumption in the same period.

**Theorem 4.16.** Let $\mathcal{E}_\omega$ be a smooth myopic economy in which consumers’ utility functions $U_i(x_i)$ are consistent and satisfy the strong survival condition. Suppose that for every $\underline{x}, \bar{x} \in \text{int } \ell^m_{\infty+}$:

i. $U_i(x_i)$ is uniformly concave on $[\underline{x}, \bar{x}]$;

ii. $D^2 U_i(x_i)$ has a uniformly dominant diagonal on $[\underline{x}, \bar{x}]$;

iii. there exists $n \in (0, 1)$ such that $\|O_i(x_i)\| \leq \frac{n}{n+1}$ for all $i$ and $x \in [\underline{x}, \bar{x}]$.

Then $\mathcal{E}_\omega$ is a Lipschitz economy.

**Proof:** I will show that the Pareto map is locally Lipschitz. Let $U \in D^0$ be given and let $V$ be a neighborhood of $U$ such that $V \subset D^0$. By the strong survival condition, there exist $\underline{x}, \bar{x} \in \text{int } \ell^m_{\infty+}$ such that $\underline{x} \leq \bar{x}$ and $x(V) \subset [\underline{x}, \bar{x}]$. Since $\lambda(U)$ is continuous (see Shannon (1994b)), there exist $\Delta, \lambda \in \mathbb{R}^m_{\infty+}$ such that $\lambda(\bar{V}) \subset [\Delta, \lambda]$.

Then since $U_i(x_i)$ is uniformly concave for each $i$, it suffices to show that the operator $S(x) = I + [\lambda_2 D^2 U_2(x_2)]^{d_1+d_2]^{-1}} D^2 U_1(x_1)^{d_1+d_2]^{-1}} \cdots + [\lambda_m D^2 U_m(x_m)]^{d_1+d_m]^{-1}} D^2 U_1(x_1)^{d_1+d_m]^{-1}}$ is uniformly bounded below on $[\underline{x}, \bar{x}] \times [\Delta, \lambda]$.

By the consistency assumption, on the interval $[\underline{x}, \bar{x}]$, $\left\{ \frac{d^2 U_i(x_i)}{d^2 U_j(x_j)} \right\}$ is uniformly bounded and uniformly bounded away from 0 for all $k = 1, \ldots, m$. Now consider $S(x)$.

$$S(x) = \left( [D^2 U_1(x_1)]^{d_1+d_2]^{-1}} + [\lambda_2 D^2 U_2(x_2)]^{d_1+d_2]^{-1}} \right) \cdots + [\lambda_m D^2 U_m(x_m)]^{d_1+d_m]^{-1}} \left( [D^2 U_1(x_1)]^{d_1+d_m]^{-1}} \right)$$

But $D^2 U_1(x_1)^{d_1+d_2} = D^2 U_1(x_1)^{d_1+d_2}$ is uniformly bounded below. Thus it suffices to show that $S(x)$ is uniformly bounded below, where

$$S(x) = [D^2 U_1(x_1)]^{d_1+d_2]^{-1}} + [\lambda_2 D^2 U_2(x_2)]^{d_1+d_2]^{-1}} \cdots + [\lambda_m D^2 U_m(x_m)]^{d_1+d_m]^{-1}} \left( [D^2 U_1(x_1)]^{d_1+d_m]^{-1}} \right)$$
First note that
\[
D^2 U_1(x_1)^{d_1 + d_2} [D^2 U_1(x_1)]^{d_1 + d_2 - 1} = D^2 U_1(x_1)^{d_1} \text{diag} \left\{ \frac{d_{i_1}(x_1)}{d_{i_1}(x_1) + d_{i_1}^*(x_k)} \right\} \left[ D^2 U_1(x_1)^{d_1} \text{diag} \left\{ \frac{d_{i_1}^*(x_1)}{d_{i_1}^*(x_1) + d_{i_1}^*(x_2)} \right\} \right]^{-1}
\]
\[
= D^2 U_1(x_1)^{d_1} \text{diag} \left\{ \frac{d_{i_1}(x_1) + d_{i_1}^*(x_2)}{d_{i_1}(x_1) + d_{i_1}^*(x_k)} \right\} [D^2 U_1(x_1)^{d_1}]^{-1}
\]
\[
= \text{diag} \left\{ \frac{d_{i_1}^*(x_1) + d_{i_1}^*(x_2)}{d_{i_1}^*(x_1) + d_{i_1}^*(x_k)} \right\}.
\]
Moreover, similar reasoning shows that
\[
[D^2 U_k(x_k)^{d_1 + d_2}]^{-1} D^2 U_1(x_1)^{d_1 + d_2} [D^2 U_1(x_1)]^{d_1 + d_2 - 1}
\]
\[
= \left[ D^2 U_k(x_k)^{d_2} \text{diag} \left\{ \frac{d_{i_2}^*(x_k)}{d_{i_2}^*(x_1) + d_{i_2}^*(x_k)} \right\} \right]^{-1} \text{diag} \left\{ \frac{d_{i_1}^*(x_1) + d_{i_1}^*(x_2)}{d_{i_1}^*(x_1) + d_{i_1}^*(x_k)} \right\}
\]
\[
= [D^2 U_k(x_k)^{d_2}]^{-1} \text{diag} \left\{ \frac{d_{i_1}^*(x_1) + d_{i_1}^*(x_2)}{d_{i_1}^*(x_k)} \right\}.
\]
Putting this together gives
\[
\hat{S}(x) = [D^2 U_1(x_1)^{d_1 + d_2}]^{-1} + \left[ \lambda_2 D^2 U_2(x_2)^{d_1 + d_2} \right]^{-1} + \left[ \lambda_3 D^2 U_3(x_3)^{d_1 + d_2} \right]^{-1} \text{diag} \left\{ \frac{d_{i_1}^*(x_1) + d_{i_1}^*(x_2)}{d_{i_1}^*(x_3)} \right\}
\]
\[
+ \cdots + \left[ \lambda_m D^2 U_m(x_m)^{d_1 + d_2} \right]^{-1} \text{diag} \left\{ \frac{d_{i_1}^*(x_1) + d_{i_1}^*(x_2)}{d_{i_1}^*(x_m)} \right\}.
\]
Also, note that
\[
[D^2 U_1(x_1)^{d_1 + d_2}]^{-1} = [D^2 U_1(x_1)^{d_1}]^{-1} \text{diag} \left\{ \frac{d_{i_1}^*(x_1) + d_{i_1}^*(x_2)}{d_{i_1}^*(x_1)} \right\}
\]
and
\[
[D^2 U_2(x_2)^{d_1 + d_2}]^{-1} = [D^2 U_2(x_2)^{d_2}]^{-1} \text{diag} \left\{ \frac{d_{i_2}^*(x_1) + d_{i_2}^*(x_2)}{d_{i_2}^*(x_1)} \right\}
\]
so letting \( \lambda_1 = 1 \),
\[
\hat{S}(x) = \sum_{i=1}^{m} \left[ \lambda_i D^2 U_i(x_i)^{d_i} \right]^{-1} \text{diag} \left\{ \frac{d_{i_1}^*(x_1) + d_{i_1}^*(x_2)}{d_{i_1}^*(x_i)} \right\}.
\]
But by assumption, for each \( k, D^2 U_k(x_k)^{d_k} = I + O_k(x_k) \), where \( ||O_k(x_k)|| < 1 \). So (see, e.g., DeVito (1990, Thm. 2 p. 83))
\[
[D^2 U_k(x_k)^{d_k}]^{-1} = \sum_{r=0}^{\infty} (O_k(x_k))^r = I + \sum_{r=1}^{\infty} (O_k(x_k))^r.
\]
Then,
\[
\| \sum_{r=1}^{\infty} (O_k(x_k))^r \| \leq \sum_{r=1}^{\infty} \| O_k(x_k) \|^r \\
= \frac{\| O_k(x_k) \|}{1 - \| O_k(x_k) \|} \\
\leq n < 1
\]
for each \( k \) by assumption (iii). Thus for each \( k \), \([D^2 U_k(x_k)^{d_k}]^{-1}\) has a uniformly dominant diagonal which is positive and uniformly bounded away from 0. Furthermore, for each \( k \), \( \{ \frac{\partial^2 h(x_k)}{\partial x^2} \} \) is positive and uniformly bounded away from 0, so for each \( k \), \( \lambda_k D^2 U_k(x_k)^{d_k} \) has a strictly positive uniformly dominant diagonal which is uniformly bounded away from 0. Thus \( \hat{S}(x) \) has a strictly positive diagonal which is uniformly bounded away from 0, i.e., \( \hat{S}(x) \) has a uniformly dominant diagonal by Theorem 4.12. Thus \( x(\hat{U}) \) is locally Lipschitz.

In an economy with habit formation preferences and more than two consumers, this result will imply that equilibria are generically determinate under the additional assumption that \( u_i(c_1, c_2) \) has a strongly dominant diagonal, i.e., that for each \( (c_1, c_2) \) there exists \( n \in (0, 1) \) such that \( |u_{12}(c_1, c_2)| \leq \frac{n}{n+1} |u_{kk}(c_1, c_2)| \) for each \( k = 1, 2 \).

**Theorem 4.17.** Suppose \( U_i(x) = v_i(x_0) + \sum_{t=1}^{\infty} \beta^t u_i(x_{t-1}, x_t) \), where \( 0 < \beta < 1, u_i, v_i \) are \( C^2 \), \( v'_i(c) > 0, v''_i(c) < 0 \) for every \( c \in R_+ \), \( D^2 u_i(c_1, c_2) \) is negative definite for every \( (c_1, c_2) \in R_+^2 \) and has a strongly dominant diagonal, and \( u_i \) satisfies the boundary condition for habit formation. Then the economy \( E \) is a Lipschitz economy.

**Proof:** First, note that consumers' utility functions are consistent in this economy, since
\[
\begin{align*}
\left\{ \frac{\partial h(x_i)}{\partial x_i} \right\} &= \left\{ \frac{\partial h(x_{i-1}, x_i)}{\partial x_{i-1}}, \frac{\partial h(x_i)}{\partial x_i} \right\} \\
&= \left\{ \frac{\partial h(x_{i-1}, x_i)}{\partial x_{i-1}}, \frac{\partial h(x_i)}{\partial x_i} \right\}
\end{align*}
\]
which is bounded and bounded away from 0 on \([\underline{x}, \overline{x}] \subset \ell_{\infty+} \). Then by Theorems 4.15 and 4.16 it suffices to show that for any \( \underline{x}, \overline{x} \in \ell_{\infty+} \) such that \( \underline{x} \leq \overline{x} \), there exists \( n \in (0, 1) \) such that \( \| O_i(x_i) \| \leq \frac{n}{n+1} \) for all \( x_i \in [\underline{x}, \overline{x}] \). Given such vectors \( \underline{x}, \overline{x} \), let \( \bar{b} = \inf \underline{x} \) and \( \underline{b} = -\| \overline{z} \| \). Then since \( D^2 u_i \) has a strongly dominant diagonal and is continuous, there exists \( n \in (0, 1) \) such that
\[
\begin{align*}
|u_{12}(c_1, c_2)| &\leq \frac{n}{n+1} |u_{11}(c_1, c_2)| \\
|u_{12}(c_1, c_2)| &\leq \frac{n}{n+1} |u_{11}(c_1, c_2)|
\end{align*}
\]
for all \((c_1, c_2), (\hat{c}_1, \hat{c}_2) \in [\underline{b}, \bar{b}] \times [\underline{b}, \bar{b}] \). Then as in the proof of Theorem 4.15,
\[
\| O_i(x_i) \| \leq \sum_{x \neq k} \left| \frac{\partial^2 h(x)}{\partial x^2} \right| = \frac{|\beta^t u_{11}(x_{i-1}, x_i)| + |\beta^{t+1} u_{12}(x_{i-1}, x_i)|}{\beta^t |u_{22}(x_{i-1}, x_i)| + |\beta^{t+1} u_{11}(x_{i+1}, x_{i+1})|}
\]

\(^{18}\)Here subscripts denote partial derivatives, and superscripts agents.
\[ \leq \frac{\beta^{t-1} u_{22}(x_{i_{t-1}}, x_{i_{t}})}{\beta^{t} u_{22}(x_{i_{t-1}}, x_{i_{t}})} + \frac{\beta^{t+1} \frac{n}{n+1} u_{11}(x_{i_{t}}, x_{i_{t+1}})}{\beta^{t} u_{22}(x_{i_{t-1}}, x_{i_{t}})} + \frac{\beta^{t+1} u_{11}(x_{i_{t}}, x_{i_{t+1}})}{\beta^{t} u_{22}(x_{i_{t-1}}, x_{i_{t}})} = \frac{n}{n+1} \]

Then the result follows from Theorem 4.16.

5 Concluding Remarks

This paper has developed a methodology for studying determinacy in general infinite horizon exchange economies by defining a natural notion of regular economy in this infinite horizon setting for which results analogous to those which hold for classical finite-dimensional economies can be established. Regular Lipschitz economies have a finite number of equilibria, each of which is locally stable with respect to perturbations in exogenous parameters. Moreover, for a Lipschitz economy, regular economies are generic in the set of economies parameterized by initial endowments. Finally, this paper establishes conditions on the economic primitives, consumer preferences, which guarantee that an economy is a Lipschitz economy, and thus that the economy has generically determinate equilibria, both by relying on dynamic programming techniques in models with stationary preferences, and by establishing a Lipschitz implicit function theorem for economies with more general preferences.
6 Appendix

Theorem 2.2. If $U(x) = \sum_{i=0}^{\infty} \beta^i u(x_i)$, where $0 < \beta < 1$ and $u : \mathcal{R}_+ \to \mathcal{R}$ is $C^2$, $u'(r) > 0$ and $u''(r) < 0$ for every $r \in \mathcal{R}_+$, then $U(x)$ is strictly monotone, strictly concave, Mackey continuous, and $C^2$ on int $\ell_{\infty+}$. Moreover, $DU(x) \in \ell_{1+}$, if $\{x^n\} \subset \text{int} \ell_{\infty+}$ is bounded then $\text{D}U(x^n) \not\rightarrow 0$ pointwise, and $D^2 U(x)$ is negative definite for every $x \in \text{int} \ell_{\infty+}$.

Proof: That $U(x)$ is $\tau(\ell_{\infty}, \ell_1)$ continuous follows from Bewley (1972, Appendix II), and clearly $U(x)$ is strictly concave and strictly monotone. Then we claim that $U$ is $C^1$ on int $\ell_{\infty+}$, and given $x \in \text{int} \ell_{\infty+}$, $DU(x) = \{\beta^i u'(x_i)\} \in \ell_{1+}$. To see this, note that by definition it suffices to show that

$$\lim_{\|h\| \to 0} \frac{\sum_{i=0}^{\infty} [\beta^i u(x_i + h_i) - \beta^i u(x_i) - \beta^i u'(x_i)h_i]}{\|h\|} = 0.$$ 

But

$$\frac{\sum_{i=0}^{\infty} [\beta^i u(x_i + h_i) - \beta^i u(x_i) - \beta^i u'(x_i)h_i]}{\|h\|} \leq \frac{\sum_{i=0}^{\infty} \beta^i |u(x_i + h_i) - u(x_i) - u'(x_i)h_i|}{\|h\|} \leq \frac{\sum_{i=0}^{\infty} \beta^i |u(x_i + h_i) - u(x_i) - u'(x_i)h_i|}{h_i},$$

by definition of $\|h\|$. Then for every $\delta > 0$, $\{x + h : \|h\| < \delta\} \subset [x - \delta e, x + \delta e]$, where $e = (1, 1, 1, \ldots)$. Since $x \in \text{int} \ell_{\infty+}$, $\xi \equiv \inf x_i > 0$. Then $[x - \delta e, x + \delta e] \subset [(\xi - \delta)e, (\|x\| + \delta)e]$. Choose $\eta > 0$ such that $\xi - \eta > 0$. On $[(\xi - \eta), (\|x\| + \eta)]$, $u'$ exists and is continuous, and thus there exists $M > 0$ such that $|u'(z)| \leq M$ for every $z \in [\xi - \eta, \|x\| + \eta]$. Then given $\epsilon > 0$, choose $T$ such that $\sum_{T}^{\infty} \beta^t < \epsilon/2M$. Since for every $t$,

$$\frac{|u(x_i + h_i) - u(x_i) - u'(x_i)h_i|}{h_i} \to 0,$$

for $t = 1, \ldots T$ there exists $\delta_t > 0$ such that for $|h_i| < \delta_t$,

$$\frac{|u(x_i + h_i) - u(x_i) - u'(x_i)h_i|}{h_i} < \frac{\epsilon}{1 - \beta}. $$

Set $\delta = \min\{\eta, \delta_t : t = 1, \ldots T\} > 0$. For $\|h\| < \delta$,

$$\frac{\sum_{i=0}^{\infty} [\beta^i u(x_i + h_i) - \beta^i u(x_i) - \beta^i u'(x_i)h_i]}{\|h\|} \leq \frac{\sum_{i=0}^{\infty} \beta^i |u(x_i + h_i) - u(x_i) - u'(x_i)h_i|}{h_i} \leq \frac{\sum_{t=0}^{T-1} \beta^t |u(x_i + h_i) - u(x_i) - u'(x_i)h_i|}{h_i} + \frac{\sum_{t=T}^{\infty} \beta^t |u(x_i + h_i) - u(x_i) - u'(x_i)h_i|}{h_i}.$$
\[
\leq \sum_{t=0}^{T-1} \beta^t \left| \frac{u(x_t + h_t) - u(x_t) - u'(x_t)h_t}{h_t} \right| + \sum_{t=T}^{\infty} \beta^t \left| \frac{u'(r_t)h_t - u'(x_t)h_t}{h_t} \right|,
\]
for some \( r_t \in (x_t, x_t + h_t) \),

\[
< \epsilon + 2M \cdot \frac{\epsilon}{2M} = 2\epsilon.
\]

So \( DU(x) = \{ \beta^t u'(x_t) \} \) for every \( x \in \ell_{\infty+} \). Since \( x \in \ell_{\infty+} \) and \( u' \) is continuous and strictly positive, \( DU(x) \in \ell_{1+} \). Moreover, if \( \{ x^n \} \subset \ell_{\infty+} \) is bounded, so \( \| x^n \| \leq b \) and \( x^n \leq \overline{b} = (b, b, \ldots) \), then since \( u(c) \) is strictly concave, \( DU(x^n) \geq DU(\overline{b}) \gg 0 \), i.e., \( DU(x^n) \neq 0 \) pointwise.

To show \( DU(x) \) is continuous, suppose \( \{ x^n \} \in \ell_{\infty+} \) and \( x^n \to x \). We must show that \( DU(x^n) \to DU(x) \) in \( \ell_1 \), i.e., that

\[
\lim_{n \to \infty} \sum_{t=0}^{\infty} |\beta^t u'(x^n_t) - \beta^t u'(x_t)| = 0.
\]

Then let \( \epsilon > 0 \) be given, and let \( 0 < \eta < \varepsilon \). Since \( \| x^n - x \| \to 0 \), there exists \( N > 0 \) such that \( n \geq N \Rightarrow \| x^n - x \| < \eta \), and thus \( x^n \in [x - \eta, x + \eta] \) for \( n \geq N \). By using the same argument as above, this implies that \( x^n \in [(x - \eta)e, (\| x \| + \eta)e] \) for all \( t \) and \( n \geq N \). Since \( u' \) is continuous, there exists \( M > 0 \) such that for \( n \geq N \), \( |u'(x^n_t)| \leq M \) for all \( t \) and \( |u'(x_t)| \leq M \) for all \( t \). Choose \( T \) such that

\[
\sum_{t=\infty}^{\infty} \beta^t < \varepsilon/2M.
\]

Then

\[
\sum_{t=0}^{\infty} |\beta^t u'(x^n_t) - \beta^t u'(x_t)| = \sum_{t=0}^{T} \beta^t |u'(x^n_t) - u'(x_t)| + \sum_{t=T+1}^{\infty} \beta^t |u'(x^n_t) - u'(x_t)|.
\]

Then for \( t = 1, \ldots, T \), there exists \( \delta_t > 0 \) such that \( |x - x_t| < \delta_t \Rightarrow |u'(z) - u'(x_t)| < \varepsilon/(1 - \beta) \), by the continuity of \( u' \). Let \( \delta = \min \{ \delta_1, \ldots, \delta_T, \eta \} \) and choose \( N' \geq N \) such that \( n \geq N' \Rightarrow \| x^n - x \| < \delta \). Then \( n \geq N' \Rightarrow \)

\[
\sum_{t=0}^{\infty} |\beta^t u'(x^n_t) - \beta^t u'(x_t)| \leq \sum_{t=0}^{T} \beta^t |u'(x^n_t) - u'(x_t)| + \sum_{t=T+1}^{\infty} \beta^t |u'(x^n_t) - u'(x_t)| \leq \frac{\varepsilon}{1 - \beta} \sum_{t=0}^{T} \beta^t + 2M \sum_{t=T+1}^{\infty} \beta^t \leq \varepsilon + \varepsilon = 2\varepsilon.
\]

Thus \( \lim_{n \to \infty} \sum_{t=0}^{\infty} |\beta^t u'(x^n_t) - \beta^t u'(x_t)| = 0 \).

Now we claim that \( D^2 U(x) : \ell_\infty \to \ell_1 \) is the operator such that for \( h \in \ell_\infty \), \( D^2 U(x)h = \{ \beta^t u''(x_t)h_t \} \). To establish this, we must show that

\[
\lim_{\| h \| \to 0} \sum_{t=0}^{\infty} \frac{|\beta^t u'(x_t + h_t) - \beta^t u'(x_t) - \beta^t u''(x_t)h_t|}{\| h \|} = 0.
\]
A straightforward variation of the argument given above to show that $DU(x) = \{ \beta^t u'(x_t) \}$ will show that this holds. Moreover, $D^2 U(\cdot)$ is continuous. To see this, let $\{x^n\} \in \text{int } \ell_{\infty}^+$ be such that $x^n \to x$. We must show that $\|D^2 U(x^n) - D^2 U(x)\| \to 0$. Let $z \in \ell_\infty$ be such that $\|z\| \leq 1$ and consider $(D^2 U(x^n) - D^2 U(x))z = \{ \beta^t (u''(x^n_t) - u''(x_t))z_t \}$.

$$\|(D^2 U(x^n) - D^2 U(x))z\| = \sum_{t=0}^{\infty} \beta^t \|u''(x^n_t) - u''(x_t)\|z_t|$$

$$\leq \sum_{t=0}^{\infty} \beta^t \|u''(x^n_t) - u''(x_t)\| |z_t|$$

$$\leq \|z\| \sum_{t=0}^{\infty} \beta^t \|u''(x^n_t) - u''(x_t)\|.$$ 

Thus $\|D^2 U(x^n) - D^2 U(x)\| \leq \sum_{t=0}^{\infty} \beta^t \|u''(x^n_t) - u''(x_t)\| \to 0$ by the same argument given above to show that $DU(x)$ is continuous.

Finally, we must show that $D^2 U(x)$ is negative definite. If $z \in \ell_\infty$ and $z \neq 0$, then $z^T D^2 U(x)z = \sum_{t=0}^{\infty} \beta^t u''(x^t_t)(z_t)^2 < 0$, so $D^2 U(x)$ is negative definite. 

\[\blacksquare\]

**Lemma 2.9.** If $E_\omega$ is an economy with smooth myopic preferences and the Pareto map $x(U)$ is locally Lipschitz, then $F(U)$ is locally Lipschitz on $\Omega$.

**Proof:** Since $DU_1(x)$ is $C^1$ for $x \in \text{int } \ell_{\infty}^+$, $DU_1(x)$ is locally Lipschitz on $\text{int } \ell_{\infty}^+$. Since $x_1(U) \in \text{int } \ell_{\infty}^+$ for $U \in \Omega$ and $x(U)$ is locally Lipschitz, $DU_1(x_1(U))$ is locally Lipschitz on $\Omega$. Similarly, $p_1(U) = (DU_1(x_1(U)))_1$ is locally Lipschitz on $\Omega$. Now let $U \in \Omega$, and choose a neighborhood $W$ of $U$ such that $W$ is bounded and $\overline{W} \subset \Omega$, and on which $p_1(\cdot)$ and $DU_1(x_1(U))$ are Lipschitz with constants $K_p$ and $K_U$. Then there exists $C > 0$ such that $|p_1(U)| \leq C$ and $\|DU_1(x_1(U))\| \leq C$ for every $U \in \overline{W}$, and hence for every $U \in W$. Moreover, $p_1(U) > 0$ for every $U \in \Omega$, so there exists $c > 0$ such that $p_1(U) \geq c$ for every $U \in \overline{W}$, and hence for every $U \in W$. Then if $U^1, U^2 \in W$,

$$\left\| \frac{DU_1(x_1(U^1))}{p_1(U^1)} - \frac{DU_1(x_1(U^2))}{p_1(U^2)} \right\|$$

$$= \left\| \frac{DU_1(x_1(U^1))}{p_1(U^1)} - \frac{DU_1(x_1(U^2))}{p_1(U^1)} + \frac{DU_1(x_1(U^2))}{p_1(U^1)} - \frac{DU_1(x_1(U^2))}{p_1(U^2)} \right\|$$

$$\leq \frac{1}{p_1(U^1)} \|DU_1(x_1(U^1)) - DU_1(x_1(U^2))\| + \|DU_1(x_1(U^2))\| \left\| \frac{1}{p_1(U^1)} - \frac{1}{p_1(U^2)} \right\|$$

$$= \frac{1}{p_1(U^1)} \|DU_1(x_1(U^1)) - DU_1(x_1(U^2))\| + \|DU_1(x_1(U^2))\| \left\| \frac{p_1(U^2) - p_1(U^1)}{p_1(U^1)p_1(U^2)} \right\|$$

$$\leq \frac{1}{c} K_U \|U^1 - U^2\| + \frac{C}{c} \|p_1(U^2) - p_1(U^1)\|$$

$$\leq K \|U^1 - U^2\|,$$

where $K = \frac{1}{c} K_U + \frac{C}{c} K_p$. So $DU_1(x_1(U))$ is locally Lipschitz on $\Omega$. Now it suffices to show that since $x_j(U)$ is locally Lipschitz on $\Omega$, then $DU_1(x_1(U)) \cdot x_j(U)$ is locally Lipschitz on $\Omega$. 

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as well. Let \( U \in \Omega \) and choose a bounded neighborhood \( W \) of \( U \) as before such that \( \overline{W} \subset \Omega \), and on which \( \hat{D}U_1(z_1(U)) \) and \( x_j(U) \) are Lipschitz with constants \( L_U \) and \( L_x \). There exists \( M > 0 \) such that \( \| \hat{D}U_1(z_1(U)) \| \leq M \) and \( \| x_j(U) \| \leq M \) for every \( U \) in \( \overline{W} \) and hence for every \( U \in W \). Let \( U^1, U^2 \in W \).

\[
\begin{align*}
\| \hat{D}U_1(z_1(U^1)) \cdot x_j(U^1) - \hat{D}U_1(z_1(U^2)) \cdot x_j(U^2) \| \\
\leq \| \hat{D}U_1(z_1(U^1)) \cdot x_j(U^1) - \hat{D}U_1(z_1(U^1)) \cdot x_j(U^2) \| \\
+ \| \hat{D}U_1(z_1(U^1)) \cdot x_j(U^2) - \hat{D}U_1(z_1(U^2)) \cdot x_j(U^2) \| \\
\leq ML_x ||U^1 - U^2|| + ML_U ||U^1 - U^2|| \\
= L ||U^1 - U^2||.
\end{align*}
\]

Thus \( F(U) \) is locally Lipschitz on \( \Omega \).

**Theorem 3.1.** For every \( U \in I_2 \), if \( x(U) \) solves the Pareto maximization problem and \( (c(U), y(U)) \) solves the recursive maximization problem, then \( x_t(U) = c(y^{-1}(U)) \) for each \( t \).

**Proof:** First, the functions \( v(U) \) and \( \hat{U}_1(U) \) are equivalent. To show this, note that both maximization problems have a unique solution for each \( U \in I_2 \), thus both \( v(U) \) and \( \hat{U}_1(U) \) are well-defined for each \( U \in I_2 \). To show these functions are equivalent, it suffices to show that \( \hat{U}_1(U) \) is also a fixed point of the operator \( T \), i.e. that

1. \( \hat{U}_1(U) \geq w_1(c, \hat{U}_1(y)) \forall (c, y) \) such that \( w_2(1 - c, y) \geq U \) and
2. for every \( \epsilon > 0 \), \( \hat{U}_1(U) \leq w_1(c, \hat{U}_1(y)) + \epsilon \) for some \( (c, y) \) such that \( w_2(1 - c, y) \geq U \).

To see that (1) holds, let \( (c, y) \) satisfying \( w_2(1 - c, y) \geq U \) be given, and let \( \epsilon > 0 \) be given. Then by definition of \( \hat{U}_1 \), there exists \( x \) such that \( U_2(1 - 2x) \geq y \) and such that \( U_1(2x) \geq U_1(y) - \epsilon \). Let \( x = (c, 2x) \). Then

\[
U_2(1 - x) = w_2(1 - c, U_2(1 - 2x)) \\
\geq w_2(1 - c, y) \geq U.
\]

So

\[
\hat{U}_1(U) \geq U_1(x) = w_1(c, U_1(2x)) \\
\geq w_1(c, U_1(y) - \epsilon).
\]

Since \( \epsilon > 0 \) was arbitrary, \( \hat{U}_1(U) \geq w_1(c, \hat{U}_1(y)) \).

To see that (2) holds, let \( U \) and \( \epsilon > 0 \) be given. There exists \( x \) satisfying \( U_2(1 - x) \geq U \) such that

\[
\hat{U}_1(U) \leq U_1(x) + \epsilon = w_1(x_1, U_1(2x)) + \epsilon.
\]

Set \( y = U_2(1 - 2x) \). Then \( \hat{U}_1(y) \geq U_1(2x) \), so

\[
\hat{U}_1(U) \leq w_1(x_1, U_1(2x)) + \epsilon \leq w_1(x_1, \hat{U}_1(y)) + \epsilon,
\]

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and by construction,

\[
w_2(1 - x, y) = w_2(1 - x_1, U_2(1 - 2x)) = U_2(1 - x) \geq U.
\]

Thus \(v(U) = \hat{U}_1(U)\).

Now we must show that if \(x(U)\) solves the Pareto problem, then \(x_t(U) = c(y^{t-1}(U))\) for every \(t\). First note that if \(x(U) = x^*\) solves the Pareto problem, then for every \(t\), \(x^*\) solves the problem \((P_t)\):

\[
\begin{align*}
\max & \quad U_1(tx) \\
\text{s.t.} & \quad U_2(1 - tx) \geq U_2(1 - tx^*).
\end{align*}
\]

Then it suffices to show that \((x_1^*, U_2(1 - 2x^*))\) solves the original recursive problem. First, \((x_1^*, U_2(1 - 2x^*))\) is feasible in the recursive problem, since \(w_2(1 - x_1^*, U_2(1 - 2x^*)) = U_2(1 - x^*) \geq U\). So \(v(U) \geq w_1(x_1^*, v(U_2(1 - 2x^*)))\). But \(v(\cdot) = \hat{U}_1(\cdot)\), and \(2x^*\) solves \((P_2)\), so \(v(U_2(1 - 2x^*)) = \hat{U}_1(U_2(1 - 2x^*)) = \hat{U}_1(2x^*)\). Thus

\[
v(U) \geq w_1(x_1^*, U_2(1 - 2x^*)) = \hat{U}_1(x^*) = \hat{U}_1(U).
\]

But again \(\hat{U}_1(U) = v(U)\), so \(v(U) = w_1(x_1^*, v(U_2(1 - 2x^*)))\). By uniqueness of the solution to the recursive problem, \((x_1^*, U_2(1 - 2x^*)) = (c(U), y(U))\), i.e., \(x_1^* = c(U)\). Now by repeating this argument, \(x_t^* = c(y^{t-1}(U))\) for every \(t\).

**Lemma 3.2.** The value function \(v(U)\) is \(C^1\) and strictly concave. The functions \(c(U)\) and \(y(U)\) are Lipschitz continuous, \(c(U)\) is nonincreasing, \(y(U)\) is nondecreasing, and for all \(t\) and \(U\), \(y^{t+1}(U) \geq y(U)\).

**Proof:** That \(v(U)\) is strictly concave follows from the fact that \(v(U) \equiv \hat{U}_1(U)\), which is strictly concave by Theorem 2.1 of Shann (1994b). Given any strictly concave, \(C^\infty\) starting point \(v_0\), the sequence \(v_n = \hat{T}v_{n-1}\) converges uniformly to \(v\). Thus \(f_n(c, y) \equiv w_1(c, v_n(y))\) converges uniformly to \(f(c, y) \equiv w_1(c, v(y))\), and since each \(v_n\) is strictly concave and \(w_1\) is strictly concave and strictly increasing, \(f_n(c, y)\) is strictly concave in \(y\) for each \(n\), and since \(v\) is strictly concave, \(f\) is strictly concave in \(y\). The constraint correspondence \(\{y \in I_2, c \in [0, 1] : w_2(1 - c, y) \geq U\}\) is compact and convex-valued, and continuous, thus by Theorem 3.8 of Stokey and Lucas (1989), \(c_n \rightarrow c\) uniformly and \(y_n \rightarrow y\) uniformly, where

\[
(c_n(U), y_n(U)) = \arg \max_{c \in [0, 1]} \max_{y \in I_2} \ w_1(c, v_n(y)) \\
\text{s.t.} \quad w_2(1 - c, y) \geq U.
\]

To show that \(c(U)\) and \(y(U)\) are Lipschitz, let \(v_0 : R \rightarrow R\) be an arbitrary \(C^\infty\) strictly concave function. It is routine to show that \(v_1 = \hat{T}v_0\) is strictly concave and \(C^\infty\), and that \(c_0(U)\) and \(y_0(U)\) are also \(C^\infty\). Also, a routine calculation using Cramer's rule shows that

\[
c'_0(U) = -\frac{u_2'(1 - c_0)[g'(y_0)](v'_0(y_0)) + g'(y_0)v'_0(y_0)}{\beta^2(u_1''(c_0) - \lambda_0 u_2''(1 - c_0) + (u_2'(1 - c_0))^2[2g''(y_0)](v_0(y_0))^2 + g'(y_0)v''_0(y_0))}
\]

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and
\[ y'_0(U) = \frac{\beta (u'_1(c_0) - \lambda_0 u'_2(1 - c_0))}{\beta^2(u'_1(c_0) - \lambda_0 u'_2(1 - c_0)) + (u'_2(1 - c_0))^2 [g''(v_0(y_0))(v'_0(y_0))^2 + g''(v_0(y_0))v'_0(y_0)]}. \]

First, note that
\[ |c'_0(U)| = \frac{1}{\frac{\beta^2(u'_1(c_0) - \lambda_0 u'_2(1 - c_0))}{u'_2(1 - c_0)[g''(v_0(y_0))(v'_0(y_0))^2 + g''(v_0(y_0))v'_0(y_0)] + u'_2(1 - c_0)}} \leq \frac{1}{u'_2(1 - c_0)} \leq \frac{1}{u'_2(1)}. \]

So \(|c'_0(U)|\) is uniformly bounded, and this bound is independent of \(v_0\). Repeating the same argument then shows that for every \(n\), \(c_n\) is Lipschitz with constant no greater than \(M = \frac{1}{u'_2(1)}\). Since \(c_n \to c\) uniformly, \(c(U)\) is also Lipschitz with constant no greater than \(M\). Moreover, note that \(c'_n(U) \leq 0\) for all \(U\) and \(n\), i.e., \(c_n\) is nonincreasing for every \(n\), so \(c\) must also be nonincreasing.

Similarly,
\[ 0 \leq y'_0(U) = \frac{1}{\beta + \frac{u'_2(1 - c_0)[g''(v_0(y_0))(v'_0(y_0))^2 + g''(v_0(y_0))v'_0(y_0)]}{\beta(u'_1(c_0) - \lambda_0 u'_2(1 - c_0))}} \leq \frac{1}{\beta}. \]

Repeating the same argument for each \(n\) shows that for each \(n\), \(y_n\) is nondecreasing and Lipschitz with constant no greater than \(\frac{1}{\beta}\). Thus \(y\) is also nondecreasing and Lipschitz with constant no greater than \(\frac{1}{\beta}\).

To see that \(v\) is \(C^1\), note that \(v_n\) is \(C^\infty\) for each \(n\), and that
\[ v'_{n+1}(U) = \lambda_n(U) = -\frac{u'_1(c_n(U))}{u'_2(1 - c_n(U))}, \]

where \(\lambda_n(U)\) is the Lagrange multiplier for the \(n\)th iteration of the recursive problem. Choose an interval \([L', U]\) such that \(L' > 0\) and \(\bar{U} < U_2(1)\). To show that \(v\) is \(C^1\), it suffices to show that \(v'_n\) converges uniformly (see, e.g., Rudin (1976), Thm. 7.17). To see this, note that for each \(U\).
\[ 1 > \bar{c} \equiv c(U) \geq c(U) \geq c(\bar{U}) \equiv c > 0. \]

If we choose \(\epsilon\) such that \(1 > \bar{c} + \epsilon\) and \(c - \epsilon > 0\), then for sufficiently large \(n\), \(1 > \bar{c} + \epsilon \geq c_n(U) \geq c - \epsilon > 0\) for all \(n\) and \(U\). Then since \(c_n \to c\) uniformly and \(u'_1(c_n(U))\) and \(u'_2(1 - c_n(U))\) are bounded by the above argument, \(\lambda_n(U) \to \lambda(U)\) uniformly, where
\[ \lambda(U) \equiv \frac{u'_1(c(U))}{u'_2(1 - c(U))}. \]

Thus \(v\) is \(C^1\), and \(v'(U) = \lambda(U)\).

Finally, to show that \(y^{t+1}(U) \geq y^t(U)\) for all \(t\) and \(U\), note that from the first order conditions for the recursive problem,
\[ v'(y(U)) = -\frac{\beta u'_1(c(U))}{g'(v(y(U)))u'_2(1 - c(U))} = \frac{\beta}{g'(v(y(U)))} \lambda(U). \]

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Since \( \beta \geq g'(r) \) for all \( r \). \( v'(y(U)) \leq v'(U) = \lambda(U) \). This implies that \( y(U) \geq U \) since \( v \) is concave. Then since \( y \) is nondecreasing, \( y^2(U) \geq y(U) \), and repeating this argument shows that for all \( t \) and \( U \), \( y^{t+1}(U) \geq y^t(U) \).

References


